Generic diffeomorphisms on compact surfaces

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Abstract

In this paper, we discuss the remaining obstacles to prove Smale's conjecture about the C^1- density of hyperbolicity among surface diffeomorphisms. Using a C^1- generic approach, we classify the possible pathologies that may obstruct the C^1- density of hyperbolicity. We show that there are essentially two types of obstruction: (i) persistence of infinitely many hyperbolic homoclinic classes and (ii) existence of a single homoclinic class which robustly presents homoclinic tangencies. In the course of our discussion, we obtain some related results about C^1- generic properties of surface diffeomorphisms involving homoclinic classes, chain-recurrence classes, and hyperbolicity. In particular, it is shown that on a connected surface the C^1-generic diffeomorphisms whose non-wandering sets have non-empty interior are the Anosov diffeomorphisms.

keywords: chain recurrence class, dominated splitting, filtration, homoclinic class, hyperbolicity, surface diffeomorphism.

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1 Introduction

1.1 Motivations

Some ideas of Mañé and new C^1- perturbation lemmas generalizing Hayashi connecting lemma [H] have recently opened the door for a global understanding of C^1- generic systems (diffeomorphisms or flows) on compact manifolds of any dimension. Many mechanisms responsible for robust non-hyperbolicity are now understood. It is not known, however, whether the best-known C^2- mechanism for generating robust non-hyperbolic behavior, the Newhouse phenomenon (of persistence of tangencies and coexistence of infinitely many sinks), has some C^1- equivalent in dimension two. In fact, this problem seems the most important open problem for the global understanding of the C^1-generic systems on compact surfaces: it remains unknown whether the (open) set of Axiom A diffeomorphisms is dense in the space of surface diffeomorphisms, for the C^1- topology. We do expect a positive answer to the following conjecture:

Conjecture (Smale, [Sm]). *Let S be a compact surface. The set of diffeomorphisms verifying the Axiom A and the no-cycle condition is (open and) dense in* $\mathrm{Diff}^1(S)$.

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This conjecture is known to be false in dimension equal to or higher than 3, for the C^1-topology, due to the co-existence of hyperbolic saddles with different indices (dimension of the unstable bundle) in the same transitive set (see [AS] and [Si] for counter-examples in dimension 4 and 3) and in dimension equal to or higher than 2 for the C^r- topology, $r \geq 2$, due to the phenomenon of persistence of tangencies (see [N] and [PV])).

Here we try to present an overview of the dynamics of C^1- generic diffeomorphisms of compact surfaces far from hyperbolic systems. If one believes in Smale's conjecture above, one can say that this work discusses the remaining obstacles to a solution of the conjecture. The difficulty of the conjecture comes from the fact that we do not know any local mechanism which could be responsible for robust non-hyperbolicity in dimension 2 for the C^1- topology (in fact, we expect that such a mechanism does not exist). For diffeomorphisms in higher dimensions, the only local phenomenon which generates C^1- robust non-hyperbolicity is associated to the unfolding of heterodimensional cycles (i.e., cycles associated to saddles having different indices), see [D, DR]. Of course, such a cycle may not appear on surfaces and the main candidate for such a mechanism on surfaces seems to be the presence of *homoclinic tangencies*, i.e. non-transverse intersections between the invariant manifolds of some hyperbolic periodic orbit: for the C^2- topology, they do imply the existence of a C^2- open set of non-hyperbolic diffeomorphisms. Moreover, Pujals and Sambarino proved in [PS] that the existence of homoclinic tangencies is a C^1- dense phenomenon among surface *diffeomorphisms far from hyperbolicity* (i.e. diffeomorphisms that can not be approximated by hyperbolic ones).

Since the result of [PS] gives only a description for a dense part of the set of diffeomorphisms far from hyperbolicity and since we are interested in robust non-hyperbolic phenomena, our aim in this paper is to study a *generic* part of this set of diffeomorphims (meaning a residual subset of $\mathrm{Diff}^1(M)$). By [PS], we know that surface diffeomorphisms that are far from hyperbolicity may be perturbed to create homoclinic tangencies. However, it is not clear whether such tangencies can be obtained in a persistent way for the same periodic point. This motivates the following definition:

Definition 1.1. *A diffeomorphism f of a compact manifold M presents a* persistent homoclinic tangency *associated to a hyperbolic periodic point p if there exist a C^1- neighborhood $\mathcal{U}$ of f and a dense subset $\mathcal{D} \subset \mathcal{U}$ such that for any diffeomorphisms $g \in \mathcal{D}$ the continuation p_g of p presents a homoclinic tangency.*

The following consequence of our results explains what occurs for generic diffeomorphisms (i.e. diffeomorphisms in a residual subset) far from hyperbolicity. Such a diffeomorphism may (in principle) exhibit a persistent tangency, but we cannot exclude a priori another possibility: that the homoclinic class of any periodic orbit is a hyperbolic basic set but that the global dynamics presents infinitely many different homoclinic classes accumulating on aperiodic classes (see Section 1.2 below for the definition of *aperiodic class*).

Theorem 1. *Let S be a closed surface. There are three disjoint open sets $\mathcal{H}$, $\mathcal{T}$, and $\mathcal{W}$ whose union is dense in* $\mathrm{Diff}^1(S)$ *such that:*

- *$\mathcal{H}$ is the set of diffeomorphisms which satisfy the Axiom A and the no-cycle condition;*
- *$\mathcal{T}$ is the set of diffeomorphisms admitting a persistent homoclinic tangency associated to some hyperbolic periodic saddle;*

- *$\mathcal{W}$ contains a residual subset $\mathcal{GW}$ such that every $f \in \mathcal{GW}$ has infinitely many homoclinic classes, all of which are hyperbolic basic sets.*

Let us discuss now how homoclinic tangencies associated to a single homoclinic class may occur persistently. Keeping in mind the classical C^2- Newhouse phenomenon (see [N]), the most intuitive way is to imagine that some hyperbolic set has a *robust tangency* in the following sense:

Definition 1.2. *Let Λ be a hyperbolic set of a diffeomorphism f on a compact manifold. We say that Λ presents* robust tangencies *if there is a constant $T > 0$ and a C^1- neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ the local stable manifold $W^s_T(\Lambda_g)$ of size T of the continuation Λ_g of Λ presents a tangency with the local unstable manifold $W^u_T(\Lambda_g)$ of size T of Λ_g.*

In fact, if a hyperbolic set Λ presents locally generically some tangency then it presents robust tangencies:

Proposition 1.3. *Let f be a diffeomorphism of a compact manifold M and Λ a hyperbolic set of f. If there exist a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{G}$ of $\mathcal{U}$ such that for every diffeomorphism $g \in \mathcal{G}$ the stable and unstable manifolds of the continuation Λ_g of Λ are tangent, then f is C^1- approached by diffeomorphisms h which exhibit robust tangencies.*

It is natural to ask whether a diffeomorphism in $\mathcal{T}$ (i.e., one that has a persistent homoclinic tangency) admits robust tangencies. The following proposition explains what else might happen.

Proposition 1.4. *There are two C^1- open disjoint subsets $\mathcal{T}_{rob}$ and $\mathcal{T}_\infty$ of $\mathcal{T}$, whose union is dense in $\mathcal{T}$, such that:*

- *$\mathcal{T}_{rob}$ is the set of diffeomorphisms admitting a robust tangency associated to some hyperbolic set;*
- *there is a residual subset $\mathcal{GT}_\infty$ in $\mathcal{T}_\infty$ such that for every $g \in \mathcal{GT}_\infty$ and for any hyperbolic set Λ of g, the invariant manifolds of Λ are transverse (even though g presents persistent tangencies in the sense of Definition 1.1).*

When trying to solve Smale's conjecture above, one usually focuses on whether the Newhouse phenomenon (robust tangencies) can occur. The discussion above shows that two other phenomena may be responsible of the non-density of Axiom A diffeomorphisms: the creation of homoclinic tangencies either "at the infinity of some homoclinic class" (corresponding to the set $\mathcal{T}_\infty$) or "in the aperiodic classes" (corresponding to the set $\mathcal{W}$). We have therefore broken Smale's conjecture into three parts:

Conjecture 1. *A generic diffeomorphism of a compact surface whose homoclinic classes are all hyperbolic satisfies the Axiom A.*

Conjecture 2 (No robust tangencies). *Let Λ be a hyperbolic set of a diffeomorphism f of a compact surface S. Then, for any $L > 0$, there is a C^1- perturbation g of f such that the local invariant manifolds of size L of the hyperbolic continuation Λ_g of Λ for g, are transverse.*

Conjecture 3 (Persistent tangencies imply robust tangencies). *Any diffeomorphism f which admits persistent tangencies associated to some hyperbolic periodic point may be C^1-approximated by a diffeomorphism which has robust tangencies associated to some hyperbolic set.*

By Theorem 1, these three conjectures imply Smale's conjecture: Conjecture 1 implies that $\mathcal{W}$ is empty, Conjecture 2 implies that $\mathcal{T}_{rob}$ is empty, and finally, Conjecture 3 shows that the open set $\mathcal{T}_{rob}$ is dense in $\mathcal{T}$ implying that $\mathcal{T}_\infty$ is empty.

The third one is equivalent to the following (a priori) weaker statement (see Proposition 6.2 at the end of the article):

Conjecture 4. *Let f be a C^1- generic diffeomorphism such that for any hyperbolic set Λ of f the stable and unstable manifolds of Λ are transverse. Then all the homoclinic classes of f are hyperbolic.*

We end this discussion with another question related to the techniques of the paper. In the context of C^1- generic dynamics of diffeomorphisms in any dimension, [ABD] formulates, and proves in some particular cases, the following conjecture:

Conjecture 5. *For any C^1- generic diffeomorphism of a compact connected manifold, if the non-wandering set has non-empty interior then the diffeomorphism is transitive, in particular, the non-wandering set coincides with the whole manifold.*

Notice that a positive answer to Smale's conjecture would imply a positive answer to this one in dimension two. The results proven in this paper do imply:

Corollary 1. *Let S be a connected closed surface. There is a residual subset $\mathcal{G}_0$ of $\mathrm{Diff}^1(S)$ consisting of diffeomorphism f such that:*

$$\mathrm{int}(\Omega(f)) \neq \emptyset \Longrightarrow S \text{ is the torus } T^2 \text{ and } f \text{ is Anosov.}$$

Moreover, in this case f is topologically conjugate to some linear Anosov diffeomorphism. In particular, f is transitive.

The conjugacy to a linear Anosov system in the previous corollary is a consequence of a result of Franks [F]. Thus in this paper we settle the two-dimensional case of the conjecture in [ABD].

We note that the residual set $\mathcal{G}_0$ in Corollary 1 will be given by Theorem 2 below.

1.2 Presentation of the results

For surface diffeomorphisms, Pujals and Sambarino have provided in [PS] a precise description of the dynamics under the hypothesis of the existence of a *dominated splitting.* This notion will be important for us since from a C^1- generic viewpoint it characterizes hyperbolicity.

Definition 1.5. *A compact set K which is invariant by a diffeomorphism f admits a* dominated splitting *if the tangent bundle T_KM over K splits into two bundles $T_KM = E \oplus F$ and if there exists an integer $\ell \geq 1$ such that for every point $x \in K$ and every pair of unit vectors $u \in E$ and $v \in F$ we have*

$$\|Df^\ell(x) \cdot u\| \leq 2\,\|Df^\ell(x) \cdot v\|.$$

Next remark states the main properties of a dominated splitting we use in this paper (see, for instance, [BDV, Section B.1.1])

Remark 1.6. (Properties of dominated splittings) *Let Λ be an f- invariant set with a dominated splitting $E \oplus F$.*

1. **Extension to the closure.** *The dominated splitting $E \oplus F$ can be extended in a dominated way to the closure of Λ.*

2. **Extension to a neighborhood.** *The dominated splitting $E \oplus F$ can be extended in a dominated way to the maximal invariant set of f in a small neighborhood of Λ.*

3. **Robustness.** *Every dominated splitting persists under C^1- perturbations: There are a C^1- neighborhood $\mathcal{U}$ of f and a neighborhood of Λ in the ambient manifold such that, for every $g \in \mathcal{U}$, the maximal invariant of g in U has a dominated splitting.*

Another important ingredient for what follows is the decomposition of the dynamics into its chain-recurrence classes, given by Conley's theory, [Co]. We recall the main results of this theory and state some relevant properties of chain-recurrence classes in Section 2.1.

Let f be a homeomorphism of a compact metric space X. For any $\varepsilon > 0$, an *ε-pseudo-orbit* of f is a sequence (x_n) in X such that, for each n, $d(f(x_n), x_{n+1}) < \varepsilon$. A point x is *chain-recurrent* if for any $\varepsilon > 0$ there exists an ε- pseudo-orbit starting and ending at x. We denote by $\mathcal{R}(f)$ the set of chain-recurrent points. It decomposes as the union of *chain-recurrence classes*, which are pairwise invariant compact sets defined as follows: two points x and y belong to the same chain-recurrence class if for any $\varepsilon > 0$, there are ε-pseudo-orbits starting at x and ending at y, and conversely starting at y and ending at x.

A third important ingredient in this paper is the notion of *homoclinic class.* The homoclinic class of a hyperbolic periodic orbit p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of its stable and unstable manifolds. This set coincides with the closure of the periodic orbits q *homoclinically related* to p, i.e., the stable manifold of p transversely meets the unstable one of q and the unstable manifold of p transversely meets the stable one of q. We observe, that a homoclinic class always is an f-invariant transitive set. An important property of homoclinic classes of C^1- generic diffeomorphisms is the following: any pair of homoclinic classes either coincide or are disjoint, see [CMP].

Quite recently, [BC] proved a perturbation lemma for pseudo-orbits and deduced from this lemma that for C^1- generic diffeomorphisms of compact manifolds, any chain-recurrence class which contains a hyperbolic periodic point p coincides with the *homoclinic class* of p (in particular, this result generalizes the one in [CMP] above). The other chain-recurrence classes of these generic diffeomorphisms (when they exist) are called *aperiodic classes* (whose existence in some locally generic regions was proven in [BD$_2$]).

Let us note that for diffeomorphisms f which satisfy the *Axiom A* (the non-wandering set $\Omega(f)$ is hyperbolic and equal to the closure of the periodic points) Smale has proven in his spectral theorem [Sm] that the non-wandering set is the union of finitely many pairwise disjoint homoclinic classes. These homoclinic classes are the *basic sets* of the spectral decomposition of the non-wandering set. Moreover, under the *no-cycles* assumption (i.e., there are no basic sets of the spectral decomposition cyclically related by intersections of the corresponding stable and unstable manifolds), the chain-recurrence classes are homoclinic classes. More precisely, in this case, each chain-recurrence class E is a *basic set*:

- E is a hyperbolic homoclinic class,
- E is the maximal invariant set in the a *level of a filtration*: there exist compact sets U and V such that $f(U) \subset \mathrm{int}(U)$, $f^{-1}(V) \subset \mathrm{int}(V)$ and E is the maximal invariant set in $U \cap V$, i.e. $E = \bigcap_{n\in\mathbb{Z}} f^n(U \cap V)$.

Our next result shows that, C^1- generically, a chain-recurrence class of a surface diffeomorphism is hyperbolic if, and only if, it is isolated, and also if, and oly if, it admits a dominated splitting (this assertion does not hold in higher dimensions, see the series of examples in [Sh, M_1, BD_1, BV]):

Theorem 2. *Let S be a compact surface. There is a residual subset $\mathcal{G}_0$ of $\mathrm{Diff}^1(S)$ such that, for any $f \in \mathcal{G}_0$ and any chain-recurrence class E of f, we have the following dichotomy:*

1. *either E is isolated in the chain-recurrent set $\mathcal{R}(f)$ of f; in this case, E is a hyperbolic homoclinic class (and so a basic set);*
2. *or E is not isolated; in this case:*
   - *(a) E does not admit any dominated splitting;*
   - *(b) E is contained in the closure of the set of sinks and sources of f;*
   - *(c) for any neighborhood U of E, there is a C^1- neighborhood $\mathcal{U}$ of f and a dense subset $\mathcal{D}$ of $\mathcal{U}$ such that any $g \in \mathcal{D}$ has a hyperbolic periodic point p_g whose homoclinic class is contained in U and presents homoclinic tangencies.*

If the chain-recurrence class E is a non-isolated homoclinic class $H(p, f)$, we obtain the announced tangencies inside the homoclinic class $H(p_g, g)$ of the continuation p_g of p:

Theorem 3. *Let f be a diffeomorphism of a compact surface S and p be a hyperbolic periodic saddle of f. If the homoclinic class $H(p, f)$ of p has no dominated splitting, then there exists a diffeomorphism g arbitrarily C^1- close to f such that the continuation p_g of p admits a homoclinic tangency.*

¿From the two previous theorems one deduces the following genericity result:

Corollary 2. *Let S be a compact surface. There is a residual subset $\mathcal{G}_1$ of $\mathrm{Diff}^1(S)$ such that, for every $f \in \mathcal{G}_1$ and every periodic point p of f whose homoclinic class is non-isolated in the recurrent set $\mathcal{R}(f)$ of f, the diffeomorphism f presents a persistent homoclinic tangency associated to p.*

2 Some results in arbitrary dimension

This section collects some results that we will use later in this paper. Most of these results were obtained in previous works. In Section 2.1, we recall the Conley's Fundamental Theorem of Dynamical Systems and state properies of chain-recurrence classes and their filtrations. A general reference for Conley's theory is [R]. In Section 2.2, we briefly study hyperbolic chain-recurrence classes. Finally, in Section 2.3, we state a result about generic chain recurrent sets (which follows from a perturbation lemma for pseudo-orbits in [BC]) and deduce properties of non-isolated chain-recurrence classes from it.

2.1 Conley's theory and filtrating sets

We begin with Conley's fundamental theorem:

Theorem 2.1 (Conley's Fundamental Theorem of Dynamical Systems, [Co]). *Let f be a homeomorphism of a compact metric space X. There exists a continuous function $\varphi\colon X \to \mathbb{R}$ such that:*

1. *φ is a Lyapunov function: for any $x \in X$, $\varphi(f(x)) \leq \varphi(x)$;*
2. *for any $x \in X$, $\varphi(f(x)) = \varphi(x)$ is equivalent to $x \in \mathcal{R}(f)$;*
3. *for every $x, y \in \mathcal{R}(f)$, $\varphi(x) = \varphi(y)$ is equivalent to x and y belonging to the same chain-recurrence class;*
4. *$\varphi(\mathcal{R}(f))$ is a totally disconnected compact subset of $\mathbb{R}$.*

Such a function φ will be called a *Lyapunov function adapted to $\mathcal{R}(f)$*.

Remark 2.1. *If X is a smooth compact manifold, the function φ may be assumed to be smooth.*

A compact set A is a *trapping region* for f if $f(A)$ is contained in the interior of A. The maximal invariant set $\bigcap_{n\in\mathbb{N}} f^n(A)$ of f in A, is the *attractor* associated to A. In the same way, $\bigcap_{n\in\mathbb{N}} f^{-n}(X \setminus A) = \bigcap_{n\in\mathbb{N}} \overline{f^{-n}(X \setminus A)}$ is the *repellor* associated to A. If B is a trapping region for f^{-1}, we define its *attractor* and *repellor* as the attractor and the repellor of the trapping region $\overline{(X \setminus B)}$ for f.

We say that a compact set U is a *filtrating set* if there are two compact sets A, B such that:

- $U = A \cap B$;
- $f(A) \subset \operatorname{int}(A)$ (that is, A is a trapping region for f);
- $f^{-1}(B) \subset \operatorname{int}(B)$ (that is, B is a trapping region for f^{-1}).

Let K be some invariant compact set of f. A *filtrating neighborhood* of K is a neighborhood of K which is a filtrating set. An *isolating filtrating neighborhood* U of K is a filtrating neighborhood of K whose maximal invariant set $\bigcap_{n\in\mathbb{Z}} f^n(U)$ is K.

Remark 2.2.

1. *Let $U = A \cap B$ be a filtrating set (where A and B are trapping regions for f and f^{-1} respectively), then the maximal invariant set of f in U is the intersection of the attractor of A with the repellor of B.*
2. *If U is a filtrating set for f, then U is also a filtrating set for any homeomorphism g of X which is C^0- close to f.*
3. *If φ is a Lyapunov function adapted to $\mathcal{R}(f)$, then for any $a, b \in (\mathbb{R} \setminus \varphi(\mathcal{R}(f)))$, $a < b$, the set $\varphi^{-1}([a, b])$ is a filtrating set.*

4. *Let $U = A \cap B$ be a filtrating set defined as above, then any compact set V contained in the interior of U and containing $f(U) \cap f^{-1}(U)$ is also a filtrating set. Moreover, the maximal invariant sets of f in U and in V coincide. Indeed, V may be written as an intersection*

$$V = A' \cap B' = (V \cup f(A)) \cap \left(V \cup f^{-1}(B)\right),$$

where A' and B' are trapping regions for f and f^{-1}, respectively.

5. *By the last item, if X is a smooth compact manifold then V can be chosen as a smooth submanifold with boundary. Hence, we can replace any filtrating set by a filtrating set which is a smooth submanifold with boundary having the same maximal invariant set.*

6. *By the definition of filtrating set, any chain-recurrence class which intersects a filtrating set is contained in the interior of the filtrating set.*

7. *Let K be the maximal invariant set in a filtrating set $U = A \cap B$ as above. Then, for any $i, j \in \mathbb{Z}$, $f^i(A) \cap f^j(B)$ is a filtrating set having K as its maximal invariant set. Moreover, $\{f^n(A) \cap f^{-n}(B), n \in \mathbb{N}\}$ is a basis of neighborhoods of K, each of which is a filtrating set.*

Proposition 2.3. *Let f be a homeomorphism of a compact metric space X, E any chain-recurrent class of f, φ a Lyapunov function adapted to $\mathcal{R}(f)$, and U a neighborhood of E. Then, for every $\varepsilon > 0$ small enough, the maximal invariant set in $V_\varepsilon = \varphi^{-1}([\varphi(E) - \varepsilon, \varphi(E) + \varepsilon])$ is contained in U.*

Proof: Note that the set of maximal invariant sets Λ_ε in V_ε is a decreasing family as ε goes to 0. We just have to prove that the intersection $\Lambda = \bigcap_\varepsilon \Lambda_\varepsilon$ is contained in the interior of U: this set is an invariant compact set on which φ is equal to $\varphi(E)$. As a consequence, $\Lambda = E$. □

Proposition 2.4. *Let f be a homeomorphism of a compact metric space X. Any chain-recurrence class E of f admits arbitrarily small filtrating neighborhoods.*

Proof: Let φ be a Lyapunov function adapted to $\mathcal{R}(f)$ and U a neighborhood of E. By Proposition 2.3, for ε small enough, the maximal invariant set Λ_ε in $V_\varepsilon = \varphi^{-1}([\varphi(E)-\varepsilon, \varphi(E)+\varepsilon])$ is contained in U. We choose ε such that $\varphi(E) - \varepsilon$ and $\varphi(E) + \varepsilon$ do not belong to $\varphi(\mathcal{R}(f))$ (this is possible since $\varphi(\mathcal{R}(f))$ has empty interior). By Remark 2.2. (3), V_ε is a filtrating set. By Remark 2.2. (7), we can choose a filtrating set, which is an arbitrarily small neighborhood of Λ_ε, hence contained in U and containing E. □

Proposition 2.5. *Let f be a homeomorphism of a compact manifold M such that its chain-recurrent set $\mathcal{R}(f)$ has non-empty interior. Then there exists a chain-recurrence class which has non-empty interior.*

Proof: Consider a connected component C of the interior of $\mathcal{R}(f)$ and a Lyapunov function φ adapted to $\mathcal{R}(f)$. The image $\varphi(C)$ is connected and contained in $\varphi(\mathcal{R}(f))$, which is totally disconnected. So, $\varphi(C)$ is a point and, by Theorem 2.1, C is contained in a chain-recurrence class E. Since a manifold is locally connected, C has non-empty interior. Hence, E has non-empty interior. □

Proposition 2.6. *Let X be a compact metric space. There exists a countable family of compact sets $\mathcal{F} = \{V_n\}$ such that for any homeomorphism f of X and any fitrating set U of f, there exists $V_n \subset U$ which is a filtrating set of f such that the maximal invariant sets of f in U and in V_n coincide.*

Proof: Let $\mathcal{O} = \{O_n\}$ be a countable basis of open sets of X. We consider the countable family $\mathcal{F}$ of compact sets that are the closure of finite unions of sets in $\mathcal{O}$.

Let f be a homeomorphism of X and U a filtrating set. We consider the compact set $f(U) \cap f^{-1}(U)$, contained in the interior of U. Each point x of $f(U) \cap f^{-1}(U)$ belongs to some open set $O_{n(x)}$, whose closure is contained in the interior of U. Let V be the closure of the union of a finite covering of $f(U) \cap f^{-1}(U)$ by such $O_{n(x)}$. The compact set V belongs to $\mathcal{F}$, is contained in the interior of U, and contains $f(U) \cap f^{-1}(U)$. Thus, by Remark 2.2. (4), the set V is a filtrating set which has the same maximal invariant set as U. $\square$

Proposition 2.7. *Let f be a homeomorphism of a compact metric space X and let U be a filtrating set. Then there exist two trapping regions A for f and B for f^{-1} such that $U = A \cap B$ and the three following equivalent properties are satisfied:*

1. *the attractor of A contains the attractor of B;*
2. *the repellor of B contains the repellor of A;*
3. *the repellor of A is disjoint from the attractor of B.*

Proof: We first prove that the three properties are equivalent. First, notice that the repellor associated to a trapping region is the largest invariant compact set disjoint from the attractor of the trapping region. If the first item is satisfied, the repellor of A is disjoint from the attractor of A, hence of B (i.e. the third item is satisfied). Conversely, if the repellor of A is disjoint from the attractor of B (item 3), then it is contained in the largest invariant compact set disjoint from the attractor of B, that is, the repellor of B (this gives item 2). The other implications follow similarly.

Let now U be a filtrating set. By definition, there are two trapping regions A_0 for f and B for f^{-1} such that $U = A_0 \cap B$. We define A as the union of A_0 with $f(X \setminus \operatorname{int}(B))$. Since the union of two trapping regions is a trapping region, A and B are trapping regions for f and f^{-1}, respectively. Since B is a trapping region for f^{-1}, $f(X \setminus \operatorname{int}(B))$ is disjoint from B so that $A \cap B = A_0 \cap B = U$.

Finally, the maximal invariant set in A contains the maximal invariant set in $f(X \setminus \operatorname{int}(B))$. Hence, the attractor of A contains the attractor of B. $\square$

2.2 Hyperbolic theory

The following result is certainly well-known and has been proven, in particular, in [ABD, Theorem 1].

Proposition 2.8. *Let M be a connected compact manifold. Any hyperbolic set Λ of a diffeomorphism $f \in \operatorname{Diff}^1(M)$, contained in the non-wandering set $\Omega(f)$ of f and with non-empty interior, is the whole manifold. Thus the diffeomorphism f is Anosov.*

We recall that a transitive hyperbolic set which is the maximal invariant set in a neighborhood is called a *basic set*. A basic set which is a chain-recurrence class is called a *basic piece.*

¿From Theorem 2.1 and the hyperbolic theory, we get:

Proposition 2.9. *Let f be a diffeomorphism of a compact manifold. Any hyperbolic chain-recurrence class of f is an isolated homoclinic class (hence, a basic piece of f).*

Remark 2.10. *In fact, any chain-recurrence class containing a hyperbolic set contains a periodic point.*

Proof: Let E be a hyperbolic chain-recurrence class of f. From Proposition 2.4, E admits arbitrarily small filtrating neighborhoods. Consider a small filtrating neighborhood U of E. The maximal invariant set Λ in U is a compact hyperbolic set. Moreover, the shadowing lemma (see, for instance, [Y]) asserts that, for any small δ, any ε-pseudo-orbit, with $\varepsilon > 0$ small enough, is δ-shadowed by a unique orbit of Λ. One deduces that Λ contains a sequence of periodic orbits $\{\gamma_n\}$ converging to E in the Hausdorff topology. As Λ is hyperbolic, the continuity of the local stable and unstable manifolds implies that there is $\mu > 0$ such that any two periodic orbits in Λ having points μ-close to each other are homoclinically related, so they have the same homoclinic class. As a consequence, all the γ_n, for n large enough, have the same homoclinic class H. As H is compact (by definition) it contains E. However, H is transitive so it is contained in a unique chain-recurrence class. Hence $E = H$, concluding the lemma. □

As a consequence of the previous proposition and the persistence of hyperbolic sets, one has the following:

Proposition 2.11. *Let f be a diffeomorphism of a compact manifold and V a filtrating set. Suppose that $\mathcal{R}(f) \cap V$ is hyperbolic. Then:*

- *$\mathcal{R}(f) \cap V$ consists of finitely many basic pieces;*
- *there is a C^1- neighborhood of f consisting of diffeomorphisms g such that V is a filtrating set of g and $\mathcal{R}(g) \cap V$ consists of finitely many basic pieces which are the continuations of the basic pieces of f. In particular, g on $\mathcal{R}(g) \cap V$ is topologically conjugate to f on $\mathcal{R}(f) \cap V$.*

2.3 Genericity results

[BC] states the following consequences of the connecting lemma for pseudo-orbits:

Theorem 2.2. *Let M be a compact manifold. The set of diffeomorphisms f for which all the periodic orbits are hyperbolic and the chain-recurrent set $\mathcal{R}(f)$ is the closure of the set of periodic points is a residual subset of* $\mathrm{Diff}^1(M)$.

This is a direct consequence of Kupka-Smale's theorem, of Pugh's theorem on the generic density of the periodic points in the non-wandering set, [P], and of [BC, Corollaire 1.2.],.

Proposition 2.12 ([BC], Remarque 1.10, 1.12 and Corollaire 1.11). *There is a residual part $\mathcal{G}_2$ of* $\mathrm{Diff}^1(M)$ *such that for any $f \in \mathcal{G}_2$, any chain-recurrence class E of f containing a periodic point is a homoclinic class.*

Remark 2.13. *The hypothesis of Proposition 2.12 are satisfied (for f in $\mathcal{G}_2$), for instance, if E has non-empty interior or if E is isolated in $\mathcal{R}(f)$.*

Combining this result with Proposition 2.9, we get:

Corollary 2.14. *For any diffeomorphism f in the residual set $\mathcal{G}_2$, any homoclinic class which is hyperbolic is a basic piece.*

Let us state a well-known property of homoclinic class which we will use later (see for instance [CMP]).

Remark 2.15. *Let M be a closed manifold. There is a residual subset of* $\mathrm{Diff}^1(M)$ *of diffeomorphisms f such that for every periodic point p of f the homoclinic class $H(p,f)$ varies continuously under small C^1- perturbations.*

[BC] shows the existence of a residual set $\mathcal{G}$ of $\mathrm{Diff}^1(M)$ such that, if $f \in \mathcal{G}$ and if p is a (hyperbolic) periodic point whose chain-recurrence class $C(p)$ is isolated in $\mathcal{R}(f)$, then this chain-recurrence class is *robustly isolated*: there are neighborhoods U of $C(p)$ in M and $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that, for any $g \in \mathcal{U}$, $\mathcal{R}(g) \cap U$ is the chain-recurrence class of the hyperbolic continuation p_g of p (in fact, the homoclinic class of p_g). The following proposition gives some converse result of this property: for generic diffeomorphisms, the non-isolated homoclinic classes are locally generically non-isolated.

Proposition 2.16. *Let M be a compact manifold. There is a residual subset $\mathcal{G}_4$ of* $\mathrm{Diff}^1(M)$ *such that any $f \in \mathcal{G}_4$ has the following property:*

Assume that p is a hyperbolic periodic point of f such that the chain-recurrence class of p is not isolated. Then there is a neighborhood $\mathcal{U}$ of f on which the hyperbolic continuation of p is well defined and such that, for any $g \in \mathcal{G}_4 \cap \mathcal{U}$, the chain-recurrence class of p_g is not isolated.

Proof: It is enough to prove the property for periodic orbits of period less than n, for any integer n (that is, we will build a residual subset $\mathcal{R}_n$ and define the residual set $\mathcal{G}_4$ as the countable intersection of the $\mathcal{R}_n$). Now fix some $n \in \mathbb{N}$, there is a dense open subset $\mathcal{O}$ of $\mathrm{Diff}^1(M)$ such that, for any connected component $\mathcal{O}_0$ of $\mathcal{O}$, the periodic orbits of period less than n are all hyperbolic and has a hyperbolic continuation on the whole $\mathcal{O}_0$. As $\mathcal{O}$ has countably many connected components, it is enough to build, for any component $\mathcal{O}_0$, a residual set $\mathcal{R}_{n,\mathcal{O}_0}$ such that the announced properties hold in $\mathcal{R}_{n,\mathcal{O}_0} \cap \mathcal{O}_0$.

So, on $\mathcal{O}_0$, the periodic orbits of period less than n are given by finitely many continuous functions $f \mapsto p(f)$. Thus it is sufficient to build, for any of these periodic orbits p, a residual set $\mathcal{R}_{n,\mathcal{O}_0,p}$ such that the property is verified on the periodic orbit p for $f \in \mathcal{R}_{n,\mathcal{O}_0,p} \cap \mathcal{O}_0$.

Let I be the set of diffeomorphisms $f \in \mathcal{O}_0$ for which the chain-recurrence class $C(p(f))$ is robustly isolated. By definition of being robustly isolated, I is an open subset of $\mathcal{O}_0$. Write $\tilde{I} = \mathcal{O} \setminus \overline{I}$, then $I \cup \tilde{I}$ is a dense open subset of $\mathcal{O}$ and therefore of $\mathrm{Diff}^1(M)$.

Denote by $\mathcal{G}$ the residual subset of $\mathrm{Diff}^1(M)$, built in [BC], on which the isolated chain-recurrence classes are robustly isolated. We define $\mathcal{R}_{n,\mathcal{O}_0,p} = \mathcal{G} \cap (I \cup \tilde{I})$. Consider $f \in \mathcal{R}_{n,\mathcal{O}_0,p} \cap \mathcal{O}_0$ and assume that the chain-recurrence class of $p(f)$ is not isolated. Then f does not belong to I; consequently, f belongs to $\tilde{I}$. By definition, $\tilde{I}$ is open and so it contains a neighborhood $\mathcal{U}$ of f. Therefore, for every $g \in \mathcal{R}_{n,\mathcal{O}_0,p} \cap \mathcal{U}$, the chain-recurrence class $C(p(g))$ is not robustly isolated (by definition of $\tilde{I}$) so it is not isolated, by definition of $\mathcal{G}$.

Now, $\mathcal{R}_{n,\mathcal{O}_0}$ is the (finite) intersection over the periodic orbits p of period less that n of the residual sets $\mathcal{R}_{n,\mathcal{O}_0,p}$ and is therefore residual. Finally, $\mathcal{R}_n$ is the intersection, over all the connected components $\mathcal{O}_0$, of $\mathcal{R}_{n,\mathcal{O}_0}$. The announced residual set $\mathcal{G}_4$ is the intersection of all the $\mathcal{R}_n$, $n \in \mathbb{N}$. □

3 Dominated splitting/homoclinic tangencies: a local version on surfaces

The aim of this section is to obtain a local version of the following theorem of [PS] and to derive some consequences of such an extension.

Theorem 3.1 (Pujals-Sambarino, [PS]). *Let S be a compact surface. Then there is a dense subset $\mathcal{D}$ of $\mathrm{Diff}^1(S)$, such that any f in $\mathcal{D}$ either is Axiom A or else exhibits a homoclinic tangency.*

In order to obtain a similar local result around a chain-recurrence class, we build in the next section some geometrical filtrating neighborhoods. Using such neighborhoods we complete the dynamics outside them by adding sinks and sources, this allows us in to apply Theorem 3.1 to the resulting global dynamics.

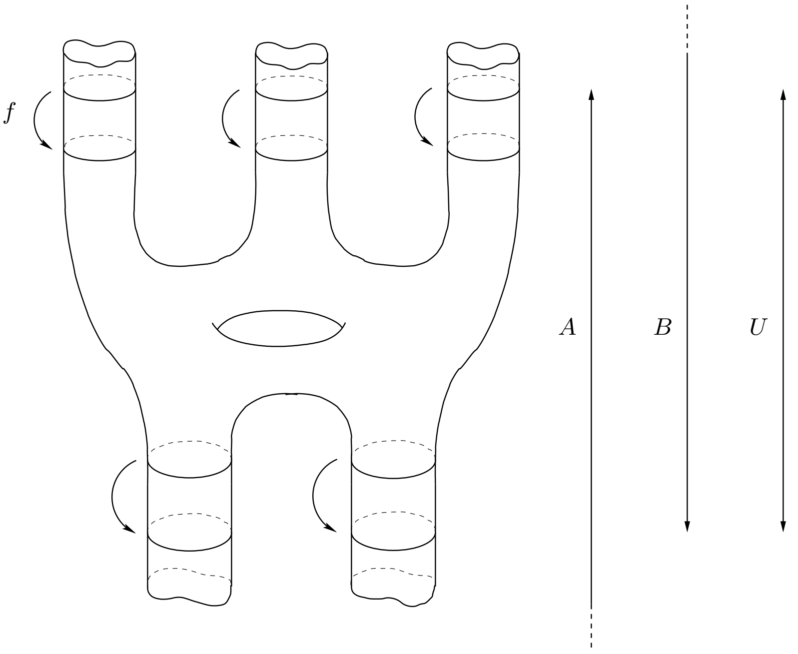
3.1 Geometrically simple filtrating neighborhoods

Definition 3.1. *Let f be a homeomorphism of a compact surface S.*

A trapping region A of f is said to be geometrically simple *if A is a compact surface with boundary and $A \setminus \mathrm{int}(f(A))$ is the finite union of disjoint annuli.*

A filtrating set U for f is said to be geometrically simple *if it is a compact surface with boundary and if there are two geometrically simple trapping regions A for f and B for f^{-1} such that:*

1. *$U = A \cap B$; and*
2. *$\mathcal{E} = A \setminus \mathrm{int}(f(A))$ and $\mathcal{S} = B \setminus \mathrm{int}(f^{-1}(B))$ are disjoint.*



Theorem 4. *Given any homeomorphism f of a compact surface S, any compact f- invariant set Λ, and any compact filtrating neighborhood V of Λ, there is a geometrically simple filtrating set $U \subset S$ whose maximal f- invariant set coincides with Λ.*

The proof uses the following lemma which is essentially proven in [BL, Lemme 1.3.2.].

Lemma 3.2. *Let f be a homeomorphism of a compact surface S and W a trapping region for f. Then there is a geometrically simple trapping region U such that the attractors and repellors (for f) of U and W coincide.*

Proof: Let Λ^+ and Λ^- be the attractor and the repellor associated to W. We define the open surface $\tilde{S} = S \setminus (\Lambda^+ \cup \Lambda^-)$. By Remark 2.2. (5), there exists a trapping region $\tilde{W}$ which is a compact surface with boundary whose attractor and repellor also are Λ^+ and Λ^-. Notice that any orbit by f in $\tilde{S}$ contains a unique point in $\tilde{W} \setminus f(\tilde{W})$ and that $\tilde{W} \setminus \operatorname{int}(f(\tilde{W}))$ is a compact surface with boundary whose boundary consists of the disjoint sets $\partial\tilde{W}$ and $f(\partial\tilde{W})$.

The orbit space of $\tilde{S}$ is canonically identified with the closed surface O obtained from $\tilde{W} \setminus \operatorname{int}(f(\tilde{W}))$ by gluing $\partial\tilde{W}$ to $f(\partial\tilde{W})$ by f. Note that O has finitely many connected components. The canonical projection $\pi\colon \tilde{S} \to O$, which associates to each point its orbit, is a covering and f is an automorphism of the covering acting transitively on the fibers: for each connected component O_0 of O, f generates the group of covering automorphisms (Deck transformations) for $\pi^{-1}(O_0) \to O_0$.

Let S_0 be a connected component of $\tilde{S}$ and $O_0 = \pi(S_0)$ be the associated connected component of O. We claim that there exists k such that $f^k(S_0) = S_0$: Otherwise S_0 would be disjoint from $f^i(S_0)$ for each i and homeomorphic to the compact surface O_0. This would imply that S contains infinitely many compact surfaces, hence connected components, which is a contradiction. As a consequence, $\tilde{S}$ has finitely many connected components. The projection $S_0 \to O_0$ is an infinite cyclic covering whose automorphism group is generated by an iterate f^ℓ of f, where ℓ is the period of S_0 under the action of f. As the genus of S_0 is bounded (by the genus of S), S_0 is homeomorphic to the open annulus $\mathbb{S}^1 \times \mathbb{R}$ and O_0 to the torus or to the Klein bottle. Moreover, f^ℓ (the automorphism of the covering) is conjugate to the translation $(x, y) \mapsto (x, y + 1)$ or else to the translation composed with the symmetry: $(x, y) \mapsto (-x, y + 1)$.

Summing up this argument, one has that $\tilde{S}$ is a finite union of disjoint cylinders S_i homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^1$ which are permuted by f and such that the expression of $f\colon S_i \to S_j$ in these coordinates writes as $(x, y) \mapsto (\pm x, y+1)$. As a consequence, there is a family of essential circles $\mathbb{S}^1 \times \{t_i\}$ in each S_i such that the union $\tilde{U}$ of the half-cylinders $\mathbb{S}^1 \times [t_i, +\infty[$ is mapped into its interior by f. Furthermore, the difference $\tilde{U} \setminus \mathrm{int}(f(\tilde{U}))$ is a finite union of disjoint annuli. Notice that the intersection of $\tilde{W}$ with any component S_i contains a half-cylinder $\mathbb{S}^1 \times]a_i, +\infty[$ and is disjoint from an other half-cylinder $\mathbb{S}^1 \times]-\infty, b_i[$.

¿From the previous arguments, we deduce that $U = \tilde{U} \cup \Lambda^+$ is a compact neighborhood of Λ^+ and hence a trapping region which is geometrically simple. Finally, the fact that the repellors of U and W coincide follows by construction. □

Proof of Theorem 4: By Proposition 2.7, one can write the filtrating neighborhood V as the intersection $A \cap B$, where A and B are trapping regions for f and f^{-1}, respectively, such that the repellor associated to A is disjoint from the attractor of B. By Lemma 3.2, there exist two geometrically simple trapping regions $\tilde{A}$ and $\tilde{B}$ for f and f^{-1}, respectively, such that A and $\tilde{A}$ (resp., B and $\tilde{B}$) have the same attractor and repellor. As a consequence, for any $i, j \in \mathbb{Z}$, the maximal invariant sets in the filtrating sets $V = A \cap B$ and $f^{-i}(\tilde{A}) \cap f^j(\tilde{B})$ coincide, see Remark 2.2. (1) and (7).

For i large enough, the fundamental domain $f^{-i}(\tilde{A}) \setminus \mathrm{int}(f^{-i+1}(\tilde{A}))$ is contained in an arbitrarily small neighborhood of the repellor of $\tilde{A}$, and hence of the repellor of A. Similarly, for j large enough, the fundamental domain $f^j(\tilde{B}) \setminus \mathrm{int}(f^{j-1}(\tilde{B}))$ is contained in an arbitrarily small neighborhood of the attractor of $\tilde{B}$, and hence of the attractor of B. Thus, defining $A_0 = f^{-i}(\tilde{A})$ and $B_0 = f^j(\tilde{B})$, the sets $A_0 \setminus f(A_0)$ and $B_0 \setminus f^{-1}(B_0)$ are disjoint. Since, by our choice, $\tilde{A}$ and $\tilde{B}$ are geometrically simple, each of the sets $A_0 \setminus \mathrm{int}(f(A_0))$ and $B_0 \setminus \mathrm{int}(f^{-1}(B_0))$ is a finite union of disjoint annuli (these sets are homeomorphic to $\tilde{A} \setminus \mathrm{int}(f(\tilde{A}))$ and $\tilde{B} \setminus \mathrm{int}(f^{-1}(\tilde{B}))$).

Consequently, $U = A_0 \cap B_0$ is a geometrically simple filtrating set having the same maximal invariant set as V. □

Proposition 3.3. *Let S be a compact surface and U a geometrically simple filtrating set of a diffeomorphism $f \in \mathrm{Diff}^1(S)$. Then there exist a compact surface S_0, a diffeomorphism f_0 of S_0, and a diffeomorphism $\varphi\colon U \to U_0 \subset S_0$ such that:*

- *for any $x \in U \cap f^{-1}(U)$, we have $\varphi \circ f(x) = f_0 \circ \varphi(x)$;*
- *the connected components of $S_0 \setminus \mathrm{int}(U_0)$ are diffeomorphic to the disk $\mathbb{D}^2$;*
- *the intersection of the chain-recurrent set $\mathcal{R}(f_0)$ of f_0 with any connected component of $S_0 \setminus \mathrm{int}(U_0)$ is a periodic point (attracting or repelling).*

Proof: Write $U = A \cap B$, where A and B are trapping regions for f and f^{-1}, and consider the disjoint sets $\mathcal{E} = A \setminus \mathrm{int}(f(A))$ and $\mathcal{S} = B \setminus \mathrm{int}(f^{-1}(B))$, which are the finite union of disjoint annuli.

Notice that $U \cup f(U)$ is equal to $U \cup f(\mathcal{S})$ and is obtained by gluing annuli along the boundary $\partial B \subset \partial U$. Consider the compact surface S_0 obtained by gluing disks on each boundary component of $U \cup f(U)$. There is a diffeomorphism f' of S_0 coinciding with f on U and satisfying the following properties:

1. each disk attached to the boundary of A (resp. $f(B)$) contains a unique periodic point, which is a source (resp. a sink).
2. $\mathcal{R}(f_0)$ is the union these periodic points and $\mathcal{R}(f) \cap f(U) \cap f^{-1}(U)$.

The proof of this fact is analogous to the argument in [BL, Lemme 1.3.4]. Finally, the proposition follows using the previous fact. □

Consider the setting and the notations of Proposition 3.3. For the sake of simplicity, let us identify U with $U_0 = \varphi(U)$. The next proposition allows us to use the perturbations of f_0 for understanding the perturbations of f.

Proposition 3.4. *Given any compact set $K \subset \operatorname{int}(U)$ there is a neighborhood $\mathcal{V} \subset \operatorname{Diff}^1(S_0)$ of the identity map of S_0 and a continuous map $\chi\colon \mathcal{V} \to \operatorname{Diff}^1(S)$ sending the identity map to the identity map, such that g and $\chi(g)$ coincide on K for any $g \in \mathcal{V}$.*

The construction uses the next lemma.

Lemma 3.5. *Let M be a compact manifold, $U \subset M$ an open set and $K \subset U$ a compact set. Denote by $\operatorname{Diff}^1_U(M) \subset \operatorname{Diff}^1(M)$ the set of diffeomorphims which are the identity outside U.*

Then there is a C^1- neighborhood $\mathcal{V} \subset \operatorname{Diff}^1(M)$ of the identity of M and a continuous map $\psi\colon \mathcal{V} \to \operatorname{Diff}^1_U(M)$ which sends the identity map to the identity map such that for any $f \in \mathcal{V}$ the diffeomorphisms f and $\psi(f)$ coincide on K.

Proof: We consider an arbitrary smooth Riemannian metric on M and denote by $\exp\colon TM \to M$ the associated exponential map. There is a small neighborhood $\tilde{\mathcal{V}}$ of the identity of M such that any $f \in \tilde{\mathcal{V}}$ is the image by exp of a C^1- vector field X_f. Furthermore, the map $f \mapsto X_f$ is a homeomorphism from $\tilde{\mathcal{V}}$ to a C^1- neighborhood $\tilde{\mathcal{X}}$ of the zero vector field of M.

Let $\theta\colon M \to [0,1]$ be a smooth map which takes the value 0 on the complement of U and the value 1 on K. The map $\mathcal{X}_\theta\colon f \mapsto \theta \cdot X_f$, defined on $\tilde{\mathcal{V}}$, is continuous for the C^1- topology and sends the identity diffeomorphism to the zero vector field. Hence there exists a neighborhood $\mathcal{V} \subset \tilde{\mathcal{V}}$ of the identity whose image by $\mathcal{X}_\theta$ is contained in $\tilde{\mathcal{X}}$. We define ψ on $\mathcal{V}$ as $f \mapsto \exp \circ \mathcal{X}_\theta(f)$.

Since θ is 1 on K and 0 outside U, the diffeomorphism $\psi(f)$ coincide with f on K and with the identity outside U. Clearly, $\psi(\mathrm{Id}) = \mathrm{Id}$. □

Proof of Proposition 3.4: We apply Lemma 3.5 to the sets $K \subset U \subset S$, obtaining a neighborhood $\mathcal{V}$ of the identity of S_0 and a map $\psi\colon \mathcal{V} \to \operatorname{Diff}^1_U(S_0)$. For each $g \in \mathcal{V}$ we define the diffeomorphism $\chi(g) \in \operatorname{Diff}^1(S)$ coinciding with $\psi(g)$ on U (using the identification of U and U_0 by φ) and with the identity map outside U. □

3.2 Dominated splitting/homoclinic tangencies: a local version

Here is the announced local version of Theorem 3.1.

Theorem 5. *Let S be a compact surface. Given any open set $\mathcal{U}$ in $\operatorname{Diff}^1(S)$ and any set V which is a filtrating set for every $f \in \mathcal{U}$, there is a dense subset $\mathcal{D} = \mathcal{H} \cup \mathcal{T}$ of $\mathcal{U}$ such that:*

1. *$\mathcal{H}$ is a C^1- open set such that for any $g \in \mathcal{H}$ the intersection $\mathcal{R}(g) \cap V$ is the union of finitely many (disjoint) hyperbolic homoclinic classes (which are basic sets);*

2. *for any $g \in \mathcal{T}$ there is a a homoclinic class contained in V which exhibits a homoclinic tangency (whose orbit is contained in V).*

Proof: Let V and $\mathcal{U}$ be as in the assumptions of the theorem. Let $\mathcal{H}$ be the set of diffeomorphisms $f \in \mathcal{U}$ such that $V \cap \mathcal{R}(f)$ is hyperbolic. The set $\mathcal{H}$ is open (see Proposition 2.11). Let $\mathcal{T}$ be the set of diffeomorphisms exhibiting a homoclinic tangency in V. We have to show that $\mathcal{H} \cup \mathcal{T}$ is dense in $\mathcal{U}$.

Consider any f in $\mathcal{U}$. Using Theorem 4, one obtains a geometrically simple filtrating neighborhood U whose maximal invariant set coincides with the maximal invariant set of f in V. So, using Remark 2.2. (2), for any g sufficiently close to f the maximal invariant sets of g in U and in V coincide. So it is sufficient to prove the theorem for the set U.

Proposition 3.3 introduces a compact surface S_0, a diffeomorphism f_0 of S_0, and a conjugation φ. For the sake of simplicity, as above, we identify $U \subset S$ with its image $\varphi(U) = U_0 \subset S_0$. We consider the compact set $K = f(U) \cap f^{-1}(U)$, which is contained in the interior of U since U is a filtrating set. We denote by $\mathcal{V}$ the neighborhood of the identity of S_0 and by $\chi\colon \mathcal{V} \to \mathrm{Diff}^1(S)$ the map given by Proposition 3.4.

Any diffeomorphism $g_0 \in \mathrm{Diff}^1(S_0)$ close enough to f_0 can be written in the form $g_0 = h_0 \circ f_0$ for some $h_0 \in \mathcal{V}$. Define $\Theta(g_0)$ by $\Theta(g_0) = \chi(h_0) \circ f$. This map is continuous and sends f_0 to f.

By Theorem 3.1, there exists $g_0 \in \mathrm{Diff}^1(S_0)$ arbitrarily C^1- close to f_0 such that g_0 either is Axiom A or else exhibits a homoclinic tangency. Let us show that $g = \Theta(g_0)$ belongs to $\mathcal{H} \cup \mathcal{T}$.

By continuity of Θ and Remark 2.2. (2), for every g_0 sufficiently close to f_0, U is a filtrating set of $g = \Theta(g_0)$. Moreover, by Remark 2.2. (4), the maximal invariant sets of g in U and K coincide. As a consequence, the intersection $\mathcal{R}(g) \cap U$ is the union of the chain-recurrence classes contained in K (see Remark 2.2. (6)). Finally, by construction, the dynamics of g and g_0 in K coincide (i.e. they are conjugate by the identification φ).

We first consider the case where g_0 is Axiom A. Then $\mathcal{R}(g) \cap U$ is hyperbolic since $\mathcal{R}(g_0) \cap U$ is hyperbolic. Thus, by Proposition 2.11, we get $g \in \mathcal{H}$.

In the case where g_0 exhibits a homoclinic tangency at some point q one has that the whole orbit of q is contained in K: note first that q is non-periodic and chain-recurrent and the chain-recurrent set of f_0 (thus of g_0) outside K is a finite union of hyperbolic sinks and sources. Since K is a filtrating set of f_0, this remains true for g_0 close enough to f_0. Therefore the orbit of q by g_0 is contained in K. Consequently, $g = \Theta(g_0)$ also exhibits a homoclinic tangency at q and its orbit remains in U, hence in V, that is, $g \in \mathcal{T}$. □

3.3 Consequences of the local dichotomy

Corollary 3. *For any compact surface S there is a residual part $\mathcal{G}_5$ of $\mathrm{Diff}^1(S)$ such that every chain-recurrence class E of any $f \in \mathcal{G}_5$ satisfies the following property:*

- *if E admits a dominated splitting then it is a hyperbolic basic piece,*
- *if E has no dominated splitting then it is accumulated by homoclinic tangencies: for every neighborhoods U of E in S and $\mathcal{U}$ of f in $\mathrm{Diff}^1(S)$ there exist a diffeomorphism $g \in \mathcal{U}$ and a homoclinic class $H(P, g)$ of g which is contained in U and exhibits a homoclinic tangency.*

The proof will use the following lemma:

Lemma 3.6. *Let V be a compact subset of a compact surface S. Then there exists a dense open subset $\mathcal{O}(V) \subset \mathrm{Diff}^1(S)$ which decomposes as $\mathcal{O}(V) = \mathcal{H}(V) \cup \mathcal{W}(V) \cup \mathcal{S}(V)$ such that:*

1. *the sets $\mathcal{H}(V)$, $\mathcal{W}(V)$ and $\mathcal{S}(V)$ are open and pairwise disjoint;*
2. *for any f in $\mathcal{H}(V)$, the set V is a filtrating set of f and the intersection of $\mathcal{R}(f)$ with V is hyperbolic;*
3. *for any $f \in \mathcal{W}(V)$, the set V is a filtrating set of f and there exists g arbitrarily C^1- close to f which exhibits a homoclinic tangency inside V; and*
4. *for any f in $\mathcal{S}(V)$, the set V is not a filtrating set of f.*

Proof: Let $\mathcal{U}(V)$ be the set of $f \in \mathrm{Diff}^1(S)$ such that V is a filtrating set of f. By Remark 2.2. (2), the set $\mathcal{U}(V)$ is open in $\mathrm{Diff}^1(S)$. The set $\mathcal{S}(V)$ is the interior of $\mathrm{Diff}^1(S) \setminus \mathcal{U}(V)$.

Let $\mathcal{H}(V)$ be the set of diffeomorphisms $f \in \mathcal{U}(V)$ such that the intersection of $\mathcal{R}(f)$ with V is hyperbolic. This set is open by Proposition 2.11.

We define $\mathcal{W}(V)$ as $\mathcal{U}(V) \setminus \overline{\mathcal{H}(V)}$. By construction, the three sets $\mathcal{H}(V)$, $\mathcal{W}(V)$, and $\mathcal{S}(V)$ are open and pairwise disjoint. Moreover, their union is dense in $\mathrm{Diff}^1(S)$. It remains to show that any diffeomorphism in $\mathcal{W}(V)$ may be approximated by a diffeomorphism which exhibits a homoclinic tangency inside V.

We apply Theorem 5 to V and the open set of diffeomorphisms $\mathcal{W}(V)$. Since for any diffeomorphism f in $\mathcal{W}(V)$ the intersection $\mathcal{R}(f) \cap V$ is not hyperbolic, Theorem 5 ensures the existence of a dense subset of diffeomorphisms of $\mathcal{W}(V)$ which exhibit a homoclinic tangency inside V. □

Proof of Corollary 3: We consider the countable family $\{V_n\}$ of compact sets of S given by Proposition 2.6. Using Lemma 3.6 we get the residual set $\mathcal{G}_5 = \bigcap_n \mathcal{O}(V_n)$. Let E be a chain-recurrence class of $f \in \mathcal{G}_5$.

First, assume that E has a dominated splitting. Since a dominated splitting always extends to a neighborhood (see Remark 1.6. (2)), there exists a neighborhood U of E such that the maximal invariant set of f in U admits a dominated splitting. By Proposition 2.4, E admits a filtrating neighborhood V contained in U. By Proposition 2.6, there exists a filtrating neighborhood V_n of E contained in U. Since the existence of a dominated splitting is a robust property (see Remark 1.6. (3)), by Remark 2.2. (2), there exists a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(S)$ such that, for any $g \in \mathcal{U}$, the set V_n is a filtrating set for g and the maximal invariant set Λ_g of g in V_n has a dominated splitting. Consequently, g does not exhibit any homoclinic tangency in V_n. This shows that f belongs neither to $\mathcal{S}(V_n)$ nor to $\mathcal{W}(V_n)$. Hence f belongs to $\mathcal{H}(V_n)$ and E is hyperbolic. Finally, by Proposition 2.9, it is a basic piece.

Let us now analyze the case where E has no dominated splitting. Consider neighborhoods U of E and $\mathcal{U}$ of f. As above, there exists a neighborhood $V_n \subset U$ of E which is a filtrating set. By construction, f does not belong to $\mathcal{S}(V_n)$. Since E has no dominated splitting, f does not belong to $\mathcal{H}(V_n)$. Hence f belongs to $\mathcal{W}(V_n)$ and may be approximated by a diffeomorphism $g \in \mathcal{U}$ which exhibits a homoclinic tangency in $V_n \subset U$. □

4 Proof of Theorem 2 and Corollary 1

Proof of Theorem 2. We need the following dichotomy for chain recurrence classes of C^1-generic diffeomorphisms which generalizes a similar dichotomy for homoclinic classes in [BDP].

Theorem 4.1 ([ABC]). *There is a residual part $\mathcal{G}_3$ of* $\mathrm{Diff}^1(M)$ *such that for any $f \in \mathcal{G}_3$ and any chain-recurrence class E of f, the following dichotomy holds:*

1. *either E admits a dominated splitting,*
2. *or E is the limit of a sequence of sinks or sources for the Hausdorff topology (Newhouse phenomenon).*

Let $\mathcal{G}_3$ and $\mathcal{G}_5$ be the residual subsets of $\mathrm{Diff}^1(S)$ given by Theorem 4.1 and Corollary 3. Let $\mathcal{G}_0$ be the intersection $\mathcal{G}_0 = \mathcal{G}_3 \cap \mathcal{G}_5$. We consider $f \in \mathcal{G}_0$ and a chain-recurrence class E of f.

First, if E has a dominated splitting then, by Corollary 3, it is a hyperbolic basic piece and hence it is isolated. If E has no dominated splitting then the same corollary implies that it is accumulated by homoclinic tangencies and Theorem 4.1 shows that E is the Hausdorff limit of a sequence of sinks or sources, hence E is not isolated in $\mathcal{R}(f)$. This concludes the proof of Theorem 2.

Proof of Corollary 1. Suppose that $\mathcal{R}(f)$ has non-empty interior for some $f \in \mathcal{G}_0$. By Proposition 2.5, there is a chain-recurrence class E of f which has non-empty interior. Note that this interior cannot contain any sink or source, so that E is not contained in the closure of sinks nor sources. By Theorem 2, this implies that E is a hyperbolic basic set (with non-empty interior). Proposition 2.8 shows that f is an Anosov diffeomorphism. By [F], the surface S is the torus $\mathbb{T}^2$ and f is conjugate to a linear Anosov map of $\mathbb{T}^2$, finishing the proof of Corollary 1.

5 Homoclinic tangencies inside a prescribed homoclinic class

5.1 Homoclinic classes with small angles

The aim of this section is to create (by C^1- perturbations) homoclinic tangencies in any homoclinic class containing saddles whose stable and unstable bundles form small angles.

Proposition 5.1. *Let g be a diffeomorphism of a closed Riemannian surface S and $H(p,g)$ a non-trivial homoclinic class.*

For any $\varepsilon \in]0, 2\pi[$, if there exists a periodic point $q \in H(p,g)$ homoclinically related to the orbit of p and such that the angle between the stable and unstable spaces at q is smaller than $\frac{\varepsilon}{8}$ then there exists a C^1- perturbation φ of the identity of S such that:

- *the support of φ is contained in an arbitrarily small neighborhood of q;*
- *the diffeomorphism φ is ε- close to the identity in the C^1- norm;*
- *the diffeomorphism $h = \varphi \circ g$ has a homoclinic tangency associated to p (thus the homoclinic class $H(p,h)$ contains a homoclinic tangency).*

The main ingredient of the proof is the following lemma.

Lemma 5.2. *Let $\varepsilon > 0$ be small enough (smaller than 2π) and A a hyperbolic automorphism of $\mathbb{R}^2$ such that*

- *the eigenvalues λ^s, λ^u of A satisfy $|\lambda^s|^{-1}, |\lambda^u| > 5$;*

- *the angle between the stable space E^s and the unstable space E^u of A is smaller than $\frac{\varepsilon}{8}$.*

Then for every pair of orbits $\mathcal{O}^s$ in the stable space E^s and $\mathcal{O}^u$ in the unstable space E^u of A there exists a perturbation φ of the identity of $\mathbb{R}^2$ such that

- *φ has compact support contained in an arbitrarily small neighborhood of $\bar{0}$ and coincides with the identity on a neighborhood of $\bar{0}$ and of the orbits $\mathcal{O}^s$ and $\mathcal{O}^u$;*
- *φ is ε- close to the identity in the C^1- norm; and*
- *the diffeomorphism $\varphi \circ A$ of $\mathbb{R}^2$ exhibits a homoclinic tangency associated to the saddle fixed point $\bar{0}$.*

Let us first prove the following lemma:

Lemma 5.3. *Let A be a hyperbolic automorphism of $\mathbb{R}^2$ such that the eigenvalues λ^s, λ^u of A satisfy $|\lambda^s|^{-1}, |\lambda^u| > 5$. Given any pair of points x in the stable space E^s of A and y in the unstable space E^u of A there exists an arbitrarily small $t > 0$ such that the strip $[\frac{t}{2}, t] \times \mathbb{R}$ is disjoint from the orbits of x and y by A.*

Proof: We consider the sequences $\{r_i\}$ and $\{s_i\}$ of the first coordinates of $A^i(x)$ and $A^i(y)$. We choose some arbitrarily small r_i and assume, for instance, that $r_i > 0$ (the argument is analogous in the other case).

By our assumption on the eigenvalues, the strip $]r_i, 5r_i[\times\mathbb{R}$ does not contain any point of the orbit of x and contains at most one point of the orbit of y. If it does not contain any point of the orbit of y then the lemma follows for $t = 2r_i$. Otherwise, $]r_i, 5r_i[$ contains some s_j, then one of the ratios $\frac{s_j}{r_i}$ and $\frac{5r_i}{s_j}$ is strictly larger than 2. We can hence choose t such that $[t, 2t]$ is contained in $]r_i, s_j[\cup]s_j, 5r_i[$. □

Proof of Lemma 5.2: We fix a smooth function $\psi \colon \mathbb{R} \to [0, 1]$ such that ψ is equal to 0 outside $]\frac{1}{2}, 1[$, its graph is tangent to the graph of the identity at the point $(\frac{3}{4}, \frac{3}{4})$, and $|D\psi|$ is bounded by 4. (One may easily build such a function by smoothing a piecewise affine function.)

We choose orthonormal coordinates of $\mathbb{R}^2$ such that in these coordinates the unstable space of A is the axis $\mathbb{R} \times \{0\}$ and the stable space is the line $y = \sigma x$, with $\sigma > 0$. By Lemma 5.3, there exists some arbitrarily small $t > 0$ such that the strip $[\frac{t}{2}, t] \times \mathbb{R}$ is disjoint from the orbits $\mathcal{O}^s$ and $\mathcal{O}^u$.

Consider the map $\tilde{\Psi}_\sigma \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x, y) \mapsto (x, y + \sigma\,\psi(x))$. By construction, this is a diffeomorphism of $\mathbb{R}^2$ which coincides with the identity outside the strip $[\frac{1}{2}, 1] \times \mathbb{R}$. We have that $\|D\tilde{\Psi}_\sigma - \mathrm{Id}\|$ and $\|D\tilde{\Psi}_\sigma^{-1} - \mathrm{Id}\|$ are strictly uniformly bounded by $4\,\sigma$ on $\mathbb{R}^2$. There exists a diffeomorphism Ψ_σ of $\mathbb{R}^2$ with compact support contained in the strip $[\frac{1}{2}, 1] \times \mathbb{R}$ such that $\|D\Psi_\sigma - \mathrm{Id}\|$ and $\|D\Psi_\sigma^{-1} - \mathrm{Id}\|$ are strictly uniformly bounded by $4\,\sigma$ on $\mathbb{R}^2$ and such that Ψ_σ and $\tilde{\Psi}_\sigma$ coincide on a neighborhood of the segment $[\frac{1}{2}, 1] \times \{0\}$.

The map φ in the proposition is now defined by $(x, y) \mapsto t\,\Psi_\sigma(\frac{x}{t}, \frac{y}{t})$. This is a diffeomorphism of $\mathbb{R}^2$ which coincides with the identity outside an arbitrarily small (for t small) neighborhood of $\bar{0}$ contained in the strip $[\frac{t}{2}, t] \times \mathbb{R}$. By construction, φ is the identity on a neighborhood of $\bar{0}$ and of the orbits $\mathcal{O}^s$ and $\mathcal{O}^u$. Finally, the image $\varphi(E^u)$ is tangent to the space E^s at the point $z = \varphi(\frac{3}{4}t, 0) = (\frac{3}{4}t, \sigma\frac{3}{4}t)$.

Since the C^1- norm is invariant by homotheties, $\|D\varphi - \mathrm{Id}\|$ and $\|D\varphi^{-1} - \mathrm{Id}\|$ are strictly uniformly bounded by $4\,\sigma$ on $\mathbb{R}^2$. Notice that σ is the tangent of the angle α between the eigenspaces of A. By assumption, the angle α is less than $\frac{\varepsilon}{8} < \frac{\pi}{4}$ so that σ is bounded by 2α, which is less than $\frac{\varepsilon}{4}$. Hence φ is an ε- perturbation of the identity for the C^1- distance.

Set $I^s = \{(x, \sigma x), x \in [\frac{t}{2}, t]\}$. Since, by construction (Lemma 5.3), for any $n > 0$, $A^n(I^s)$ is disjoint from the strip $[\frac{t}{2}, t] \times \mathbb{R}$, we have $(\varphi \circ A)^n(I^s) = A^n(I^s)$, so that I^s is contained in the stable manifold of $\bar{0}$ for $\varphi \circ A$. Similarly, we set $I^u = [\frac{t}{2}, t] \times \{0\}$. We have, for every $n > 0$, that $(\varphi \circ A)^{-n}(\varphi(I^u)) = A^{-n}(I^u)$ so that $\varphi(I^u)$ is contained in the unstable manifold of $\bar{0}$ by $\varphi \circ A$. It follows that the stable and unstable manifolds of $\bar{0}$ for $\varphi \circ A$ have a tangency at z (in a small neighborhood of $\bar{0}$). $\square$

Proof of Proposition 5.1: Since $H(p, g)$ is a non-trivial homoclinic class there exists some non-trivial hyperbolic basic set K contained in $H(p, g)$ that contains q. Hence there is a family of distinct periodic orbits $\{\gamma_n, n \in \mathbb{N}\}$ contained in K such that for each n there is a point $q_n \in \gamma_n$ such that the sequence $\{q_n\}$ converges to q. This implies that

- the periods π_n of the orbits γ_n go to ∞; hence, for n large enough, the stable eigenvalue λ_n^s and the unstable eigenvalue λ_n^u of γ_n satisfy $|\lambda_n^s|^{-1}, |\lambda_n^u| > 5$;
- the angle between the stable and the unstable manifold of γ_n at q_n is smaller than $\frac{\varepsilon}{8}$ for n large enough.

We consider one q_n with n large enough. By an arbitrarily small perturbation in an arbitrarily small neighborhood of q_n, we can assume that g^{π_n} is linear in a neighborhood of q_n. After this small perturbation q_n and p remain homoclinically related. We choose $x \in W^s(\gamma_n) \pitchfork W^u(p)$ and $y \in W^u(\gamma_n) \pitchfork W^s(p)$. Notice that there is a small neighborhood U of q_n on which g^{π_n} is linear such that the intersection of the orbit of x with U is contained in the local stable manifold of q_n and has the form $\{g^{k\pi_n}(x_0), k \in \mathbb{N}\}$. Similarly, the intersection of the orbit of y with U is contained in the local unstable manifold of q_n and has the form $\{g^{-k\pi_n}(y_0), k \in \mathbb{N}\}$.

By Lemma 5.2, there exists a perturbation φ_0 of the identity of S, with support in U, disjoint from the forward orbit of x_0 and the backward orbit of y_0 and such that $\varphi_0 \circ g^{\pi_n}$ has a homoclinic tangency z in U associated to the point q_n. As the support of this perturbation is disjoint from the forward orbit of x, the backward orbit of y, and the whole orbits of p and q_n, the points p and q_n remain homoclinically related for the diffeomorphism $\varphi_0 \circ g$. By a classical argument, an unfolding of a homoclinic tangency associated to q_n (that is a small perturbation in a neighborhood of z) creates a homoclinic tangency associated to p.

The C^1- norm of the perturbation φ_0 was strictly smaller than ε so that the perturbation remains ε- close to the identity. Finally, by construction, the support of the perturbation is arbitrarily close to q. $\square$

5.2 Homoclinic classes with no dominated splitting

Proposition 5.4. *Let f be a diffeomorphism of a compact surface S and p a hyperbolic periodic point of f whose homoclinic class $H(p, f)$ does not admit any dominated splitting. Fix a C^1-neighborhood $\mathcal{U}$ of f and any $\alpha > 0$.*

Then there exists a periodic point $q \in H(p, f)$ whose orbit by f is homoclinically related to p such that for any neighborhood U of the orbit of q there exists a perturbation $g \in \mathcal{U}$ of f, with support in U, such that

- *the orbit of q remains homoclinically related to p;*

- *the angle between the stable and unstable spaces of q is bounded by α.*

Proof: One fixes some constant $\eta > 0$ such that for any two diffeomorphisms h, h' which are η-C^1- close to the identity the composed diffeomorphism $h \circ f \circ h'$ is in $\mathcal{U}$.

Since p is hyperbolic and $H(p, f)$ has no dominated splitting, the homoclinic class of p is non-trivial. If the angles between the stable and the unstable spaces of the periodic points whose orbits are homoclinically related to p are not (uniformly) bounded from below then the proposition follows taking $g = f$. Thus, from now on, we can assume that all these angles are greater than some $\delta > 0$.

Consider a sequence of periodic orbits $\{\gamma_n\}$ in $H(p, f)$ which are homoclinically related to p, whose periods π_n go to infinity, and which are dense in $H(p, f)$. Since dominated splittings extend to closures (see Remark 1.6. (1)), there is no dominated splitting on the union of the orbits γ_n. Thus, for any $N > 0$ there are a periodic orbit γ_n with arbitrarily large period π_n and a point $x \in \gamma_n$ such that for any unit vectors $u \in E^s(x)$ and $v \in E^u(x)$ we have

$$2\,\|Df^N(x) \cdot u\| > \|Df^N(x) \cdot v\|.$$

At each point $y \in \gamma_n$, $n \in \mathbb{N}$, one chooses unit vectors $u_y \in E^s(y)$ and $v_y \in E^u(y)$. Recall that the angle between the invariant spaces at each point $y \in \gamma_n$, $n \in \mathbb{N}$, is greater than δ. Hence the basis transformation maps from an orthonormal basis to the basis (u_y, v_y), as well as their inverses, are uniformly bounded in $y \in \gamma_n$ and $n \in \mathbb{N}$ by some constant $K > 0$. If one writes $Df(y)$ in the basis $\{u_y, v_y\}$ of T_yM and $\{u_{f(y)}, v_{f(y)}\}$ of $T_{f(y)}M$ one obtains a diagonal matrix

$$\begin{pmatrix} a_y & 0 \\ 0 & b_y \end{pmatrix}.$$

In order to work in the basis (u_y, v_y) one needs to adapt the constants η and α as follows: For $\varepsilon > 0$ small enough, any pair of linear maps $P, H \in GL(2, \mathbb{R})$ with $\|P\| < K$, $\|P^{-1}\| < K$, and $\|H - Id\| < \varepsilon$ satisfies $\|P \circ H \circ P^{-1} - Id\| < \eta$. For $\beta > 0$ small enough and $P \in GL(2, \mathbb{R})$ with $\|P\| < K$, $\|P^{-1}\| < K$, if u, v are non-zero vectors whose angle is less than β, then the angle between $P(u)$ and $P(v)$ is less than α.

We will use the following lemma, which explains how to perturb linear maps (and indeed diagonal linear maps) of $\mathbb{R}^2$ by diffeomorphisms in order to create small angles between the invariant manifolds while keeping a control of the invariant manifolds outside a small neighborhood of $\bar{0}$. The proof of this lemma will be postponed until after the end of the proof of the proposition:

Lemma 5.5. *For any $\varepsilon > 0$ and any $\beta > 0$ there exists $L > 0$ such that for any integers $N \geq L$ and $M \geq N + 2L$ we have the following property:*

Let (a_i) and (b_i) be two sequences of non-zero real numbers such that

- $2\left|\prod_{i=1}^N a_i\right| > \left|\prod_{i=1}^N b_i\right|$*;*
- $\left|\prod_{i=1}^M a_i\right| < \left|\prod_{i=1}^M b_i\right|$.

For each i, let A_i be the matrix given by $\begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix}$.

Then there exists a sequence of diffeomorphisms $(\varphi_i)_{i \in \{1,\dots,M\}}$ of $\mathbb{R}^2$ with compact support such that, denoting by B_i the diffeomorphism $A_i \circ \varphi_i$ for $i \in \{1, \dots, M-1\}$ and by B_M the diffeomorphism $\varphi_M \circ A_M$, we have the following properties:

1. *for each i the diffeomorphism φ_i is a perturbation of the identity which sends $\bar{0}$ to $\bar{0}$ and whose derivative is ε- close to the identity;*
2. *the point $\bar{0}$ is a hyperbolic fixed point of $B_M \circ \cdots \circ B_1$ whose stable and unstable manifolds are the lines $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$, respectively; and*
3. *the angle between the stable and unstable spaces of $\bar{0}$ for $B_N \circ \cdots \circ B_1 \circ B_M \circ \cdots \circ B_{N+1}$ is less than β.*

We fix $\varepsilon > 0$ and $\beta > 0$ associated to η and α, and let K be the constant which bounds the basis transformation maps, as discussed above. We choose L given by Lemma 5.5, $N = L$, a periodic orbit γ_n with period π_n larger that $N+L$, and a point $x \in \gamma_n$ such that $2\,\|Df^N(x) \cdot u_x\| > \|Df^N(x) \cdot v_x\|$. Finally, we fix $M = \pi_n > N + L$.

By an arbitrarily small C^1- perturbation of f in a small neighborhood of γ_n, one can assume that f is linear in a neighborhood of each point of γ_n. One defines $a_i = a_{f^{i-1}(x)}$ and $b_i = b_{f^{i-1}(x)}$, for $i \in \{1, \ldots, M\}$. By construction, these numbers verify the hypotheses of Lemma 5.5. The lemma yields some diffeomorphisms φ_i of $\mathbb{R}^2$. Notice that the conclusions of the lemma remain true if we conjugate all the diffeomorphisms φ_i by a common homothety. Hence we can assume that the supports of the φ_i's are contained in an arbitrarily small neighborhood of $\bar{0} \in \mathbb{R}^2$. Consequently, using the local coordinates around the points of γ_n, one can consider each φ_i as a diffeomorphism of S coinciding with the identity outside a small neighborhood of $f^{i-1}(x)$. Denote by g the diffeomorphism coinciding with f outside the neighborhood of γ_n, with $f \circ \varphi_i$ in the neighborhood of $f^{i-1}(x)$ for $i \in \{1, \ldots, M-1\}$, and with $\varphi_M \circ f$ in the neighborhood of $f^{M-1}(x) = f^{-1}(x)$.

We verify now that g satisfies the properties in the proposition with $q = f^N(x)$, provided that we choose the support of the φ_i small enough:

- g coincides with f on the orbit γ_n;
- by the choice of ε, the diffeomorphisms g belongs to $\mathcal{U}$;
- by the choice of β, the angle between the invariant spaces of Dg^M at $f^N(x)$ is smaller than α: just note that in the basis $\{u_{f^N(x)}, v_{f^N(x)}\}$ this differential has the expression $B_N \circ \cdots \circ B_1 \circ B_M \circ \cdots \circ B_{N+1}$, the assertion follows from Lemma 5.5. (3);
- recall that one can take the support of the φ_i arbitrarily small; Lemma 5.5. (2) asserts that the local invariant manifolds at x of g coincide with those of f (after linearization of f along γ_n), in a neighborhood of x whose size does not depend on the support of the φ_i; and
- consider two points x_s, x_u such that x_s (resp. x_u) is a transverse intersection point of the local stable (resp. unstable) manifold of x for f with the unstable (resp. stable) manifold of the orbit of p for f. If the support of the perturbation φ_i of the identity is small enough, the positive orbit of x_u (resp. the negative orbit of x_s) for f and for g coincide. As a consequence x (and then q) is homoclinically related to p for g.

This completes the proof of the proposition. □

Proof of Lemma 5.5: In order to get small angles between the stable and unstable bundles the proof follows Mañé's arguments (see [M$_2$]) which deals with matrices. However, in order to control the local invariant manifolds, we need to adapt these arguments using perturbations with compact support.

Let Φ be the matrix

$$\Phi = \begin{pmatrix} \frac{1}{1+\frac{\varepsilon}{2}} & 0 \\ 0 & 1+\frac{\varepsilon}{2} \end{pmatrix}.$$

This matrix is ε-close to the identity.

Denote by D_i, $i \in \{1, \dots, M-1\}$, the matrix defined by $D_i = A_i \circ (\Phi^{-1})$ if $i \in \{2, \dots, L-1\}$, $D_i = A_i \circ \Phi$ for $i \in \{N+1, \dots, M-2\}$, and $D_i = A_i$ in the other cases. The D_i's are diagonal matrices $\begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}$ verifying:

- $2 \left| \prod_{i=1}^{N} \alpha_i \right| > (1+\frac{\varepsilon}{2})^{2(L-2)} \left| \prod_{i=1}^{N} \beta_i \right|$;
- $|\alpha| < |\beta|$, where $\alpha = \prod_{i=1}^{M} \alpha_i$ and $\beta = \prod_{i=1}^{M} \beta_i$.

One defines φ_1 as a diffeomorphism on the form $(x, y) \mapsto (x + \psi(x, y), y)$, where $\psi \colon \mathbb{R}^2 \to \mathbb{R}$ has support in the unit ball, $\psi(0, 0) = 0$, $\|D\psi(x, y)\| < \varepsilon$ at each point (x, y), and $\frac{\partial}{\partial y}\psi(0, 0) = \varepsilon$.

Consider the image of the vertical axis $\{0\} \times \mathbb{R}$ by $D_M \circ \cdots \circ D_1 \circ \varphi_1$. This is a graph over the second coordinate, which is mapped on the vertical axis $\{0\} \times \mathbb{R}$ by a diffeomorphism $\tilde{\varphi}_M$ of the form $(x, y) \mapsto (x + \theta(y), y)$, where $\theta(y) = -\alpha\, \psi(0, \frac{y}{\beta})$. The function θ has support in the interval $[-|\beta|, |\beta|]$ and the modulus of its derivative $|D\theta(y)|$ is less that $|\frac{\alpha}{\beta}|\, \varepsilon \le \varepsilon$.

This allows us to build a diffeomorphism φ_M, coinciding with the identity outside some compact set and with $\tilde{\varphi}_M$ on $D_M \circ \cdots \circ D_1 \circ \varphi_1(\{0\} \times \mathbb{R})$, which is ε- C^1- close to the identity.

We define $B_1 = A_1 \circ \varphi_1$ and $B_M = \varphi_M \circ A_M$. Notice that, by construction, $\bar{0}$ is a hyperbolic fixed point of $B_M \circ D_{M-1} \circ \cdots \circ D_2 \circ B_1$, whose stable and unstable manifolds are the horizontal and vertical axes, respectively. As a consequence, $\bar{0}$ is a hyperbolic fixed point of $D_N \circ \cdots \circ D_2 \circ B_1 \circ B_M \circ D_{M-1} \circ \cdots \circ D_{N+1}$ whose invariant manifolds are the images of the axes by $D_N \circ \cdots \circ D_2 \circ B_1$. It follows that its invariant directions are generated by the vectors

$$u = (1, 0) \quad \text{and} \quad v = (1, v_y) = (1, \frac{\prod_{i=1}^{N} \beta_i}{\varepsilon \prod_{i=1}^{N} \alpha_i}).$$

Note that the modulus of v_y is bounded by $\frac{2}{\varepsilon}\, (1+\frac{\varepsilon}{2})^{-2(L-2)}$, which is arbitrarily small for large L: the angle between u and v is therefore smaller than β.

