Summary of my research

Jairo Bochi

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This is a commented list of my papers, in approximate chronological order (which not necessarily agrees with the order of publication). The papers are available in my homepage.

Paper 1

Discontinuity of the Lyapunov exponent for non-hyperbolic cocycles. Unpublished preprint *(around 2000).*

I prove the following result: *A C^0-generic* $\mathrm{SL}(2,\mathbb{R})$*-cocycle either is uniformly hyperbolic or has zero Lyapunov exponents Lebesgue almost everywhere.*

This nice paper was kept hidden for a while, because I was on the track of proving the more important diffeomorphism version, which I did in paper [3]. Since the method of [3] was more flexible and also yielded the cocycle result above, I eventually lost interest in publishing the preprint. (But later me and other people found some applications of [1]'s method, which was inspired in some ideas of Knill.)

Paper 2 (*with A. Avila*)

A formula with some applications to the theory of Lyapunov exponents. Israel Journal of Mathematics*, 131 (2002), 125–137.*

In his celebrated 1983 paper "Une métode pour minorer les exposants de Lyapounov ...", M. Herman uses complex analysis to show the positivity of certain $\mathrm{SL}(2,\mathbb{R})$-cocycles. We discovered that a key inequality of his can be essentially rewritten in a more elegant form (with no limits and no matrix entries), and that it's in fact an equality. The formula is this: for any matrices $A_1, \ldots, A_n \in \mathrm{SL}(2,\mathbb{R})$

$$\frac{1}{2\pi}\int_0^{2\pi} N(A_n R_\theta \ldots A_2 R_\theta A_1 R_\theta)\, d\theta = \sum_{j=1}^{n} N(A_j),$$

where R_θ is the rotation matrix, $\|A\|$ is the euclidian norm of the matrix A, and

$$N(A) = \frac{1}{2\pi}\int_0^{2\pi} \log\|A(\cos\theta, \sin\theta)\|\, d\theta = \log\frac{\|A\| + \|A\|^{-1}}{2}$$

is the "mean expansion rate" of A.

The following consequence of the formula helps to understand its significance: While $\|A_n \ldots A_1\|$ can be much smaller that $\|A_n\| \ldots \|A_1\|$, this ceases to be true if each A_j is replaced with $A_j R_\theta$, for most values of θ.

Remark. It's easy to guess similar formulas for higher dimensional matrix groups, however computer experiments shows that these guesses are false. We still don't know any correct generalization of the formula.

The appendix of this paper contained a (unrelated) result about the spectral radii of cocycle products. This result was recently improved and extended by Ian Morris.

Paper 3

> *Genericity of zero Lyapunov exponents.* Ergodic Theory and Dynamical Systems, *22 (2002), 1667–1696.*

This is still my most cited paper. The main result is: *A C^1 area-preserving diffeomorphism of a compact surface either is Anosov or has zero Lyapunov exponents Lebesgue almost everywhere.*

This theorem was stated by Mañé in his ICM 1983 lecture. A sketch of the proof was published in 1996, shortly after his death. My proof follows his ideas.[1]

Paper 4 (*with F. Abdenur and A. Avila*)

> *Robust transitivity and topological mixing for C^1 flows.* Proceedings of the American Mathematical Society, *132 (2003), 699–705.*

In this short paper, we prove that non-trivial homoclinic classes of C^r-generic flows are topologically mixing, thus generalizing a theorem by Bowen on the basic sets of generic Axiom A flows.

The discrete-time case turns out to be much more difficult, and only recently F. Abdenur and S. Crovisier are obtaining substantial progress in this direction.

Paper 5 (*with M. Viana*)

> *Uniform (projective) hyperbolicity or no hyperbolicity: a dichotomy for generic conservative maps.* Annales de l'Institut Henri Poincaré – Analyse non linéaire, *19 (2002), 113–123.*

This was an announcement of a result that were published later in 6. (Actually it also announces the symplectic result which I was only able to prove much later in 16.)