The family $B_1, D_2, \dots, D_{M-1}, B_M$ satisfies all the announced properties in the lemma, except that the D_i are linear maps and their supports are not compact. For $i \in \{2, \dots, M-1\}$, we choose φ_i with compact support, of the form $(x, y) \mapsto (\lambda(x)\, x, \lambda^{-1}(y)\, y)$, coinciding with Φ (for $i \in \{N+1, \dots, M-1\}$) or Φ^{-1} (for $i \in \{2, \dots, N\}$) on a very large ball Γ centered at the origin, and whose derivative at each point is ε- close to the identity. We now let, for $i = 2, \dots, M-1$, $B_i = A_i \circ \varphi_i$.

Taking the ball Γ large enough, one verifies that $\bar{0}$ is a hyperbolic fixed point of $B_M \circ \cdots \circ B_1$ whose stable and unstable manifolds are the lines $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$, respectively: the images $B_1(\{0\} \times \mathbb{R})$, $D_2 \circ B_1(\{0\} \times \mathbb{R})$, $\dots$, $D_{M-1} \circ \cdots \circ D_2 \circ B_1(\{0\} \times \mathbb{R})$ of the vertical axis coincide with the vertical axis outside a compact domain; one simply chooses Γ large enough to contain this compact domain.

Now the family φ_i, $i \in \{1, \cdots, M\}$, satisfies all the announced properties. □

5.3 Proof of Theorem 3

Let f be a diffeomorphism of a compact surface and p a hyperbolic periodic saddle whose homoclinic class $H(p,f)$ admits no dominated splitting. Fix some $\varepsilon > 0$ and $\alpha = \frac{\varepsilon}{8}$. By Proposition 5.4, there is an ε- perturbation g_1 of f (coinciding with f in a neighborhood of the orbit of p), such that there is a periodic saddle q of g_1 homoclinically related to p such that the angle between the invariant manifolds of q is smaller than α. Proposition 5.1 now yields an ε-perturbation g of g_1 (coinciding with g_1, therefore with f, in a neighborhood of the orbit of p) with a homoclinic tangency associated to p. So we obtain an arbitrarily small C^1- perturbation g of f with a homoclinic tangency associated to p, concluding the proof of Theorem 3.

6 Proof of the other results

Proof of Proposition 1.3: Let f be a diffeomorphism of a compact surface. Assume that there is some hyperbolic set Λ of f which locally generically presents some tangency: there is a C^1- open neighborhood $\mathcal{U}$ of f such that for every g in a residual subset of $\mathcal{U}$ the stable and the unstable manifolds of the continuation Λ_g of Λ are tangent at some point x. Notice that x belongs to $W^s_T(\Lambda_g) \cap W^u_T(\Lambda_g)$ for some integer $T > 0$. Consider, for $T \in \mathbb{N}$, the set $\mathcal{F}_T$ of diffeomorphisms $g \in \mathcal{U}$ having a tangency between the local invariant manifold of size T of Λ_g. These sets $\mathcal{F}_T$ are closed and their union, for $T \in \mathbb{N}$, is a residual subset in $\mathcal{U}$. One deduces that the union of the interiors of the sets $\mathcal{F}_T$ is a dense open subset of $\mathcal{U}$: that is, there are robust tangencies for diffeomorphims close to f. $\square$

Proof of Proposition 1.4: Recall that $\mathcal{T}$ is the C^1-open set of diffeomorphisms admitting a persistent tangency associated to some hyperbolic periodic saddle and that $\mathcal{T}_{rob} \subset \mathcal{T}$ is the open subset of diffeomorphisms admitting a robust tangency associated to some hyperbolic set. We denote by $\mathcal{T}_\infty$ the set $\mathcal{T} \setminus \overline{\mathcal{T}_{rob}}$.

Let $\mathcal{O} = \{O_n\}$ be a countable basis of open sets of S and $\mathcal{F}$ the (countable) family of the closures of finite unions of sets in $\mathcal{O}$. For each $V \in \mathcal{F}$, one defines $\mathcal{U}_V$ as the set of diffeomorphisms f for which the maximal invariant set $\Lambda(V,f)$ of f in V is hyperbolic. This set is open: let f be in $\mathcal{U}_V$ and g a diffeomorphism C^1- close to f. Then $\Lambda(V,g)$ is contained in a small neighborhood of $\Lambda(V,f)$ and hence it is hyperbolic. Let $\mathcal{V}_V$ be the C^1- interior of the complement of $\mathcal{U}_V$. The open set $\mathcal{U}_V \cup \mathcal{V}_V$ is dense in $\mathrm{Diff}^1(S)$.

For each $V \in \mathcal{F}$ and each integer $L > 0$, we denote by $\mathcal{U}^{trans}_{V,L}$ the set of diffeomorphisms $f \in \mathcal{U}_V$ such that the local stable and local unstable manifolds of $\Lambda(V,f)$ of size L are transverse. We will use the following lemma whose proof we postpone.

Lemma 6.1. *Each $\mathcal{U}^{trans}_{V,L}$ is open and dense in $\mathcal{U}_V \setminus \overline{\mathcal{T}_{rob}}$.*

By this lemma, each set $\mathcal{U}^{trans}_{V,L} \cup \mathcal{V}_V$ is open and dense in $(\mathrm{Diff}^1(S) \setminus \overline{\mathcal{T}_{rob}})$. Thus the intersection of this countable family (for $V \in \mathcal{F}$ and $L \in \mathbb{N}$) is a residual subset $\mathcal{G}$ of $(\mathrm{Diff}^1(S) \setminus \overline{\mathcal{T}_{rob}})$. We define $\mathcal{G}\mathcal{T}_\infty$ as the residual subset of $\mathcal{T}_\infty$ obtained as the intersection of $\mathcal{G}$ with $\mathcal{T}_\infty$.

Let $f \in \mathcal{G}\mathcal{T}_\infty$ and Λ be a hyperbolic set of f. There exists $V \in \mathcal{F}$ such that the maximal invariant set $\Lambda(V,f)$ contains Λ and is hyperbolic. Thus the diffeomorphism f belongs to $\mathcal{U}_V$. Moreover, by our choice of $\mathcal{G}\mathcal{T}_\infty$, for each L, the local invariant manifolds of $\Lambda(V,f)$ of size L are transverse. Consequently the (global) invariant manifolds of $\Lambda \subset \Lambda(V,f)$ are transverse. This completes the proof of the proposition. $\square$

Proof of Lemma 6.1: Let f be a diffeomorphism, Λ a hyperbolic set of f, and $\mathcal{U}$ a neighborhood of f of diffeomorphisms g such that the hyperbolic continuation Λ_g of Λ is defined. For each integer $L > 0$, we denote by $\mathcal{U}^{trans}_{\Lambda,L}$ the set of diffeomorphisms $g \in \mathcal{U}$ such that the local stable and local unstable manifolds of Λ_g of size L are transverse. This set is open in $\mathcal{U}$ (by continuity of the local invariant manifolds). Moreover, it is also dense in $\mathcal{U} \setminus \overline{\mathcal{T}_{rob}}$: any diffeomorphism g in the interior of $\mathcal{U} \setminus \mathcal{U}^{trans}_{\Lambda,L}$ admits a robust tangency associated to Λ_g (thus it belongs to $\mathcal{T}_{rob}$).

Let us come to the proof of the lemma: we consider $V \in \mathcal{F}$, an integer $L > 0$, and some $f \in \mathcal{U}_V \setminus \overline{\mathcal{T}_{rob}}$. We choose a small neighborhood $V' \in \mathcal{F}$ of $\Lambda(V, f)$ such that the maximal invariant set $\Lambda = \Lambda(V', f)$ is contained in a small neighborhood of $\Lambda(V, f)$, and hence it is hyperbolic. There is a small neighborhood $\mathcal{U}_0$ of f such that for any $g \in \mathcal{U}_0$ the hyperbolic continuation Λ_g of Λ is defined.

We claim that there exists a small neighborhood $\mathcal{U} \subset \mathcal{U}_0$ of f such that for any $g \in \mathcal{U}$ the set $\Lambda(V, g)$ is contained in Λ_g. By upper-semi-continuity of $\Lambda(V, f)$, any orbit O in $\Lambda(V, g)$ is a pseudo-orbit (with arbitrarily small jumps if g is close enough to f) of $\Lambda(V, f)$. Now, by the shadowing lemma (see for example [Y]), this orbit is shadowed by an (unique) orbit O' of f contained in V', that is, an orbit of Λ. The orbit $O \in \Lambda(V, g)$ is shadowed by the orbit $O' \in \Lambda$ and so, by definition and unicity of the continuation of Λ (see [Y]), it is an orbit of the continuation Λ_g of Λ. We have proved that $\Lambda(V, g) \subset \Lambda_g$ for g close to f.

By the argument above, for each integer $L > 0$, the set $\mathcal{U}^{trans}_{\Lambda,L}$ is dense in $\mathcal{U} \setminus \overline{\mathcal{T}_{rob}}$. For each diffeomorphism $g \in \mathcal{U}^{trans}_{\Lambda,L}$, since $\Lambda(V, g)$ is contained is Λ_g, the local invariant manifolds of $\Lambda(V, g)$ of size L are transverse, so that g belongs also to $\mathcal{U}^{trans}_{V,L}$. This proves the density of $\mathcal{U}^{trans}_{V,L}$ in $\mathcal{U}_V \setminus \overline{\mathcal{T}_{rob}}$.

Let us prove now the openness. We consider f in $\mathcal{U}^{trans}_{V,L} \setminus \overline{\mathcal{T}_{rob}}$ and choose a small neighborhood $V' \in \mathcal{F}$ of V such that the maximal invariant set $\Lambda = \Lambda(V', f)$ is contained in a small neighborhood of $\Lambda(V, f)$. Hence Λ is hyperbolic and its local invariant manifolds of size L are transverse (each local manifold of Λ is C^1-close to a local manifold of $\Lambda(V, f)$). Thus there exists a small neighborhood $\mathcal{U}$ of f such that for each $g \in \mathcal{U}$ the hyperbolic continuation Λ_g of Λ is defined, contains the maximal invariant set $\Lambda(V, g)$ (see the arguments above), and the local invariant manifolds of size L of Λ_g are transverse. As a consequence, $\mathcal{U}$ is contained in $\mathcal{U}^{trans}_{V,L}$, which proves the openness and ends the proof of the lemma. $\square$

Proof of Corollary 2: Consider the residual set $\mathcal{G}_1$ of $\operatorname{Diff}^1(S)$ obtained as the intersection of the following residual sets of $\operatorname{Diff}^1(S)$: $\mathcal{G}_0$, given by Theorem 2, $\mathcal{G}_4$ given by Proposition 2.16, $\mathcal{G}_2$ given by Proposition 2.12, and the residual set of Kupka-Smale diffeomorphisms.

Consider $f \in \mathcal{G}_1$ and a (hyperbolic) periodic point p of f whose homoclinic class $H(p, f)$ is non-isolated in $\mathcal{R}(f)$. This homoclinic class coincides with the chain-recurrence class of p, by Proposition 2.12. By Proposition 2.16, there exists a neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{G}_1 \cap \mathcal{U}$ the homoclinic class of p_g coincides with the chain-recurrence class of p_g and is not isolated. By Theorem 2, for any $g \in \mathcal{G}_1 \cap \mathcal{U}$ the homoclinic class of p_g has no dominated splitting. By Theorem 3, there are diffeomorphisms $\tilde{g}$ arbitrarily C^1- close to g such that $p_{\tilde{g}}$ admits a homoclinic tangency. This yields the density in $\mathcal{U}$ of diffeomorphisms exhibiting a homoclinic tangency associated to the continuation of p, thus concluding the proof. $\square$

Proof of Theorem 1: Consider the open set $\mathcal{W} = \operatorname{Diff}^1(S) \setminus \overline{\mathcal{H} \cup \mathcal{T}}$ and let $\mathcal{G}_0$ and $\mathcal{G}_1$ be the residual subsets built in Theorem 2 and Corollary 2. Consider $f \in \mathcal{W} \cap \mathcal{G}_0 \cap \mathcal{G}_1$ and a hyperbolic periodic point p of f. If the homoclinic class of p is non-isolated then, by the definition of $\mathcal{G}_1$, f

presents a persistent homoclinic tangency associated to p, contradicting $f \notin \mathcal{T}$. Consequently, the homoclinic class of p is isolated and is thus a hyperbolic basic piece, by the definition of $\mathcal{G}_0$.

We denote by $\mathcal{GW}$ the set of diffeomorphisms $f \in \mathcal{W} \cap \mathcal{G}_0 \cap \mathcal{G}_1$ for which each periodic point is hyperbolic and the set of periodic points is dense in the chain-recurrent set $\mathcal{R}(f)$. This set is residual in $\mathcal{W}$ by Theorem 2.2. Assume that the number of homoclinic classes of such a diffeomorphisms $f \in \mathcal{GW}$ is finite. As the homoclinic classes are compact and dense in $\mathcal{R}(f)$, one gets that $\mathcal{R}(f)$ is the union of these homoclinic classes, hence of finitely many basic sets. This shows that f satisfies the Axiom A and the no-cycle condition, contradicting $f \notin \mathcal{H}$. So for f in $\mathcal{GW}$ the number of homoclinic classes is infinite and each homoclinic class is a hyperbolic basic piece. $\square$

We conclude this paper with a discussion of the conjectures formulated in the introduction.

Proposition 6.2. *Conjectures 3 and 4 are equivalent.*

Proof: We first assume that Conjecture 3 is satisfied so that the open set $\mathcal{T}_{rob}$ is dense in $\mathcal{T}$. By Theorem 1, the open set $\mathcal{H} \cup \mathcal{W} \cup \mathcal{T}_{rob}$ is dense in $\mathrm{Diff}^1(S)$. We consider a diffeomorphism f in this dense open set such that for any hyperbolic set the stable and the unstable manifolds are transverse. By definition, f does not belong to $\mathcal{T}_{rob}$ so that $f \in \mathcal{H} \cup \mathcal{W}$. In either case all of the homoclinic classes of f are hyperbolic.

We now assume that Conjecture 4 is satisfied: there exists a residual subset $\mathcal{G} \subset \mathrm{Diff}^1(S)$ of diffeomorphisms f such that, if for any hyperbolic set of f its invariant manifolds are transverse, then all the homoclinic classes of f are hyperbolic. We can assume that this set $\mathcal{G}$ is contained in the residual set $\mathcal{G}_0$ given by Theorem 2: that is, hyperbolic homoclinic classes of diffeomorphisms in $\mathcal{G}$ are (robustly) isolated.

We can also assume that the generic diffeomorphism f is such that, for every (hyperbolic) periodic point p of f, the continuation $H(p_g, g)$ of the homoclinic class $H(p, f)$ varies continuously under small C^1- perturbations g of f (see Remark 2.15).

We shall prove that $\mathcal{GT}_\infty \cap \mathcal{G}$ is empty. Suppose, by contradiction, that there is $f \in \mathcal{GT}_\infty \cap \mathcal{G}$. By definition of $\mathcal{GT}_\infty$, for each hyperbolic set of f its invariant manifolds are transverse, so every homoclinic class is hyperbolic, by definition of $\mathcal{G}$. This also implies that every homoclinic class $H(p, f)$ is isolated. Therefore, there are a C^1- neighbourhood $\mathcal{W}$ of f and a neighbourhood U of $H(p, f)$ such that the map $g \mapsto H(p_g, g)$ is continuous and $H(p_g, g)$ is hyperbolic and equal to the maximal invariant set of g in U.

We claim that this implies that every diffeomorphism $g \in \mathcal{W}$ exhibits no tangencies associated to p_g. Arguing by contradiction, assume that there is $g \in W$ such that p_g has a homoclinic tangency x. First, due to the hyperbolicity, the point x does not belong to the homoclinic class $H(p_g, g)$. As this set is the maximal invariant set in U, this implies that we can take $x \notin U$. Unfolding this tangency, we obtain an explosion of the homoclinic class of p_g, contradicting that this class is contained in U.

This contradicts the existence of a persistent tangency for some hyperbolic periodic point of f. We deduce that $\mathcal{T}_\infty$ is empty (since $\mathcal{G} \cap \mathcal{GT}_\infty$ is residual in $\mathcal{T}_\infty$) so that $\mathcal{T}_{rob}$ is dense in $\mathcal{T}$, which is Conjecture 3 $\square$

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Periodic points and homoclinic classes

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Abstract

We prove that there is a residual subset $\mathcal{I}$ of $\mathrm{Diff}^1(\mathrm{M})$ such that any homoclinic class of a diffeomorphism $f \in \mathcal{I}$ having saddles of indices α and β contains a dense subset of saddles of index τ for every $\tau \in [\alpha, \beta] \cap \mathbb{N}$. We also derive some consequences from this result about the Lyapunov exponents of periodic points and the sort of bifurcations inside homoclinic classes of generic diffeomorphisms.

Keywords: chain recurrence class, dominated splitting, heterodimensional cycle, homoclinic class, homoclinic tangency, index of a saddle.
MSC 2000: 37C05, 37C20, 37C25, 37C29, 37D30.

1 Introduction

Hyperbolic periodic points play a key role in the study of the dynamics of diffeomorphisms. Naively speaking, one tries to structure the dynamics using these points as a spine. This strategy is supported by the Closing Lemma of Pugh ([Pu]), which claims that, for generic C^1-diffeomorphisms (i.e., diffeomorphisms in a *residual* subset of $\mathrm{Diff}^1(\mathrm{M})$, that is, a set containing a countable intersection of open and dense subsets of $\mathrm{Diff}^1(\mathrm{M})$), hyperbolic periodic points form a dense subset of the limit set of f. Using hyperbolic periodic points one attempts to split the dynamics of C^1-generic diffeomorphisms into elementary pieces (whose archetypal models are the basic sets of the Smale's theory [Sm]). Ideally, these elementary pieces should be pairwise disjoint, indecomposable (i.e., each piece is in some sense a dynamical unity) and not contained in bigger pieces (this corresponds to the notion of maximality).

Recently, in the C^1-generic context, substantial progress has been made in the direction of finding good candidates for the role of these elementary pieces. The results in [BC] state that, for generic diffeomorphisms of $\mathrm{Diff}^1(\mathrm{M})$, these elementary pieces are the *chain recurrence classes*. Moreover, every chain recurrence class containing a periodic point p is the *homoclinic class* of p. See also some preliminary results in [Ab, Ar, GW, We_1]. Furthermore, due to the closing lemma

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mentioned above and the results in [CMP], the homoclinic classes constitute a partition of a dense part of the limit set of generic diffeomorphisms. Furthermore, [Cr] states that chain recurrence classes of generic diffeomorphisms are Hausdorff limits of homoclinic classes. These results evidence the importance of the homoclinic classes in dynamics. For a discussion of the notion of elementary piece of dynamics and a survey on recent progress in the study of C^1-generic dynamics we refer to [BDV, Chapter 10].

The previous results motivate our interest in obtaining as complete as possible description of homoclinic classes, especially in the non-hyperbolic context. Homoclinic classes were introduced by Newhouse in [Ne_1] as a generalization of the basic sets in the Smale Decomposition Theorem (see [Sm]). The homoclinic class of a (hyperbolic) saddle p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit of p. A homoclinic class can be also (equivalently) defined as the closure of the set of (hyperbolic) saddles q *homoclinically related* to p (the stable manifold of the orbit of q transversely meets the unstable one of the orbit of p and vice-versa). This implies that the saddles having the same *index* (dimension of the stable bundle) as p form a dense subset of the whole class $H(p, f)$.

Every homoclinic class $H(p, f)$ is f-invariant and transitive (in fact, the points of $H(p, f)$ whose orbits are dense in the whole class form a residual subset of it): the homoclinic class is the ω-limit set of some $z \in H(p, f)$ (see for instance, [Ne_4]). But a homoclinic class may fail to be uniformly hyperbolic (for instance, it may contain in a robust way hyperbolic saddles having different indices as p, see the constructions in [$\mathrm{Dí}_1$, $\mathrm{Dí}_2$, DR] and the examples of non-hyperbolic robustly transitive diffeomorphisms in [BD_1, BV]) and locally maximal (for instance, a class can be contained with some persistence in the closure of an infinite set of sinks or sources, see [BD_2]).

Bearing in mind that homoclinic classes may fail to be hyperbolic and may indeed contain hyperbolic periodic points having different indices in a robust way, it is natural to wonder whether these indices form an interval in $\mathbb{N}$. We give a positive answer to this question for homoclinic classes of C^1-generic diffeomorphisms. Our main result is the following:

Theorem 1. *There is a residual subset $\mathcal{I}$ of $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f such that, for every $f \in \mathcal{I}$, any homoclinic class $H(p, f)$ containing hyperbolic saddles of indices α and β contains a dense subset of saddles of index τ for all $\tau \in [\alpha, \beta] \cap \mathbb{N}$.*

This paper can be viewed as a continuation of [BDPR], where an analogous result was proved for *robustly transitive sets*[1]: among the diffeomorphisms $f \in \mathrm{Diff}^1(\mathrm{M})$ having a robustly transitive set Λ_f, the property of the indices of the saddles of Λ_f forming an interval in $\mathbb{N}$ holds open and densely. Our proof involves a lot of the C^1-generic machinery developed recently. Moreover, as in [BDPR], a key ingredient in our constructions is the notion of *heterodimensional cycle* (i.e., there are saddles p and q of different indices such that the stable manifold of p intersects the unstable one of q and vice-versa). We analyze the creation of new saddles (of intermediate indices) via heterodimensional cycles associated to saddles of the homoclinic class (see Sections 3.2 and 3.3).

Let us briefly explain this point and summarize the main ingredients of our proof. Suppose that a diffeomorphism f has a saddle p_f of index α whose homoclinic class contains a saddle q_f of index β. In our generic context, we can assume that this property holds *locally residually* (i.e., in a

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[1] Recall that an f-invariant compact set Λ_f is *transitive* if there is some $x \in \Lambda_f$ whose forward orbit is dense in the whole Λ_f. The set Λ_f is a *robustly transitive set* of the diffeomorphism f if there are an open neighborhood U of Λ_f in the ambient manifold M and a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that, for every $g \in \mathcal{U}_f$, the set $\Lambda_g(U) = \bigcap_{k \in \mathbb{Z}} g^k(\overline{U})$ is transitive, contained in U, and $\Lambda_f(U) = \Lambda_f$.

residual set of an open neighborhood of f in $\mathrm{Diff}^1(\mathrm{M})$) and that, in fact, the homoclinic classes of p_f and q_f locally residually coincide (this follows from [CMP], see Section 2.2). Using the Connecting Lemma of Hayashi (see Lemma 2.7), this allows us to create heterodimensional cycles associated to the continuations of p_f and q_f (see Section 2.4). We then show that the unfolding of these heterodimensional cycles leads to diffeomorphisms g having saddles r_g of any index τ in between α and β (Section 3.2). In principle, this new saddle r_g might not belong to the homoclinic class $H(p_g, g) = H(q_g, g)$. Let us point out that in the context of robustly transitive sets in [BDPR] this problem does not appear: the new saddle r_g automatically belongs to the robustly transitive set. So an extra difficulty of this paper is to see that the saddle r_g, obtained after a perturbation, can be taken inside the homoclinic class.

We see that, after an appropriate perturbation, the new saddle r_g can be taken in $H(p_g, g)$. Generically, this implies that $H(r_g, g) = H(p_g, g)$. This is done in two steps. We first see that the new saddle can be obtained in such a way $W^u(p_g, g)$ transversely meets $W^s(r_g, g)$ and that $W^u(r_g, g)$ transversely meets $W^s(q_g, g)$ (this is done in Section 3.3). We see that this implies that the three saddles p_g, r_g and q_g are in the same chain recurrence class (see Section 2.1 for the precise definition). Finally, a genericity argument we borrow from [BC] assures that, for generic diffeomorphisms, any chain recurrence class containing a periodic point is the homoclinic class of that point. This guarantees that generically $H(p_g, g) = H(q_g, g) = H(r_g, g)$, which implies Theorem 1 (see Section 4 for details).

The proof of Theorem 1 allows us to obtain the following property about Lyapunov exponents of the saddles inside the homoclinic classes of generic diffeomorphisms. Let us define the *Lyapunov exponent vector* of a hyperbolic point p of period $\pi(p)$ by

$$v = \left(\frac{\log(|\mu_1|)}{\pi(p)}, \dots, \frac{\log(|\mu_n|)}{\pi(p)} \right),$$

where $\mu_1, \dots, \mu_n$ are the eigenvalues of $Df^{\pi(p)}(p)$ ordered by their moduli.

Corollary 2. *Let $\mathcal{I}$ be the residual subset of* $\mathrm{Diff}^1(\mathrm{M})$ *in Theorem 1. For every $f \in \mathcal{I}$ and any homoclinic class $H(p, f)$ of f, the closure of the set of Lyapunov vectors of the saddles of $H(p, f)$ is convex.*

Using Markov partitions, it is easy to see that this result holds for any hyperbolic homoclinic class.

We now derive further consequences from our main result and state some related problems. We first study the sort of bifurcations associated to a homoclinic class in terms of its dominated splitting. A *dominated splitting* of an f-invariant set K is a Df-invariant splitting

$$T_K M = E_1 \oplus \cdots \oplus E_k$$

over K such that the dimensions of the fibers $E_i(x)$ of E_i do not depend on the base point $x \in K$, and the expansion rate of Df on E_i is uniformly smaller than the expansion rate on E_{i+1}, that is, there are $C > 0$ and $\lambda > 1$ such that for any integer $n > 0$, any $x \in K$, one has:

$$\frac{\|Df^n(u)\|}{\|Df^n(v)\|} < C\lambda^{-n}, \qquad \text{for every unitary vectors } u \in E_i(x),\ v \in E_j(x) \text{ with } i < j.$$

In this case one writes $E_1 \prec \cdots \prec E_k$.

Let us recall two results relating dominated splitting and homoclinic tangencies. Consider a diffeomorphism f and an f-invariant compact set K contained in the closure of a set of periodic saddles having the same index τ. [We$_2$] proved that if the natural splitting over this set of periodic orbits given by the stable and unstable bundles cannot be extended to a dominated splitting over the whole K, then there is g arbitrarily C^1-close to f having a homoclinic tangency (in an arbitrarily small neighborhood of K).

The second result deals with robustly transitive sets. First, every robustly transitive set $\Lambda_f(U)$ has a finest dominated splitting $T_{\Lambda_f(U)}M = E_1 \oplus \cdots \oplus E_m$, (see [BDP]): that is, a dominated splitting whose bundles cannot be split in a dominated way. Let now $\Lambda_f(U)$ be a robustly transitive set defined for any f in an open set $\mathcal{U}$ of $\mathrm{Diff}^1(\mathrm{M})$. Then, open and densely in $\mathcal{U}$, the maximum β and the minimum α of the indices of the saddles in $\Lambda_f(U)$ are locally constant functions. Now, if for some k the dimension of $E = E_1 \oplus \cdots \oplus E_k$ is less than or equal to α then this bundle E is uniformly contracting. Similarly, if the dimension of $F = E_{m-k} \oplus \cdots \oplus E_m$ is less than or equal to $(\dim(M) - \beta)$ then the bundle F is uniformly expanding. Finally, suppose that $j \in [\alpha, \beta] \cap \mathbb{N}$ is such that there is $k \in \{1, \dots, m-1\}$ with

$$\dim(E_1 \oplus \cdots \oplus E_k) < j < \dim(E_1 \oplus \cdots \oplus E_{k+1}).$$

Then there exists g arbitrarily close to f with a homoclinic tangency associated to a saddle of index j of $\Lambda_g(U)$. This result is stated in [BDPR, Theorem F].

These two results motivate the following conjecture[2]:

Conjecture 1. *For every C^1-generic diffeomorphism f and every homoclinic class $H(p_f, f)$ of f there is the following dichotomy:*

- *either there is g arbitrarily C^1-close to f having a homoclinic tangency associated to the continuation of some saddle of $H(p_f, f)$;*
- *or there is a dominated splitting*

  $$T_{H(p_f,f)}M = E^s \oplus E^c_1 \oplus \cdots \oplus E^c_k \oplus E^u,$$

  where E^s is uniformly contracting, E^u is uniformly expanding, and every E^c_i is not hyperbolic and has dimension one.

Let us observe that the arguments in [BDPR] in fact prove this conjecture for isolated homoclinic classes. For non-isolated homoclinic classes one of the main difficulties comes from the fact that the homoclinic class is necessarily accumulated by periodic saddles which do not belong to the class. Moreover, the indices of some of these saddles may not belong to the interval of indices of the saddles of the class. Thus one of the difficulties for proving this conjecture is that the homoclinic tangency that one obtains may correspond to a saddle outside of the homoclinic class. This difficulty illustrates the importance of the periodic saddles in a neighborhood of the homoclinic class.

As a direct consequence of Theorem 1 and the result in [We$_2$] above we obtain a partial answer to this conjecture:

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[2] This conjecture was proposed by the second author during the Second Latin American Conference of Mathematicians (Cancún, Mexico, 2004).

Corollary 3. *Let $\mathcal{I}$ be the residual subset of* $\mathrm{Diff}^1(\mathrm{M})$ *in Theorem 1 and f a diffeomorphism in $\mathcal{I}$ having a homoclinic class $H(p_f, f)$ which contains hyperbolic saddles of indices α and β. Then at least one of the two following possibilities holds:*

1. *For any neighborhood U of $H(p_f, f)$ and any C^1-neighborhood $\mathcal{U}$ of f there is a diffeomorphism $g \in \mathcal{U}$ with a homoclinic tangency associated to a periodic orbit contained in U.*

2. *There is a dominated splitting*

$$T_{H(p_f,f)}M = E \oplus F_1 \oplus \cdots \oplus F_{\beta-\alpha} \oplus G,$$

with $\dim(E) = \alpha$ *and* $\dim(F_i) = 1$ *for all i.*

A result recently announced by Gourmelon implies that the homoclinic tangency in the first item of the corollary is associated to a saddle of the homoclinic class $H(p_f, f)$ having an index in between α and β; this implies in particular that the two cases of the conjecture cannot occur simultaneously.

Let us state some questions related to Theorem 1 and to the conjecture, in the broader setting of *chain recurrence classes*, which we now briefly recall.

A point x is *chain recurrent* if for any $\varepsilon > 0$, there are periodic ε-pseudo-orbits containing x. The set of chain recurrent points is denoted by $\mathcal{R}(f)$. This set admits a natural partition into invariant compact sets called the *chain recurrence classes*: two points $x, y \in \mathcal{R}(f)$ are in the same class if, for any $\varepsilon > 0$, there are ε-pseudo-orbits going from x to y and from y to x. The Conley theory, [Co, Ro], associates a filtration to this partition. [BC] proved that, for C^1-generic diffeomorphisms, the chain recurrent set $\mathcal{R}(f)$ is the closure of the hyperbolic periodic orbits. Furthermore, for C^1-generic f, there are two types of chain recurrence classes: those containing periodic points (which are the homoclinic classes), and those without periodic orbits (called therefore *aperiodic classes*).

Moreover, [Cr] proved that any chain recurrence class of a generic diffeomorphisms is the Hausdorff limit of a sequence of periodic orbits. This result, however, does not yield any information on the indices of these periodic orbits. One may ask many questions regarding chain recurrence classes, indices of saddles, Hausdorff limits of periodic orbits, and the closure of periodic points of a prescribed index. Let us propose two such questions. First, we do not know whether chain recurrence classes Σ of generic diffeomorphisms are *index complete*, that is, whether if whenever Σ is the Hausdorff limit of periodic points of indices α and β, $\alpha < \beta$, then it is also the Hausdorff limit of periodic points of index τ for any $\tau \in [\alpha, \beta] \cap \mathbb{N}$.

Question 1. *Are all the chain recurrence classes of C^1-generic diffeomorphism index complete?*

The next question relates the local dynamics (the dynamics around an invariant compact set of the class) and the global dynamics in the whole class:

Question 2. *Let f be a C^1-generic diffeomorphism and Σ any chain recurrence class of f such that every neighborhood of the whole set Σ contains periodic orbits of index τ.*

Is Σ the Hausdorff limit of periodic orbits of index τ? Or a weaker question: is Σ contained in the closure of the saddles of index τ?

We observe that these two questions have positive answers in the case of isolated chain recurrence classes (which are also the isolated homoclinic classes). On the other hand, they are open for non-isolated homoclinic classes.

Let us explain briefly how the previous questions are related to the conjecture and why positive answers to both of them would imply a weaker version of the conjecture: in this weaker version, the first case involves the creation of some tangency associated to a saddle *near* – but not necessarily inside – the homoclinic class $H(p_f, f)$.

Let us assume that we are not in the first case: that is, that there are no homoclinic tangencies associated to saddles whose orbit is contained in a neighborhood of the class $H(p_g, g)$, for g close to f. We consider the set of indices of periodic points arbitrarily close to the class. By Question 2, the class is the Hausdorff limit of periodic orbits of these indices. By Question 1 this set of indices is an interval $\{\alpha, \alpha+1, \dots, \alpha+k\}$). By [We$_2$] this implies a dominated splitting of the form

$$T_{H(p_f,f)}M = E \oplus E_1^c \oplus \cdots \oplus E_k^c \oplus F,$$

where E has dimension α and every E_i^c is not hyperbolic and has dimension one. It now follows, essentially from Mañé's Ergodic Closing Lemma, [Ma], (see also [BDPR, Theorem B and Section 4] and [We$_3$]), that E is uniformly contracting (the argument for F is similar); the crucial assumption here is that it is forbidden to decrease the index of a periodic point of index α by perturbation. This means that we are in the second case of the conjecture, as claimed.

We finish this introduction discussing briefly how homoclinic classes may be accumulated by periodic orbits.

Non-isolated homoclinic classes of C^1-generic diffeomorphisms are accumulated by infinitely many disjoint homoclinic classes and are called *wild homoclinic classes.* There are no known examples of C^1-wild homoclinic classes for surface diffeomorphisms (there are, however, examples of C^2-wild classes associated to homoclinic tangencies, see [Ne$_2$, Ne$_3$]). Moreover, the only known examples of C^1-wild homoclinic classes occur in dimension equal to or higher than three. In such examples the classes are either (a) contained in the closure of an infinite set of sinks or sources, or else (b) obtained by considering the product of a three dimensional diffeomorphism f exhibiting a wild class accumulated by (say) sinks by a strong expansion. In case (b) one obtains a diffeomorphism F having a normally hyperbolic wild homoclinic class accumulated by saddles (in this case the dimension of the ambient manifold is at least 4), see [BD$_2$, BDP, CM]. But the existence of wild homoclinic classes accumulated only by infinitely many *true* saddles (i.e., not obtained via a product) remains an open problem. As the nature of this problem is three dimensional (and to avoid tricky solutions considering multiplications by a strong expansion/contraction), we formulate the following question:

Question 3. *Let M be a closed manifold of dimension 3. Do there exist locally generic diffeomorphisms $f \in \mathrm{Diff}^1(M)$ having wild homoclinic classes which are (a) not contained in a normally hyperbolic locally invariant submanifold and (b) disjoint from the closure of the set of sinks and sources?*

Finally, there is a special type of non-isolated homoclinic classes, which we call *pelliculaire classes*: those classes which are accumulated by *true* periodic saddles whose indices do not belong to the index interval corresponding to the class given by Theorem 1. We do not know whether these classes exist locally generically. The archetypal model of a pellicular class in dimension 3 is the following: suppose that the index interval of the class $H(p_f, f)$ is $\{1\}$ but that the class is accumulated simultaneously by saddles of index 2. In fact, this type of homoclinic classes needs to be considered when trying to solving the questions we posed above.

We close this introduction by explaining the organization of this paper and giving an outline of the proof of Theorem 1. This paper is organized as follows. In Section 2.1 we summarize the basic properties of homoclinic and chain recurrence classes of generic diffeomorphisms we use throughout the paper. In Section 2.2, we state a generic dichotomy result: generically homoclinic class either persistently coincide or are persistently disjoint. In Section 2.4, we explain how saddles of a homoclinic class having different indices can be related via a heterodimensional cycle. In Section 2.3, we state a key technical result (Proposition 2.3): *for homoclinic classes of generic diffeomorphisms, the saddles having positive real multipliers of multiplicity one form a dense subset of the class.* The importance of this proposition is the following: the saddles of intermediate indices are obtained through heterodimensional cycles associated to saddles in the homoclinic class. But the creation of such saddles is only well understood when the saddles in the cycle have real multipliers with multiplicity one. This is the reason we need this preparatory step. In fact, in Section 3.1, we show how heterodimensional cycles can be perturbed to create new cycles along which the dynamics is essentially affine. This allows us to analyze the dynamics in a heterodimensional cycle in a rather simple way. Using these affine heterodimensional cycles we obtain the saddles having intermediate indices in Section 3.2. In Sections 3.3 and 4, we see that these saddles of intermediate indices can be taken inside the original homoclinic class. This ends the proof of Theorem 1.

2 Homoclinic and chain recurrence classes of C^1-generic diffeomorphisms

In this section we collect some properties of homoclinic and chain recurrence classes of C^1-generic diffeomorphisms.

2.1 Summary of generic properties of C^1-diffeomorphisms

The results in [Ku, Sm_1, Pu, CMP] give a residual subset $\mathcal{G}$ of $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f verifying the following properties:

- **(G1)** f is Kupka-Smale (hyperbolicity of the periodic points and general position of the invariant manifolds);
- **(G2)** the periodic points of f are dense in the non-wandering set of f;
- **(G3)** for any pair of saddles p and q of f, either $H(p, f) = H(q, f)$ or $H(p, f) \cap H(q, f) = \emptyset$; and
- **(G4)** for every saddle p of f, the homoclinic class $H(p_g, g)$ depends continuously on $g \in \mathcal{G}$, where p_g is the continuation of the saddle p of f for g close to f.

To state the generic conditions (G5) and (G6) we need the notion of *chain recurrence class.* A point y is *f-chain attainable* from the point x if for every $\varepsilon > 0$ there is an *ε-pseudo-orbit* going from x to y (i.e., there is a finite sequence $(x_i)_{i=0}^m$, $m \geq 1$, such that $x_0 = x$, $x_m = y$, and $\mathrm{dist}(f(x_i), x_{i+1}) < \varepsilon$ for all $i = 0, \dots, (m-1)$). The points x and y are *f-bi-chain attainable* if

x is chain attainable from y and vice-versa. An f-invariant set Λ is *chain recurrent* if every pair of points of Λ are bi-chain attainable. The bi-chain attainability relation defines an equivalence relation on the *chain recurrent set* $R(f)$ of f (i.e., the set of points x which are chain attainable from themselves). The *chain recurrence classes* are the equivalence classes of $R(f)$ for the bi-chain attainability relation. These sets are the maximal recurrent sets.

We also have the following two generic properties:

(G5) The chain recurrence classes of f form a partition of the chain recurrent set of f (i.e., they are pairwise disjoint and cover $R(f)$)[3].

(G6) Every chain recurrence class Λ containing a (hyperbolic) periodic point p satisfies $\Lambda = H(p, f)$, see [BC, Remarque 1.10].

2.2 Coincidence of homoclinic classes

It is well-known that any hyperbolic periodic point p_f of f has a unique *continuation*: there are a neighborhood U of the orbit of p_f in the ambient manifold M and a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that every g in $\mathcal{U}$ has a unique periodic point p_g close to p_f whose orbit is contained in U. The point p_g is called the *continuation of* p_f.

The next lemma is perhaps well-known in C^1-dynamics. We include its proof for completeness.

Lemma 2.1. *There is a residual subset* $\mathcal{G}_0 \subset \mathcal{G}$ *of* $\mathrm{Diff}^1(\mathrm{M})$ *such that, for every diffeomorphism* $f \in \mathcal{G}_0$ *and every pair of saddles* p_f *and* q_f *of* f, *there is a neighborhood* $\mathcal{U}_f$ *of* f *in* $\mathcal{G}_0$ *such that either* $H(p_g, g) = H(q_g, g)$ *for all* $g \in \mathcal{U}_f$, *or* $H(p_g, g) \cap H(q_g, g) = \emptyset$ *for all* $g \in \mathcal{U}_f$.

Proof: We first fix some integer $N \geq 1$. Given $f \in \mathcal{G}$ let $\mathrm{Per}_N(f)$ be the (finite) set of periodic points of f of period less than N. This set varies continuously in a C^1-neighborhood $\mathcal{U}_N(f)$ of f: let $\mathrm{Per}_N(f) = \{p_f^1, \dots, p_f^{k(N)}\}$, then for every $g \in \mathcal{U}_N(f)$ one has $\mathrm{Per}_N(g) = \{p_g^1, \dots, p_g^{k(N)}\}$ (each p_g^i is the continuation of p_f^i).

We now fix $f \in \mathcal{G}$ and its neighborhood $\mathcal{U}_N(f)$. For each $i, j \in \{1, \dots, k(N)\}$ let

- $\mathcal{V}_{i,j} = \{g \in \mathcal{U}_N(f) \cap \mathcal{G} \colon H(p_g^i, g) \cap H(p_g^j, g) = \emptyset\}$;
- $\mathcal{B}_{i,j} = (\mathcal{U}_N(f) \setminus \overline{\mathcal{V}_{i,j}}) \cap \mathcal{G}$.

Claim 2.2. *The set* $\mathcal{V}_{i,j}$ *is open in* $\mathcal{G}$.

We postpone the proof of the claim and finish the proof of the lemma. First, note that for every $g \in \mathcal{B}_{i,j}$ the homoclinic classes $H(p_g^i, g)$ and $H(p_g^j, g)$ have non-empty intersection so that, by (G3), the two homoclinic classes coincide.

By Claim 2.2, the set $\mathcal{V}_{i,j} \cup \mathcal{B}_{i,j}$ is open and dense in $\mathcal{U}_N(f)$. Let $\mathcal{O}_N(f)$ be the finite intersection $\mathcal{O}_N(f) = \cap_{i,j}(\mathcal{V}_{i,j} \cup \mathcal{B}_{i,j})$. By construction, this set is open and dense in $\mathcal{U}_N(f)$.

Observe that for every diffeomorphism $g \in \mathcal{O}_N(f)$ we have the conclusions of the lemma for the saddles of period less than N of g. Let now $\mathcal{O}_N$ be the union of all the sets $\mathcal{O}_N(f)$, $f \in \mathcal{G}$. The set $\mathcal{O}_N$ is open and dense in $\mathcal{G}$. Finally, the set $\mathcal{G}_0$ is the intersection of the open and dense sets $\mathcal{O}_N$. By construction, this set is residual in $\mathrm{Diff}^1(\mathrm{M})$ and every $g \in \mathcal{G}_0$ satisfies the conclusions in the lemma.

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[3] In fact, by the Conley Theory, see [Co, Ro], this property holds for all C^1 diffeomorphisms.

Proof of the claim: Let $g \in \mathcal{V}_{i,j}$. By (G6) the disjoint homoclinic classes $H(p^i_g, g)$ and $H(p^j_g, g)$ are also disjoint chain recurrence classes. By the Conley Theory, these sets are separated by a filtration (see [Co, Ro]): there is an open set U of M with $g(\overline{U}) \subset U$, such that one of the classes, say $H(p^i_g, g)$, is contained in U, and the other one is contained in $M \setminus \overline{U}$. Clearly, for every h close to g it holds $h(\overline{U}) \subset U$. Therefore U contains the stable manifold of the orbit of p^i_h. Similarly, $M \setminus \overline{U}$ contains the unstable manifold of the orbit of p^j_h. Thus, by definition of homoclinic class, $H(p^i_h, h) \subset U$ and $H(p^j_h, h) \subset (M \setminus \overline{U})$. Hence $H(p^i_h, h) \cap H(p^j_h, h) = \emptyset$. This ends the proof of the claim. □

The proof of the lemma is now complete. □

2.3 Saddles of homoclinic classes

Given a homoclinic class $H(p, f)$, denote by $\mathrm{Per}_h(H(p, f))$ the set of hyperbolic saddles q homoclinically related to p, and by $\mathrm{Per}_{\mathbb{R}}(H(p, f))$ the subset of $\mathrm{Per}_h(H(p, f))$ of points q such that all the eigenvalues of the derivative $Df^{\pi(q)}(q)$ are real, positive, and have multiplicity one; here $\pi(q)$ denotes the period of q.

Proposition 2.3. *There is a residual subset $\mathcal{G}_1$ of* $\mathrm{Diff}^1(\mathrm{M})$ *consisting of diffeomorphisms f such that* $\mathrm{Per}_{\mathbb{R}}(H(p_f, f))$ *is dense in $H(p_f, f)$ for every nontrivial homoclinic class $H(p_f, f)$ of f.*

Proof of the proposition: This proposition is just a translation of the results in [BDP] for *periodic linear systems (cocycles) with transitions* to the context of homoclinic classes. Recall that a periodic linear system is a 4-uple $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$, where f is a diffeomorphism, Σ is an infinite set of periodic points of f, $\mathcal{E}$ an Euclidean vector bundle defined over Σ, and $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ is such that $A(x) \colon \mathcal{E}_x \to \mathcal{E}_{f(x)}$ is a linear isomorphism for each x ($\mathcal{E}_x$ is the fiber of $\mathcal{E}$ at x). We refer to [BDP, Section 1] for the precise definition (we do not need it here). Naively speaking, such systems are cocycles where the extra structure of the transitions allows us to compose linear maps at different points. In a rough terms, this guarantees the existence of new periodic orbits visiting certain prescribed different periodic orbits of the system (i.e., there is a shadowing-like property in the bundle). Let us now explain how Proposition 2.3 follows:

Lemma 2.4. ([BDP, Lemma 1.9]) *Let $H(p_f, f)$ be a nontrivial homoclinic class. Then the derivative Df of f induces a periodic linear system with transitions over* $\mathrm{Per}_h(H(p_f, f))$.

We say that a periodic linear system with transitions $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ is *diagonalizable* at the point $x \in \Sigma$ if the linear map

$$M_A(x) \colon \mathcal{E}_x \to \mathcal{E}_x, \quad M_A(x) = A(f^{\pi(x)-1}(x)) \circ \cdots \circ A(f^2(x)) \circ A(x),$$

only has positive real eigenvalues of multiplicity one.

Lemma 2.5. ([BDP, Lemma 4.16]) *For every periodic linear system with transitions $\mathcal{P} = (\Sigma, f, \mathcal{E}, A)$ and every $\varepsilon > 0$ there is a dense subset Σ' of Σ and an ε-perturbation A' of A defined on Σ' which is diagonalizable, that is, $M_{A'}(x)$ has positive real eigenvalues of multiplicity one for every $x \in \Sigma'$.*

The next result allows us to perform dynamically the perturbations of a cocycle:

Lemma 2.6. (Franks, [Fr]). *Consider a diffeomorphism f and an f-invariant finite set Σ. Let A be an ε-perturbation of the derivative of f in Σ (i.e., the linear maps $Df(x)$ and $A(x)$ are ε-close for all $x \in \Sigma$). Then, for every neighborhood U of Σ, there is g ε-C^1-close to f such that*

- *$f(x) = g(x)$ for every $x \in \Sigma$ and every $x \notin U$,*
- *$Dg(x) = A(x)$ for all $x \in \Sigma$.*

We are now ready to prove the proposition. By Lemma 2.4, the derivative of f induces a periodic linear system with transitions over $\Sigma = \mathrm{Per}_h(H(p_f, f))$. Applying Lemma 2.5 to such a system, we get that fixing any $\varepsilon > 0$ there is a dense subset Σ' of $\mathrm{Per}_h(H(p_f, f))$ such that for every $r_f \in \Sigma'$ there is an ε-perturbation A of Df throughout the orbit of r_f such that

$$M_A(r_f) = A(f^{\pi(r_f)-1}(r_f)) \circ A(f^{\pi(r_f)-2}(r)) \circ \cdots \circ A(f(r_f)) \circ A(r_f)$$

has positive real eigenvalues of multiplicity one. Applying Lemma 2.6 to the orbit of r_f and the perturbation A of Df, we get g close to f such that $r_f = r_g$ is a periodic point of g (of period $\pi(r_f)$) and $Dg^{\pi(r_f)}(r_g) = M_A(r_f)$. Thus, r_g is a periodic point of g having real positive eigenvalues of multiplicity one.

Note that, from the proof of [BDP, Lemma 4.16], the orbit of r_f can be taken arbitrarily close in the Hausdorff metric to the orbit of a transverse homoclinic point x of p_f. More precisely, let $\Lambda = \{\mathcal{O}(p_f) \cup \mathcal{O}(x)\}$, then the orbit of r_f can be taken arbitrarily close to Λ in the Hausdorff metric. Since Λ is a hyperbolic set of f, there is $\delta > 0$ such that for any r_f whose orbit $\mathcal{O}(r_f, f)$ is Hausdorff δ-close to Λ and any g that is C^1-ϵ close to f, r_g and p_g are homoclinically related. Thus $r_g \in H(p_g, g)$.

The proof of the proposition concludes as follows. For each n, take a finite covering $\mathbb{B}_n$ of $H(p_f, f)$ by open balls B_i of radius $1/n$ (each ball intersecting $H(p_f, f)$). Arguing as above, in each ball B_i we will obtain, after a small C^1-perturbation, $g \in \mathcal{G}$ with a saddle $r_g \in \mathrm{Per}_g(H(p_g, g))$ in B_i having only positive real eigenvalues of multiplicity one.

Considering perturbations at each ball B_i (there are finitely many) and recalling condition (G4) (in $\mathcal{G}$ the homoclinic classes depend continuously), we get $g \in \mathcal{G}$ close to f such that $\mathrm{Per}_{\mathbb{R}}(H(p_g, g))$ is $2/n$-dense in $H(p_g, g)$ (by continuity, $H(P_g, g)$ is contained in the union of the balls B_i). This also implies that $\mathrm{Per}_{\mathbb{R}}(H(p_h, h))$ is $2/n$-dense in $H(p_h, h)$ for all h close to g in $\mathcal{G}$. Thus there are a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}$ and an open and dense subset $\mathcal{D}_n$ of $\mathcal{U}_f$ of diffeomorphisms g such that $\mathrm{Per}_{\mathbb{R}}(H(p_g, g))$ is $2/n$-dense in $H(p_g, g)$. The proof of the proposition now follows using a genericity argument identical to the one in Lemma 2.1. The proof of the proposition is now complete. $\square$

2.4 Creation of intersections between invariant manifolds

Let $\mathcal{G}_2 = \mathcal{G}_0 \cap \mathcal{G}_1$, where $\mathcal{G}_0$ and $\mathcal{G}_1$ are the residual subsets of $\mathrm{Diff}^1(\mathrm{M})$ in Lemma 2.1 and Proposition 2.3. Consider f in $\mathcal{G}_2$ having a homoclinic class $H(p_f, f)$ and a saddle q_f of different index from that of p_f with $q_f \in H(p_f, f)$. In particular, $H(p_f, f)$ is nontrivial. Without loss of generality we can assume that the indices of p_f and q_f are α and β, with $\alpha < \beta$. By Lemma 2.1, there is a neighborhood $\mathcal{U}_f$ of f in $\mathcal{G}_2$ such that $H(p_g, g) = H(q_g, g)$ for all $g \in \mathcal{U}_f$.

By hypotheses, recall Proposition 2.3, there is a saddle p_g^1 (resp. q_g^1) homoclinically related to p_g (resp. q_g) such that $Dg^{\pi(p_g^1)}(p_g^1)$ (resp. $Dg^{\pi(q_g^1)}(q_g^1)$) has positive real eigenvalues of multiplicity one. Observe that the indices of p_g^1 and q_g^1 are α and β.

We need the following lemma about the creation of cycles:

Lemma 2.7. (Hayashi's Connecting Lemma, [Ha]) *Let a_f and b_f be a pair of saddles of a diffeomorphism f such that there are sequences of points y_n and of natural numbers k_n such that*

- $y_n \to y \in W^u_{loc}(a_f, f)$, $y \neq a_f$, *and*
- $f^{k_n}(y_n) \to z \in W^s_{loc}(b_f, f)$, $z \neq b_f$.

Then there is a diffeomorphism g arbitrarily C^1-close to f such that $W^u(a_g, g)$ and $W^s(b_g, g)$ have an intersection arbitrarily close to y.

Lemma 2.8. *Consider a homoclinic class $H(a_f, f)$, any saddle $b_f \in H(a_f, f)$, and any transverse homoclinic point y of a_f. Then there is g arbitrarily C^1-close to f such that $W^u(a_g, g)$ and $W^s(b_g, g)$ have an intersection arbitrarily close to y.*

Proof: Recall that $H(a_f, f)$ is the ω-limit set of some $w \in H(a_f, f)$. Thus the forward orbit of w passes arbitrarily close to a_f, b_f and y and it accumulates to some $z \in W^s_{loc}(b_f)$. Hence there are sequences $f^{m_n}(w)$ converging to some $z \in W^s_{loc}(b_f)$ and $f^{r_n}(w) \to y$, where $m_n > r_n$ and $m_n \to \infty$. Taking $y_n = f^{r_n}(w)$ and $k_n = m_n - r_n$ we obtain the hypotheses of Lemma 2.7, which implies the result. □

By Lemma 2.8 (taking $a_f = p_f$, $b_f = q_f$ and any homoclinic point y of a_f), there is g arbitrarily C^1-close to f such that $W^s(q_g, g)$ and $W^u(p_g, g)$ have some intersection close to y. Observe that

$$\dim(W^s(q_g, g)) + \dim(W^u(p_g, g)) = \beta + (n - \alpha) > \beta + (n - \beta) = n.$$

Thus we can assume (after a perturbation) that the previous intersection between $W^u(p_g, g)$ and $W^s(q_g, g)$ is transverse. Since the property of having a transverse intersection is open, using a Baire argument similar to the one in the proof of Lemma 2.1, one immediately obtains:

Lemma 2.9. *There is a residual subset $\mathcal{G}_3$ of $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f such that for every pair of saddles p_f and q_f of f of indices α and β, $\alpha < \beta$, with $H(p_f, f) = H(q_f, f)$ it holds that $W^u(p_f, f)$ and $W^s(q_f, f)$ have some transverse intersection.*

3 Heterodimensional cycles and creation of periodic orbits

Let us recall that, by convention, the *index* of a hyperbolic periodic point x is the dimension of its stable manifold. Given a pair of hyperbolic points p and q we write $p <_{\mathrm{us}} q$ if the unstable manifold $W^u(\mathcal{O}_p)$ of the orbit $\mathcal{O}_p$ of p intersects transversally the stable manifold $W^s(\mathcal{O}_q)$ of the orbit $\mathcal{O}_q$ of q: there exists a point $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ such that $T_xM = T_xW^u(\mathcal{O}_p) + T_yW^s(\mathcal{O}_q)$.

Remark 3.1. *The property $<_{\mathrm{us}}$ is open in $\mathrm{Diff}^1(\mathrm{M})$: let p_f and q_f be hyperbolic periodic points of a diffeomorphism f with $p_f <_{\mathrm{us}} q_f$, then there is a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(\mathrm{M})$ such that $p_g <_{\mathrm{us}} q_g$ for every $g \in \mathcal{U}_f$ (here p_g and q_g are the continuations of p_f and q_f).*

Recall that a diffeomorphism f has a *heterodimensional cycle* associated to the saddles p and q if p and q have different indices and both intersections $W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ and $W^s(\mathcal{O}_q) \cap W^u(\mathcal{O}_p)$ are non-empty.

We say that a periodic point p of period $\pi(p)$ of a diffeomorphism f has *real eigenvalues* if every eigenvalue of the linear isomorphism $Df^{\pi(p)}(p)\colon T_pM \to T_pM$ is real.

Theorem 3.2. *Let f be a diffeomorphism having a heterodimensional cycle associated to periodic saddles p_f and q_f, of indices α and β with $\alpha < \beta - 1$, with real eigenvalues. Then, for any C^1-neighborhood $\mathcal{U}$ of f and for any integer τ with $\alpha \leq \tau \leq \beta$, there exists $g \in \mathcal{U}$ having a periodic point r_g of index τ such that $p_g <_{\mathrm{us}} r_g <_{\mathrm{us}} q_g$ (p_g and q_g are the continuations of p_f and q_f).*

In fact, we will see that under some additional hypotheses (in our case, $H(p_f, f) = H(q_f, f)$), the saddle r_g can be taken such that $r_g \in H(p_g, q) = H(q_g, g)$, see Propositions 3.10 and 4.1.

The proof of Theorem 3.2 is the aim of this whole section.

3.1 Affine heterodimensional cycles

Definition 3.3 (Affine heterodimensional cycle). *Let f be a diffeomorphism having a heterodimensional cycle associated to periodic points p, q of indices α and β, $\alpha < \beta$, and to heteroclinic points $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ and $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$. We say that the* heterodimensional cycle is affine *if all the following properties are satisfied:*

- **(A1)** *The eigenvalues associated to the saddles p and q are all real and different in modulus and have multiplicity one: denote the eigenvalues of $Df^{\pi(p)}(p)$ and $Df^{\pi(q)}(q)$ by $\lambda_1, \dots, \lambda_n$ and $\sigma_1, \dots, \sigma_n$, respectively, so that we have $0 < |\lambda_1| < \cdots < |\lambda_\alpha| < 1 < |\lambda_{\alpha+1}| < \cdots < |\lambda_n|$ and $0 < |\sigma_1| < \cdots < |\sigma_\beta| < 1 < |\sigma_{\beta+1}| < \cdots < |\sigma_n|$.*
- **(A2)** *There are local charts $\varphi_p \colon U_p \to \mathbb{R}^n$ and $\varphi_q \colon U_q \to \mathbb{R}^n$ centered at the points p and q such that the open sets $U_p, f(U_p), \dots, f^{\pi(p)-1}(U_p), U_q, f(U_q), \dots, f^{\pi(q)-1}(U_q)$ are pairwise disjoint. Moreover, these charts linearize the dynamics locally: in these local coordinates, the maps*

  $$f^{\pi(p)} \colon U_p \cap f^{-\pi(p)}(U_p) \to U_p \quad \text{and} \quad f^{\pi(q)} \colon U_q \cap f^{-\pi(q)}(U_q) \to U_q$$

  are the diagonal linear maps whose k^{th} diagonal entries are λ_k and σ_k, respectively. Furthermore, $\varphi_p\left(U_p \cap f^{-\pi(p)}(U_p)\right)$ and $\varphi_q\left(U_q \cap f^{-\pi(q)}(U_q)\right)$ contain the cube $[-2,2]^n$ of $\mathbb{R}^n$.
- **(A3)** *The heteroclinic point $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ has two iterates $x^- = f^{n_x^-}(x) \in U_p$ and $x^+ = f^{n_x^+}(x) \in U_q$, where $n_x^- < n_x^+$, whose local coordinates are*

  $$x^- = (\underbrace{0, \dots, 0}_{\alpha}, 1, \underbrace{0 \dots, 0}_{n-\alpha-1}), \quad \text{and} \quad x^+ = (\underbrace{0, \dots, 0}_{\beta-1}, 1, \underbrace{0 \dots, 0}_{n-\beta}).$$
- **(A4)** *There is a neighborhood $U_x^- \subset U_p$ of x^- such that the neighborhood $U_x^+ = f^{n_x}(U_x^-)$ of x^+ is contained in U_q, where $n_x = n_x^+ - n_x^-$. Furthermore, in the correspondig local coordinates, U_x^- and U_x^+ are contained in the cube $[-2,2]^n$, the map $T_x = f^{n_x} \colon U_x^- \to U_x^+$ is affine and its linear part $\mathcal{T}_x$ is diagonal. We denote the diagonal entries of $\mathcal{T}_x$ by $(t_{x,1}, \dots, t_{x,n})$. Note that by definition of T_x one has*

  $$T_x(\underbrace{0, \dots, 0}_{\alpha}, 1, \underbrace{0 \dots, 0}_{n-\alpha-1}) = (\underbrace{0, \dots, 0}_{\beta-1}, 1, \underbrace{0 \dots, 0}_{n-\beta}).$$

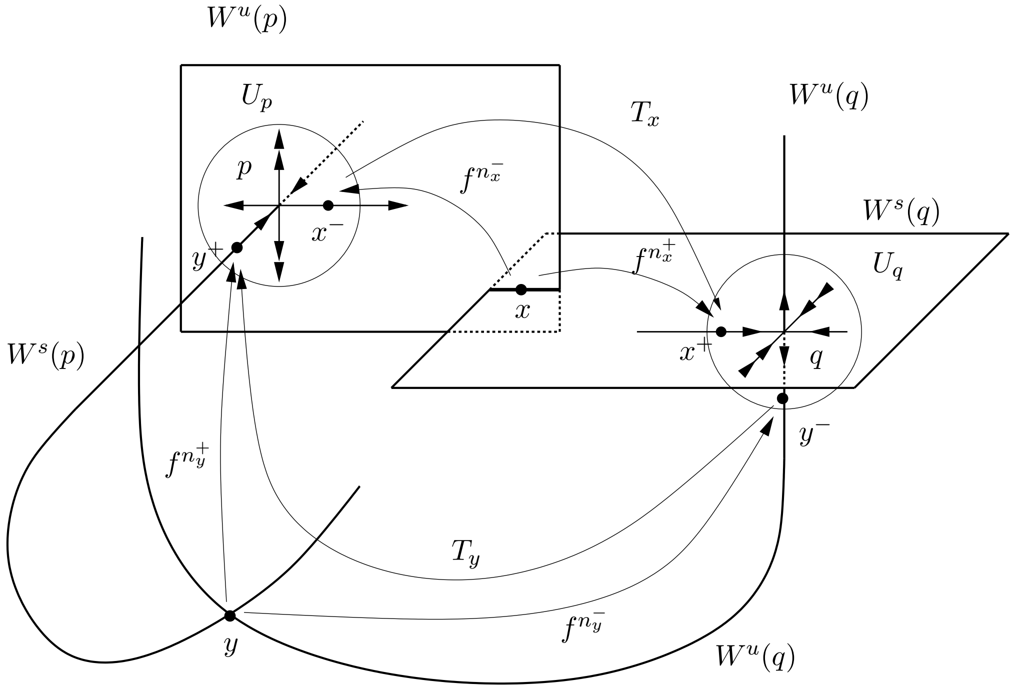

Figure 1: An affine heterodimensional cycle

(A5) *The heteroclinic point $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$ has two iterates $y^- = f^{n_y^-}(y) \in U_q$ and $y^+ = f^{n_y^+}(y) \in U_p$, where $n_y^- < n_y^+$, whose local coordinates are*

$$y^- = (\underbrace{0,\dots,0}_{\beta},1,\underbrace{0\dots,0}_{n-\beta-1}) \quad \textit{and} \quad y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,\underbrace{0\dots,0}_{n-\alpha}).$$

(A6) *There is a neighborhood $U_y^- \subset U_q$ of y^- such that the neighborhood $U_y^+ = f^{n_y}(U_y^-)$ of y^+ is contained in U_p, where $n_y = n_y^+ - n_y^-$. Moreover, in the corresponding local coordinates, U_y^- and U_y^+ are contained in the cube $[-2,2]^n$, the map $T_y = f^{n_y} \colon U_y^- \to U_y^+$ is affine and its linear part $\mathcal{T}_y$ is diagonal. We denote the diagonal entries of $\mathcal{T}_y$ by $(t_{y,1},\dots,t_{y,n})$. Observe that by definition of T_y,*

$$T_y(\underbrace{0,\dots,0}_{\beta},1,\underbrace{0\dots,0}_{n-\beta-1}) = (\underbrace{0,\dots,0}_{\alpha-1},1,\underbrace{0\dots,0}_{n-\alpha}).$$

Lemma 3.4. *Let f be a diffeomorphism having a heterodimensional cycle associated to the saddles p and q, having real eigenvalues and indices α and β, $\alpha < \beta$, and to the heteroclinic points $x \in W^u(\mathcal{O}_p) \cap W^s(\mathcal{O}_q)$ and $y \in W^s(\mathcal{O}_p) \cap W^u(\mathcal{O}_q)$. Then, for any C^1-neighborhood $\mathcal{U}$ of f, there is $g \in \mathcal{U}$ having an affine heterodimensional cycle associated to the saddles p and q and to the heteroclinic points x and y.*

Proof: The proof is essentially the same as that of [BDPR, Lemma 3.2], so we just sketch it. We first note that, after an arbitrarily small C^1-perturbation, we can assume that f is linearizable in some small neighborhoods of the saddles p and q, while keeping the heteroclinic points x and y. Moreover, we can also suppose that all the eigenvalues of $Df^{\pi(p)}(p)$ and of $Df^{\pi(q)}(q)$ have different moduli. In this way we obtain local coordinates verifying (A1) and (A2) above.

Next, by a local perturbation in the neighborhood of the heteroclinic points x and y, one makes the intersection $W^u(\mathcal{O}_p)\cap W^s(\mathcal{O}_q)$ at x transverse and the intersection $W^s(\mathcal{O}_p)\cap W^u(\mathcal{O}_q)$ at y quasi-transverse (i.e., $T_yW^s(\mathcal{O}_p)\cap T_yW^u(\mathcal{O}_q)=\bar{0}$). Furthermore, one may prevent these intersections from belonging to the strong stable or strong unstable manifolds, so that the heteroclinic intersections belong to the weak stable and the weak unstable manifolds. Let us explain this point more precisely. In the linearizing local coordinates we define local weak unstable and stable manifolds by

$$W^{cu}_{loc}(\mathcal{O}_q)=\{(\underbrace{0,\dots,0}_{\beta},s,\underbrace{0,\dots,0}_{n-\beta-1}),s\in[-2,2]\},\quad W^{cs}_{loc}(\mathcal{O}_p)=\{(\underbrace{0,\dots,0}_{\alpha-1},s,\underbrace{0,\dots,0}_{n-\alpha}),s\in[-2,2]\}.$$

Considering iterations by f we obtain a global invariant weak unstable manifold $W^{cu}(\mathcal{O}_q)$. Similarly, iterating by f^{-1} we obtain a weak stable manifold $W^{cs}(\mathcal{O}_p)$.

Analogously, we first define local strong unstable and stable manifolds by

$$W^{uu}_{loc}(\mathcal{O}_q)=\{(\underbrace{0,\dots,0}_{\beta+1})\}\times[-2,2]^{n-\beta-1},\quad W^{ss}_{loc}(\mathcal{O}_p)=[-2,2]^{\alpha-1}\times\{(\underbrace{0,\dots,0}_{n-\alpha+1})\}.$$

We next extend these manifolds by iterations of $f^{\pm1}$ to global strong unstable and strong stable manifolds $W^{uu}(\mathcal{O}_q)$ and $W^{ss}(\mathcal{O}_p)$.

After a new perturbation, we can assume that $y\notin W^{ss}(\mathcal{O}_p)$, therefore (using domination) its forward iterates *accumulate* to $W^{cs}(\mathcal{O}_p)$. Therefore, after a new arbitrarily small perturbation, we can assume that $y\in W^{cs}(\mathcal{O}_p)$. Using similar arguments, but now considering backward iterations, we can assume that (after an arbitrarily small perturbation) $y\in W^{cu}(\mathcal{O}_q)$. Hence $y\in W^{cs}(\mathcal{O}_p)\cap W^{cu}(\mathcal{O}_q)$. This implies (A5) above. To obtain (A3) one proceeds analogously.

A more subtle point consists in perturbing f in such a way that the differential of f along the heteroclinic orbits preserves the ordered eigenspaces. This was done in [BDPR, Lemma 3.2], so we will just sketch this point. One first shows (using domination) that (after an arbitrarily small perturbation) it is possible to preserve the one-dimensional bundles corresponding to the strong stable eigenvalues λ_1 and σ_1 (say $E_1(p)$ and $E_1(q)$) and the $(n-1)$-dimensional bundles corresponding to the remainder eigenvalues $\lambda_2,\dots,\lambda_n$ and $\sigma_2,\dots,\sigma_n$ (say $E_1^{n-1}(p)$ and $E_1^{n-1}(q)$). Now, a new perturbation, keeping invariant the previous bundles $E_1(p)$, $E_1(q)$, $E_1^{n-1}(p)$ and $E_1^{n-1}(q)$ but now focusing on $E_1^{n-1}(p)$ and $E_1^{n-1}(q)$, allows us to preserve the one-dimensional bundles corresponding to λ_2 and σ_2 (say $E_2(p)$ and $E_2(q)$) and the $(n-2)$-dimensional bundles corresponding to the remaining eigenvalues $\lambda_3,\dots,\lambda_n$ and $\sigma_3,\dots,\sigma_n$ (say $E_2^{n-2}(p)$ and $E_2^{n-2}(q)$). This step is identical to the first one and also involves domination. The proof now follows inductively.

In this way we obtain a diffeomorphism having a cycle verifying (A1)–(A6) above, ending the proof of the lemma. □

3.2 Creation of saddles of intermediate indices

Consider a diffeomorphism f with an affine heterodimensional cycle associated to the saddles p and q, of indices α and β, $\alpha<\beta$, and the heteroclinic points $x\in W^u(\mathcal{O}_p)\cap W^s(\mathcal{O}_q)$ and $y\in W^s(\mathcal{O}_p)\cap W^u(\mathcal{O}_q)$. We use the notations introduced in Section 3.1.

Consider a sequence of points $(r_{\ell,m})_{\ell,m\in\mathbb{N}}$ having local coordinates in U_p as follows:

$$r_{\ell,m} = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},1,\lambda_{\alpha+1}^{-\ell},0,\dots,0}_{\beta},\lambda_{\beta+1}^{-\ell}\,t_{x,\beta+1}^{-1}\,\sigma_{\beta+1}^{-m},\underbrace{0,\dots,0}_{n-\beta-1}).$$

We next define the points

$$s_{\ell,m} = f^{\pi_{\ell,m}}(r_{\ell,m}), \quad \text{where} \quad \pi_{\ell,m} = \ell\,\pi(p) + n_x + m\,\pi(q) + n_y.$$

We perturb f in a small neighborhood D_y^+ of $y^+ = f^{n_y^+}(y)$ which is relatively compact in U_y^+. We can assume that D_y^+ is disjoint from the cube $f^{\pi(p)}([-1-\varepsilon, 1+\varepsilon]^n) \subset U_p$, where $\varepsilon > 0$ is a small constant: this is possible because $f^{\pi(p)}([-1-\varepsilon,1+\varepsilon]^n)$ is a small neighborhood of the cube whose expression in the coordinates of U_p is $[-|\lambda_1|,|\lambda_1|]\times\cdots\times[-|\lambda_n|,|\lambda_n|]$ and because $|\lambda_\alpha| < 1$ and D_y^+ is a small neighborhood of the point $y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,\underbrace{0\dots,0}_{n-\alpha})$. Moreover, we can also choose D_y^+ disjoint from the orbits $(f^k(x^-))$, $k\in\{0,\dots,n_x-1\}$, and $f^k(y^-)$, $k\in\{0,\dots,n_y-1\}$ and from the open sets $f(U_p),\dots,f^{\pi(p)-1}(U_p)$, $U_q, f(U_q),\dots,f^{\pi(q)-1}(U_q)$.

Proposition 3.5. *For every ℓ and m large enough, the points $r_{\ell,m}$ and $s_{\ell,m}$ are well-defined, belong to U_p, and the sequences $(r_{\ell,m})$ and $(s_{\ell,m})$ converge to the point $y^+ = f^{n_y^+}(y)$ as $\ell, m\to\infty$. Moreover, the intermediate iterates $f^k(r_{\ell,m})$, $k\in\{1,\dots,\pi_{\ell,m}-1\}$, do not intersect the set D_y^+.*

Proof: As $|\lambda_{\alpha+1}|, |\lambda_{\beta+1}|$ and $|\sigma_{\beta+1}|$ are greater than 1, the sequence $(r_{\ell,m})$ converges to the point $y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,0\dots,0)$. In particular, the point $r_{\ell,m}$ is well-defined for ℓ,m large, and belongs to U_p. Now, for ℓ,m large, $f^{\ell\,\pi(p)}(r_{\ell,m})$ is the point in U_p whose coordinates are

$$f^{\ell\,\pi(p)}(r_{\ell,m}) = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},\lambda_\alpha^\ell,1,0,\dots,0}_{\beta},t_{x,\beta+1}^{-1}\,\sigma_{\beta+1}^{-m},\underbrace{0,\dots,0}_{n-\beta-1}).$$

The intermediate points $f^k(r_{\ell,m})$, $k\in\{1,\dots,\ell\,\pi(p)\}$, do not belong to U_p when k is different from 0 modulo $\pi(p)$ (just note that the sets $U_p,\dots,f^{\pi(p)-1}(U_p)$ are pairwise disjoint) and belong to $f^{\pi(p)}([-1-\varepsilon,1+\varepsilon]^n)$ when k is equal to 0 modulo $\pi(p)$. In all these cases, these iterates do not meet D_y^+.

Thus, for ℓ and m large enough, the point $f^{\ell\,\pi(p)}(r_{\ell,m})$ is very close to $x^- = (\underbrace{0,\dots,0}_{\alpha},1,0,\dots,0)$. Hence $T_x\left(f^{\ell\,\pi(p)}(r_{\ell,m})\right)$ is well-defined and close to $x^+ = (\underbrace{0,\dots,0}_{\beta-1},1,0,\dots,0)\in U_q$. Moreover, by the definition of T_x in (A4), in the local coordinates we have

$$T_x\left(f^{\ell\,\pi(p)}(r_{\ell,m})\right) = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},t_{x,\alpha}\,\lambda_\alpha^\ell,,0,\dots,0}_{\beta-1},1,\sigma_{\beta+1}^{-m},\underbrace{0,\dots,0}_{n-\beta-1}).$$

Moreover, the points $f^k\left(f^{\ell\,\pi(p)}(r_{\ell,m})\right)$, $k \in \{0,\dots,n_x\}$, are very close to the points $f^k(x^-)$ and are disjoint from D_y^+.

Repeating these arguments, one shows that, for ℓ, m large enough, $f^k\left(T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right)$, $k \in \{0,\dots,m\,\pi(q)\}$, is disjoint from D_y^+ and $f^{m\,\pi(q)}\left(T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right) \in U_q$ and has coordinates

$$f^{m\,\pi(q)}\left(T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right) = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},\sigma_\alpha^m\,t_{x,\alpha}\,\lambda_\alpha^\ell,0,\dots,0}_{\beta-1},\sigma_\beta^m,1,\underbrace{0,\dots,0}_{n-\beta-1}).$$

As $|\sigma_\beta|, |\sigma_\alpha|, |\lambda_\alpha| < 1$, this point is arbitrarily close to $y^- = (\underbrace{0,\dots,0}_{\beta},1,0,\dots,0)$ if m is large enough. So, the points

$$s_{\ell,m} = T_y\left(f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right)$$

are well-defined and close to $y^+ \in U_p$. Finally, they belong to D_y^+ and their local coordinates are:

$$s_{\ell,m} = (\underbrace{\overbrace{0,\dots,0}^{\alpha-1},1+t_{y,\alpha}\,\sigma_\alpha^m\,t_{x,\alpha}\,\lambda_\alpha^\ell,0,\dots,0}_{\beta-1},t_{y,\beta}\,\sigma_\beta^m,\underbrace{0,\dots,0}_{n-\beta}).$$

Moreover, the intermediate points $f^k\left(f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})\right)$, $k \in \{0,\dots,n_y-1\}$, are close to the points $f^k(y^-)$ and do not meet D_y^+.

As $|\lambda_\alpha|, |\sigma_\alpha|, \sigma_\beta| < 1$, one easily checks that, when $m \to \infty$, the sequence $(s_{\ell,m})$ converges to $y^+ = (\underbrace{0,\dots,0}_{\alpha-1},1,0,\dots,0)$. This completes the proof of the proposition. □

For ℓ, m large enough, we use the coordinates of the chart φ_p to define the vector $\theta_{\ell,m} \in \mathbb{R}^n$ by $r_{\ell,m} = s_{\ell,m} + \theta_{\ell,m}$. We denote by $\Theta_{\ell,m}$ the local diffeomorphism defined on D_y^+ whose expression in the local coordinates of U_p is the translation $z \mapsto z + \theta_{\ell,m}$.

We fix a neighborhood V_y of y^+, relatively compact in the interior of D_y^+. The following lemma yields a sequence $(g_{\ell,m})$ of local C^1-perturbations of f, $g_{\ell,m} \to f$ as $\ell, m \to \infty$, each one closing the orbit of the corresponding point $r_{\ell,m}$.

Lemma 3.6. *For any C^1-neighborhood $\mathcal{U}$ of f and for every ℓ, m large enough, there is a diffeomorphism $h_{\ell,m}$ coinciding with $\Theta_{\ell,m}$ on V_y and with the identity map outside of D_y^+, such that the diffeomorphism $g_{\ell,m} = h_{\ell,m} \circ f$ belongs to $\mathcal{U}$.*

Proof: This comes from the fact that the vectors $\theta_{\ell,m}$ go to $\bar{0}$ as ℓ and m go to ∞. □

One now obtains the announced periodic points for $g_{\ell,m}$:

Proposition 3.7. *For every ℓ, m large enough, the $r_{\ell,m}$ is a periodic point of $g_{\ell,m}$ whose period is $\pi_{\ell,m}$. Furthermore, the derivative $Dg^{\pi_{\ell,m}}(r_{\ell,m})$ is*

$$Dg^{\pi_{\ell,m}}(r_{\ell,m}) = \mathcal{T}_y \circ \left(Df^{\pi(q)}(q)\right)^m \circ \mathcal{T}_x \circ \left(Df^{\pi(p)}(p)\right)^\ell.$$

In other words, $Dg^{\pi_{\ell,m}}(r_{\ell,m})$ is the diagonal linear map whose k^{th} diagonal entry is

$$t_{y,k}\,\sigma_k^m\,t_{x,k}\,\lambda_k^\ell.$$

Proof: For ℓ and m large enough, the points $r_{\ell,m}$ and $s_{\ell,m}$ belong to V_y so that $\Theta_{\ell,m}(s_{\ell,m}) = r_{\ell,m}$. As $r_{\ell,m}$ and $s_{\ell,m} = f^{\pi_{\ell,m}}(r_{\ell,m})$ are the only points of the segment of orbit $r_{\ell,m}, f(r_{\ell,m}), \dots, f^{\pi_{\ell,m}}(r_{\ell,m})$ in the support D_y^+ of the perturbation $h_{\ell,m}$, the point $r_{\ell,m}$ is $\pi_{\ell,m}$-periodic for $g_{\ell,m}$.

In the local coordinates, the derivative $Df^{\pi_{\ell,m}}(r_{\ell,m})$ is $\mathcal{T}_y \circ \left(Df^{\pi(q)}(q)\right)^m \circ \mathcal{T}_x \circ \left(Df^{\pi(p)}(p)\right)^\ell$. In the coordinates of $V_y \subset U_p$, the map $h_{\ell,m}$ is a translation, hence, one obtains the same expression for the derivative of $Dg^{\pi_{\ell,m}}(r_{\ell,m})$. □

By choosing carefully the integers ℓ and m, one obtains any index τ between α and β:

Corollary 3.8. *For any integer $\tau \in \{\alpha, \dots, \beta\}$, there exists a sequence (ℓ_k, m_k), with $\lim_{k\to+\infty} \ell_k = \lim_{k\to+\infty} m_k = +\infty$ such that, for every k, the point r_{ℓ_k,m_k} is a hyperbolic saddle of g_{ℓ_k,m_k} having index τ.*

Proof: Let us assume that τ belongs to $\{\alpha+1, \dots, \beta\}$, the case $\tau = \alpha$ follows similarly. As $|\sigma_\tau| < 1 < |\lambda_\tau|$, one can choose ℓ and m arbitrarily large such that the modulus of the τ^{th} eigenvalue of $r_{\ell,m}$, which is $|t_{y,\tau}\, \sigma_\tau^m\, t_{x,\tau}\, \lambda_\tau^\ell|$, belongs to $[|\sigma_\tau^2|, |\sigma_\tau|]$, in particular is less than one. On the other hand, the ratio between the moduli of the $(\tau+1)^{\text{th}}$ and $(\tau)^{\text{th}}$ eigenvalues of $r_{\ell,m}$ is

$$\frac{|t_{y,\tau+1}\, t_{x,\tau+1}|}{|t_{y,\tau}\, t_{y,\tau}|} \left(\frac{|\lambda_{\tau+1}|}{|\lambda_\tau|}\right)^m \left(\frac{|\sigma_{\tau+1}|}{|\sigma_\tau|}\right)^\ell .$$

Since $|\lambda_{\tau+1}| > |\lambda_\tau|$ and $|\sigma_{\tau+1}| > |\sigma_\tau|$, if ℓ and m are big enough, the ratio between the $(\tau+1)^{\text{th}}$ and the $(\tau)^{\text{th}}$ eigenvalues above is strictly bigger than $|\sigma_\tau^{-3}|$. This implies that the modulus of the $(\tau+1)^{\text{th}}$ eigenvalue of $r_{\ell,m}$ is greater that $|\sigma_\tau^{-1}| > 1$. This shows that the index of $r_{\ell,m}$ is exactly τ, ending the proof of the lemma. □

Remark 3.9. *The arguments in the proof of Corollary 3.8 imply the following: for every $\varepsilon > 0$ and every $\rho \in (0,1)$, there is a saddle $r = r_{\ell,m}$ of $g = g_{\ell,m}$ (for some appropriate large ℓ and m) whose Lyapunov exponents $(\log|\mu_1|/\pi(r)), \dots, (\log|\mu_n|/\pi(r))$ (here the μ_i are the eigenvalues of $Dg^{\pi(r)}(r)$) verify*

$$\left|\frac{\log|\mu_i|}{\pi(r)} - \left(\rho\, \frac{\log|\lambda_i|}{\pi(p)} + (1-\rho)\, \frac{\log|\sigma_i|}{\pi(q)}\right)\right| < \varepsilon.$$

This remark and Theorem 1 immediately imply Corollary 2.

3.3 Control of the heteroclinic intersections

In this section we will complete the proof of Theorem 3.2.

Note that the diffeomorphisms $g_{\ell,m}$ coincide with f in the neighborhood of the orbits of p and q, thus the orbits of p and q by f and $g_{\ell,m}$ coincide. Thus, according to Corollary 3.8, to prove Theorem 3.2 it is enough to see the following:

Proposition 3.10. *For ℓ and m large enough, the diffeomorphism $g_{\ell,m}$ satisfies*

$$p <_{\text{us}} r_{\ell,m} <_{\text{us}} q.$$

Proof: Given ℓ and m big enough, denote by $\Delta^s_{\ell,m}$ the α-dimensional disk contained in U_p defined in the corresponding local coordinates by

$$\Delta^s_{\ell,m} = [-1-\varepsilon, 1+\varepsilon]^\alpha \times (\lambda^{-\ell}_{\alpha+1}, \underbrace{0,\dots,0}_{\beta-\alpha-1}, \lambda^{-\ell}_{\beta+1}\, t^{-1}_{x,\beta+1}\, \sigma^{-m}_{\beta+1}, \underbrace{0,\dots,0}_{n-\beta-1}).$$

Similarly, consider the $(n-\beta)$-dimensional cube $\Delta^u_{\ell,m}$ defined in the local coordinates of U_q by

$$\Delta^u_{\ell,m} = (\underbrace{0,\dots,0}_{\alpha-1}, t_{x,\alpha}\,\lambda^\ell_\alpha\,\sigma^m_\alpha, \underbrace{0,\dots,0}_{\beta-\alpha-1}, \sigma^m_\beta) \times [-1-\varepsilon, 1+\varepsilon]^{n-\beta}.$$

Note that the disks $\Delta^s_{\ell,m}$ and $\Delta^u_{\ell,m}$ contain the points $r_{\ell,m}$ and $f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m}) = T_y^{-1}(r_{\ell,m})$, respectively.

Lemma 3.11. *For every ℓ, m large enough,*

- $\Delta^s_{\ell,m} \subset W^s(r_{\ell,m}, g_{\ell,m})$;
- $\Delta^u_{\ell,m} \subset W^u(T_y^{-1}(r_{\ell,m}))$.

Before proving this lemma let us complete the proof of the proposition assuming it. Observe that when ℓ and m go to $+\infty$, the disks $\Delta^s_{\ell,m}$ converge (in the C^1 topology) to the disk

$$\Delta^s_p = [-1-\varepsilon, 1+\varepsilon]^\alpha \times \{0\}^{n-\alpha},$$

which is a local stable manifold of p. This implies that $\Delta^s_{\ell,m}$ intersects transversally the unstable manifold of p when ℓ and m are large enough. Lemma 3.11 implies that the unstable manifold of p transversally intersects the stable one of $r_{\ell,m}$, thus $p <_{\mathrm{us}} r_{\ell,m}$. In the same way, one shows that the disk $\Delta^u_{\ell,m}$, containing $f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m}) = T_y^{-1}(r_{\ell,m})$, intersects transversally the local stable manifold of q. Therefore $r_{\ell,m} <_{\mathrm{us}} q$. Thus to complete the proof of the proposition it remains to prove Lemma 3.11.

Proof of the lemma: We just prove the first item of the lemma, the second one follows similarly by considering backward iterates of f.

The images of $\Delta^s_{\ell,m}$ by f^k, $k \in \{1, \dots, \ell\,\pi(p)\}$, are disjoint from D^+_y: when k is different from 0 modulo $\pi(p)$, the image is disjoint from U_p and, when k is equal to 0 modulo $\pi(p)$, the image is a disk contained in the cube $f^{\pi(p)}([-1-\varepsilon, 1+\varepsilon]^n) \subset U_p$. In particular, this shows that $g^{\ell\,\pi(p)}_{\ell,m}$ coincides with $f^{\ell\,\pi(p)}$ on $\Delta^s_{\ell,m}$ and that $g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ is an α-dimensional disk contained in U^-_x of size *less* than $|(\lambda_\alpha)^\ell(1+\varepsilon)|$ containing $g^{\ell\,\pi(p)}_{\ell,m}(r_{\ell,m}) = f^{\ell\,\pi(p)}(r_{\ell,m})$. As the diameter of this disk tends to 0 when ℓ goes to $+\infty$, one has that $g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m}) \subset U^-_x$, so that T_x is defined on $g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ and $T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m}) = g^{\ell\,\pi(p)+n_x}_{\ell,m}(\Delta^s_{\ell,m})$ is an arbitrarily small disk containing the point $T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})$ in U^+_x.

In the same way, one verifies that $g^{m\,\pi(q)}_{\ell,m} \circ T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ is a very small disk that contains $f^{m\,\pi(q)} \circ T_x \circ f^{\ell\,\pi(p)}(r_{\ell,m})$ in U^-_y. So (for ℓ, m large enough) $g^{m\,\pi(q)}_{\ell,m} \circ T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$ is contained in the small neighborhood $T_y^{-1}(V_y)$ of y^-. As a consequence, $g^{n_y}_{\ell,m}$ coincides with $\Theta_{\ell,m} \circ T_y$ on $g^{m\,\pi(q)}_{\ell,m} \circ T_x \circ g^{\ell\,\pi(p)}_{\ell,m}(\Delta^s_{\ell,m})$.

The previous arguments show that $g_{\ell,m}^{\pi_{\ell,m}}(\Delta^s_{\ell,m})$ is an α-dimensional disk containing $r_{\ell,m} = g_{\ell,m}^{\pi_{\ell,m}}(r_{\ell,m})$ and whose diameter tends to 0 as ℓ, m tend to $+\infty$. So for ℓ, m large enough, the map $g_{\ell,m}^{\pi_{\ell,m}}$ maps $\Delta^s_{\ell,m}$ into itself and is a linear contraction on this disk, having $r_{\ell,m}$ as a fixed point. This concludes the proof of the lemma. □

The proof of Proposition 3.10 is now complete. □

4 Heterodimensional cycles and periodic points. Proof of Theorem 1

In this section we finish the proof of Theorem 1, which follows immediately from the proposition below: just note that if r_f is a saddle of index τ then the saddles of index τ in $H(r_f, f)$ constitute a dense subset of $H(r_f, f)$.

Proposition 4.1. *There is a residual subset $\mathcal{G}_4$ of* $\mathrm{Diff}^1(\mathrm{M})$ *with the following property: for every $f \in \mathcal{G}_4$ and every pair of saddles p_f and q_f of f having indices α and β, $\alpha < \beta$, such that $H(p_f, f) = H(q_f, f)$, given any $\tau \in (\alpha, \beta) \cap \mathbb{N}$ there is a hyperbolic periodic saddle r_f of index τ such that*

$$H(p_f, f) = H(q_f, f) = H(r_f, f).$$

The proposition immediately follows from a standard Baire argument (analogous to the one in the proof of Lemma 2.1) and the lemma below.

Lemma 4.2. *Let $\mathcal{U}$ be an open subset of $\mathcal{G}_3$ (the residual subset of* $\mathrm{Diff}^1(M)$ *in Lemma 2.9) such that for every $f \in \mathcal{U}$ there are saddles p_f and q_f of indices α and β, $\alpha < \beta$, depending continuously on f such that $H(p_f, f) = H(q_f, f)$. Then for every $\tau \in (\alpha, \beta) \cap \mathbb{N}$ there is an open and dense subset $\mathcal{V}_\tau$ of $\mathcal{U}$ such that every $g \in \mathcal{V}_\tau$ has a saddle r_g of index τ with $H(p_g, g) = H(q_g, g) = H(r_g, g)$.*

Proof: Let $f \in \mathcal{U}$. By Proposition 2.3, we can assume (after replacing the initial saddles by saddles homoclinically related to them) that p_f and q_f both have real positive eigenvalues of multiplicity one (i.e., $p_f, q_f \in \mathrm{Per}_{\mathbb{R}}(H(p_f, f))$). As $f \in \mathcal{G}_3$, one has that $W^u(p_f, f)$ and $W^s(q_f, f)$ have some non-empty transverse intersection. This property holds for every g close to f. Using Lemma 2.8, we get h close to f having a heterodimensional cycle associated to p_h and q_h.

Fix now $\tau \in (\alpha, \beta) \cap \mathbb{N}$. Using Theorem 3.2 we obtain some g close to h, thus close to f, having a saddle r_g of index τ with $p_g <_{\mathrm{us}} r_g <_{\mathrm{us}} q_g$. By Remark 3.1 this relation persists under C^1-perturbations. This gives an open and dense subset $\mathcal{V}_\tau$ of $\mathcal{U}$ such that every $g \in \mathcal{V}_\tau$ has a saddle r_g of index τ with $p_g <_{\mathrm{us}} r_g <_{\mathrm{us}} q_g$.

Claim 4.3. *Let $g \in \mathcal{V}_\tau$. For every $\varepsilon > 0$ there is a closed ε-pseudo-orbit containing p_g, r_g and q_g.*

This claim implies that the saddles p_g, q_g and r_g are in the same chain recurrence class Λ of g. By (G6),

$$\Lambda = H(p_g, g), \quad \Lambda = H(r_g, g), \quad \Lambda = H(q_g, g).$$

Thus the three homoclinic classes coincide. To end the proof of the lemma it remains only to prove the claim.

Proof of the claim: Fix any $\varepsilon > 0$. Since $H(p_g, g) = H(q_g, g)$ is a transitive set, there is a (finite) segment $S_{q,p}$ of ε-pseudo-orbit going from q_g to p_g. On the other hand, since $W^u(p_g, g)$ intersects $W^s(r_g, g)$ there is an orbit going from p_g to r_g. Thus there is is a (finite) segment $S_{p,r}$ of ε-pseudo-orbit going from p_g to r_g. Similarly, as $W^u(r_g, g)$ intersects $W^s(q_g, g)$ there is a (finite) segment $S_{r,q}$ of ε-pseudo-orbit going from r_g to q_g. The announced pseudo-orbit is obtained concatenating the segments of ε-pseudo-orbits $S_{q,p}$, $S_{p,r}$ and $S_{r,q}$. □

The proof of Lemma 4.2 is now complete. □

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A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources

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A Ricardo Mañé (1948-1995), por todo su trabajo

Abstract

We show that, for every compact n-dimensional manifold, $n \geq 1$, there is a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms for which the homoclinic class of any periodic saddle of f verifies one of the following two possibilities: Either it is contained in the closure of an infinite set of sinks or sources (Newhouse phenomenon), or it presents some weak form of hyperbolicity called *dominated splitting* (this is a generalization of a bidimensional result of Mañé [Ma$_3$]). In particular, we show that any C^1-robustly transitive diffeomorphism admits a dominated splitting.

Résumé

Généralisant un résultat de Mañé sur les surfaces [Ma$_3$], nous montrons que, en dimension quelconque, il existe un sous-ensemble résiduel de $\mathrm{Diff}^1(M)$ de difféomorphismes pour lesquels la classe homocline de toute selle périodique hyperbolique possède deux comportements possibles : ou bien elle est incluse dans l'adhérence d'une infinité de puits ou de sources (phénomène de Newhouse), ou bien elle présente une forme affaiblie d'hyperbolicité appelée une *décomposition dominée.* En particulier nous montrons que tout difféomorphisme C^1-robustement transitif possède une décomposition dominée.

Introduction

Context

The Anosov-Smale theory of uniformly hyperbolic systems has played a double role in the development of the qualitative theory of dynamical systems. On one hand, this theory shows that chaotic and random behavior can appear in a stable way for deterministic systems depending on a very small number of parameters. On the other hand, the chaotic

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systems admit in this theory a quasi-complete description from the ergodic point of view, moreover the hyperbolic attractors satisfy very simple statistical properties (see [Si], [Ru], and [BoRu]): For Lebesgue almost every point in the topological basin of the attractor, the time average of any continuous function along its orbit converges to the spatial average of the function by a probability measure whose support is the attractor.

However, since the end of the 60s, one knows that this hyperbolic theory does not cover a dense set of dynamics: There are examples of open sets of non-hyperbolic diffeomorphisms. More precisely,

- For every compact surface S there exist non-empty open sets of $\mathrm{Diff}^2(S)$ of diffeomorphisms whose non-wandering set is not hyperbolic (see [N_1]).
- Given any compact manifold M, with $\dim(M) \geq 3$, there are non-empty open subsets of $\mathrm{Diff}^1(M)$ of diffeomorphisms whose non-wandering set is not hyperbolic (see, for instance, [AS] and [So] for the first examples).

In the $2-$dimensional case, at least in the C^1-topology, the heart of this phenomenon is closely related with the appareance of homoclinic tangencies: For every compact surface S the set of C^1-diffeomorphisms with homoclinic tangencies is C^1-dense in the complement in $\mathrm{Diff}^1(S)$ of the closure of the Axiom A diffeomorphisms (that is a recent result in [PuSa]).

Even if in this work we are concerned with the C^1-topology, let us recall that Newhouse showed (see [N_1]) that generic unfolding of homoclinic tangencies create C^2-locally residual subsets of $\mathrm{Diff}^2(S)$ of diffeomorphisms having an infinite set of sinks or sources. Along this paper, by *C^r-Newhouse phenomenon* we mean the coexistence of infinitely many sinks or sources in a C^r-locally residual subset of $\mathrm{Diff}^r(M)$.

The main motivation of this article comes from the following result of Mañé (see [Ma_3] (1982)), which gives, for C^1-generic diffeomorphisms of surfaces, a dichotomy between hyperbolic dynamics and Newhouse phenomenon:

Theorem (Mañé) *Let S be a closed surface. Then there is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(S)$, $\mathcal{R} = \mathcal{R}_1 \coprod \mathcal{R}_2$, such that every $f \in \mathcal{R}_1$ verifies the Axiom A and every $f \in \mathcal{R}_2$ has infinitely many sinks or sources.*

Recall that a diffeomorphism of a manifold M is *transitive* if it has a dense orbit in the whole manifold. Such a diffeomorphism is called *C^r-robustly transitive* if it belongs to the C^r-interior of the set of transitive diffeomorphisms. Since transitive diffeomorphisms have neither sinks nor sources, a direct consequence from Mañé's result is the following:

Every C^1-robustly transitive diffeomorphism on a compact surface admits a hyperbolic structure on the whole manifold, i.e. it is an Anosov diffeomorphism.

Let us observe that Mañé's result has no direct generalization to higher dimensions: For every $n \geq 3$ there are compact $n-$dimensional manifolds supporting C^1-robustly transitive non-hyperbolic diffeomorphisms (in particular, without sources and sinks). All the examples of such diffeomorphisms, successively given by [Sh] (1972) on the torus T^4, by [Ma_2] (1978) on T^3, by [BD_1] (1996) in many other manifolds (those supporting a transitive Anosov flow or of the form $M \times N$, where M is a manifold with an Anosov diffeomorphism and N any compact manifold), by [B] (1996) and [BoVi] (1998) examples in T^3 without any hyperbolic expanding direction and examples in T^4 without any hyperbolic direction, present some weak form of hyperbolicity, the newer the examples the weaker the form of hyperbolicity, but always exhibiting some remaining weak form of hyperbolicity. Let us be more precise.

Recall first that an invariant compact set K of a diffeomorphism f on a manifold M is hyperbolic if the tangent bundle $TM|_K$ of M over K admits a f_*-invariant continuous splitting $TM|_K = E^s \oplus E^u$, such that f_* uniformly contracts the vectors in E^s and uniformly expands the vectors in E^u. This means that there is $n \in \mathbb{N}$ such that $\|f_*^n(x)|_{E^s(x)}\| < 1/2$ and $\|f_*^{-n}(x)|_{E^u(x)}\| < 1/2$ for any $x \in K$ (where $||\cdot||$ denotes the norm).

The examples of C^1-robustly transitive diffeomorphisms f in [Sh] and [Ma$_2$] let invariant a splitting $TM = E^s \oplus E^c \oplus E^u$, where f_* contracts uniformly the vectors in E^s and expands uniformly the vectors in E^u. Moreover, this splitting is dominated (roughly speaking, the contraction (resp. expansion) in E^s (resp. E^u) is stronger than the contraction (resp. expansion) in E^c, for details see Definition 0.1 below), and the central bundle E^c is one dimensional. The examples in [BD$_1$] admit also such a nonhyperbolic splitting, but the central bundle may have any dimension. The diffeomorphisms in [B] have no stable bundle E^s and admit a splitting $E^c \oplus E^u$, where the restriction of f_* to E^c is not uniformly contracting, but it uniformly contracts the area. Finally, [BoVi] gives examples of robustly transitive diffeomorphisms on T^4 without any uniformly stable or unstable bundles: They let invariant some dominated splitting $E^{cs} \oplus E^{cu}$, where the derivative of f contracts uniformly the area in E^{cs} and expands uniformly the area in E^{cu}.

Roughly speaking, in this paper we see that, if a transitive set does not admit a dominated splitting, then one can create as many sinks or sources as one wants in any neighbourhood of this set. In particular, C^1-robustly transitive diffeomorphisms always admit some dominated splitting.

Before stating our results more precisely, let us mention two previous results on $3-$manifolds which are the roots of this work:

- [DPU] shows that there is an open and dense subset of C^1-robustly transitive $3-$dimensional diffeomorphisms f admitting a dominated splitting $E^1 \oplus E^2$ such that at least one of the two bundles is uniformly hyperbolic (either stable or unstable). In that case, by terminology, f is *partially hyperbolic.* Moreover, [DPU] also gives a semi-local version of this result defining C^1-robustly transitive sets. Given a C^1-diffeomorphism f, a compact set K is C^1-*robustly transitive* if it is the maximal $f-$invariant set in some neighbourhood U of it and if, for every g C^1-close to f, the maximal invariant set $K_g = \bigcap_{\mathbb{Z}} g^n(U)$ is also transitive.

- [BD$_2$] gives examples of diffeomorphisms f on $3-$manifolds having two saddles P and Q with a pair of contracting and expanding complex (non-real) eigenvalues, respectively, which belong in a robust way to the same transitive set Λ_f. Clearly, this simultaneous presence of complex contracting and expanding eigenvalues prevents the transitive set Λ_f from admitting a dominated splitting! Then [BD$_2$] shows that, for a C^1-residual subset of such diffeomorphisms, the transitive set Λ_f is contained in the closure of the (infinite) set of sources or sinks.

The results of these two papers seem to go in opposite directions, but here we show that they describe two sides of the same phenomenon. In fact,we give here a framework which allows us to unify these results and generalize them in any dimension: In the absence of weak hyperbolicity (more precisely, of a dominated splitting) one can create an arbitrarily large number of sinks or sources.

In the non-hyperbolic context, the classical notion of basic pieces (of the Smale's spectral decomposition theorem) is not defined and an important problem is to understand what could be a good substitute of it. The elementary pieces of non-trivial transitive dynamics are the homoclinic classes of hyperbolic periodic points, which are exactly the basic sets in Smale theory. Actually, [BD$_2$] shows that, C^1-generically, two periodic points belong to the same transitive set if and only if their two homoclinic classes are equal [1]. The hyperbolic-like properties of these homoclinic classes are the main subject of this paper.

Finally, we also see that some of our arguments can be adapted almost straightforwardly in the volume preserving setting. Let us now state our results in a precise way.

Statement of the results

Our first theorem asserts that given any hyperbolic saddle P its homoclinic class either admits an invariant dominated splitting or can be approximated (by $\mathcal{C}^1-$perturbations) by arbitrarily many sources or sinks.

Definition 0.1 *Let f be a diffeomorphism defined on a compact manifold M, K an $f-$invariant subset of M, and $TM|_K = E \oplus F$ an f_*-invariant splitting of TM over K, where the fibers E_x of E have constant dimension. We say that $E \oplus F$ is a* dominated splitting *(of f over K) if there exists $n \in \mathbb{N}$ such that*

$$\|f_*^n(x)|_E\| \, \|f_*^{-n}(f^n(x))|_F\| < 1/2.$$

We write $E \prec F$, or $E \prec_n F$ if we want to emphasize the role of n, and we speak of $n-$dominated splitting.

Let us make two comments on this definition. First, the invariant set K is not supposed to be compact and the splitting is not supposed to be continuous. However, if K admits a dominated splitting, it is always continuous and can be extended uniquely to the closure $\bar{K}$ of K (these are classical results, for details see Lemma 1.4 and Corollary 1.5). Moreover, the existence of a dominated splitting is equivalent to the existence of some continuous strictly-invariant cone field over $\bar{K}$; this cone field can be extended to some neighbourhood U of $\bar{K}$ and persists by C^1-perturbations. Thus there is an open neighbourhood of f of diffeomorphisms for which the maximal invariant set in U admits a dominated splitting. In that sense, the existence of a dominated splitting is a C^1-robust property.

Given a hyperbolic saddle P of a diffeomorphism f we denote by $H(P, f)$ the *homoclinic class* of P, i.e. the closure of the transverse intersections of the invariant manifolds of P.

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[1] Recently, some substantial progress has be done for understanding the elementary pieces of dynamics of nonhyperbolic diffeomorphisms. In [Ar$_1$] and [CMP] it is showed that, for C^1-generic diffeomorphims or flows, any homoclinic class is a maximal transitive set. Moreover, any pair of homoclinic classes are either equal or disjoint. On the other hand, [BD$_3$] constructs examples of C^1-locally generic $3-$dimensional diffeomorphisms with maximal transitive sets without periodic orbits.

This set is transitive and the set Σ of hyperbolic periodic points $Q \in H(P,f)$ of f, whose stable and unstable manifolds intersect transversally the invariant manifolds of P, is dense in $H(P,f)$.

Theorem 1 *Let P be a hyperbolic saddle of a diffeomorphism f defined on a compact manifold M. Then*

- *either the homoclinic class $H(P,f)$ of P admits a dominated splitting,*
- *or given any neighbourhood U of $H(P,f)$ and any $k \in \mathbb{N}$ there is g arbitrarily C^1-close to f having k sources or sinks arbitrarily close to P, whose orbits are included in U.*

If P is a hyperbolic periodic point of f then, for every g C^1-close to f, there is a hyperbolic periodic point P_g of g close to P (this point is given by the implicit function theorem), called the *continuation* of P for g. From Theorem 1 we get the following two corollaries.

Corollary 0.2 *Under the hypotheses of Theorem 1, one of the following two possibilities holds,*

- *either there are a C^1-neighbourhood $\mathcal{U}$ of f and a dense open subset $\mathcal{V} \subset \mathcal{U}$ such that $H(P_g,g)$ has a dominated splitting for any $g \in \mathcal{V}$,*
- *or there exist diffeomorphisms g arbitrarily C^1-close to f such that $H(P_g,g)$ is contained in the closure of infinitely many sinks or sources.*

Corollary 0.3 *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{R}$ and any hyperbolic periodic saddle P of f, the homoclinic class $H(P,f)$ satisfies one of the following possibilities:*

- *either $H(P,f)$ has a dominated splitting,*
- *or $H(P,f)$ is included in the closure of the infinite set of sinks and sources of f.*

Problem *Is there a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that the homoclinic class of any hyperbolic periodic point P is the maximal transitive set containing P (i.e. every transitive set containing P is included in $H(P,f)$)? Moreover, when is $H(P,f)$ locally maximal?* [2]

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[2] Observe that the first part of the problem was positively answered in [Ar$_1$] and [CMP], (recall footnote 1). Using these results, [Ab] shows that there is a C^1-residual set of diffeomorphisms such that the number of homoclinic classes is well defined and locally constant. Moreover, if this number is finite, the homoclinic classes are locally maximal sets and there is a filtration whose levels correspond to homoclinic classes.

Actually, we prove a quantitative version of Theorem 1 relating the strength of the domination with the size of the perturbations of f that we consider to get sinks or sources (see Proposition 2.6). This quantitative version is one of the keys for the next two results.

Note first that the creation of sinks or sources is not compatible with the C^1-robust transitivity of a diffeomorphism. We apply Hayashi's connecting lemma (see [Ha] and Section 2) to get, by small perturbations, a dense homoclinic class in the ambient manifold. Then using the quantitative version of Theorem 1 we show:

Theorem 2 *Every C^1-robustly transitive set (or diffeomorphism) admits a dominated splitting.*

Mañé's theorem for surface diffeomorphisms mentioned before gives a C^1-generic dichotomy between hyperbolicity and C^1-Newhouse phenomenon. It is now natural to ask what happens, in any dimension, *far from* C^1-Newhouse phenomenon.

Theorem 3 *Let f be a diffeomorphism such that the cardinal of the set of sinks and sources is bounded in a C^1-neighbourhood of f. Then there exist $l \in \mathbb{N}$ and a C^1-neighbourhood $\mathcal{V}$ of f such that, for any $g \in \mathcal{V}$ and every periodic orbit P of g whose homoclinic class $H(P,g)$ is not trivial, $H(P,g)$ admits an $l-$dominated splitting.*

A long term objective is to get a spectral decomposition theorem in the nonuniformly hyperbolic case for diffeomorphisms far from Newhouse phenomenon. Having this goal in mind, we can reformulate Theorem 3 as follows:

Under the hypotheses of Theorem 3, for every g sufficiently C^1-close to f there are compact invariant sets $\Lambda_i(g)$, $i \in \{1, \dots, \dim(M)-1\}$, such that:

- *Every $\Lambda_i(g)$ admits an $l-$dominated splitting $E_i(g) \prec_l F_i(g)$ with $\dim(E_i(g)) = i$,*
- *every non-trivial homoclinic class $H(Q,g)$ is contained in some $\Lambda_i(g)$.*

Unfortunately, this result has two disadvantages. First, the $\Lambda_i(g)$ are supposed to be neither transitive nor disjoint. Moreover, the nonwandering set $\Omega(g)$ is not *a priori* contained in the union of the $\Lambda_i(g)$ (but every homoclinic class of a periodic orbit in $(\Omega(g) \setminus \bigcup_i \Lambda_i(g))$ is trivial). So we are still far away from a completely satisfactory spectral decomposition theorem[3]. In view of these comments the following problem arises in a natural way.

Problem *Let $\mathcal{U} \subset \mathrm{Diff}^1(M)$ be an open set of diffeomorphisms for which the number of sinks and sources is uniformly bounded. Is there some subset $\mathcal{U}_0 \subset \mathcal{U}$, either dense and open or residual in $\mathcal{U}$, of diffeomorphisms g such that $\Omega(g)$ is the union of finitely many disjoint compact sets $\Lambda_i(g)$ having a dominated splitting?*

Let us observe that Newhouse phenomenon is not incompatible with the existence of a dominated splitting if we do not have any additional information on the action of f_* on the subbundles of the splitting. Actually, using Mañé's ergodic closing lemma (see [Ma$_3$])

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[3] Fortunately, the results in the footnote 2 gave a spectral decomposition for generic diffeomorphisms with finitely many homoclinic classes.

we will get some control on the action of the derivative f_* on the volume induced on the subbundles. For that we need to introduce dominated splittings having more than two bundles. An invariant splitting $TM|_K = E_1 \oplus \cdots \oplus E_k$ is *dominated* if $\bigoplus_1^j E_i \prec \bigoplus_{j+1}^k E_i$ for every j. In this case we use the notation $E_1 \prec E_2 \prec \cdots \prec E_k$.

By Proposition 4.11, there is a unique dominated splitting, called *finest dominated splitting,* such that any dominated splitting is obtained by regrouping its subbundles by packages corresponding to intervals.

Theorem 4 *Let $\Lambda_f(U)$ be a C^1-robustly transitive set and $E_1 \oplus \cdots \oplus E_k$, $E_1 \prec \cdots \prec E_k$, be its finest dominated splitting. Then there exists $n \in \mathbb{N}$ such that $(f_*)^n$ contracts uniformly the volume in E_1 and expands uniformly the volume in E_k.*

This result synthesizes previous results in lower dimensions of [Ma$_3$] and [DPU] on robustly transitive diffeomorphisms (or sets) and it shows that, in the list of robustly transitive diffeomorphisms above, each example corresponds to the worst pathological case in the corresponding dimension. Observing that if E_1 or E_k has dimension one, then it is uniformly hyperbolic (contracting or expanding) then, for robustly transitive diffeomorphisms, we have:

- In dimension 2 the dominated splitting has necessarily two one dimensional bundles, so the diffeomorphism is hyperbolic and then Anosov (Mañé's result above).
- In dimension 3 at least one of the bundles has dimension 1 and so it is hyperbolic and the diffeomorphism is partially hyperbolic (see [DPU]). In this dimension, the finest decomposition can contain *a priori* two or three bundles and in the list above there are examples of both of these possibilities.
- In higher dimensions the extremal subbundles may have dimension strictly bigger than one and so they are not necessarily hyperbolic: This is exactly what happens in the examples in [BoVi].

Theorem 4 motivates us to introduce the notions of *volume hyperbolicity* and *volume partial hyperbolicity,* as the existence of dominated splittings, say $E \prec G$ and $E \prec F \prec G$, respectively, such that the volume is uniformly contracted on the bundle E and expanded on G. We think that this notion could be the best substitute for the hyperbolicity in a nonuniformly hyperbolic context.

The volume partial hyperbolicity is clearly incompatible with the existence of sources or sinks. However, in the proof of Theorem 4 , at least for the moment, we need the robust transitivity to obtain the partial volume hyperbolicity. Bearing in mind this comment and our previous results, let us pose some questions:

Problems

1. *In Theorem 1, is it possible to replace the notion of dominated splitting by the notion of volume partial hyperbolicity?* [4]

---

[4] In this direction, using the techniques in this paper, [Ab] shows the volume hyperbolicity of the homoclinic classes of generic diffeomorphisms having finitely many homoclinic classes.

2. *Is the notion of volume hyperbolicity (or volume partial hyperbolicity) sufficient to assure the generic existence of finitely many Sinai-Ruelle-Bowen (SRB) measures whose basins cover a total Lebesgue measure set? More precisely:*

   Let f be a C^1-robustly transitive diffeomorphism of class C^2 on a compact manifold M. Does it exist g close to f having finitely many SRB measures such that the union of their basins has total Lebesgue measure in M?

For ergodic properties of partially hyperbolic systems (mainly existence of SRB measures) we refer the reader to [BP], [BoVi], and [ABV].

Let us observe that in the measure preserving setting (also volume preserving) the notion of stably ergodicity (at least in the case of C^2-diffeomorphisms) seems to play the same role that the robust transitivity in the topological setting. See the results in [GPS] and [PgSh] which, in rough terms, show that weak forms of hyperbolicity may be necessary for stably ergodicity and go a long way in guaranteeing it. Actually, in [PgSh] it is conjectured that stably ergodic diffeomorphisms are open and dense among the partially hyperbolic C^2-volume preserving diffeomorphisms. See [BPSW] for a survey on stably ergodicity and [DW] for recent progress on the previous conjecture. Our next objective is precisely the study of C^1-volume preserving diffeomorphisms.

Although this paper is not devoted to conservative diffeomorphisms some of our results have a quite straightforward generalization into the conservative context. This means that the manifold is endowed with a smooth volume form ω; then we can speak of *conservative* (i.e. volume preserving) diffeomorphisms. We denote by $\mathrm{Diff}^1_\omega(M)$ the set of C^1-conservative diffeomorphisms.

A first challenge is to get a suitable version of the generic spectral decomposition theorem by Mañé (dichotomy between hyperbolicity and Newhouse phenomenon) for conservative diffeomorphisms. Obviously, since conservative diffeomorphisms have neither sinks nor sources, one needs to replace sinks and sources by *elliptic points* (i.e. periodic points whose derivatives have some eigenvalue of modulus one). Very little is known in this context. First, there is a unpublished result by Mañé (see [Ma$_1$]) which says that C^1-generically, area preserving diffeomorphisms of compact surfaces are either Anosov or have Lyapunov exponents equal to zero for almost every orbit (see also [Ma$_4$] for an outline of a possible proof)[5]. Mañé also announced a version of his theorem in higher dimensions for symplectic diffeomorphisms, see [Ma$_1$] [6]. Unfortunately, as far as we know, there are no available complete proofs of these results. See also the results by Newhouse in [N$_2$] where he states a dichotomy between hyperbolicity (Anosov diffeomorphisms) and existence of elliptic periodic points.

Related to the announced results of Mañé, there is the following question in [He]:

Conjecture (Herman) *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative diffeomorphism of a compact manifold M. Assume that there is a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_\omega(M)$ such that for any $g \in \mathcal{U}$ and every periodic orbit x of g the matrix $g^n_*(x)$ (where n is the period of x) has at least one eigenvalue of modulus different from one. Then f admits a dominated splitting.*

---

[5] Recently, [Bc] gives a complete proof of this result. For a generalization to higher dimensions see [BcVi$_1$].

[6] See [Ar$_2$] for progress on this subject in dimension four.

The following results give partial answers to this question:

Theorem 5 *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative diffeomorphisms of an $N-$dimensional manifold M. Then there is $l \in \mathbb{N}$ such that,*

- *either there is a conservative $\varepsilon - C^1-$perturbation g of f having a periodic point x of period $n \in \mathbb{N}$ such that $g^n_*(x) = Id$,*
- *or for any conservative diffeomorphism g $\varepsilon - C^1-$close to f and every periodic saddle x of g the homoclinic class $H(x, g)$ admits an $l-$dominated splitting.*

Theorem 6 *Let f be a conservative diffeomorphism defined on a compact $N-$dimensional manifold. Then there are two possibilities:*

- *Either given any $k \in \mathbb{N}$ there is a conservative diffeomorphism g arbitrarily C^1-close to f having k periodic orbits whose derivatives are the identity.*
- *Or the manifold M is the union of finitely many (less than $N-1$) invariant compact (*a priori *nondisjoint) sets having a dominated splitting.*

Observe that if in Theorem 6 above the diffeomorphism f is transitive and the second possibility of the dichotomy occurs, then one of the invariant compact sets has to be the whole manifold (one of them contains a dense orbit). This means that, in the transitive case, Theorem 6 gives a complete positive answer to Herman's conjecture:

Corollary 0.4 *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative transitive diffeomorphism of a manifold M. Assume that there is a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_\omega(M)$ such that for any $g \in \mathcal{U}$ and every periodic orbit x of g the matrix $g^n_*(x)$ (where n is the period of x) has at least one eigenvalue of modulus different from one. Then f admits a dominated splitting.*

Let us observe that if f is transitive and there is some periodic point x of f such that $f^n_*(x) = Id$, n is the period of x, then given any $\varepsilon > 0$ there is a C^1-perturbation $g \in \mathrm{Diff}^1_\omega(M)$ of f such that its totally elliptic points (derivative equal to the identity) are $\varepsilon-$dense in M.

A conservative diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$ is *robustly transitive in* $\mathrm{Diff}^1_\omega(M)$ if there is $\varepsilon > 0$ such that every $\varepsilon-$ perturbation $g \in \mathrm{Diff}^1_\omega(M)$ of f is transitive. Observe that *a priori* the robust transitivity in $\mathrm{Diff}^1_\omega(M)$ does not imply the robust transitivity in $\mathrm{Diff}^1(M)$.

Conjecture *Let f be a robustly transitive diffeomorphism in* $\mathrm{Diff}^1_\omega(M)$. *Then f admits a nontrivial dominated splitting defined on the whole M.*

In view of Corollary 0.4, to prove this conjecture one needs to to show that a robustly transitive diffeomorphism f can not have periodic points x whose derivative $f^n_*(x)$ is the identity.

Finally, we observe that the control of the volume in the subbundles is almost straightforward for conservative systems:

Proposition 0.5 *Let f be a conservative diffeomorphism and $E \oplus F$, $E \prec F$, be a dominated splitting of TM. Then f_* contracts uniformly the volume in E and expands uniformly the volume in F.*

Main ideas of the proofs

Mañé's paper [Ma$_3$] combines two main ingredients: Systems of matrices and the Ergodic Closing Lemma. He first considers the linear maps induced by the derivative of a diffeomorphism f over the orbits of its periodic points, thus obtaining a *system of matrices.* He shows that (in his context) a system of matrices admits a dominated splitting if it is not possible to perturb it to get a matrix with some eigenvalue of modulus one. By a lemma of Franks', see [F] and Section 1, each perturbation of the system of matrices over a finite number of periodic orbits corresponds to a C^1-perturbation of f and vice versa. Hence the existence of a dominated splitting also holds for C^1-diffeomorphisms. Finally, to get the uniform expansion and contraction on the subbundles of the splitting he uses his Ergodic Closing Lemma (see [Ma$_3$]).

Our proof uses these two tools introduced by Mañé. Using Franks' Lemma we translate the problem of the existence of a dominated splitting for diffeomorphisms into the same problem for abstract linear systems. However, the systems of matrices in [Ma$_3$] do not contain some relevant dynamical informations about f that we need. Actually, the solution of this difficulty is probably the subtlest point of our arguments, so let us be somewhat more precise:

On one hand, in the context in [Ma$_3$] all the periodic points have the same *index* (dimension of the stable bundle), thus the system of matrices has a natural splitting (the one corresponding to the stable and unstable bundles of f_*). Then if this splitting is not dominated one gets a perturbation of it having one eigenvalue of modulus 1. On the other hand, in our case there are points having different indices. Moreover, points having eigenvalues of modulus 1 are not forbidden. So we need some extra arguments to conclude our proof. In fact, we need to control *all* the eigenvalues to create sources or sinks.

The additional argument, that comes from the dynamics, is a property of our linear systems called *transitions.* Given two periodic points P and Q in the same *homoclinic class* (i.e. their invariant manifolds intersect transversally) there are periodic orbits passing first arbitrarily close to P, and thereafter arbitrarily close to Q, and so on. These orbits can be chosen upon arbitrary sequence of times (the orbit spends k_1 iterates close to P then, after a bounded number of iterates, it becomes close to Q and remains k_2 iterates close to Q, and so on). So we define a structure we call transitions which translates this dynamical behavior into the world of the abstract linear systems. This property allows us consider the product of matrices of the system corresponding to different orbits as a matrix of the system. In fact, the transitions endow the linear system with a "semigroup-like" structure. Clearly, this is not the case for general linear systems.

Finally, after introducing the linear systems with transitions, the proof of the existence of a dominated splitting involves only arguments of linear algebra. Precisely, this algebraic approach has allowed us to improve previous results by stating them in higher dimensions and by eliminating the robust transitivity hypothesis.

The problem of the existence of points with different indices already appeared in [DPU], where it has been solved by considering only robustly transitive sets, thus any perturbation of the dynamics remains transitive. This additional hypothesis in [DPU] allows to jump from the dynamical world to the abstract linear world, here do some perturbation, and then jump back to the dynamical world do a new perturbation, and so on. In our context we have no control of the variation of a homoclinic class after dynamical perturbations. So it is crucial for Theorem 1 that all the perturbations we do "live in the world of abstract linear systems" and do not modify the underlying dynamics (that is here possible because Mañé's linear systems have been enriched with the transitions).

In our proof, assuming that there is no dominated splitting, we perform a series of perturbations of the linear system, as a final result of such perturbations we get a linear system having a homothety. It is only then that we realize this linear system as a diffeomorphism using Franks' Lemma, and the point corresponding to the homothety becomes a sink or a source of the diffeomorphism.

Finally, for the control of the volume in the extremal subbundles (Theorem 4) we use the Ergodic Closing Lemma, which gives a dynamical perturbation having a periodic point reflecting the lack of volume expansion or contraction of the bundles. Unfortunately, without any additional hypothesis, this point has *a priori* nothing to do with the initial homoclinic class. This explains why Theorem 4 only holds for robustly transitive systems.

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Contents

1 Linear systems with transitions

Let f be a diffeomorphism. By Franks' Lemma below (see for instance [F]), to any perturbation A of the derivative f_* along the orbits of finitely many periodic points corresponds a diffeomorphism g, C^1-close to f, such that $g_* = A$ along these orbits. This lemma allows us to consider perturbations of the derivative f_* keeping unchanged the dynamics of f, in order to get a suitable derivative along some periodic orbits. The aim of this section is to define in details the framework (*periodic linear systems*) which gives a precise meaning of this kind of perturbations, and to translate into this language the dynamical properties that we will need (specially the notion of *transitions,* see Definition 1.6). Finally, we prove that the homoclinic classes define periodic linear system with transitions (Lemma 1.9) and we state an easy (but typical) consequence of the existence of transitions (Lemma 1.10).

Before beginning this section let us state precisely the Franks' Lemma:

Lemma (Franks) *Suppose that E is an $f-$invariant finite set and B is an $\varepsilon-$perturbation of f_* along E, then there is a diffeomorphism g $\varepsilon-\mathcal{C}^1-$close to f, coinciding with f out of an arbitrarily small neighbourhood of E, equal to f in E, and such that g_* coincides with B in E.*

Let us point out that Franks' Lemma is the key which allows to translate results on linear systems to the dynamical context and it will be ofen used in this paper.

1.1 Linear systems: Topology and linear changes of coordinates

Let Σ be a topological space and f a homeomorphism defined on Σ. Consider a locally trivial vector bundle (of finite dimension) $\mathcal{E}$ over Σ. Denote by $\mathcal{E}_x$ the fiber of $\mathcal{E}$ at $x \in \Sigma$. We assume that the dimension of the fibers $\mathcal{E}_x$, $\dim(\mathcal{E}_x)$, does not depend on $x \in \Sigma$. In what follows, we call this number *dimension of the bundle* $\mathcal{E}$, denoted by $\dim(\mathcal{E})$.

An *euclidian metric* $|\cdot|$ *on the bundle* $\mathcal{E}$ is a collection of euclidian metrics on the fibers $\mathcal{E}_x$, $x \in \Sigma$, *a priori* not depending continuously on x.

We denote by $\mathcal{GL}(\Sigma, f, \mathcal{E})$ the set of maps $A\colon \mathcal{E} \to \mathcal{E}$ such that for every $x \in \Sigma$ the induced map $A(x,\cdot)$ is a linear isomorphism from $\mathcal{E}_x \to \mathcal{E}_{f(x)}$, thus $A(x,\cdot)$ belongs to $\mathcal{L}(\mathcal{E}_x, \mathcal{E}_{f(x)})$ and is invertible. For each $x \in \Sigma$ the euclidian metrics on $\mathcal{E}_x$ and $\mathcal{E}_{f(x)}$ induce a norm (always denoted by $|\cdot|$) on $\mathcal{L}(\mathcal{E}_x, \mathcal{E}_{f(x)})$:

$$|B(x,\cdot)| = \sup\{|B(x,v)|, v \in \mathcal{E}_x, |v| = 1\}.$$

Let now $A \in \mathcal{GL}(\Sigma, f, \mathcal{E})$ and define $|A| = \sup_{x\in\Sigma} |A(x,\cdot)|$. Observe that, for any $A \in \mathcal{GL}(\Sigma, f, \mathcal{E})$ its inverse A^{-1} belongs to $\mathcal{GL}(\Sigma, f^{-1}, \mathcal{E})$. So we can define $|A^{-1}|$ in the same way. Finally, the *norm* of a $A \in \mathcal{GL}(\Sigma, f, \mathcal{E})$ is $\|A\| = \sup\{|A|, |A^{-1}|\}$.

Definition 1.1 *A* linear system[7] *is a* $4-$*uple* $(\Sigma, f, \mathcal{E}, A)$ *where* Σ *is a topological space,* f *is a homeomorphism of* Σ*,* $\mathcal{E}$ *is an Euclidean bundle over* Σ*,* A *belongs to* $\mathcal{GL}(\Sigma, f, \mathcal{E})$*, and* $\|A\| < \infty$*.*

In what follows, for the sake of simplicity, we sometimes denote by A a linear system $(\Sigma, f, \mathcal{E}, A)$ if there is no ambiguity on Σ, f, and $\mathcal{E}$.

Example 1 *Let* f *be a diffeomorphism defined on a riemannian manifold* M *and* $\Sigma \subset M$ *an* $f-$*invariant subset. Consider the restriction to* Σ *of the tangent bundle,* $\mathcal{E} = TM|_\Sigma$*. The riemannian metric on* M *induces an Euclidean structure on* $\mathcal{E}$*. Then* $(\Sigma, f|_\Sigma, \mathcal{E}, f_*|_\mathcal{E})$ *is the* natural linear system induced by f over Σ.

We denote by $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ the space of linear systems over $(\Sigma, f, \mathcal{E})$ such that $||A|| < \infty$ endowed with the distance defined by

$$d(A,B) = \sup\{|A - B|, |A^{-1} - B^{-1}|\}, \quad A,\, B \in \mathcal{GL}^\infty(\Sigma, f, \mathcal{E}).$$

We can now define an $\varepsilon-$*perturbation* of A as a linear system $\tilde{A}$, defined over $(\Sigma, f, \mathcal{E})$, such that $d(A, \tilde{A}) < \varepsilon$.

Very elementary arguments of linear algebra show that any perturbation of a linear system can be obtained by composing it with linear maps close to identity. More precisely, let $A \in \mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ and consider some linear system $E \in \mathcal{GL}^\infty(\Sigma, Id_\Sigma, \mathcal{E})$. Then $E \circ A$ and $A \circ E$ (defined in the obvious way) belong to $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$. Moreover, if E is close to the identity linear system $(\Sigma, Id_\Sigma, \mathcal{E}, Id_\mathcal{E})$, then $E \circ A$ and $A \circ E$ are also close to A.

Consider now some change of the Euclidean metrics on the fibers. Assume that the matrices of the changes of coordinates (from an orthonormal basis of the initial metric to an orthonormal basis of the new metric) and their inverses are uniformly bounded on Σ.

---

[7] After writing this paper, we realized that this notion corresponds to the classical concept of *linear cocycle* over the homeomorphism f.

Then every linear system in the initial metric induces a new system (for the new metric). Moreover, this change of metrics keeps invariant the topology of the set of linear systems. Let us be a little bit more precise.

Let $\mathcal{E}$ denote an Euclidean bundle on a topological space Σ endowed with the Euclidean metric $|\cdot|$. Denote by $\mathcal{E}_1$ the same bundle, but now endowed with a different Euclidean metric $|\cdot|_1$. Denote by $P\colon \mathcal{E} \to \mathcal{E}_1$ the identity map considered as a morphism of bundles. Using the metrics $|\cdot|$ and $|\cdot|_1$ we can define the norms $|P|$ and $|P^{-1}|$. Write $\|P\| = \sup\{|P|, |P^{-1}|\}$. If $\|P\| < \infty$, the canonical bijection $Id\colon \mathcal{GL}(\Sigma, f, \mathcal{E}) \to \mathcal{GL}(\Sigma, f, \mathcal{E}_1)$ induces an homeomorphism from $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ (with the distance d) to $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E}_1)$ (with the corresponding distance d_1). These two simple facts are put together in the following lemma.

Lemma 1.2

1. *Given $K > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that for any linear system $(\Sigma, f, \mathcal{E}, A)$, $A \in \mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ and $\|A\| < K$, and every $\delta-$perturbation of the identity $(\Sigma, id_\Sigma, \mathcal{E}, E)$, one has that $E \circ A$ and $A \circ E$ are $\varepsilon-$perturbations of A.*

2. *For every $K > 0$, $K_0 > 0$, and $\varepsilon > 0$ there are $K_1 > 0$, $\delta > 0$ satisfying the following property:*

   Consider a pair of Euclidean bundles $\mathcal{E}$ and $\mathcal{E}_1$ over Σ endowed with the metrics $|\cdot|$ and $|\cdot|_1$, and the isomorphism of bundles $P\colon \mathcal{E} \to \mathcal{E}_1$ induced by the identity on Σ (i.e. given $x \in \Sigma$ the map $P(x, \cdot)$ is a linear isomorphism from $\mathcal{E}_x$ to $(\mathcal{E}_1)_x$). Assume that $\|P\| < K_0$. Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system such that $\|A\|$ is bounded by K.

   Let $B = P \circ A \circ P^{-1}$, then $(\Sigma, f, \mathcal{E}_1, B)$ is a linear system bounded by K_1. Moreover, any $\delta-$perturbation of B is conjugate by P to some $\varepsilon-$perturbation of A.

Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and $n \in \mathbb{N}$. The *$n-th$ iterate of A,* denoted by $A^{(n)}$, is the linear system over $(\Sigma, f^n, \mathcal{E})$ defined by $A^{(n)}(x) = A(f^{n-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x)$.

Consider an $f-$invariant subset Σ' of Σ and the restriction of the linear bundle $\mathcal{E}$ to Σ', then A induces canonically a linear system over $(\Sigma', f|_{\Sigma'}, \mathcal{E}|_{\Sigma'})$ called *linear subsystem induced by A over Σ'.*

1.2 Special linear systems

Along this work, the linear systems we consider will be often endowed with some additional structures: In some cases they are *continuous,* and most of them are *periodic.* We also consider *systems of matrices.* Finally, the most important additional structure we will introduce is the notion of *transitions.* Let us now present the three first quite natural structures. Due to its specific and subtle nature we postpone to the next paragraph the notion of transition. This key definition will deserve special attention and care.

In the sequel, Σ is a topological space, f a homeomorphism of Σ, and $\mathcal{E}$ a locally trivial vector bundle over Σ endowed with an Euclidean metric $|\cdot|$ on the fibers. A linear system $(\Sigma, f, \mathcal{E}, A)$ is *continuous* if the Euclidean structure on the fibers varies continuously and the function $A\colon \mathcal{E} \to \mathcal{E}$ is continuous.

The linear system $(\Sigma, f, \mathcal{E}, A)$ is *periodic* if all the orbits of f are periodic. In this case we let $M_A(x)\colon \mathcal{E}_x \to \mathcal{E}_x$ the product of the $A(f^i(x))$ along the orbit of x. More precisely, let $p(x)$ be the period of $x \in \Sigma$, then

$$M_A(x) = A(f^{p(x)-1}(x)) \circ \cdots \circ A(x) = A^{(p(x))}(x).$$

Finally, $(\Sigma, f, \mathcal{E}, A)$ is a *system of matrices* if the Euclidean bundle $\mathcal{E}$ is the trivial bundle $\Sigma \times \mathbb{R}^N$, where $\mathbb{R}^N$ is endowed with the canonical Euclidean metric. In this case every linear map $A(x)$ is canonically identified with an element of $GL(N, \mathbb{R})$.

Let $(\Sigma, f, \mathcal{E}, A)$ be an (*a priori* non-continuous) linear system. It will be sometimes useful to fix an orthonormal basis on each fiber $\mathcal{E}_x$ (this basis does not depend, in general, continuously on the point $x \in \Sigma$). These basis give an (a priori non-continuous) trivialization of the Euclidean bundle $\mathcal{E}$. So in these new coordinates A can be considered as a system of matrices. Two systems of matrices *define the same linear system* if at each point there exists an orthonormal change of coordinates conjugating the two systems.

1.3 Dominated splittings

The definition of dominated splitting for an invariant set of a diffeomorphism (see Definition 0.1) can be directly generalized for linear systems as follows. Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system, an *invariant subbundle* is a collection of linear subspaces $F(x) \subset \mathcal{E}_x$ whose dimensions do not depend on x and such that $A(F(x)) = F(f(x))$. An $A-$*invariant splitting* $F \oplus G$ is given by two invariant subbundles such that $\mathcal{E}_x = F(x) \oplus G(x)$ at each $x \in \Sigma$.

Definition 1.3 *Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and $\mathcal{E} = F \oplus G$ an $A-$invariant splitting. We say that $F \oplus G$ is a* dominated splitting *if there exists $n \in \mathbb{N}$ such that*

$$\|A^{(n)}(x)|_F\| \, \|A^{(-n)}(f^n(x))|_G\| < 1/2$$

for every $x \in \Sigma$. We write $F \prec G$.

*If we want to emphasize the role of n then we say that $F \oplus G$ is an $n-$*dominated splitting *and write $F \prec_n G$.*

Finally, the dimension of the dominated splitting *is the dimension of the subbundle F.*

Suppose now that $(\Sigma, f, \mathcal{E}, A)$ is a **continuous** linear system, then any dominated splitting can be obtained by considering subsystems induced by A over dense subsets $\Sigma' \subset \Sigma$. More precisely,

Lemma 1.4 *Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous linear system such that there is a dense $f-$invariant subset $\Sigma_1 \subset \Sigma$ whose corresponding linear subsystem admits an $l-$dominated splitting. Then $(\Sigma, f, \mathcal{E}, A)$ admits an $l-$dominated splitting.*

More generally, suppose that there is a sequence of (non necessarily continuous) systems $(\Sigma, f, \mathcal{E}, A_k)$ converging to $(\Sigma, f, \mathcal{E}, A)$ such that for every k there is a dense invariant subset $\Sigma_k \subset \Sigma$ where A_k admits an $l-$dominated splitting. Then A admits an $l-$dominated splitting in the whole Σ.

Finally, any dominated splitting of a continuous linear system is continuous.

Proof: Given $x \in \Sigma$ consider a sequence (x_k), $x_k \in \Sigma_k$, converging to x. For a fixed k we have an $l-$dominated splitting $E_k \oplus F_k$. Taking a subsequence we can assume that the dimensions of these spaces are independent of k and that the sequences $E_k(x_k)$ and $F_k(x_k)$ converge to some subspaces $E(x)$ and $F(x)$.

By definition of $l-$dominance, given any k, $u_k \in E_k(x_k)$, and $v_k \in F_k(x_k)$, we have

$$2\,\frac{\|A_k^l(u_k)\|}{\|u_k\|} \leq \frac{\|A_k^l(v_k)\|}{\|v_k\|}.$$

By the continuity of A and the convergences of $A_k \to A$, $x_k \to x$, $E_k(x_k) \to E(x)$, and $F_k(x_k) \to F(x)$, we get

$$2\,\frac{\|A^l(u)\|}{\|u\|} \leq \frac{\|A^l(v)\|}{\|v\|}$$

for every $u \in E(x)$ and $v \in F(x)$. So these two spaces are transverse.

Finally, it remains to check that these two spaces are uniquely defined and give an invariant splitting. Observe first that $A(E(x))$ and $A(F(x))$ are the limits of the (same) subsequences before of $E_k(f(x_k))$ and $F_k(f(x_k))$. Then for any $m \in \mathbb{Z}$ we get

$$2^m\,\frac{\|A^{ml}(u)\|}{\|u\|} \leq \frac{\|A^{ml}(v)\|}{\|v\|}$$

for every $u \in E(x)$ and $v \in F(x)$. Now a standard dynamical argument asserts that the spaces $E(x)$ and $F(x)$ verifying this inequality are uniquely determined by their dimensions.

To complete the proof, observe that the unicity of the dominated splitting above gives the continuity. $\square$

Corollary 1.5 *Let f be a diffeomorphism defined on a compact manifold M and Λ an $f-$invariant set. Assume that there are $l \in \mathbb{N}$, $i \in 1, \ldots, \dim(M) - 1$, and a sequence of diffeomorphisms f_n converging to f in the C^1-topology such that*

- *every f_n has a periodic orbit x_n such that $H(x_n, f_n)$ admits an $l-$dominated splitting of dimension i,*
- *the set Λ is included in the topological upper limit set of the $H(x_n, f_n)$, i.e.*

$$\limsup_{n\to\infty}(H(x_n, f_n)) = \bigcap_{n=1}^{\infty} \operatorname{closure}(\bigcup_{i>n} H(x_n, f_n)).$$

Then Λ admits an $l-$dominated splitting of dimension i.

Proof: Consider the topological set

$$I = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\}.$$

In $M \times I$ we consider the union

$$(\Lambda \times \{0\}) \cup \bigcup_{1}^{+\infty} (H(x_n, f_n) \times \{\frac{1}{n}\}).$$

The differentials of f and f_n define in a natural way a linear system on this set, which is continuous because the f_n converge to f in the C^1-topology. Moreover, $\bigcup_1^{+\infty}(H(x_n, f_n) \times \{\frac{1}{n}\}$ is a dense subset (because Λ is contained in the topological upper limit set of the $H(x_n, f_n)$) and the system over $\bigcup_1^{+\infty}(H(x_n, f_n) \times \{\frac{1}{n}\}$ admits an $l-$dominated splitting. To finish the proof it is now enough to apply Lemma 1.4. □

1.4 Periodic linear systems with transitions

To saddles P and Q of the same index which are linked by transverse intersections of their invariant manifolds (i.e. they are *homoclinically related*) belong to the same transitive hyperbolic set. So they are accumulated by other periodic orbits whose orbits spend an arbitrarily long time close to P, thereafter close to Q, and so on. In fact, the existence of Markov partitions shows that fixed any finite sequence of times there is a periodic orbit expending alternately the times of the sequence close to P and Q, respectively. Moreover, the transition time (between a neighbourhood of P and a neighbourhood of Q) can be chosen bounded. This property will allow us to scatter in the whole homoclinic class of P some properties of the periodic points Q of this class.

The aim of this section is to translate this property into the language of linear systems, introducing the concept of *linear system with transitions.* Then we shall deduce some direct consequences of the existence of such transitions. Let us go into the details of our constructions. We begin by giving some definitions.

Given a set $\mathcal{A}$, a *word with letters* in $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$, its *length* is the number of letters composing it. The set of words admits a natural a semi-group structure: The product of the word $[a] = (a_1, \ldots, a_n)$ by $[b] = (b_1, \ldots, b_k)$ is $[a][b] = (a_1, \ldots, a_n, b_1, \ldots, b_k)$. We say that a word $[a]$ *is not a power* if $[a] \neq [b]^k$ for every word $[b]$ and $k > 1$.

In this section $(\Sigma, f, \mathcal{E}, A)$ is a periodic linear system of dimension N: Recall that every $x \in \Sigma$ is periodic for f, $p(x)$ denotes its period, and $M_A(x)$ denotes the product $A^{(p(x))}(x)$ of A along the orbit of x.

If (Σ, f, A) is a periodic system of matrices (in $GL(N, \mathbb{R})$), then for any $x \in \Sigma$ we write $[M]_A(x) = (A(f^{p(x)-1}(x)), \ldots, A(x))$; which is a word with letters in $GL(N, \mathbb{R})$. Hence the matrix $M_A(x)$ is the product of the letters of the word $[M]_A(x)$.

Definition 1.6 *Given $\varepsilon > 0$, a periodic linear system $(\Sigma, f, \mathcal{E}, A)$ admits $\varepsilon-$transitions if for every finite family of points $x_1, \ldots, x_n = x_1 \in \Sigma$ there is an orthonormal system of coordinates of the linear bundle $\mathcal{E}$ (so that* $(\Sigma, f, \mathcal{E}, A)$ *can be now considered as a system of matrices* (Σ, f, A)*), and for any $(i, j) \in \{1, \ldots, n\}^2$ there exist $k(i, j) \in \mathbb{N}$ and a finite word $[t^{i,j}] = (t_1^{i,j}, \ldots, t_{k(i,j)}^{i,j})$ of matrices in $GL(N, \mathbb{R})$, satisfying the following properties:*

1. *For every $m \in \mathbb{N}$, $\iota = (i_1, \ldots, i_m) \in \{1, \ldots, n\}^m$, and $a = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$ consider the word*

   $$[W(\iota, a)] = [t^{i_1,i_m}][M_A(x_{i_m})]^{\alpha_m}[t^{i_m,i_{m-1}}][M_A(x_{i_{m-1}})]^{\alpha_{m-1}} \cdots [t^{i_2,i_1}][M_A(x_{i_1})]^{\alpha_1},$$

   where the word $w(\iota, a) = ((x_{i_1}, \alpha_1), \ldots, (x_{i_m}, \alpha_m))$ with letters in $M \times \mathbb{N}$ **is not a power**. *Then there is $x(\iota, a) \in \Sigma$ such that*

- *The length of* $[W(\iota,a)]$ *is the period* $p(x(\iota,a))$ *of* $x(\iota,a)$.
- *The word* $[M]_A(x(\iota,a))$ *is* $\varepsilon-$*close to* $[W(\iota,a)]$ *and there is an* $\varepsilon-$*perturbation* $\tilde{A}$ *of* A *such that the word* $[M]_{\tilde{A}}(x(\iota,a))$ *is* $[W(\iota,a)]$.

2. *One can choose* $x(\iota,a)$ *such that the distance between the orbit of* $x(\iota,a)$ *and any point* x_{i_k} *is bounded by some function of* α_k *which tends to zero as* α_k *goes to infinity.*

Given ι and a as above, the word $[t^{i,j}]$ is an $\varepsilon-$*transition from* x_j *to* x_i. We call $\varepsilon-$*transition matrices* the matrices $T_{i,j}$ which are the product of the letters composing $[t^{i,j}]$.

Remark 1.7 *Consider points* $x_1,\dots,x_{n-1},x_n = x_1 \in \Sigma$ *and* $\varepsilon-$*transitions* $[t^{i,j}]$ *from* x_j *to* x_i. *Then we have that*

1. *for every positive* $\alpha \geq 0$ *and* $\beta \geq 0$ *the word* $([M]_A(x_i))^{\alpha}\,[t^{i,j}]\,([M]_A(x_j))^{\beta}$ *is also an* $\varepsilon-$*transition from* x_j *to* x_i*,*
2. *for any* i*,* j*, and* k *the word* $[t^{i,j}][t^{j,k}]$ *is an* $\varepsilon-$*transition from* x_k *to* x_i*,*
3. *As a consequence of the two items above, the words* $W(\iota,\alpha)$ *in Definition 1.6 are* $\varepsilon-$*transitions from* x_{i_1} *to itself. Moreover, the set of such* $\varepsilon-$*transitions forms a semigroup.*

Definition 1.8 *We say that a periodic linear system* admits transitions *if for any* $\varepsilon > 0$ *it admits* $\varepsilon-$*transitions.*

The following lemma justifies the introduction of the notion of transition for studying homoclinic classes:

Lemma 1.9 *Let* P *be a hyperbolic saddle of index* k *(dimension of its stable manifold). The derivative* f_* *induces a continuous periodic linear system with transitions on the set* Σ *of hyperbolic saddles in* $H(P,f)$ *of index* k *and homoclinically related to* P.

Proof: Fix any $\varepsilon > 0$ and a finite family $x_1,\dots,x_n$ in Σ. As the x_i are homoclinically related to P, there is a compact transitive hyperbolic subset K of $H(P,f)$ containing all the x_i. So this set K can be covered by a Markov partition with arbitrarily small rectangles. We can now choose orthonormal systems of coordinates in $T_x(M)$, $x \in K$, such that the orthonormal basis depend continuously on x when the points are in the same rectangle.

Let (K,f,A) be the system of matrices defined on K by writing f_* in this system of coordinates. Now, using the continuity of f_*, and by subdividing if necessary the rectangles of the Markov partition, we can assume that, for any x and y in the same rectangle, one has

$$\|A(x) - A(y)\| < \varepsilon \quad \text{and} \|A^{-1}(x) - A^{-1}(y)\| < \varepsilon.$$

The transitions from x_i to x_j are now obtained considering the derivative of f along any orbit in K going from the rectangle containing x_i to the rectangle containing x_j. $\square$

The next lemma shows how a property at one point of a system with transitions can be scatter to a dense subset:

Lemma 1.10 (Scattering Property) *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Fix $\varepsilon > \varepsilon_0 > 0$ and assume that there exist an ε_0−perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is either a dilatation (i.e. all its eigenvalues have modulus bigger than 1) or a contraction (i.e. all its eigenvalues have modulus less than 1).*

Then there are a dense f−invariant subset $\tilde{\Sigma}$ of Σ and an ε−perturbation $\hat{A}$ of A such that for any $y \in \tilde{\Sigma}$ the linear map $M_{\hat{A}}(y)$ is either a dilatation or a contraction (according to the choice before).

Proof: Write $\varepsilon_1 = \varepsilon - \varepsilon_0$, take some point z in Σ, and consider two ε_1−transitions $T_{x,z}$ (from z to x) and $T_{z,x}$ (from x to z). For a fixed $\delta > 0$, by definition of transitions, there is $n(z, \delta)$ such that for any $n > 0$ there are $y_n \in \Sigma$, with $d(y_n, z) < \delta$, and an ε_1−deformation A' of A along the orbit of y_n such that

$$M_{A'}(y_n) = T_{z,x} \circ M_A(x)^n \circ T_{x,z} \circ M(z)^{n(z,\delta)}.$$

Define $\hat{M}_n$ by

$$\hat{M}_n = T_{z,x} \circ M_{\tilde{A}}(x)^n \circ T_{x,z} \circ M(z)^{n(z,\delta)}.$$

We can now choose n big enough so that $\hat{M}_n$ is either a dilatation or a contraction (according to $M_{\tilde{A}}(x)$). Thus by an ε_1−perturbation $\hat{A}$ of A' along the orbit of y_n we can get $M_{\hat{A}}(y) = \hat{M}_n$.

Since we are not requiring the continuity of $\hat{A}$, we can build it as above, that is, orbit by orbit considering points in a dense subset. This ends the proof of the lemma. □

2 Quantitative results: Proofs of the theorems.

In this section we state, in terms of linear systems (Proposition 2.1) and in terms of diffeomorphisms (Proposition 2.6), quantitative results on the existence of dominated splittings (giving the strength of the dominance).

Proposition 2.1 gives a dichotomy between the existence of a dominated splitting for a linear system and the existence of perturbations of the system with homotheties. This proposition is divided into two main steps: Proposition 2.4, asserting that the lack of dominance allows to create complex eigenvalues, and Proposition 2.5, which says that sufficiently many complex eigenvalues allow us to get homotheties. These propositions will be proven in the next two sections.

In this section we deduce from Proposition 2.1 most of the results announced in the introduction.

2.1 Reduction of the study of the dynamics to a problem on linear systems

Proposition 2.1 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $l > 0$ such that any continuous periodic N−dimensional linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K (i.e. $\|A\| < K$) and having transitions satisfies the following,*

- *either A admits an l−dominated splitting,*

- *or there are an $\varepsilon-$perturbation $\tilde{A}$ of A and a point $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is an homothety.*

The proof of Proposition 2.1 is divided in two main steps: In the first one, we show that, if $(\Sigma, f, \mathcal{E}, A)$ is a linear system with transitions such that no dense-subsystem of it admits an $l-$dominated splitting, then we can perturb A to get *a lot of complex eigenvalues.* In the second step, we see that, if we can obtain sufficiently many complex eigenvalues, then we can perturb the system to get a homothety (which will be either a contraction or a dilatation). Let us state precisely these two steps. We begin with some definitions.

Definition 2.2 *Let $M \in GL(N, \mathbb{R})$ be a linear isomorphism of $\mathbb{R}^N$ such that M has some complex eigenvalue λ, i.e. $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We say that λ has* rank $(i, i+1)$ *if there is a $M-$invariant splitting of $\mathbb{R}^N$, $F \oplus G \oplus H$, such that:*

- *Every eigenvalue σ of $M|_F$ (resp. $M|_H$) has modulus $|\sigma| < |\lambda|$ (resp. $|\sigma| > |\lambda|$),*
- $\dim(F) = i - 1$ *and* $\dim(H) = N - i - 1$,
- *the plane G is the eigenspace of λ.*

Definition 2.3 *A periodic linear system $(\Sigma, f, \mathcal{E}, A)$* has a complex eigenvalue of rank $(i, i+1)$ *if there is $x \in \Sigma$ such that the matrix $M_A(x)$ has a complex (non-real) eigenvalue of rank $(i, i+1)$.*

Proposition 2.1 is a direct consequence of Propositions 2.4 and 2.5 below:

Proposition 2.4 *For every $\varepsilon > 0$, $N \in \mathbb{N}$, and $K > 0$ there is $l \in \mathbb{N}$ satisfying the following property:*

Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic $N-$dimensional linear system with transitions such that its norm $\|A\|$ is bounded by K. Assume that there exists $i \in \{1, \dots, N-1\}$ such that every $\varepsilon-$perturbation $\tilde{A}$ of A has no complex eigenvalues of rank $(i, i+1)$.

Then $(\Sigma, f, \mathcal{E}, A)$ admits an $l-$dominated splitting $F \oplus G$, $F \prec_l G$, with $\dim(F) = i$.

Proposition 2.5 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Given $\varepsilon > \varepsilon_0 > 0$ assume that, for any $i \in \{1, \dots, N-1\}$, there is an ε_0-perturbation of A having a complex eigenvalue of rank $(i, i+1)$. Then there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is a homothety with ratio of modulus different from* 1.

The key of the proof of Proposition 2.4 is a $2-$dimensional argument of Mañé that we present in Section 3. The proof in higher dimensions consists of an inductive argument which allows us to reduce the dimension of the linear space by considering some quotients (roughly speaking, considering projections). Using this inductive procedure we finally arrive at a two-dimensional space. The lemmas in Section 4.1 allow us to make these successive reductions of dimension. The proofs of Propositions 2.4 and 2.5 are in Section 5.

Now using Proposition 2.1 we proof most of the results announced in the introduction.

2.2 Proofs of the Theorems

Let us first explain why Proposition 2.1 implies Theorem 1. Actually, this proposition implies the following quantitative version of Theorem 1, which is our main (but a little bit technical) result:

Proposition 2.6 *For every* $K > 0$*,* $N > 0$*, and* $\varepsilon > 0$ *there is* $l(\varepsilon, K, N) \in \mathbb{N}$ *such that for any diffeomorphism* f *defined on a riemannian* $N-$*dimensional manifold* M *such that the derivatives* f_* *and* f_*^{-1} *are bounded by* K*, and any saddle* P *of* f *with a non-trivial homoclinic class* $H(P, f)$*, the following holds:*

- *Either the homoclinic class* $H(P, f)$ *admits an* $l(\varepsilon, K, N)-$*dominated splitting,*
- *or for every neighbourhood* U *of* $H(P, f)$ *and* $k \in \mathbb{N}$ *there is* g $\varepsilon-C^1-$*close to* f *having* k *sources or sinks whose orbits are contained in* U*.*

2.2.1 Proofs of Theorem 1 and Proposition 2.6

As Proposition 2.6 implies directly Theorem 1, it remains to see that Proposition 2.6 follows from Proposition 2.1.

For that consider a diffeomorphism f, such that $\|f_*\|$ and $\|f_*^{-1}\|$ are bounded by K, and a periodic saddle P of f with a nontrivial homoclinic class. Let

$$\Sigma = H(P, f), \quad \mathcal{E} = TM|_\Sigma, \quad \text{and} \quad A = (f_*)|_\Sigma.$$

Then $(\Sigma, f, \mathcal{E}, A)$ is a continuous linear system. Denote by $\Sigma' \subset \Sigma$ the set of saddles homoclinically related to P (in particular, having the same index as P). Observe that Σ' is a dense $f-$invariant subset of Σ. Moreover, by Lemma 1.9, the subsystem induced by A over Σ' admits transitions.

If A admits an $l-$dominated splitting over Σ' then, by Lemma 1.4, such a splitting can be extended to an $l-$dominated splitting on the whole $\Sigma = H(P, f)$, and we are done.

Now take the constant $l > 0$ given by Proposition 2.1 corresponding to K, $N = \dim(M)$, and $\varepsilon/2$. If A does not admit an $l-$dominated splitting over Σ', then Proposition 2.1 says that there is an $\varepsilon/2-$perturbation $\tilde{A}$ of A and a point $x \in \Sigma'$ such that $M_{\tilde{A}}(x)$ is a homothety. We can suppose that (up to an arbitrarily small perturbation) this homothety is either a dilatation or a contraction. Assume, for instance, the first possibility.

As the system admits transitions, by Lemma 1.10, there is a dense subset of Σ' of points y admitting $\varepsilon-$deformations $\hat{A}$ along their orbits such that the corresponding linear map $M_{\hat{A}}(y)$ is a dilatation. Choose now an arbitrarily large (but finite) number of such points y, and denote by E this set of periodic orbits.

The proofs of Proposition 2.6 (thus of Theorem 1) follows now immediately from the Franks' Lemma. □

We now prove the corollaries of Theorem 1 in the Introduction.

2.2.2 Proof of Corollary 0.2

Let P be a hyperbolic saddle of a diffeomorphism f and $\mathcal{U}$ a neighbourhood of f where P admits a continuation P_g for every $g \in \mathcal{U}$.

Denote by $\mathcal{DS}$ the set of diffeomorphisms $g \in \mathcal{U}$ for which $H(P_g, g)$ admits a dominated splitting. If the closure of the interior of $\mathcal{DS}$, $cl(int(\mathcal{DS}))$, is a neighbourhood of f then the first possibility in the corollary holds and we are done. Otherwise, for any $\varepsilon > 0$ there is g_1, $\varepsilon/2-$close to f in the C^1-topology, in the complement of $cl(int(\mathcal{DS}))$. Thus g_1 has an open neighbourhood $\mathcal{U}_1$ such that for any $g \in \mathcal{U}_1$ there is h arbitrarily close to g such that $H(P_h, h)$ does not admit any dominated splitting.

Given a set $E \subset M$ and $\delta > 0$, let $V(E, \delta)$ be the set of points of M at distance strictly less than δ from E. We now construct inductively sequences $\varepsilon_i > 0$ and $g_i \in \mathcal{U}_1$ satisfying the following properties:

1. $H(P_{g_i}, g_i)$ has no dominated splitting,
2. g_{i+1} is $\varepsilon_i/2-$close to g_i in the C^1-topology,
3. there is a finite set $\mathcal{S}_{i+1}$ of periodic sinks or sources of g_{i+1} such that $H(P_{g_i}, g_i) \subset V(\mathcal{S}_{i+1}, \varepsilon_i/2)$,
4. $\varepsilon_{i+1} < \varepsilon_i/2$,
5. for all g $\varepsilon_{i+1}-$close to g_{i+1} the set of sinks or sources $\mathcal{S}_{i+1}$ has a continuation $\mathcal{S}_{i+1}(g)$ such that $H(P_{g_i}, g_i) \subset V(\mathcal{S}_{i+1}(g), \varepsilon_i)$.

Let us first end the proof of the corollary using the sequences (ε_i) and (g_i) above. The sequence (g_i) is a Cauchy sequence in $\mathrm{Diff}^1(M)$, so it converges to some C^1-diffeomorphism h. Moreover, from $\varepsilon_{i+1} < \varepsilon_i/2$ and the $\varepsilon_i/2-$proximity of g_{i+1} to g_i, we get that h is ε_i-close to g_i for all i. Therefore, by item (5), the set of sources or sinks $\mathcal{S}_{i+1}(h)$ is well defined and $H(P_{g_i}, g_i) \subset V(\mathcal{S}_{i+1}(h), \varepsilon_i)$ for every i.

Consider now the set $\mathcal{S}(h) = \bigcup_1^\infty (\mathcal{S}_i(h))$ consisting of sinks or sources. By construction, the closure of $\mathcal{S}(h)$ contains the topological upper limit set of the $H(P_{g_i}, g_i)$, that is

$$\text{closure}\,(\mathcal{S}(h)) \supset \limsup_{i\to\infty} H(P_{g_i}, g_i) = \bigcap_{i=1}^{\infty} \text{closure}\,(\bigcup_{j>i} H(P_{g_j}, g_j)).$$

Finally, by definition of homoclinic class and since the transverse intersections vary continuously, this upper limit set contains $H(P_h, h)$, so $H(P_h, h)$ is contained in the closure of the set of sinks or sources of h. Thus h is the diffeomorphism in the statement of the corollary.

To end the proof of the corollary it remains to build the sequences (ε_i) and (g_i) above. We proceed inductively, assume that ε_j and g_j are defined for every $j \leq i$. Consider some finite set $\Sigma_i \subset H(P_{g_i}, g_i)$ of saddles such that $H(P_{g_i}, g_i) \subset V(\Sigma_i, \varepsilon_i)$. By item (1), applying Theorem 1 finitely many times, we can create a sink or a source close to each point of Σ_i, obtaining a diffeomorphism g_{i+1} which is $\varepsilon_i/2-$close to g_i and has a set of sinks or sources $\mathcal{S}_{i+1}$ containing $H(P_i, g_i)$ in its $\varepsilon_i/2-$neighbourhood $V(\mathcal{S}_{i+1}, \varepsilon_i/2)$. Thus g_{i+1} satisfies items (2) and (3). Having in mind the definition of $\mathcal{U}_1$, we can suppose (after a new perturbation if necessary) that $H(P_{g_{i+1}}, g_{i+1})$ has no dominated splitting, i.e. g_{i+1} satisfies item (1). Then, using the continuous variation of the finite set $\mathcal{S}_{i+1}(g)$ in a small neighbourhood of g_{i+1}, we can choose $\varepsilon_{i+1} < \varepsilon_i/2$ (item (4)) verifying item (5) above. This ends the proof of the corollary. □

2.2.3 Proof of Corollary 0.3

Recall first that there are dense open subsets $\mathcal{O}_n$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f for which the set $\mathcal{P}(f,n)$ of periodic points of period less than n is finite and hyperbolic. Remark also that the cardinal of $\mathcal{P}(f,n)$ is locally constant in $\mathcal{O}_n$ and that the set $\mathcal{P}(f,n)$ depends continuously on f.

Denote by $\Sigma(n,f) \subset \mathcal{P}(f,n)$ the set of saddles with non-trivial homoclinic class. Then there is a dense open subset $\tilde{\mathcal{O}}_n \subset \mathcal{O}_n$ of diffeomorphisms f such that $\Sigma(n,f)$ has locally constant cardinal and depends continuously on f.

Claim *There is a residual subset $\mathcal{R}_n \subset \tilde{\mathcal{O}}_n$ of diffeomorphisms f such that for any $P \in \Sigma(n,f)$ either $H(P,f)$ admits a dominated splitting or P belongs to the closure of the set of sinks or sources.*

Proof of the claim: Consider any open subset $\mathcal{O} \subset \tilde{\mathcal{O}}_n$ where the periodic points in $\Sigma(n,f)$ are continuous functions of f. So let us denote by Σ_n the (finite) set of these functions: Given $P \in \Sigma_n$ and $f \in \mathcal{O}$ we denote by P_f the corresponding periodic point of f.

For a fixed $P \in \Sigma_n$ let $\mathcal{DS}(P)$ be the set of $f \in \mathcal{O}$ such that $H(P_f, f)$ admits a dominated splitting. Let $\mathcal{U}(P)$ be the complement in $\mathcal{O}$ of the closure of the interior of $\mathcal{DS}(P)$. Let $\mathcal{U}(P,i)$ be the set of $f \in \mathcal{U}(P)$ for which there is a sink or a source Q_f (of any period) with $d(P_f, Q_f) < 1/i$: This set is open and, by Theorem 1, dense in $\mathcal{U}(P)$. Therefore the intersection

$$\mathcal{R}_n(P) = \bigcap_{i=1}^{\infty} (\mathcal{U}(P,i) \cup \mathrm{int}(\mathcal{DS}(P)))$$

is a residual subset of $\mathcal{O}$. Write

$$\mathcal{R}_n(\mathcal{O}) = \bigcap_{P \in \Sigma_n} \mathcal{R}_n(P),$$

by construction, the set $\mathcal{R}_n(\mathcal{O})$ is a residual subset of $\mathcal{O}$ consisting of diffeomorphisms f satisfying the conclusion in the claim. Thus to end the proof of the claim it suffices to consider the set $\mathcal{R}_n$ obtained as the union (over all the open sets $\mathcal{O} \subset \tilde{\mathcal{O}}_n$) of the $\mathcal{R}_n(\mathcal{O})$. □

We are now ready to end the proof of the corollary. Consider $\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{R}_n$. By the claim, the set $\mathcal{R}$ is a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that for any saddle P of f there are two possibilities, either $H(P,f)$ has a dominated splitting, or P is in the closure of the set $\mathcal{S}(f)$ of sinks or sources of f (remark that if the homoclinic class of P is trivial then it admits a dominated splitting because P is hyperbolic).

In the first case we are done. In the second one, we need to check that the whole homoclinic class of P is contained in the closure of the sinks or sources. So assume that $H(P, f)$ does not admit any dominated splitting. Observe that for every saddle Q homoclinically related to P one has $H(Q, f) = H(P, f)$, thus $H(Q, f)$ has no dominated splitting. As $f \in \mathcal{R}$, we have just seen that Q is in the closure of $\mathcal{S}(f)$. Since the set of saddles homoclinically related with P is dense in $H(P, f)$ we have that $H(P, f)$ itself is contained in the closure of $\mathcal{S}(f)$. So $\mathcal{R}$ is the residual set announced in the Corollary 0.3 and the proof is complete. □

2.2.4 Proof of Theorem 2

Let us first prove this theorem in the case of robustly transitive diffeomorphisms. There are two reasons for that. First, the proof of this case is simpler than the proof in the case of transitive sets (i.e. the general case). Second, proceeding in this way can emphasize the additional difficulties and subtleties of the proof for transitive sets.

Proof of Theorem 2 for robustly transitive diffeomorphisms. Consider a C^1−robustly transitive diffeomorphism f and an open neighbourhood $\mathcal{U}$ of f such that any $g \in \mathcal{U}$ is transitive. Reducing the size of $\mathcal{U}$ if necessary, we can assume that there are $K > 0$ and $\varepsilon > 0$ such that every ε−perturbation h of any $g \in \mathcal{U}$ is transitive and the differentials h_* and h_*^{-1} are bounded by K.