The paper was also intended to advertise the concept of *projective hyperbolicity*, also known as *dominated splitting*, which in many situations is the right replacement for uniform hyperbolicity in higher dimensions.

Paper 6 (*with M. Viana*)

> *The Lyapunov exponents of generic volume preserving and symplectic maps.* Annals of Mathematics, *161 (2005), No. 3, 1423–1485.*

This is a higher-dimensional version of the result form paper 3. The main result is: *For C^1-generic volume-preserving diffeomorphisms of compact manifolds, the Oseledets splitting along almost every orbit is dominated (or trivial).*

[1] At that time, it wasn't clear if it was possible to fill the gaps in Mañé's sketch. Indeed, when I first presented the proof in a lecture around 2000, M. Herman said "It'll never work."

Although Mañé never stated it, the theorem follows his program – quoting from his 1983 ICM lecture:

> Oseledets' theorem is essentially a measure theoretical result and therefore the information it provides holds only in that category. For instance, the Lyapunov splitting is just a measurable function of the point and the limits defining the Lyapunov exponents are not uniform. It is clear that this is not a deficiency of the theorem but the natural counterweight to its remarkable generality. However, one can pose the problem ... of whether these aspects can be substantially improved by working under generic conditions.

Unfortunately the theorem is not so clean as its two-dimensional version from [3], since different types and strengths of domination may appear in the manifold. How to improve this is still open; some results in this direction were obtained recently in [21].

The paper also proves the (easier) cocycle version of the result.

Remark. As an example that cocycles can be useful in the study of smooth dynamics (i.e., diffeomorphisms and flows), some results of Bonatti–Díaz–Pujals on the creation of sinks and sources can be deduced from the cocycle theorem from [6]: see the appendix of [19].

Paper 7 (*with M. Viana*)

Lyapunov exponents: How frequently are dynamical systems hyperbolic? Modern dynamical systems and applications, *271–297, Brin, Hasselblatt, Pesin (eds.) Cambridge Univ. Press, 2004.*

This expository paper compared the main result of [6] with results (by Bonatti, Gomez-Mont and Viana) which point in the opposite direction when higher regularity is assumed.

It also contained a new result: *In the symplectic case, every dominated splitting is partially hyperbolic.*

Paper 8

Inequalities for numerical invariants of sets of matrices. Linear Algebra and its Applications, *368 (2003), 71–81.*

The *joint spectral radius* (JSR) of a set of square matrices is (loosely speaking) the maximal Lyapunov exponent for the products of the matrices in the set.

In this paper I prove some inequalities that relate the growth of the products in a finite time scale with the asymptotic growth rate (i.e. the JSR).

The subject of JSR is getting more fashionable and this paper is getting a few applications recently.

Paper 9 (*with A. Arbieto*)

L^p-generic cocycles have one-point Lyapunov spectrum. Stochastics and Dynamics, *3 (2003), 73–81.*

In this short paper we show that for *L^p-generic cocycles, all Lyapunov exponents are equal.* Actually, denseness was previously proven by Arnold and Cong, essentially using (without saying) that the existence of a dominated splitting ceases to be an open condition in the L^p topology. What was missing to get a residual set was to show an upper semicontinuity property of Lyapunov exponents. Notice that the coarser the topology, the harder it is for a real function to be semicontinuous; that's why it's tricky to work in the L^p topology.