Recall that by the Pugh's Closing Lemma [P] there is a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms whose non-wandering set is the closure of the hyperbolic periodic points. So there is a residual subset $\mathcal{R}_0$ of $\mathcal{U}$ of diffeomorphisms g having a dense set of hyperbolic saddles (note that due to the transitivity the diffeomorphisms in $\mathcal{U}$ have neither sinks nor sources).

Moreover, [BD$_2$, Theorem B] says that there is a residual set $\mathcal{R}_1$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that two periodic points of f belong to the same transitive set if and only if their homoclinic classes are equal. Thus for any $g \in \mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1$ and every periodic point P_g of g the homoclinic class $H(P_g, g)$ is the whole manifold M.

By the robust transitivity of the g, given by the choice of ε, it is not possible to create a sink or a source by an ε−perturbation of any $g \in \mathcal{R}$. So Proposition 2.6 gives l such that every $g \in \mathcal{R}$ admits an l−dominated splitting on $M = H(P_g, g)$. Finally, choosing a sequence $g_n \in \mathcal{R}$ converging to f, Corollary 1.5 ensures that f admits an l−dominated splitting, ending the proof of the theorem for robustly transitive diffeomorphisms. □

Proof of Theorem 2, general case. Let $\Lambda_f = \bigcap_{-\infty}^{+\infty} f^i(\bar{U}) \subset U$ be a C^1−robustly transitive set in some open set U. Assume that Λ_f is not reduced to a single hyperbolic orbit (in that case we have nothing to do).

The proof follows essentially along the arguments in the robust transitive case before, but we need to pay special attention to the following fact: In the transitive case ($U = M$) all the orbits we consider are automatically in $\Lambda_f = M$ (that is a tautology), but *a priori* this does not happen when $U \neq M$. Let us go into the details of the proof of this case.

Let $\mathcal{U}$ be a C^1−open neighbourhood of f such that, for every $g \in \mathcal{U}$, the maximal invariant set $\Lambda_g = \bigcap_{-\infty}^{+\infty} g^i(\bar{U})$ is transitive. As above, using the Pugh's Closing Lemma and [BD$_2$, Theorem B], we get a residual subset $\mathcal{R}_0$ of $\mathcal{U}$ of diffeomorphisms g such that the hyperbolic periodic points of Λ_g are dense in Λ_g and have the same homoclinic classes. Thus, for every $g \in \mathcal{R}_0$, the set Λ_g is included in the homoclinic class $H(P, g)$ of some periodic point P. However, that is the special difficulty of this case, we do not know *a priori* if $H(P,g)$ is contained in U (in the robust transitive case that is obvious: $U = M$!)

To solve this problem denote by $H(P,g,U)$ the points of the closure of the transverse intersections of the invariant manifolds of P whose orbits remain in U. We call this set the *homoclinic class of P in U*. So $H(P,g,U)$ is a transitive compact subset of Λ_g and, since $\Lambda_f \subset U$, it is far from the boundary of U.

Lemma 2.7 *There is a residual set $\mathcal{R}_2$ of $\mathcal{U}$ such that for any $g \in \mathcal{R}_2$ there is a periodic point P such that $H(P,g,U) = \Lambda_g$.*

Proof: The proof is identical to [BD$_2$, Theorem B] using the following version of Hayashi's Connecting Lemma. So we do not go into the details.

Theorem (Hayashi's Connecting Lemma) *Let M be a compact manifold, U an open set of M, V an open set relatively compact in U, and f a diffeomorphism defined on M.*

Assume that there are periodic saddles P and Q whose orbits are contained in U, a sequence of points x_i converging to some point $x \in W^u_{loc}(P,f)$, and a sequence of positive integers n_i such that $f^{n_i}(x_i)$ converges to some $y \in W^s_{loc}(Q,f)$. Suppose also that for any i and every $m \in \{0,\dots,n_i\}$ one has $f^m(x_i) \in V$.

Then there is g arbitrarily C^1-close to f such that $x \in W^u(P,g) \cap W^s(Q,g)$. Moreover, the whole orbit of x is contained in U and $g^n(x) = y$ for some $n > 0$.

□

To prove Theorem 2 we apply Proposition 2.6, so we need to see that the sets $\Lambda_g = H(P_g,g,U)$ given by Lemma 2.7 are not all reduced to the single periodic orbit P_g.

Lemma 2.8 *There is g arbitrarily C^1-close to f having at least two different hyperbolic periodic orbits contained in U.*

Using Lemma 2.7, this implies that one can choose g such that $\Lambda_g = H(P_g,g,U)$ is not reduced to the periodic orbit P_g.

Proof: We are assuming that Λ_f is not reduced to a single periodic orbit (this is just the trivial case). Hence, if f has at least two periodic orbits the result is immediate: After perturbation we can make these orbits hyperbolic ones.

So it remains to consider the case where f has not periodic orbits. We know that there are diffeomorphisms g close to f having periodic orbits. We argue by contradiction, suppose that every g (with periodic points) close to f has only one periodic orbit Q. Then considering an isotopy from g to f we get a bifurcation of this periodic orbit. After a perturbation, we get a saddle-node, a flip, or a Hopf bifurcation. In these three cases, a new perturbation gives two periodic orbits: A saddle-node and a flip split into two hyperbolic periodic points, and a Hopf point into a periodic point and an invariant circle (in this case to get a new periodic point it is enough modify the rotation number of the restriction of the map to the invariant circle). □

By definition of robust transitivity, there are no perturbations of f having sinks or sources whose orbits are contained in U. Take a sequence $g_n \to f$ of diffeomorphisms such that $\Lambda_{g_n} = H(P_{g_n}, g_n, U)$ is nontrivial. Proposition 2.6 implies that there is $l \in \mathbb{N}$ such that

Λ_{g_n} admits an $l-$dominated splitting for any n large enough. Corollary 1.5 now implies that these dominated splittings induce an $l-$dominated splitting on $\limsup_{n\to\infty}(\Lambda_{g_n})$, and so on Λ_f. Now the proof of the theorem is complete. $\Box$

2.2.5 Proof of Theorem 3

This theorem is a direct consequence of Proposition 2.6. We argue as follows, let $m \in \mathbb{N}$, $K > 0$, and $\varepsilon > 0$ such that any diffeomorphism g which is $\varepsilon-$close to f has less than m sinks and sources, and g_* and $(g_*)^{-1}$ are both uniformly bounded by K. Let l_0 be the constant $l(K, \varepsilon/2, \dim(M))$ given by Proposition 2.6. Then, for every g $\varepsilon/2-$close to f and any saddle P of g having a non-trivial homoclinic class $H(P, g)$, we have that $H(P, g)$ has an l_0-dominated splitting. $\Box$

To prove Theorem 4 we need new arguments of a very different nature: The notion of *finest dominated splitting* and the *ergodic closing lemma* of Mañé, so let us postpone its proof until Section 6 of this paper.

We also postpone until the end of the paper (Section 7) the results about conservative diffeomorphisms.

3 Two-dimensional linear systems

In this section we give a version (Proposition 3.1) of Proposition 2.1 for two dimensional systems (without requiring transitions) following an argument essentially due to Mañé in [Ma$_3$]. In the next sections, using an argument of reduction of the dimension of the system (quotients and restrictions of linear systems, see Section 4.1) we deduce from this two dimensional result the general version of it (see Section 5).

Proposition 3.1 *Given any $K > 0$ and $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that for every two-dimensional linear system $(\Sigma, f, \mathcal{E}, A)$, with norm $\|A\|$ bounded by K and such that the matrices $M_A(x)$ preserve the orientation, one has that*

- *either A admits an $l-$dominated splitting,*
- *or there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ has a complex (non-real) eigenvalue.*

The difference between Proposition 3.1 and Proposition 2.1 (in the case of $2-$dimensional systems) is that here we get a complex eigenvalue instead of a homothety. In fact, if the system admits transitions then one can use this complex eigenvalue to get homotheties (this will be done later in any dimension, see Proposition 2.5).

We begin the proof of Proposition 3.1 by a very elementary lemma whose proof we omit:

Lemma 3.2 *For every $\alpha > 0$ and every matrix $M \in GL_+(2, \mathbb{R})$ having two different eigenspaces E_1 and E_2 whose angle is less than α there is $s \in [-1, 1]$ such that $R_{s\,\alpha} \circ M$ has a complex (non-real) eigenvalue (here $R_{t\,\alpha}$ denotes the rotation of angle $t\,\alpha$).*

In what follows, for notational simplicity, let us write $I_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$.

Lemma 3.3 *For every $\alpha > 0$ and $\mu > 0$ there is $c > 1$ verifying the following property: Consider the matrices*

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \textit{ and} C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \textit{ such that } \frac{|b_1|}{|b_2|} > c \textit{ and } \frac{|b_1\, c_1|}{|b_2\, c_2|} < 1.$$

Then the angle between the eigenvectors of the matrix $D = B \circ I_\mu \circ C$ is less than α.

Proof: Observe first that $(1, 0)$ is an eigenvector of the matrix D. The heuristic idea of the proof is very simple: Consider the vector $(1, \beta)$, for some *small* $\beta \leq 2/(c\,\mu)$ fixed. As $|b_1/b_2|$ and $|c_2/c_1|$ are large (i.e. greater than c) the vectors $B^{-1}(1, \beta)$ and $C(1, \beta)$ are almost vertical (angle with the vertical less than μ). The role of the matrix I_μ now is to send the direction of $C(1, \beta)$ into the direction of $B^{-1}(1, \beta)$, thus $(1, \beta)$ is an eigenvector of D.

The precise calculations are not more complicated: Let $(1, \beta)$, $\beta \neq 0$, be some eigenvector of D not parallel to $(1, 0)$ (i.e. associated to the eigenvalue $b_2\, c_2$). Then β satisfies

$$|\beta| = \frac{|b_2\, c_2 - b_1\, c_1|}{|\mu\, b_1\, c_2|} = \frac{|b_2/b_1 - c_1/c_2|}{\mu} < \frac{2}{c\,\mu}.$$

This completes the proof of the lemma. □

Consider a periodic system of matrices (Σ, f, A) in $GL_+(2, \mathbb{R})$ such that all the matrices of the system are diagonal. Thus the canonical splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is invariant. Given $x \in \Sigma$ denote by $\sigma(x)$ and $\lambda(x)$ the eigenvalues of $M_A(x)$ associated to the vertical direction $(\{0\} \times \mathbb{R})$ and the horizontal direction $(\{0\} \times \mathbb{R})$, respectively. Up to a trivial change of coordinates, one can assume that for any $x \in \Sigma$, the eigenvalue $\sigma(x)$ of $M_A(x)$ is bigger in modulus than the eigenvalue $\lambda(x)$.

Lemma 3.4 *For any $\varepsilon > 0$, $\alpha > 0$, and $K > 0$ there is $l \in \mathbb{N}$ with the following property:*

Consider a periodic system (Σ, f, A) of diagonal matrices in $GL_+(2, \mathbb{R})$ as above, bounded by K such that $|\sigma(x)| \geq |\lambda(x)|$ for every $x \in \Sigma$.

Suppose that the splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is not $l-$dominated. Then there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that the angle between the eigenspaces of $M_{\tilde{A}}(x)$ is less than α.

Proof: Let us write

$$A(x) = \begin{pmatrix} a(x) & 0 \\ 0 & b(x) \end{pmatrix}.$$

Observe that if the splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is $l-$dominated then for every $x \in \Sigma$ one has

$$2\, |\prod_0^{l-1} a(f^i(x))| < |\prod_0^{l-1} b(f^i(x))|.$$

Recall that, by Lemma 1.2, there is $\mu > 0$ (depending on ε and K) such that multiplying matrices $A(x)$ by diagonal matrices $\mu-$close to identity one gets $\varepsilon/3-$perturbations of A.

Suppose first that there is x in Σ such that

$$|\sigma(x)| \leq (1+\mu)^{2\,p(x)}\,|\lambda(x)|, \qquad \text{where } p(x) \text{ is the period of } x.$$

Then multiplying the matrices $A(f^i(x))$ by some matrix of the form

$$\begin{pmatrix} 1+\nu & 0 \\ 0 & 1/(1+\nu) \end{pmatrix}, \quad \text{for some } \nu \in [0,\mu],$$

we get an $\varepsilon/3-$perturbation A' of A such that $M_{A'}(x)$ is an homothety. So given any pair of (different) directions of $\mathbb{R}^2$, there is an arbitrarily small perturbation $\tilde{A}$ of A' such that $M_{\tilde{A}}(x)$ has two eigenvectors parallel to such directions. This ends the proof of the lemma in this first case.

So we can now assume that

$$|\sigma(x)| > (1+\mu)^{2p(x)}\,|\lambda(x)| \qquad \text{for every } x \in \Sigma.$$

Consider the constant c, given by Lemma 3.3, associated to α and μ, and l such that

$$(1+\mu)^l > 2\,c.$$

We show that, for any system of matrices (Σ, A, f) bounded by K, l is the constant announced in the statement of the lemma.

Recall the observation in the beginning of the proof of the lemma, since the canonical splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is not $l-$dominated, then there is $x \in \Sigma$ such that

$$2\,|\prod_0^{l-1} a(f^i(x))| \geq |\prod_0^{l-1} b(f^i(x))|.$$

Assume first that $l < p(x)$. Given $y \in \Sigma$ let

$$\tilde{a}(y) = (1+\mu)\,a(y) \text{ if } y = f^i(x), i \in \{0,\cdots,l-1\}, \quad \tilde{a}(y) = a(y) \text{ if } i \in \{l,\ldots,p(x)-1\}.$$

Consider now the $\varepsilon/3-$perturbation $\tilde{A}$ of A given by

$$\tilde{A}(y) = \begin{pmatrix} \tilde{a}(y) & 0 \\ 0 & b(y) \end{pmatrix}.$$

Let $B = \tilde{A}^{(l)}(x)$ (the product of the matrices of the system $\tilde{A}$ along the orbit of x from time 0 to time $l-1$) and $C = \tilde{A}^{(p(x)-l)}(f^l(x))$. Then, we get

$$M_{\tilde{A}}(x) = C \circ B \quad \text{and} \quad M_{\tilde{A}}(f^l(x)) = B \circ C.$$

Observe that B and C verify the hypotheses of Lemma 3.3, so the angle between the eigenvectors of the matrix $D = B \circ I_\mu \circ C$ is less than α.

Denote by $\bar{A}$ the $\varepsilon/3-$perturbation of $\tilde{A}$ obtained modifying only the matrix $\tilde{A}(f^{-1}(x))$ $= A(f^{-1}(x))$ by replacing this matrix by $\bar{A}(f^{-1}(x)) = I_\mu \circ A(f^{-1}(x))$. Then the angle between the eigenvectors of $M_{\bar{A}}(f^l(x)) = D$ is less than α.

To finish the proof of lemma it remains the case $l \geq p(x)$. Remark that l can not be a multiple of $p(x)$, hence $l = k\,p(x) + l_0$, for some $k \geq 1$ and $1 \leq l_0 < p(x)$. By hypothesis,

$$(1+\mu)^{2\,p(x)}\,|\prod_{0}^{p(x)-1} a(f^i(x))| < |\prod_{0}^{p(x)-1} b(f^i(x))|.$$

Since we are assuming that the splitting is not $l-$dominated, we also have

$$2\,|\prod_{0}^{l-1} a(f^i(x))| \geq |\prod_{0}^{l-1} b(f^i(x))|.$$

So we get that

$$|\prod_{0}^{l_0-1} a(f^i(x))| > \frac{(1+\mu)^{2\,k\,p(x)}}{2}\,|\prod_{0}^{l_0-1} b(f^i(x))|.$$

Finally, note that $2\,k\,p(x) > l$ and recall the choice of l (i.e. $(1+\mu)^l > 2\,c$), thus

$$\frac{(1+\mu)^{2\,k\,p(x)}}{2} > \frac{(1+\mu)^l}{2} > c.$$

Thus, as in the previous case, we can apply Lemma 3.3 to the matrices $B = A^{(l_0)}(x)$ and $C = A^{(k\,p(x)-l_0)}(f^{l_0}(x))$. This completes the proof of the lemma. $\square$

3.1 Proof of Proposition 3.1

The proof of Proposition 3.1 follows almost directly from Lemmas 1.2, 3.2, and 3.4. Let us go into the details.

For a fixed $\varepsilon > 0$ and $K > 0$, consider a system of matrices (Σ, f, A) in $GL_+(2,\mathbb{R})$, bounded by K, such that it is not possible to create a complex eigenvalue by an $\varepsilon-$perturbation of A. We prove that such a system is $l-$dominated. This clearly implies the proposition.

By Lemma 1.2, there is $\alpha = \alpha(K,\varepsilon) > 0$ such that the composition of the system with a rotation of angle less than α gives an $\varepsilon/2-$perturbation of the system. So Lemma 3.2 ensures that for any $x \in \Sigma$ the angle between the eigenspaces of $M_A(x)$ is bigger than α. This means that there are $K_0 = K_0(\alpha)$ and a family of matrices $P(x)$, $x \in \Sigma$, with $P(x)$ and $P^{-1}(x)$ bounded by K_0, such that the eigenspaces of $P(x) \circ M_A(x) \circ P(x)^{-1}$ are orthogonal.

Let $B = P \circ A \circ P^{-1}$ be the system defined by $B(x) = P(f(x)) \circ A(x) \circ P(x)^{-1}$. By Lemma 1.2, there are $K_1 = K_1(K, K_0)$ and $\delta = \delta(K, K_0, \varepsilon)$ such that B and B^{-1} are bounded by K_1 and any $\delta-$perturbation of B is obtained by conjugating by P some $\varepsilon-$perturbation of A.

By construction, all the eigenspaces of the matrices $M_B(x)$ are orthogonal. Thus, by an orthonormal change of coordinates, we can assume that B let invariant the canonical splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$, and that the eigenvalue of $M_B(x)$ corresponding to the vertical direction is bigger (in modulus) than the eigenvalue associated to the horizontal direction. Finally, there is no possible to create a complex eigenvalue by a $\delta-$perturbation of B

Fix $K_2 = K_2(K_1)$ such that any $\delta-$perturbation of B is bounded by K_2. Consider now $\alpha_2 = \alpha_2(K_2)$ such that the composition by a rotation of angle at most α_2 of a linear system bounded by K_2 gives a $\delta/3-$perturbation. Then (following Lemma 3.2) given any $\delta/3-$perturbation $\tilde{B}$ of B the angles between the eigenspaces of $M_{\tilde{B}}(x), x \in \Sigma$, are bigger than α_2. So, from Lemma 3.4, we get $l_0(\delta, K_1, \alpha_2)$ such that B is l_0-dominated.

Using that $A = P^{-1} \circ B \circ P$, where P is bounded by K_0, we get $l(l_0, K_0)$ such that A is $l-$dominated.

Now, to conclude the proof it is enough to remark that all the constants introduced in the proof are functions of K and ε. □

4 Invariant subbundles: Reduction of the dimension and the finest dominated splitting

4.1 Quotient of linear systems and restriction to subbundles

The proof of Proposition 2.4 uses successively Proposition 3.1 on $2-$dimensional subbundles. Our construction also involves an argument (considering quotient spaces) that allows us to reduce the dimension of the ambient space. We pay special attention to the invariant subbundles of a linear system, and we will often need to compare the action of the system on them. This motivates the introduction of the notions of *restriction* and *quotient* of a linear system.

Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and F an invariant subbundle of $\mathcal{E}$ (with constant dimension). We denote by A_F the *restriction of A to F* and by A/F the *quotient of A along F* endowed with the metric of the orthogonal complement $F^\perp$ of F, i.e. given a class $[v]$ we let

$$|[v]| = |v_F^\perp|, \quad \text{where } v = v_F^\perp + v_F,\ v_F^\perp \in F^\perp, \text{ and } v_F \in F.$$

Write A in blocks of the form
$$\begin{pmatrix} A_F & B \\ 0 & C \end{pmatrix},$$
since $C = A/F = (P_{F^\perp} \circ A)/F$, where $P_{F^\perp}$ is the orthogonal projection on $F^\perp$, we have the following lemma:

Lemma 4.1 *Given any $\varepsilon > 0$ one has that*

- *every $\varepsilon-$perturbation of A_F is the restriction of an $\varepsilon-$perturbation of A keeping invariant the other eigenvalues (but not necessarily the eigenspaces). Actually, A/F is not modified.*
- *Any $\varepsilon-$perturbation of A/F is the quotient of an $\varepsilon-$perturbation of A keeping invariant A_F.*

The definition of domination of a linear system, recall Definition 1.3, has a direct generalization for pairs of invariant subbundles. Suppose that E and F are two invariant subbundles of a linear system $(\Sigma, f, \mathcal{E}, A)$, we say that E *is* $l-$*dominated by* F if for every $x \in \Sigma$ one has

$$\|A_E^{(l)}(x)\|\,\|A_F^{(-l)}(f^l(x))\| < 1/2.$$

In this case we write $E \prec F$ or $E \prec_l F$. It is easy to see that this dominance implies that for any $x \in \Sigma$ the intersection $E(x) \cap F(x)$ is reduced to the zero-vector. So this definition is equivalent to say that $E \oplus F$ is an $l-$dominated splitting of the system $A_{E\oplus F}$.

In what follows we will often use the following very easy lemma whose proof we omit:

Lemma 4.2 *For any $K > 0$ and $l > 0$ there are $K_0 > 0$, $l_0 > 0$, and $K_1 > 0$ satisfying the following property.*

Consider a linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K with an invariant splitting $E \prec_l F$. Then there is a linear change of coordinates P, bounded by K_0, such that the bundles $P(E)$ and $P(F)$ are orthogonal and invariant by the system defined by $B = P \circ A \circ P^{-1}$. Moreover, $P(E) \prec_{l_0} P(F)$ (for B) and B is bounded by K_1.

One of the main difficulties in the proof of Proposition 2.4 comes from the fact that the dominance has not a good behavior if one consider sums of dominated subbundles. The following remark illustrates this difficulty:

Remark 4.3 *There exist linear maps and invariant bundles $\mathcal{E} = E \oplus F \oplus G$ such that $E \prec F$ and $E \prec G$, but the splitting $E \oplus (F \oplus G)$ is not dominated.*

This difficulty comes from the relationship between dominance and *angles* (the angles between two bundles of a dominated splitting can not be *very* small), and the following easy geometric observation: One can have simultaneously *big* angles between E and F, $\angle(E, F)$, and E and G, $\angle(E, G)$, but an arbitrarily small angle $\angle(E, F \oplus G)$. See the next example, and Lemma 4.2 and Remark 4.5 below.

Example 2 *Consider the quotient Σ of $\mathbb{Z} \times \mathbb{N}$ by the relation $(n, m) = (n + 3m, m)$ and the map $f\colon \Sigma \to \Sigma$ given by $(n, m) \mapsto (n + 1, m)$. Denote by $\mathcal{E}$ the trivial bundle of fiber $\mathbb{R}^3$. Define the linear map A (acting on $\mathbb{R}^3$) by*

$$A(n,m) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{if } -m \le n < 0, \, , \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{if } 0 \le n < m, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{if } m \le n < 2m. \end{cases}$$

Then we have that $(\Sigma, f, \mathcal{E}, A)$ is a periodic linear system.

Observe that the directions $e_1\mathbb{R}$, $(e_2 + e_3)\mathbb{R}$, and $e_3\mathbb{R}$ define a splitting of $\mathcal{E}(0, m)$ by eigenspaces of the matrix $M_A(0, m)$. We now define E, F, and G at the point (n, m) as the images of these spaces by A^n.

By construction, $\mathcal{E}_0 = E \oplus F \oplus G$ is an invariant splitting and A induces an isometry on E and a dilatation of ratio bigger than 2 on G and F. So we get $E \prec_1 F$ and $E \prec_1 G$. However, this system does not admit any dominated splitting: There are vectors $e_2 \in F \oplus G$ which are contracted during an arbitrarily large time.

The following lemma explains how we solve the difficulty above.

Lemma 4.4 *Consider any $K > 0$ and $l \in \mathbb{N}$. Then there exists L with the following property: Given any linear system $(A, f, \mathcal{E}, \Sigma)$ such that $\|A\|$ is bounded by K with an invariant splitting $E \oplus F \oplus G$, one has*

1. *$(E \prec_l F$ and $E/F \prec_l G/F) \Rightarrow E \prec_L (F \oplus G)$,*
2. *$(F \prec_l G$ and $E/F \prec_l G/F) \Rightarrow (E \oplus F) \prec_l G)$.*

Proof: Observe that Lemma 4.2 allows us to assume that (up to a bounded change of coordinates) E is orthogonal to F and E/F is orthogonal to G/F (in the quotient space). This means that E is ortoghonal to $P_{F^\perp}(G)$, thus E is also orthogonal to G.

Hence at each point $x \in \Sigma$ we can choose an orthonormal basis $(e_i(x))$ of $\mathcal{E}_x$ such that $e_i(x) \in E(x)$ for $i \leq \dim(E(x))$, and $e_i(x) \in F(x)$ for $\dim(E(x)) < i \leq \dim(E(x)) + \dim(F(x))$. Moreover, $G(x)$ is contained in the subspace spanned by the $e_i(x)$ for $i > \dim(E(x))$. So this subspace is $F(x) \oplus G(x)$ and it is invariant by A. Using the basis $(e_i(x))$ we can write the matrices of the system A as follows,

$$A(x) = \begin{pmatrix} A_E(x) & 0 & 0 \\ 0 & A_F(x) & B(x) \\ 0 & 0 & C(x) \end{pmatrix},$$

where C is $A/(E \oplus F)$ and B is bounded by a constant K_1 depending only on K and l. Write

$$D = A_{F \oplus G} = \begin{pmatrix} A_F & B \\ 0 & C \end{pmatrix}.$$

We can now consider D as a linear system over $F \oplus G$, which allows us to iterate D. We prove that there is L (depending on K and l) such that

$$\|A_E^L(x)\| \, \|D^{-L}(x)\| < 1/2,$$

that is exactly the first assertion in the lemma ($E \prec_L F \oplus G$). To prove this claim observe that

$$D^{-l} = \begin{pmatrix} A_F^{-l} & H \\ 0 & C^{-l} \end{pmatrix}, \quad \text{where} \quad H = -\sum_{j=1}^{l} A_F^{-j} \circ B \circ C^{j-l-1}.$$

Therefore all the matrices in the definition of D^{-l} are bounded by some constant K_2 depending on K and K_1, and so depending on K and l. An elementary calculation shows that for $i \geq 1$

$$D^{-il} = \begin{pmatrix} A_F^{-il} & \sum_0^{i-1} A_F^{-jl} \circ H \circ C^{(j+1-i)l} \\ 0 & C^{-il} \end{pmatrix}.$$

By hypotheses,

$$\|A_E^l\|\,\|A_F^{-l}\| < 1/2 \quad \text{and} \quad \|A_E^l\|\,\|C^{-l}\| < 1/2,$$

the last inequality follows immediately from $E/F \prec_l G/F$. Thus

$$\sup\{\|A_F^{-i\,l}\|,\, \|C^{-i\,l}\|\} < \frac{1}{2^i\,\|A_E^{i\,l}\|}.$$

An elementary estimate shows that the norm of a matrix of the form $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ is bounded by $\sup\{\|X\|, \|Z\|\} + \|Y\|$. So we get that

$$\|D^{-i\,l}\| \leq \frac{1}{2^i\,\|A_E^{i\,l}\|}\;(1 + 2\,i\,K_1)$$

It is now immediate that

$$\|A_E^{i\,l}\|\,\|D^{-i\,l}\| < \frac{1}{2^i}(1 + (2\,i\,K_1)).$$

Taking i big enough, this product is clearly less than 1/4, ending the proof of our claim. This completes the first part of the lemma, the second one follows analogously, so we omit it. $\square$

Remark 4.5

- *The relations $\prec$ and $\prec_l$ are (strict and partial) order relations on the set of $A-$invariant subbundles of a linear system $(\Sigma, f, \mathcal{E}, A)$: The transitivity and the strict anti-symmetry of these relations are clear.*

- *Let E and G be two subbundles of $\mathcal{E}$, the angle $\angle(E, G)$ between E and G is the infimum of the angles (u, v), where $u \in E_x, v \in G_x, x \in \Sigma$. By Lemma 4.2, if $E \prec_l G$ then the angle $\angle(E, G)$ between E and G is greater than some constant α depending on K and l.*

- *Let E and F be two subbundles such that $E \prec F$ for $A_{E\oplus F}$. Let G be an $A-$invariant subbundle such that $G \cap E = G \cap F = 0$ and that the angles $\angle(E, G)$ and $\angle(F, G)$ are bigger than some $\alpha > 0$. Then E/G is dominated by F/G. Moreover, the constant of domination depends only on the bound K of A, the constant of domination of the splitting $E \prec F$, and the angle α. The easy idea of the proof is the following: As the angle $\angle(E, G)$ is greater than α, then the metrics of E/G (i.e. the metric in $G^\perp$) and of E are obtained by a bounded change of coordinates. The same is true for F. It is now enough to go back to the definition of domination.*

As a corollary we now get,

Lemma 4.6 *Let $K > 0$ and $l \in \mathbb{N}$. There is L such that for any linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K and any invariant subbundles E, F, and G, such that $E \prec_l F$ and $F \prec_l G$, one has $E \prec_L F \oplus G$ and $E \oplus F \prec_L G$.*

Proof: As the $E \prec_l F$ and $F \prec_l G$ there is a constant α (depending only on K and l) such that the angles $\angle(E,F)$ and $\angle(F,G)$ are greater than α. By transitivity of the relation $\prec_l$ one has $E \prec_l G$. So, by the third part of the remark, we get l_1 (depending on K, l, and α) such that $E/F \prec_{l_1} G/F$. Now the corollary follows directly from Lemma 4.4 $\square$

4.2 The finest dominated splitting

Lemma 4.4 allows us to define the notions of *undecomponible* and *finest dominated splittings.* Let us recall that an invariant splitting $\mathcal{E} = E_1 \oplus \cdots \oplus E_k$ is dominated if for any $1 \leq j \leq k-1$ one has $\bigoplus_1^j E_i \prec \bigoplus_{j+1}^k E_i$.

Corollary 4.7 *A splitting $\mathcal{E} = \bigoplus_{i=1}^k E_i$ is dominated if and only if $E_i \prec E_{i+1}$ for every $i \in \{1, \ldots, k-1\}$.*

Proof: Observe that if $\mathcal{E} = \bigoplus_{i=1}^k E_i$ is dominated then one has $E_i \prec E_{i+1}$ straightforwardly from the definition. To prove the converse we argue inductively (on the number k of subbundles of the splitting). For $k = 3$ the corollary is exactly Lemma 4.6. Assume now that the lemma is true for $k-1$.

We have to prove that if $E_i \prec E_j$ for every $i < j$ then the splitting is dominated, that is $\bigoplus_1^i E_j \prec \bigoplus_{i+1}^k E_j$ for all $0 < i < k$. Assume first that $i > 1$. Applying the induction hypothesis to the restrictions of A to $\bigoplus_2^k E_j$ and $\bigoplus_1^i E_j$, we get $\bigoplus_2^i E_j \prec \bigoplus_{i+1}^k E_j$ and $E_1 \prec \bigoplus_2^i E_j$. Now the dominance $\bigoplus_1^i E_j \prec \bigoplus_{i+1}^k E_j$ follows applying Lemma 4.6 to the subbundles E_1, $\bigoplus_2^i E_j$, and $\bigoplus_{i+1}^k E_j$.

Finally, if $i = 1$ we apply the induction hypothesis to $\bigoplus_1^{k-1} E_j$ and $\bigoplus_2^k E_j$. This gives $E_1 \prec \bigoplus_2^{k-1} E_j$ and $\bigoplus_2^{k-1} E_j \prec E_k$. Now the conclusion follows as above applying Lemma 4.6 to these three subbundles. $\square$

Corollary 4.7 above allows us to use the notation $E_1 \prec E_2 \prec \cdots \prec E_k$ to denote a dominated splitting $\bigoplus_1^k E_j$.

Lemma 4.8 *Let $E_1 \prec \cdots \prec E_k \prec F \prec G_1 \cdots \prec G_m$ be a dominated splitting such that A_F admits a dominated splitting $F_1 \prec F_2$. Then $E_1 \oplus \cdots \oplus E_k \oplus F_1 \oplus F_2 \oplus G_1 \oplus \cdots \oplus G_m$ is a dominated splitting.*

Proof: It suffices to verify that $E_k \prec F_1$ and $F_2 \prec G_1$. $\square$

Definition 4.9 *Let $\bigoplus_1^k E_i$ and $\bigoplus_1^m F_j$ be two dominated splittings of $\mathcal{E}$, $\mathcal{E} = \bigoplus_1^k E_i$ and $\bigoplus_1^m F_j$. We say that the splitting $\bigoplus_1^m F_j$ is* finer *than $\bigoplus_i^k E_i$ if every E_i is the direct sum of some of the F_j, i.e. $E_i = \bigoplus_{j_1}^{j_2} F_j$. In this case we write $\bigoplus_1^m F_j \sqsubseteq \bigoplus_1^k E_i$.*

Remark 4.10 *Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system of dimension N.*

- *The relation $\sqsubseteq$ is an order relation on the set of dominated splittings of a linear system $(\Sigma, f, \mathcal{E}, A)$.*

- *Given any finite sequence n_i with $\sum_i n_i = N$, there is at most one dominated splitting $E_1 \prec \cdots \prec E_k$ such that $\dim E_i = n_i$ for every i (actually this easy assertion is a direct consequence of the next lemma). As a consequence, the set of dominated splittings of $(\Sigma, f, \mathcal{E}, A)$ is finite. So there are splittings which are minimal for the relation $\sqsubseteq$, these splittings are called* undecomponible. *Moreover, given any splitting we can subdivide it to get a minimal one, i.e. every splitting is bigger, $\sqsupseteq$, than at least one undecomponible splitting.*

- *Following Lemma 4.8, a dominated splitting $\bigoplus_1^k E_i$ is undecomponible if and only if none of the subsystems A_{E_i} admits dominated decomposition.*

Proposition 4.11 *For every linear system $(\Sigma, f, \mathcal{E}, A)$ there is a unique undecomponible dominated splitting, called* the finest dominated splitting*.*

Proof: The first step to prove this proposition is the following lemma.

Lemma 4.12 *Let $E_1 \prec \cdots \prec E_k$ and $F_1 \prec \cdots \prec F_m$ be two dominated splittings of $\mathcal{E}$. Then either $E_1 \subset F_1$ and $\bigoplus_2^m F_i \subset \bigoplus_2^k E_i$, or $F_1 \subset E_1$ and $\bigoplus_2^k E_i \subset \bigoplus_2^m F_i$. As a consequence, if $E_1 \neq F_1$ then either E_1 or F_1 admits a dominated splitting.*

Proof: First we prove that for any $x \in \Sigma$ one of the linear spaces $E_1(x)$ and $F_1(x)$ is contained in the other one. We argue by contradiction, suppose, contrary to our hypotheses, that there are

$$x \in \Sigma, \quad u \in (E_1(x) \setminus F_1(x)), \quad \text{and} \quad v \in (F_1(x) \setminus E_1(x)).$$

As $\bigoplus_1^k E_i$ is dominated, and $u \in E_1$ and $v \notin E_1$, for every i big enough we get

$$\frac{\|A^{(i)}(x)(u)\|}{\|u\|} \leq \frac{\|A^{(i)}(x)(v)\|}{2\,\|v\|}.$$

Similarly, the dominance of $\bigoplus_1^m F_i$, $u \notin F_1(x)$, and $v \in F_1(x)$ imply (for big i)

$$\frac{\|A^{(i)}(x)(v)\|}{\|v\|} \leq \frac{\|A^{(i)}(x)(u)\|}{2\|u\|}.$$

These two inequalities give a contradiction. So we have proven that for any $x \in \Sigma$ one has $E_1(x) \subset F_1(x)$ or vice versa. As the dimensions of these spaces do not depend on x, we have that either $E_1 \subset F_1$ or $F_1 \subset E_1$.

The same arguments give the other inclusion, that is either $\bigoplus_2^m F_i \subset \bigoplus_2^k E_i$, or $\bigoplus_2^k E_i \subset \bigoplus_2^m F_i$.

Finally, assume now that $E_1 \neq F_1$ and, for instance, $E_1 \subset F_1$. Then F_1 is transverse to $\bigoplus_2^k E_i$, so we get a splitting $F_1 = E_1 \oplus (F_1 \cap \bigoplus_2^k E_i)$. This splitting is clearly invariant and dominated. This finishes the proof. □

We are now ready to prove Proposition 4.11:

Let $\bigoplus_1^k E_i$ and $\bigoplus_1^m F_j$ be two undecomponible dominated splittings. Using Lemma 4.12 above we get that $E_1 = F_1$ and that $\bigoplus_2^k E_i = \bigoplus_2^m F_j$. Consider now the restriction $B = A_{\bigoplus_2^m F_j}$. Now $E_2 \prec \cdots \prec E_k$ and $F_2 \prec \cdots \prec F_m$ are two undecomponible dominated splittings of B, so $E_2 = F_2$. We conclude the proof repeating k times this argument. □

4.3 Transitions and invariant spaces

Lemma 4.13 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with $\varepsilon-$transitions and $\varepsilon_0 > \varepsilon$. Consider an invariant subset $\Sigma_0 \subset \Sigma$ such that there is a dominated splitting $E_1 \prec \cdots \prec E_k$ defined over Σ_0. Then for every finite subset $\Lambda = \{x_1, \dots, x_m\}$ of Σ_0 there are ε_0-transitions such that the corresponding linear maps (transitions) T_{x_j,x_i} map $E_l(x_i)$ on $E_l(x_j)$ for any $l \in \{1, \dots, k\}$ and x_j, $x_i \in \Lambda$.*

Before proving this lemma let us observe that as a direct consequence of it we get the following corollary.

Corollary 4.14 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transition and $E \oplus F$ a dominated splitting. Then the induced systems A_E and A_F admit transitions.*

Proof: By hypothesis, there are $\varepsilon-$transitions $[t^0_{x_j,x_i}]$ from x_i to x_j for every pair of points in Λ. By Remark 1.7, for any n_1, $n_2 \geq 0$ the word

$$[M]_A(x_j)^{n_2}\, [t^0_{x_j,x_i}]\, [M]_A(x_i)^{n_1}$$

is also a transition . Moreover, any $(\varepsilon_0 - \varepsilon)-$perturbation of an $\varepsilon-$transition is also a ε_0-transition.

Taking an arbitrarily small perturbation of $[t^0_{x_j,x_i}]$ we can assume that its corresponding matrix $\tilde{T}^0_{j,i}$ maps $E_k(x_i)$ transversally to $\bigoplus_1^{k-1} E_m(x_j)$. Thus, by the dominance $\bigoplus_1^{k-1} E_m \prec E_k$, taking n_2 large enough, we have that

$$\left((M_A(x_j))^{n_2} \circ \tilde{T}^0_{j,i}\right)(E_k(x_i)) \quad \text{is arbitrarily close to} \quad E_k(x_j).$$

So there is a small perturbation $M_{\tilde{A}}(x_j)$ of $M_A(x_j)$ such that

$$\left(M_{\tilde{A}}(x_j) \circ (M_A(x_j))^{n_2} \circ \tilde{T}^0_{j,i}\right)(E_k(x_i)) = E_k(x_j).$$

Denote by $T^1_{j,i}$ the linear map corresponding to this new transition. Now the consider pre-image of $E_k(x_j)$ by $T^1_{j,i}$ and observe that

$$\left(M_A(x_i)^{-n_1} \circ (T^1_{j,i})^{-1}\right)(E_k(x_j)) = E_k(x_i).$$

As above, since $\bigoplus_1^{k-1} E_m \prec E_k$, for n_1 sufficiently large we get that

$$\left(M_A(x_i)^{-n_1} \circ (T^1_{j,i})^{-1}\right)(\bigoplus_1^{k-1} E_m(x_j)) \quad \text{is arbitrarily close to} \quad \bigoplus_1^{k-1} E_m(x_i).$$

So, arguing as before, the announced transition $T_{j,i}$ is obtained composing (at the right) $T^1_{j,i} \circ M_A(x_i)^{n_1}$ with a small perturbation of $M_A(x_i)$ mapping $E_k(x_i)$ into itself and $\bigoplus_1^{k-1} E_m(x_i)$ into $\left(M_A(x_i)^{-n_1} \circ (T^1_{j,i})^{-1}\right)(\bigoplus_1^{k-1} E_m(x_j))$.

Next we consider the restriction of A to $\bigoplus_1^{k-1} E_m$ to get a new perturbation of the transition mapping $E_k(x_i)$, $E_{k-1}(x_i)$, and $\bigoplus_1^{k-2} E_m(x_i)$ into the corresponding spaces for x_j. The inductive pattern of our construction is now clear. □

4.4 Diagonalizable systems.

Definition 4.15 *A periodic linear system* $(\Sigma, f, \mathcal{E}, B)$ *of dimension* N *is* diagonalizable *if for every* $x \in \Sigma$ *the matrix* $M_B(x)$ *has only real positive eigenvalues with multiplicity* 1.

Denote by $\lambda_1(x) < \cdots < \lambda_N(x)$ *the eigenvalues of* $M_B(x)$ *and by* $E_i(x)$ *the one dimensional eigenspace corresponding to* $\lambda_i(x)$*. Then* $\mathcal{E} = \bigoplus_1^N E_i$ *is an invariant splitting of* B.

The next result says that every periodic system with transitions can be approximated by a diagonalizable subsystem (defined on a dense subset). More precisely,

Lemma 4.16 *Let* $(\Sigma, f, \mathcal{E}, A)$ *be a periodic linear system with transitions. Then for any* $\varepsilon > 0$ *there is a diagonalizable* $\varepsilon-$*perturbation* $\tilde{A}$ *of* A *defined on a dense invariant subset* $\tilde{\Sigma}$ *of* Σ.

Proof: Observe first that any matrix $M \in GL(n, \mathbb{R})$ can be perturbed (by an arbitrarily small perturbation) to get $\tilde{M}$ having only eigenvalues (complex or real) with multiplicity one and such that any pair of eigenvalues with the same modulus are complex and conjugate, having rational argument. So there is k such that $\tilde{M}^k$ has only real positive eigenvalues, but some of them may have multiplicity 2 (those coming from a complex eigenvalue of $\tilde{M}$). As above, by a small perturbation of $\tilde{M}^k$, we get a matrix M_1 having only real positive eigenvalues with multiplicity 1.

Take $x \in \Sigma$ and write $M = M_A(x)$. By the comments above, there is an $\varepsilon/10-$perturbation $\tilde{M}$ of M having only multiplicity one eigenvalues. Denote by M_1 the perturbation of an appropriate power of $\tilde{M}$ as the one built above (i.e having only real eigenvalues with multiplicity one). Let T be an $\varepsilon/10-$transition matrix from x into itself. So for any positive n_1 and n_2 there are $y \in \Sigma$ and a $3\varepsilon/10-$perturbation $\tilde{A}$ of A along the orbit of y such that

$$M_{\tilde{A}}(y) = M_1^{n_2} \circ T \circ M_1^{n_1}.$$

So, taking n_1 and n_2 sufficiently big, we can now repeat the proof of Lemma 4.13 to get a $4\varepsilon/10-$transition T_1 preserving all the eigenspaces of M_1. These spaces are $1-$dimensional, thus they are the eigenspaces of T_1 and their corresponding eigenvalues are real.

Recall (see Remark 1.7) that if T_1 is a transition from x into itself then T_1^2 is also a transition. Therefore, replacing if necessary T_1 by T_1^2, we can assume that T_1 has only real positive eigenvalues with multiplicity one. Observe that T_1 and M_1 have the same one dimensional eigenspaces, thus T_1 commutes with M_1. So, for k big enough, $M_1^k \circ T_1$ has only positive real eigenvalues with multiplicity 1. Moreover, there are some point z (whose orbit passes arbitrarily close to x) and an $\varepsilon/2-$perturbation $\hat{A}$ of A along the orbit of z such that

$$M_{\hat{A}}(z) = M_1^k \circ T_1.$$

This proves that the set $\tilde{\Sigma}$ of points $z \in \Sigma$ admitting an $\varepsilon-$perturbation along their orbits such that $M_{\hat{A}}(z)$ has only positive real eigenvalues with multiplicity 1 is dense in Σ, ending the proof of the lemma. $\square$

In what follows, to prepare the proof of Theorem 4, we will give some volume controlling versions of our lemmas, as we do in the next remark:

Remark 4.17 *Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system with transitions such that there is some point $x_0 \in \Sigma$ such that the modulus of the jacobian of the matrix $M_A(x_0)$, denoted by $J(M_A(x_0))$, is greater than one. Then we can choose the diagonalizable $\varepsilon-$perturbation $\tilde{A}$ of A given by Lemma 4.16 such that the modulus of the jacobian $J(M_{\tilde{A}}(p))$ is bigger than one for every p in the dense subset $\tilde{\Sigma}$ of Σ.*

Proof: First, by using the transitions and the existence of the point x_0 with $J(M_A(x_0)) > 1$, we get a dense subset $\hat{\Sigma}$ of Σ such that $J(M_A(x)) > 1$ for all $x \in \hat{\Sigma}$. Now we can repeat the proof of Lemma 4.16 above: Taking the exponents n_1 and n_2 large enough we get that (in the proof of the lemma) the moduli of the jacobian of the corresponding matrices $M_{\tilde{A}}(y)$ are greater than one. □

5 Dominated splittings, complex eigenvalues of rank $(i, i+1)$, and homotheties

In this section we prove Propositions 2.4 and 2.5. To prove Proposition 2.4, we use Lemma 4.4 and consider successive quotients of linear systems in order to get complex eigenvalues (obtained by using the arguments of the 2−dimensional case, see Section 3). Then the transitions and the existence of complex eigenvalues of any rank allow us to "mix all the eigenvalues" of some matrix, obtaining the homothety announced in Proposition 2.5.

5.1 Getting complex eigenvalues of any rank

Lemma 5.1 *Given $K > 0$ and $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that for any diagonalizable linear periodic system $(\Sigma, f, \mathcal{E}, B)$ of dimension N and bounded by K, and any $1 \leq i \leq N-1$ one has:*

- *Either there is an $\varepsilon-$perturbation of B having a complex eigenvalue of rank $(i, i+1)$,*
- *or*
$$E_j/(E_{j+1} \oplus \cdots \oplus E_k) \prec_l E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k)$$
for every $j \leq i \leq k$.

Proof: Fix $\varepsilon > 0$ and let l be the dominance constant given by Proposition 3.1.

If $E_j/(E_{j+1} \oplus \cdots \oplus E_k)$ is $l-$dominated by $E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k)$ we are done. Otherwise, by Proposition 3.1, we can perturb the quotient to get eigenvalues $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} = \alpha$. Moreover, by Lemma 4.1, this perturbation of the quotient gives a perturbation $\tilde{B}$ of B having a pair of eigenvalues $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} = \alpha$ and preserving the eigenvalues of the restriction of B to $E_{j+1} \oplus \cdots \oplus E_k$ (we denote such eigenvalues by λ_i).

Consider a small isotopy B_t producing this perturbation (i.e. $B_0 = B$ and $B_1 = \tilde{B}$) and denote by $\lambda_{j,t}$ and $\lambda_{k+1,t}$ the continuations of the eigenvalues λ_j and λ_{k+1} at time t, so

$$\lambda_{j,0} = \lambda_j, \quad \lambda_{j,1} = \tilde{\lambda}_j, \quad \lambda_{k+1,0} = \lambda_{k+1}, \quad \text{and} \quad \lambda_{k+1,1} = \tilde{\lambda}_{k+1}.$$

Moreover, we can assume that for every $0 \leq t < 1$ one has $\lambda_{j,t} < \lambda_{k+1,t}$ (otherwise we stop the isotopy at the first t such that $\lambda_{j,t} = \lambda_{k+1,t}$). Then, by continuity, following this isotopy there are three possibilities:

- at some stage t of the isotopy, $\lambda_{j,t} = \lambda_{i+1} \leq \lambda_{k+1,t}$,
- at some stage t of the isotopy, $\lambda_{j,t} \leq \lambda_i = \lambda_{k+1,t}$,
- for $t = 1$, $\lambda_i < \lambda_{j,1} = \alpha = \lambda_{k+1,1} < \lambda_{i+1}$.

In each of these cases a small perturbation gives a complex eigenvalue of rank $(i, i+1)$, ending the proof of the lemma. □

To deduce Proposition 2.4 from Lemma 5.1 above we will apply successively Lemma 4.4. Even if the arguments in the proof of the proposition are rather simple and the main difficulty of it is of combinatorial type, for clearness and to organize this combinatorics, we divide the proof into two steps:

Lemma 5.2 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $L_0 \in \mathbb{N}$ such that for every diagonalizable periodic linear system $(\Sigma, f, \mathcal{E}, B)$, of dimension N and bounded by K, and for any $1 \leq i \leq N-1$ one has:*

- *Either B admits an ε−perturbation having a complex eigenvalue of rank $(i, i+1)$*
- *or*

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_{L_0} \bigoplus_{i+1}^{N} E_k/_{\bigoplus_{j+1}^{i} E_m}$$

  for every $j \leq i$.

Before proving the lemma observe that if $j = i$ the second item of the lemma means that $E_i \prec_{L_0} \bigoplus_{i+1}^{N} E_k$.

Proof: Assume that every ε−perturbation of B has no complex eigenvalues of rank $(i, i+1)$. From Lemma 5.1, taking $k = i$ and $k = i+1$ we get l such that for every $j < i$ one has

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_l E_{i+1}/_{\bigoplus_{j+1}^{i} E_m} \quad \text{and} \quad E_j/_{\bigoplus_{j+1}^{i+1} E_m} \prec_l E_{i+2}/_{\bigoplus_{j+1}^{i+1} E_m}.$$

We now apply Lemma 4.4 taking

$$E = E_j/_{\bigoplus_{j+1}^{i} E_m}, \quad F = E_{i+1}/_{\bigoplus_{j+1}^{i} E_m}, \quad \text{and} \quad G = E_{i+2}/_{\bigoplus_{j+1}^{i} E_m}.$$

Observe first that

$$E/F = E_j/_{\bigoplus_{j+1}^{i+1} E_m} \quad \text{and} \quad G/F = E_{i+2}/_{\bigoplus_{j+1}^{i+1} E_m}.$$

By the comments above, $E \prec_l F$ and $E/F \prec_l G/F$. Thus by Lemma 4.4 there is l_1 such that

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_{l_1} (E_{i+1} \oplus E_{i+2})/_{\bigoplus_{j+1}^{i} E_m}.$$

Moreover, this holds for every $j \leq i$. Taking $k = i+2$ in Lemma 5.1, one gets

$$E_j/_{\bigoplus_{j+1}^{i+2} E_m} \prec_l E_{i+3}/_{\bigoplus_{j+1}^{i+2} E_m}.$$

As before, applying Lemma 4.4 to

$$E = E_j/_{\bigoplus_{j+1}^{i} E_m}, \quad F = (E_{i+1} \oplus E_{i+2})/_{\bigoplus_{j+1}^{i} E_m}, \quad \text{and} \quad G = E_{i+3}/_{\bigoplus_{j+1}^{i} E_m},$$

one gets l_2 such that

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_{l_2} (E_{i+1} \oplus E_{i+2} \oplus E_{i+3})/_{\bigoplus_{j+1}^{i} E_m}.$$

The inductive procedure to prove the lemma is now clear. □

Lemma 5.3 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $L \in \mathbb{N}$ such that for every diagonalizable periodic periodic system $(\Sigma, f, \mathcal{E}, B)$, of dimension N and bounded by K, and any $1 \leq i \leq N-1$ one has that:*

- *Either B admits an $\varepsilon-$perturbation having a complex eigenvalue of rank $(i, i+1)$,*
- *or*

$$\bigoplus_{1}^{i} E_j \prec_L \bigoplus_{i+1}^{N} E_j.$$

Proof: Assume that it is not possible to perturb B to get a complex eigenvalue of rank $(i, i+1)$. Once more, the proof is an inductive argument using alternately Lemmas 5.2 and 4.4 above.

By Lemma 5.2, applied to $j = i$ and to $j = i-1$, one knows

$$E_i \prec_{L_0} \bigoplus_{i+1}^{N} E_k \quad \text{and} \quad E_{i-1}/E_i \prec_{L_0} \bigoplus_{i+1}^{N} E_k/E_i.$$

So Lemma 4.4 gives L_1 such that $(E_{i-1} \oplus E_i) \prec_{L_1} \bigoplus_{i+1}^{N} E_k$. By Lemma 5.2, taking $j = i-2$, one gets

$$E_{i-2}/(E_{i-1} \oplus E_i) \prec_{L_0} \bigoplus_{i+1}^{N} E_k/(E_{i-1} \oplus E_i).$$

Thus applying Lemma 4.4 to E_{i-2}, $E_{i-1} \oplus E_i$, and $\bigoplus_{i+1}^{N} E_k$ one obtains

$$(E_{i-2} \oplus E_{i-1} \oplus E_i) \prec_{L_1} \bigoplus_{i+1}^{N} E_k.$$

The lemma now follows by a very simple induction. □

5.2 End of the proof of Proposition 2.4

Recall that Proposition 2.4 says that if there is no perturbation of the system with a complex eigenvalue of rank $(i, i+1)$ then the system has an $l-$dominated splitting of dimension i.

Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic linear system, of dimension N and bounded by K, having transitions. Assume that there are $\varepsilon > 0$ and $i \in \{1, \dots, N-1\}$ such that every $\varepsilon-$perturbation of A has no complex eigenvalues of rank $(i, i+1)$.

Choose a sequence ε_n such that $\varepsilon/2 > \varepsilon_n \to 0$. As the system $(\Sigma, f, \mathcal{E}, A)$ has transitions, using Lemma 4.16 we get dense subsets Σ_n of Σ and diagonalizable ε_n-perturbations B_n of A defined on Σ_n. Note that B_n is a diagonalizable periodic linear system of dimension N bounded by $K_1 = K + \varepsilon/2$. Thus, by hypotheses, it is impossible to create complex eigenvalues of rank $(i, i+1)$ by $(\varepsilon/2)-$perturbations of it.

By Lemma 5.3, there is $L = L(K_1, \varepsilon/2, N)$ such that every system B_n above admits an $L-$dominated splitting $E_n \oplus F_n$, with $E_n \prec_L F_n$ and $\dim(E_n) = i$. Finally, as the system A is continuous, the sets Σ_n are dense in Σ, and $\|B_n - A\| \to 0$, Lemma 1.4 ensures that A admits an $L-$dominated splitting $E \prec_L F$ with $\dim(E) = i$. The proof of the proposition is now complete. $\square$

5.3 Proof of Proposition 2.5

The naive idea of the proof of Proposition 2.5 is to use the transitions to multiply matrices corresponding to different points of Σ having complex eigenvalues of different rank. In this way one distributes homogeneously the action of the eigenvalues in the whole fibers, obtaining homotheties. The main step of the proof is the following lemma:

Lemma 5.4 *Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic linear system of dimension N with transitions. Fix $0 < \varepsilon_0$ and assume that there are $i \in \{1, \dots, N-1\}$ and an ε_0-perturbation of A having a complex eigenvalue of rank $(i, i+1)$.*

Then for every $0 < \varepsilon_1 < \varepsilon_0$ there is a point $p \in \Sigma$ such that for every $1 \leq i < N$ there is an ε_1-transition $[t^i]$ from p to itself with the following properties:

- *There is an ε_1-perturbation $[M]_{\tilde{A}}(p)$ of the word $[M]_A(p)$ such that the corresponding matrix $M_{\tilde{A}}(p)$ has only real positive eigenvalues with multiplicity 1. Denote by $\tilde{\lambda}_1 < \dots < \tilde{\lambda}_N$ such eigenvalues and by $E_i(p)$ their respective (1−dimensional) eigenspaces.*
- *There is an $(\varepsilon_0+\varepsilon_1)-$perturbation $[\tilde{t^i}]$ of the transition $[t^i]$ such that the corresponding matrix $\tilde{T}^i$ satisfies*
  - $\tilde{T}^i(E_j(p)) = E_j(p)$ *if* $j \notin \{i, i+1\}$,
  - $\tilde{T}^i(E_i(p)) = E_{i+1}(p)$ *and* $\tilde{T}^i(E_{i+1}(p) = E_i(p)$.

In fact we have also a stronger (volume controlling) version of this lemma:

Remark 5.5 *Under the hypotheses of Lemma 5.4, suppose in addition that there is a point $x \in \Sigma$ such that the modulus of the jacobian $J(M_A(x))$ is bigger than one. Then we can choose the point p and the perturbation $\tilde{A}$ in the lemma such that $J(M_{\tilde{A}}(p)) > 1$.*

Before proving Lemma 5.4 and Remark 5.5 let us deduce Proposition 2.5 from them.

5.3.1 End of the proof of Proposition 2.5

The hypothesis of the proposition (existence of ε_0-perturbations with complex eigenvalues of any rank) imply that we can apply Lemma 5.4 to the system $(\Sigma, f, \mathcal{E}, A)$ for all $i \in \{1, \dots, N-1\}$. To prove the proposition we show that for any $\varepsilon > \varepsilon_0$ there are an $\varepsilon-$perturbation $\hat{A}$ of A and a point $x \in \Sigma$ for which $M_{\hat{A}}(x)$ is an homothety.

Chosen $0 < \varepsilon_1 < (\varepsilon - \varepsilon_0)/10$, let p and $[t^i]$ be the point and the ε_1-transition from p to itself given by Lemma 5.4.

Observe that the action of the perturbed transition $\tilde{T}^i$ (which are $(\varepsilon_0+\varepsilon_1)-$perturbations of T_i) on the finite set $\{E_i(p)\}_{1\leq j\leq N}$ of eigenspaces of $M_{\tilde{A}}(p)$ is the transposition $(i, i+1)$ which interchanges $E_i(p)$ and $E_{i+1}(p)$, keeping invariant the others $E_j(p)$. Recall that a *transposition* is an order 2 permutation, thus it is equal to its inverse. Moreover, the transpositions $(i, i+1)$, $i \in \{1, \dots, N-1\}$, generate the group of (all) permutations of the finite set $\{E_j(p)\}_{1\leq j\leq N}$.

Given $0 \leq k < N$ denote by σ_k the cyclic permutation defined by $\sigma_k(E_j(p)) = E_{j+k}(p)$, where the sum $i+j$ is considered in the cyclic group $\mathbb{Z}/N\mathbb{Z}$.

As a direct consequence of the previous comments, we have that for every $0 < k < N$ there exists an element $[\tilde{S}_k]$ in the semi-group generated by the transitions $[\tilde{t^i}]$ such that its action on the finite set $\{E_i(p)\}_{1\leq j\leq N}$ is the permutation σ_k, i.e. if $\tilde{S}_k$ is the matrix corresponding to the word $[\tilde{S}_k]$ then one has $\tilde{S}_k(E_i(p)) = \sigma_k(E_i(p))$.

Let $[S_k]$ be the word of matrices corresponding to the perturbation $[\tilde{S}_k]$ in the semi-group generated by the initials $[t^j]$. As the $[t^j]$ are ε_1-transitions from p to itself, any word in the semi-group generated by the $[t^j]$ is also an ε_1-transition (recall Remark 1.7). In particular, the $[S_k]$ are ε_1-transitions from p to itself. (For completeness let us write $[S_0] = [S_N]$ the empty word whose corresponding matrix is by convention the identity).

By definition of transitions, for any $n \in \mathbb{N}$ there is a point $x_n \in \Sigma$ such that the word $[M]_A(x_n)$ is ε_1-close to the word $[W_n]$ defined by

$$\begin{aligned} [W_n] = \quad & [S_1]\,[M]^n_A(p)\,[S_{N-1}]\,[S_1]\,[S_{N-1}]\,[S_2]\,[M]^n_A(p)\,[S_{N-2}]\,[S_2]\,[S_{N-2}]\cdots \\ & \cdots[S_{N-2}]\,[M]^n_A(p)\,[S_2]\,[S_{N-2}]\,[S_2]\,[S_{N-1}]\,[M]^n_A(p)\,[S_1]\,[S_{N-1}]\,[S_1]\,[M]^n_A(p). \end{aligned}$$

Let us state some properties justifying the introduction of this word:

i) Given any i, the matrix $\tilde{S}_{N-i} \circ \tilde{S}_i$ acts trivially on the set of spaces $\{E_j(p)\}$. Let us denote by $\mu_{i,j}$ the eigenvalue of $\tilde{S}_{N-i} \circ \tilde{S}_i$ corresponding to the eigenspace $E_j(p)$.

ii) Recall that $\tilde{S}_i$ maps $E_j(p)$ into $E_{i+j}(p)$ and that $M_{\tilde{A}}(p)$ is diagonal in the basis corresponding to the directions $E_k(p)$, denote by $\tilde{\lambda}_k$ the eigenvalue of $M_{\tilde{A}}(p)$ corresponding to such a direction. Hence every $E_j(p)$ is an eigenspace of $\tilde{S}_{N-i} \circ (M_{\tilde{A}}(p))^n \circ \tilde{S}_i$ whose corresponding eigenvalue is $\mu_{i,j}\,\tilde{\lambda}^n_{j+i}$.

iii) By the two items before, for every j the space $E_j(p)$ is an eigenspace of the matrix

$$\tilde{W}_{i,n} = \tilde{S}_{N-i} \circ (M_{\tilde{A}}(p))^n \circ \tilde{S}_i \circ \tilde{S}_{N-i} \circ \tilde{S}_i$$

whose corresponding eigenvalue is $\mu^2_{i,j}\,\tilde{\lambda}^n_{j+i}$. (Recall that $\tilde{W}_{0,n} = M_{\tilde{A}}(p)^n$ by convention).

Proposition 2.5 is now an immediate consequence of the following claim:

Claim *For every $n > n_0$ sufficiently large there is an $\varepsilon-$perturbation $\hat{A}$ of A along the orbit of x_n such that $M_{\hat{A}}(x_n)$ is a homothety of ratio $\tilde{\Lambda}^n$, where*

$$\tilde{\Lambda} = \prod_1^N \tilde{\lambda}_i = J(M_{\tilde{A}}(p)).$$

Proof: Denote by $[\tilde{W}_n]$ the word obtained from $[W_n]$ by putting a $\tilde{}$ above any letter S and A; this word is an $(\varepsilon_0 + \varepsilon_1)-$perturbation of the word $[W_n]$, so it is an $(\varepsilon_0 + 2\,\varepsilon_1)-$perturbation of the word $[M]_A(x_n)$. Moreover, using the notation in the item (iii) above, the corresponding matrix $\tilde{W}_n$ is the product

$$\tilde{W}_n = \tilde{W}_{N-1,n} \circ \cdots \circ \tilde{W}_{1,n} \circ \tilde{W}_{0,n}.$$

So, by item (iii) above, for every j, the one-dimensional space $E_j(p)$ is an eigenspace of $\tilde{W}_n$ and its corresponding eigenvalue $\tilde{\lambda}_{j,n}$ is

$$\tilde{\lambda}_{j,n} = \prod_{i=0}^{N-1} \tilde{\lambda}_{j+i}^n \prod_{i=0}^{N-1} \mu_{i,j}^2 = \prod_{i=1}^{N} \tilde{\lambda}_i^n \prod_{i=0}^{N-1} \mu_{i,j}^2.$$

Writing

$$C_j = \prod_{i=0}^{N-1} \mu_{i,j}^2 > 0 \quad \text{and} \quad \tilde{\Lambda} = \prod_1^N \tilde{\lambda}_i,$$

we get that, for any $n \in \mathbb{N}$, the eigenvalue $\tilde{\lambda}_{j,n}$ is

$$\tilde{\lambda}_{j,n} = C_j\, \tilde{\Lambda}^n.$$

This means that the matrix $\tilde{W}_n$ is the product of a homothety $(\tilde{\Lambda}^n \cdot Id)$ with a matrix B which does not depend on n and let invariant every one-dimensional space $E_i(p)$. So the matrix B commutes with every $\tilde{W}_{i,n}$. Finally, by construction, all the eigenvalues of B are positive.

Denote by

$$C_{n,j} = (C_j)^{-\frac{1}{n}}.$$

Clearly, when n becomes very large the $C_{n,j}$ are arbitrarily close to 1. Consider the matrix B_n having the $E_j(p)$ as eigenspaces and the $C_{n,j}$ as the corresponding eigenvalues. Denote by $[M]_{\hat{A}}(p)$ the word obtained from $[M]_{\tilde{A}}(p)$ by replacing its first letter $\tilde{A}(p)$ (at the right) by $\tilde{A}(p) \circ B_n$. For n large enough this new word is an ε_1-perturbation of $[M]_{\tilde{A}}(p)$, so by item (i) of Lemma 5.4 it is also a $2\,\varepsilon_1-$perturbation of $[M]_A(p)$. Now, the matrix corresponding to $[M]_{\hat{A}}(p)$ is $M_{\tilde{A}}(p) \circ B_n$. As B_n commutes with $M_{\tilde{A}}(p)$, and recalling the definitions of B_n and $C_{n,j}$, we get that

$$(M_{\tilde{A}}(p) \circ B_n)^n = M_{\tilde{A}}^n(p) \circ B^{-1}.$$

As a conclusion, the word $[\hat{W}_n]$ obtained by changing the initial subword $[M]_{\tilde{A}}^n(p)$ of $\tilde{W}_n$ by $[M]_{\hat{A}}(p)$ is $(\varepsilon_0 + 2\,\varepsilon_1) < \varepsilon$ close to the word $[M]_A(x_n)$, and its corresponding matrix $\hat{W}_n = \tilde{W}_n \circ B^{-1} = \tilde{\Lambda}^n \cdot Id$ is a homothety. This ends the proof of the claim. □

After reading carefully the proof before, one has the following remark which will play a key role to control the volume in the next section.

Remark 5.6 *Under the hypotheses of Proposition 2.5, suppose in addition that there is* $x_0 \in \Sigma$ *such that the modulus of the jacobian* $J(M_A(x_0))$ *is bigger than one. Then we can choose the perturbation* $\tilde{A}$ *of* A *and the point* $x \in \Sigma$ *in the proposition such that* $M_{\tilde{A}}(x)$ *is a homothety with ratio of modulus bigger than one.*

Proof: It is enough to repeat the proof of Proposition 2.5 bearing in mind the volume controlling Remark 5.5: So the point p and the first perturbation $\tilde{A}$ can be chosen such that $1 < J(M_{\tilde{A}}(p)) = \tilde{\Lambda}$. To conclude the proof, it is now enough to recall that (with the notation of the proof of Proposition 2.5) $J(M_{\hat{A}}(x_n)) = \tilde{\Lambda}^n$ (see the claim in this proof). $\square$

To end the proof of Proposition 2.5 it remains to prove Lemma 5.4. This is done in the next section.

5.3.2 Proof of Lemma 5.4 (and Remark 5.5)

The proof of Lemma 5.4 follows from the ideas of the proof of Lemma 4.13, based on the following fact: Given a vector v, a pair of matrices T and M, and the eigenvector w associated to the largest (in modulus) eigenvalue of M, it is very easy to map v into w by an arbitrarily small perturbation of $M^n \circ T$, if n is large enough. So, using this fact, given a dominated splitting, a simple inductive argument allows us to get transitions preserving it. The difficulty here is that we want to get a transition interchanging two spaces of a dominated splitting (in our case the eigenspaces of a diagonal matrix). For that we will use the complex eigenvalues, which enable us to map an arbitrary vector into the eigenvector corresponding to the weaker eigenvalue. Let us explain all that in details.

Fix any $0 < \varepsilon_1 < \varepsilon_0$. We now build some $\varepsilon_1/10-$perturbation $\tilde{A}$ of A, modifying the initial system along a finite number of orbits. In the next paragraphs we describe this perturbation along each orbit.

First, by Lemma 4.16, there are a point $p \in \Sigma$ and a perturbation $\tilde{A}$ of A along the orbit of p such that the corresponding matrix $M_{\tilde{A}}(p)$ is diagonalizable and has only positive real eigenvalues with multiplicity 1. Denote by $\lambda_1 < \cdots < \lambda_N$ such eigenvalues and by $E_j(p)$ the corresponding eigenspaces. Moreover, by Remark 4.17, if the linear system A satisfies the hypothesis of Remark 5.5, i.e. existence of some point with jacobian greater than one, one can choose the point p and the perturbation $\tilde{A}$ such that $J(M_{\tilde{A}}(p)) = \prod_{i=1}^{N} \lambda_i$ is strictly bigger than one.

By hypotheses, there is a point $p_i \in \Sigma$ (whose orbit is disjoint from the one of p) and an ε_0-perturbation $\tilde{A}$ (along the orbit of p_i) such that the matrix $M_{\tilde{A}}(p_i)$ has a complex eigenvalue of rank $(i, i+1)$. Now we fix $\varepsilon_1/10-$transitions $[t_{i,0}]$ and $[t_{0,i}]$ from p to p_i and from p_i to p.

To simplify the notations, we write $[M] = [M]_A(p)$, $[M_i] = [M]_A(p_i)$, and M and M_i for the corresponding matrices. We also write $[\tilde{M}]$, $[\tilde{M}_i]$, $\tilde{M}$, and $\tilde{M}_i$ for the corresponding perturbations of the words and matrices.

Observe that, once more by Remark 1.7, for every i and positive n_1, n_2, and n_3 the word

$$[t_i(n_1, n_2, n_3)] = ([M])^{n_3}[t_{0,i}]([M_i])^{n_2}[t_{i,0}][M]^{n_1}$$

is also an $\varepsilon_1/10-$transition from p to itself of the system A.

The proof of Lemma 5.4 (and of Remark 5.5) now follows immediately from the next result:

Lemma 5.7 *There are* $n_1 \geq 0$, $n_2 \geq 0$, *and* $n_3 \geq 0$ *such that the word* $[t_i(n_1, n_2, n_3)]$ *defined above admits an* $(\varepsilon_0+\varepsilon_1)-$*perturbation such that the corresponding matrix* $\tilde{T}_i(n_1, n_2, n_3)$ *let invariant every* E_j, $j \notin \{i, i+1\}$, *and interchanges* E_i *and* E_{i+1}.

Proof: For a fixed $i \in \{1, \ldots, N\}$ consider the ε_0-perturbation $\tilde{A}$ of A along the orbit of p_i constructed above and denote by $E(p_i) \oplus F(p_i) \oplus G(p_i)$ its invariant splitting (over the orbit of p_i) where $F(p_i)$ is the $2-$dimensional eigenspace corresponding to the complex (conjugate) eigenvalues of $\tilde{M}_i$ and $E(p_i) \prec F(p_i) \prec G(p_i)$. Since $M_{\tilde{A}}(p)$ is diagonalizable we have the splitting

$$E(p) = \bigoplus_{1}^{i-1} E_j(p), \quad F(p) = E_i(p) \oplus E_{i+1}(p), \quad \text{and} \quad G(p) = \bigoplus_{i+1}^{N} E_j(p),$$

where $E_j(p)$ is the eigenspace associated to λ_j.

As in Lemma 4.13, replacing, if necessary, the transitions $[t_{0,i}]$ and $[t_{i,0}]$ by words of the form $[M]^n[t_{0,i}][M_i]^n$ and $[M_i]^n[t_{i,0}][M]^n$ for some big n, we can assume that $[t_{i,0}]$ admits a $(\varepsilon_0+\varepsilon_1/10)-$perturbation $[\tilde{t}_{i,0}]$ such that the corresponding matrix $\tilde{T}_{i,0}$ maps the splitting $E(p) \prec F(p) \prec G(p)$ into $E(p_i) \prec F(p_i) \prec G(p_i)$. Conversely, we can also suppose that the matrix $\tilde{T}_{0,i}$ of the $(\varepsilon_0+\varepsilon_1/10)-$perturbation $[\tilde{t}_{0,i}]$ of $[t_{i,0}]$ maps $E(p_i) \prec F(p_i) \prec G(p_i)$ into $E(p) \prec F(p) \prec G(p)$.

Our next objective is to get two different one-dimensional subspaces of $F(p_i)$ and a perturbation of $\tilde{M}_i$ interchanging such subspaces. For that write

$$E_i(p_i) = \tilde{T}_{0,i}^{-1}(E_i(p)) \quad \text{and} \quad E_{i+1}(p_i) = \tilde{T}_{i,0}(E_{i+1}(p)).$$

As $E_i(p)$ is a subspace of $F(p)$, $E_i(p_i)$ is a (non-invariant!) subspace of $F(p_i)$. The same argument shows that $E_{i+1}(p_i)$ is also a non-invariant subspace of $F(p_i)$.

Recall that $\tilde{M}_i$ has a pair of complex (non-real) eigenvalues whose eigenspace is $F(p_i)$. So it is an exercise to get $m > 0$ and an $\varepsilon_1/10-$perturbation $\hat{M}_i$ of $\tilde{M}_i$, preserving the splitting $E(p_i) \oplus F(p_i) \oplus G(p_i)$, such that $\hat{M}_i^m(E_{i+1}(p_i)) = E_i(p_i)$.

Consider now the linear map

$$B_0 = \tilde{T}_{0,i} \circ \hat{M}_i^m \circ \tilde{T}_{i,0}: \mathcal{E}_p \to \mathcal{E}_p,$$

recall that $\mathcal{E}_p$ denotes the fiber of p. By construction

- **i)** B_0 preserves the splitting $E(p) \oplus F(p) \oplus G(p)$,
- **ii)** $B_0(E_{i+1}(p)) = E_i(p)$.
- **iii)** $B_0(E_i(p))$ is a straightline (in the plane $F(p)$) different from $E_i(p)$ and so transverse to $E_i(p)$

Observe that these three properties also hold for $\tilde{M}^k \circ B_0$ for every $k > 0$.

Recalling that $E_i(p) \oplus E_{i+1}(p)$ is a dominated splitting over the orbit of p (for the perturbated system whose matrix is $\tilde{M}$), using the transversality in item (iii), we have that if $k > 0$ is large enough then $\tilde{M}^k(E_i(p))$ is arbitrarily close to $E_{i+1}(p)$. So we can choose $k > 0$ and an $\varepsilon_1/10-$perturbation $\hat{M}$ of $\tilde{M}$ such that $B_1 = \hat{M} \circ \tilde{M}^k \circ B_0$ satisfies the following two properties:

- B_1 preserves the splitting $E(p) \oplus F(p) \oplus G(p)$,
- $B_1(E_i(p)) = E_{i+1}(p)$ and $B_1(E_{i+1}(p)) = B_1(E_i(p)$.

Note that these properties of B_1 are also verified by every map of the form $\tilde{M}^{k_1} \circ B_1 \circ \tilde{M}^{k_2}$ (k_1 and $k_2 > 0$). Applying the arguments of the proof of Lemma 4.13 to the restrictions of B_1 to $E(p)$ and $G(p)$ we get k_1 and $k_2 > 0$, and $\varepsilon_1/10-$perturbations $[N_1]$ of the word $[\tilde{M}]^{k_1}$, and $[N_2]$ of $[\tilde{M}]^{k_2}$, coinciding with $[\tilde{M}]^{k_1}$ and $[\tilde{M}]^{k_2}$ on $F(p)$, such that

$$N_1 \circ B_1 \circ N_2(E_j(p)) = E_j(p) \quad \text{for every } j \notin \{i, i+1\}.$$

To finish the proof of the lemma it suffices to observe that, by construction, the matrix $N_1 \circ B_1 \circ N_2$ corresponds to a word which is a $(\varepsilon_0 + 3\,\varepsilon_1/10)-$perturbation $(\varepsilon_0 + 3\varepsilon_1/10 < \varepsilon_0 + \varepsilon_1)$ of

$$[\tilde{M}]^{k_1}\,[\tilde{M}]\cdots[\tilde{M}]\,[t_{0,i}]\,[\tilde{M}_i]^k\,[t_{i,0}]\,[\tilde{M}]\cdots[\tilde{M}]\,[\tilde{M}]^{k_2}.$$

Thus this word is an ε_1-perturbation of $[t_i(n_1, n_2, n_3)]$ for some n_1, n_2, and n_3. Now the proof of the lemma is complete. □

6 Finest dominated splitting and control of the jacobian in the extremal bundles: Proof of Theorem 4

6.1 Control of jacobian over periodic points

Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Suppose that $F_1 \oplus F_2 \oplus \cdots \oplus F_{k-1} \oplus F_k$, $F_1 \prec F_2 \prec \cdots \prec F_{k-1} \prec F_k$, is the finest dominated splitting of this system. We call F_1 and F_k *extremal bundles of the dominated splitting.* Denote by A_i the restriction of A to the subbundle F_i. The goal of this section is to prove some estimates on the determinant of A_i. Let us begin with the following result:

Lemma 6.1 *For any $K > 0, N \in \mathbb{N}, L \in \mathbb{N}$, and $\varepsilon > 0$ there is $l > 0$ with the following property:*

Consider a periodic linear system $(\Sigma, f, \mathcal{E}, A)$, of dimension N and bounded by K, having an $L-$dominated splitting $E \oplus F$ such that

- *the subbundle E does not admit any nontrivial $l-$dominated splitting, and*
- *there is a point $p \in \Sigma$ such that $\det(M_{A_E}(p)) > 1$.*

Then there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that all the eigenvalues of $M_{\tilde{A}}(x)$ have modulus strictly bigger than 1.

Clearly, there is a version of this lemma where the bundle F has no $l-$dominated splitting and $\det(M_{A_F}(p)) < 1$, then the moduli of the eigenvalues of the perturbation $\tilde{A}$ are strictly smaller than 1.

We will apply twice this lemma to the finest dominated splitting of a system, first taking $E = F_1$ and $F = F_2 \oplus \cdots \oplus F_k$, and second taking $E = F_1 \oplus \cdots \oplus F_{k-1}$ and $F = F_k$. Let us now prove the lemma.

Proof: Note first that there is $\delta_L > 0$ such that, for every system C δ_L-close to A, we can define the continuation $E_C \oplus F_C$ of the splitting $E \oplus F$, which is also dominated (recall the comments after Definition 0.1).

Let $\tau = \inf\{\varepsilon, \delta_L\}$. Consider the $l_0 > 0$ given by Proposition 2.4 associated to $\tau/2$, and fix $l = 2\,l_0$.

Now take a system $(\Sigma, f, \mathcal{E}, A)$ satisfying the hypotheses of the lemma. ¿From Corollary 4.14, the system A_E induced by A on the subbundle E admits transitions and, by hypothesis, does not admit any $l-$dominated splitting. In particular, by Proposition 2.4, for every $0 \leq i < \dim(E)$, there is a $\tau/2-$perturbation B_i of A_E having a complex eigenvalue of rank $(i, i+1)$, i.e. there is $x_i \in \Sigma$ such that $M_{B_i}(x_i)$ has a complex eigenvalue of rank $(i, i+1)$.

Moreover, by hypothesis, there is a point $p \in \Sigma$ for which the modulus of the jacobian $J(M_{A_E})(p)$ is strictly bigger than one.

The previous comments mean that taking $\varepsilon_0 = \tau/2$, the system (Σ, f, E, A_E) satisfies all the hypotheses of Proposition 2.5 and Remark 5.6. So there is a $\tau-$perturbation $\tilde{A}_E$ of A_E and a point $x \in \Sigma$ such that $M_{\tilde{A}_E}(x)$ is a homothety of ratio bigger than one.

Finally, by Lemma 4.1, $\tilde{A}_E$ is the restriction to E of some $\tau-$perturbation $\tilde{A}$ of A which coincides with A over F. Since the splitting $E \oplus F$ is dominated for $\tilde{A}$, we get that all the eigenvalues of $M_{\tilde{A}}(x)$ associated to F have necessarily modulus bigger than one. This ends the proof of the lemma. $\square$

Theorem 4 follows from the next proposition that is a consequence of the Ergodic Closing Lemma in [Ma3] and whose proof (very similar to Mañé's argument in [Ma3] to get hyperbolicity) we postpone until the next subsection:

Proposition 6.2 *Let f be a diffeomorphism and $\Lambda_f(U)$ an $f-$invariant compact set which is maximal invariant in some neighbourhood U of it. Suppose that $E \oplus F$, $E \prec F$, is a dominated splitting of $T_{\Lambda_f(U)}M$ for f_*.*

By shrinking U, if necessary, there is a C^1-neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the maximal invariant set $\Lambda_g(U)$ of $g \in \mathcal{U}$ has a dominated splitting $E_g \oplus F_g$ which is the continuation of $E \oplus F$. Then one has that

- *either there are an arbitrarily small C^1-perturbation g of f and a hyperbolic periodic point $p \in \Lambda_g(U)$ such that g_* expands the volume on $E_g(p)$,*
- *or $f_*|_E$ contracts uniformly the volume.*

Let us now end the proof of Theorem 4.

Proof of Theorem 4: Let $\Lambda_f(U)$ be a (non-trivial) robustly transitive set. Denote by N the dimension of the ambient manifold and let K be a strict upper bound of the norms of f_* and f_*^{-1}. Take $\varepsilon > 0$ such that for every g $\varepsilon - C^1-$close to f the set $\Lambda_g(U)$ is transitive and the norms of g_* and g_*^{-1} are bounded by K.

The proof is by contradiction. Let $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ be the finest dominated splitting of f_*. Write $E = F_1$ and $F = F_2 \oplus \cdots \oplus F_k$, and fix L such that the splitting $E \oplus F$ is $2L-$dominated. Now let $l > 0$ be the constant associated to K, N, $2L$, and $\varepsilon/2$ in Lemma 6.1.

If $f_*|E$ does not contract uniformly the volume in E then, by Proposition 6.2, there are g arbitrarily close to f and a hyperbolic periodic point $p \in \Lambda_g(U)$ such that g_* expands the volume in $E_g(p)$. Since, by hypothesis, $\Lambda_g(U)$ is C^1-robustly transitive, using Lemma 2.7 we can assume (after a perturbation) that the relative homoclinic class $H(p, g, U)$ of p is the whole $\Lambda_g(U)$.

Consider the dense subset $\Sigma \subset \Lambda_g(U)$ consisting of all the hyperbolic periodic points of $\Lambda_g(U)$ homoclinically related to p. Then g induces the periodic linear system $(\Sigma, g, T_\Sigma M, g_*)$, of dimension N and bounded by K, with transitions, see Lemma 1.9.

Moreover, if g is close enough to f, $E(g) \oplus F(g)$ is an $L-$dominated splitting of this system and the bundle $E(g)$ does not admit any $l-$dominated splitting (this last assertion follows from Lemma 1.4). So applying Lemma 6.1, we get an $\varepsilon/2-$perturbation B of $(\Sigma, g, T_\Sigma M, g_*)$ and a periodic point $q \in \Sigma$ such that all the eigenvalues of $M_B(q)$ are bigger than one (in modulus). Using Franks' Lemma we get that the (non-trivial) maximal invariant set in U of some $\varepsilon-$perturbation g of f contains a repeller (precisely the point q). This contradicts the choice of ε (robust transitivity of $\Lambda_g(U)$).

To end the proof of the theorem it remains to get the uniform expansion of the volume in F_k, this follows as above by replacing f by f^{-1}. □

6.2 Mañé's Ergodic Closing Lemma: Proof of Proposition 6.2

Definition 6.3 *Let f be a diffeomorphism defined on a compact manifold M endowed with a Riemannian metric d. A point x is* well closable *(for f) if for every $\varepsilon > 0$ there are g $\varepsilon - C^1-$close to f and a periodic point y of g such that $d(f^i(x), g^i(y)) < \varepsilon$ for every $0 \leq i < k(y)$, where $k(y)$ is the period of y. We denote by $\mathcal{W}(f)$ the set of well closable points of f.*

We have the following result (see [Ma$_3$]),

Theorem (Ergodic Closing Lemma) *Let f be a diffeomorphism and μ an $f-$invariant probability. Then $\mu-$almost every point is well closable, i.e. $\mu(\mathcal{W}(f)) = 1$.*

Suppose now that $\Lambda_f(U)$ is a locally maximal set in a neighborhood U of it and that $E \oplus F$, $E \prec F$, is a dominated splitting of f_* over $T_{\Lambda_f(U)}M$. Recall that the bundle E is continuous (see Lemma 1.4), thus $(\Lambda_f(U), f, E, f_*|_E)$ is a continuous linear system.

Since E is endowed with a continuous Euclidian metric, we can define the modulus of the determinant of $f_*|_E$, the *jacobian of f on E,* denoted by $|J(f, E)|\colon \Lambda_f(U) \to \mathbb{R}$, which is a continuous (and so integrable) positive function. Thus $\log(|J(f, E)|)\colon \Lambda_f(U) \to \mathbb{R}$ is well defined and continuous. Moreover, by shrinking U, if necessary, we have that for every g close enough to f there is defined a dominated continuation $E_g \oplus F_g$ of the splitting $E \oplus F$. Thus we can define the function $\log(|J(g, E_g)|)$ depending continuously on g (observe that the subbundles E_g and F_g depend continuously on g).

The first step to prove Proposition 6.2 is the following lemma.

Lemma 6.4 *Assume there is an $f-$invariant probability measure μ supported on $\Lambda_f(U)$ such that*

$$\int \log(|J(f, E)|)\, d\mu \geq 0.$$

Then there are g arbitrarily C^1-close to f and a periodic orbit $y \in \Lambda_g(U)$ of g where g_* expands the volume of E_g, i.e.

$$|J(g^k, E_g)(y)| > 1, \quad \textit{where } k \textit{ is the period of } y.$$

Proof: Observe first that we can assume that μ is ergodic: Otherwise it is enough to consider the decomposition of μ into ergodic invariant measures, then for at least one of them the integral of $\log(|J(f, E)|)$ is also positive.

By the Ergodic Closing Lemma, there is a $\mu-$generic point x which is well closable. If x is periodic we have nothing to do. In the other case, there are sequences of diffeomorphisms g_n converging to f in the C^1-topology, of periodic points y_n (of period k_n) of g_n, and of numbers $\varepsilon_n \to 0$, such that

$$d(f^i(x), g_n^i(y_n)) < \varepsilon_n, \quad \text{for every } 0 \leq i < k_n - 1.$$

In particular, if ε_n is small enough, this ensures that the point y_n belongs to $\Lambda_{g_n}(U)$. Moreover,

$$\frac{1}{k_n} \sum_0^{k_n-1} \log(|J(g_n, E_{g_n}|)(g_n^i(y_n)) \to \int \log(|J(f, E)|)\, d\mu \geq 0.$$

So given any $\delta > 1$, taking n large enough, we have that the linear system defined over the orbit of y_n obtained by multiplying $(g_n)_*|E_{g_n}$ by the scalar δ expands the volume on $E_{g_n}(y_n)$. The conclusion in the lemma now follows immediately using the Franks' Lemma (see the very beginning of Section 1). □

Using Lemma 6.4 we have that Proposition 6.2 (thus Theorem 4) is a direct consequence of:

Lemma 6.5 *Let $E \oplus F$, $E \prec F$, be a dominated splitting of f over $\Lambda_f(U)$. Assume that for any $N \in \mathbb{N}$ the jacobian $|J(f^N, E)|$ is not bounded from above by one uniformly. Then there is an $f-$invariant measure μ such that*

$$\int \log(|J(f, E)|)\, d\mu \geq 0.$$

Proof: By hypothesis, given any $N > 0$ there exists some point $x_N \in \Lambda_f(U)$ such that $|J(f^N, E)(x)| \geq 1$. Write

$$\mu_N = \frac{1}{N} \sum_0^{N-1} \delta(f^i(x_N)),$$

where $\delta(z)$ is the Dirac measure at the point z. As the space of probabilities is compact for the weak topology, there is a subsequence N_i such that μ_{N_i} converges weakly to some probability measure μ.

A classical elementary argument proves that μ is $f-$invariant: $f_*(\mu) - \mu$ is the weak limit of $\frac{1}{N_i}(\delta(f^{N_i}(x_{N_i})) - \delta(x_{N_i}))$, which converges to 0. Finally, observing that

$$\int \log(|J(f, E)|)\, d\mu_N = \frac{1}{N} \sum_0^{N-1} \log(|J(f, E)(f^i(x_N))|) = \frac{1}{N} \log(|J(f^N, E)(x_N)|) \geq 0,$$

one deduces immediately that $\int \log(|J(f,E)|\,d\mu \geq 0$. $\square$

7 The conservative case

In this section we translate some of our constructions into the conservative context (volume preserving C^1-diffeomorphisms).

In what follows the compact manifold M is endowed with a smooth volume form ω, and we denote by $\mathrm{Diff}^1_\omega(M)$ the set of C^1-diffeomorphisms preserving this volume form ω endowed with the usual C^1-topology. Following the traditional terminology, a *conservative diffeomorphism* is an element of $\mathrm{Diff}^1_\omega(M)$. Here we only consider manifolds of dimension N strictly bigger than one.

Observe that, given a linear system $(\Sigma, f, \mathcal{E}, A)$, using the Euclidian metric on the bundle $\mathcal{E}$, we can define the modulus of the determinant of the linear maps $A(x)$, denoted by $J_A(x) = |\det A(x)|$.

Definition 7.1 *A periodic linear system* $(\Sigma, f, \mathcal{E}, A)$ *is* conservative *if* $J_A(x) = 1$ *for all* $x \in \Sigma$.

Remark 7.2 *Given a linear system* $(\Sigma, f, \mathcal{E}, A)$ *of dimension* N *we define its* conservative part A^c *by*

$$A^c(x) = J_A(x)^{\frac{-1}{N}} \cdot A(x).$$

Clearly, if A *is conservative then* $A = A^c$. *The map* $A \mapsto A^c$ *is continuous. Therefore for any* $K > 0$ *and* $\varepsilon > 0$ *there is* $\varepsilon_1 > 0$ *such that if* A *is a conservative system bounded by* K *and* B *is a (*a priori *nonconservative)* ε_1-*perturbation of it, then* B^c *is* $\varepsilon-$*close to* A.

Finally, if the initial system A *is continuous, periodic, and with transitions, then the same holds for its conservative part* A^c.

We begin with a straightforward corollary of Proposition 2.1:

Proposition 7.3 *For any* $K > 0$, $N > 0$, *and* $\varepsilon > 0$ *there is* $l > 0$ *such that for every continuous periodic conservative linear system* $(\Sigma, f, \mathcal{E}, A)$, *of dimension* N *and bounded by* K, *with transitions one has that,*

- *either* A *admits an* $l-$*dominated splitting,*
- *or there is a conservative* $\varepsilon-$*perturbation* $\tilde{A}$ *of* A *such that* $M_{\tilde{A}}(x)$ *is the identity for some* $x \in \Sigma$.

Proof: According to Proposition 2.1, there is l with the following property:

Let A be a continuous periodic conservative linear system, of dimension N and bounded by K, and admitting transitions. If A does not admit any $l-$dominated splitting then there is an ε_1-perturbation B of A such that the matrix $M_B(x)$ is a homothety for some $x \in \Sigma$. Then, by Remark 7.2, B^c is a conservative $\varepsilon-$perturbation of A and $M_{B^c}(x)$ is a homothety with determinant 1 (because the system is conservative), thus it is the identity. This completes the proof of the proposition. $\square$

To proceed our proofs in the conservative setting we need suitable versions of our perturbations lemmas (Franks' Lemma and Hayashi's Connecting Lemma) for volume preserving diffeomorphisms.

Proposition 7.4 (Conservative Franks' Lemma) *Consider a conservative diffeomorphism f and a finite $f-$invariant set E. Assume that B is a conservative $\varepsilon-$perturbation of f_* along E. Then for every neighbourhood V of E there is a conservative diffeomorphism g arbitrarily C^1-close to f coinciding with f on E and out of V, and such that g_* is equal to B on E.*

As we did not find any precise reference for this probably well known result, so let us give here the sketch of its proof:

Proof: The proof is based on the following elementary fact of linear algebra:

Lemma 7.5 *For any $N > 1$ and $\varepsilon > 0$ there is a neighbourhood $\mathcal{G}$ of the identity in $SL(N, \mathbb{R})$ such that any matrix $A \in \mathcal{G}$ can be written as a product $B_1 \circ B_2 \circ \cdots \circ B_{4N-4}$, where the $B_i = P_i \circ R_i \circ P_i^{-1}$, P_i and R_i are $\varepsilon-$close to the identity in $SL(N, \mathbb{R})$, and R_i is a rotation.*

Proof: We just give the main steps of the proof. All the matrices we consider will be in $SL(N, \mathbb{R})$. We have the following properties:

- Every matrix A close to identity can be written in the form $L \circ L^{-1} \circ A$, where L is diagonal with real eigenvalues of multiplicity one and close to 1, $L^{-1} \circ A$ is diagonalizable and has eigenvalues close to 1, and the matrix of change of coordinates is close to identity.
- Any diagonal matrix in $SL(N, \mathbb{R})$ can be written as the product of $N - 1$ diagonal matrices whose eigenvalues are all equal to one, except two of them (inverse one to the other).
- Any diagonal matrix D close to the identity, $D \in SL(2, \mathbb{R})$, can be written as $R \circ R^{-1} \circ D$, where R is a rotation, $R^{-1} \circ D$ is conjugate to a rotation by some matrix P, and the matrices R, $R^{-1} \circ D$, and P are close to identity.

The lemma follows immediately from these three properties. □

Observe that Franks' Lemma consists of local perturbations around finitely many points. Taking local charts at these points we can consider that the volume is the Lebesgue measure (Leb), see for instance [Mo]. So the conservative Franks' Lemma follows from the next lemma:

Lemma 7.6 *For every $N \in \mathbb{N}$ and $\varepsilon > 0$ there is a neighbourhood $\mathcal{G}$ of the identity in $SL(N, \mathbb{R})$ such that for every $A \in \mathcal{G}$ there is $h \in \mathrm{Diff}^1_{Leb}(\mathbb{R}^N)$ satisfying the following properties:*

- *h coincides with the identity outside the unit ball at the origin,*
- *$h(0) = 0$ and $h_*(0) = A$,*

- $\|h_* - Id\| < \varepsilon$.

To prove this lemma it is enough to see that its proof is very easy if A is a rotation or conjugate to a rotation. Since Lemma 7.5 allows us to write A as the product of $4N-4$ of such maps, all them close to identity, the general case follows from the simple first case. $\square$

Exactly as Proposition 2.6 follows from Franks' Lemma and Proposition 2.1, we deduce from Proposition 7.4 (conservative Franks' lemma) and Propositions 7.3 the following conservative version of Proposition 2.6:

Proposition 7.7 *Given any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $l(\varepsilon, K) \in \mathbb{N}$ such that for every conservative diffeomorphism f on a Riemannian $N-$dimensional manifold M, with derivatives f_* and f_*^{-1} bounded by K, and any saddle p of f having a non-trivial homoclinic class $H(p, f)$, one has that:*

- *Either the homoclinic class $H(p, f)$ admits an $l(\varepsilon, K)-$dominated splitting,*
- *or for every neighbourhood U of $H(p, f)$ and $k \in \mathbb{N}$ there are a conservative diffeomorphism g $\varepsilon-C^1-$close to f and k periodic points x_i of g arbitrarily close to p, whose orbits are contained in U, such that the derivatives $g_*^{n_i}(x_i)$ are the identity (n_i is the period of x).*

Observe that this proposition implies Theorem 5 (in fact, this proposition is a quantitative version of this theorem).

7.1 Proof of Theorem 6

The first step to prove the theorem is the following lemma,

Lemma 7.8 *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M)$ of diffeomorphisms f such that the non-trivial homoclinic classes of hyperbolic periodic points of f are dense in M.*

Proof: Let us begin by recalling that, for conservative diffeomorphisms, the recurrent points are dense in M. Moreover, using the C^1-closing lemma in the conservative case (see [Rb]), one has that the conservative diffeomorphisms whose periodic orbits are hyperbolic and dense in the ambient manifold form a dense subset of $\mathrm{Diff}^1_\omega(M)$. Moreover, by the continuity of the hyperbolic periodic orbits, this dense subset is in fact residual.

Suppose that p is a hyperbolic periodic point of a diffeomorphism g such that its periodic points are hyperbolic and dense in the manifold. Then, given fundamental domains D^s and D^u of $W^s(p, g)$ and $W^u(p, g)$, there is a sequence of periodic points q_i converging to some $z \in D^u$ such that $g^{k_i}(q_i) \to y \in D^s$ for some sequence $k_i \to +\infty$. This implies that the invariant manifolds of any periodic point p of g satisfy the hypotheses of the Conservative Connecting Lemma of Xia [X]:

Theorem (Conservative Connecting Lemma) *Let M be a compact manifold, g a conservative diffeomorphism, and p a hyperbolic periodic point of $g \in \mathrm{Diff}^1_\omega(M)$.*

Suppose that there are sequences of points (x_i), $x_i \to z \in W^u(p)$, and integers (n_i) such that $n_i \to \infty$ and $g^{n_i}(x_i) \to y \in W^s(p)$.

Then there is $h \in \mathrm{Diff}^1_\omega(M)$ arbitrarily C^1-close to g such that $W^u(p_h, h)$ intersects transversely $W^s(p_h, h)$ at z and $h^k(z) = y$ for some $k > 0$.

Now a classical argument (see, for instance, the proof of [BD$_2$, Proposition 1.1]) gives that there is a dense open subset $\mathcal{R}_n$ of $\mathrm{Diff}^1_\omega(M)$ of diffeomorphism whose non-trivial homoclinic classes are $\frac{1}{n}-$dense. To end the proof of the lemma it is enough to take $\mathcal{R} = \bigcap_{n>0} \mathcal{R}_n$. □

Now to end the proof of Theorem 6 consider a diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$ and $\varepsilon > 0$ such that there is no $\varepsilon-$perturbation of f having periodic points whose derivative is the identity. Thus, by Proposition 7.7, there is l such that every non-trivial homoclinic class of any conservative $\varepsilon/2-$perturbation g of f admits an $l-$dominated splitting. For such a g we define $\Lambda_i(g)$, $i = 1, \ldots, N-1$, as the closure of union of the non-trivial homoclinic classes with an $l-$dominated splitting $E \prec F$ of dimension i (i.e. $\dim(E) = i$). By Lemma 1.4, this invariant compact set $\Lambda_i(g)$ has an $l-$dominated splitting.

Using Lemma 7.8, we can take a sequence of diffeomorphisms $g_n \in \mathcal{R}$ as above converging to f. Then, for each n, the union of the $\Lambda_i(g_n)$ is the whole manifold. We define $K_i(f)$ as the topological upper limit set of the $\Lambda_i(g_n)$, i.e.

$$K_i(f) = \limsup_{n\to\infty} \Lambda_i(g_n) = \bigcap_k \mathrm{closure}(\bigcup_{n\geq k} \Lambda_i(g_n)).$$

Again by Lemma 1.4, the set $K_i(f)$ admits an $l-$dominated splitting. Finally, by construction, the manifold M is the union of the $K_i(f)$.

The arguments above shows that, if f does not admit any $\varepsilon-$pertubation with a periodic orbit whose derivative is the identity, then M is the union of finitely many invariant compact sets with $l-$dominated splittings. Otherwise, there is a conservative perturbation g of f and a homoclinic class of a periodic point of g whose induced periodic linear system can be perturbed to get one point such that its linear map is the identity. Using the transitions we can get an arbitrarily large number of such points. Now the result follows from the conservative Franks' Lemma (Proposition 7.4). □

7.2 Volume properties of dominated splittings of conservative systems

We end this paper by giving some volume properties of dominated splitting of conservative linear systems.

Proposition 0.5 is a direct consequence of the following lemma:

Lemma 7.9 *Let $(\Sigma, f, \mathcal{E}, A)$ be a conservative linear system with an $l-$dominated splitting $E \oplus F$, $E \prec_l F$. Then*

$$|\det(A^l_E)(x))| \leq \frac{1}{\sqrt{2}} \quad \text{and} \quad |\det(A^l_F)(x))| \geq \sqrt{2},$$

for every $x \in \Sigma$, recall that $A^l(x) = A(f^{l-1}(x)) \circ \cdots \circ A(x)$.

Proof: Since the system is conservative,

$$|\det(A_E^l)(x)\det(A_F^l)(x)| = 1.$$

In particular, the second inequality in the lemma follows from the first one.

We argue by contradiction. If the conclusion in the lemma does not hold then there is $x \in \Sigma$ such that $|\det((A_E^l)(x))| > \frac{1}{\sqrt{2}}$. Thus the modulus of the determinant of the matrix $(\sqrt{2})^{\frac{1}{\dim(E)}} \cdot A_E^l(x)$ is bigger than 1. So this matrix expands at least one vector. Thus there is some unit vector $u \in E(x)$ such that

$$\|A_E^l(x)(u)\| > 2^{-\frac{1}{2\dim(E)}}.$$

As the system is $l-$dominated, given any unit vector $v \in F(x)$ we have

$$\|A_F^l(x)(v)\| > 2^{1-\frac{1}{2\dim(E)}} \geq \sqrt{2}.$$

In particular,

$$\Big(|\det A_F^l(x)| > \sqrt{2}^{\dim(F)} \geq \sqrt{2}\Big) \Longrightarrow \Big(|\det(A_E^l)(x)\ \det(A_F^l)(x)| > 1\Big),$$

contradicting that the system is conservative.

□

The next lemma implies immediately Proposition 0.5:

Lemma 7.10 *Let f be a conservative diffeomorphism with a dominated splitting $E \oplus F$, $E \prec F$. Then there is ℓ such that*

$$|\det(f_*^\ell(x)|_E)| < \frac{1}{2} \quad \textit{and} \quad |\det(f_*^{-\ell}(x))|_F| > \frac{1}{2} \quad \textit{for every } x \in M.$$

Proof: Observe first that, using the conservative version of the Closing Lemma, after a perturbation, we can assume that for every k the periodic points of f of period bigger than k are dense in M.

The proof now is by contradiction. Suppose that the thesis is false, then for every $t > 0$ there is $g_t \in \mathrm{Diff}_\omega^1(M)$ close to f and a periodic point P_t such that

$$g_t^n(P_t) = P_t \quad \text{and} \quad \det((g_t^n)_*(P_t)|_E) > (1-t)^n.$$

It is not hard to see that we can assume that the periods of the points P_t go to infinite as $t \to 0$ (otherwise, after perturbation, one gets a linear system A and periodic point x such that $M_A(x)$ has an eigenvalue of modulus bigger or than 1 in E and an eigenvalue of modulus less than 1 in F, contradicting the dominance of the splitting).

Taking $t = 1/n$, arguing as in Lemma 7.10, we perturb each $(g_{1/n})_*$ along the orbit of $P_{1/n}$ to get a linear system B close to f_* such that $E \oplus F$ is a dominated splitting of B which does not satisfies Lemma 7.9. This gives a contradiction. □

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Partial hyperbolicity, heterodimensional cycles, and transitivity

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In these notes we study the interplay between the notions of heterodimensional cycles, weak forms of hyperbolicity, and transitivity.

We begin by studying in Section 1 the *topological explosion* of non-normally hyperbolic planar horsehsoes. The explosion of these horseshoes explain how invariant manifolds (stable or unstable) associated to a hyperbolic bundle of dimension k may topologically behave as a manifold of dimension $k+1$. Using these ideas we study in Section 2 the unfolding of heterodimensional cycles and explain how (in a very simple situations) the unfolding of these cycles may lead to non-hyperbolic dynamics (the phenomenon of intermingled homoclinic classes of different indices).

The constructions in Sections 1 and 2 lead to the notion of blender introduced and discussed in Section 3. We discuss the role of blender in the context of robustly transitive dynamics.

In Section 4, we study robust transitive dynamics. We first present some examples of non-hyperbolic robustly trnasitive diffeomorphisms. Then we formulate the main result about theses systems claiming that they always are weakly transitive. We deduce some dynamical consequences from this. In this constructions, we review the argument of Mañé that asserts that in absence of domination one can create either sinks or sources after perturbations.

1 Topological explosions of non-normally hyperbolic horseshoes

In this section, we see how a planar horseshoe in $\mathbb{R}^3$ (i.e., a horseshoe contained in a plane) can be perturbed to obtain a *thick* horseshoe. Suppose, for instance, that the stable direction of the horseshoe has dimension one, then there is a small three dimensional cube around the saddle of the horseshoe such that the stable manifold of the saddle of the horseshoe separates this cube into two parts: roughly, it is not possible to join the top and the bottom of the cube by a cylinder without intersecting this stable manifold. In other words, the one dimensional stable manifold behaves, in some topological sense, as a two dimensional one. This enables to suppose that both invariant manifolds of the horseshoe are two-dimensional.

The argument for proving the assertions above involves a reduction to one-dimensional dynamics and is the key of the blenders-like constructions in Section 2 and 3 for obtaining robust transitivity.

1.1 Non-normally hyperbolic horseshoes

We begin by recalling a few basic facts concerning hyperbolicity and homoclinic classes. For other forms of weak hyperbolicity (partial hyperbolicity, dominated splittings and their properties, see the appendix in Section 5). The construction in this section is exported from [6].

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Given a $\mathcal{C}^r$-diffeomorphism $f\colon M \to M$, $1 \leq r \leq \infty$, we say that a compact f-invariant set $\Lambda_f \subset M$ is *hyperbolic* if there is a splitting $E^s_{\Lambda_f} \oplus E^u_{\Lambda_f}$ of the tangent bundle $T_{\Lambda_f}M$ and there are constants $C > 0$, $\lambda < 1$, such that

$$||D_x f^n(v)|| \leq C\lambda^n ||v|| \quad \text{and} \quad ||D_x f^{-n}(w)|| \leq C\lambda^n ||w||$$

for all $v \in E^s_x$, $w \in E^u_x$, $x \in \Lambda_f$ and $n \geq 1$. We call Λ_f a *basic set* if it is *transitive* (i.e. there is a dense orbit of f in Λ_f) and *isolated* (i.e. $\Lambda_f = \cap_{i\in\mathbb{Z}} f^i(U)$ for some neighbourhood U of Λ_f) and contains a dense subset of periodic points. An important feature of basic sets is their persistence under perturbation of the dynamics: there is a neighbourhood $\mathcal{V}^r$ of f in $\mathrm{Diff}^r(M)$ such that for all $g \in \mathcal{V}^r$

$$\Lambda_g = \bigcap_{i\in\mathbb{Z}} g^i(U) \quad \text{(the \textit{continuation} of } \Lambda_f)$$

is a basic set of g and, moreover, $g|\Lambda_g$ is *conjugate* to $f|\Lambda_f$: there is a homeomorphism $h\colon \Lambda_f \to \Lambda_g$ such that $(g|\Lambda_g) \circ h = h \circ (f|\Lambda_f)$.

The homoclinic class of a hyperbolic periodic orbit p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of its stable and unstable manifolds. This set coincides with the closure of the periodic orbits q *homoclinically related* to p, i.e., the stable manifold of p transversely meets the unstable one of q and the unstable manifold of p transversely meets the stable one of q. We observe, that a homoclinic class always is an f-invariant transitive set.

Since our construction is local, it is no restriction to consider $M = \mathbb{R}^{n+1}$, $n \geq 2$, and we do so from now on. We begin by taking a $\mathcal{C}^r$-diffeomorphism F of $\mathbb{R}^n$ with a basic set Σ (a *horseshoe*) such that $F|\Sigma$ is conjugate to the full shift on two symbols. Next, we let $1 < \lambda < 2$ and consider the diffeomorphism

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad (X, x) \mapsto (F(X), \lambda x).$$

Note that f also has a horseshoe Let P be some fixed point of F in Σ. Then $(P, 0)$ is a hyperbolic fixed point of f and

$$W^s((P,0), f) = W^s(P, F) \times \{0\} \text{ and } W^u((P,0), f) = W^u(P, F) \times \mathbb{R}.$$

For simplicity, we assume that every expanding eigenvalue of $DF(P)$ is larger than 2 and then the strong unstable manifold of $(P, 0)$ is

$$W^{uu}((P,0), f) = W^u(P, F) \times \{0\}.$$

Since $W^s(P, F)$ and $W^u(P, F)$ meet transversely at some $Q \in \mathbb{R}^n$, the manifolds $W^s((P,0), F)$ and $W^{uu}((P,0), F)$ have a quasi-transverse intersection at $(Q, 0)$. Now we consider arcs of $\mathcal{C}^r$-diffeomorphisms $\{f_t\}_{t\in[-1,1]}$, with $f_0 = f$, unfolding generically this intersection and we prove that the continuation Λ_t of $\Lambda_0 = \Lambda_f$ satisfies the following:

Geometrical property: *There is a cube $C = [-1,1]^s \times [-1,1]^u \times [-1,1]$ and a constant $\kappa > 0$ such that the stable manifold of P transversely intersects any stip of the form $x_0^s \times [-1,1] \times [r_1, r_2]$ for all $r_1, r_2 \in (0, \kappa)$.* This implies, that $W^s(P)$ transversely meets any *vertical* cylinder contained in C and joining the top and the bottom of C. See Lemma 1.1

As we already said, we start with a diffeomorphism F of $\mathbb{R}^n$, $n \geq 2$, exhibiting a horseshoe Σ

$$\Sigma = \bigcap_{i\in\mathbb{Z}} F^i(R), \qquad \text{with } R = [-1,1]^n \text{ say,}$$

and $F^{-1}(R)\cap R$ consisting of two connected components $\hat{D}_1$, $\hat{D}_2$. We take F to be affine on each of these components: there are s, $u \geq 1$, with $s+u = n$, and linear maps $S_i\colon \mathbb{R}^s \to \mathbb{R}^s$, $U_i\colon \mathbb{R}^u \to \mathbb{R}^u$, such that

$$DF|\hat{D}_i = \begin{pmatrix} S_i & 0 \\ 0 & U_i \end{pmatrix} \text{ and } ||S_i||, ||U_i^{-1}|| < 1 \text{ for } i = 1,\, 2.$$

In particular, $\hat{D}_i = [-1,1]^s \times D_i$, $D_i \subset [-1,1]^u$, for $i = 1$, 2. In what follows (x^s, x^u) are the usual coordinates in $R = [-1,1]^s \times [-1,1]^u$ and we suppose the fixed point P of F in $\hat{D}_1$ to be located at $(0^s, 0^u)$.

Define the smooth arc $\{f_t\}_{t\in[-1,1]}$ of diffeomorphisms of $\mathbb{R}^{n+1}$ by

$$f_t(x^s, x^u, x) = \begin{cases} (F(x^s,x^u), \lambda x) \text{ if } x^u \in D_1, \\ (F(x^s,x^u), \lambda x - t) \text{ if } x^u \in D_2. \end{cases}$$

We let $1 < \lambda < 2$ and $||U_1^{-1}|| < 1/2$ so that the fixed point $\mathcal{O} = (0^s, 0^u, 0)$ of f_t has

$$W^{uu}_{\text{loc}}(\mathcal{O}, f_t) = \{0^s\} \times [-1,1]^u \times \{0\} \text{ and } W^s_{\text{loc}}(\mathcal{O}, f_t) = [-1,1]^s \times \{0^u\} \times \{0\}.$$

On the other hand, given any $x^s \in [-1,1]^s$ there are x_1^s, $x_2^s \in [-1,1]^s$ such that

$$\begin{aligned} f_t(\{x^s\} \quad &\times [-1,1]^u \times \{x\}) \supset \\ &\supset (\{x_1^s\} \times [-1,1]^u \times \{\lambda x\}) \cup (\{x_2^s\} \times [-1,1]^u \times \{\lambda x - t\}) \end{aligned}$$

for every $x \in \mathbb{R}$. Hence,

$$f_t(W^{uu}_{\text{loc}}(\mathcal{O}, f_t)) \supset \{0_2^s\} \times [-1,1]^u \times \{-t\}$$

(note that $0_1^s = 0^s$) and so the arc $\{f_t\}_{t\in[-1,1]}$ unfolds generically the quasi-transverse intersection of $W^s(\mathcal{O}, f_0)$ and $W^{uu}(\mathcal{O}, f_0)$ at $(0_2^s, 0^u, 0)$.

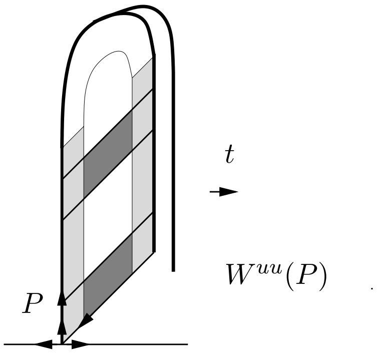

Figure 1: Non-normally hyperbolic horseshoe and its unfolding

Denote by Λ_t the continuation for f_t, small t, of the basic set $\Lambda_0 = \Sigma \times \{0\}$ of f_0. Observe that Λ_t coincides with the closure of the set $H(\mathcal{O}, f_t) = W^s(\mathcal{O}, f_t) \cap W^u(\mathcal{O}, f_t) \cap R$ of all (transverse) homoclinic points of $\mathcal{O}$ in R.

Lemma 1.1. *Let $t > 0$ (resp. $t < 0$) be close to zero and $J \subset [0,t]$ (resp. $J \subset [t,0]$) be an open interval. Then there are $x^s \in [-1,1]^s$ and $j \geq 0$ such that*

$$f_t^j(\{0^s\} \times [-1,1]^u \times J) \supset \{x^s\} \times [-1,1]^u \times \{\lambda^{-1}t\}.$$

As a consequence, $(\{0^s\} \times [-1,1]^u \times J) \cap H(\mathcal{O}, f_t) \neq \emptyset$.

This Lemma gives,

$$\pi\left(\overline{H(\mathcal{O}, f_t)}\right) \supset [0,t], \quad \text{where } \pi: \ [-1,1]^s \times [-1,1]^u \times \mathbb{R} \to \mathbb{R}, \ \pi(x^s, x^u, x) = x.$$

Proof of the lemma: We suppose $t > 0$, the case $t < 0$ being completely analogous. Define s by

$$\lambda\, s - t = s, \qquad s = \frac{t}{\lambda - 1} > t.$$

Consider the affine functions

$$\begin{aligned} \pi_{1,t}: &\ [0, \lambda^{-1}s] \to [0,s], \ \pi_{1,t}(y) = \lambda y, \\ \pi_{2,t}: &\ [\lambda^{-1}t, t] \to [0,s], \ \pi_{2,t}(y) = \lambda y - t. \end{aligned}$$

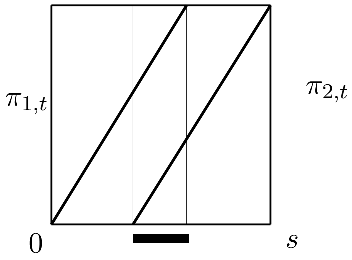

Figure 2: One-dimensional associated dynamics

Note that $\pi_{2,t}$ is well defined since $1 < \lambda < 2$ and that there is an overlapping interval $[\lambda^{-1}t, \lambda^{-1}s]$ where the two maps are defined. By (1)

$$\begin{aligned} f_t(\{x^s\} \times [-1,1]^u \times \{x\}) \supset \{x_1^s\} \times [-1,1]^u \times \{\pi_{1,t}(x)\} \quad & \text{if } x \in [0, \lambda^{-1}t], \\ f_t(\{x^s\} \times [-1,1]^u \times \{x\}) \supset \{x_2^s\} \times [-1,1]^u \times \{\pi_{2,t}(x)\} \quad & \text{if } x \in [\lambda^{-1}t, t]. \end{aligned}$$

The dynamical interpretation of this maps is the following. Take any vertical disk $V = \{x^s\} \times [-1,1]^u \times \{\kappa\}$ in the cube $C = [-1,1]^s \times [-1,1]^u \times [0,s]$. Then the image $f_t(V) \cap C$ has one or two connected components (if the point κ is in the overlapping region one has two componentes, otherwise one has just one component) which are vertical disks and whose central coordinate is given by the maps $\pi_{i,t}$.

Write $I_0 = J$ and $z_0 = 0^s$. If I_0 contains $\lambda^{-1}t$ then there is nothing to prove. Hence, we may assume that either $I_0 \subset (0, \lambda^{-1}t)$ or $I_0 \subset (\lambda^{-1}t, t)$. We let $i = 1$ in the first case and $i = 2$ in the second one and we write $I_1 = \pi_{i,t}(I_0)$ and $z_1 = 0_i^s$. Then, by (2),

$$f_t(\{0^s\} \times [-1,1]^u \times I_0) \supset \{z_1\} \times [-1,1]^u \times I_1$$

As above, if I_1 contains $\lambda^{-1}t$ then we are done. Otherwise we apply the previous procedure inductively: for each $j \geq 1$, if $\lambda^{-1}t \notin I_{j-1}$ then we construct an open interval $I_j \subset [0,t]$ and a point $z_j \in [-1,1]^s$ so that

$$I_j = \pi_{1,t}(I_{j-1}) \text{ if } I_{j-1} \subset [0, \lambda^{-1}t] \quad \text{and} \quad I_j = \pi_{2,t}(I_{j-1}) \text{ if } I_{j-1} \subset [\lambda^{-1}t, t] \quad \text{and}$$
$$f_t(\{z_{j-1}\} \times [-1,1]^u \times I_{j-1}) \supset (\{z_j\} \times [-1,1]^u \times I_j).$$

Since $\text{length}\,(I_j) = \lambda \cdot \text{length}\,(I_{j-1})$ and $\lambda > 1$ there must be a first j such that $\lambda^{-1}t \in I_j$. This ends the proof of the first part of the Lemma.

As for the second one, it is now a direct consequence. Observe that

$$f_t(\{x^s\} \times D_2 \times \{\lambda^{-1}t\}) \cap W^s_{\text{loc}}(\mathcal{O}, f_t) \neq \emptyset$$

and so $\{x^s\} \times [-1,1]^u \times \{\lambda^{-1}t\}$ intersects $W^s(\mathcal{O}, f_t)$ at some point Q_t. Then, taking $j \geq 0$ as above,

$$f_t^{-j}(Q_t) \in (\{0^s\} \times [-1,1]^u \times J) \subset W^u(\mathcal{O}, f_t),$$

which also means that $f_t^{-j}(Q_t) \in H(\mathcal{O}, f_t)$. This completes our argument. □

2 Heterodimensional cycles and robust transitivity

In this section we study heterodimensional cycles and see how the unfolding of these cycles may lead to the phenomenon of robust transitive dynamics. The ideas in this section are, in some sense, close to the ones in the previous section. The idea is to reduce the dynamics to the analysis of a system of (one-dimensional) iterated functions. Using these functions we see how an invariant manifold of dimension one may topologically behave as a manifold of dimension two.

We call *heterodimensional cycle* the geometric configuration corresponding to two periodic points P and Q of Morse unstable index (dimension of the unstable bundle) $p < q$, respectively, such that the stable manifold of each point meets the unstable manifold of the other. In most cases we consider $q = p + 1$. Such cycles were first considered by Newhouse, Palis in [26], and were systematically studied in the series of papers [9, 10, 12, 13, 14, 15, 16]. This elementary phenomenon, which occurs for diffeomorphisms in any dimension larger than 2, is one of the main mechanisms for non-hyperbolicity. For instance, the homoclinic classes of P and Q may coincide, for open sets in parameter space, in which case they can not be hyperbolic.

At the heart of heterodimensional cycles and the kinds of dynamics they yield, are certain transitive hyperbolic sets that we call *blenders*. This was introduced in [2], motivated by [10]. Topologically blenders are Cantor sets, but their distinctive geometric feature is that their embedding in the manifold is such that their stable set behaves as if it were of dimension higher than its actual topological dimension. More precisely, if p is the unstable index of the blender Γ, then the closure of the $(n-p)-$dimensional stable manifold of a periodic point $P \in \Gamma$ (in other words, the stable set of Γ) may intersect an unstable manifold of dimension $p-1$ in a robust fashion. This property as a key ingredient in a semi-local argument for building robustly transitive dynamics.

In Section 3 we present an affine model of a blender, and then explore various generalizations of this model. In Section 3.2 we show that the unfolding of heterodimensional cycles yields blenders, leading to periodic orbits of different indices being heteroclinically related in a robust way.

2.1 Heterodimensional cycles

Consider a closed manifold M and a diffeomorphism f defined on M and having a pair of hyperbolic saddles P and Q with different indices, that is, different dimensions of their unstable subspaces. Assume $W^s(P)$ and $W^u(Q)$ have non-empty intersection, and the same for $W^u(P)$ and $W^s(Q)$. We say that f has a *heterodimensional cycle* associated to P and Q. Of course, heterodimensional cycles can only exist in dimension bigger or equal to 3.

2.1.1 Explosion of homoclinic classes

The main result in this section depicts heterodimensional cycles as a powerful mechanism for explosion of homoclinic classes of periodic points and creation of robust non-hyperbolic transitive sets. The assumptions are not minimal, we shall see later that this kind of behavior is typical of a large class of cycles.

For simplicity, we suppose that the periodic points P and Q in the cycle are actually fixed points. In addition, we always assume

- (codimension 1) the saddles P and Q have indices p and $q = p + 1$, respectively,
- (quasi-transversality) the manifolds $W^s(P)$ and $W^u(Q)$ intersect transversely, and the intersection between $W^u(P)$ and $W^s(Q)$ is quasi-transverse: $T_xW^s(Q) \cap T_xW^u(P) = \{0\}$ for every intersection point x.

The next theorem asserts that the homoclinic classes of P and Q often explode, and become intermingled (non-empty intersection) when the cycle is unfolded.

A parametrized family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms *unfolds generically* a heterodimensional cycle of $f = f_0$ if there are open disks $K_t^u \subset W^u(P_t)$ and $K_t^s \subset W^s(Q_t)$, depending continuously on t, such that $K_0^u \cap K_0^s$ contains a point of quasi-transverse intersection, and the distance between K_t^s and K_t^u increases with positive velocity when t increases. Here P_t and Q_t denote the continuations for f_t of the periodic points P and Q.

Theorem 1 ([10]). *There is a non-empty open set of parametrized C^∞ families of diffeomorphisms $(f_t)_{t\in[-1,1]}$ unfolding generically a heterodimensional cycle of $f = f_0$, such that for all small positive t*

1. *the transverse intersection between $W^s(P_t)$ and $W^u(Q_t)$ is contained in the homoclinic class of Q_t;*
2. *and the homoclinic class of P_t is contained in the homoclinic class of Q_t.*

Let d be the dimension of the ambient manifold M. The key fact behind Theorem 1 is that, for every small positive t, the $(d-p)$-dimensional manifold $W^s(P_t)$ is contained in the closure of $W^s(Q_t)$. This makes that the stable manifold of Q_t, which has dimension $(d-p-1)$, behaves in fact as a manifold of dimension $d-p$, that is one unit bigger. This type of phenomenon has been synthesized in the notion of blender, which we discuss in Section 3.

The proof of the theorem, which we are going to sketch next, relies on a reduction of the dynamics to a family of iterated functions systems on the interval. Besides the standing hypotheses of codimension 1 and quasi-transversality above, the main conditions defining the open set in the theorem are *connectedness* of the transverse intersection γ, *non-criticality* or partial hyperbolicity of the dynamics on γ, and *small distortion* of the transition from the neighborhood of Q to that of P. This will be explained in the sequel, but let us point out right away that these conditions may hold simultaneously for a parametrized family f_t and for its inverse f_t^{-1}: one gets open sets of families for which the homoclinic classes of P_t and Q_t coincide after the bifurcation.

2.1.2 A model heterodimensional cycle

To explain the ideas of the proof of Theorem 1 while avoiding technical difficulties, we consider here a simple situation of a cycle in $\mathbb{R}^3$, which displays the main features of the general case we are dealing with. In this section, we construct a model one-parameter family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms unfolding a heterodimensional cycle. The study of the semi-local dynamics of f_t will be reduced to the analysis of a one-parameter family of endomorphisms with infinitely many discontinuities which describe the dynamics of f_t in the *central direction,* see Section 2.1.3.

Consider a diffeomorphism f with a heterodimensional cycle having the following dynamical configuration. In local coordinates in $\mathbb{R}^3$, the cycle is associated to saddle fixed points $Q = (0,0,0)$ and $P = (0,1,0)$ of indices 2 and 1, respectively, verifying the following conditions:

Partially hyperbolic (semi-local) dynamics of the cycle:

- In the cube $[-1,1]\times[-1,2]\times[-1,1]$ the diffeomorphism has the form
$$f(x,y,z) = (\lambda_s x, F(y), \lambda_u z),$$
where $F\colon [-1,2] \to (-1,2)$ is an increasing map with exactly two fixed points, a source at 0 and a sink at 1, and $0 < \lambda_s < d_m < 1 < d_M < \lambda_u$, where $0 < d_m < F'(x) < d_M$ for all $x \in [0,1]$.
- There is $\delta > 0$ such that F is linear in $[-\delta,\delta]$ and affine in $[1-\delta, 1+\delta]$. We denote by $\beta > 1$ and $0 < \lambda < 1$, the derivative of F at 0 and 1, respectively.

Observe that $[-1,1]\times\{(0,0)\} \subset W^s(Q)$, $\{0\}\times[0,1)\times[-1,1] \subset W^u(Q)$, $\{(0,0)\}\times[-1,1] \subset W^u(P)$, and $[-1,1]\times(0,1]\times\{0\} \subset W^s(Q)$. Thus the curve $\gamma = \{0\}\times(0,1)\times\{0\}$ (called *connexion*) is a normally hyperbolic curve contained in $W^u(Q) \pitchfork W^s(P)$.

Existence and unfolding of the cycle:

- **The cycle:** There exist $k_0 \in \mathbb{N}$ and a small neighborhood U of $(0,1,-1/2) \in W^u(P)$ such that the restriction of f^{k_0} to U is a translation,
$$f^{k_0}(x,y,z) = (x - 1/2, y - 1, z + 1/2).$$
Thus $f^{k_0}(0,1,-1/2) = (-1/2,0,0) \in W^u(Q)$ and $W^s(Q)$ and $W^u(P)$ meet throughout the orbit of the heteroclinic point $(-1/2,0,0)$. By construction, $(-1/2,0,0)$ is a quasi-transverse heteroclinic point.
- **The unfolding of the cycle:** Consider the family $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ of diffeomorphisms coinciding with f in $[-1,1]\times[-1,2]\times[-1,1]$ and such that the restriction of $f_t^{k_0}$ to U is of the form
$$f_t^{k_0}(x,y,z) = (x - 1/2, y - 1 + t, z + 1/2) = f^{k_0}(x,y,z) + (0,t,0).$$
So, for $t > 0$, $\{(-1/2,t)\}\times[-1,1] \subset W^u(P,f_t)$ and $x_t = (-1/2,t,0)$ is a transverse homoclinic point of P (for f_t). Similarly, $y_t = (-1/2,0,0)$ is a transverse homoclinic point of Q.

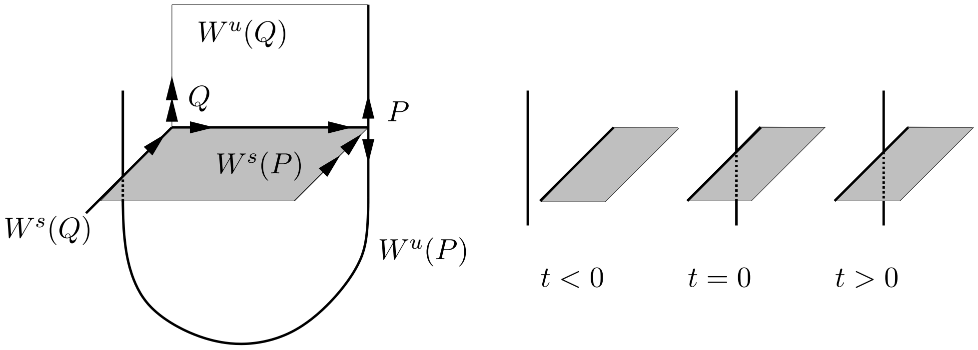

Figure 3: The model cycle and its unfolding

Consider a small *neighborhood of the heterodimensional cycle associated with* f_0, that is, an open set W containing the connexion $\gamma = \{0\} \times [0,1] \times \{0\}$ and the f_0-orbit of the heteroclinic point $(-1/2, 0, 0)$. For small t, let Λ_t be the *maximal* f_t*-invariant set in* W, $\Lambda_t = \cap_{n\in\mathbb{Z}} f_t^n(W)$. Consider also the forward and backward f_t-invariant sets in W, $\Lambda_t^+ = \cap_{n\geq 0} f_t^{-n}(W)$ and $\Lambda_t^- = \cap_{n\geq 0} f_t^n(W)$.

Fix a small positive ρ and consider the fundamental domains of F given by $D^+ = [\beta^{-1}\rho, \rho]$ and $D^- = [1-\rho, \lambda(1-\rho)]$ contained in the neighborhoods of 0 and 1 where F is affine. We can choose ρ such that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$ (for notational simplicity let us put $N = 1$). The map F^N is the *transition from* 0 *to* 1 [1]. Assume that that $F'(1) = \lambda$ and $F'(0) = \beta$ verify:

(T1) $F'(x) \geq \dfrac{1}{2}\,\dfrac{1-\lambda}{1-\beta^{-1}}$, for all $x \in D^+$,

(T2) $(1-\lambda) < \beta^{-1}$, and

(T3) $\dfrac{(1-\lambda)\,\lambda}{2\,(1-\beta^{-1})\,\beta} = \ell > 1.$

To get condition (T1) it is enough to consider F with small distortion. For conditions (T2) and (T3) it is enough to take β close enough to 1^+. Our first result is:

Theorem 2.1. *For every $t > 0$ sufficiently small, $H(P, f_t) \subset H(Q, f_t)$.*

In fact, it is possible also prove the inclusion $\Lambda_t \subset H(Q, f_t)$.

---

[1] This transition plays a key role for determining the dynamics after unfolding the cycle and it is determined by the *Mather invariant* of F: in a neighborhood of 0, the map F is the time-one of the vector field $X(y) = (\log\beta)\, y\, \frac{\partial}{\partial y}$, whose flow is $y \mapsto \beta^t\, y$. Similarly, in a neighborhood of 1, F is the time-one of $Y(y) = (\log\lambda)\,(y-1)\,\frac{\partial}{\partial y}$. Consider, for y close to 0^+, an large n such that $F^n(x)$ is close to 1^- and write $DF^n(y)(X(y)) = \mu(y)\,Y(F^n(y))$. Using the local F-invariance of X and Y (near 0 and 1), one has that $\mu(x)$ does not depend on n and that $\mu(x) = \mu(F(x))$. The function μ is the Mather invariant of F which describes its distortion. For instance, if μ is identically 1, then F is exactly the exponential of a vector field.

2.1.3 Expanding one-dimensional dynamics associated to the cycle

For each small $t > 0$, consider the *scaled fundamental domains* $D_t^\pm$ defined as follows: let $D_t^- = [1-t, \lambda(1-t)]$ and define k_t as the smaller $k \in \mathbb{N}$ with $F^{-k}(D_t^-) \subset [0,t]$. We define

$$D_t^+ = [a_t, b_t] = [\beta^{-1}(b_t), b_t] = F^{-k_t}(D_t^-), \quad \text{where} \quad \beta^{-2} t < a_t < b_t \le t.$$

We next define an expanding map R_t, $R_t\colon D_t^+ \to D_t^+$, with discontinuities describing the central dynamics of the return map of f_t defined on $[-1,1] \times D_t^+ \times [-1,1]$. First, for each small $t > 0$ define the *transition map T_t from D_t^+ to D_t^-* by

$$T_t\colon D_t^+ \to D_t^-, \quad x \mapsto T_t(x) = F^{k_t}(x).$$

Lemma 2.2. *The map T_t verifies $T_t'(x) > \ell > 1$ for all $x \in D_t^+$, where ℓ is as in condition* **(T3)**.

Proof: Given $x \in D_t^+$ let $k_t = n_t(x) + 1 + m_t(x)$, where $F^{n_t(x)}(x) \in D^+$ and $F^i(x) \notin D^+$ for all $1 \le i < n_t(x)$. We claim that

$$\frac{1}{\beta\, t} \le \beta^{n_t(x)} \le \frac{\beta^2}{t} \quad \text{and} \quad \lambda\, t \le \lambda^{m_t(x)} \le \frac{t}{\lambda}. \tag{1}$$

For first inequalities just note $x \in D_t^+ \subset (\beta^{-2}\, t, t]$ and $\beta^{n_t(x)}\, x \in [\beta^{-1}, 1]$. The other ones follow analogously. As $T_t(x) = \lambda^{m_t(x)}\, F(\beta^{n_t(x)}\, x)$, hypotheses (T1) and (T3) and the estimates in (1) give

$$|T_t'(x)| = \lambda^{m_t(x)}\, |F'(x)|\, \beta^{n_t(x)} \ge (\lambda\, t) \left(\frac{1}{2}\, \frac{1-\lambda}{1-\beta^{-1}}\right) \left(\frac{1}{\beta\, t}\right) = \frac{1}{2}\, \frac{\lambda\,(1-\lambda)}{\beta\,(1-\beta^{-1})} = \ell > 1,$$

as we claimed. $\square$

Since $T_t(x) \in D_t^- = [1-t, 1-\lambda t]$ for all $x \in D_t^+$, we can define the map G_t by

$$G_t\colon D_t^+ \to [0,t], \quad x \mapsto G_t(x) = T_t(x) + (t-1).$$

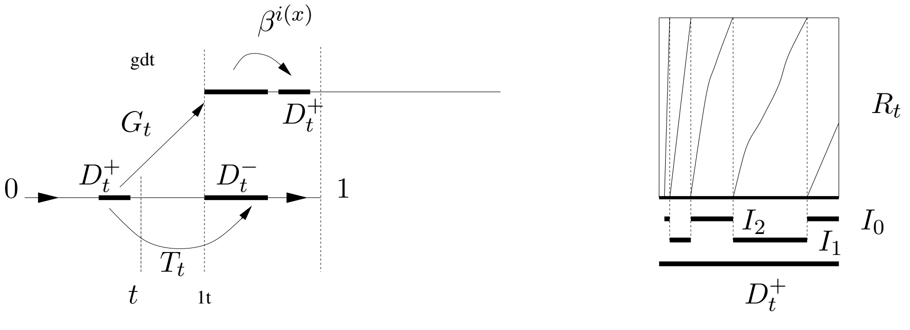

Figure 4: The expanding maps T_t, G_t and R_t

Remark 2.3. *The map G_t is monotone increasing and $G_t(D_t^+) = [0, t(1-\lambda)]$.*

Claim 2.4. *Let $(a_t, b_t] = \tilde{D}_t^+ \subset D_t^+$. Given $x \in \tilde{D}_t^+$ let $i(x) \in \mathbb{Z}$ be the minimum i with $\beta^i(G_t(x)) \in D_t^+$. Then there is $i_0 > 0$ (maximum with such property) such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: Recall first that $b_t \in (\beta^{-1}\, t, t]$. On the other hand, from $(1-\lambda) < \beta^{-1}$ (condition (T2)) and Remark 2.3, $G_t(\tilde{D}_t^+) = (0, t\,(1-\lambda)] \subset (0, \beta^{-1}\, t)$. Thus the right extreme of $G_t(\tilde{D}_t^+)$ is less than the left extreme of D_t^+, hence $i(x) > 0$ for all $x \in \tilde{D}_t^+$, ending the proof of the claim. $\square$

Finally, the return map R_t is defined by

$$R_t \colon \tilde{D}_t^+ \to D_t^+, \quad R_t(x) = \beta^{i(x)}(G_t(x)) = \beta^{i(x)}\,(T_t(x) + (t-1)).$$

Next we study the dynamics of R_t: the map R_t is uniformly expanding and has (infinitely many) discontinuities where the lateral derivatives are well defined. These discontinuities will play a key role in our constructions. The definition of $i_0 \in \mathbb{N}$ in Claim 2.4 implies that $\beta^{-i_0}(a_t) \in G_t(D_t^+)$. For each $i \geq i_0$ define $d_i \in \tilde{D}_t^+$ by $G_t(d_i) = \beta^{-i}(a_t)$. By construction, the sequence $(d_i)_{i \geq i_0}$ corresponds to the discontinuities of R_t and verifies the following:

- $d_{i+1} < d_i$ and $d_i \to a_t$,
- Let $[d_{i+1}, d_i] = I_i$, $i > i_0$, and $I_{i_0} = [d_{i_0}, b_t]$. The map R_t is continuous and strictly increasing in the interior of each interval I_i. We continuously extend R_t to the whole I_i, obtaining a bi-valuated return map R_t with $R_t(d_i) = \{a_t, F(a_t) = b_t\}$ for all $i > i_0$. In particular, the restriction of R_t to any I_i, $i > i_0$, is onto. We let $R_t(b_t) = c_t \leq b_t$.

The main properties of R_t are summarized in the next lemma.

Lemma 2.5. *The restriction of R_t to each interval $I_i, i > i_0$, is onto and $R_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = d_i$ this means that the lateral derivatives of R_t at x are greater than ℓ). Moreover, $0 \in G_t(R_t(d_i))$ for all $i \geq i_0$.*

The expansiveness of R_t follows from Lemma 2.2 and Claim 2.4. Condition $0 \in G_t(R_t(d_i))$ follows from $a_t \in R_t(d_i)$ and $0 \in G_t(a_t)$.

Lemma 2.6. *Consider small $t > 0$ and an open subinterval J of $\tilde{D}_t^+$. Then there is $k \in \mathbb{N} \cup \{0\}$ such that $R_t^k(J)$ contains a discontinuity of R_t. In particular, there is $x \in J$ such that $G_t(R_t^k(x)) = 0$.*

Proof: If the interval J contains a discontinuity we are done. Otherwise, let $i > 0$ be such that the intervals J, $R_t(J), \dots, R_t^i(J)$ do not contain discontinuities. Thus, for each $k \in \{0, \dots, i\}$, there is $i_k \geq i_0$ such that $R_t^k(J) \subset I_{i_k}$. Lemma 2.5 implies that $|R_t^k(J)| \geq \ell^k |J|$, $\ell > 1$, for all $k \in \{0, \dots, i\}$. Since the size of the intervals I_i is upper bounded, this inequality implies that there is a first $m \in \mathbb{N}$ such that $R_t^m(J)$ is not contained in any I_i, thus it intersects the set of discontinuities of R_t. $\square$

The maximal invariant set: Proof of Theorem 2.1 Next proposition is the main technical result of this section. Heuristically, it means that the one-dimensional stable manifold of Q topologically behaves as a two-dimensional manifold.

Proposition 2.7. *For every small $t > 0$ and every two-disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^s(Q, f_t) \pitchfork \chi \neq \emptyset$. In particular, $W^s(P, f_t)$ is contained in the closure of $W^s(Q, f_t)$.*

Proof of the inclusion $H(P,f_t) \subset H(Q,f_t)$ in Theorem 2.1. By the definition of $H(P,f_t)$, it suffices to see that any $x \in W^s(P,f_t) \pitchfork W^u(P,f_t)$ is accumulated by homoclinic points of Q. By the configuration of the cycle, $W^u(P,f_t) \subset \text{closure}\,(W^u(Q,f_t))$, thus, given any $x \in H(P,f_t)$ and any $n>0$, there is a disk Δ_n, contained in $W^u(Q,f_t)$ and in the ball of radius $1/n$ centered at x, whose interior meets transversely $W^s(P,f_t)$. By Proposition 2.7, $\Delta_n \pitchfork W^s(Q,f_t) \neq \emptyset$. Thus there is $y_n \in \Delta_n \cap H(Q,f_t)$. By construction, $y_n \to x$, proving the inclusion $H(P,f_t) \subset H(Q,f_t)$.

Proof of Proposition 2.7 We now go into the details of the proof of Proposition 2.7. We first introduce some definitions.

- A set $\Delta \subset [-1,1]\times[-1,2]\times[-1,1]$ is a *vertical strip* if $\Delta = \{x_1\}\times[l_1,l_2]\times[r_1,r_2]$, where $l_1<l_2$ and $r_1<0<r_2$. The segment $\{x_1\}\times[l_1,l_2]\times\{0\}$ is the *basis* of Δ. The *width* and the *height* of Δ are $w(\Delta)=(l_2-l_1)$ and $h(\Delta)=(r_2-r_1)$. The strip Δ is *complete* if $r_2=1$ and $r_1=-1$, *well located* if $[l_1,l_2]$ is contained in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.
- A subset $J \subset [-1,1]\times[-1,2]\times[-1,1]$ is a *vertical segment* if $J=\{x_1\}\times\{l_1\}\times[r_1,r_2]$, where $r_1<0<r_2$. The point $(x,\ell_1,0)$ is the basis of J. The height of J is $h(J)=(r_2-r_1)$. As above, the segment J is *complete* if $r_1=-1$ and $r_2=1$, *well located* if l_1 is in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.
- A vertical segment J (resp. strip Δ) is *at the right of Q* if $l_1 \in (0,1]$.

Given an interval $\alpha \subset [-1,2]$, let $\Delta(\{x\}\times\alpha\times\{0\}) = \{x\}\times\alpha\times[-1,1]$ be the *unique* complete vertical strip with basis $\{x\}\times\alpha\times\{0\}$. Similarly, $J(x,y,0)$ is the unique complete vertical segment with basis $(x,y,0)$. The next algorithm associates to perfect segments and strips their successors:

Algorithm 2.8. *Let $\Delta=\Delta(\{x\}\times\alpha\times\{0\})$ be a perfect strip and define $\mathcal{G}_t(\Delta)$ as the perfect strip such that:*

- *the basis of $\mathcal{G}_t(\Delta)$ is of the form $(\{x'\}\times G_t(\alpha)\times\{0\})$, where $x'=\lambda_s^{k_t}\,x-1/2$,*
- *$\mathcal{G}_t(\Delta)$ is contained in $f_t^{k_t}(\Delta)$ (where $F^{k_t}(D_t^+)=D_t^-$).*

Suppose now that α does not contain discontinuities, i.e. $\alpha\subset(d_{i+1},d_i)$ for some i. Define $\mathcal{R}_t(\Delta)$ as the perfect strip such that:

- *the basis of $\mathcal{R}_t(\Delta)$ is of the form $(\{\hat{x}\}\times R_t(\alpha)\times\{0\})$, where $\hat{x}=\lambda_s^i(\lambda_s^{k_t}\,x-1/2)$,*
- *$\mathcal{R}_t(\Delta)$ is contained in $f_t^{k_t+i}(\Delta)$.*

Similarly, to a perfect segment $J=J(x,y,0)$ we associate perfect segments $\mathcal{G}_t(J)$ and $\mathcal{R}_t(J)$ (provided $y\neq d_i$ for all i).

The strips $\mathcal{G}_t(\Delta)$ and $\mathcal{R}_t(\Delta)$ in Algorithm 2.8 are obtained as follows. Given a set A and a point $x\in A$, let $C(x,A)$ be the connected component of A containing x. Take a small neighborhood V of $f_0^{k_0}(\{0,1\}\times[-1,1])\subset W^u(P,f_0)$, k_0 as in the definition of the cycle in Section 2, then

$$\begin{aligned}
\mathcal{G}_t(\Delta) &= C(f_t^{k_t}(x,y,0), f_t^{k_t}(\Delta)\cap V)\cap[-1,1]^3),\\
\mathcal{R}_t(\Delta) &= C(f_t^{i}(x',y',0), f_t^{k_t+i}(\Delta)\cap V)\cap[-1,1]^3),
\end{aligned}$$

where $(x, y, 0)$ is any point in the basis of Δ, $x' = (\lambda_s^{k_t} x - 1/2)$, and $y' \in G_t(\alpha)$. The construction for the successors of segments is analogous.

Lemma 2.9. *The manifold* $W^u(P, f_t)$ *contains a perfect segment for all small* $t > 0$.

Proof: Consider the transverse homoclinic point $x_t = (-1/2, t, 0)$ of P. Recall that $t \geq b_t$ and $\beta^{-1} t \in D_t^+ = [a_t, b_t]$. Let us assume that $t > b_t$, and thus $\beta^{-1} t \in (a_t, b_t)$ (the case $t = b_t$ follows similarly, so it will be omitted). Consider the complete vertical segment $\mathcal{R}_t(J)$, where $J = J(-(\lambda_s^{-1}/2), \beta^{-1} t, 0) \subset W^u(P, f_t)$. If $R_t(\beta^{-1} t)$ belongs to the interior of D_t^+, then $\mathcal{R}_t(J) \subset W^u(P, f_t)$ is the announced segment. Otherwise, $R_t(\beta^{-1} t) = b_t$ and there is a homoclinic point of P of the form $(x', b_t, 0)$. Using the λ-lemma and the product structure of the cycle, one gets homoclinic points $(x_n, y_n, 0)$ of P and complete segments $J_n = J(x_n, y_n, 0) \subset W^u(P, f_t)$ such that $x_n \to x'$, $y_n \to b_t$, and y_n is increasing. Thus y_n is in the interior of D_t^+ for every big n and $J_n \subset W^u(P, f_t)$ is perfect. ☐

For clearness we first prove Proposition 2.7 in the following special case:

Proposition 2.10. *Let* $\chi \subset [-1, 1] \times [-1, 2] \times [-1, 1] = \{x\} \times A$, *where* A *is a disk of* $\mathbb{R}^2$ *whose interior contains a point of the form* $(y, 0)$ *with* $y \in (0, 2)$. *Then* χ *intersects transversely* $W^s(Q, f_t)$.

We claim that is enough to prove the Proposition 2.10 for perfect strips:

Lemma 2.11. *Let* Δ *be a perfect strip. Then there is* $k \in \mathbb{N}$ *such that* $f_t^k(\Delta) \pitchfork W^s(Q, f_t) \neq \emptyset$.

Proof: Suppose that $\Delta = \Delta(\{x_0\} \times \alpha \times \{0\})$, α in the interior of D_t^+. By Lemma 2.6, there exist y_0 in the interior of α and $k \in \mathbb{N}$ such that $G_t(R_t^k(y_0)) = 0$. Thus the vertical strip $\mathcal{G}_t(\mathcal{R}_t^k(\Delta))$ (contained in the forward orbit of Δ) intersects transversely $[-1, 1] \times \{(0, 0)\} \subset W^s(Q, f_t)$. ☐

Proof of Proposition 2.10: By Lemma 2.9, $W^u(P, f_t)$ contains a perfect vertical segment J. Since, by definition, χ meets transversely $W^s(P, f_t)$, the λ-lemma implies that forward orbit of χ contains a sequence of complete strips χ_n accumulating to J. Thus χ_n contains a perfect strip for all n large. Lemma 2.11 implies that χ_n (and thus χ) transversely meets $W^s(Q, f_t)$. ☐

Proof of Proposition 2.7: We can assume that χ is transverse to $W^s_{loc}(P, f_t)$ and contained in $[-1, 1] \times [-1, 2] \times [-1, 1]$. If χ contains a subset of the form $\{x\} \times A$, where A is an open subset of $\mathbb{R}^2$ containing a point $(0, y)$ with $y \in (0, 2)$, Proposition 2.10 implies the result. For the general case, consider a point $(x_0, y_0, 0)$, $y_0 \in (0, 2)$, in the interior of $\chi \pitchfork W^s_{loc}(P, f_t)$ and for every big n the vertical strip

$$\Sigma_n = \{x_0\} \times [y_0 - 1/n, y_0 + 1/n] \times [-1/n, 1/n].$$

The strips Σ_n verify the hypotheses of Proposition 2.10, hence there is $(x_0, y_n, z_n) \in \Sigma_n \pitchfork W^s(Q, f_t)$ such that $H_n = [-1, 1] \times \{(y_n, z_n)\} \subset W^s(Q, f_t)$. Since $(x_0, y_n, z_n) \to (x_0, y_0, 0)$, it is immediate that H_n meets transversely χ for all large n, ending the proof of the proposition. ☐

2.2 Unfolding heterodimensional cycles

The dynamics associated to the unfolding of general heterodimensional cycles is very rich, and still far from being completely understood. However, a good number of results are available, especially for the (codimension-1) quasi-transversal case. Here we review some of those results.

The example of heterodimensional cycle in the previous section is *connected*, meaning that the transverse intersection $W^s(P) \cap W^u(Q)$ contains a smooth invariant curve γ joining P to Q, and *non-critical*, meaning that the closed invariant set $\gamma \cup \{P, Q\}$ admits a Df-invariant dominated splitting of the form $E^u \oplus T\gamma \oplus E^s$. Finally, the restriction to the connection γ has *small distortion*. Due to non-criticality, no homoclinic tangencies related to either P or Q may occur in the unfolding of those cycles. As we are going to see, other types of cycles may exhibit very different dynamical behavior. For instance, [13] shows that the unfolding of *critical* cycles may lead to the phenomenon of persistence of tangencies. We observe that this construction does not involve fractal dimensions. On the other hand, in the same paper it is shown that the unfolding of *non-connected* cycles may lead to Morse-Smale dynamics after the bifurcation: essentially, one considers cycles such that the connected components of γ are small, compared to the length of fundamental domains of f near P and Q.

Finally, [14] exhibits connected non-critical cycles with *large distortion*, and for which there are sets of parameters t with positive Lebesgue density at $t = 0$ (the density can be made arbitrarily close to 1) such that the limit set of f_t in the neighborhood of the cycle is hyperbolic and coincides with the disjoint union of the homoclinic classes of P_t and Q_t.

3 Blenders

While the arguments in Section 2.1.2 require a fair amount of differentiability, the conclusion of Theorem 1 does extend to the C^1 topology, as we shall see in Section 3.2. One way to prove this is using the notion of blender, which we discuss here.

Blenders were introduced in [3], and have been used to obtain persistence of cycles and transitivity. In this regard, they play a role similar to that of thick Cantor sets in the phenomenon of persistence of tangencies in [24, 25]. With the advantage that, unlike thick Cantor sets, blenders are robust already in the C^1 category.

Let us point out that what makes blenders so useful is not so much the dynamics they carry, but rather their geometry. Indeed, in simple terms, *a blender is a hyperbolic basic set such that convenient projections of its stable, or unstable, set have larger topological dimension than the stable, or unstable, set itself.* One such model appeared already in Section 1. This was a hyperbolic Cantor $H \subset \mathbb{R}^3$, with 1-dimensional stable manifolds and such that the projection of its stable set $W^s(H)$ onto the plane generated by the stable and the weak unstable directions contained a whole open set. In fact, the topologically 1-dimensional set $W^s(H)$ intersects any curve roughly parallel to the strong unstable direction. See also the construction in Section 2.1.2 Blenders are occur naturally in the unfolding of non-critical heterodimensional cycles. For instance, in the model considered in Section 2.1.2, a blender is obtained by considering those points having an expanding return to the reference fundamental domain D^t_Q. These points form a saddle hyperbolic set with 1-dimensional stable direction. The main step of the proof of Theorem 1 consisted in checking that the (one-dimensional) local stable manifold of the saddle Q transversely intersects any disk transverse to the 2-dimensional manifold $W^s(P)$. Since the points in the blender are homoclinically related to Q, it is not hard to deduce the same assertion for the stable manifold of any point of the blender.

3.1 A simplified model

We start by describing a simple example displaying the main features of blenders. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism of $\mathbb{R}^2$ such that the rectangle $R = [0,1]^2$ is a Smale horseshoe for f:

- $R \cap f(R)$ is the union of the vertical sub-rectangles $A = I_a \times [0,1]$ and $B = I_b \times [0,1]$, where I_a and I_b are closed intervals contained in $(0,1)$.
- $R \cap f^{-1}(R)$ is the union of the horizontal sub-rectangles $f^{-1}(A) = [0,1] \times J_a$ and $f^{-1}(B) = [0,1] \times J_b$, where J_a and J_b are closed intervals contained in $(0,1)$.
- The restrictions of f to $f^{-1}(A)$ and to $f^{-1}(B)$ are affine maps with linear part $\begin{pmatrix} \pm\frac{1}{3} & 0 \\ 0 & \pm 3 \end{pmatrix}$. In particular, such restrictions preserve the horizontal and vertical directions.

Denote by $q = (x_0, y_0)$ the unique fixed point of f in A. Consider now a diffeomorphism $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that in $[-1,1] \times R$ is of the form

$$F(t,x,y) = \begin{cases} (\frac{5}{4}\,t, f(x,y)) & \text{if } (x,y) \in f^{-1}(A) \text{ and } t \in [-1,1], \\ (\frac{5}{4}\,t - \frac{1}{2}, f(x,y)) & \text{if } (x,y) \in f^{-1}(B) \text{ and } t \in [-1,1]. \end{cases}$$

The dynamics of F in the cube $\Gamma = [-1,1] \times R$ is hyperbolic, with $\mathbb{T} \oplus \mathbb{Y}$ as unstable direction ($\mathbb{T}$ is weak unstable and $\mathbb{Y}$ is strong unstable), and $\mathbb{X}$ as stable direction. The *blender* is the maximal invariant set inside Γ. See Figure 5.

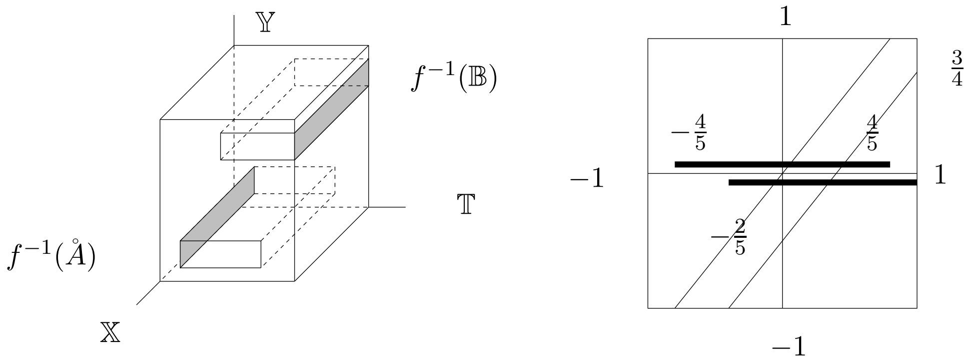

Figure 5: Blender with one-dimensional reduction

The distinctive geometric feature of blenders is stated in the next lemma. For the statement we need some terminology. Note that $\Gamma \cap F(\Gamma)$ contains the cubes $\mathring{A} = [-1,1] \times A$ and $\mathbb{B} = [-1, \frac{3}{4}] \times B$ and that $Q = (0, x_0, y_0) = (0, q) \in \mathring{A}$ is a hyperbolic fixed saddle of F of index 2. Let $W^s_{loc}(Q) = \{0\} \times [0,1] \times \{y_0\}$ be the connected component of $W^s(Q) \cap \Gamma$ that contains Q.

- A *vertical segment to the right of* $W^s_{loc}(Q)$ is a segment $\sigma_{t,x} = \{t\} \times \{x\} \times [0,1]$, where $x \in [0,1]$ and $0 < t \leq 1$.
- A *vertical strip to the right of* $W^s_{loc}(Q)$ is a rectangle $\Delta = [t_1, t_2] \times \{x\} \times [0,1]$, where $x \in [0,1]$ and $0 < t_1 < t_2 \leq 1$. The number $w(\Delta) = (t_2 - t_1)$ is the *width of* Δ.

Lemma 3.1. *(distinctive property) Every vertical strip* Δ *to the right of* $W^s_{loc}(Q)$ *intersects* $W^s(Q)$*.*

Proof: We show that if Δ is a vertical strip to the right of $W^s_{loc}(Q)$ then either $F(\Delta)$ intersects $W^s_{loc}(Q)$, or else $F(\Delta) \cap \Gamma$ contains a vertical strip Δ_1 to the right of $W^s_{loc}(Q)$ with $w(\Delta_1) = (5/4)\, w(\Delta)$.

To deduce the lemma from this claim, start with Δ as in the statement and consider its forward iterates. For as long as these iterates do not intersect the local stable manifold of Q, they contain vertical strips whose widths grow exponentially fast. Since the width is bounded above by 1, some iterate $F^n(\Delta)$ must intersect $W^s_{loc}(Q)$. This implies that Δ intersects the (global) stable manifold of Q. Hence, we are left to prove the claim in the previous paragraph.

Consider any vertical strip $\Delta = [t_1, t_2] \times \{x\} \times [-1, 1]$ to the right of $W^s_{loc}(Q)$. Write $\Delta_A = F(\Delta) \cap \mathring{A}$ and $\Delta_B = F(\Delta) \cap \mathbb{B}$.

- If $t_2 \leq \frac{4}{5}$ then $[\frac{5}{4}\, t_1, \frac{5}{4}\, t_2] \subset (0, 1]$. It follows that Δ_A is a vertical strip to the right of $W^s_{loc}(Q)$, with width $w(\Delta_A) = \frac{5}{4}\, w(\Delta)$. Thus the second possibility in the claim holds.
- If $t_2 \in (\frac{4}{5}, 1]$ then $\frac{5}{4}\, t_2 - \frac{1}{2} \in (\frac{1}{2}, \frac{3}{4}]$. By the definition of F, there is x' such that

$$\Delta_B = F(\Delta) \cap \mathbb{B} = \{[\frac{5}{4}\, t_1 - \frac{1}{2}, \frac{5}{4}\, t_2 - \frac{1}{2}]\} \times \{x'\} \times [0, 1].$$

  There are now two subcases:

- If $\frac{5}{4}\, t_1 - \frac{1}{2} > 0$ then Δ_B is a vertical strip to the right of $W^s_{loc}(Q)$, with width $w(\Delta_B) = (5/4)\, w(\Delta)$. So, the second possibility in the claim holds in this case too.
- Otherwise $\frac{5}{4}\, t_1 - \frac{1}{2} \leq 0$ and $\frac{5}{4}\, t_2 - 1/2 \geq 0$. Then Δ_B meets $W^s_{loc}(Q) = \{0\} \times [0, 1] \times \{y_0\}$, which corresponds to the first possibility in the claim.

The proof is complete. □

As we did in the previous section, we may consider the quotient of the dynamics by the sum of the strong stable and strong unstable directions. This yields an iterated functions system $\mathcal{G}(\phi_1, \phi_2)$ with two generators, as follows:

- for $t \in [-\frac{4}{5}, \frac{4}{5}]$ consider $\phi_1(t) = \frac{5}{4}\, t$, corresponding to the restriction of F to the domain $F^{-1}(\mathring{A}) = [-\frac{4}{5}, \frac{4}{5}] \times f^{-1}(A)$,
- for $t \in [-\frac{2}{5}, 1]$ consider $\phi_2(t) = \frac{5}{4}\, t - \frac{1}{2}$, corresponding to the restriction of F to the domain $F^{-1}(\mathbb{B}) = [-\frac{2}{5}, 1] \times f^{-1}(B)$.

Notice that both maps ϕ_1 and ϕ_2 are expanding, and their domains of definition overlap on the interval $[-\frac{2}{5}, \frac{4}{5}]$. This reflects the fact that given any vertical segment $\sigma = \{t\} \times \{x\} \times [0, 1]$ with t in the overlapping interval, there are x_1 and x_2 such that the vertical segments $\{\phi_1(t)\} \times \{x_1\} \times [0, 1]$ and $\{\phi_2(t)\} \times \{x_2\} \times [0, 1]$ are the two connected components of $F(\sigma) \cap \Gamma$.

Lemma 3.1 may now be reformulated for the iterated functions system, as follows. The proof is similar to the previous case.

Lemma 3.2. *Given any open interval* $\alpha \subset [0, 1]$ *there is* $\phi \in \mathcal{G}(\phi_1, \phi_2)$ *such that* $0 \in \phi(\alpha)$*.*

To illustrate the dynamical usefulness of blenders, we close this section with the following direct consequence of Lemma 3.1:

Lemma 3.3. *Suppose that there is a hyperbolic periodic point* P *of* F *of index* 1 *whose one-dimensional unstable manifold crosses* Γ *along a vertical segment to the right of* $W^s_{loc}(Q)$. *Then* $W^s(P) \subset \overline{W^s(Q)}$.

So the one-dimensional stable manifold of Q looks like a 2-dimensional manifold, as its closure contains the 2-dimensional $W^s(P)$.

3.1.1 Relaxing the construction

So far we discussed a very simple affine model of blender. Of course, for most applications we need more flexible versions of the previous constructions, robust under C^1 small perturbations. Just as an example, in the context of Lemma 3.3 we would like to have the conclusion when the segment is just approximately vertical. We are going to explain how this flexibilization is carried out, starting with perturbations of the affine model, and to indicate other ways in which the construction may be generalized. Motivated by this, we discuss possible definitions of the notion of blender.

In the affine model before, $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{Y}$ is a dominated splitting on Γ, with $\mathbb{X}$ as strong stable, $\mathbb{Y}$ as strong unstable, and $\mathbb{T}$ as central (weak unstable) direction. Hence, there are cone fields $\mathcal{C}^s$, $\mathcal{C}^{uu}$, $\mathcal{C}^u$ around the directions $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{T} \oplus \mathbb{Y}$, respectively, strictly invariant under the derivative DF. These cone fields allow us to define *almost vertical segments (to the right of* $W^s_{loc}(Q)$*)*, as curve segments tangent to $\mathcal{C}^{uu}$ and crossing Γ (on the right hand side of the local stable manifold), as well as *almost vertical strips (to the right of* $W^s_{loc}(Q)$*)*, as 2-dimensional disks through Γ that are tangent to $\mathcal{C}^u$ and foliated by almost vertical segments (to the right of $W^s_{loc}(Q)$). The notion of width extends to these almost vertical strips: the length of the shortest curve transverse to $\mathcal{C}^{uu}$ joining the two boundary components inside the strip.

The families of almost vertical segments and strips are invariant under forward iteration, in the same sense as before for truly vertical segments and strips. Moreover, the widths of almost vertical strips not intersecting $W^s_{loc}(Q)$ increase exponentially under iteration, again in the same sense as before. This means that the proof of Lemma 3.1 carries through to this context, only with extra technical difficulties: *every almost vertical strip at the right of* $W^s_{loc}(Q)$ *intersects* $W^s(Q)$.

Now, the cone fields remain invariant for any diffeomorphism in a C^1 neighborhood of F. Moreover, the width estimate is also robust under any small C^1 perturbation, because it depends only on the fact that the derivative expands all vectors in the cone $\mathcal{C}^u$. This means that the conclusion of the previous paragraph is true for any map near F: *the distinctive blender property is a robust property!*

Blenders from a constructive viewpoint. One way the construction of blenders may be relaxed, with respect to the affine model in Section 3.1, is allowing for more complicated combinatorics: any number of dynamical components, with variable return times to the ambient box.

Consider a C^1 diffeomorphism F and a box $\Gamma = D^k \times [-1,1] \times D^l$, where $D^i = [-1,1]^i$, in the ambient space, endowed with cone fields $\mathcal{C}^s$ and $\mathcal{C}^{uu} \subset \mathcal{C}^u$ around the directions of D^k, D^l, and $[-1,1] \times D^l$, respectively. Assume that there are sub-boxes $(\Gamma_i)_{i=1}^k$ of Γ, and return times $(n_i)_{i=1}^k$ such that every return map $F^{n_i}\colon \Gamma_i \to \Gamma$ preserves both cone fields and expands uniformly all vectors in $\mathcal{C}^u$. Define as above almost vertical (l-dimensional) disks, tangent to $\mathcal{C}^{uu}$, and almost vertical ($l+1$-dimensional) strips, tangent to $\mathcal{C}^u$ and foliated by almost vertical disks crossing Γ.