Paper 10 (*with B. Fayad and E. Pujals*)

A remark on conservative diffeomorphisms. Comptes Rendus Acad. Sci. Paris, *Ser. I 342 (2006), 763–766.*

In this short paper we put together the main result of [6] with some (then) recent results by other people to obtain the following theorem: *C^1-open and densely, stably ergodic diffeomorphisms are nonuniformly hyerbolic and have a global dominated splitting, which separates the positive and negative Lyapunov exponents.*

Diffeomorphisms satisfying the conclusion of the theorem were later dubbed (e.g. in [21]) *nonuniformly Anosov.*

Paper 11 (*with B. Fayad*)

Dichotomies between uniform hyperbolicity and zero Lyapunov exponents for $\mathrm{SL}(2,\mathbb{R})$ *cocycles.* Bulletin of the Brazilian Mathematical Society, *37, no. 3 (2006), 307–349.*

In this paper we try to obtain [3]-like generic dichotomies (uniform hyperbolicity $\times$ zero exponents) for $\mathrm{SL}(2,\mathbb{R})$-cocycles. The difference is that we perturb the base dynamics and not the cocycle.

More precisely, the dynamical systems in the base are volume-preserving homeomorphisms, and the matrix maps need to satisfy some "richness" property. The techniques here are very different from those from [3], [6].

The paper also poses an elementary and difficult problem about growth of matrix products, which motivated later work by Fayad–Krikorian and Avila–Roblin. However the problem is still open.

Paper 12 (*with A. Avila*)

A generic C^1 map has no absolutely continuous invariant probability measure. Nonlinearity, *19 (2006), 2717–2725.*

The result proven in this nice short paper is stated in its title. As corollaries, we obtain that:

- The C^1-generic expanding map has no a.c.i.m.
- The C^1-generic diffeomorphism has no a.c.i.m.

It's seems likely that the second corollary holds in higher topologies, but I know of no result in this direction (except for Anosov maps).

Paper 13 (*with A. Avila*)

> *Generic expanding maps without absolutely continuous invariant σ-finite measure.* Mathematical Research Letters, *14 (2007), 721–730.*

In this sequel of [12], we prove that *the C^1-generic expanding map of the circle has no a.c.i.σ (absolutely continuous invariant σ-finite measure)*, thus answering a question of Quas. Such maps are classically called *type III.*

It's seems likely that the same is true in higher dimension, but it's not clear how to prove it.

Paper 14 (*with A. Avila*)

> *A uniform dichotomy for generic* $\mathrm{SL}(2,\mathbb{R})$ *cocycles over a minimal base.* Bulletin de la Société Mathématique de France, *135 (2007), 407–417.*

This is an extension of the cocycle result from [1], [3]. We consider $\mathrm{SL}(2,\mathbb{R})$-cocycles over a minimal base dynamics, and we show that if a cocycle is not uniformly hyperbolic then we can perturb it and make the Lyapunov exponents *with respect to all invariant measures* vanish.

Paper 15 (*with A. Avila and D. Damanik*)

> *Cantor spectrum for Schrödinger operators with potentials arising from generalized skew-shifts.* Duke Mathematical Journal, *146, no. 2 (2009), 253–280.*

This paper contains some new results about $\mathrm{SL}(2,\mathbb{R})$-cocycles and then applies them to Schrödinger operators.

More precisely, we take as base dynamics an uniquely ergodic transformation that fibers over an irrational rotation. We show that uniform hyperbolicity is (open and) dense in every homotopy classes of cocycles that do not present an obstruction. For cocycles arising from Schrödinger operators, the obstruction vanishes, and we conclude that uniform hyperbolicity is dense, which implies that for a generic continuous potential, the spectrum of the corresponding Schrödinger operator is a Cantor set.

Some novelties in this paper in the use of hyperbolic geometry[2] to find suitable perturbations, and a general method of reducing perturbative problems for Schrödnger operators to the more flexible $\mathrm{SL}(2,\mathbb{R})$ case.

Paper 16

> *C^1-generic symplectic diffeomorphisms: partial hyperbolicity and zero center Lyapunov exponents.* Journal of the Institute of Mathematics of Jussieu, *9, no. 1 (2010), 49–93.*

In this paper I prove the symplectic counterpart of [6], which was stated by Mañé in his 1983 ICM lecture: *For C^1-generic symplectic diffeomorphisms, the Oseledets splitting along almost every orbit is either trivial or partially hyperbolic.* (Actually [6] contains a weaker version of this result.)

[2]For more about this, see the summary of [27].

The main technical novelty is a probabilistic method for the construction of perturbations, using random walks. This is a paper I'm particularly proud of.

Combining the main result of this paper with Dolgopyat and Wilkinson, I show that for generic partially hyperbolic maps, all Lyapunov exponents in the (minimal) center bundle are zero almost everywhere. This kind of nonuniform center-bunching motivated was the starting point for a later work [19].

Paper 17 (*with A. Avila and J.-C. Yoccoz.*)

Uniformly hyperbolic finite-valued $\mathrm{SL}(2,\mathbb{R})$ *cocycles.* Commentarii Mathematici Helvetici, *85, no. 4 (2010), 813–884.*

This is a long paper which took us a long time to complete. Basically, we deal with finite subsets of $\mathrm{SL}(2,\mathbb{R})$ (say, of cardinality k) all whose products grow exponentially. This is equivalent to the uniform hyperbolicity of the obvious finite-valued cocycle over the full shift in k symbols.[3]

The aim of the paper is to study the *hyperbolicity locus*, that is, the subset $\mathcal{H}_k$ of $\mathrm{SL}(2,\mathbb{R})^k$ formed by uniformly hyperbolic k-tuples of matrices. The main results of the paper can be summarized as follows:

- We gave a simple geometric characterization of uniform hyperbolicity of a set of matrices: there is a finite union of cones in $\mathbb{R}^2$ (called a *multicone*) that is strictly invariant by each matrix in the set. Multicones allow the combinatorial study of uniformly hyperbolic sets.

- We found a complete description of the uniformly hyperbolic sets formed by $k = 2$ matrices. This case already shows richness of possibilities: the hyperbolicity locus $\mathcal{H}_2$ has an infinite number of connected components, corresponding to an infinite family of multicone combinatorics.

- We studied bifurcation points, that is, k-tuples in the boundary of a connected component of $\mathcal{H}_k$. We obtained a complete classification of the possible bifurcations.

- We gave an example with $k = 3$ showing that a bifurcation called *homoclinic connection* may give birth to complicated phenomena, namely the accumulation of an infinite number of hyperbolicity components. Such examples lead us to believe that a complete description of the components for $k = 3$ is unfeasible.

Despite all these results, there are many basic questions about the hyperbolicity locus that we are unable to answer. For example: Is every connected component unbounded (modulo conjugacy)?

The components that have no homoclinic connections in their boundaries are better-behaved: see [23].

[3]Equivalent points of view for this setting are: (uniformly hyperbolic) finitely-generated semigroups of $\mathrm{SL}(2,\mathbb{R})$; iterated function systems (IFS) of projective transformations of $P(\mathbb{R}^2) \simeq \mathbf{S}^1$. (Actually a few of the results of the paper also apply to the more general setting of finite-valued cocycles over shifts of finite type.)

Paper 18 (*with N. Gourmelon*)

Some characterizations of domination. Mathematische Zeitschrift, *263, no. 1 (2009), 221–231.*

This is a nice little paper about dominated splittings. It contains two results:

- The first result is a characterization of dominated splittings in terms of separation of singular values. This generalizes a result of Yoccoz for $\mathrm{SL}(2,\mathbb{R})$. Despite the purely "topological" statement of the result, its proof (differently from Yoccoz) uses ergodic theory.
- The second result gives a multicone characterization of "dominated" sets of matrices, thus generalizing the multicone theorem from [17].

Remark. A particular case of this result were (I suppose) independently obtained later by Barnsley and Vince, who are interested in IFS (iterated function systems).