Define the width $w(\mathcal{S})$ of an almost vertical strip $\mathcal{S}$, again as before. We turn the type of property in Lemma 3.1 into our first definition of a blender:

Definition 3.4. *(Γ, F) is a* blender associated to a periodic saddle Q of index $l+1$ *if for every almost vertical strip $\mathcal{S}$ crossing Γ there is $i \in \{1, \ldots, k\}$ such that*

- *either $F^{n_i}(\mathcal{S} \cap \Gamma_i)$ contains an almost vertical strip $\mathcal{S}'$ intersecting $W^s_{loc}(Q)$*
- *or $F^{n_i}(\mathcal{S} \cap \Gamma_i)$ contains an almost vertical strip $\mathcal{S}'$ with $w(\mathcal{S}') \geq \lambda\, w(\mathcal{S})$, where $\lambda > 1$ is independent of the strip $\mathcal{S}$.*

The periodic point Q in the definition does not need to belong to the cube Γ: our arguments just require that $W^s(Q)$ contains a disk Δ crossing Γ and tangent to the cone field $\mathcal{C}^s$. The reader may check that the unfolding of the heterodimensional cycle in Section 2.1.2 yields a blender inside a box close to the fixed point Q but not necessarily containing it.

In the particular case where the maps F^{n_i} preserve the D^k and D^l directions, as was the case for the affine blender, one may consider the quotient dynamics of F by the sum $D^k \oplus D^l$. In this way we obtain, for each i, a one-dimensional expanding map $\phi_i : I_i \to [0,1]$ induced by F^{n_i} on a compact $I_i \subset [0,1]$. (Γ, F) being a blender corresponds to the maps ϕ_i having the covering property: $\cup_i \mathrm{inter} I_i = (0,1)$. A point $x \in [0,1]$ is *periodic* for the iterated functions system $\mathcal{G}(\phi_1, \ldots, \phi_k)$ if $\phi(x) = x$ for some $\phi \in \mathcal{G}$. Extending Lemma 3.2, we have

Proposition 3.5. *Given any system $\mathcal{G}(\phi_1, \ldots, \phi_k)$ of iterated expanding functions with the covering property, there is a finite set $\mathcal{F} = \{x_1, \ldots, x_s\}$, $1 \leq s < k$, of periodic points such that, for any open interval $\alpha \subset [0,1]$, there is $\phi \in \mathcal{G}$ with $\phi(\alpha) \cap \mathcal{F} \neq \emptyset$.*

Blenders from an operational viewpoint. We now adopt a different point of view: we will think of them as tools for proving transitivity. Thus we propose another definition which takes into account their properties, rather than their construction.

Definition 3.6. *Let $F : M \to M$ be a diffeomorphism and Q be a hyperbolic saddle of index $l+1$. We say that F* has a blender associated to Q *if there is a C^1 neighborhood $\mathcal{U}$ of F, and a C^1 open set $\mathcal{D}$ of embeddings of an l-dimensional disk D^l into M, such for every $G \in \mathcal{U}$, every disk $D \in \mathcal{D}$ intersects the closure of $W^s(Q_G)$, where Q_G is the continuation of the periodic point Q for G.*

The set $\mathcal{D}$ in the definition is the *superposition region* of the blender. For instance, in the simplified model, $l = 1$ and the superposition region is the family of all almost vertical segments (that is, tangent to the vertical cone field and crossing the box Γ) to the right of $W^s_{loc}(Q)$. Bearing in mind the distinctive property of blenders, one says that the blender is *activated* by a saddle P of index l if the unstable manifold of P contains a disk of the superposition region.

Lemma 3.7. *Let F be a diffeomorphism having a blender associated to a saddle Q of index $l+1$. Suppose that the blender is activated by a saddle P of index l. Then, for every diffeomorphism G in a C^1 neighborhood of F, the closure of $W^s(Q_G)$ contains $W^s(P_G)$, where P_G and Q_G are the continuations of P and Q for G.*

The point with this result is that it means that the $(d-l-1)$-dimensional manifold $W^s(Q_G)$ of Q_G looks like it contains a manifold with higher dimension, namely $W^s(P_G, G)$.

Proof: Let $\mathcal{D}$ the superposition region of the blender. As compact parts of invariant manifolds depend continuously on the diffeomorphism, $W^u(P_G)$ contains a disk $D_G \in \mathcal{D}$, for all G close to F. Let U be any open set intersecting $W^s(P_G)$. By the λ-lemma, for every n big enough, $G^n(U)$ contains a C^1-open set of disks arbitrarily close to D_G. Then these disks are in the superposition region. By definition of blender, it follows that $W^s(Q_G)$ intersects $G^n(U)$. As the stable manifold is invariant, it also intersects U. □

3.2 Partially hyperbolic cycles

We are going to see that the formation of intermingled homoclinic classes is a frequent phenomenon in the unfolding of cycles, even in the C^1 topology. Consider a diffeomorphism f with a cycle associated to saddles P and Q of indices p and $p+1$, respectively. We say that the cycle is *C^k-far from homoclinic tangencies* if every diffeomorphism g which is C^k close to f does not have homoclinic tangencies associated either of the continuations P_g and Q_g of P and Q.

Theorem 2 (D., Rocha [15]). *Let f be a C^1 diffeomorphism with a heterodimensional cycle associated to saddles P and Q of indices p and $q = p+1$. Suppose that the cycle is C^1-far from homoclinic tangencies. Then there is an open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ whose closure contains f and such that the homoclinic classes of P_g and Q_g coincide for all $g \in \mathcal{U}$.*

The conclusion of this theorem also holds for non-critical connected cycles, without any assumption on the distortion. In that case we even have the following version for one-parameter families generically unfolding the cycle: *Given $t > 0$, let $B(t)$ be the interior of the set of parameters $s \in [-t, t]$ such that the homoclinic classes of P_s and Q_s coincide. Then $B(t)$ has positive Lebesgue measure for every $t > 0$.* Moreover, if the cycle corresponds to a first bifurcation, the relative density of $B(t)$ at the bifurcation is positive.

Theorem 2 is a consequence of the following proposition, which uses blenders with several generators.

Proposition 3.8. *Let f be a diffeomorphism as in Theorem 2. Then there is an open set $\mathcal{V} \subset \mathrm{Diff}^1(M)$ whose closure contains f such that for every g in $\mathcal{V}$ there are blenders $\Gamma(g)$ and $\Gamma(g^{-1})$, defined for g and g^{-1}, respectively, such that:*

- *$\Gamma(g)$ is associated to a hyperbolic periodic point R_g homoclinically related to Q_g and is activated by P_g.*
- *$\Gamma(g^{-1})$ is associated to a hyperbolic periodic point S_g homoclinically related to P_g and is activated by Q_g.*

Observe that $\Gamma(g^{-1})$ is defined for g^{-1}: to be activated by Q_g means that the stable manifold of Q_g contains a (p-dimensional) disk of the superposition region. Since R_g and Q_g (respectively, S_g and P_g) are homoclinically related, Lemma 3.7 and Proposition 3.8 imply that, for every g in the open set $\mathcal{V}$,

$$W^s(P_g) \subset \overline{W^s(R_g)} = \overline{W^s(Q_g)} \quad \text{and} \quad W^u(Q_g) \subset \overline{W^u(S_g)} = \overline{W^u(P_g)}.$$

This is the main step towards showing that the homoclinic classes of P_g and Q_g coincide.

4 Robust transitivity

A diffeomorphisms $f : M \to M$ is *robustly transitive* if every diffeomorphism in a C^1 neighborhood has dense orbits. More generally, an invariant compact set Λ is robustly transitive if there is an open neighborhood U of Λ and a neighborhood $\mathcal{V}$ of the diffeomorphism f in $\mathrm{Diff}^1(M)$ such that for every $g \in \mathcal{V}$ the maximal invariant set Λ_g of g in $\bar{U}$, $\Lambda_g = \cap_{i \in \mathbb{Z}} g^i(U)$, is a transitive set contained in U and $\Lambda_f = \Lambda$. These sets were introduced in [11] as a generalization of the basic sets of the Smale theory. The description of these systems is an important challenge: on the one hand, being robust, they can not be ignored in any global picture of dynamical systems; on the other hand, their dynamics is often not stable, indeed many different bifurcations may occur inside the set, without affecting transitivity. However, such bifurcations can never produce sinks nor sources. In the case of surface diffeomorphisms this implies that the periodic points inside a robustly transitive set are persistently hyperbolic, in a uniform fashion:

Theorem 3 (Mañé [23]). *Every robustly transitive set of a surface C^1 diffeomorphism is a hyperbolic set.*

Section 4.1 presents several constructions of robustly transitive diffeomorphisms. A point to be remarked about all these examples is that, while they need not be uniformly hyperbolic, they all have some weak form of hyperbolicity: at least, a dominated splitting of the tangent bundle. Indeed, the main conclusion in this chapter is that *existence of a dominated splitting is a necessary condition for robust transitivity.* (See Theorem 4). The basic idea behind this statement is the observation that in the absence of a dominated splitting one can perturb the derivative at some periodic orbit, to turn that orbit into an attractor or a repeller, thus, breaking transitivity. A consequence of this main result is that homoclinic classes of C^1-generic diffeomorphisms either admit a dominated splitting or are contained in the closure of (infinitely many) periodic attractors or repellers.

In fact, in some situations (especially in low dimensions) one has the stronger property of partial hyperbolicity, and that is better understood. The main reason is that may use strong stable and strong unstable foliations as very useful tools for understanding topological and ergodic properties of the system.

4.1 Examples of robust transitivity

The best known examples of robustly transitive diffeomorphisms are the Anosov diffeomorphisms, especially the linear ones, that is the diffeomorphisms induced on the torus $\mathbb{T}^n$ by a hyperbolic map in $GL(n, \mathbb{Z})$. By focus here on non-hyperbolic examples. The first examples of these diffeomorphisms were constructed in the seventies by Shub [31] and Mañé [22], respectively, on $\mathbb{T}^4$ and on $\mathbb{T}^3$. Their examples are partially hyperbolic with a dominated splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ into 3 non-trivial subbundles, the central subbundle E^c being one-dimensional and non-hyperbolic. Much later, [2] and [7] gave new classes of robustly transitive diffeomorphisms, on much more general manifolds and also satisfying weaker hyperbolicity conditions.

Let us list the three classes of known examples:

(A) Perturbations of the time-1 of any transitive Anosov flow.

These examples are isotopic to identity, and they exist in many manifolds, other than the tori, already in dimension 3. It is not known which 3-manifolds do support such diffeomorphisms.

(B) Skew-products: $\mathbb{T}^n \times N \to \mathbb{T}^n \times N$, $(x, y) \mapsto (A(x), B(x, y))$ and their perturbations, where A is a transitive Anosov diffeomorphism, N is any compact manifold, and the behavior in the y direction is dominated by that of A along its stable and unstable directions.

In these examples the non-hyperbolic central direction may be chosen tangent to the fibers $\{x\} \times N$ and so $\dim E^c = \dim N$, providing examples of robustly transitive diffeomorphism having a central direction of any dimension.

(C) Robustly transitive diffeomorphisms derived from Anosov, that is, obtained by deforming Anosov torus diffeomorphisms by isotopy. They may be constructed

- (in dim ≥ 3) without any stable subbundle, that is, $TM = E^u \oplus E^{cs}$, where E^{cs} is indecomposable and non-hyperbolic. Analogously, without any unstable subbundle.
- (in dim ≥ 4) without any hyperbolic (stable or unstable) subbundle, that is, $TM = E^{cu} \oplus E^{cs}$ is a dominated splitting where E^{cs} and E^{cu} are indecomposable and non-hyperbolic.

We see in Theorem 4 that these two last constructions correspond to the weakest possible hyperbolicity for robustly transitive systems.

4.1.1 An example of Shub

Consider Anosov diffeomorphisms A and B defined on the torus $\mathbb{T}^2$ such that the product map $(A, B)\colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ is partially hyperbolic with central bundle E^c tangent to the fibers $\{x\} \times \mathbb{T}^2$ (for that it is enough to choose $A = B^n$ for any $n > 1$). Assume that A has fixed points p and q and B has a fixed point r.

Consider a map $f\colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ of the form $f(x, y) = (A(x), F_x(y))$, partially hyperbolic with E^c tangent to the fibers $\{x\} \times \mathbb{T}^2$, and also that $F_p = B$.

Proposition 4.1. *The diffeomorphism f is robustly transitive.*

Proof: As the central foliation $\mathcal{F}^c$ (tangent to the subbundle E^c) is differentiable, Theorem 14 implies that the pair $(\mathcal{F}^c, f)$ is structurally stable. It follows that the stable manifold and the unstable manifold of the continuation of the fiber $\{p\} \times \mathbb{T}^2$ are both dense in $\mathbb{T}^4$ for every C^1 perturbation of f. One deduces that the invariant manifolds of the fixed point (p, r) are both robustly dense in $\mathbb{T}^4$. This implies that f is robustly transitive: the stable manifold of the fixed point intersect any open set U of $\mathbb{T}^4$, thus the iterates $f^n(U)$, $n > 0$ accumulate arbitrarily large disks in the unstable manifold, which, on their turn, intersect any open set V. That is, $f^n(U) \cap V \neq \emptyset$. $\square$

To ensure that these examples are non-hyperbolic, it is enough to take $F_q\colon \mathbb{T}^2 \to \mathbb{T}^2$ having periodic points with different Morse indices (a saddle and a sink): then f also has points with different Morse indices, and transitivity implies that it can not be hyperbolic.

4.1.2 An example of Mañé

Consider a linear Anosov map $A\colon \mathbb{T}^3 \to \mathbb{T}^3$ having three real eigenvalues, $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$ and $3 < \lambda_3$. Let $\mathcal{F}^c$ be the linear foliation corresponding to the eigenvalue λ_2. Fix small domains $C' \subset C$

with C' much smaller than C. Consider a partially hyperbolic perturbation f of A preserving the foliation $\mathcal{F}^c$, having it as its central foliation, and coinciding with A outside C'.

Assume that the expansion of f in the strong-unstable direction remains bigger than 3. If C is small enough, there is a constant $L > 0$ such that, for every unstable segment σ of length $(\sigma) \geq L$, the set $(f(\sigma) \setminus C)$ contains an unstable segment of length bigger than L. On the other hand, given any unstable segment σ the lengths of its forward iterates $f^n(\sigma)$ are eventually bigger than L. This shows that every unstable segment contains some point x_0 whose forward orbit intersects C finitely many times only.

We claim that, given any small central segment α through such a point x_0, the lengths of its forward iterates increase exponentially. This is obvious while the iterates are shorter than the distance from C' to the complement of C, just because $f^n(\alpha)$ remains outside C' (discard finitely many iterates if necessary), where the central direction is actually expanding. When the segments are long just observe that the fraction of $f^n(\alpha)$ inside C' is very small: expansion of the big sub-segments outside C' suffices for expansion of the total length.

Given an open set $U \subset M$ take an unstable segment $\sigma \subset U$, a point $x_0 \in \sigma$ visiting C finitely many times only, and a central segment $\alpha \subset U$ through x_0. By Theorem 14 each leaf of $\mathcal{F}^c$ is dense in $\mathbb{T}^3$, in a robust fashion. So, the fact that the length of $f^n(\alpha)$ goes to infinity implies that the orbit of α is dense, and so the same is true for the orbit of U. This gives (robust) transitivity of f.

To asure that this construction yields non-hyperbolic maps, Mañé considered a fixed point p of A and performed a saddle-node bifurcation along a central leaf $\mathcal{F}^c$ inside a small neighborhood C' of p. In this way, f has periodic points with different Morse indices 2 and 1.

4.1.3 A local criterium for robust transitivity

In this section we explain the method used in [2] and [15] for proving robust transitivity. Let us start with the following simple criterium, which we used in Shub's example.

Lemma 4.2. *(Basic transitivity criterium) Suppose f has a hyperbolic periodic point p whose invariant manifolds are both (robustly) dense in M. Then f is (robustly) transitive.*

In what follows, we restrict ourselves to 3-dimensional manifolds. See Section 3 for details and notation about blenders. In the next statement note that $W^s(p)$ and $W^u(q)$ are two-dimensional.

Proposition 4.3. *(Blender criterium) Assume that*

1. *f is partially hyperbolic on the whole manifold and there is a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ into three non-trivial subbundles such that E^u is uniformly expanding and E^s is uniformly contracting.*

2. *There are a constant $L > 0$ and hyperbolic periodic points p and q with Morse indices 1 and 2, respectively, such that the stable manifold $W^s(p)$ of p intersects every unstable segment of length bigger than L, and the unstable manifold $W^u(q)$ of q intersects every stable segment of length bigger than L.*

3. *The point q is homoclinically related to a blender Γ such that $E^c \mid \Gamma$ is uniformly expanding, and the one-dimensional unstable manifold $W^u(p)$ of p intersects every superposition region of Γ (we say that p activates Γ).*

Then f is robustly transitive.

Proof: From assumption 2 it follows that $W^s(p)$ intersects all unstable segments, since the length of forward iterates of unstable segments goes to infinity, and the stable manifold is invariant. This implies that the stable manifold of p is dense in M.

By compactness of the family of unstable segments of length $2L$, and the fact that the unstable foliation and the stable manifold of p depend continuously on the dynamics, one easily gets that $W^s(p,g)$ intersects every unstable segment of length $2L$ for all g close to f. As before, this implies the density of $W^s(p,g)$. In other words, the stable manifold of p is robustly dense in M. The same arguments give robust density of the unstable manifold of q.

Using the principle of superposition of the blenders from Section 3, we deduce that the closure of the stable manifold of q robustly contains the stable manifold of p, thus it is robustly dense in M. At this point we know that both invariant manifolds of q are robustly dense in M, and so the basic criterium in Lemma 4.2 implies that f is robustly transitive. $\square$

The blender criterium is used in [2] to prove that both

- the time-1 map of a transitive Anosov flow and
- the product of an Anosov map by the identity on a compact manifold N

are approximated by (non-hyperbolic) robustly transitive diffeomorphisms. For the first situation one perturbs the dynamics along a closed orbit of the flow to get a pair of periodic orbits with different Morse index. In the second situation, one uses a chain of blenders to find a periodic point of index u (the dimension of the unstable manifold of the Anosov map) such that the closure of its unstable manifold contains the unstable manifold of a point of index $\dim N + u$.

4.2 Robust transitivity without hyperbolic directions

All robustly transitive diffeomorphisms described above have hyperbolic directions, that is, either uniformly contracting or uniformly expanding. Here we describe the construction in [7] of robustly transitive diffeomorphisms without any hyperbolic direction. This construction follows closely the one in Section 4.1.2, so we just emphasize the main differences.

Consider a linear Anosov map A of the torus $\mathbb{T}^4$ having 4 real eigenvalues, $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 < \lambda_4$. Fix small domains $C_1' \subset C_1$ and $C_2' \subset C_2$, with C_1 and C_2 disjoint. Fix an A-invariant cone field $\mathcal{C}^{cu}$ and an A^{-1}-invariant cone field $\mathcal{C}^{cs}$, around the planes corresponding to the expanding eigenvalues λ_3, λ_4 and the contracting eigenvalues λ_1, λ_2, respectively. Now consider diffeomorphisms f coinciding with A outside $C_1 \cup C_2$ and satisfying

1. Df preserves $\mathcal{C}^{cu}$ and expands area uniformly in this cone field.
2. Df^{-1} preserves $\mathcal{C}^{cs}$ and expands area uniformly in this cone field.
3. Df expands uniformly the norm of vectors in $\mathcal{C}^{cu}$ outside C_1'.
4. Df^{-1} expands uniformly the norm of vectors in $\mathcal{C}^{cs}$ outside C_2'.

By choosing sufficiently thin cone fields, we verify that there exists a constant $L > 0$ such that every center unstable disk D_1 (tangent to $\mathcal{C}^{cu}$) of radius bigger than L intersects any center stable disk D_2 (tangent to $\mathcal{C}^{cs}$) of radius larger than L.

Arguing as in Mañé's example, but using uniform expansion of area instead of uniform expansion of norms, we show that every center unstable disk D^1 contains a point whose forward orbit meets C_1 only finitely many times. This allows us to show that $f^n(D_1)$ contains a center unstable disk of radius bigger than L, for every large $n > 0$. The same argument shows that the large negative iterates of any center stable disk D_2 contains a center stable disk of radius bigger that L.

Finally, given any pair of open sets U and V, we may choose a center unstable disk $D_1 \subset U$ and a center stable disk $D_2 \subset V$. Thus, forward iterates of U and backward iterates of V intersect eventually: $f^n(U) \cap f^{-m}(V)$ for all large n and m. This implies that f is transitive.

In order to ensure that f does not preserve any hyperbolic subbundle, we choose the diffeomorphism such that C_1 contains a fixed point p_1 of f of index 2 with non-real contracting eigenvalues together with a fixed point q_1 of index 3. This prevents the existence of uniformly contracting subbundles: the presence of p_1 implies that a stable subbundle would have to be of dimension at least 2, while q_1 implies that the dimension is at most 1. We also choose f so that C_2 contains a fixed point p_2 of index 2 with non-real expanding eigenvalues together with a fixed point q_2 of index 1. Arguing as before, this prevents the existence of any uniformly expanding subbundle.

4.3 Consequences of robust transitivity

The main result in this section is the following generalization of Theorem 3 to arbitrary dimension:

Theorem 4 (Bonatti, D., Pujals, Ures [4, 11]). *Every robustly transitive set of a C^1 diffeomorphism is volume hyperbolic.*

An invariant set Λ is called *volume hyperbolic* if there exists a splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ into Df-invariant subbundles such that

1. the splitting is *dominated*: there exists $m \geq 1$ such that
$$\|Df^n \mid E^i(x)\| \, \|(Df^n \mid E^j(x))^{-1}\| \leq 1/2$$
for every $n \geq m$, $i > j$, and $x \in \Lambda$.
2. the extreme subbundles E^1 and E^k are uniformly volume expanding and volume contracting: for all $n \geq m$ and $x \in \Lambda$,
$$|\det(Df^{-n} \mid E^1(x))| \leq 1/2 \quad \text{and} \quad |\det(Df^n \mid E^k(x))| \leq 1/2$$

In dimension 2, the only possibility is for the splitting to be into two 1-dimensional subbundles $(2 = 1 + 1)$, and then volume hyperbolicity is the same as hyperbolicity. Therefore, Theorem 4 contains Theorem 3. Moreover, in dimension 3 there are two possibilities, $3 = 2 + 1$ and $3 = 1 + 1 + 1$. In either case, at least one of the subbundles is 1-dimensional and hyperbolic: either uniformly expanding or uniformly contracting. Thus, in this case robust transitivity implies partial hyperbolicity [11]: at least one of the invariant subbundles is hyperbolic. These results are best possible, by the examples in Section 4.1.

As mentioned before, the way to prove Theorem 4 is by showing that the lack of a dominated splitting leads to the creation of a sink or source, after a convenient perturbation. If the lack of dominated splitting is persistent, this yields infinitely many coexisting sinks or sources! We may apply this idea to a transitive set naturally associated to every hyperbolic saddle point p of a diffeomorphism f, its *homoclinic class*. This is defined as the closure of the transverse intersections between its invariant manifolds and denoted by $H(p, f)$.

Theorem 5 (Bonatti, D., Pujals [4])). *There exists a residual subset $\mathcal{R}$ of $\operatorname{Diff}^1(M)$ such that for every diffeomorphism $f \in \mathcal{R}$ and periodic point p of f,*

- *$H(p, f)$ is contained in the closure of infinitely many sources or sinks,*
- *or else $H(p, f)$ admits a dominated splitting.*

In the rest of this section we explain the main ideas in the proofs of the theorems above. Roughly speaking, Theorem 4 follows from an idea in linear algebra: in the absence of domination, the eigenspaces of a linear map obtained by multiplication of several linear maps are very unstable: by small perturbation of each of the factors, one can mix the eigenvalues in order to get a homothetic transformation, which then leads to a new sink or a new source.

4.3.1 Lack of domination and creation of sinks or sources

We begin the exposition with the, simpler, 2-dimensional case. The argument we are going to present, due to Mañé [23], shows the following: *if a set $\mathcal{P}$ of hyperbolic periodic saddles of a diffeomorphism of a compact surface does not admit a dominated splitting then it is possible to turn one of such periodic points to be a new sink or source, by a C^1 small perturbation.*

We begin with an elementary property of matrices in $GL(2, \mathbb{R})$. Denote by R_α the rotation of angle α in $\mathbb{R}^2$.

Lemma 4.4. *Consider an orientation preserving matrix $A \in GL(2, \mathbb{R})$ having two different real eigenvalues and whose eigenspaces form an angle less than $\varepsilon > 0$. Then there is $t \in [-\varepsilon, \varepsilon]$ such that $R_t \circ A$ has a pair of conjugate complex eigenvalues.*

A way to prove this is to consider the action h_t of $R_t \circ A$ on the projective space $\mathbb{RP}^1$, identified with the circle S^1. By hypothesis, the map h_0 has two hyperbolic fixed points, at distance less than ε. Moreover, outside the small interval between these fixed points, the sign of $h_0(x) - x$ is constant, say positive. So, $h_0(x) - x > -\varepsilon$ for every x in S^1. Consequently, $h_\varepsilon(x) - x > 0$ for all x in S^1. This implies that the rotation number of the circle map h_t is not constant in $[0, \varepsilon]$. Choosing t in this interval such that h_t has non-zero rotation number, we get that the eigenvalues of $R_t \circ A$ cannot be real, as claimed.

Now consider the natural splitting $T_pM = E^u(p) \oplus E^s(p)$ into eigenspaces, defined over $\mathcal{P}$.

Lemma 4.5. *Suppose that the angle $\operatorname{angle}(E^s(p), E^u(p))$ between $E^s(p)$ and $E^u(p)$ is not bounded away from 0 for $p \in \mathcal{P}$. Then there are g arbitrarily C^1-close to f and $p \in \mathcal{P}$ such that p is either a sink or a source of g.*

We need the following result that allows to perform dynamically perturbations of the linear co-cycle defined by the derivative of a diffeomorphism f. It may also be stated uniformly for any g in a neighborhood of f.

Theorem 6 (Franks [17]). *Let f be a C^1-diffeomorphism defined on a closed manifold M and consider any $\delta > 0$. Then there is $\varepsilon > 0$ such that, given any finite set Σ, any neighborhood U of Σ, and any linear maps*

$$A_x \colon T_xM \to T_{f(x)}M \,, \quad x \in \Sigma$$

such that every A_x is ε-close to the derivative of f at x, there is a diffeomorphism g which is δ-close to f in the C^1 topology, coinciding with f on $M \setminus U$ and on Σ, and such that the derivative of g at any point $x \in \Sigma$ is A_x.

By the previous theorem, any small perturbation of the derivative of a diffeomorphism f along a periodic orbit p can be realized dynamically, that is, as the derivative of a C^1-nearby diffeomorphism g for which p is still a periodic point. Then, it is enough to see that there are arbitrarily small perturbations of the derivative of f along the orbit of some $p \in \mathcal{P}$ such that the resulting matrix has non-real eigenvalues of absolute value different from 1.

This is deduced from Lemma 4.4 as follows. Choose a point $p \in \mathcal{P}$ such that $\operatorname{angle}(E^s(p), E^u(p))$ is less than some small fixed ε and let n be the period of p. Then compose the derivative $Df(f^{n-1}(p))$ with a rotation $R_t \colon T_p(M) \to T_p(M)$ for an appropriate t, so that the resulting linear map

$$R_t \cdot Df(f^{n-1}(p)) \cdot Df(f^{n-2}(p)) \cdots Df(p)$$

has complex conjugate eigenvalues. Up to a final perturbation, the absolute value may be taken different from 1.

Lemma 4.5 completes the argument when the angle between E^s and E^u is not bounded away from 0. So, we are left to consider the case in which this angle is uniformly bounded from below on $\mathcal{P}$. In this case, up to a uniformly bounded change of the Riemannian metric, there are unitary vectors $e^s(p) \in E^s(p)$ and $e^u(p) \in E^u(p)$ forming an orthonormal basis of T_pM. In this basis, the derivative $Df(p)$ is given by a diagonal matrix. The set $\mathcal{P}$ is called *hyperbolic in the period* if there is $0 < \lambda < 1$ such that

$$\|Df^{n_p}(p)e^s(p)\| \leq \lambda^{n_p} \quad \text{and} \quad \|Df^{-n_p}(p)e^u(p)\| \leq \lambda^{n_p}$$

for every $p \in \mathcal{P}$, where n_p is the period of p.

Lemma 4.6. *Suppose $\mathcal{P}$ is not hyperbolic in the period. Then there exists a small C^1-perturbation g of f and a periodic point $p \in \mathcal{P}$ such that p is either a sink or a source of g.*

This follows by multiplying each matrix $Df(p)$ by a homothetic transformation of ratio $(1+\varepsilon)$ or $(1-\varepsilon)$, depending on which direction hyperbolicity fails, and then applying Theorem 6.

Finally, consider the case where $\mathcal{P}$ is hyperbolic in the period. As we are assuming that $\mathcal{P}$ does not admit a dominated splitting, there are arbitrarily large n and $p \in \mathcal{P}$ such that $n < n_p$ and

$$\frac{\|Df^n(p)e^u(p)\|}{\|Df^n(p)e^s(p)\|} < 2.$$

Since $\mathcal{P}$ is hyperbolic in the period, $n_p - n$ must go to infinite as n goes to infinity.

Lemma 4.7. *Assume the splitting $E^u \oplus E^s$ over $\mathcal{P}$ is not dominated. Then there is an arbitrarily small perturbation of the derivative of f along the orbit of some point $p \in \mathcal{P}$ such that the angle between the eigenspaces of the corresponding matrix is arbitrarily small.*

This reduces the problem to the situation in Lemma 4.5, thus ending the proof of the statement made at the beginning of this section. So let us now explain the proof of this lemma.

Consider n and p as in the inequality above. Since n is large and the matrices of the derivatives are diagonal, by multiplying each $Df(f^i(p))$, $0 \leq i < n$ by

$$C = \begin{pmatrix} \dfrac{1}{1+\varepsilon} & 0 \\ 0 & 1+\varepsilon \end{pmatrix}$$

we can make the quotient $\|Df^n(p)(e^u(p))\|/\|Df^n(p)(e^s(p))\|$ arbitrarily small, Denote $A(f^i(p)) = C \cdot Df(f^i(p))$. Then consider

$$B = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$$

and replace $A(p)$ by $A'(p) = A(p) \cdot B$. At this stage the direction $E^u(p)$ is no longer invariant, but our perturbations do keep $E^s(p)$ invariant. The linear map $A(f^{n-1}(p)) \circ \cdots \circ A(f(p)) \circ A'(p)$ sends $E^s(p)$ to $E^s(f^n(p))$, and $E^u(p)$ to some straight line $\tilde{E}(f^n(p))$ forming a small angle with $E^s(f^n(p))$, provided n is large enough. Through a final perturbation at times $> n$ we are going to recover the invariance of E^u, while keeping E^s invariant. This means they will again be the eigenspaces and so, since at time n their angle is small, the proof will be complete.

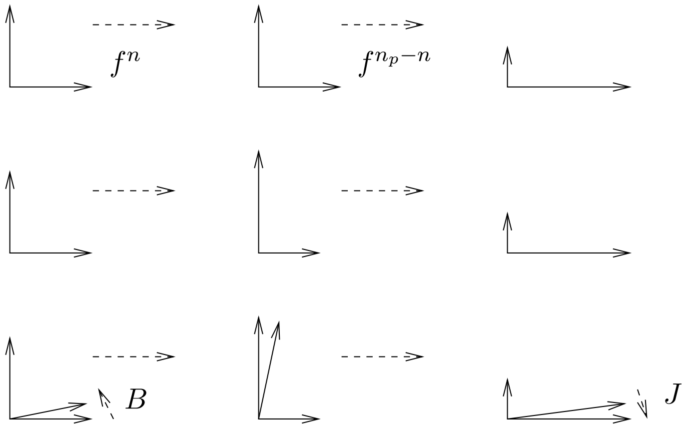

Figure 6: Getting small angles between eigenspaces

So, let us explain this last step, with the aid of Figure 6. Since the periodic orbit is hyperbolic at the period, the image of $e^u(p) + \varepsilon e^s(p)$ under $A^{n_p}(p)$ makes a small angle with $E^u(p)$. In other words, $A(f^{n_p-1}(p)) \circ \cdots A(f(p)) \circ A'(p)$ maps $E^s(p)$ to itself and $E^u(p)$ to a subspace $\tilde{E}^u(p)$ forming a small angle with $E^u(p)$. Then there exists a matrix J close to the identity mapping $E^s(p)$ into $E^s(p)$ and $\tilde{E}^u(p)$ into $E^u(p)$. Thus, $E^s(p)$ and $E^u(p)$ are eigenspaces of

$$J \circ A(f^{n-1}(p))) \circ A(f^{n-2}(p))) \circ \cdot \circ A(f(p)) \circ A'(p).$$

This is the final perturbation mentioned above, and so the proof of the lemma is complete.

4.3.2 Dominated splittings *vs.* homothetic transformations

The previous ideas extend to arbitrary dimension. One gets that, given a set $\mathcal{P}$ of hyperbolic periodic saddles all with the same Morse index (the same dimension of the unstable direction), if the natural splitting induced by the stable and unstable directions is not dominated then it is possible to change the index of some saddle by a small C^1 perturbation.

However, in dimension larger than 2 such bifurcations do not break transitivity, and they do occur in robustly transitive sets. Towards proving Theorem 4, we are going to show that *in the absence of any dominated splitting it is possible to perturb the derivative of f along a periodic orbit* *in order to turn into a homothetic transformation and, after an additional small perturbation, to create a sink or a source for a diffeomorphism close to f.* An important tool is Franks' lemma (Theorem 6), according to which C^1 small perturbations of the derivative can always be realized dynamically.

The main novelty with respect to the previous arguments is the introduction of *transitions* between different periodic orbits, which permit a kind of concatenation of such orbits and corresponding multiplication of eigenvalues. In a few words, while in the 2-dimensional situation presented above we deal with a linear cocycle defined over period points and their perturbations only, the arguments that follow use strongly the dynamics on a homoclinic class and the continuity of the derivative: the transitions correspond to heteroclinic orbits.

We begin with another argument of linear algebra which is crucial in the proof. Consider a basis $\beta = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ and let L_i be the subspace spanned by v_i. Consider a linear map $A \in GL(\mathbb{R}, n)$ which is diagonal relative to the basis β, and $(n-1)$ linear maps $A_1, A_2, \ldots, A_{n-1} \in GL(\mathbb{R}, n)$ such that each A_j

- leaves invariant the subspaces $L_1, \ldots, L_{j-1}, L_{j+2}, \ldots, L_n$ and
- permutes L_j and L_{j+1}, that is, $A_j(L_j) = L_{j+1}$ and $A_j(L_{j+1}) = L_j$.

For arbitrarily small neighborhoods $\mathcal{A}$ of A and $\mathcal{A}_i$ of each A_i, consider all possible products of matrices in $\mathcal{A} \cup (\cup_i \mathcal{A}_i)$. Then some of these products are homothetic transformations. The idea is that, as the matrices A_i generate all permutations of the eigenspaces of A, it is possible to mix the eigenvalues of A to obtaining a matrix having essentially the same rate of expansion in all directions.

In the model we have in mind, the matrices A and A_i correspond to the derivatives $Df^k(p)$, $k = \operatorname{per}(p)$ at periodic points p of a diffeomorphism f. Of course, to multiply derivatives corresponding to different points does not make dynamical sense. However, properties of homoclinic classes $H(p, f)$ allow us to give a sense to this procedure.

First of all, there is a dense subset K in $H(p, f)$ consisting of periodic points homoclinically related to p. As a dominated splitting extends to the closure of its domain (see Appendix 5), absence of dominated splitting on $H(p, f)$ implies the same for K. Thus, it is enough to consider this set K of periodic orbits.

Next, given two fixed or periodic hyperbolic saddles x and y in K, and given any family $\{n_1, m_1, \ldots, n_k, m_k\}$ of arbitrarily big natural numbers, there are a natural number r (independent of n_i and m_i) and a saddle $z \in H(p, f)$ of period $t = kr + \sum_i (n_i + m_i)$ whose orbit spends alternately n_i consecutive iterates close to x and m_i consecutive iterates close to y. This means that the derivative of $Df^t(z)$ is, roughly, of the form

$$Df^t(z) = (T_2 \circ Df^{m_k}(q) \circ T_1 \circ Df^{n_k}(p)) \circ \cdots \circ (T_2 \circ Df^{m_1}(q) \circ T_1 \circ Df^{n_1}(p))$$

where T_1 and T_2 are called the *transitions* from x to y and from y to x, respectively. These transitions correspond to a bounded number of iterations, independent of n_i and m_i, so that their contribution to the product becomes negligible as n_i and m_i go to infinite.

Thus, it makes sense to multiply derivatives corresponding to different periodic points if they are homoclinically related. Besides the resulting linear maps correspond, roughly, to the derivative at another periodic saddle in the homoclinic class. Similar arguments show that in any homoclinic class there is a dense subset of periodic points whose derivative at the period can be made diagonal, with positive eigenvalues of multiplicity 1, by a small perturbation of the derivative along the orbit.

The construction above shows that a robust transitive set always admits a dominated splitting. Then the *finest* such splitting is well-defined. To get volume hyperbolicity it remains to prove uniform volume expansion and volume contraction along the two extreme subbundles. This is an application of the next results. We first state the Pugh closing lemma:

Theorem 7 (Pugh [29, 30]). *Let f be a C^1 diffeomorphism defined on a compact manifold M and $x \in M$ a non-wandering point of f. There exists $g \in \mathrm{Diff}^1(M)$ arbitrarily close to f in the C^1 topology for which x is a periodic point.*

Let $\Sigma(f)$ be the set of points $x \in M$ such that given any C^1 neighborhood $\mathcal{U}$ of f and any $\varepsilon > 0$ there exist $g \in \mathcal{U}$, coinciding with f outside the ε-neighborhood of the f-orbit of x, and there exists a g-periodic point $y \in M$ such that

$$d(f^i(x), g^i(y)) \leq \varepsilon \quad \text{for all } 0 \leq i \leq \mathrm{per}(y).$$

Theorem 8 (Mañé [23]). *For any C^1 diffeomorphism f of a compact manifold, the set $\Sigma(f)$ has full probability for f.*

A major application of this statement is to express in terms of periodic orbits dynamical behaviors that are detected by invariant measures. The following argument, which is part of Mañé's proof of the 2-dimensional version of Theorem 4 in [23], is a good illustration:

Let f be a surface diffeomorphism with a dominated splitting $E \oplus F$ and suppose the sub-bundle F is *not* uniformly expanding. Then there are arbitrarily long orbit segments over which expansion along F is arbitrarily weak. Any weak accumulation point of Dirac averages along such orbit segments is an invariant measure for which the sub-bundle F is not expanding. Pugh's closing lemma permits to close a typical orbit of such a measure. However, not knowing whether the periodic orbit remains close to the original one, one loses control on the dilation along F. On the contrary, the ergodic closing lemma ensures that the derivative of the perturbed map g is, at best, weakly expanding along the continuation F_g of F for g over the orbit of the periodic point y. By another small perturbation (this is Franks' lemma, that we shall discuss in a while) we can now turn any slight expansion into a contraction, which means that y becomes a sink.

We are not ready to consider the finest dominated splitting above. Since E^1 does not admit any dominated sub-splitting, the previous arguments ensure that by a small perturbation we can produce a homothetic transformation along E^1. If E^1 is not uniformly volume expanding, by using the ergodic closing lemma we get a homothetic transformation along E^1 at a periodic point whose volume expansion rate is less than $1 + \delta$. So, maybe after a new perturbation, we get a homothetic transformation with contraction rate strictly smaller than 1. By domination, this implies that all the subbundles E_i are contracting. This means that the periodic point is really a source, which is a contradiction.

4.3.3 On the dynamics of robustly transitive sets

Here our goal is to deduce dynamical properties of robustly transitive sets, such as the kind of bifurcations they may exhibit, or the types of saddles they may contain, from the analysis of the finest dominated splitting. The arguments in the previous section imply that *if it is not possible* *to change, by perturbation, the index of a periodic point in a robustly transitive set Λ, then Λ is hyperbolic.* In the same spirit, the following result explains what happens when it is not possible to increase the indices of the periodic saddle points of Λ: it relates the maximal and minimal indices with the hyperbolicity of the subbundles in the finest dominated splitting.

Theorem 9 (Bonatti, D., Pujals, Rocha [5]). *Consider $k \geq 1$ and an open subset U of a closed manifold. Let $\mathcal{U}$ be an open subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive, has a dominated splitting $E_f \oplus F_f$ with $\dim E_f = k$, and does not contain saddles of Morse index less than k. Then E_f is uniformly expanding for every diffeomorphism f in $\mathcal{U}$.*

Theorem 10 (Bonatti, D., Pujals, Rocha [5]). *There is an open and dense subset $\mathcal{I}(U)$ of $\mathcal{R}(U)$ such that for all $g \in \mathcal{I}(U)$ the set of indices of the hyperbolic saddles of $\Lambda_g(U)$ is an interval in $\mathbb{N}$.*

The main ingredient in the proof is a notion of transitions associated to a heterodimensional cycle. These are similar to the transitions associated to homoclinically related periodic points in Section 4.3.2, but there are also some subtle differences that we now explain. We first recall the connecting lemma:

Theorem 11 (Hayashi [18]). *Given two periodic points p and q, if there are orbits visiting the neighborhoods of some point in the unstable manifold of p and some point in the stable manifold of q then, by an arbitrarily C^1-small perturbation, one can make the two invariant manifolds intersect each other.*

The connecting lemma was a decisive ingredient in the recent progress towards understanding C^1 generic dynamical systems. On the other hand, many of these applications require variations of the original statement. In particular, the following question posed itself quite naturally: Assume the unstable manifold of p and the stable manifold of q accumulate on some point x. *Is there a C^1 perturbation of the system for which the two invariant manifolds intersect each other ?* This has been answered recently, assuming x is not periodic itself, independently by Arnaud [1], Hayashi [19], and Wen, Xia [32]. Their statements are slightly different, here we quote [1]:

Theorem 12. *Let f be a diffeomorphism on a compact manifold, $x \in M$ be a non-periodic point, and $\mathcal{U}$ be a C^1 neighborhood of f. Then there exist $N \geq 1$ such that for any neighborhood V of x there exists another neighborhood $W \subset V$ such that given any points y and z outside $\cup_{0 \leq n < N} f^n(V)$ and such that W contains some forward iterate of y and some backward iterate of z, there exists $g \in \mathcal{U}$ coinciding with f outside $\cup_{0 \leq n < N} f^n(V)$ and for which z is in the forward orbit of y, passing through V.*

Given two periodic points x and y of Λ with different indices, using robust transitivity and the connecting lemma, Theorem 12, we may create a heterodimensional cycle involving x and y. Then there is a perturbation of f having a periodic orbit which remains arbitrarily large periods close to x and close to y, with bounded transition times from one saddle point to the other. The derivative on this periodic orbit is a small perturbation of the product of arbitrarily large powers of the derivatives at x and y by a pair of bounded matrices, the *transitions*. One difference with respect to Section 4.3.2 is that here the new periodic orbits passes close to x and y only once, it closes the period already at the first return. A second difference is that in the homoclinic case the transitions are for the initial diffeomorphism, whereas here they are defined for perturbations of it.

Another important topic is the occurrence of homoclinic tangencies inside robustly transitive sets. A heuristic principle (see [5] for precise statements) is that *if the finest dominated splitting of* $\Lambda_f(U)$ *involves a non-hyperbolic subbundle of dimension bigger than* 1 *then there is* g *close to* f *such that* $\Lambda_g(U)$ *contains a homoclinic tangency.* Moreover, it is possible to determine the index of the hyperbolic saddle associated to the tangency. On the other hand, if all non-hyperbolic subbundles of the finest dominated splitting are 1-dimensional then it is not possible to create homoclinic tangencies inside the robustly transitive set by small perturbations of the system. In this case the prototypical bifurcations are heterodimensional cycles and saddle-node and period-doubling bifurcations.

Let us observe that homoclinic tangencies are indeed compatible with robust transitivity because, in higher dimensions, they need not lead to the creation of sinks or sources, unless the saddle point involved in the tangency is sectionally dissipative, see [27].

5 Appendix: Weak forms of hyperbolicity

There are several notions extending the concept of uniform hyperbolicity in the following spirit: one still requires the existence of an invariant splitting of the tangent bundle, but one allows one or more of the subbundles involved to be such that the derivative exhibits mixed contracting/neutral/expanding behavior along them. Typically, these non-hyperbolic subbundles concentrate the dynamical difficulties: often one can project the dynamics along the remaining hyperbolic directions thus reducing the study to the central part.

Brin, Pesin [8] and Hirsch, Pugh, Shub [20] considered *partially hyperbolic systems*, that is, admitting an invariant splitting of the tangent bundle as the sum $E^s \oplus E^c \oplus E^u$ of three subbundles, where E^s is uniformly contracting, E^u is uniformly expanding, and the central direction E^c is "in between" the other two (one of the subbundles may be trivial). Examples include the time-1 map of an Anosov flow and the product of an Anosov diffeomorphism by the identity. In the context of the stability conjecture, Liao [21], Mañé [23], Pliss [28] were led to the more general notion of *dominated splitting* into two subbundles: one of them is definitely more contracted (or less expanded) than the other, after a uniform number of iterates.

5.1 Dominated splittings

5.1.1 Definition and properties

Definition 5.1. *Let* $f : M \to M$ *be a diffeomorphism on a closed manifold* M *and* K *be any* f*-invariant set. A splitting* $T_xM = E_1(x) \oplus \cdots \oplus E_k(x)$, $x \in K$ *of the tangent bundle over* K *is* dominated *if it is invariant under the derivative* Df *and there exists* $\ell \in \mathbb{N}$ *such that for every* $i < j$, *every* $x \in K$, *and every pair of unit vectors* $u \in E_i(x)$ *and* $v \in E_j(x)$, *one has*

$$\frac{\|Df^\ell(x)u\|}{\|Df^\ell(x)v\|} < \frac{1}{2} \tag{2}$$

and the dimension of $E_i(x)$ *is independent of* $x \in K$ *for every* $i \in \{1, \ldots, k\}$.

We write $E_i \prec E_j$. In some contexts we need to specify the strength of the domination, measured by the number ℓ of iterates needed to get the inequality in the definition. Then we write $E_i \prec_\ell E_j$ and say that the splitting is ℓ*-dominated.*

The previous definition may be formulated, equivalently, as follows: there exist $C > 0$ and $\lambda < 1$ such that

$$\frac{\|Df^n(x)u\|}{\|Df^n(x)v\|} < c\lambda^n \quad \text{for all } n \geq 1. \tag{3}$$

When using this notion, λ and c may be taken as measures of the strength of the domination.

Let us list some useful elementary properties of dominated splittings. An important ingredient in the proofs of the properties *extension to a neighborhood* and *persistence* is a characterization of existence of a dominated splitting in terms of invariant cone-fields.

Uniqueness: The dominated splitting is unique if one fixes the dimensions of the subbundles: Let d be the dimension of the ambient manifold and let

$$d = n_1 + n_2 + \cdots + n_k\,.$$

Then there is at most one dominated splitting $T_K M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ over K with dimension of E_i equal to n_i for all i. It is clear that without the dimension requirement, there may be more than one dominated splitting. For instance, if $E^s \oplus E^c \oplus E^u$ is dominated then so are $E \oplus E^u$ and $E^s \oplus F$, where $E = E^s \oplus E^c$ and $F = E^c \oplus E^u$.

Transversality: The angles between any two subbundles of a dominated splitting are uniformly bounded from zero. In fact, the angle between the spaces $\oplus_{i \in I} E_i$ and $\oplus_{j \in J} E_j$ is uniformly bounded from zero, for any disjoint sets $I, J \subset \{1, \ldots, k\}$.

Continuity: Every dominated splitting is continuous, meaning that the subbundles $E_i(x)$ depend continuously on the point x.

Extension to the closure: Every ℓ-dominated splitting over a set K may be extended to an ℓ-dominated splitting over the closure of K. Even more, if g_n is a sequence of diffeomorphisms converging to f and K_n is a g_n-invariant set with an ℓ-dominated splitting, with dimensions independent of n, then the f-invariant set

$$K = \limsup_n K_n = \bigcap_N \overline{\bigcup_{n \geq N} K_n}$$

admits an ℓ-dominated splitting.

Extension to a neighborhood: Let K be an f-invariant set with a dominated splitting. Then such a splitting can be extended in a dominated way to the maximal invariant set of f in a neighborhood of K.

Persistence: Every dominated splitting persists under C^1-perturbations. More precisely, consider an f-invariant set K with an ℓ-dominated splitting. Then, for every $\varepsilon > 0$ there is a neighborhood U of K and a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, the maximal g-invariant set in the closure of U admits an $(\ell - \varepsilon)$-dominated splitting ($(\ell - \varepsilon)$-dominated splitting meaning that $\|Df^\ell(x)u\|/\|Df^\ell(x)v\| < \frac{1}{2} + \varepsilon$), having the same dimensions of the initial dominated splitting over K.

Clustering: Suppose that $E_1 \oplus \cdots \oplus E_k$ is a dominated splitting and let $1 \leq i < j \leq k$, with either $i > 1$ or $j < k$. Then

$$E_1 \oplus \cdots \oplus E_{i-1} \oplus F \oplus E_{j+1} \oplus \cdots \oplus E_k\,,$$

where $F = E_i \oplus \cdots \oplus E_j$, is a dominated splitting.

Sub-splittings: Let $E_1 \oplus \cdots \oplus E_k$ be a dominated splitting and suppose that $E_j = F_1 \oplus \cdots \oplus F_m$ is a dominated splitting. Then

$$E_1 \oplus \cdots \oplus E_{j-1} \oplus F_1 \oplus \cdots \oplus F_m \oplus E_{j+1} \oplus \cdots \oplus E_k$$

also is a dominated splitting.

These properties motivate the following notion introduced in [4]:

Proposition 5.2. *Let K be an f-invariant set admitting a dominated splitting. Then there is a* unique finest dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_k$ *on K such that no E_i admits a dominated splitting. Any other invariant splitting is obtained by considering a partition of $\{1, \ldots, k\}$ into sub-intervals, and clustering the corresponding subbundles.*

The following notions are relevant in the study of robustly transitive dynamics:

Definition 5.3. *Let f be a diffeomorphism and K be an f-invariant set having a dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_k$. We say that the splitting and the set K are*

- partially hyperbolic *if there is $n \geq 1$ such that the restriction of Df^n either contracts uniformly E_1 or expands uniformly E_k.*
- volume hyperbolic *if there is $n \geq 1$ such that the Jacobian of $Df^n \mid E_1$ is strictly less than 1 (uniform volume contraction) and the Jacobian of $Df^n \mid E_k$ is strictly greater than 1 (uniform volume expansion).*

We say that the diffeomorphism f is partially/volume hyperbolic if the ambient space M is a partially/volume hyperbolic set for f.

Remark 5.4. *An invariant set is volume hyperbolic if and only if its finest dominated splitting is volume hyperbolic. If a volume hyperbolic splitting has at least one of the extremal bundles E_1 or E_k of dimension 1 then it is partially hyperbolic.*

Just to give some flavor of the dynamical consequences of these properties, let us quote a result on bifurcations of invariant sets with a dominated splitting, from [5]: *Let f be a diffeomorphism having a dominated splitting into one-dimensional subbundles, defined on the whole ambient manifold. Then no diffeomorphism close to f exhibits homoclinic tangencies.*

5.1.2 Proofs of the elementary properties:

Consider a diffeomorphism $f : M \to M$ on a closed manifold and let K be an f-invariant set.

Transversality: We consider the case of two subbundles. Assume that the tangent bundle over K contains two subbundles E and F, not necessarily continuous, and some $\ell \geq 1$ such that $E \prec_\ell F$ on K. We first show that the angle between E and F is uniformly bounded from zero. For doing that it is enough to prove that the difference $\|u - v\|$, $u \in E(x)$, $v \in F(x)$, $\|u\| = \|v\| = 1$, is larger than some constant independent of $x \in K$, u and v. Assume otherwise. Then there are sequences

$u_n \in E(x_n)$ $v_n \in F(x_n)$ of unit vector with $u_n - v_n$ going to 0. As the derivative of f is bounded, one may find a sequence $m_n \to +\infty$ such that

$$\frac{1}{2} < \frac{\|Df^{m_n}(u_n)\|}{\|Df^{m_n}(v_n)\|} < 2,$$

which contradicts the ℓ-domination. This proves the first part of the transversality property.

Clustering: Assume now that K contains 3 subbundles E, F, and G, and there exits $\ell \geq 1$ such that $E \prec_\ell F$ and $F \prec_\ell G$. As the angle between E and F is uniformly bounded from zero, by the previous argument, there is $\alpha > 0$ such that any unit vector $u \in E(x) \oplus F(x)$, $x \in K$ may be written as $u = u_E + u_F$ with $\|u_E\| + \|u_F\| < \alpha$. Fix $k \geq 1$ such that $2^k > 2\alpha$. Then, for any unit vector $v \in G(x)$,

$$\frac{\|Df^{k\ell}(u)\|}{\|Df^{k\ell}(v)\|} < \frac{1}{2^k}\alpha < \frac{1}{2}.$$

So we proved $(E \oplus F) \prec_{k\ell} G$. The same proof give that $E \prec (F \oplus G)$. Now a simple induction argument gives the complete clustering property.

Uniqueness: Assume that $E \prec F$ and $G \prec H$ are two dominated splittings over K such that $\dim E \leq \dim G$. We argue, by contradiction, that $E \subset G$. Actually, this is a stronger result than what we claimed, useful for proving uniqueness of the finest dominated splitting. A simpler proof of the uniqueness statement we claimed is contained in the proof of the continuity and extension properties below.

So, assume $E(x) \not\subset G(x)$ for some $x \in K$ and consider some unit vector $u \in E(x) \setminus G(x)$. Write $u = u_G + u_H$ with $u_H \neq 0$. Then the positive iterates of u grow at the same rate as those of u_H. Write also $u_H = v_E + v_F$. If v_F did not vanish then the positive iterates of v_F would grow at the same rate as those of u_H, that is, at the same rate as the iterates of u, which would contradict $E \prec F$. Therefore, $v_F = 0$, which means that $u_H \in E(x) \cap H(x)$. Since $\dim E \leq \dim G$ and we are assuming that they do not coincide, there is some unit vector $w \in G(x) \setminus E(x)$. Write $w = w_E + w_F$ with $w_F \neq 0$. Then the positive iterates of $w \in G$ grow at the same rate as those of w_F, and so they grow exponentially faster than the iterates of u_H (because $u_H \in E(x)$). Since u_H is also in $H(x)$, this contradicts the domination $G \prec H$, and completes our argument. This gives the uniqueness of a dominated splitting into two subbundles, once dimensions are fixed.

Continuity and extension to the closure: Let $E \oplus F$ be an ℓ-dominated splitting over K. Let $x_n \in K$ be a sequence converging to some point x. Up to choosing subsequences one may assume that the spaces $E(x_n)$ and $F(x_n)$ converge to subspaces $E(x)$ and $F(x)$. A simple continuity argument show that, for any $k \in \mathbb{N}$, for any unit vectors $u \in E_x$ $v \in F(x)$ one has

$$\frac{\|Df^{k\ell}(u)\|}{\|Df^{k\ell}(v)\|} < \frac{1}{2^\ell} \quad \text{and} \quad \frac{\|Df^{-k\ell}(v)\|}{\|Df^{-k\ell}(u)\|} < \frac{1}{2^\ell}. \tag{4}$$

This directly proves that $E(x) \oplus F(x) = T_xM$. The first part of (4) characterizes $E(x)$ uniquely (once its dimension is fixed): the iterates of any unit vector u in $E(x)$ grow slower than those of any unit vector $w \notin E(x)$. In the same way, the second part of (4) characterizes $F(x)$ uniquely.

This now implies that, for any other sequence y_n converging to x, any accumulation point of $E(y_n)$ coincides with $E(x)$ and, in the same way, any accumulation point of $F(y_n)$ coincides with $F(x)$. As a consequence, one gets that the splitting on K extends by continuity to the closure of K. At the same time, this argument proves that the subbundles are continuous on K. The assumption that the splitting is invariant on K implies that invariance on the closure as well. Finally, (4) shows that this invariant extension is still dominated.

The argument extends easily to any dominated splitting with any number of subbundles, using clustering: the previous argument shows that each splitting $(E_1 \oplus \cdots \oplus E_i) \oplus (E_{i+1} \oplus \cdots \oplus E_k)$, and by intersecting the subbundles in all these extensions one obtains the extension of the original splitting to the closure of K.

Extension to a neighborhood and persistence Consider a ℓ-dominated splitting $E \oplus F$ on K. Extend it to a continuous splitting $TM = E \oplus F$ in a neighborhood U of K, not necessarily invariant. Then consider the cone fields $\mathcal{C}_\alpha$ on U consisting of all the vectors such that the norm of the component in the F-direction is larger than α times the norm of the component in the E-direction. The domination property implies that, for any $x \in K$, $Df^\ell(\mathcal{C}_1(x)) \subset \mathcal{C}_2(f^\ell(x))$. As a consequence, there is a neighborhood $V \subset U$ of K and a C^1-neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ and any $x \in V$ one has $Dg^\ell(\mathcal{C}_1(x)) \subset \mathcal{C}_{2-\varepsilon}(g^\ell(x))$. This implies that the maximal invariant set of g in V has a dominated splitting with the same dimensions of the initial one, and with almost the same strength.

5.1.3 Invariant foliations

Let f be a C^r diffeomorphism on a closed manifold M admitting a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ where Df is uniformly expanding along E^u and uniformly contracting along E^s (f is partially hyperbolic).

Theorem 13 (Brin, Pesin [8], Hirsch, Pugh, Shub [20]). *Assuming* $\dim E^u > 0$*, there exists a unique family* $\mathcal{F}^u$ *of injectively* C^r *immersed submanifolds* $\{\mathcal{F}^u(x) : x \in M\}$ *such that* $x \in \mathcal{F}^u(x)$ *and* $\mathcal{F}^u(x)$ *is tangent to* E^u_x *at every* $x \in M$*. This family is invariant, in the sense that* $f(\mathcal{F}^u(x)) = \mathcal{F}^u(f(x))$ *for all* $x \in K$*, and the leaves* $\mathcal{F}^u(x)$ *are uniformly contracted by some negative iterate of* f*. Moreover,* $\mathcal{F}^u$ *is a continuous lamination of* M*, meaning that every point has a continuous local chart which trivializes the leaves and is uniformly* C^r *along each leaf.*

$\mathcal{F}^u$ is the *strong unstable foliation* and its leaves $\mathcal{F}^u(x)$ are called strong unstable manifolds. Local strong unstable manifolds are neighborhoods of the point inside the corresponding leaf. By reversing time one gets strong stable foliations and (local) strong stable manifolds, assuming $\dim E^s > 0$. More generally, strong stable and strong unstable foliations exist for any invariant subset $K \subset M$ admitting a dominated splitting as above. The family $\mathcal{F}^u$ is indexed by $x \in K$. These strong invariant foliations are continuous laminations, in the same sense as before.

The following general structural stability result assumes that the initial central foliation is *plaque expansive*: given ε there exists δ such that if x_n and y_n are δ-pseudo orbits with all jumps along the central leaves, and they remain within distance ε for all times, then x_n and y_n are in the same local central leaf for all n.

Theorem 14 (Hirsch, Pugh, Shub [20]). *If f admits a plaque expansive central foliation $\mathcal{F}^c$ then $(f, \mathcal{F}^c)$ is structurally stable. The hypothesis of plaque expansiveness holds, in particular, if $\mathcal{F}^c$ is of class C^1.*

The ergodic properties of orbits propagate along stable and unstable sets, since forward time averages are constant on stable sets and backward time averages are constant on unstable sets. In order to use this idea, often called the Hopf argument, one needs the holonomies of foliations to preserve zero measure sets. This *absolute continuity* property does hold for the strong stable and strong unstable foliations of partially hyperbolic C^2 diffeomorphisms [8].

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Robustly transitive sets and heterodimensional cycles

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Abstract

It is known that non-hyperbolic robustly transitive sets Λ_φ have all a dominated splitting and contain generically periodic points of different indices. We show that for a C^1-dense open subset of diffeomorphisms φ the indices of periodic points in a robust transitive set Λ_φ form an interval in $\mathbb{N}$. We also proof that the homoclinic classes of two periodic points in Λ_φ are robustly equal. Finally, we describe what kind of homoclinic tangencies can appear in Λ_φ by analyzing the dominated splittings of Λ_φ.

1 Introduction

When a diffeomorphism ϕ is *hyperbolic*, i.e., it verifies the Axiom A, the Spectral Decomposition Theorem of Smale says that its limit set (set of non-wandering points) is the union of finitely many *basic pieces* satisfying nice properties: they are invariant, compact, *transitive* (there is a dense orbit), pairwise disjoint and isolated (each piece is the maximal invariant set in a neighborhood of itself). Moreover, by construction, a basic piece is the *homoclinic class* of a hyperbolic periodic point, i.e., the closure of the transverse intersections of its invariant manifolds.

Even if the dynamics is non-hyperbolic, the homoclinic classes of hyperbolic periodic points seem to be the natural elementary pieces of the dynamics, satisfying many of the properties of the basic sets of the Smale's theorem: invariance, compactness, transitivity and density of hyperbolic periodic points. Recent results in [BD$_2$], [Ar] and [CMP] show that, for $\mathcal{C}^1$-generic diffeomorphisms (i.e., those belonging to a residual subset of $\mathrm{Diff}^1(M)$) two homoclinic classes are either disjoint or equal and they are maximal transitive sets (i.e., every transitive set intersecting a homoclinic class is contained in it). Let us observe that, in general, the homoclinic classes fail to be hyperbolic, isolated and pairwise disjoint.

In [BDP] it is shown that, for $\mathcal{C}^1$-generic diffeomorphisms, a homoclinic class is either contained in the closure of an infinite set of sinks or sources, or satisfies some weak form of hyperbolicity (partial hyperbolicity or, at least, existence of a *dominated splitting*). The first situation (called the Newhouse phenomenon) can be locally generic (in the residual sense): there are open sets in $\mathrm{Diff}^r(M)$ where the diffeomorphisms with infinitely many sinks or sources are (locally) residual for the $\mathcal{C}^r$-topology, see [N] for $r \geq 2$ for surface diffeomorphisms, [PV] for $r \geq 2$ in higher dimensions, and [BD$_1$] for $r = 1$ in dimensions greater or equal than 3. Certainly, the Newhouse phenomenon

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exhibits very wild behavior and it is conjectured that (in some sense) diffeomorphisms satisfying this phenomenon are very rare (for instance, for generic parametrized families of diffeomorphisms, the Lebesgue measure of the parameters corresponding to diffeomorphisms satisfying the Newhouse phenomenon is zero), see [Pa].

We focus here on the opposite behavior, more precisely, on the so-called *robustly transitive sets* introduced in [DPU] as a non-hyperbolic generalization of the basic sets of the Spectral Decomposition of Smale: a robustly transitive set Λ of a diffeomorphism ϕ is a transitive set which is locally maximal in some neighbourhood U of it and such that, for every C^1-perturbation ψ of the diffeomorphism ϕ, the maximal invariant set of ψ in U is transitive. From the results in [M$_2$], [DPU] and [BDP] every robustly transitive set Λ admits a *dominated splitting,* say $T_\Lambda M = E_1 \oplus \cdots \oplus E_k$, and by [BD$_2$], $\mathcal{C}^1$-generically, it is a *homoclinic class.* An invariant set may admit more than one dominated splitting. The reason is that, in some cases, one can sum some bundles of a dominated splitting, obtaining a new dominated splitting with less bundles, or, conversely, split some bundle of the splitting in a dominated way. So it is natural to consider the *finest dominated splitting* of the set Λ (i.e., one of which it is not possible to split any bundle of the splitting to get a new dominated splitting).

In this paper we study the interplay between the dominated splittings (especially the finest one) of a robustly transitive set Λ and its dynamics, answering questions about the *indices* (dimension of the stable manifold) of the periodic points of Λ, the possible bifurcations (saddle-node and homoclinic tangencies) occurring in this set as well as its dynamical structure.

In order to present our results we need to give some precise definitions.

In what follows, M denotes a compact, closed Riemannian manifold and $\mathrm{Diff}^1(M)$ the space of $\mathcal{C}^1$-diffeomorphisms of M endowed with the usual topology.

Let Λ be a compact invariant set of a diffeomorphism ϕ. A ϕ_*-invariant splitting $T_\Lambda M = E \oplus F$ over Λ is *dominated* if the fibers of E and F have constant dimension and there exists a k such that for every $x \in \Lambda$ one has

$$||\phi_*^k|_{E(x)}|| \cdot ||\phi_*^{-k}|_{F(\phi^k(x))}|| < \frac{1}{2},$$

that is, ϕ_*^k expands the vectors in F uniformly more than the vectors in E. Then we say that F *dominates* E and write $E \prec F$.

An invariant bundle E over Λ is *uniformly contracting* if there exists a k such that for every $x \in \Lambda$ one has:

$$||\phi_*^k|_{E(x)}|| < \frac{1}{2}.$$

An invariant bundle is *uniformly expanding* if it is uniformly contracting for ϕ_*^{-1}.

Let $T_\Lambda M = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ be a ϕ_*-invariant splitting over Λ such that the fibers of the bundles E_i have constant dimension. Denote by $E_i^j = \bigoplus_i^j E_k$. Observe that $E_1^{k-1} \oplus E_k^m$ is a splitting of $T_\Lambda M$ for all $k \in \{2, \ldots, m\}$. We say that $E_1 \oplus E_2 \oplus \cdots \oplus E_m$ is the *finest dominated splitting* of Λ if $E_1^{k-1} \oplus E_k^m$ is a dominated splitting for each $k \in \{2, \ldots, m\}$ and every E_k is *indecomposable* (i.e., it does not admit any (nontrivial) dominated splitting). See [BDP] for the existence and uniqueness of the finest dominated splitting.

Consider a set $V \subset M$ and a diffeomorphism $\varphi \colon M \to M$. We denote by $\Lambda_\varphi(V)$ the *maximal invariant set* of φ in V, i.e., $\Lambda_\varphi(V) = \bigcap_{i \in \mathbb{Z}} \varphi^i(V)$. Given an open set U of M the set $\Lambda_\varphi(U)$ is *robustly transitive* if $\Lambda_\psi(U)$ is equal to $\Lambda_\psi(\bar{U})$ and is *transitive* for all ψ in a $\mathcal{C}^1$-neighbourhood of

φ. We say that a ψ-invariant closed set K is transitive if there is some $x \in K$ having a positive dense orbit in the whole set K.

If a robustly transitive set $\Lambda_\phi(U)$ is not (uniformly) hyperbolic then, by a $\mathcal{C}^1$-small perturbation of ϕ, one can create non-hyperbolic periodic points, and thus hyperbolic periodic points with different indices in $\Lambda_\phi(U)$ (see [M$_2$]). Our first two results describe the possible indices of the periodic points of $\Lambda_\phi(U)$, in terms of the finest dominated splitting of $\Lambda_\phi(U)$:

Theorem A. *Let U be an open set of M and $\mathcal{M}(U)$ a $\mathcal{C}^1$-open set of* $\mathrm{Diff}^1(M)$ *such that $\Lambda_\varphi(U)$ is robustly transitive for every $\varphi \in \mathcal{M}(U)$. Then there is a dense open subset $\mathcal{N}(U)$ of $\mathcal{M}(U)$ such that, for every $\varphi \in \mathcal{N}(U)$, the set of indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$ is an interval of integers (i.e., if P and Q are hyperbolic periodic points of indices p and q, $p \geq q$, of $\Lambda_\varphi(U)$, $\varphi \in \mathcal{N}(U)$, and $j \in [q,p]$, then $\Lambda_\varphi(U)$ has a hyperbolic periodic point of index j).*

In the next result we use the arguments in [M$_2$] to relate the uniform contraction or dilatation of the extremal bundles of the finest dominated splitting of a robustly transitive set with the indices of the periodic points of this set.

Theorem B. *Consider an open set U of a compact manifold M and an integer $q \in \mathbb{N}^*$. Let $\mathcal{U}$ be a $\mathcal{C}^1$-open set of* $\mathrm{Diff}^1(M)$ *such that for every $\phi \in \mathcal{U}$ the maximal invariant set $\Lambda_\phi(\bar{U})$ satisfies the following properties:*

1. *the set $\Lambda_\phi(\bar{U})$ is contained in U and admits a dominated splitting $E_\phi \oplus F_\phi$, $E_\phi \prec F_\phi$, with $\dim E_\phi(x) = q$,*
2. *the set $\Lambda_\phi(\bar{U})$ has no periodic points of index $k < q$.*

Then the bundle E_ϕ is uniformly contracting for every $\phi \in \mathcal{U}$.

We can summarize the two results above to get a characterization of the set of indices of the periodic points of the set $\Lambda_\phi(\bar{U})$, as follows.

Let U be an open set of M and φ a diffeomorphism such that $\Lambda_\varphi(U)$ is robustly transitive with a finest dominated splitting of the form $T_{\Lambda_\varphi(U)}M = E_1 \oplus \cdots \oplus E_{k(\varphi)}$, $E_i \prec E_{i+1}$. Denote by E^s the sum of all the uniformly contracting bundles of this splitting and let E_α be the first non-uniformly contracting bundle, i.e., $E^s = E_1 \oplus \cdots \oplus E_{\alpha-1}$. In the same way, denote by E^u the sum of all the uniformly expanding bundles of the splitting and let E_β be the last non-uniformly expanding bundle, i.e., $E^u = E_{\beta+1} \oplus \cdots \oplus E_{k(\varphi)}$. Let $\mathcal{U}$ be a $\mathcal{C}^1$-neighborhood of φ such that for every $\psi \in \mathcal{U}$ the set $\Lambda_\psi(U)$ has the same properties as $\Lambda_\varphi(U)$ (i.e., robustly transitive and the number $k(\psi)$ of bundles of the finest dominated splitting is equal to $k(\varphi)$) and the dimensions of bundles $E^s(\psi)$, $E_\alpha(\psi)$, $E_\beta(\psi)$ and $E^u(\psi)$, defined in the obvious way, are constant in $\mathcal{U}$ and equal to corresponding bundles for ϕ.

Corollary C. *With the notation above, there are a C^1-open and dense subset $\mathcal{V}$ of $\mathcal{U}$ and locally constant functions $i, j\colon \mathcal{V} \to \mathbb{N}^*$ such that*

$$
\begin{aligned}
i(\psi) &\in [\dim(E^s), \dim(E^s) + \dim(E_\alpha)] \cap \mathbb{N}^*,\\
j(\psi) &\in [\dim(E^u), \dim(E^u) + \dim(E_\beta)] \cap \mathbb{N}^*,
\end{aligned}
$$

and, for every $\psi \in \mathcal{V}$, the set of indices of the hyperbolic periodic points of $\Lambda_\psi(U)$ is the interval $[i(\psi), \dim(M) - j(\psi)] \cap \mathbb{N}^*$.

The first known examples of non-hyperbolic robustly transitive sets had a one-dimensional central direction, see [M_1] and [Sh]. As a consequence, these examples do not present *homoclinic tangencies* (non-transverse homoclinic intersections between the invariant manifolds of some periodic point). Let us observe that if a periodic point has a homoclinic tangency then, after a perturbation of the diffeomorphism, one create a Hopf bifurcation (a periodic point whose derivative has a pair of conjugate nonreal eigenvalues of modulus one), see [YA] and [R], and hence points whose central direction has dimension at least two. Currently examples of robustly transitive sets having a central direction of dimension two or more are known, see [BD_1], [B] and [BV]. Moreover, in some cases these sets exhibit homoclinic tangencies, see [B] and [BV]. Our next result explains what kind of dominated splitting of a robustly transitive set prevents homoclinic bifurcations.

We say that a robustly transitive set $\Lambda_\varphi(U)$ is *$\mathcal{C}^1$-far from homoclinic tangencies* if there are no *homoclinic tangencies* in $\Lambda_\psi(U)$ for any ψ in a C^1-neighbourhood of φ.

Theorem D. *Given an open set U of M let $\mathcal{P}(U) \subset \mathrm{Diff}^1(M)$ be an open set of diffeomorphisms φ such that:*

1. *The set $\Lambda_\varphi(U)$ is robustly transitive and the minimum and the maximum of the indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$ are constant in $\mathcal{P}(U)$. Denote these numbers by i_s and i_c, respectively.*
2. *The set $\Lambda_\varphi(U)$ is $\mathcal{C}^1$-far from homoclinic tangencies.*

Then there is an open and dense subset $\mathcal{O}(U)$ of $\mathcal{P}(U)$ such that, for every $\varphi \in \mathcal{O}(U)$, the set $\Lambda_\varphi(U)$ has a dominated splitting $T_{\Lambda_\varphi(U)} = E^s \oplus E_1 \oplus \cdots \oplus E_r \oplus E^u$, such that

- *E^s is uniformly contracting and has dimension $i_s \geq 1$,*
- *E^u is uniformly expanding and has dimension $\dim(M) - i_c \geq 1$,*
- *$r = i_c - i_s$ and the bundle E_i has dimension one and is not uniformly hyperbolic for every $i = 1, \ldots, r$.*

Actually, from the proof of this theorem we get somewhat more: given any robustly transitive set $\Lambda_\phi(U)$ the dimensions of the non-hyperbolic bundles of its finest dominated splitting determine, for diffeomorphisms in a C^1-neighbourhood of ϕ, the *ranks* of the homoclinic tangencies (that is, the index of the periodic point exhibiting the tangency) that can occur in $\Lambda_\psi(U)$. The precise statement of this result is in Section 6, see Theorem F.

Finally, for robustly transitive sets which are far from homoclinic tangencies, we prove that the *(relative) homoclinic classes* of two periodic points of this set are equal in a $\mathcal{C}^1$-robust way. More precisely, let P_φ be a hyperbolic periodic point of a diffeomorphism φ. We denote by H_{P_φ} the set of transverse intersections of the invariant manifolds of P_φ. Observe that the homoclinic class of P_φ is the closure of H_{P_φ}. Given an open set U, the *relative homoclinic class* of P_φ in U is the closure of the set $H_{P_\varphi}(U)$ of transverse homoclinic points of P_φ whose orbits are contained in U.

Theorem E. *Let U be an open set of M and $\mathcal{S}(U) \subset \mathrm{Diff}^1(M)$ an open set of diffeomorphisms φ such that the set $\Lambda_\varphi(U)$ is robustly transitive and there are no homoclinic tangencies (in the whole manifold) associated to periodic points of $\Lambda_\varphi(U)$.*

Consider any pair of hyperbolic periodic points P_φ and Q_φ of $\Lambda_\varphi(U)$ with indices p and q whose continuations are defined for every ψ in $\mathcal{S}(U)$. Then there is an open and dense subset $\mathcal{D}(U)$ of $\mathcal{S}(U)$ such that

$$\overline{H_{P_\psi}(U)} = \overline{H_{Q_\psi}(U)}$$

for every ψ in $\mathcal{D}(U)$.

Unfortunately, in the theorem above we cannot ensure that the relative homoclinic classes of P_ψ and Q_ψ are equal to $\Lambda_\psi(U)$, although by the results in [BD$_2$] this is true for a residual subset of $\mathcal{S}(U)$.

Let us now say a few words about the proofs of our results. One of the main tools is the notion of *heterodimensional cycle.* Given a diffeomorphism ϕ with two hyperbolic periodic points P_ϕ and Q_ϕ with different indices, say index$(P_\phi) >$ index(Q_ϕ), we say that ϕ has a *heterodimensional cycle associated to P_ϕ and Q_ϕ,* denoted by $\Gamma(\phi, P_\phi, Q_\phi)$, if $W^s(P_\phi)$ and $W^u(Q_\phi)$ have a (nontrivial) transverse intersection and $W^u(P_\phi)$ and $W^s(Q_\phi)$ have a quasi-transverse intersection along the orbit of some point x, i.e., $T_xW^u(P_\phi) + T_xW^s(Q_\phi)$ is a direct sum. Observe that in this case $\dim(M) - \dim(T_xW^u(P_\phi) + T_xW^s(Q_\phi))$ is equal to index$(P_\phi)-$ index(Q_ϕ), this number being the *codimension of the cycle.*

The proof of Theorem A has two main ingredients. The first is Theorem 3.1, which implies that, by unfolding a heterodimensional cycle associated to points of indices q and p as above, one gets hyperbolic periodic points of some index in between q and p (a priori, we do not know the index of such a point). The second ingredient of the proof is the Connecting Lemma of Hayashi (see Theorem 2.1 and [H]) which allows us to create (after a $\mathcal{C}^1$-perturbation) heterodimensional cycles associated to any pair of periodic points of a robustly transitive set.

Two other important tools are the constructions in [M$_2$] and in [BDP], specially the periodic linear systems with transitions of [BDP]. In this paper we need to introduce *transitions between points of different indices* in the same homoclinic class, generalizing the construction in [BDP], in which only transitions between points with the same index were considered.

Finally, to prove Theorem E, the main ingredient, besides the Connecting Lemma, is the proposition below concerning the structure of the homoclinic classes of hyperbolic points having a heterodimensional cycle.

We say that a hyperbolic periodic point R_ϕ is *$\mathcal{C}^1$-far from tangencies* if there is a $\mathcal{C}^1$-neighbourhood $\mathcal{W}$ of ϕ in Diff$^1(M)$ such that every $\psi \in \mathcal{W}$ has no homoclinic tangencies associated to R_ψ. A heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ is *$\mathcal{C}^1$-far from homoclinic tangencies* if the points P_ϕ and Q_ϕ in the cycle are $\mathcal{C}^1$-far from homoclinic tangencies.

Finally, we say that two points x and y are *transitively related* for ϕ if there exists a transitive set of ϕ containing x and y. The points x and y are *transitively related in an open set U* if there exists a transitive set of ϕ contained in U that contains x and y.

Proposition 1.1. *Let U be an open set, φ a diffeomorphism and P_φ and Q_φ a pair of hyperbolic periodic points of φ of indices p and $q = p - 1$, respectively. Consider a neighbourhood $\mathcal{W} \subset$* Diff$^1(M)$ *of φ of diffeomorphisms ψ such that,*

- *the continuations of P_ψ and Q_ψ are defined and $\mathcal{C}^1$-far from tangencies,*
- *the points P_ψ and Q_ψ are transitively related in U.*

Then there is a $\mathcal{C}^1$-open subset $\mathcal{W}_\varphi$ of $\mathcal{W}$ whose closure contains φ such that the relative homoclinic classes of P_ψ and Q_ψ in U are equal for every $\psi \in \mathcal{W}_\varphi$.

[DR, Theorem A] guarantees that given any heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ of codimension one far from homoclinic tangencies, there is a $\mathcal{C}^1$-open set, whose closure contains ϕ, of diffeomorphisms φ such that P_φ and Q_φ are transitively related. Thus, for any diffeomorphism ϕ with a heterodimensional cycle which is far from homoclinic tangencies, there are diffeomorphisms φ arbitrarily close to ϕ satisfying the hypotheses of the proposition. The proof of Proposition 1.1 follows from the results in [DR] and the Connecting Lemma of Hayashi.

This paper is organized as follows. In Section 2 we get some results concerning heterodimensional cycles, robustly transitive sets and homoclinic classes using the Hayashi's Connecting Lemma. In Section 3 we prove Theorem A. For that we need to study the creation of periodic points in the unfolding of heterodimensional cycles (of any codimension). In Section 4 we prove Theorem B, for that we recall some folklore results concerning dominated splittings and remember and reformulate some results in [M$_2$]. In Sections 5 and 6 we study the interplay between the finest dominated splitting of a robustly transitive set and the creation of homoclinic tangencies inside this set. Finally, in Section 7 we prove the results concerning (relative) homoclinic classes.

2 Transitively related points

We begin the proofs of our results by recalling the Hayashi's Connecting Lemma and deducing some consequences from it.

2.1 Connecting lemma and transitively related points

Theorem 2.1. (Hayashi's Connecting Lemma, [H]) *Let P_φ and Q_φ be a pair of hyperbolic periodic points of a $\mathcal{C}^1$-diffeomorphism φ such that there are sequences of points x_n and of natural numbers k_n such that the sequences x_n and $\varphi^{k_n}(x_n)$ accumulate on $W^s_{loc}(Q_\varphi)$ and on $W^u_{loc}(P_\varphi)$, respectively.*

Then there is a diffeomorphism ψ arbitrarily $\mathcal{C}^1$-close to φ such that $W^s(Q_\psi)$ and $W^u(P_\psi)$ have a nonempty intersection.

Remark 2.2. *Every pair of hyperbolic periodic points P_φ and Q_φ which are transitively related satisfy the hypotheses of the Connecting Lemma (Theorem 2.1).*

Proof of the remark: Consider a transitive set Λ containing P_φ and Q_φ and a point x of Λ whose positive orbit is dense in Λ. Then there are sequences of natural numbers m_n and r_n, $m_n, r_n \to \infty$ as $n \to \infty$, such that $\varphi^{m_n}(x) \to P_\varphi$ and $\varphi^{r_n}(x) \to Q_\varphi$. Then it is immediate to get new sequences m'_n and r'_n, with $m'_n, r'_n \to \infty$, such that $\varphi^{m'_n}(x)$ and $\varphi^{r'_n}(x)$ converge to some point of $W^u_{loc}(P_\varphi)$ and of $W^s_{loc}(Q_\varphi)$, respectively. Taking subsequences, if necessary, we can assume that $r'_n = m'_n + k_n$ for some $k_n > 0$. Now it suffices to take $x_n = \varphi^{m'_n}(x)$ and consider the sequences x_n and k_n. $\square$

2.2 Homoclinic relative classes and robustly transitive sets

By [BD$_2$, Theorem B] there is a residual subset of $\mathrm{Diff}^1(M)$ consisting of diffeomorphisms such that the homoclinic classes of any pair of transitively related hyperbolic periodic points are equal. The proof of this result is based on the Hayashi's Connecting Lemma. Using the *relative version* of the connecting lemma we get a relative version of [BD$_2$, Theorem B] whose prove we omit here.

Theorem 2.3. (Relative version of [BD$_2$, Theorem B]). *Given an open set U of M there is a residual subset $\mathcal{G}(U)$ of $\mathrm{Diff}^1(M)$ such that for every $\varphi \in \mathcal{G}(U)$ a pair of hyperbolic periodic points P_φ and Q_φ of φ, are transitively related in U if and only if the relative homoclinic class in U of P_φ and Q_φ are equal, i.e., $\overline{H_{P_\varphi}(U)} = \overline{H_{Q_\varphi}(U)}$.*

Let $\mathcal{A}(U)$ be an open set of $\mathrm{Diff}^1(M)$ such that $\Lambda_\varphi(U)$ is robustly transitive for all φ. By the Pugh closing lemma (see [Pu]) and a Kupka-Smale argument, there is a residual subset $\mathcal{R}(U)$ of $\mathcal{A}(U)$ of diffeomorphisms φ such that the hyperbolic periodic points form a dense subset of $\Lambda_\varphi(U)$. Taking $\mathcal{T}(U) = \mathcal{G}(U) \cap \mathcal{R}(U)$, where $\mathcal{G}(U)$ and $\mathcal{R}(U)$ are as above we get the following:

Proposition 2.4. *Let U and $\mathcal{A}(U)$ be open sets of M and of $\mathrm{Diff}^1(M)$, respectively, such that $\Lambda_\varphi(U)$ is robustly transitive for all $\varphi \in \mathcal{A}(U)$. Then there is a residual subset $\mathcal{T}_{\mathcal{A}}(U)$ of $\mathcal{A}(U)$ such that*

$$\overline{H_{P_\varphi}(U)} = \Lambda_\varphi(U)$$

for every $\varphi \in \mathcal{T}_{\mathcal{A}}(U)$ and every hyperbolic periodic point P_φ of $\Lambda_\varphi(U)$.

2.3 Heterodimensional cycles

We will use the following lemma which is a consequence of the Connecting Lemma and an argument of transversality:

Lemma 2.5. *Let P_φ and Q_φ be a pair of hyperbolic periodic points of a diffeomorphism φ of indices p and q, $p \geq q$. Suppose that P_ψ and Q_ψ are transitively related for every ψ in a neighbourhood $\mathcal{V}$ of φ. Then there is a dense subset $\mathcal{W}$ of $\mathcal{V}$ such that every ϕ in $\mathcal{W}$ has a heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ of codimension $(p-q)$.*

Proof: Consider any $\psi \in \mathcal{V}$, since P_ψ and Q_ψ are transitively related, by Remark 2.2, we can apply Theorem 2.1 to get ξ arbitrarily close to ψ (hence ξ is in $\mathcal{V}$) such that $W^s(P_\xi) \cap W^u(Q_\xi) \neq \emptyset$. Since

$$\dim(W^s(P_\xi)) + \dim(W^u(Q_\xi)) = p + (\dim(M) - q) \geq \dim(M),$$

we can assume that this intersection is transverse.

Since ξ belongs to $\mathcal{V}$ the points P_ξ and Q_ξ are transitively related. Thus, again by Remark 2.2, we can apply Theorem 2.1 to get ϕ arbitrarily close to ξ (ϕ in $\mathcal{V}$) such that $W^s(P_\phi)$ and $W^u(Q_\phi)$ have (non empty) transverse intersection and $W^u(P_\phi) \cap W^s(Q_\phi) \neq \emptyset$. After a new perturbation, if necessary, we can assume that the last intersection is quasi-transverse, obtaining a heterodimensional cycle $\Gamma(\phi, P_\phi, Q_\phi)$ of codimension $(p-q)$, ending the proof of the lemma. □

Let us state two remarks of the proof above that we will use in Section 7.

Remark 2.6. *Let P_φ and Q_φ be a pair of hyperbolic periodic points of a diffeomorphism φ of indices p and q, $p \geq q$. Suppose that P_ψ and Q_ψ are transitively related for every ψ in a neighbourhood $\mathcal{V}$ of φ. Then there is a dense and open subset $\mathcal{D}$ of $\mathcal{V}$ such that $W^s(P_\psi)$ and $W^u(Q_\psi)$ have a nontrivial transverse intersection for every ψ in $\mathcal{D}$.*

If in Theorem 2.3 one assumes that the points P_φ and Q_φ have the same index, one has the following stronger version of it:

Remark 2.7. *Let P_φ and Q_φ be a pair of hyperbolic periodic points of the same index of a diffeomorphism φ and U an open set containing the orbits of P_φ and Q_φ. Suppose that P_ψ and Q_ψ are transitively related for every ψ in a neighbourhood $\mathcal{V}$ of φ. Then there is a dense and open subset $\mathcal{O}$ of $\mathcal{V}$ such that, for every ψ in $\mathcal{O}$, the relative homoclinic classes of P_ψ and Q_ψ in U are equal.*

3 Proof of Theorem A: unfolding heterodimensional cycles

3.1 Transitions for heterodimensional cycles

We begin this section stating a somewhat technical result introducing the notion of *transition between periodic points of different indices.*

Theorem 3.1. *Let P and Q be two hyperbolic periodic points of a diffeomorphism φ of indices p and q, $p > q$, and periods $n(P)$ and $n(Q)$, respectively. Denote by M_P and M_Q the linear maps*

$$\varphi_*^{n(P)}(P)\colon T_PM \to T_PM \quad \textit{and} \quad \varphi_*^{n(Q)}(Q)\colon T_Q \to T_QM.$$

Assume that there are dominated splittings

$$T_PM = E_1(P) \oplus E_2(P) \oplus E_3(P) \quad \textit{and} \quad T_QM = E_1(Q) \oplus E_2(Q) \oplus E_3(Q),$$

with $\dim(E_1(P)) = \dim(E_1(Q)) = q$ and $\dim(E_3(P)) = \dim(E_3(Q)) = \dim(M) - p$, which are invariant by M_P and M_Q, respectively. Assume, in addition, that there is a heterodimensional cycle $\Gamma(\varphi, U, P, Q)$ in some open subset U of M.

Then, fixed any $\varepsilon > 0$, there are matrices T_0 and T_1 and $\delta > 0$ such that, for every n and $m \geq 0$, and every family matrices $(I_i), i = 0, \dots, (n+m)+2$, δ-close to identity, there is a diffeomorphism ψ ε-C^1-close to φ having a periodic orbit R of period $n(R)$ such that the linear map $M_R = \psi_*^{n(R)}$ is conjugate to

$$I_{n+m+2} \circ T_1 \circ I_{n+m+1} \circ M_Q \circ I_{n+m} \circ \cdots \circ I_{n+2} \circ M_Q \circ I_{n+1} \circ T_0 \circ I_n \circ M_P \circ I_{n-1} \circ \cdots \circ I_1 \circ M_P \circ I_0.$$

Moreover, $n(R) = t_1 + t_2 + n \cdot n(P) + m \cdot n(Q)$, where t_1 and t_2 are constants depending only on the choice of T_0 and T_1.

The maps T_0 and T_1 are called *transitions* (from P to Q and from Q to P, respectively). These maps are a generalization of the transitions introduced in [BDP] for hyperbolic periodic points which are homoclinically related.

Theorem 3.1 is the main step in the proof of Theorem A. Taking appropriate n and m and assuming that $\mathrm{index}(P) > \mathrm{index}(Q) + 1$, using the theorem one gets that the index of R is in

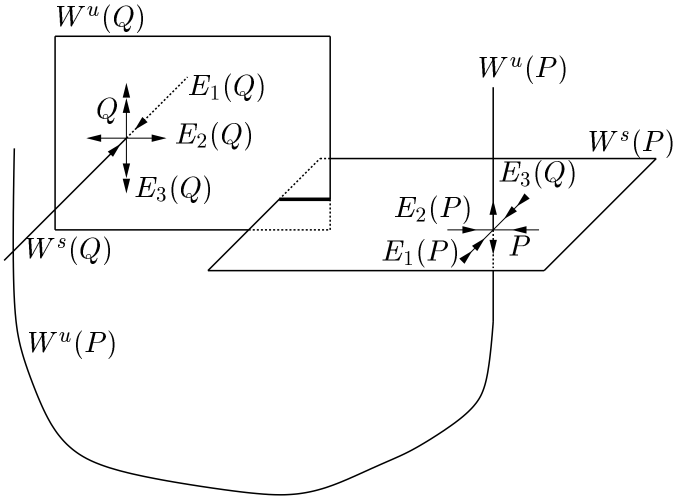

Figure 1: A heterodimensional cycle

between the indices of P and Q, see Corollary 3.6. This construction will also allow us to get points R corresponding to saddle-node bifurcations.

Proof: For simplicity let us assume that P and Q are fixed points. Notice that $E_1(Q)$ is the stable direction of Q, $E_1(P)$ is the strong stable direction of P, $E_3(Q)$ is the strong unstable direction of Q and $E_3(P)$ is the unstable direction of P.

We now perform a C^1-perturbation of the diffeomorphism φ to get appropriate linearizing coordinates of the cycle. The properties of this linearization are summarized in the next lemma:

Lemma 3.2. *Let φ be a diffeomorphism satisfying the hypotheses of Theorem 3.1. Then there is ϕ arbitrarily C^1-close to φ having a heterodimensional cycle $\Gamma(\phi, U, P, Q)$ such that:*

1. *There are smooth linearizing charts*

$$U_P,\, U_Q \simeq [-1,1]^q \times [-1,1]^{p-q} \times [-1,1]^{\dim(M)-p}$$

   defined on neighbourhoods of P and Q where ϕ is a linear map such that, for every $x \in U_P \cap \phi^{-1}(U_P)$ or $x \in U_Q \cap \phi^{-1}(U_Q)$, one has:

   (a) *In these charts P and Q correspond to the point $\{0\}^{\dim(M)}$ and $\phi_*(P) = \varphi_*(P)$ and $\phi_*(Q) = \varphi_*(Q)$,*

   (b) *The foliation by q-planes parallel to $[-1,1]^q \times \{0\}^{p-q} \times \{0\}^{\dim(M)-p}$ (called* strong stable foliation, *$\mathcal{F}^s$) is locally invariant and corresponds to the smaller (in modulus) eigenvalues of the linear maps induced by ϕ in U_P and U_Q.*

   (c) *The foliation by $(p-q)$-planes parallel to $\{0\}^q \times [-1,1]^{p-q} \times \{0\}^{\dim(M)-p}$ (called* central foliation, *$\mathcal{F}^c$) is locally invariant.*

   (d) *The foliation by $(n-p)$-planes parallel to $\{0\}^q \times \{0\}^{p-q} \times [-1,1]^{\dim(M)-p}$ (called* strong unstable foliation, *$\mathcal{F}^u$) is locally invariant and corresponds to the bigger (in modulus) eigenvalues of the linear maps induced by ϕ in U_P and U_Q.*

2. *There are points* $X_0 \in (W^u(Q) \pitchfork W^s(P)) \cap U_Q$ *and* $Y_0 = \phi^{k_0}(X_0) \in U_P$, $k_0 > 0$, *such that, in these coordinates,* $X_0 \in \{0\}^q \times [-1,1]^{p-q} \times \{0\}^{\dim(M)-p}$ *(the* local center-unstable manifold *of* Q*, denoted by* $W^{cu}_{loc}(Q)$*) and* $Y_0 \in \{0\}^q \times [-1,1]^{p-q} \times \{0\}^{\dim(M)-p}$ *(the* local center-stable manifold *of* P*, denoted by* $W^{cs}_{loc}(P)$*).*

3. *There are points* $X_1 \in (W^s(Q) \cap W^u(P)) \cap U_P$ *and* $Y_1 = \phi^{k_1}(X_1) \in U_Q$, $k_1 > 0$, *such that, in these coordinates,* $X_1 \in \{0\}^q \times \{0\}^{p-q} \times [-1,1]^{\dim(M)-p}$ *(the* local unstable manifold *of* P, $W^u_{loc}(P)$*) and* $Y_1 \in [-1,1]^q \times \{0\}^{p-q} \times \{0\}^{\dim(M)-p}$ *(the* local stable manifold $W^s_{loc}(Q)$ *of* Q*).*

4. *There are small cubes* $C_0 \subset U_Q$ *and* $C_1 \subset U_P$ *centered at* X_0 *and* X_1*, respectively, such that*
   (a) $\phi^{k_0}(C_0) \subset U_P$ *and* $\phi^{k_1}(C_1) \subset U_Q$,
   (b) *the restrictions* $T_0 = \phi^{k_0}|_{C_0}$ *and* $T_1 = \phi^{k_1}|_{C_1}$ *are affine maps which preserve the strong stable, central and strong unstable foliations above.*

Proof: To get a point of the heteroclinic intersection $W^u(Q) \pitchfork W^s(P)$ in the central direction just observe that, generically, there are points X of such an intersection which are not in the strong unstable manifold of Q nor in the strong stable manifold of P. Thus, after an arbitrarily small perturbation of φ, we can assume that this is our case. Considering a point X with this property and using the domination, we have that the backward orbit of X approaches to the center-unstable manifold of Q. Similarly, the forward iterates of X approach to the center-stable manifold of P. Now by two local small $\mathcal{C}^1$-perturbations one gets the announced points X_0 and $Y_0 = \phi^{k_0}(X_0)$. Observe that we can perform these two perturbations without breaking the cycle (i.e., preserving the non-transverse intersection between $W^s(Q)$ and $W^u(P)$). Observe now that the points X_1 and $Y_1 = \phi^{k_1}(X_1)$ in the lemma are directly given by the intersection $W^s(Q) \cap W^u(P)$.

After a new perturbation, we can assume that ϕ is linear in small neighbourhoods of P and of Q and that ϕ^{k_0} and ϕ^{k_1} are both affine in small neighbourhoods of X_0 and X_1. The only difficulty is to see that these affine maps can be chosen preserving the foliations (strong stable, central and strong unstable). This fact follows along the lines of the proof of [BDP, Lemma 4.13] using the domination. Let us explain all that in details.

In our linearizing charts there are foliations $\mathcal{F}^{cs}$ (resp., $\mathcal{F}^{cu}$) tangent to the sum $E_1 \oplus E_2$ of the stable and central directions (resp., the sum $E_2 \oplus E_3$ of the central and unstable directions). By genericity, we can assume that the images by ϕ^{k_0} of the foliations $\mathcal{F}^s$, $\mathcal{F}^u$, $\mathcal{F}^c$, $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ are in general position. Now, one checks that the forward iterates of the images by ϕ^{k_0} of the leaves of $\mathcal{F}^{cu}$ become close to the center-unstable leaves in U_P. Replacing the initial k_0 by $k_0 + \ell$, for some big positive ℓ, and doing a small perturbation, one gets an invariant center-unstable foliation.

To get the invariance of the strong stable foliation we consider negative iterates of the foliations in the neighbourhood of Y_0. By the previous construction, the center-unstable foliation is preserved by negative iterations. So the negative iterates of the strong stable foliation are transverse to the center-unstable one. As above, the backwards iterates of the strong stable leaves approach to the leaves of the strong stable foliation in U_Q. So we can replace X_0 by some (large) negative iterate of it, say $-\ell'$, and perform a small perturbation (preserving the center unstable foliation) in such a way the transition map $\phi^{k_0+\ell+\ell'}$ from a neighbourhood of X_0 to a neighbourhood of Y_0 preserves the strong stable and center-unstable foliations.

To get the invariance of the strong unstable and center foliations (keeping the invariance of the strong stable one) one repeats all the arguments above inside the center-unstable foliation. We omit the details of this construction. This gives the transition T_0.

The transition T_1 is obtained using the same arguments, so we do not go into the details. The proof of the lemma is now complete. $\square$

Definition 3.1. *Consider a* $\dim(M)$*-cube* $C = I^s \times I^c \times I^u$*, where* I^s *is a* q*-cube,* I^c *a* $(p-q)$*-cube and* I^u *a* $(\dim(M)-p)$*-cube, where there are defined coordinates* (x^s, x^c, x^u) *as above.*

A subset Δ *of* C *is* s*-complete if, for every* $Z = (z^s, z^c, z^u) \in \Delta$*, the horizontal* q*-cube* $I^s \times \{(z^c, z^u)\}$ *is contained in* Δ*. Similarly, a subset* Δ *of* C *is* u*-complete if, for every point* $Z \in \Delta$*, the vertical* $(\dim(M)-p)$*-cube* $\{z^s, z^c\} \times I^u$ *is contained in* Δ*.*

By shrinking, if necessary, the size of the neighbourhood U_Q in the strong unstable direction and taking an appropriate cube C_1 around X_1, we can assume that the image by T_1 of any u-complete disk Δ of C_1 (contained in a leaf of the strong unstable foliation) is a u-complete disk of U_Q.

For simplicity let us denote by A and B the restrictions of ϕ to U_Q and U_P, respectively.

Lemma 3.3. *There is* $\ell_0 \geq 0$ *such that:*

1. *Consider any* $Z \in W^u_{loc}(Q)$ *and any* s*-complete disk* Δ^s *of* C_0 *(contained in a leaf of the strong stable foliation) containing* Z*. Then the connected component of* $A^{-n}(\Delta^s) \cap U_Q$ *containing* $A^{-n}(Z)$ *is a* s*-complete disk in* U_Q *for all* $n \geq \ell_0$*.*
2. *Consider any* u*-complete disk* Δ^u *of* C_0 *(in a leaf of the strong unstable foliation). Then the intersection between* Δ^u *and* $T_0^{-1}(W^s_{loc}(P))$ *is a unique point* W*. Let* Δ^u_m *be the connected component of* $(B^m \circ T_0(\Delta^u)) \cap U_P$ *containing* $B^m \circ T_0(W)$*. Then* $\Delta^u_m \cap C_1$ *is a complete* u*-disk (in* C_1*) for every* $m \geq \ell_0$*.*

Proof: For instance to see the first item just observe that A^{-1} expands the s-direction and recall the A-invariance of the foliations. The second item follows using that B expands in the u-direction and the B-invariance of the foliations. $\square$

We are now ready to end the proof of Theorem 3.1. Given $\varepsilon > 0$ there is an $\varepsilon/2$-perturbation ϕ of φ satisfying Lemmas 3.2 and 3.3. We will now obtain the final diffeomorphism considering a perturbation of ϕ obtained by composing the transition T_1 with a small translation T_v in the direction of a vector v parallel to the central direction (in U_Q). Let us now explain the details of this construction.

In our coordinates, $X_0 = (0^s, x_0^c, 0^u)$. Consider now the su-disk

$$\Delta = \Big([-1,1]^q \times \{x_0^c\} \times [-1,1]^{\dim(M)-p}\Big) \cap C_0.$$

With the terminology above, the disk Δ is u and s-complete in C_0.

Given n and m bigger than ℓ_0, ℓ_0 as in Lemma 3.3, let Δ^{-m} and Δ^n_0 be the connected components of $A^{-m}(\Delta) \cap U_Q$ containing $A^{-m}(X_0)$ and of $(B^n \circ T_0(\Delta)) \cap U_P$ containing $B^n(T_0(X_0))$, respectively. Let $\Delta^n_1 = \Delta^n_0 \cap C_1$. Write $\Delta^n = T_1(\Delta^n_1)$. By Lemma 3.3 and the observation before, Δ^{-m} and Δ^n are a s-complete and u-complete disks in U_Q and C_1, respectively.

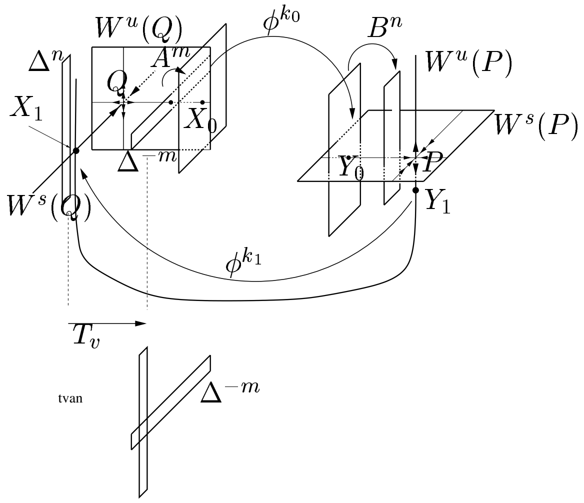

Figure 2: A periodic orbit

Observe that there is a unique vector v parallel to the central direction such that the intersection between $T_v(\Delta^n)$ and Δ^{-m} is not empty. Moreover, since these sets are both su-disks of U_Q, such an intersection is a sub-rectangle R intersecting *completely* Δ^{-m} in the u-direction and Δ^n in the s-direction. Here by a *complete intersection in the u-direction* we mean that, for every $Z \in R$, the leaf $F^u(Z)$ of the strong unstable foliation containing Z is such that the connected components of $F^u(Z) \cap R$ and of $\Delta^{-m} \cap F^u(Z)$ containing Z are equal. The definition of *complete intersection in the s-direction* is totally analogous (considering strong stable leaves).

Now a classical argument of hyperbolicity implies that the map $T = T_v \circ T_1 \circ B^n \circ T_0 \circ A^m$ has a fixed point W in Δ^{-m}. Observe that the derivative of T at W is $\tilde{T}_1 \circ B^n \circ \tilde{T}_0 \circ A^m$ (where $\tilde{T}_i$ is the linear part of the affine map T_i).

So it remains to see that the size of the translation T_v can be taken smaller than $\varepsilon/2$. For that first observe that the disks Δ^{-m} and Δ^n can be taken passing arbitrarily close to the heteroclinic intersection Y_1, for that it is enough to take n and m large enough. Thus there is n_0 such that the distance between Δ^{-m} and Δ^n is less than $\varepsilon/2$ for every n and m greater n_0. Fixed such an n_0 and replacing T_0 by $T_0 \circ A^{n_0}$ and T_1 by $T_1 \circ B^{n_0}$, we get that for every positive n and m there is a translation T_v, $v = v(n, m)$, such that the modulus of v is less than $\varepsilon/2$.

The diffeomorphism ψ in the statement of the theorem is obtained from ϕ by composing T_1 with T_v. By construction, ψ has a periodic point R of period $n_R = t_0 + t_1 + n + m$, where $t_0 = k_0 + n_0$ and $t_1 = k_1 + n_1$, such that

$$\psi_*^{n_R}(R) = \tilde{T}_1 \circ B^n \circ \tilde{T}_0 \circ A^m.$$

Observe that t_0 and t_1 depend exclusively on the transitions T_0 and T_1. The theorem now follows from the definition of A and B and the lemma below, that allows us to perform any small perturbation of the derivative of a diffeomorphism along the orbit of a periodic point in a dynamical way.

Lemma 3.4. **([F], [M$_2$])** *Consider a $\mathcal{C}^1$-diffeomorphism φ and a φ-invariant finite set Σ. Let A be an ε-perturbation of φ_* along Σ (i.e., the linear maps $A(x)$ and $\varphi_*(x)$ are ε-close for all $x \in \Sigma$). Then for every neighbourhood U of Σ there is a diffeomorphism ϕ C^1-ε-close to φ such that*

- *$\varphi(x) = \phi(x)$ if $x \in \Sigma$ or if $x \notin U$,*
- *$\phi_*(x) = A(x)$ for all $x \in \Sigma$.*

The proof of Theorem 3.1 is now complete. $\square$

We end this subsection by stating a lemma that follows from the proof of [BDP, Lemma 4.13]:

Lemma 3.5. *Let M_P and M_Q be linear maps as in the statement of Theorem 3.1. Suppose that M_P and M_Q preserve the dominated splittings $T_P M = E^1_P \oplus \cdots \oplus E^k_P$ and $T_Q M = E^1_Q \oplus \cdots \oplus E^k_Q$, where $\dim(E^i_P) = \dim(E^i_Q)$ for every i. Then one can choose the matrices T_0 and T_1 in Theorem 3.1 such that*

$$T_0(E^i_P) = E^i_Q \text{ and } T_1(E^i_Q) = E^i_P, \quad \text{for every } i \in \{1, \ldots, k\}.$$

3.2 Periodic points in the unfolding of heterodimensional cycles

Using Lemma 3.4 we get the following two corollaries of Theorem 3.1. First we use the notation $\Gamma(\varphi, U, P, Q)$ to localize a cycle, that is, if we are only concerned with the intersection between the invariant manifolds of P and Q whose orbit is contained in U.

Corollary 3.6. *Consider a heterodimensional cycle $\Gamma(\varphi, U, P, Q)$ associated to the hyperbolic periodic points P and Q of indices p and q, where $p > q$, having positive real eigenvalues of multiplicity one. Then, for every integer $\ell \in [q, p]$, there is a diffeomorphism ϕ arbitrarily close to φ with a hyperbolic periodic point of index ℓ in $\Lambda_\phi(U)$.*

Proof: This corollary is trivial when $\ell = p$ or q. So let us fix some $\ell \in]q, p[$. Define the matrices M_P and M_Q as in the statement of Theorem 3.1 and denote by $\lambda^1_P, \cdots, \lambda^{\dim(M)}_P$ the eigenvalues of M_P, where $0 < \lambda^1_P < \cdots < \lambda^{\dim(M)}_P$, and by $\lambda^1_Q, \cdots, \lambda^{\dim(M)}_Q$ the eigenvalues of M_Q, where $0 < \lambda^1_Q < \cdots < \lambda^{\dim(M)}_Q$.

For each $i \in \{1, \ldots, \dim(M)\}$ let $E^i(P)$ and $E^i(Q)$ the eigenspaces corresponding to λ^i_P and λ^i_Q, respectively. We now consider the invariant splittings (of M_P and M_Q) given by

$$\begin{aligned} E_1(P) &= E^1(P) \oplus \cdots E^{\ell-1}(P), \quad E_2(P) = E^\ell(P), \quad E_3(P) = E^{\ell+1}(P) \oplus \cdots E^{\dim(M)}(P), \\ E_1(Q) &= E^1(Q) \oplus \cdots E^{\ell-1}(Q), \quad E_2(Q) = E^\ell(Q), \quad E_3(Q) = E^{\ell+1}(Q) \oplus \cdots E^{\dim(M)}(Q). \end{aligned}$$

Observe that, by the hypotheses on the eigenvalues of P and Q, the splittings $E_1(R), E_2(R)$ and $E_3(R)$, $R = P, Q$, are dominated (for M_P and M_Q), thus they satisfy the hypotheses of Theorem 3.1.

Since $q < \ell < p$ we have that $\lambda^\ell_P < 1 < \lambda^\ell_Q$. Thus there are constants C and C', $0 < C < 1 < C'$, and arbitrarily big natural numbers n_0 and m_0 such that

$$(\lambda^{\ell-1}_P)^{n_0} (\lambda^{\ell-1}_Q)^{m_0} < C < (\lambda^\ell_P)^{n_0} (\lambda^\ell_Q)^{m_0} < C' < (\lambda^{\ell+1}_P)^{n_0} (\lambda^{\ell+1}_Q)^{m_0}.$$

Applying Theorem 3.1 to the matrices M_P and M_Q, $n = n_0$, $m = m_0$, and the matrices $I_0, \ldots, I_{n+m+2}$ equal to the identity, we get transitions T_0 and T_1 and a diffeomorphism ϕ close to φ having a periodic point $R \in \Lambda_\phi(U)$ of period $n(R) \simeq n_0 + m_0$ such that $\phi^{n(R)}_*$ is conjugate to

$$M_R = T_1 \circ M^{m_0}_Q \circ T_0 \circ M^{n_0}_P.$$

By Lemma 3.5 we can suppose that T_0 and T_1 preserve the splittings $E_1 \oplus E_2 \oplus E_3$. Hence the ℓ^{th}-eigenvalue λ_R^ℓ of M_R is such that

$$\frac{C}{k_1} < |\lambda_R^\ell| < k_2\, C',$$

where k_1 is the product of the norms of T_0^{-1} and T_1^{-1} and k_2 is the product of the norms of T_0 and T_1. Observe that a priori we can not guarantee that this eigenvalue is positive (we do not know if the transitions preserve the orientation). Thus, taking n_0 and m_0 big enough, we can assume that $|\log(\lambda_R^\ell)|/(n_0+m_0)$ is arbitrarily close to zero.

Applying now Lemma 3.4 to the derivative of ϕ along the orbit of R, we can assume that the eigenvalues $\lambda_R^1, \dots, \lambda_R^{\dim(M)}$ of $\phi_*^{n(R)}(R)$ satisfy

$$0 < |\lambda_R^1| < \cdots < |\lambda_R^{\ell-1}| < 1 = |\lambda_R^\ell| < |\lambda_R^{\ell+1}| < \cdots < |\lambda_R^{\dim(M)}|. \tag{1}$$

After a final perturbation, we have that R has index ℓ, ending the proof of the corollary. $\square$

Finally, a minor modification of the proof of Corollary 3.6 gives the following:

Corollary 3.7. *Consider a heterodimensional cycle $\Gamma(\varphi, U, P, Q)$ satisfying the hypothesis of Theorem 3.1. Moreover suppose that there is a dominated splitting $F_1 \oplus \cdots \oplus F_i \oplus \cdots \oplus F_k$ over $\Lambda_\varphi(U)$ such that the moduli of the Jacobians of φ restricted to F_i along the orbits of Q and P are strictly bigger and less than one, respectively.*

Then there is a diffeomorphism ϕ arbitrarily C^1-close to φ with a hyperbolic periodic point $R \in \Lambda_\phi(U)$ such that the modulus of the Jacobian of $\phi^{n(R)}$ over F_i at R is equal to one.

Proof: Consider the dominated splittings

$$E_1 = F_1 \oplus \cdots \oplus F_{i-1}, \quad E_2 = F_i \quad E_3 = F_{i+1} \oplus \cdots \oplus F_k.$$

Just observe that by Lemma 3.5 we can choose the transitions T_i preserving the dominated splitting $E_1 \oplus E_2 \oplus E_3$. The result follows arguing as in Corollary 3.6. $\square$

3.3 End of the proof of Theorem A

We need the following lemma,

Lemma 3.8. ([BDP, Lemma 5.4]) *Let V be an open set of M and R_φ a hyperbolic periodic point of a diffeomorphism φ such that its relative homoclinic class in V, $\overline{H_{R_\varphi}(V)}$, is non trivial. Then there is a diffeomorphism ϕ arbitrarily $\mathcal{C}^1$-close to φ such that $\overline{H_{R_\phi}(V)}$ contains a hyperbolic periodic point of the same index of R_ϕ whose eigenvalues are all real, positive and of multiplicity one.*

Under the hypothesis of Theorem A, this lemma allows us to assume that, after perturbing the original diffeomorphism and replacing the initial points P_φ and Q_φ by other points of $\Lambda_\varphi(U)$ of the same index, we can assume that the points P_φ and Q_φ of $\Lambda_\varphi(U)$ have real positive eigenvalues of multiplicity one. To see why this is so just observe that, by Theorem 2.3, after a $\mathcal{C}^1$-perturbation of φ we can assume that $\overline{H_{P_\varphi}(U)} = \overline{H_{Q_\varphi}(U)} \subset \Lambda_\varphi(U)$, thus these two relative homoclinic classes are non-trivial. Hence we can now apply Lemma 3.8 to such homoclinic classes to get the periodic points (of indices p and q) in $\Lambda_\varphi(U)$ with real positive eigenvalues of multiplicity one. So we lose no generality assuming that the points P_φ and Q_φ in Theorem A have real positive eigenvalues of multiplicity one. Using Lemma 2.5 and Corollary 3.6 one gets:

Lemma 3.9. *Given $p > q$ and $\ell \in]q,p]$ let $\varphi \in \mathcal{M}(U)$ be a diffeomorphism with two hyperbolic periodic points P_φ and Q_φ in $\Lambda_\varphi(U)$ of indices p and q having positive real eigenvalues of multiplicity one. Then there is $\phi \in \mathcal{M}(U)$ arbitrarily C^1-close to φ having a hyperbolic periodic point of index ℓ in $\Lambda_\phi(U)$.*

Proof: By hypothesis, the continuations P_ϕ and Q_ϕ of P_φ and Q_φ are transitively related for every ϕ in a neighbourhood of φ in $\mathcal{M}(U)$ (just observe that set $\Lambda_\phi(U)$ is robustly transitive and P_ϕ and Q_ϕ belong to $\Lambda_\phi(U)$). Hence we can apply Lemma 2.5 to P_φ and Q_φ to create a heterodimensional cycle $\Gamma(\psi, U, P_\psi, Q_\psi)$ for some ψ arbitrarily close to φ. Corollary 3.6 now gives ϕ close to ψ (thus close to φ) with a periodic point of index ℓ in $\Lambda_\phi(U)$, ending the proof of the lemma. $\square$

Given $\varphi \in \mathcal{M}(U)$ consider a neighbourhood $\mathcal{U}_\varphi$ of φ in $\mathcal{M}(U)$ such that every $\psi \in \mathcal{U}_\varphi$ has hyperbolic periodic points of indices q and p. Let $\mathcal{H}_j$ be the set of diffeomorphisms $\psi \in \mathcal{U}_\varphi$ having some hyperbolic periodic point of index j in $\Lambda_\varphi(U)$. Applying Lemma 3.9 finitely many times, one gets that the sets $\mathcal{H}_j$, $j \in [q,p]$, are dense in $\mathcal{U}_\varphi$.

Theorem A now follows by observing that, for every j, the set $\mathcal{H}_j$ is open. Now it is enough to consider the set $\cap_q^p \mathcal{H}_j$, which is a dense open subset of $\mathcal{U}_\varphi$. This ends the proof of Theorem A.

4 Hyperbolicity of the extremal bundles

In this section we prove Theorem B. For that, as in the hypotheses of this theorem, consider an open set U of a compact manifold M and $q \in \mathbb{N}^*$ and let $\mathcal{U}$ be a C^1-open set of $\mathrm{Diff}^1(M)$ such that for every diffeomorphism $\phi \in \mathcal{U}$ the set $\Lambda_\phi(\bar{U})$ has a dominated splitting $E_\phi \oplus F_\phi$ with $\dim(E_\phi(x)) = q$ for all $x \in \Lambda_\phi(\bar{U})$. Suppose that every $\phi \in \mathcal{U}$ has no periodic points of index $r < q$. Then we prove that the bundle E_ϕ is uniformly contracting for every $\phi \in \mathcal{U}$.

The proof of this result follows using the arguments in [M$_2$] after some small technical modifications. So here we will just sketch this proof, emphasizing the main modifications that we need to introduce.

The results in [M$_2$] are formulated in terms of *families of periodic sequences of linear maps.* It is considered the family obtained by taking all the diffeomorphism ϕ in an open set of $\mathrm{Diff}^1(M)$ and the restrictions of the derivatives of these diffeomorphisms to their periodic orbits. He consider perturbations of this system of linear maps without caring if such perturbations come from perturbations of the initial diffeomorphism. However, a Lemma of Franks' (see Lemma 3.4 above) allows one to perform dynamically the perturbation of the derivative: given a diffeomorphism φ and a periodic point x of φ, to each perturbation A of the derivative φ_* throughout the orbit of x corresponds a diffeomorphism ψ C^1-close to φ which preserves the φ-orbit of x and such that $A(z) = \psi_*(z)$ for all z in the φ-orbit of x.

We begin by recalling some results about dominated splittings, see next section. In Section 4.2 we recall the terminology about families of periodic linear systems and some results in [M$_2$]. Finally, in Section 4.3 we prove Theorem B.

4.1 Remarks on dominated splittings

In this subsection we state precisely some folklore results on dominated splittings. Before that let us observe that if $\Lambda_\varphi(U)$ is a robustly transitive, then, by definition, it is a φ-invariant compact subset of U which is the maximal φ-invariant set of $\bar{U}$. This implies that, for any neighbourhood V of $\Lambda_\varphi(U)$ and every diffeomorphism ϕ close to φ, the set $\Lambda_\phi(U)$ coincides with $\Lambda_\phi(\bar{U})$ and is contained in V. Thus $\Lambda_\phi(U)$ depends lower-semi-continuously on ϕ. We say that $\Lambda_\phi(U)$ is the *continuation* of $\Lambda_\varphi(U)$ for ϕ.

Lemma 4.1. *Let φ be a diffeomorphism and U an open subset of M such that $\Lambda_\varphi(U)$ coincides with $\Lambda_\varphi(\bar{U})$ and admits a dominated splitting $T_{\Lambda_\varphi(U)}M = E\oplus F$, $E \prec F$. Then, for every diffeomorphism ψ close enough to φ, there is a unique dominated splitting $E_\psi \oplus F_\psi$, $E_\psi \prec F_\psi$, defined over $\Lambda_\psi(U)$ such that* $\dim(E_\psi) = \dim(E)$.

The splitting $E_\psi \oplus F_\psi$ above is the *continuation* of $E \oplus F$. Moreover, the continuations E_ψ and F_ψ depend continuously on ψ. This lemma also holds for dominated splitting with an arbitrary number of bundles.

Proof: Let us just sketch the proof of the lemma. By the definition of domination, there is a strictly φ_*-invariant continuous cone field $\mathcal{C}^+$ defined over $\Lambda_\varphi(U)$ such that the bundle F is obtained as the intersection of the forward φ_*-iterates of the cones of $\mathcal{C}^+$. Similarly, there is a strictly (φ_*^{-1})-invariant continuous cone field $\mathcal{C}^-$ defined over $\Lambda_\varphi(U)$ such that the intersections of the backward iterates of $\mathcal{C}^-$ define E. These cone fields can be extended continuously to invariant cone fields $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ defined on a compact neighbourhood V of $\Lambda_\varphi(U)$.

Observe that every ψ close to φ let invariant the cone fields $\mathcal{C}_0^+$ and $\mathcal{C}_0^-$ and recall that $\Lambda_\psi(U) \subset V$. We now define the bundles E_ψ and F_ψ as the intersection of the (backward and forward, respectively) iterates by ψ_* of the cones of $\mathcal{C}_0^-$ and $\mathcal{C}_0^+$, respectively. By construction, the splitting $E_\psi \oplus F_\psi$ is dominated and satisfies $\dim E_\psi = \dim E$.

For the continuous dependence of the bundles E_ψ and F_ψ on the diffeomorphism ψ we refer the reader to [BDP, Lemma 1.4], for instance. This ends the sketch of the proof. □

Lemma 4.2. ([BDP, Lemma 1.4]). *Let ϕ be a diffeomorphism and Σ a ϕ-invariant set having a dominated splitting $E \oplus F$. Then this splitting can be extended (in a dominated way) to the closure of Σ.*

Remark 4.3. *Let φ be a diffeomorphism, K a transitive φ-invariant compact set, $T_KM = E_1 \oplus E_2 \oplus \cdots \oplus E_m$ the finest dominated splitting of φ over K, and $\Sigma \subset K$ a φ-invariant dense subset of K. Then the finest dominated splitting of φ over Σ is given by the restriction to Σ of the bundles E_i.*

Proof of the remark: We argue by contradiction, suppose that there is a dominated splitting over Σ which refines the splitting given by the restrictions to Σ of the bundles E_i. Then, by Lemma 4.2, such a splitting can be extended to the whole K, contradicting that the splitting $E_1 \oplus \cdots \oplus E_m$ is the finest one. □

Let us state a final result whose proof we omit.

Remark 4.4. *Let φ be a diffeomorphism and E a φ_*-invariant bundle defined on a φ-invariant compact set K_1. Consider any φ-invariant dense subset K_2 of K_1. Then we have the following:*

- *The bundle E is uniformly hyperbolic over K_1 if and only if its restriction to K_2 is uniformly hyperbolic.*
- *The diffeomorphism φ contracts (resp. expands) uniformly the volume in E over K_1 if and only if it contracts uniformly (resp. expands) the volume in E over K_2.*

4.2 Families of periodic sequences of linear maps and dominated splittings

We begin this section by recalling some definitions in [M$_2$].

Definition 4.1.

1. *A* periodic sequence of a linear maps *is a periodic map $\xi\colon \mathbb{Z} \to GL(N,\mathbb{R})$, $n \mapsto \xi_n$, such that the sequence of norms $\|\xi_n\|$ and $\|\xi_n^{-1}\|$ are uniformly bounded (independently of n). We denote this family by $\{\xi_n\}$.*
2. *A periodic sequence of linear maps $\{\xi_n\}$ of period n is called* contracting *if the product $\xi_{n-1} \circ \cdots \circ \xi_0$ is an uniform contraction, i.e., all its eigenvalues have modulus strictly less than* 1.
3. *A family $\Xi = \{\xi^{(\alpha)}\}_{\alpha\in\mathcal{A}}$ of periodic sequences of linear maps is* robustly contracting[1] *if there is $\varepsilon > 0$ such that any family $\Theta = \{\theta^{(\alpha)}\}_{\alpha\in\mathcal{A}}$ having the same period function $n(\alpha)$ and ε-close to Ξ (i.e., $\|\theta^\alpha_n - \xi^\alpha_n\| < \varepsilon$ for all $\alpha \in \mathcal{A}$ and $n \in \mathbb{Z}$) is contracting.*

The example of family of periodic sequence of linear maps that will be play a key role in the proof of Theorem B is obtained as follows. Let $\phi \in \mathcal{U}$, $\mathcal{U}$ as in Theorem B, and $\delta > 0$ such that every diffeomorphism ψ which is 2δ-C^1-close to ϕ belongs to $\mathcal{U}$. Now let $\mathcal{A}_\phi$ be the set of pairs $\alpha = (x, \psi)$ such that ψ is δ-close to ϕ and the ψ-orbit of x is contained in U and periodic. Consider now some trivialization of the bundles E_ψ (as in Theorem B) over the set of periodic points (by choosing an orthonormal basis of $E_\psi(x)$) and for each $\alpha = (x, \psi) \in \mathcal{A}_\phi$ define ξ^α by the restrictions of the differential ψ_* to $\{E_\psi(\psi^i(x))\}_{i\in\mathbb{Z}}$. We now have that $\Xi_\phi = \{\xi^\alpha\}_{\alpha\in\mathcal{A}_\phi}$ is a family of periodic sequences of linear maps.

Lemma 4.5. *The family Ξ_ϕ defined above is robustly contracting.*

Proof: The proof if by contradiction. Otherwise, there are $(x, \psi) \in \mathcal{A}_\phi$ and a linear map ν corresponding to a perturbation of the restriction of the differential of ψ to E_ψ along the periodic ψ-orbit of x, having an eigenvalue of modulus bigger or equal than one, i.e.,

$$\nu(\psi^{n(x)-1}(x)) \circ \cdots \circ \nu(x)\colon E_\psi(x) \to E_\psi(x)$$

has an eigenvalue λ such that $|\lambda| \geq 1$, where $n(x)$ is the ψ-period of x.

Using Lemma 3.4 we get a diffeomorphism ζ close to ψ, thus in $\mathcal{U}$, such that x is a periodic point of $\Lambda_\zeta(\bar{U})$ and

$$\zeta_*^{n(x)}(x) = \zeta_*(\zeta^{n(x)-1}) \circ \cdots \zeta_*(x) = \nu(\psi^{n(x)-1}(x)) \circ \cdots \circ \nu(x).$$

---

[1] This notion is called *uniformly contracting* in [M$_2$], but we rename it to avoid ambiguity with the now usually accepted notion of uniform hyperbolicity or uniform contraction.

Thus the restriction of $\zeta_*^{n(x)}(x)$ to $E_\zeta(x)$ has at most $(q-1)$ eigenvalues of modulus (strictly) less than one. On the other hand, by the domination $E_\psi \prec F_\psi$, the eigenvalues of the restriction of $\zeta_*^{n(x)}(x)$ to $F_\zeta(x)$ are all strictly bigger than one in modulus. This implies that there is a periodic point x in $\Lambda_\zeta(\bar{U})$ of index (strictly) less than q, contradicting the definition of $\mathcal{U}$. This contradiction ends the proof of the lemma. $\square$

We now borrow the following lemma from [M$_2$].

Lemma 4.6. ([M$_2$, Lemma II.7]). *Let $\{\xi^{(\alpha)}, \alpha \in \mathcal{A}\}$ be a robustly contracting family of periodic sequences of isomorphisms of $\mathbb{R}^N$. Then there exist $K > 0$, $0 < \lambda < 1$ and $m \in \mathbb{N}^*$ such that:*

a) if $\alpha \in \mathcal{A}$ and ξ^α has minimum period $n \geq m$, then

$$\prod_{j=0}^{k-1} \left\| \prod_{i=0}^{m-1} \xi_{i+mj}^{(\alpha)} \right\| \leq K\lambda^k,$$

where k is the integer part of n/m;

b) for all $\alpha \in \mathcal{A}$

$$\limsup_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log \left(\left\| \prod_{i=0}^{m-1} \xi_{i+mj}^{(\alpha)} \right\| \right) < 0.$$

Applying Lemma 4.6 to the family Ξ_ϕ defined above we get the next proposition which is a reformulation of [M$_2$, Proposition II.1]:

Proposition 4.7. *Let $\phi \in \mathcal{U}$ ($\mathcal{U}$ as in Theorem B). Then there are a neighborhood $\mathcal{V}$ of ϕ and constants $K > 0$, $m \in \mathbb{N}^*$ and $0 < \lambda < 1$ such that for every $g \in \mathcal{V}$ and every periodic point x of ψ whose orbit is contained in U one has:*

a) If x has minimum period $n \geq m$ then

$$\prod_{i=0}^{k-1} \left\| (\psi^m)_*(\psi^{mi}(x))|_{E_\psi(\psi^{mi}(x))} \right\| \leq K\lambda^k,$$

where k is the entire part of n/m.

b) Moreover,

$$\limsup_{r\to+\infty} \frac{1}{r} \sum_{i=0}^{r-1} \log \left(\left\| (\psi^m)_*(\psi^{mi}(x))|_{E_\psi(\psi^{mi}(x))} \right\| \right) < 0.$$

Theorem B will be a consequence of Proposition 4.7 and the Mañé's Ergodic Closing Lemma that we now recall, for completeness:

Theorem 4.8. (Ergodic Closing Lemma, [M$_2$, Theorem A]). *Consider a diffeomorphism ϕ defined on a compact manifold. Then there is a ϕ-invariant set $\Sigma(\phi)$ (named set of* well closable points*of ϕ) such that:*

1. *The set* $\Sigma(\phi)$ *has total measure (i.e.* $\mu(\Sigma(\phi)) = 1$ *for every* ϕ*-invariant probability measure* μ*).*

2. *For every* $x \in \Sigma(\phi)$ *and* $\varepsilon > 0$ *there is a diffeomorphism* ψ*, which is* ε*-close to* ϕ *in the* C^1*-topology, such that* x *is periodic for* ψ *and the distance* $\mathrm{dist}(\phi^i(x), \psi^i(x)) < \varepsilon$ *for all* $i \in [0, n(x,\psi)]$*, where* $n(x,\psi)$ *is the period of* x *for* ψ*.*

4.3 End of the proof of Theorem B

The proof of the theorem now follows almost exactly as the proof of [M$_2$, Theorem B], see pages 520-524. Let us recall the main steps of this proof and point out the changes we need to introduce.

Proof: Let $\phi \in \mathcal{U}$. By compactness of the set $\Lambda_\phi(\bar{U})$, as in [M$_2$] to get the uniform contraction of the bundle E_ϕ it is enough to see that

$$\liminf_{n\to+\infty} ||\phi_*^n|_{E_\phi(x)}|| = 0.$$

We argue by contradiction. If ϕ_* is not uniformly contracting on E_ϕ over $\Lambda_\phi(\bar{U})$ then there are a constant $\kappa > 0$, point $x \in \Lambda_\phi(\bar{U})$ and $n_0 \in \mathbb{N}$ such that

$$||\phi_*^n|_{E_\phi(x)}|| > \kappa > 0$$

for every $n > n_0$. We now choose a sequence j_n, $j_n \to +\infty$, such that the sequence of probabilities μ_n defined by

$$\mu_n = \frac{1}{j_n}\sum_{i=0}^{j_n-1} \delta(\phi^{mi}(x))$$

converges (in the weak topology) to a probability μ, where $\delta(z)$ is the Dirac measure at the point z and m is as in Proposition 4.7.

Let $\varphi^\phi = \log||\phi_*^m|_{E_\phi}||$. By Lemma 4.1 the bundle E_ϕ is continuous on $\Lambda_\phi(\bar{U})$, so φ^ϕ is continuous on $\Lambda_\phi(\bar{U})$. By the choice of x, one has $\int \varphi^\phi\, d\mu_n \geq 0$ for every large enough n, so $\int \varphi^\phi\, d\mu \geq 0$. Using the Birkhoff's Theorem and the Ergodic Closing Lemma we get a point $p \in \Lambda_\phi(\bar{U}) \cap \Sigma(\phi)$ such that

$$\lim_{n\to+\infty} \frac{1}{j_n}\sum_{i=0}^{j_n-1} \log ||\phi_*^m|_{E_\phi(\phi^{mi}(p))}|| \geq 0.$$

By item (b) of Proposition 4.7, the point p is not periodic. Now, by Theorem 4.8, there is ψ arbitrarily C^1-close to ϕ (so $\psi \in \mathcal{V} \subset \mathcal{U}$, $\mathcal{V}$ as in Proposition 4.7) such that p is a periodic point of ψ of period $n(p)$ and the distance $\mathrm{dist}(\phi^i(p), \psi^i(p))$ is less than an arbitrarily small $\varepsilon > 0$ for every $i \in [0, n(p)]$. Observe that since p is not periodic for ϕ, the period $n(p)$ goes to infinity as ε goes to zero, i.e., ψ tends to ϕ.

Since the fibers $E_\psi(y)$ vary continuously with (y, ψ), recall Lemma 4.1, the function

$$\varphi^\psi(y) = \log||\psi_*^m|_{E_\psi(y)}||$$

is continuous. Now for λ as in Proposition 4.7 take λ_0 and $n_0 \in \mathbb{N}^*$ such that $\lambda < \lambda_0 < 1$ and for every $n \geq n_0$ one has

$$\frac{1}{n}\sum_{i=0}^{n-1} \varphi^\phi(\phi^{mi}(p)) \geq \frac{1}{2}\log(\lambda_0).$$

We can also assume that $K\lambda^n < \lambda_0^n$ for every $n \geq n_0$. So if ψ is close enough to ϕ then

$$|\varphi^{\psi}(\psi^i(p)) - \varphi^{\phi}(\phi^i(p))| < |\frac{1}{2}\log(\lambda_0)|$$

for every $i \in [0, n(p)]$. Moreover, the entire part k of $n(p)/m$ is greater than n_0. Thus

$$\frac{1}{k}\sum_{i=0}^{k-1} \varphi^{\psi}(\psi^{mi}(p)) \geq \log(\lambda_0) > \frac{1}{k}\log(K\lambda^k),$$

contradicting item (a) of Proposition 4.7. This contradiction ends the proof of Theorem B. □

5 Proof of Theorem D

5.1 Perturbation of the derivative at periodic points

In this section we recall some results from [BDP]. These results are formulated in terms of *families of periodic linear systems,* that is, considering the differential of the diffeomorphism as an abstract linear cocycle over the set $\Lambda_\varphi(U)$ and perturbations of this cocycle, without caring if such perturbations come from perturbations of the diffeomorphism. However, as in Section 4, Lemma 3.4 allows us to perform dynamically the final abstract cocycle. Let us explain these results in a detailed way.

Given a diffeomorphism φ and a hyperbolic periodic point P_φ of φ of index p denote by Σ_{P_φ} the subset of $\overline{H(P_\varphi, U)}$ of hyperbolic periodic points R of index p homoclinically related to P_φ, i.e., $W^s(R) \pitchfork W^u(P_\varphi) \neq \emptyset$ and $W^u(R) \pitchfork W^s(P_\varphi) \neq \emptyset$. Observe that in our setting we can assume that Σ_{P_φ} is not trivial (different to the orbit of P_φ).

As above, given $x \in \Sigma_{P_\varphi}$ denote by M_x the matrix $M_x = \varphi_*^{n(x)}(x)\colon T_xM \to T_xM$, where $n(x)$ is the period of x. The first important property formalized in [BDP] is that the matrices M_x corresponding to different points of Σ_{P_φ} (the derivatives of $\varphi^{n(x)}$ at these points x) can be *essentially* multiplied how many times as one wants, and the resulting product corresponds to a matrix of the system at some different point. More precisely,

Lemma 5.1. *Let φ be a diffeomorphism and P_φ a hyperbolic periodic point of φ. Consider any pair of periodic points of x and y of φ in Σ_{P_φ} and $\varepsilon > 0$. Suppose that M_x and M_y let invariant dominated splittings*

$$T_xM = E_x^1 \oplus \cdots \oplus E_x^k, \quad E_i(x) \prec E_{i+1}(x), \quad \textit{and} \quad T_yM = E_y^1 \oplus \cdots \oplus E_y^k, \quad E_i(y) \prec E_{i+1}(y),$$

such that $\dim(E_x^i) = \dim(E_y^i)$ *for every i. Then there is $\delta \in]0, \varepsilon[$ satisfying the following property:*

Given any pair of δ-perturbations $\tilde{M}_x$ and $\tilde{M}_y$ of M_x and M_y, respectively, $\tilde{M}_x\colon T_xM \to T_xM$ and $\tilde{M}_y\colon T_yM \to T_yM$, there are linear maps

$$T_1\colon T_xM \to T_yM \quad \textit{and} \quad T_2\colon T_yM \to T_xM$$

preserving the dominated splittings above (i.e., $T_1(E_x^i) = E_y^i$ and $T_2(E_y^i) = E_x^i$ for every i) such that, for any $n \geq 0$ and $m \geq 0$, there are a periodic point $z \in \Sigma_{P_\varphi}$ and an ε-perturbation of φ_* along the orbit of z,

$$A^i\colon T_{\varphi^i(z)}M \to T_{\varphi^{i+1}(z)}M, \quad i = 0, \ldots, n(z) - 1,$$

such that

$$\tilde{M}_z = A^{n(z)-1} \circ \cdots \circ A^0 \colon T_z M \to T_z M$$

is conjugate to the product $T_2 \circ M_y^m \circ T_1 \circ M_x^n$.

Remark 5.2. *In fact, in* [BDP] *it is shown that Lemma 5.1 holds for any finite number of orbits* $x_1, \dots, x_k$ *of* Σ_{P_φ}. *This allows us to get linear maps* $T_i \colon T_{x_i} M \to T_{x_{i+1}} M$ *preserving a dominated splitting such that, for every* $n_1, \dots, n_k$, *there are a point* $z \in \Sigma_{P_\varphi}$ *and perturbations* A^i *of the derivative of* φ_* *at* $\varphi^i(z)$ *such that* $\tilde{M}_z = A^{n(z)-1} \circ \cdots \circ A^0$ *is conjugate to* $T_k \circ M_{x_k}^{n_k} \circ \cdots \circ T_2 \circ M_{x_2}^{n_2} \circ T_1 \circ M_{x_1}^{n_1}$.

The maps T_i correspond to the so called *transitions*, recall also Theorem 3.1. The fact that the transitions can be taken preserving a dominated splitting has been proved in [BDP, Lemma 4.13]. This property is the basis of the proof of the following result:

Lemma 5.3. *Let* $E_1 \oplus \cdots \oplus E_m$, $E_i \prec E_{i+1}$, *be the finest dominated splitting of* TM *over* Σ_{P_φ} *of* φ_*. *Then, for every* $\varepsilon > 0$, *there are a dense subset* Σ_ε *of* Σ_{P_φ} *and an* ε*-perturbation* A_ε *of* φ_* *preserving the splitting* $E_1 \oplus \cdots \oplus E_m$ *such that, for every* $R \in \Sigma_\varepsilon$, *the restriction of the linear maps*

$$M_{A_\varepsilon}(R) = A_\varepsilon(\varphi^{n(R)-1}(x)) \circ \cdots \circ A_\varepsilon(\varphi(x)) \circ A_\varepsilon(x)$$

to $E_i(R)$ *is a homothety.*

Moreover, if there are $i \in \{1, \dots, m\}$ *and* $Q \in \Sigma_{P_\varphi}$ *such that the modulus of the Jacobian of the restriction of* $\varphi_*^{n(Q)}$ *to* $E_i(Q)$ *is one then* R *can be taken such that the restriction of* $M_{A_\varepsilon}(R)$ *to* $E_i(R)$ *is identity.*

This lemma is a consequence of [BDP, Propositions 2.4 and 2.5]. To see that these propositions can be applied in our context, one just needs to observe that the restriction of φ_* to each bundle E_i (over Σ_{P_φ}) defines a periodic linear system with transitions. For that it is enough to recall that the transitions of φ_* can be chosen preserving the bundles E_j of the dominated splitting (see [BDP, Section 4]).

Given a hyperbolic linear map A of an Euclidean space (i.e., without eigenvalues of modulus equal to 1) the *index of* A is the number of eigenvalues of A of modulus less than 1, counted with multiplicity.

Lemma 5.4. ([BDP, Lemma 4.16]) *Given* $\varepsilon > 0$ *there exist* $x \in \Sigma_{P_\varphi}$ *and an* ε*-perturbation of* φ_* *along the orbit of* x *such that the corresponding matrix* M_x *has index* p, $p = \text{index}(P_\varphi)$, *and all the eigenvalues of* M_x *are real, positive and with multiplicity* 1.

5.2 Tangencies and codimension one heterodimensional cycles

The existence of non-real eigenvalues in the *central direction* of the saddles in a (codimension one) heterodimensional cycle produces homoclinic tangencies. That is formalized in the following result we export from [DR].

Let A be a linear map of an n-dimensional Euclidean space E, we say that a non-real eigenvalue $\lambda \in (\mathbb{C} \setminus \mathbb{R})$ of A has *rank* ℓ if there are $(\ell - 1)$ eigenvalues (counted with multiplicity) of A of modulus strictly less than $|\lambda|$ and $(n - \ell - 1)$ eigenvalues of modulus strictly bigger than $|\lambda|$. A periodic point P of a diffeomorphism φ has a *non-real eigenvalue of rank* ℓ if its derivative $\varphi_*^{n(P)}(P)$ has a a non-real eigenvalue of rank ℓ.

Lemma 5.5. *Let $\Gamma(\phi, U, R^1_\phi, R^2_\phi)$ be a codimension one heterodimensional cycle associated to hyperbolic periodic points of indices $(r+1)$ and r. Suppose that R^1_ϕ (resp. R^2_ϕ) has a non-real eigenvalue of rank r (resp. $r+1$). Then there is ψ arbitrarily close to ϕ with a homoclinic tangency associated to R^2_ψ (resp. R^1_ψ) in $\Lambda_\psi(U)$.*

Proof: Just observe that if R^1_ϕ has a non-real eigenvalue of rank r then the unstable manifold of R^2_ϕ spiralizes around $W^u(R^1_\phi)$. Now unfolding the cycle $\Gamma(\phi, U, R^1_\phi, R^2_\phi)$ one gets a homoclinic tangency associated to the continuation of R^2_ϕ. See [DR, Section 8.1] for details. $\square$

5.3 Proof of Theorem D

Consider $\varphi \in \mathcal{P}(U)$ and its finest dominated splitting $E_1(\varphi) \oplus \cdots \oplus E_{m(\varphi)}(\varphi)$ over $\Lambda_\varphi(U)$. By Lemma 4.1, the continuation of this splitting over $\Lambda_\phi(U)$ is uniquely defined for every ϕ close to φ. Denote such a continuation by $E_1(\phi) \oplus \cdots \oplus E_{m(\varphi)}(\phi)$. By Lemma 4.1, the the number $m(\varphi)$ of bundles of the finest dominated splitting of $\Lambda_\varphi(U)$ is lower semi-continuous, thus locally constant in an open and dense subset $\mathcal{P}_1(U)$ of $\mathcal{P}(U)$. Moreover, the dimensions of the bundles of the finest dominated splitting are also locally constant in $\mathcal{P}_1(U)$. So there is an open and dense subset $\mathcal{O}(U)$ of $\mathcal{P}(U)$ where $m(\varphi)$ and the dimensions of the bundles of the finest dominated splitting are continuous functions. This set $\mathcal{O}(U)$ is the open and dense subset of $\mathcal{P}(U)$ announced in Theorem D.

Observe that it is enough to prove the theorem for a connected component of $\mathcal{O}(U)$. So from now on we restrict our attention to a fixed connected component $\mathcal{O}_0$ of $\mathcal{O}(U)$.

Given $\varphi \in \mathcal{O}_0$ consider the finest dominated splitting of $\Lambda_\varphi(U)$, say $T_{\Lambda_\varphi(U)}M = E_1(\varphi) \oplus E_2(\varphi) \oplus \cdots \oplus E_{m(\varphi)}(\varphi)$, as the dimensions and the number of bundles of the splitting do not depend on $\varphi \in \mathcal{O}_0$ from now on we will omit such a (in)dependence.

Let us now introduce some notations. For simplicity write $p = i_c$ and $q = i_s$ (the maximum and minimum indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$). Given i and j in $\{1, \dots, m\}$, with $i < j$, let

$$E_i^j = E_i \oplus E_{i+1} \oplus \cdots \oplus E_j.$$

Denote by d_i and d_i^j the dimensions of E_i and E_i^j, respectively, thus $d_i^j = \sum_{k=i}^j d_k$. We define i_q and i_p by the relations

$$d_1^{i_q-1} < q \le d_1^{i_q} \quad \text{and} \quad d_1^{i_p-1} < p \le d_1^{i_p}.$$

To prove Theorem D it is enough to see the following:

- **(A)** $d_1^{i_q} = q$ and $d_{i_p+1}^m = \dim(M) - p$,
- **(B)** $d_j = 1$ and the bundle E_j is not uniformly hyperbolic for all $j \in \{i_q+1, \dots, i_p\}$,
- **(C)** $E_1^{i_q}$ and $E_{i_p+1}^m$ are uniformly contracting and expanding, respectively.

In Lemmas 5.6, 5.7 and 5.8 we will prove these items.

Lemma 5.6. (Proof of (A)). *$d_1^{i_q} = q$ and $d_{i_p+1}^m = \dim(M) - p$.*

Proof: Let us prove the first part of the lemma. The proof is by contradiction, assume that $d_1^{i_q} > q$, then, by definition of $d_1^{i_q}$, one has

$$d_1^{i_q-1} < q < q+1 \le d_1^{i_q} = d_1^{i_q-1} + d_{i_q},$$

thus

$$d_{i_q} \ge 2. \tag{2}$$

By Proposition 2.4 and the definition of $\mathcal{O}_0$, there is a diffeomorphism $\varphi \in \mathcal{O}_0$ with a hyperbolic periodic point Q_φ of index q such that Σ_{Q_φ} is dense in $\Lambda_\varphi(U)$. By Remark 4.3, the finest dominated splitting of φ over Σ_{Q_φ} is the restriction to Σ_{Q_φ} of the bundles E_i.

By equation (2), E_{i_q} is undecomposable and has dimension d_{i_q} equal or bigger than 2. Applying Lemma 5.3 to the set Σ_{Q_φ} and the bundle E_{i_q}, we get $R_\varphi \in \Sigma_{Q_\varphi}$ of period $n(R_\varphi)$ and a perturbation A of φ_* throughout the φ-orbit of R_φ such that

$$M_A(R_\varphi)) = A(\varphi^{n(R_\varphi)-1}(R_\varphi)) \circ \cdots \circ A(\varphi(R_\varphi)) \circ A(R_\varphi)$$

is a homothety in $E_{i_q}(R_\varphi)$. We observe that the perturbation A of φ_* can be obtained (and we do so) such that its restrictions to the bundles $E_k(R_\varphi)$, $k \neq i_q$, coincide with φ_*. Thus, since all points of Σ_{Q_φ} have index q, one has that, for every $T_\varphi \in \Sigma_{Q_\varphi}$, the bundles $E_j(T_\varphi)$, $j > i_q$, correspond to expanding eigenvalues of $\varphi_*^{n(T_\varphi)}$. Hence the number of contracting eigenvalues of $M_A(R_\varphi)$ is at most $d_1^{i_q}$.

First, if the ratio of this homothety (the restriction of $M_A(R_\varphi)$ to $E_{i_q}(R_\varphi)$) is bigger or equal than one, using Lemma 3.4, one gets ϕ close to φ ($\phi \in \mathcal{O}_0$) with a hyperbolic periodic point $R_\phi \in \Lambda_\phi(U)$ having at most $d_1^{i_q-1}$ contracting eigenvalues. By hypothesis, $d_1^{i_q-1} < q$, thus the index of R_ϕ is strictly inferior than q, contradicting the definition of q (minimality of the index of the points of $\Lambda_\phi(U)$, $\phi \in \mathcal{P}(U)$).

So we can assume that the ratio of the homothety $M_A(R_\varphi)|_{E_{i_q}(R_\varphi)}$ is less than one. As the restriction of $\varphi_*^{n(R_\varphi)}$ to each $E_i(R_\varphi)$, $i > i_q$, has expanding eigenvalues, the index of R_ϕ is exactly $d_1^{i_q}$. Now, the definition of p implies that $d_1^{i_q} \le p$.

Write $\ell = d_1^{i_q} \le p$. Since all the eigenvalues of the restriction of $\phi_*^{n(R_\varphi)} = M_A(R_\varphi)$ to $E_{i_q}(R_\phi)$ are equal and $\dim(E_{i_q}(R_\phi)) \ge 2$, using again Lemma 3.4, one gets a diffeomorphisms υ close to ϕ such that R_υ has index ℓ and $\upsilon_*^{n(R_\upsilon)}(R_\upsilon)$ has a contracting non-real eigenvalue of rank $(\ell - 1)$.

By Theorem A, since $q \le \ell - 1$, there is a diffeomorphism ζ close to υ with a periodic point $S_\zeta \in \Lambda_\zeta(U)$ of index $(\ell - 1)$. Using Lemma 2.5, we obtain η close to ζ with a codimension one heterodimensional cycle in U associated to R_η and S_η, say $\Gamma(\eta, U, R_\eta, S_\eta)$. Since η can be taken arbitrarily close to υ we can assume that R_η has index ℓ and a non-real eigenvalue of rank $\ell - 1$ and that S_η has index $(\ell - 1)$. Finally, by Lemma 5.5 there is a diffeomorphism $\xi \in \mathcal{O}_0$ arbitrarily close to η with a homoclinic tangency in $\Lambda_\xi(U)$ associated to the point S_ξ of index $(\ell - 1)$, contradicting the definition of $\mathcal{P}(U)$. This ends the proof of the first assertion in the lemma.

Using the same arguments one gets that $d_{i_p+1}^m = (\dim(M) - p)$, so we omit this proof. □

Lemma 5.7. (Proof of (B)). *The bundle* E_i *is one dimensional and non-uniformly hyperbolic for all* $i \in \{i_q + 1, \ldots, i_p\}$.

Proof: Given $k \in \{i_q+1, \cdots, i_p\}$ let $\ell = d_1^k = \dim(E_1^k)$. Observe that by, Lemma 5.6, $q < \ell \leq p$.

The bundle E_k is not uniformly hyperbolic: We argue by contradiction. Otherwise, since E_k is undecomposable, it would be either uniformly contracting or expanding. In the first case, using the domination of the splitting, one has that every periodic point of $\Lambda_\varphi(U)$ has index bigger or equal than $\ell > q$, contradicting the definition of q. In the second case, again by the domination of the splitting, every periodic point of $\Lambda_\varphi(U)$ has index strictly less than $\ell \leq p$, contradicting the definition of p.

The bundle E_k is one-dimensional: The proof is by contradiction, assuming that $\dim(E_k) = d_k \geq 2$. By Theorem A and Proposition 2.4, there is $\varphi \in \mathcal{O}_0$ having a hyperbolic periodic point $R_\varphi \in \Lambda_\varphi(U)$ of index ℓ such that Σ_{R_φ} is dense in $\Lambda_\varphi(U)$. By Lemma 5.3, there are a perturbation A of φ_* and a point $S_\varphi \in \Sigma_{R_\varphi}$ such that the restriction of $M_A(S_\varphi)$ to $E_k(S_\varphi)$ is a homothety. Moreover, as above we can take A such that its restrictions to the bundles $E_i(S_\varphi)$, $i \neq k$, coincide with the one of φ_*.

Suppose, for instance, that the ratio of such a homothety is bigger than one. From $S_\varphi \in \Sigma_{R_\varphi}$ and the definition of Σ_{R_φ}, the restrictions of $\varphi_*^{n(S_\varphi)}$ to the bundles $E_i(S_\varphi)$, $i > k$, have only expanding eigenvalues. Thus the matrix $M_A(S_\varphi)$ has exactly $r = d_1^{k-1}$ contracting eigenvalues, where

$$q \leq d_1^{i_q} \leq d_1^{k-1} = r \leq d_1^{i_p-1} < d_1^{i_p} = p \quad \text{and} \quad r < r + d_k \leq r + 2 \leq p.$$

Using Lemma 3.4, we get $\phi \in \mathcal{O}_0$ with a hyperbolic periodic point $S_\phi \in \Lambda_\phi(U)$ of index r such that the restriction of ϕ_* to $E_k(S_\phi)$ is equal to A. After a new perturbation, if necessary, we can assume that $\phi_*^{n(S_\phi)}(S_\phi)$ has a expanding non-real eigenvalue of rank $(r+1)$.

As in the proof of Lemma 5.6, by Theorem A and Lemma 2.5, there is $\psi \in \mathcal{O}_0$ close to ϕ with a periodic point $T_\psi \in \Lambda_\psi(U)$ of index $(r+1) < p$ and a heterodimensional cycle $\Gamma(\psi, U, T_\psi, S_\psi)$, where S_ψ has index r and a (expanding) non-real eigenvalue of rank $(r+1)$. Finally, by Lemma 5.5, there is $\xi \in \mathcal{O}_0$ close to ψ with a homoclinic tangency associated to T_ξ, contradicting the definition of $\mathcal{O}_0$. This ends the proof of the lemma in this case. If the homothety given by the restriction of $M_A(S_\varphi)$ to E_k has ratio less than one the proof follows similarly.

□

Lemma 5.8. (Proof of (C)). *The bundles $E_1^{i_q}$ and $E_{i_p+1}^m$ are uniformly volume contracting and volume expanding, respectively.*

Proof: This lemma follows from Theorem B. To see, for instance, that $E = E_1^{i_q}$ is uniformly contracting just observe that the set $\mathcal{O}_0$ and the dominated splitting $E_1^{i_q} \oplus E_{i_q+1}^m$ satisfy the hypotheses of Theorem B, recall that, by Lemma 5.6, $q = d_1^{i_q} = \dim(E_1^{i_q})$.

The uniform expansion of $E_{i_p+1}^m$ follows analogously. This completes the proof of the lemma and of the theorem. □

6 Homoclinic tangencies

We now analyze the dimensions of the bundles of finest dominated splitting of a robust transitive set to deduce the different types of homoclinic bifurcations that that this set may exhibit.

We consider an open set U of M and an open set of diffeomorphisms $\mathcal{N}(U)$ such that, for every $\varphi \in \mathcal{N}(U)$, the set $\Lambda_\varphi(U)$ is robustly transitive and

- the maximum and the minimum of the indices of the periodic points of $\Lambda_\varphi(U)$ are constant, equal to p and q, respectively,
- the dimensions of the bundles of the finest dominated splitting of $\Lambda_\varphi(U)$ do not depend on $\varphi \in \mathcal{N}(U)$.

Let us observe that in this section we do not assume that there are no homoclinic tangencies in $\Lambda_\varphi(U)$, as in the previous section.

We use the notation introduced in Section 5.3 for the dimensions of the bundles of the finest dominated splitting. Recall that, with this notation and by definition, $q \le d_1^{i_q}$ and $p \le d_1^{i_p}$.

We say that a robustly transitive set $\Lambda_\varphi(U)$ has a *homoclinic tangency of rank r* if there is a periodic point $R_\varphi \in \Lambda_\varphi(U)$ of index r having a homoclinic tangency and such a point of tangency belongs to $\Lambda_\varphi(U)$.

Theorem F. *Let U, $\mathcal{N}(U)$, p and q as above. Consider any $\varphi \in \mathcal{N}(U)$.*

- *If $d_1^{i_q} > q$ then there is ϕ arbitrarily close to φ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank $(d_1^{i_q} - 1)$.*
- *If $d_1^{i_p} > p$ then there is ϕ arbitrarily close to φ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank $(d_1^{i_p - 1} + 1)$.*
- *If $d_j \ge 2$ for some $j \in \{i_q + 1, \ldots, i_p\}$ then, for every $k \in [d_1^{j-1} + 1, d_1^j)$, there is ϕ arbitrarily close to φ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank k.*

This theorem is a generalization of the result [DPU, Corollary G] for three dimensional robustly transitive sets, which says that the existence of an undecomposable bundle of dimension strictly bigger than one leads to the creation of homoclinic tangencies in a (non-hyperbolic) robustly transitive set.

The proof of Theorem F follows from a small modification of the the proofs of Lemmas 5.6 and 5.7 and involves heterodimensional cycles.

Denote by $\mathcal{T}_k(U)$, $k = 1, \ldots, \dim(M) - 1$, the subset of $\mathcal{N}(U)$ of diffeomorphisms ϕ such that $\Lambda_\phi(U)$ has a homoclinic tangency of rank k. Theorem F now follows from the next two lemmas.

Lemma 6.1. *Under the hypothesis of Theorem F, we have the following*

- *If $d_1^{i_q} > q$ then $\mathcal{T}_{d_1^{i_q} - 1}(U)$ is dense in $\mathcal{N}(U)$.*
- *If $d_1^{i_p} > p$ then $\mathcal{T}_{d_1^{i_p - 1} + 1}(U)$ is dense in $\mathcal{N}(U)$.*

Proof: First, observe that, by definition, if $d_1^{i_q} > q$ (resp. $d_1^{i_p} > p$) then $d_{i_q} > 1$ (resp. $d_{i_p} > 1$).

To prove the first part of the lemma it is enough to see that if $\varphi \in \mathcal{N}(U)$ and $d_1^{i_q} > q$ then there is υ arbitrarily close to φ such that $\Lambda_\upsilon(U)$ has a homoclinic tangency of rank $(d_1^{i_q} - 1)$. Let us recall that in the proof of Lemma 5.6, under the assumption that $\ell = d_1^{i_q} > q$, we got υ close to φ having a hyperbolic periodic point $R_v \in \Lambda_\nu(U)$ of index ℓ with a non-real eigenvalue of rank $(\ell-1)$.

Since $q \leq \ell-1 < p$, by Theorem A and Lemma 2.5, after a $\mathcal{C}^1$-perturbation of v, we can assume that v has a periodic point S_v of index $(\ell-1)$ and a (codimension one) heterodimensional cycle $\Gamma(v, U, R_v, S_v)$ (R_v of index ℓ with a non-real eigenvalue of rank $(\ell-1)$). By Lemma 5.5 there is ξ close to v with a homoclinic tangency associated to S_ξ. This ends the first part of the lemma.

The second part of the lemma follows similarly. □

Lemma 6.2. *Under the hypotheses of Theorem F, suppose that* $d_j \geq 2$, $j \in \{i_q+1, \ldots, i_p-1\}$. *Then, for every* $k \in [d_1^{j-1}+1, d_1^j)$, *the set* $\mathcal{T}_k(U)$ *is dense in* $\mathcal{N}(U)$.

Proof: As in the previous lemma, given any $\varphi \in \mathcal{N}(U)$ with $d_j \geq 2$ and $k \in [d_1^{j-1}+1, d_1^j)$ we will obtain ϕ arbitrarily close to φ such that $\Lambda_\varphi(U)$ has a homoclinic tangency or rank k. By Theorem A and since

$$q \leq d_1^{j-1} < d_1^j \leq d_1^{i_p-1} < p,$$

after perturbing φ, we can assume that φ has a pair of hyperbolic periodic points $S_\varphi, T_\varphi \in \Lambda_\varphi(U)$ of indices d_1^j and d_1^{j-1}, respectively.

By Lemma 2.5, there is ψ close to φ with a heterodimensional cycle $\Gamma(\psi, U, S_\psi, T_\psi)$. Observe that the modulus of the restriction of the Jacobian of $\psi_*^{n(T_\psi)}$ to $E_j(T_\psi)$ is bigger than one and the modulus of the restriction of the Jacobian of $\psi_*^{n(S_\psi)}$ to $E_j(S_\psi)$ is less than one. By Corollary 3.7, unfolding this cycle, we get ϕ close to φ with a hyperbolic periodic point $R_\phi \in \Lambda_\phi(U)$ with index r, $r \in [d_1^{j-1}, d_1^j]$, such that the modulus of the Jacobian of $\phi_*^{n(R_\phi)}$ to $E_j(R_\phi)$ is exactly one.

By Proposition 2.4, after a perturbation of ϕ, we can assume that $\Sigma_{R_\phi}(\phi)$ is dense in $\Lambda_\phi(U)$. Since $E_j(R_\phi)$ is undecomposable and has dimension equal or bigger than 2, arguing exactly as in the proof of Lemma 5.7, but now applying the final part of Lemma 5.3, we get ξ (arbitrarily close to ϕ) with a periodic point $A_\xi \in \Lambda_\xi(U)$ such that the restriction of $\xi_*^{n(A_\xi)}$ to $E_j(A_\xi)$ is the identity.

Take now any $k \in [d_1^{j-1}+1, d_1^j)$, after a perturbation of ξ we can assume that the index of A_ξ is $k-1$, and that $\xi_*^{n(A_\xi)}(A_\xi)$ has an expanding non-real eigenvalue of rank k. Again, by Theorem A, we can assume there is a periodic point $B_\xi \in \Lambda_\xi(U)$ of index k, where $k > q$. Finally, by Lemma 2.5, there is η close to ξ with a codimension one cycle $\Gamma(\eta, U, B_\eta, A_\eta)$, A_η of index $(k-1)$ and with an expanding non-real eigenvalue of rank k and B_η of index k. Now the lemma follows from Lemma 5.5. □

7 Proof of Theorem E

As we have mentioned in the introduction, Theorem E follows from Proposition 1.1. So before proving the proposition let us deduce the theorem from it.

Recall that U and $\mathcal{S}(U)$ are open sets of M and $\mathrm{Diff}^1(M)$ such that for every diffeomorphism $\varphi \in \mathcal{S}(U)$ the set $\Lambda_\varphi(U)$ is robustly transitive and has no homoclinic tangencies (in the whole manifold). By Theorem D, there is an open and dense subset $\mathcal{I}(U)$ of $\mathcal{S}(U)$, such that if φ belongs to $\mathcal{I}(U)$ and $\Lambda_\varphi(U)$ contains periodic points on indices q and p, $q < p$, then $\Lambda_\varphi(U)$ contains points of every index in between q and p. So it is enough to prove the theorem for the subset $\mathcal{I}(U)$ of $\mathcal{S}(U)$.

Consider the maps $i^+, i^- : \mathcal{I}(U) \to \mathbb{N}^*$ that associate to each $\varphi \in \mathcal{I}(U)$ the maximum and the minimum of the indices of the hyperbolic periodic points of $\Lambda_\varphi(U)$, respectively. These two functions are semi-continuous, so they are continuous in an open and dense subset $\mathcal{I}_0(U)$ of $\mathcal{I}(U)$. Now it is enough to fix a connected component $\mathcal{I}_0$ of $\mathcal{I}(U)$ where i^+ and i^- are both constant and to prove the theorem for this set. Suppose that $i^+(\varphi) = p$ and $i^-(\varphi) = q$ for all $\varphi \in \mathcal{I}_0$, $q \leq p$.

Let us assume that $q < p$ (the case $q = p$ follows from Remark 2.7, so we omit it). Let Q_φ and P_φ be points of indices q and p of $\Lambda_\varphi(U)$. For notational simplicity let us assume that their continuations are defined in the whole $\mathcal{I}_0$. Since P_φ and Q_φ are transitively related in $\mathcal{I}_0$, by Remark 2.6, there is an open and dense subset $\mathcal{I}_1$ of $\mathcal{I}_0$ such that $W^s(P_\phi)$ and $W^u(Q_\phi)$ have nonempty transverse intersection for all $\phi \in \mathcal{I}_1$. So it is enough to prove the theorem for $\mathcal{I}_1$.

For each $j \geq 0$ with $q + j \leq p$, let $\mathcal{A}(j)$ be the subset of $\mathcal{I}_1$ of diffeomorphisms ψ such that $\Lambda_\psi(U)$ contains hyperbolic periodic points $R^0_\psi, R^1_\psi, \dots, R^j_\psi$ such that

- $\text{index}(R^i_\psi) = q + i$,
- $\overline{H_{R^0_\varphi}(U)} = \overline{H_{R^1_\varphi}(U)} = \cdots = \overline{H_{R^j_\varphi}(U)}$ for every φ in a neighbourhood of ψ

To end the proof of Theorem E it is enough to see the following.