Paper 19 (*with A. Avila and A. Wilkinson*)

Nonuniform center bunching and the genericity of ergodicity among C^1 partially hyperbolic symplectomorphisms. Annales Scientifiques de l'École Normale Supérieure, *42, n. 6 (2009), 931–979.*

This paper introduces the concept of *nonuniform center bunching* for partially hyperbolic diffeomorphims (not necessarily volume-preserving). This is a weaker version of (uniform) center bunching in the sense of Burns and Wilkinson. A large part of the paper consists in extending Burns–Wilkinson's arguments to show that nonuniform center-bunching enjoys the technical saturation properties which make the concept useful for the study of ergodicity.

In the symplectic case, a result of [16] (mentioned above) implies that C^1-generically, nonuniform center bunching holds at almost every point. This permits us to prove the following nice result: *C^1-generic partially hyperbolic symplectic diffeomorphisms are ergodic.* (Actually the argument is a little involved, since Burns–Wilkinson properties require C^2 regularity; also we need to "insert" an Anosov–Katok example as a local source of ergodicity.)

Paper 20 (*with A. Avila and D. Damanik*)

Opening gaps in the spectrum of strictly ergodic Schrödinger operators. To appear in Journal of the European Mathematical Society. `arXiv:0903.2281`

Here we continue the study initiated in [15] and prove results for $\mathrm{SL}(2,\mathbb{R})$ cocycles having in mind applications to Schrödinger operators.

Let me summarize the main result. We consider Schrödinger operators over uniquely ergodic base dynamics (in a more general setting than [15]). The Gap Labeling Theorem (from the 1980's) states that the possible gaps in the spectrum can be canonically labelled by an at most countable set defined purely in terms of the dynamics. Which labels actually appear depends on the choice of the potential function; the missing labels are said to correspond to collapsed gaps. Here we show that for any collapsed gap, the potential may be continuously deformed so that the gap immediately opens. As a corollary, we conclude that for generic potentials, all gaps are open.

These results follow from a cocycle theorem: We characterize when a $\mathrm{SL}(2,\mathbb{R})$-cocycle A_0 can be "accessed" by uniformly hyperbolic cocycles, i.e. there is a continuous path of uniformly hyperbolic cocycles converging to A_0. The condition is that twice the rotation number of the cocycle must be in a certain dynamically defined set (related to the Schwartzman asymptotic cycle).

We use some ideas from [1] (Knill's method), together with many new arguments.

Paper 21 (*with A. Avila*)

Nonuniform hyperbolicity, global dominated splittings and generic properties of volume-preserving diffeomorphisms. To appear in Transactions of the American Mathematical Society. `arXiv:0912.2699`

As commented before, the main theorem of [6] concerns individual orbits and thus does not give global properties (i.e. properties concerning the whole manifold). The paper [21] gives a global improvement of [6].

Again we consider C^1-generic volume-preserving diffeomorphisms. In short, we show that *the presence of nonuniform hyperbolicity on a positive volume subset implies the existence of a global dominated splitting.* In particular, for manifolds where no dominated splitting can exist (for example, even-dimensional spheres), we conclude that for almost every point there is at least one zero Lyapunov exponent.

More precisely, we show that generically either there is at least one zero Lyapunov exponent at almost every point, or the set of points with only nonzero exponents forms an ergodic component. Moreover, if this nonuniformly hyperbolic component has positive measure then it is essentially dense in the manifold (that is, it has a positive measure intersection with any nonempty open set) and there is a global dominated splitting.

Our results give some support for the following conjecture: *A C^1-generic volume-preserving diffeomorphism either has all exponents zero at Lebesgue almost every point, or is ergodic and nonuniformly Anosov.* For 3-dimensional manifolds, a proof of this conjecture was announced recently by Jana R. Hertz (using our result together with other tools and arguments).

In this paper we also establish some new properties of independent interest that hold C^r-generically for any $r \geq 1$, namely: the continuity of the ergodic decomposition, the persistence of invariant sets, and the L^1-continuity of Lyapunov exponents. The proof of the main theorem itself also uses other ingredients: a bunch of C^1-generic properties; Pesin manifolds for C^1 maps with dominated splittings; Hertz–Hertz–Tahzibi–Ures Pesin-theoretic criterion for ergodicity; and Avila's smoothening result to bridge the gap between C^1 and C^2.

Paper 22 (*with C. Bonatti*)

Perturbation of the Lyapunov spectrum of periodic orbits. To appear in Proceedings of the London Mathematical Society. `arXiv:1004.5029`

This is another paper which deals with the idea of mixing Lyapunov exponents in the absence of dominated splitting. Here we deal with general (i.e. not necessarily volume-preserving) diffeomorphisms, and we are only interested in what perturbations we can make on the periodic orbits.

We describe all Lyapunov spectra that can be obtained by perturbing the derivatives along periodic orbits of a diffeomorphism. The description is expressed in terms of the finest dominated splitting and Lyapunov exponents that appear in the limit of a sequence of periodic orbits, and involves the majorization partial order.

This extends (and maybe gives the "optimal" form) of previous results of Bonatti–Díaz–Pujals and Bonatti–Gourmelon–Vivier.

Among the applications, we give a simple criterion for the occurrence of universal dynamics.

I believe this paper (together with Gourmelon's version of Frank's lemma) will have several applications in ("dissipative") C^1-generic dynamics.

Paper 23 (*with A. Avila and J.-C. Yoccoz – IN PROGRESS*)

Group-type components of the hyperbolicity locus. (working title)

We will consider the connected components of the hyperbolicity locus that have no homoclinic connections in their boundaries. We know that these special components are naturally related with Riemann surfaces, which allows us to use complex analysis to study them.

Paper 24 (*with N. Gourmelon – IN PROGRESS*)

Universal nonsingular control for generic semilinear systems.

This paper deals with controllability of products of matrices. It was (more precisely, will be) written having in mind the applications to [25]. We hope it will be of interest for people in Control Theory.

We consider control systems of the form $x_{n+1} = A(u_n)x_n$, where the next state x_{n+1} depends linearly on the current state x_n but non-linearly on the control variable u_n. We are interested in the local controllability of these systems. We show that a strong controllability property holds for generic maps A (more precisely, for A in a C^2-open, C^∞ -dense set). The proof relies on transversality arguments together with a delicate estimate of certain singular sets (involving some algebraic geometry).

Paper 25 (*with N. Gourmelon – IN PROGRESS*)

Cocycles over generic volume preserving dynamics.

We've been working in this paper for a long time. In fact, the original motivation of [18] and [24] was to apply the results to this paper.

Here we improve the results of [11] in several ways, also making the dimension arbitrary.

Paper 26 (*with C. Bonatti and L. J. Díaz – IN PROGRESS*)

Robust vanishing of all central Lyapunov exponents. (working title)

This paper extends to arbitrary dimension some results of Gorodetski, Ilyashenko, Kleptsyn, Nalsky. We describe C^1-open sets of IFS (iterated funcion systems) on arbitrary compacts manifolds (of any dimension) admitting invariant measures all whose Lyapunov exponents vanish.

We plan to use this to construct a C^1-robust examples of a partially hyperbolic diffeomorphisms (with center fiber of arbitrary dimension) that have ergodic fully-supported invariant measures all whose Lyapunov exponents vanish.

Paper 27 (*with A. Navas – IN PROGRESS*)

A geometric path from zero Lyapunov exponents to bounded products. (working title)

The Oseledets theorem has been extended to much more general setting by Kaimanovich, Karlsson, Margulis, Ledrappier.

In this paper we consider cocycles of isometries of a negatively curved symmetric space. Using purely geometrical techniques, we extend a few (not all!) results of [15], [20] to this general setting. The new proofs are not only more powerful but also conceptually simpler.

Paper 28 (*with M. K. Schnoor – IN PROGRESS*)

A generic C^1 flow has no a.c.i.p. (working title)

We prove the continuous-time version of [12].