Lemma 7.1. *The set $\mathcal{A}(j)$ is open and dense in $\mathcal{I}_1$ for every $j \in (0, r]$, $r = p - q$.*

Before proving this lemma let us end the proof of the theorem.

Observe that by Lemma 7.1, $\mathcal{A}(r)$ is open and dense in $\mathcal{I}_1$ and for every ψ in $\mathcal{A}(r)$ there are hyperbolic periodic points R^0_ψ and R^r_ψ of $\Lambda_\psi(U)$ of indices q and $q + r = p$ such that

$$\overline{H_{R^r_\psi}(U)} = \overline{H_{R^0_\psi}(U)}.$$

As above, for notational simplicity, let us assume that the continuations of R^0_ψ and R^r_ψ are defined in the whole $\mathcal{A}(r)$. The points Q_ψ and R^0_ψ have index q and are transitively related in $\mathcal{A}(r)$. Thus, by Remark 2.7, there is an open and dense subset $\mathcal{D}_1$ of $\mathcal{A}(r)$ of diffeomorphisms ζ such that the relative homoclinic classes of Q_ψ and R^0_ψ in U are equal. Similarly, there is an open and dense subset $\mathcal{D}_2$ of $\mathcal{A}(r)$ of diffeomorphisms ζ such that the relative homoclinic classes of P_ψ and R^r_ψ in U are equal. Thus, for all $\zeta \in \mathcal{D}_1 \cap \mathcal{D}_2$, one has that

$$\overline{H_{P_\zeta}(U)} = \overline{H_{R^r_\zeta}(U)} = \overline{H_{R^0_\zeta}(U)} = \overline{H_{Q_\zeta}(U)}.$$

Since $\mathcal{D}_1 \cap \mathcal{D}_2$ is open and dense in $\mathcal{A}(r)$, thus in $\mathcal{I}_1$, this ends the proof of the theorem.

Proof of the lemma: The proof of Lemma 7.1 is by induction. To see that $\mathcal{A}(1)$ is open and dense in $\mathcal{I}_1$ it suffices to see that given any $\phi \in \mathcal{I}_1$ there is an open subset $\mathcal{A}_\phi$ of $\mathcal{I}_1$ such that

- ϕ belongs to the closure of $\mathcal{A}_\phi$,
- for every $\psi \in \mathcal{A}_\phi$ there is a hyperbolic periodic point $R^1_\psi \in \Lambda_\psi(U)$ of index $(q + 1)$ such that $\overline{H_{Q_\psi}(U)} = \overline{H_{R^1_\psi}(U)}$ (here we take $R^0_\psi = Q_\psi$).

Since ϕ is in $\mathcal{I}_1$ there is a periodic point $R^1_\phi \in \Lambda_\phi(U)$ of index $(q+1)$. Observe that Q_ϕ and R^1_ϕ are transitively related and $\mathrm{index}(Q_\psi)+1 = \mathrm{index}(R^1_\psi)$. Thus, by Lemma 2.5, after a perturbation of ϕ, we can assume that ϕ has a (codimension one) cycle $\Gamma(\phi, U, R^1_\phi, Q_\phi)$. By hypothesis, this cycle is far from homoclinic tangencies. Thus, by Proposition 1.1, there is an open set $\mathcal{B}_\phi$, whose closure contains ϕ, such that $\overline{H_{Q_\zeta}(U)} = \overline{H_{R^1_\zeta}(U)}$ for all $\zeta \in B_\phi$. The first inductive step follows taking $\mathcal{A}_\phi = \mathcal{B}_\phi \cap \mathcal{I}_1$.

Suppose now defined inductively the open and dense subsets $\mathcal{A}(1), \mathcal{A}(2), \dots, \mathcal{A}(j-1)$, $q+j \le p$, of $\mathcal{I}_1$ satisfying the properties above. Then the set

$$\mathcal{A}'(j-1) = \mathcal{A}(1) \cap \cdots \cap \mathcal{A}(j-1)$$

is open and dense in $\mathcal{I}_1$. Now it is enough to get an open and dense subset $\mathcal{A}(j)$ of $\mathcal{A}'(j-1)$ with the announced properties. For that we argue exactly as in the step $j=1$.

Consider any $\phi \in \mathcal{A}'(j-1)$. Since $\phi \in \mathcal{I}_1$ the set $\Lambda_\phi(U)$ contains a hyperbolic periodic point R^j_ϕ of index $(q+j)$. As in the first step of the induction, using Lemma 2.5, we can assume (after a perturbation of ϕ) that ϕ has a (codimension one) cycle $\Gamma(\phi, U, R^j_\phi, R^{j-1}_\phi)$, where R^{j-1}_ϕ is the point of index $(q+j-1)$ in the inductive step $(j-1)$. By hypothesis, this cycle is far from homoclinic tangencies. Thus, by Proposition 1.1, there is an open set $\mathcal{B}_\phi \subset \mathcal{A}'(j-1)$ containing ϕ in its closure such that

$$\overline{H_{R^{j-1}_\zeta}(U)} = \overline{H_{R^j_\zeta}(U)}$$

for all $\zeta \in \mathcal{B}_\phi$. Since $\mathcal{B}_\phi \subset \mathcal{A}'(j-1)$, we have

$$\overline{H_{R^0_\zeta}(U)} = \overline{H_{R^1_\zeta}(U)} = \overline{H_{R^{j-1}_\zeta}(U)} = \overline{H_{R^j_\zeta}(U)}$$

for all $\zeta \in \mathcal{B}_\phi$, ending the proof of the lemma. $\square$

7.1 Proof of Proposition 1.1

Suppose now that (as in the hypotheses of Proposition 1.1) the indices of P_φ and Q_φ are p and q with $p = (q+1)$. By [BDP, Lemma 5.4], we can assume that the robustly transitive set $\Lambda_\varphi(U)$ contains a pair of hyperbolic periodic points of indices q and $p+1$ having only real eigenvalues with multiplicity one and different modulos. For notational simplicity, let us assume that Q_φ and P_φ verify these hypotheses. In particular, these points verify the hypotheses of Corollary 3.6. By (1) in the proof of the corollary, after a small perturbation, we can assume that φ has a saddle-node periodic point (a point with an eigenvalue equal to one) with q contracting eigenvalues and $(\dim(M) - q - 1)$ expanding eigenvalues. After a new perturbation, by unfolding the saddle-node, we can assume that φ has a pair of periodic points A_φ and B_φ of indices p and q, respectively, such that there is a curve γ whose extremes are A_φ and B_φ and whose interior is contained in $W^s(A_\varphi) \pitchfork W^u(B_\varphi)$. By Remark 2.7, we can assume that there is an open set $\mathcal{V}$ containing φ in its closure such that $\overline{H_{P_\psi}(U)} = \overline{H_{A_\psi}(U)}$ and $\overline{H_{Q_\psi}(U)} = \overline{H_{B_\psi}(U)}$ for all ψ in $\mathcal{V}$.

By Remark 2.6, there is a sequence of diffeomorphisms φ_k, $\varphi_k \to \varphi$ in the $\mathcal{C}^1$-topology, such that φ_k has a codimension one heterodimensional cycle $\Gamma(\varphi_k, U, A_{\varphi_k}, B_{\varphi_k})$. By construction, these cycles are *connected* ones, i.e., $W^s(A_{\varphi_k}) \pitchfork W^u(B_{\varphi_k})$ has a periodic connected component whose extremes are contained in the orbits of A_{φ_k} and B_{φ_k} (here the connected component is the continuation of the curve γ above).

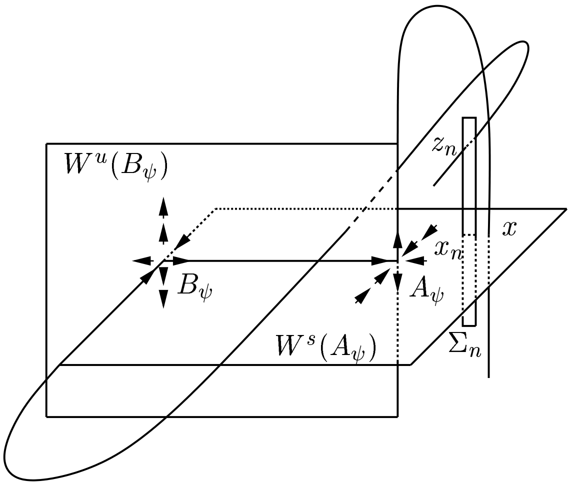

Figure 3: Homoclinic points

The proposition now follows directly from [DR]. For completeness let us state these results.

Lemma 7.2. *Let ζ be a $\mathcal{C}^1$-diffeomorphism with a codimension one connected heterodimensional cycle $\Gamma(\zeta, U, A_\zeta, B_\zeta)$ as above. Then given any $\mathcal{C}^1$-neighbourhood $\mathcal{A}$ of ζ there is a $\mathcal{C}^1$-open subset $\mathcal{U}(\zeta)$ of $\mathcal{A}$ such that $\overline{H_{A_\psi}(U)} = \overline{H_{B_\psi}(U)}$ for every $\psi \in \mathcal{U}(\zeta)$.*

By the lemma, for each φ_k as above there is an open set $\mathcal{U}(\varphi_k) \subset \mathcal{V}$ containing φ_k in its closure, such that, for every $\psi \in \mathcal{U}(\varphi_k)$, one has $\overline{H_{A_\psi}(U)} = \overline{H_{B_\psi}(U)}$. Since $\psi \in \mathcal{V}$, we have that $\overline{H_{A_\psi}(U)} = \overline{H_{P_\psi}(U)}$ and $\overline{H_{Q_\psi}(U)} = \overline{H_{B_\psi}(U)}$. Proposition 1.1 now follows taking $\mathcal{W}_\varphi = \bigcup_k \mathcal{U}(\varphi_k)$.

Proof of the lemma: Observe that the cycle $\Gamma(\zeta, U, A_\zeta, B_\zeta)$ is connected and far from homoclinic tangencies. In [DR], see the comments after Theorem A, it is proved that given any neighbourhood $\mathcal{U}$ of ζ there is an open subset $\mathcal{U}_0$ of $\mathcal{U}$ such that every $\psi \in \mathcal{U}_0$ has a transitive set Λ_ψ containing A_ψ and B_ψ such that $\Lambda_\psi \subset \overline{H(B_\psi)}$. The main step to prove this result is the fact we will state below.

Let us first observe that, by construction, there is a multiplicity one contracting eigenvalue $\lambda_c \in \mathbb{R}$ of the derivative of ζ at A_ζ such that $1 > |\lambda_c| > |\lambda|$ for every contracting eigenvalue λ of A_ζ different from λ_c (see condition (CE) in [DR, Section 3.1]). Thus for every ψ close to ζ the (codimension one) strong stable foliation $\mathcal{F}^s_\psi$ of $W^s(A_\psi)$ is defined. Similarly, we have that the (codimension one) strong unstable foliation $\mathcal{F}^u_\psi$ of $W^u(A_\psi)$ is defined. Now the lemma will follow form the following fact.

Fact: *Let $\mathcal{A}$ be as in Lemma 7.2.*

- *Let $u = (\dim(M) - p)$ be the dimension of the unstable bundle of A_ζ. There is an open subset $\mathcal{A}_0$ of $\mathcal{A}$ of diffeomorphisms ψ such that $W^s(B_\psi)$ meets transversely every $(u+1)$-disk Σ transverse to $\mathcal{F}^s_\psi$.*
- *Let s be the dimension of the stable bundle of A_ζ. There is an open subset $\mathcal{A}_0$ of $\mathcal{A}$ of diffeomorphisms ψ such that $W^u(A_\psi)$ meets transversely every $(s+1)$-disk Σ transverse to $\mathcal{F}^u_\psi$.*

This fact is a non-technical reformulation of [DR, Proposition 3.6 (b)]. Let us observe that (due to the context) in [DR] this proposition is stated for parametrized families of diffeomorphisms unfolding a connected cycle corresponding to a first bifurcation. But, as mentioned in [DR, Section 6], it holds in a much more general setting (including the case under consideration).

To see, for instance, that $\overline{H_{A_\psi}(U)}$ is contained in $\overline{H_{B_\psi}(U)}$ we use the first part of the fact. Take any x in $H_{A_\psi}(U)$. By the cycle configuration $W^u(A_\psi)$ is contained in the closure of $W^u(B_\psi)$, thus there is a sequence $x_n \to x$ with $x_n \in W^s(A_\psi) \pitchfork W^u(B_\psi)$ for all n. Associate to each x_n we have a $(u+1)$-disk Σ_n of diameter less than $1/n$ which is contained in $W^u(B_\psi)$ and transverse to $W^s(A_\psi)$ at x_n (see figure). The fact implies that for each n there is $z_n \in W^s(B_\psi) \pitchfork \Sigma_n$. By construction $z_n \in H_{A_\psi}$ (in fact one can take $z_n \in H_{B_\psi}(U)$) and $\lim z_n = \lim x_n = x$.

The inclusion $\overline{H_{B_\varphi}(U)} \subset \overline{H_{B_\varphi}(U)}$ follows similarly using the second part of the fact. This ends the sketch of the proof of the lemma. $\square$

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Educational Summary

1985-1990	PhD in Sciences (Doutor em Ciências), Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil. Title of the thesis: *Bifurcações e ciclos heterodimensionais (Bifurcations and heterodimensional cycles)*, advisor: J. Palis.
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Past Positions

1998-	Associate Professor, Pontifícia Universidade Católica de Rio de Janeiro (PUC-Rio), Rio de Janeiro, Brazil.
1994-1998	Assistant Professor, Pontifícia Universidade Católica de Rio de Janeiro (PUC-Rio), Rio de Janeiro, Brazil.
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1984-1985 Assistant Professor, Escuela Técnica Superior de Ingenieros de Telecomunicaciones, Universidad Politécnica de Madrid, (ETSIT), Madrid, Spain.

Fellowships and Grants

CNPq – Brazil (Research Grant): August 93 - July 97 (level 2B), August 97 - July 99 (2A), August 99 - July 01 (1C), August 01 - December 04 (1B).

Fellowship for research in Japan of the Japan Society for the Promotion of the Science (October 15th - November 15th, 1999).

Travel Grant for Young Mathemathicians (International Mathematics Union) for attending the International Congress of Mathematics ICM-94 at Zurich (Switzerland).

CNPq – Brazil (Associate Researcher): August 91 - July 93.

Grant for studies in foreign countries, Bank of Spain – Spain: September 89 - September 90.

Ph.D. fellowship, CNPq – Brazil: August 85 - May 87.

Publications

Articles

(24) Non-wandering set with non-empty interior, (with F. Abdenur and Ch. Bonatti), *Nonlinearity,* **17**(1), 175-903, (2004).

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(18) Heterodimensional cycles, partial hyperbolicity and limit dynamics, (with J. Rocha), *Fundamenta Mathematicae,* **184**, 127-186, (2002).

(17) Critical saddle-node cycles: Persistence of tangencies and Haussdorf dimension, (with R. Ures), *Ergodic Theory and Dynamical Systems,* **22**, 1117-1140, (2002).

(16) The intermittancy route to chaotic dynamics, (with I. Rios and M. Viana), *Global Analysis of Dynamical Systems, Festschrift dedicated to F. Takens for his 60th birthday,* Broer, Krauskopf and Vegter editors, IOP, 309-327, (2001).

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(11) Cubic tangencies and hyperbolic diffeomorphisms, (with Ch. Bonatti and F. Villeumin), *Boletim Sociedade Matemática Brasileira* **29** (1), 99-144, (1998).

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Submitted Articles

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Articles in Proceedings of Conferences

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(1) *Periodic points and homoclinic classes,* (with L. Wen).

(2) *Robustly transitive diffeomorphisms are not shadowable.* (with F. Abdenur).

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(1) *Ferraduras parcialmente hiperbólicas (Partially hyperbolic horseshoes), Matemática Universitária,* **28**, 33-66, (2000).

Translations

(1) *Análisis Real,* Elon Lima, Colección Textos del INCA, **vol. 1**, (1997), Lima, Perú (original title: Análise Real).

Scientific Activities

Invited Talks (Conferences, Congress and Workshops)

2004 11–20 March, *International Congress on Dynamical Systems (in memory of J. L. Massera),* IMERL-CMAT Universidad de la República, Montevideo, Uruguay, (Scientific Committee: A. Katok, J. Lewowicz, S. Newhouse, J. Palis, D. Ruelle, A. Verjovsky, J.-C. Yoccoz), *Periodic points and homoclinic classes.* (`http://iemrl.fing.edu.uy/congress/`)

20–30 January, *Workshop em Geometria das subvariedades e Dinâmica caótica (em homenagem a M. do Carmo),* (Organizing committee: H. Alencar, M. Dajzer, M. Viana), Universidade Federal de Alagoas, Maceió, Brazil, *Colissão, explosão e colapso de classes homoclínicas.* (`http://www.impa.br/Conferencias/Geometria_Dinamica/index.html`)

2003 7–11 July, *Recent trends in dynamics 2003,* (Scientific Committee: M. Lyubich, J. Palis, Ya. Siani, S. van Strien, M. Viana, J.-C. Yoccoz), Centro de Matemática da Universidade do Porto, Porto, Portugal, *Periodic points and homoclinic classes.*

18–21 June, *Joint meeting American Mathematical Society and Real Sociedad Española de Matemáticas, Dynamical Systems Session,* (Organizers: G. Haller, X. Jarque, Z. Nitecki, E. Ponce, T. Seara), ETSII, Universidad de Sevilla, Spain, *Cycles and systems of iterated functions.*

2002 4–7 September, *International Conference on Dynamical Methods in Differential Equations,* (Scientific Committee: A. Delshams, R. A. Johnson, R. Obaya, R. Ortega), Castillo de la Mota, Valladolid, Spain, *Tame and wild C^1-dynamics.*

5–15 August, *New Directions in Dynamical Systems,* Universities of Kyoto and Ryukoku, Kyoto, (Japan), *The end of domination.*

6–10 May, *Recent trends in Dynamical Systems III,* (Organizers: J. Alves, V. Araujo, M. Carvalho, F.J. Moreira, J. Rocha), Departamento de Matemática Pura da Universidade do Porto, Porto, Portugal, *Wild non-dominated C^1-dynamics.*

2001 26 July – 16 August, *School and Workshop on Dynamical Systems,* (Scientific Committee: J. Palis, Ya. Sinai, M. Viana, J.-C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste, Italy, *Tame and wild dynamics.*

22–27 July, *23 Colóquio Brasileiro de Matemática, Dynamical Systems Session,* (Organizer: M. J. Dias Carneiro) Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, *Genericidade local de dinâmicas selvagens em dimensão três.*

25–29 June, *International Conference Global Analysis and Dynamical Systems (in honor of the 60th birthday of F. Takens),* (Programm Committee: J. Mather, S. Newhouse, J. Palis, D. Ruelle, Y. Sinai, J.-C. Yoccoz) Lorentz Center, Leiden, Netherland, *Spectral decompositions for nonhyperbolic diffeomorphisms and transitive sets.*

29 May–2 June, *International Conference on Partial Hyperbolicity (in honor of the 60th birthday of C. Pugh),* (Organizers: K. Burns, A. Wilkinson) Northwestern University, Evanston, Illinois, *Robust transitivity I – III* (series of three joint lectures with Ch. Bonatti and R. Ures).

7–11 May, *Recent trends in Dynamical Systems II,* (Organizers: J. Alves, V. Araujo, M. Carvalho, F.J. Moreira, J. Rocha), Departamento de Matemática Pura da Universidade do Porto, Porto, Portugal, *Minimality of strong foliations for partially hyperbolic diffeomorphisms.*

2000 19–28 July, *International Conference on Dynamical Systems, Dedicated to the 60th-anniversary of J. Palis,* (Scientific Committe: J. Mather, W. de Melo, J. Milnor, J. Moser, S. Newhouse, R. Roussarie, Ya. Sinai, F. Takens, M. Viana, J.-C. Yoccoz), Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, *Robust transitivity and heterodimensional cycles.*

18–21 May, *International Conference on Dynamical Systems and Differential Equations,* (Organized by *Journal of Discrete and Continuous Dynamical Systems,* session *Non-uniformly hyperbolic dynamics,* organized by M. Viana), Kennesaw State University, Georgia, United States, *Creation of partially hyperbolic transitive sets at heteroclinic cycles.*

8–13 May, *International Conference on Dynamical Systems,* (Scientific Commitee: M. Benedicks, J. Palis, Ya. Sinai, J. van Strien, J.-C. Yoccoz), Centro Internacional de Matemática (CIM, Coimbra) and Centro de Matemática Aplicada Universidade do Porto, Porto, Portugal, *Robustly transitive sets and heterodimensional cycles.*

4 March, *VII Encuentro de Topologos Españoles,* (Scientific Committee: J.M. Sanjurjo, E. Outerelo) Universidad Complutense of Madrid, El Escorial, Madrid, Spain, *Transitividad robusta de conjuntos y difeomorfismos.*

1999 18–20 October, *Mini-Workshop on Dynamical Systems,* (Organizer: N. Aoki), Tokyo Metropolitan University, Tokyo, Japan, *Stable Transitivity and Weak Hyperbolicity I and II.*

4–11 July, *International Conference on Dynamical Systems,* (Organizadores: V. Baladi, Ch. Bonatti, J.M. Gambaudo, E. Ghys, B. Schmidtt), Laboratoire de Topologie, Departament de Mathemátiques, Université de Bourgogne, Abbaye de La Bussière, Burgundy, France, *A C^1-genercic dichotomy: Weak forms of hyperbolicity or infinitely many sinks or sources.*

5–9 April, *Escuela de Sistemas Dinámicos,* (Organizer: G. Corach), Instituto Argentino de Matemática (IAM), Buenos Aires, Argentina, *Transitividad e hiperbolicidade parcial.* and *Minicurso: Introducción a la dinámica no hiperbólica.*

1998 31 August–18 September, *Workshop on Dynamical Systems,* (Scientific Committe: J. Palis, Ya. Sinai, J.-C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste, Italy, *Cycles, transitivity, and partial hyperbolicity.*

11–14 May, *Joint meeting IMPA - KTH (STINT - cooperation in Dynamical Systems),* (Organizers: M. Benedicks, H. Eliasson), Kungl Teknisks Högskolan (KTH), Stockholm, Sweden, *Cycles, transitivity, and partial hyperbolicity.*

12–13 February, *Encontro da Sociedade Portuguesa de Matemática (S.P.M), Dynamical System Session,* Universidade do Minho, Braga, Portugal, *Transitividade e hiperbolicidade parcial.*

1997 3–5 December, *Encuentro de Matemáticas (10 años del Centro de Matemática, CMAT, Univ. de la República, Montevideo),* Centro de Matemática (CMAT), Montevideo, Uruguay, *Transitividad e hiperbolicidad.*

24–28 November, 4^{th} *International School on Dynamical Systems,* (Scientific Committes: R. Bamón, R. Labarca, S. Plaza, I. Szántó), Universidad de Santa Maria, Valparaiso, Chile, *Infinitely many sinks in the C^1-topology.*

27 July–9 August, 4^{th} *Quadrienial International Conference on Dynamical Systems,* (Scientific Committee: J. Mather, W. de Melo, J. Milnor, J. Moser, S. Newhouse, J. Palis, R. Roussarie,

Ya. Sinai, M. Viana, J.-C. Yoccoz), Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, *Infinitely many sinks in the C^1-topology.*

7–11 July, *Cursos de Verano de la Universidad Complutense de Madrid, Introducción a los Sistemas Dinámicos,* (Organizers: R. de la Llave and A. Delshamps), El Escorial, Madrid, Spain, *Conjuntos hiperbólicos y parcialmente hiperbólicos,* (mini-course).

1–13 June, *Conferencia Internacional en Sistemas Dinámicos,* (Scientific Committes: E. Ghys, X. Gomez-Mont, Y. Ilyashenko, A. Katok, J. Lewowics, J-P. Ramis, J. Seade, Y. Sinai, A. Verjovsky, M. Viana), Cuernavaca, Mexico, *Rubust intersections and Newhouse's Phenomenon.*

1996 28 – 30 November, *Tercer Coloquio Chileno de Sistemas Dinámicos,* (Organizadores: J. Billeke and J. Ramos), Univ. de Los Lagos, Osorno, Chile, *Ciclos e Hiperbolicidad Parcial.*

1995 August, *Workshop em Sistemas Dinâmicos,* (Organizer: J. Palis), Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brazil, *Ciclos sela-nó: dimensão e hiperbolicidade.*

17 May–2 June, *Workshop on Dynamical Systems,* (Scientific Committee: J. Palis, Ya. Sinai, J.-C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste, Italy, *Persistence of partially hyperbolic transitive attractors.*

27 March–1 April 95, *Workshop on Dynamical Systems,* (Scientic Committe: M. Hermann, J. Lewowicz, R. Mañé, J. Mather, S. Newhouse, J. Palis, Ya. Sinai), Instituto de Matemática y Estadística Rafael Laguardia (IMERL), Montevideo, Uruguay, *Persistence of partially hyperbolic transitive attractors.*

1994 31 July–5 August, *Reunião de Sistemas Dinâmicos,* (Organizers: M. Carvalho, J. Rocha) Departamento de Matemática Pura, Univ. Porto, Portugal, *Dimensão de conjuntos hiperbólicos.*

5–8 July, *Reunião de Algebra, Sistemas Dinâmicos e Geometria,* (Organizer: A. Simis) Instituto de Matemática, Univ. Federal da Bahia (IM – UFBa), Salvador, Brazil, *Persistência de difeomorfismos transitivos.*

1992 2–8 August, *Workshop on Dynamical Systems,* (Scientific Committee: A. Douady, J. Moser, S. Newhouse, J. Palis, W. Parry, Ya. Sinai, J.-C. Yoccoz), Departamento de Matemática Pura, Univ. Porto, Portugal, *Hyperbolicity and the creation of heterodimensional cycles.*

1991 13–16 May, *Verbo da Dinámica Hiperbólica,* (Organizer: A. Rodriguez) Departamento de Análise Matemática, Univ. de Santiago de Compostela, Santiago, Spain, *Bifurcaciones y ciclos.*

Organizing Activities

2003 3–8 November, *International Workshop on robustness and partial hyperbolicity,* Hotel Collona Park, Búzios, Rio de Janeiro, Brazil. Organizing Committe: L. J. Díaz, M.J. Pacifico (UFRJ), M. Viana (IMPA).

22–24 October, *Oktobermat,* Departamento de Matemática PUC-Rio, Rio de Janeiro, Brazil.

27 July – 1 August, *Dynamical Systems Session, 24 Colóquio Brasileiro de Matemática,* IMPA, Rio de Janeiro, Brazil. Organizers: L.J. Díaz, M. Viana (IMPA).

2000 9–28 Janeiro, *Summer School of the Departamento de Matemática PUC-Rio,* Dept. de Matemática PUC-Rio, Rio de Janeiro, Brazil. Organizer.

1996 2–3 December, *30 Anos do Departamento de Matemática PUC-Rio,* Departamento de Matemática PUC-Rio, Rio de Janeiro, Brazil. Organizing Committee: L.J. Díaz, C.F. Palmeira.

31 July–10 August, *International Workshop on Dynamical Systems and Geometry,* Depto. Matemática PUC-Rio, Rio de Janeiro, Brazil. Organizing Committee: G.Contreras, L.J. Díaz, R. O. Ruggiero, P. Schweitzer.

1995 12–16 September, *International Conference on Dynamical Systems (celebrating the 25th Anniversary of IMPA's Dynamical Systems Seminar),* IMPA, Rio de Janeiro, Brazil. Scientific and Organizing Committes: M. J. Dias Carneiro L.J. Díaz, A.Lopes, R. Markarian, M. Viana

Academic Visits

2004	Departamento de Matemática Pura, Fac. de Ciências, Univ. do Porto, Porto, Portugal (23 – 28 February); Instutut de Mathematiques de Bourgogne, Dijon, France (02 – 21 February).
2003	Centro Internacional de Matemáticas (CIMAT), Guanajuato, Mexico (18-23 August); Departamento de Matemática Pura, Fac. de Ciências, Univ. do Porto, Porto, Portugal (9 – 15 June and 7 – 20 December); Instutut de Mathematiques de Bourgogne, Dijon, France (20 January – 10 February); School of Mathematics, Peking University, Peking, China (11 – 28 February); Institute of Science and Technology of Hefei, Anhui, China (1 – 3 March).
2002	Instituto de Matemática, Estatística e Ciências da Computação (IMECC), Unicamp, Campinas, Brazil (30 October); Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Portugal (7–14 January, 6 – 17 May and 9 – 20 December); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (14 January – 2 February); Instituto de Matemática e Ciencias da Computação (ICMC), USP-São Carlos, São Carlos, Brazil, (23 – 24 April and 28 – 29 October).
2001	Dep. de Matemática Aplicada i Análise, Univ. de Barcelona, Barcelona, Spain, (17 – 22 December); Dto. de Matemática, Fac. de Ciências, Univ. do Porto, Porto, Portugal, (3 – 25 January and 2 – 11 May); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (29 January – 23 February and 15 – 27 October); Fac. Ciencias Matemáticas, Dto. Geometría y Topología, Univ. Complutense de Madrid, Spain (2 – 6 July).
2000	Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Porto, Portugal (11 – 18 February); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (21 February – 3 March); Instituto Venezolano de Investigaciones Científicas (IVIC), Caracas, Venezuela (21 – 23 June); Fac. Ciencias Matemáticas, Dto. Geometría y Topología, Univ. Complutense de Madrid, Spain (6 – 10 March).
1999	Tokyo Metropolitan University, Hachiochi, Japan (18 – 24 October and 8 – 13 November); Kitami Institute of Technology, Kitami, Hokkaido, Japan (25 – 31 October); Kyoto University, Kyoto, Japan, (1 – 7 November); Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Portugal (11 – 29 January and 26 – 31 July); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (8 – 24 February); Depto. de Matemática, UFMG, Belo Horizonte, Brazil (3 – 4 May).
1998	Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Portugal (26 January – 13 February); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (5 – 23 January); Départament de Mathématiques, Univ. de Genéve, Switzerland (12 – 13 January); Dep. of Mathematics, KTH, Stockholm, Sweden (2 – 10 May).
1997	Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Portugal (26 January – 14 and 14 – 21 July); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (5 – 24 January); Fac. de Ingieneria, Univ. de Montevideo, Uruguay(5 – 14 May); Départament de Mathématiques, Univ. de Genéve, Switzerland (16 – 19 January); Depto. de Matemáticas, Univ. de Oviedo, Spain (24 – 26 February); Depto. de Matemática, Univ. de São Paulo (USP), São Paulo, Brazil (27 – 28 October).
1996	Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Porto, Portugal (2 – 23 January); Universidad de Santiago de Chile (USACH) (21 – 27 November).
1995	Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Porto, Portugal (January); Universidad de Santiago de Chile (USACH) (July).
1994	Fac. de Ciências, Dto. de Matemática, Univ. do Porto, Porto, Portugal (3 – 24 January); Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (5 – 20 December),
1993	Départament de Mathématiques, Laboratoire de Topologie, Univ. de Bourgogne, Dijon, France (February); Fac. de Ingieneria, Univ. de Montevideo, Uruguay (9 – 15 May).
1990	Fac. Ciencias Matemáticas, Dto. Análise Matemática, Univ. de Santiago de Compostela, Spain (30 January – 6 February).

Invited Talks

Brazil	*O fim da dominação,* (ICMC-USP São Carlos, October 2002); *Ferraduras Parcialmente Hiperbólicas,* (ICMC-USP São Carlos, April 2002, IMECC-Unicamp, October 2002); *Transitividade robusta e hiperbolicidade parcial: à procura de um teorema de decomposição espectral,* (IM-UFMG, May 99); *Ciclos e transitividade: um modelo semi-global,* (IMPA, October 98); *Ciclos, transitividade e hiperbolicidade parcial,* (IMPA, March 98); *Transitividade e hiperbolicidade,* (IME-USP, October 97); *Persistência de difeomorfismos transitivos,* (IM-UFF, May 97); *Ciclos, conjuntos parcialmente hiperbólicos e um modelo de quase-renormalização,* (IMPA, February 96); *Discontinuity of Haussdorf dimension and persistence of partially hyperbolic attractors,* (IMPA October 94); *Persistence of partially hyperbolic attractors* (IMPA, October 93).
Chile	*Persitencia de atratores parcialmente hiperbólicos transitivos* (series of 3 talks), (Fac. de Matemáticas y C. Computación, Univ. de Santiago de Chile (USACH) and Depto. Matemática Univ. de Chile, July).
China	*Heterodimensional cycles,* (mini-course of 5 lectures: *Bifurcations via heterodimensional cycles: an overview, Bifurcations and heterodimensional cycles: some models, Heterodimensional cycles and blenders, Typical dynamics when unfolding heterodimensional cycles,* and *The end of domination: infinitely many sinks and more*) (School of Mathematics, University of Peking, Pekim, February 03); *Tame and wild C^1-dynamics* (Colloquim, School of Mathematics, University of Pekim, Peking, February 03); *Critical saddle-node cycles,* (School of Mathematics, Tsingua University, Pekim, February 03); *Minimal C^1-generic dynamics,* (School of Mathematics, Institute of Science and Technology, Hefei, Anhui, March 03).
France	*To shadow or not to shadow,* (Séminaire Géneral, Institut de Mathématiques Université de Bourgogne, February 04); *Cycles and collisions of homoclinic classes,* (Séminaire Systèmes Dynamiques, Institut de Mathématiques Université de Bourgogne, January 03); *Heterodimensional cycles, partial hyperbolicity and limit dynamics,* (Univ. de Paris-Sud, Centre d'Orsay - Mathématique, January 02); *Maximal transitive sets with minimal dynamics,* (Univ. Denis Diderot (Paris VII) Institut de Mathématiques et Dynamique, Séminaire de Systèmes Dinamiques (L.H. Eliasson – J.-C.Yoccoz), January 02); *Collision, explosion and collapse of homoclinic classes,* (Univ. de Paris-Sud, Centre d'Orsay - Mathématique, February 01); *A la recherche de la decomposition spectral perdue,* (Univ. de Bourgogne, Dép. de Mathématiques, Lab. de Topologie, Dijon, January 99); *Partially hyperbolic diffeomorphisms in three manifolds,* (Dijon, January 98); *The robustly transitive diffeomorphisms are normally hyperbolic,* (Dijon, January 97); *Stability of transitive dynamics on three manifolds,* (Dijon, January 97); *Derived from Anosov Diffeomorphisms: Discountinuity of Haussdorf dimension,* (Dijon, February 93); *Prevalence of strange attractors and saddle-node cycles,* (Dijon, July 92); *Hyperbolicity and the creation of cycles,* (Dijon, June 92).
Italy	*Strange attractors ma non troppo,* (Scoula Internazionale Superiore di Studi Avanzati - International School for Advanced Studies SISSA-ASIS, Trieste, February 92)
Japan	*Stable Transitivity and weak hyperbolicity* (mini-course of 6 hours) (Kyoto University, Kyoto, November 99); *Robust transitive sets, homoclinic classes, and strong partially hyperbolic dynamics,* (Kanagawa University, Yokohama, November 99); *Mechanisms for robust transitive dynamics,* (Tokyo Metropolitan University, Hachiochi, November 99); *Robust transitivity and partial hyperbolicity generated by heteroclinic bifurcations,* (Tokyo Institute of Technology, Tokyo, October 99); *Stable Transitivity and Weak Hyperbolicity I – III,* (Kitami Institute of Technology, Hokkaido, October 99).
Mexico	*Clases homoclínicas y puntos periódicos,* (Centro Internacional de Matemáticas (CIMAT), Guanajuato, México, August 03).
Portugal	*Sem sombra de sombremaneto (para as dinâmicas robustamente transitivas não hiperbólicas),* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, June 03); *Peças elementares de dinâmica,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, June 03); *Difeomorfismos robustamente não-errantes,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, December 02); *Colisão, explosão e colapso de classes homoclínicas,* (Fac. de Ciências, Dto. de Matemática

	Pura, Univ. Porto, January 02); *Ferraduras sela-nó,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, January 01); *Mecanismos de transitividade robusta.* (Porto, February 00); *Hiperbolicida fraca das classes homoclinicas,* (Porto, January 99); *Perseguindo um teorema de decomposição espectral,* (Fac. de Ciências, Dto. de Matemática Aplicada, Univ. Porto, January 99); *Ligações persistentes,* (Porto, February 98); *Infinidade de poços na topologia* C^1*,* (Porto, July 97); *Persistência de difeomorfismos transitivos,* (Porto, February 97); *Frequência de hiperbolicidade e bifurcações sela-nó,* (Porto, January 96); *Frequência de hiperbolicidade e bifurcações sela-nó,* (Instituto Superior Técnico, Departamento de Matemática, Lisboa, January 96); *Bifurcações de difeomorfismos de Anosov,* (Porto, January 95); *Atratores não hiperbólicoss,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, March 93); *Ciclos: hiperbolicidade versus não-hiperbolicidade,* (Porto, June 91); *Ciclos em dimensão maior do que dos,* (Fac. de Ciências, Dto. de Matemática (Porto, February 90).
Spain	*Genericidade de conjuntos maximales com dinamica minimal,* (Fac. de Ciencias Matemáticas, Depto. de Matemática Aplicada i Analise, Universitat Central de Barcelona, December 01); *Em busca de la descoposición espectral perdida,* (Fac. de Ciencias Matemáticas, Universidad de Valladolid, June 01); *Em busca de la descoposición espectral perdida,* (Dep. de Geometría y Yopología, Fac. de Ciencias Matemáticas, Univ. Complutense de Madrid, June 01); *Mecanismos de transitividad robusta,* (UCM, March 00); *Persistência de transitividad e hiperbolicidad normal,* (Fac. de Matemáticas, Univ. de Oviedo, February 97); *Bifurcações de difeomorfismos de Anosov* (Univ. de Oviedo, January 95); *Persistencia de difeomorfismos transitivos,* (UCM, July 94); *Bifurcaciones homoclínicas: resultados recentes,* (Fac. de Ciências Matemáticas, Univ. Autónoma de Madrid, February 90); *Teoria geométrica de las bifurcaciones homoclínicas y heteroclínicas,* (Fac. Ciencias, Dto. Matemáticas, Univ. Pais Vasco, February 90); *Bifurcaciones y ciclos heterodimensionales* (Fac. Ciências Matemáticas, Dto. Análise Matemática, Univ. Santiago de Compostela, February 90); *Continuidad de estados de equilibrio y discontinuidad de dimension para arcos de difeomorfismos,* (Univ. Santiago de Compostela, February 90).
Switzerland	*Partially hyperbolic diffeomorphisms in three manifolds,* (Départament de Mathématiques, Université de Genéve, January 98).
Uruguay	*Intersecciones robustas y fenómeno de Newhouse,* (Facultad de Ingieneria, Univ. de Montevideo, May 97); *Persitencia de difeomorfismos transitivos* (Montevideo, May 94).
Venezuela	*Transitividad estable e hiperbolicidad débil (mini-curso, 4h 30min),* (Instituto Venezolano de Investigaciones Científicas (IVIC), Caracas, June 00); *Transitividad e hiperbolicidad,* (IVIC, October 97); *Transitividad robusta e hiperbolicidad parcial,* (IVIC, October 97).

Advising Activities

Carlos Carballo (PhD), *Propriedades genéricas de classes homoclínicas* (April 2001). Thesis published in Bol. S.B.M, **31**, 287-303, (2000), Ergodic Theory Dynam. Systems, **23**(2), 403–415, (2003) and Bull. London Math. Soc., **35**(1), 85–91, (2003).

Bianca Santoro (MSc), *Colisão, colapso e explosão de classes homoclínicas,* (August 2001). Thesis accepted in Nonlinearity.

Armando de Castro (PostDoc Student), Faperj (Foundation for the Promotion of the Science of the State of Rio de Janeiro) fellowship, May 99–March 2000.

Member of Examining Commites

PhD Thesis

2004	Instituto de Matemática Pura e Aplicada, advisor M. Viana, Carlos Matheus Silva Santos, *Contribuições à teoria ergódica de sistemas não hiperbólicos;*

	Departamento de Matemática, PUC-Rio, advisor R. O. Ruggiero, José Barbosa Gomes, *Rigidez de superfícies cujos fluxos geodésicos preservam folheações de codimensão 1.*
2001	Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, advisor Francisco Romero, José Manuel Salazar Crespo, *Indice de punto fijo en hiperespacios e índice de Conley;* Instituto de Matemática Pura e Aplicada, advisor M. Viana, Jairo Bochi, *Expoentes de Lyapunov nulos em sistemas conservativos.*
2000	Instituto de Matemática Pura e Aplicada, advisor M. Viana, Alexandre Baraviera, *Robust non uniform hyperbolicity for volume preserving maps;*
	Departamento de Matemática, PUC-Rio, advisor R. O. Ruggiero, Wladimir Rosas *Sobre fluxos geodésicos expansivos: Conjunto de Pesin e E Teorema de Livsic.*
1999	Instituto de Matemática Pura e Aplicada, advisor M. Viana, Wanderley Horita, *Dimensão de Hausdorff de repulsores de não hiperbólicos de derivados difeomorfismos de Anosov.*
1998	Instituto de Matemática Pura e Aplicada, advisor J. Palis, Fernando José Sánchez-Salas, *Propriedades geometricas de atratores ergódicos;*
	Instituto de Matemática Pura e Aplicada, advisor M. Viana, Isabel Rios, *Desdobramentos de tangências homoclínicas internas de ferraduras: hiperbolicidade, dimensoões fracionárias e tangencias persistentes;*
	Instituto de Matemática Pura e Aplicada, advisor M. Viana, Augusto Armando de Castro *Backward inducing and exponential decay of correlations for partially hyperbolic attractors with mostly contracting central direction;*
	Instituto de Matemática Pura e Aplicada, advisor M. Viana, José F. Alves, *Medidas SRB para atratores multidimensionais.*
1996	Instituto de Matemática Pura e Aplicada, advisor J. Palis, Eduardo Colli, *Coexisteência de infinitos atratores estranhos.*
1995	Instituto de Matemática Pura e Aplicada, advisor R. Mañé, Heber Henrich, *Bifurcações Heteroclínicas de difeomorfismos de Anosov.*
1992	Instituto de Matemática Pura e Aplicada, advisor J. Palis, Carlos Martín Rivas, *Bifurcações Homoclínicas e Cascatas de Duplicação de Periodo em Qualquer Dimensão.*

MSc Thesis

2002	Departamento de Matemática, PUC-Rio, advisor M. Souza, Cristiane Aparecida Monteiro, *Análise de modelos inmunológicos de rede.*

Administrative Activities

Departamento de Matemática:

Coordinator of the Post-Graduation Programm of the Departamento de Matemática, PUC-Rio (97 – 98) and (01-04).

Member of the General Committee of the Departamento de Matemática PUC-Rio (97, 99, 00, 01, 03)

Member of the Research and Post-Graduation Committee of the Departamento de Matemática PUC-Rio (97, 98 and 04).

Centro Tecnológico Científico (CTC) PUC-Rio:

Member of the Faculty Committee of the CTC (01 and 02).

Member of the Career Committee do Comissão of the CTC (03 – 04).

Miscellaneous

Reviewer of the Mathematical Reviews da American Mathematical Society (40 published reviews).

Refereeing (for journals): *Inventiones Mathematicae, Ergodic Theory and Dynamical Systems, Boletim SBM, Nonlinearity, Journal of Dynamics and Stability of Systems, Asterisque, Proceedings of the American Mathematical Society,* and *Duke Mathematical Journal.*

Refereeing for Research Agencies: CAPES (Brazil), CNPq (Brazil), Faperj (Brazil), Facepe (Brazil), UFBa (Brazil), Fondecyt (Chile) Agencia Nacional de Evaluacion y Prospectiva (ANEP), Ministerio de Ciencia y Tecnología (Spain), Programa de Desarrollo Tecnológico (PDT) del Ministerio de Educación y Cultura de Uruguay (Uruguay).

Member of the Editorial Board Matemática Universitaria (Brazilian Mathematical Society), since January 2001.

Rio de Janeiro, April 5, 2004

Lorenzo J. Díaz

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CURRICULUM VITAE

LORENZO J. DIAZ

Dados pessoais

Lorenzo Justiniano **Díaz** Casado
Departamento de Matemática, PUC-Rio,
Marquês de São Vicente 225, Gávea, Rio de de Janeiro, 22453-900, RJ, Brasil
Tel. (55 21) 3114-1745, Fax: (55 21) 2259-5495 E-mail: lodiaz@mat.puc-rio.br

Local e data de nascimento: Madri, Espanha, 27 Junho 1961

Posição atual

Professor Associado (S15), Departamento de Matemática, Pontifícia Universidade Católica de Rio de Janeiro (PUC-Rio), Rio de Janeiro, Brasil.

Bolsa Produtividade em Pesquisa CNPq (nivel 1B)

Formação acadêmica

1985-1990 Doutor em Ciências, Instituto de Matemática Pura e Aplicada (IMPA-CNPq) Rio de Janeiro, Brasil
Título da Tese: *Bifurcações e ciclos heterodimensionais* (orientador: J. Palis).

1979-1984 Licenciado com grau em Matemática (mestrado), Universidad Complutense de Madrid (UCM), Madri, Espanha.
Título da Disertação (tesina): *Conjuntos salvajes de difeomorfismos* (orientador: E. Outerelo).

Bolsas e distinções

Cientista do Nosso Estado Faperj (2005–2006)

Bolsa de Pesquisa CNPq, Agosto 95–Julho 97 (nivel 2B), Agosto 97–Julho 99 (nivel 2A), Agosto 99–Julho 01 (nivel 1C), Agosto 01–Março 05 (nivel 1B).

Bolsa da Japan Society for the Promotion of the Science, JSPS, no programa *JSPS Invitation Fellowship Program for Research in Japan,* (15 Out. – 14 Nov. 1999).

Travel Grant for Young Mathemathicians (International Mathematical Union) para assistir ao International Congress of Mathematics (ICM–94) em Zurique (Suiça).

Bolsa de Pesquisador Associado, CNPq, Agosto 91 – Julho 93.

Bolsa de estudos no estrangeiro, Banco de España – Espanha, Setembro 89 – Septembro 90.

Bolsa de doutorado, CNPq – Brasil, Agosto 85 – Maio 87.

Publicações

Artigos publicados

(25) Collision, explosion and collapse of homoclinic classes, (com B. Santoro), *Nonlinearity.* **17**, 1001-1032, (2004).

(24) Non-wandering set with non-empty interior, (com F. Abdenur e Ch. Bonatti), *Nonlinearity,* **17**(1), 175-903, (2004).

(23) A C^1-generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources, (com Ch. Bonatti e E.R. Pujals), *Annals of Mathematics,* **158**(2), 355-418, (2003).

(22) Robustly transitive sets and heterodimensional cycles, (com Ch. Bonatti, E.R. Pujals e J. Rocha), *Astérisque,* **286**, 187-222, (2003).

(21) Critical saddle-node horseshoes: bifurcations and entropy, (com I. Rios), *Nonlinearity,* **16**(3) 897-929, (2003).

(20) On maximal transitive sets of generic diffeomorphisms, (com Ch. Bonatti), *Publications Mathématiques I.H.É.S.,* **96**, 171-197, (2002).

(19) Minimality of the strong foliations for partially hyperbolic diffeomorphisms, (com Ch. Bonatti and R. Ures), *Journal de l'Institut de Mathématiques de Jussiieu,* **1**(4), 513-541, (2002).

(18) Heterodimensional cycles, partial hyperbolicity and limit dynamics, (com J. Rocha), *Fundamenta Mathematicae,* **184**, 127-186, (2002).

(17) Critical saddle-node cycles: Persistence of tangencies and Haussdorf dimension, (com R. Ures), *Ergodic Theory and Dynamical Systems,* **22**, 1117-1140, (2002).

(16) The intermittancy route to chaotic dynamics, (com I. Rios e M. Viana), *Global Analysis of Dynamical Systems, Festschrift dedicated to F. Takens for his 60th birthday,* Broer, Krauskopf and Vegter editors, IOP, 309-327, 2001.

(15) Partially hyperbolic and transitive dynamics generated by heteroclinic cycles, (com J. Rocha), *Ergodic Theory and Dynamical Systems,* **21**, 25-76, (2001).

(14) Pas de Shadowing Lemma pour les dynamiques partiellment hyperboliques, (com Ch. Bonatti e G. Turcat), *Comptes Rendus de Academie de Sciencies de Paris,* t 330, 587-592, 2000.

(13) Partial hyperbolicity and robust transitivity, (com E.R. Pujals e R. Ures), *Acta Mathematica,* **183**, 1-43, (1999). (**Featured AMS review**).

(12) Connexions hétérocliniques et généricité d'une infinité de puits ou de sources, (com Ch. Bonatti), *Annales Scientifiques de l'École Normal Soupériore de Paris,* **32** (4), 135-150, (1999)

(11) Cubic tangencies and hyperbolic diffeomorphisms, (com C. Bonatti e F. Villeumin), *Boletim Sociedade Matemática Brasileira* **29** (1), 99-144, (1998).

(10) Large measure of hyperbolic dynamics when unfolding heteroclinic cycles (com J. Rocha), *Nonlinearity,* **10**, 857 - 884, (1997).

(9) Noncritical saddle-node cycles and robust nonhyperbolic dynamics (com J. Rocha), *Journal of Dynamics and Stability of Systems,* **12** (2), 109-135, (1997).

(8) Saddle-node cycles and strange attractors: prevalence and globality (com J.Rocha e M.Viana), *Inventiones Math.,* **125** (1), 37-74, (1996).

(7) Persistent Transitive Diffeomorphisms, (com C. Bonatti), *Annals of Mathematics,* **143**(2), 357-396, (1996).

(6) Persistence of cycles and nonhyperbolic dynamics at the unfolding of heteroclinic bifurcations, *Nonlinearity,* **8**, 693-715, (1995).

(5) Discountinuity of Hausdorff dimension of hyperbolic sets (com Ch. Bonatti e M. Viana), *Comptes Rendus de Academie de Sciencies de Paris,* **t320-I**, 713-718, (1995).

(4) Robust nonhyperbolic dynamics and heterodimensional cycles, *Ergodic Theory and Dynamical Systems,* **15**, 291-315, (1995).

(3) Persistent homoclinic tangencies and the unfolding of cycles, (com R. Ures), *Publ. Inst. Henri Poincaré Ann. Nonlineare,* **6**, 643-689, (1994).

(2) Non-connected heterodimensional cycles: bifurcation and stability, (com J. Rocha), *Nonlinearity,* **5**, 1315-1341, (1992).

- Bifurcações e ciclos heterodimensionais, Tese de Doutorado IMPA, Informes de Matemática IMPA.

(1) Discountinuity of Hausdorff dimension and limit capacity on arcs of diffeomorphism, (com M. Viana), *Ergodic Theory and Dynamical Systems,* **9**, 403 -425, (1989).

Livros

(i) *Dynamics beyond uniform hyperbolicity,* (com Ch. Bonatti e M. Viana), Encyclopaedia of Mathematical Sciences (Mathematical Physics), **102**, Springer Verlag.

Capítulos de livros

(1) *Routes to wild dynamics,* 127–143, Contibuciones Matemáticas, Homenaje al Porfesor Enrique Outerelo Dominguez, (editor J. Ruiz) Editorial Complutense (UCM), (2004).

Trabalhos em atas de congressos

(3) *Cycles and systems of iterated functions: the role of secondary bifurcations,* cd-rom do First Joint Meeting rsme-ams, Sevilla, Espanha, 18-21 junho, (2003), (Dynamical Systems, session 15th, page 4).

(2) *The end of domination,* Abstracts Booklet of the conference New Directions in Dynamical Systems, Kyoto, Japão, 204-211,(2002).

(1) *Transitividad robusta de conjuntos y difeomorfismos,* Actas VII Reunión de Topólogos Españoles, 1-20, (2000).

Artigos submetidos

(2) *Generic diffeomorphisms on compact surfaces,* (com F. Abdenur, C. Bonatti e S. Crovisier).

(1) *How do homoclinic classes collide at heterodimensional cycles?* (com J. Rocha).

Pré-publicações

(2) *Generic diffeomorphisms on compact surfaces,* (com F. Abdenur, C. Bonatti e S. Crovisier), prepublicação PUC-Rio 2005.

(1) *Collision, explosion and collapse of hyperbolic sets at heterodimensional cycles classes,* (com J. Rocha), prepublicação PUC-Rio 2004.

Trabalhos em Preparação

(3) *Periodic points and homoclinic classes,* (com L. Wen).

(2) *Robustly transitive diffeomorphisms are not shadowable.* (com F. Abdenur).

(1) *Robust heterodimensional cycles.* (com C. Bonatti e N. Gourmelon).

Notas de cursos

(a) *Cycles, partial hyperbolicity, and transitivity,* V Escuela Internacional de Sistemas Dinámicos, Facultad de Ciencias de la Universidad de Chile (3-7 janeiro 2005).

Artigos de divulgação

- *Ferraduras parcialmente hiperbólicas, Matemática Universitária,* **28**, 33-66, (2000).

Traduções

- *Análisis Real,* Elon Lima, Colección Textos del INCA, **vol. 1**, (1997), Lima, Perú (original: Análise Real).

Atividades científicas

Conferências Convidadas em Congressos

2004 30 agosto–03 setembro, *Geometric and Ergodic Theory of Dynamical Systems (a workshop in honor of the 60th birthday of C. Gutierrez and M. A. Teixeira),* (comité Científico: R. Garcia, J. Llibre, M. J. Pacifico, M. Peixoto, J. Sotomayor e M. Viana) ICMC - USP, São Carlos (Brasil), *Finitiness of homoclinic classes at heterodimensional cycles* `(http://www.icmc.usp.br/ getds/)`

19 julho–06 agosto, *Summer School and Conference on Dynamical Systems,* (comité Científico: M. Viana e J. -C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste (Italia), *Collisions of hyperbolic homoclinic classes.* `(http://www.ictp.trieste.it)`

20–26 junho, *Segundo Congreso Latinoamericano de Matemáticos, UMALCA,* Cancún, México, (comité científico: J. Seade, M. Viana, C. Camacho, G. Corach, J. A. de la Peñam D. Eisenbaud, E. Friedman, L. D. Trang e S. R. S. Varahdan) Sessão temática Sistemas Dinâmicos e Teoria Ergódica (comité científico: R. Markarian, L. Mora e M. J. Pacifico) *Heterodimensional Cycles.* `(http://www.matmor.unam.mx/eventos/latino/)`

26–28 maio, *I Reunião Regional de Sistemas Dinâmicos da UNESP,* IBILCE, São Josá do Rio Preto, São Paulo, (comité científico: C. Gutierrez, C. Ragazzo, M. A. Teixeira), *Colisões de classes homoclínicas e persistência de ciclos,* `(http://www.mat.ibilce.unesp.br/RRSD2004/)`

3–8 maio, *School and Workshop on Dynamical Systems and its Applications,* Thematic Term on Mathematics and Envirnment, Centro Internacional de Mamtemática (CIM), (comité científico e organizador J. F. Alves e M. Viana), *Shadowing in the C^1-topology,* `(http://www.mat.uc.pt/-tt2004/dynsystems/)`

11–20 março, *International Congress on Dynamical Systems (in memory of J. L. Massera),* IMERL-CMAT Universidad de la República, Montevideo, (comité científico: A. Katok, J. Lewowicz, S. Newhouse, J. Palis, D. Ruelle, A. Verjovsky, J.-C. Yoccoz), *Periodic points and homoclinic classes,* `(http://iemrl.fing.edu.uy/congress/)`

20–30 janeiro, *Workshop em Geometria das subvariedades e Dinâmica caótica (em homenagem a M. do Carmo),* (Comissão Organizadora: H. Alencar, M. Dajzer e M. Viana), Universidade Federal de Alagoas, Maceió (Brasil), *Colissão, explosão e colapso de classes homoclínicas.* `(www.impa.br/Conferencias/Geometria_Dinamica/index.html).`

2003 7–11 julho, *Recent trends in dynamics 2003,* Centro de Matem'atica da Universidade do Porto, (comité científico M. Lyubich, J. Palis, Ya. Siani, S. van Strien, M. Viana, J.-C. Yoccoz), *Periodic points and homoclinic classes.*

18–21 junho, *Joint meeting American Mathematical Society and Real Sociedad Española de Matemáticas, Sessão de Sistemas Dinâmicos,* ETSII, Universidad de Sevilla, Espanha, (Organizadores: G. Haller, X. Jarque, Z. Nitecki, E. Ponce, T. Seara), *Cycles and systems of iterated functions.*

2002 4–7 setembro, *International Conference on Dynamical Methods in Differential Equations,* Castillo de la Mota, Valladolid, Espanha, (Comité Científico: A. Delshams, R. A. Johnson, R. Obaya e R. Ortega), *Tame and wild C^1-dynamics.*

5–15 agosto, *New Directions in Dynamical Systems,* Universidades de Kyoto e Ryukoku, Kyoto, (Japão), *The end of domination.*

6–10 maio, *Recent trends in Dynamical Systems III,* (Organizadores: J. Alves, V. Araujo, M. Carvalho, F.J. Moreira, e J. Rocha), Departamento de Matemática Pura da Universidade do Porto, Porto, (Portugal), *Wild non-dominated C^1-dynamics.*

2001 26 julho–16 agosto, *School and Workshop on Dynamical Systems,* (Comité Científico: J. Palis, Ya. Sinai. M. Viana e J. -C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste (Italia), *Tame and wild dynamics.*

22–27 julho, *23 Colóquio Brasileiro de Matemática, Sessão Sistemas Dinâmicos,* (Organizador: M. J. Dias Carneiro) Instituto de Matemática Pura e Aplicada, Rio de Janeiro, *Genericidade local de dinâmicas selvagens em dimensão três.*

25–29 junho, International Conference Global Analysis and Dynamical Systems (in honor of the 60th birthday of F. Takens), (Comité de Programa: J. Mather, S. Newhouse, J. Palis, D. Ruelle, Y. Sinai e J. C. Yoccoz) Lorentz Center, Leiden, Holanda, *Spectral decompositions for nonhyperbolic diffeomorphisms and transitive sets.*

29 maio–2 junho, *International Conference on Partial Hyperbolicity (in honor of the 60th birthday of C. Pugh),* (Organizadores: K. Burns e A. Wilkinson) Northwestern University, Evanston, Illinois, *Robust transitivity I – III* (série de três palestras conjuntas com C. Bonatti e R. Ures).

7–11 maio, *Recent trends in Dynamical Systems II,* (Organizadores: J. Alves, V. Araujo, M. Carvalho, F.J. Moreira, e J. Rocha), Departamento de Matemática Pura da Universidade do Porto, Porto, (Portugal), *Minimality of strong foliations for partially hyperbolic diffeomorphisms.*

2000 19–28 julho, *International Conference on Dynamical Systems, Dedicated to the* 60^{th}*-anniversary of J. Palis,* (Comitê Científico: J. Mather, W. de Melo, J. Milnor, J. Moser, S. Newhouse, R. Roussarie, Ya. Sinai, F. Takens,M. Viana e J.-C. Yoccoz), Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brasil, *Robust transitivity and heterodimensional cycles.*

18–21 maio, *International Conference on Dynamical Systems and Differential Equations,* (Organizada pelo *Journal of Discrete and Continuous Dynamical Systems,* sessão *Non-uniformly hyperbolic dynamics,* organizada por M. Viana), Kennesaw State University, Georgia, Estados Unidos, *Creation of partially hyperbolic transitive sets at heteroclinic cycles.*

8–13 maio, *International Conference on Dynamical Systems,* (Comité Científico: M. Benedicks, J. Palis, Ya. Sinai, J. van Strien e J. -C. Yoccoz), Centro Internacional de Matemática (CIM, Coimbra) e Centro de Matematica Aplicada Universidade de Porto), Porto, (Portugal), *Robustly transitive sets and heterodimensional cycles.*

4 março, *VII Encuentro de Topologos Españoles,* (Comité Científico: J.M. Sanjurjo, E. Outerelo) Universidad Complutense de Madri, El Escorial, *Transitividad robusta de conjuntos y difeomorfismos.*

1999 18–20 outubro, *Mini-Workshop on Dynamical Systems,* (Organizador: N. Aoki), Tokyo Metropolitan University, *Stable Transitivity and Weak Hyperbolicity I and II.*

4–11 julho, *International Conference on Dynamical Systems,* (Organizadores: V. Baladi, C. Bonatti, J.M. Gambaudo, E. Ghys e B. Schmidtt), Laboratoire de Topologie, Departament de Mathemátiques, Universite de Bourgogne, Abbaye de La Bussière, *A* C^1*-genercic dichotomy: Weak forms of hyperbolicity or infinitely many sinks or sources.*

5–9 abril, *Escuela de Sistemas Dinámicos,* (Organizador: G. Corach), Instituto Argentino de Matemática (IAM), Buenos Aires (Argentina), *Transitividade e hiperbolicidade parcial.* e *Minicurso: Introducción a la dinámica no hiperbólica.*

1998 31 agosto–18 setembro, *Workshop on Dynamical Systems,* (Comité Científico: J. Palis, Ya. Sinai e J. -C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste (Italia), *Cycles, transitivity, and partial hyperbolicity.*

11–14 maio, *Joint meeting IMPA - KTH (STINT - cooperation in Dynamical Systems),* (Organizadores: M. Benedicks e H. Eliasson), Kungl Teknisks Högskolan (KTH), Estocolmo, Suecia, *Cycles, transitivity, and partial hyperbolicity.*

12–13 fevereiro, *Encontro da Sociedade Portuguesa de Matemática (S.P.M), Secção de Sistemas Dinâmicos,* Universidade do Minho, Braga, *Transitividade e hiperbolicidade parcial.*

1997 3–5 dezembro, *Encuentro de Matemática (10 anos do Centro de Matemática, CMAT, Univ. de la República, Montevideu),* Centro de Matemática (CMAT), Montevideu, Uruguai, *Transitividad e hiperbolicidad.*

24–28 novembro, 4^{th} *International School on Dynamical Systems,* (Comitê Científico: R. Bamón, R. Labarca, S. Plaza, I. Szántó), Universidade de Santa Maria, Valparaiso, Chile, *Infinitely many sinks in the* C^1*-topology.*

27 julho–9 agosto, 4^{th} *Quadrienial International Conference on Dynamical Systems,* (Comitê Científico: J. Mather, W. de Melo, J. Milnor, J. Moser, S. Newhouse, J. Palis, R. Roussarie, Ya. Sinai, M. Viana e J.-C. Yoccoz) Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brasil, *Infinitely many sinks in the* C^1*-topology.*

1–13 junho, *Conferencia Internacional en Sistemas Dinámicos,* (Comité Científico: E. Ghys, X. Gomez-Mont, Y. Ilyashenko, A. Katok, J. Lewowics, J-P. Ramis, J. Seade, Y. Sinai, A. Verjovsky, M. Viana), Cuernavaca, México, *Rubust intersections and Newhouse's Phenomenon.*

1996 28 – 30 novembro 96, *Tercero Coloquio Chileno de Sistemas Dinámicos,* (Organizadores: J. Billeke e J. Ramos), Univ. de Los Lagos, Osorno, Chile, *Ciclos e Hiperbolicidad Parcial.*

1995	agosto, *Workshop em Sistemas Dinâmicos,* (Organizador: J. Palis) Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brasil, *Ciclos sela-nó: dimensão e hiperbolicidade.*
	17 maio–2 junho, *Workshop on Dynamical Systems,* (Comité Científico: J. Palis, Ya. Sinai e J. -C. Yoccoz), International Center of Theoretical Physics (ICTP), Trieste (Italia), *Persistence of partially hyperbolic transitive attractors.*
	27 março–1 abril 95, *Workshop on Dynamical Systems,* (Comitê Científico: M. Hermann, J. Lewowicz, R. Mañé, J. Mather, S. Newhouse, J. Palis e Ya. Sinai), Instituto de Matemática y Estadística Rafael Laguardia (IMERL), Montevideu, Uruguai, - *Persistence of partially hyperbolic transitive attractors* (sessão paralela).
1994	31 julho–5 agosto, *Reunião de Sistemas Dinâmicos,* (Organizadores: M. Carvalho e J. Rocha) Departamento de matemática Pura, Univ. Porto, Portugal, *Dimensão de conjuntos hiperbólicos.*
	5–8 de julho, *Reunião de Algebra, Sistemas Dinâmicos e Geometria,* (Organizador: A. Simis) Instituto de Matemática, Univ. Federal da Bahia (IM – UFBa), *Persistência de difeomorfismos transitivos.*
1992	2–8 de agosto, *Workshop on Dynamical Systems,* (Comitê Científico : A. Douady, J. Moser, S. Newhouse, J. Palis, W. Parry, Ya. Sinai e J.-C. Yoccoz), Departamento de Matemática Pura, Univ. Porto, Portugal, 2 – 8 Agosto 92, *Hyperbolicity and the creation of heterodimensional cycles* (sessão paralela).
1991	13–16 maio, *Verbo da Dinámica Hiperbólica,* (Organizador: A. Rodriguez) Departamento de Análise Matemática, Univ. de Santiago de Compostela, Espanha, 13 – 16 Maio 91, *Bifurcaciones y ciclos.*

Cursos Convidadas em Congressos e Escolas

1997	7–11 julho, *V Escula Internacional de Sistemas Dinámicos,* (Organizadores: S. Plaza, R. Bamón, J. Kiwi, J. Rivera e B. San Martín), Faculatad de Ciencias, Universidad de Chile (Santiago de Chile), *Ciclos, hiperbolicidad parcial y transitividad,* (minicurso).
1997	7–11 julho, *Cursos de Verano de la Universidad Complutense de Madrid, Introducción a los Sistemas Dinámicos,* (Organizadores: Rafael de la Llave e Amadeu Delshamps), El Escorial, Madrid (Espanha), *Conjuntos hiperbólicos y parcialmente hiperbólicos.*

Organização de eventos

2004	8–12 novembro, *Jornadas de Iniciao Cientfica no IMPA,* Instituto de Matemática Pura e Aplicada, Rio de Janeiro, Brasil, *membro do comité científico.*
	20–22 outubro, *II Oktobermat,* Departamento de Matemática PUC-Rio, Rio de Janeiro, Brazil
2003	3–8 novembro, *International Workshop on robustness and partial hyperbolicity,* Hotel Collona Park, Búzios, Rio de Janeiro, Brasil, *co-organizador, em colaboração com M.J. Pacifico (UFRJ) e M. Viana (IMPA).*
	22–24 outubro, *Oktobermat,* Departamento de Matemática PUC-Rio, Rio de Janeiro, Brazil.
	27 julho – 1 agosto, *Sessão de Sistemas Dinâmicos,* 24^0 *Colóquio Brasileiro de Matemática,* IMPA, Rio de Janeiro, Brasil, *co-organizador, em colaboração com M. Viana (IMPA).*
2000	9–28 janeiro, *Programa de Verão 2000 do Departamento de Matemática PUC-Rio,* Depto. Matemática PUC-Rio, Rio de Janeiro, Brasil, *Organizador.*
1996	2–3 dezembro, *30 Anos do Departamento de Matemática PUC-Rio,* Depto. Matemática PUC-Rio, Rio de Janeiro, Brasil, *Co-organizador.*
	31 julho–10 agosto, *International Workshop on Dynamical Systems and Geometry,* Depto. Matemática PUC-Rio, Rio de Janeiro, Brasil, *membro do comitê organizador.*
1995	12–16 setembro, *International Conference on Dynamical Systems (celebrating the 25th Anniversary of IMPA's Dynamical Systems Seminar),* IMPA, Rio de Janeiro, Brasil, *membro dos comitês científico e organizador.*

Visitas Acadêmicas

2005	Departamento de Matemáticas, Pontificia Universidad Católica de Santiago de Chile, Chile, (10 – 12 de janeiro), Departament of Mathematics, University of Toronto, Canadá, (23 janeiro – 6 fevereiro), School of Mathematical Sciences, Peking University, China, (14 fevereiro – 5 março).
2003	Centro Internacional de Matemáticas (CIMAT), Guanajuato, México (18-23 agosto), Departamento de Matemática Pura, Fac. de Ciências, Univ. do Porto, Portugal, (9–15 de junho e 7–20 dezembro), Instutut de Mathemátiques de Bourgogne, Dijon, França (20 janeiro – 10 fevereiro), School of Mathematics, Peking University, Pequin, China, (11 – 28 fevereiro), Institut of Science and Technology of Hefei, Anhui, China (1 – 3 março).
2002	Instituto de Matemática, Estatística e Ciências da Computação IMECC, Unicamp, Campinas (30 de outubro). Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (7–14 janeiro, 6–17 maio e 9–20 dezembro), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (14 janeiro–2 fevereiro), Instituto de Matemática e Ciencias da Computação ICMC, USP-São Carlos (23 e 24 abril e 28 e 29 de outubro).
2001	Univ. de Barcelona, Dep. de Matemática Aplicada i Análise, Barcelona, Espanha, (17–22 dezembro), Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (3–25 janeiro e 2–11 maio), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (29 janeiro–23 fevereiro e 15–27 de outubro), Univ. Complutense de Madrid, Espanha, Fac. Ciencias Matemáticas, Dto. Geometría y Topología (2–6 julho).
2000	Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (11–18 fevereiro), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (21 fevereiro–3 março), Instituto Venezolano de Investigaciones Científicas (IVIC), Caracas, Venezuela, (21–23 junho, Univ. Complutense de Madrid, Espanha, Fac. Ciencias Matemáticas, Dto. Geometría y Topología (6–10 março).
1999	Tokyo Metropolitan University, Hachiochi, Japão, (18–24 outubro e 8–13 novembro), Kitami Institute of Technology, Kitami, Hokkaido, Japão, (25–31 outubro), Kyoto University, Kyoto, Japão, (1–7 novembro), Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (11–29 janeiro e 26-31 julho), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (8–24 fevereiro), Depto. de Matemática, UFMG, Belo Horizonte, Brasil, (3–4 maio).
1998	Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (26 janeiro–13 fevereiro), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (5–23 janeiro), Départament de Mathèmatiques, Univ. de Genéve, Suiça, (12–13 janeiro), Dep. of Mathematics, KTH, Estocolmo, Suecia, (2–10 maio).
1997	Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (26 janeiro–14 e 14–21 julho), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (5–24 janeiro), Fac. de Ingieneria, Univ. de Montevideo, Uruguai: (5 – 14 maio), Départament de Mathèmatiques, Univ. de Genéve, Suiça: (16–19 janeiro), Depto. de Matemáticas, Univ. de Oviedo, Espanha (24–26 fevereiro), Depto. de Matemática, Univ. de São Paulo (USP), São Paulo, Brasil, (27–28 outubro).
1996	Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (2–23 janeiro), Universidad de Santiago de Chile (USACH) (21–27 novembro).
1995	Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (janeiro), Universidad de Santiago de Chile (USACH) (julho),
1994	Univ. do Porto, Portugal, Fac. de Ciências, Dto. de Matemática (3–24 janeiro), Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (5–20 dezembro),
1993	Univ. de Bourgogne, Dijon, França, Départament de Mathèmatiques, Laboratoire de Topologie (fevereiro), Fac. de Ingieneria, Univ. de Montevideo, Uruguai: (9–15 maio),
1990	Univ. de Santiago de Compostela, Espanha, Fac. Ciências Matemáticas, Dto. Análise Matemática (30 janeiro–6 fevereiro).

Conferências em universidades

Brasil	*O fim da dominação,* (ICMC-USP São Carlos, outubro 2002), *Ferraduras Parcialmente Hiperbólicas,* (ICMC-USP São Carlos, abril 2002, IMECC-Unicamp, outubro 2002), *Transitividade robusta e hiperbolicidade parcial: à procura de um teorema de decomposição espectral,* (IM-UFMG, maio 99), *Ciclos e transitividade: um modelo semi-global,* (IMPA, outubro 98), *Ciclos, transitividade e hiperbolicidade parcial,* (IMPA, março 98), *Transitividade e hiperbolicidade,* (IME-USP, outubro 97), *Persistência de difeomorfismos transitivos,* (IM-UFF, maio 97) *Ciclos, conjuntos parcialmente hiperbólicos e um modelo de quase-renormalização,* (IMPA, fevereiro 96) *Discontinuity of Haussdorf dimension and persistence of partially hyperbolic attractors,* (IMPA outubro 94), *Persistence of partially hyperbolic attractors* (IMPA, outubro 93).
Canadá	*Robust cycles,* (Dep. of Mathematics, Univ. of Toronto).
Chile	*Persitencia de atratores parcialmente hiperbólicos transitivos* (serie de 3 palestras), (Fac. de Matemáticas y C. Computación, Univ. de Santiago de Chile (USACH) e Depto. Matemática Univ. de Chile, julho), *Ciclos robustos,* (Dep. de Matemáticas, PUC – Santiago).
China	*Heterodimensional cycles,* (mini-curso de 5 palestras: *Bifurcations via heterodimensional cycles: an overview, Bifurcations and heterodimensional cycles: some models, Heterodimensional cycles and blenders, Typical dynamics when unfolding heterodimensional cycles,* e *The end of domination: infinitely many sinks and more*) (School of Mathematics, University of Peking, Pekim, fevereiro 03), *Tame and wild C^1-dynamics* (Colloquim, School of Mathematics, University of Pekim, Peking, fevereiro 03), *Critical saddle-node cycles,* (School of Mathematics, Tsingua University, Pekim, fevereiro 03), *Minimal C^1-generic dynamics,* (School of Mathematics, Institute of Science and Technology, Hefei, Anhui, março 03).
Espanha	*Genericidade de conjuntos maximales com dinamica minimal,* (Fac. de Ciencias Matemáticas, Depto. de Matematica Aplicada i Analise, Universitat Central de Barcelona, dezembro 01), *Em busca de la descoposición espectral perdida,* (Fac. de Ciencias Matemáticas, Universidad de Valladolid, junho 01), *Em busca de la descoposición espectral perdida,* (Dep. de Geometría y Yopología, Fac. de Ciencias Matemáticas, Univ. Complutense de Madrid, junho 01), *Mecanismos de transitividad robusta,* (UCM, março 00), *Persistência de transitividad e hiperbolicidad normal,* (Fac. de Matemáticas, Univ. de Oviedo, fevereiro 97), *Bifurcações de difeomorfismos de Anosov* (Univ. de Oviedo, janeiro 95), *Persistencia de difeomorfismos transitivos,* (UCM, julho 94), *Bifurcaciones homoclínicas: resultados recentes,* (Fac. de Ciências Matemáticas, Univ. Autónoma de Madrid, fevereiro 90), *Teoria geométrica de las bifurcaciones homoclínicas y heteroclínicas,* (Fac. Ciencias, Dto. Matemáticas, Univ. Pais Vasco, fevereiro 90), *Bifurcaciones y ciclos heterodimensionales* (Fac. Ciências Matemáticas, Dto. Análise Matemática, Univ. Santiago de Compostela, fevereiro 90), *Continuidad de estados de equilibrio y discontinuidad de dimension para arcos de difeomorfismos,* (Univ. Santiago de Compostela, fevereiro 90).
França	*To shadow or not to shadow,* (Séminaire Géneral, Institut de Mathématiques Université de Bourgogne, fevereiro 04), *Cycles and collisions of homoclinic classes,* (Séminaire Sistémes Dynamiques, Institut de Mathématiques Université de Bourgogne, janeiro 03), *Heterodimensional cycles, partial hyperbolicity and limit dynamics,* (Univ. de Paris-Sud, Centre d'Orsay - Mathématique, janeiro 02), *Maximal transitive sets with minimal dynamics,* (Univ. Denis Diderot (Paris VII) Institut de Mathématiques et Dynamique, Séminaire de Systemes Dinamiques (L.H. Eliasson – J.C.Yoccoz), janeiro 02), *Collision, explosion and collapse of homoclinic classes,* (Univ. de Paris-Sud, Centre d'Orsay - Mathématique, fevereiro 01), *A la recherche de la decomposition spectral perdue,* (Univ. de Bourgogne, Dép. de Mathèmatiques, Lab. de Topologie, Dijon, janeiro 99), *Partially hyperbolic diffeomorphisms in three manifolds,* (Dijon, janeiro 98) *The robustly transitive diffeomorphisms are normally hyperbolic,* (Dijon, janeiro 97), *Stability of transitive dynamics on three manifolds,* (Dijon, janeiro 97) *Derived from Anosov Diffeomorphisms: Discountinuity of Haussdorf dimension,* (Dijon, fevereiro 93) *Prevalence of strange attractors and saddle-node cycles,* (Dijon, julho 92), *Hyperbolicity and the creation of cycles,* (Dijon, junho 92).
Italia	*Strange attractors ma non troppo,* (Scoula Internazionale Superiore di Studi Avanzati - International School for Advanced Studies SISSA-ASIS, Trieste,fevereiro 92)

Japão	*Stable Transitivity and weak hyperbolicity* (minicurso de 6 horas) (Kyoto University, Kyoto, novembro 99), *Robust transitive sets, homoclinic classes, and strong partially hyperbolic dynamics,* (Kanagawa University, Yokohama, novembro 99), *Mechanisms for robust transitive dynamics,* (Tokyo Metropolitan University, Hachiochi, novembro 99), *Robust transitivity and partial hyperbolicity generated by heteroclinic bifurcations,* (Tokyo Institute of Technology, Tokyo, outubro 99), *Stable Transitivity and Weak Hyperbolicity I – III,* (Kitami Institute of Technology, Hokkaido, outubro 99).
México	*Clases homoclínicas y puntos periódicos,* (Centro Internacional de Matemáticas (CIMAT), Guanajuato, México, agosto 03).
Portugal	*Sem sombra de sombremaneto (para as dinâmicas robustamente transitivas não hiperbólicas),* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, junho 03), *Peças elementares de dinâmica,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, junho 03), *Difeomorfismos robustamente não-errantes,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, dezembro 02), *Colisão, explosão e colapso de classes homoclínicas,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, janeiro 02), *Ferraduras sela-nó,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, janeiro 01), *Mecanismos de transitividade robusta.* (Porto, fevereiro 00), *Hiperbolicida fraca das classes homoclinicas,* (Porto, janeiro 99), *Perseguindo um teorema de decomposição espectral,* (Fac. de Ciências, Dto. de Matemática Aplicada, Univ. Porto, janeiro 99), *Ligações persistentes,* (Porto, fevereiro 98), *Infinidade de poços na topologia* C^1*,* (Porto, julho 97), *Persistência de difeomorfismos transitivos,* (Porto, fevereiro 97), *Frequência de hiperbolicidade e bifurcações sela-nó,* (Porto, janeiro 96), *Frequência de hiperbolicidade e bifurcações sela-nó,* (Instituto Superior Técnico, Departamento de Matemática, Lisboa, janeiro 96), *Bifurcações de difeomorfismos de Anosov,* (Porto, janeiro 95), *Atratores não hiperbólicoss,* (Fac. de Ciências, Dto. de Matemática Pura, Univ. Porto, março 93), *Ciclos: hiperbolicidade versus não-hiperbolicidade,* (Porto, junho 91), *Ciclos em dimensão maior do que dos,* (Fac. de Ciências, Dto. de Matemática (Porto, fevereiro 90).
Suiça	*Partially hyperbolic diffeomorphisms in three manifolds,* (Départament de Mathèmatiques, Universitè de Genéve, janeiro 98).
Uruguai	*Intersecciones robustas y fenómeno de Newhouse,* (Facultad de Ingieneria, Univ. de Montevideo, maio 97), *Persitencia de difeomorfismos transitivos* (Montevideo, maio 94).
Venezuela	*Transitividad estable e hiperbolicidad débil (mini-curso, 4h 30min),* (Instituto Venezolano de Investigaciones Científicas (IVIC), Caracas, junho 00), *Transitividad e hiperbolicidad,* (IVIC, outubro 97), *Transitividad robusta e hiperbolicidad parcial,* (IVIC, outubro 97).

Atividades profissionais

Posições anteriores

1995-1998	Professor Assistente, Departamento de Matemática, Pontifícia Universidade Católica de Rio de Janeiro (PUC-Rio), Rio de Janeiro, Brasil.
1994	Professor Vínculo Extraordinário, Departamento de Matemática, Pontifícia Universidade Católica de Rio de Janeiro (PUC-Rio), Rio de Janeiro, Brasil.
1993	Professor Associado, Laboratorio de Matemática, Universidade Estadual do Norte Fluminense (UENF), Campos, Rio de Janeiro, Brasil.
1991-1993	Pesquisador Associado ao Conselho Nacional de Pesquisas (CNPq - Brasil), Instituto de Matemática, Universidade Federal de Rio de janeiro (IM-UFRJ), Rio de Janeiro, Brasil.
1990-1991	Professor Assistente, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid (UCM), Madri, Espanha.
1987-1990	Assistente de Pesquisa, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro, Brasil (1987 – 1990).
1984-1985	Professor Assistente, Escuela Técnica Superior de Ingenieros de Telecomunicaciones, Universidad Politécnica de Madrid (ETSIT-UPM), Madri, Espanha (1984 – 1985).

Participação em projetos

Coordenador do Programa de Cientistas do Nosso Estado da Faperj, *Sistemas Dinâmicos reais e complexos* (equipe: J. C. Rebelo e R. O. Ruggiero) (2005-2006).

Membro do Programa de Cientistas do Nosso Estado da Faperj, *Sistemas Dinâmicos Caôticos* (Coord. M. Viana) (1999 – 2000), (2000-2001),(2002-2003), (2004-2005).

Rede Sul Americana de Sistemas Dinâmicos (Prosul) (coordenador de subprojeto) (Coord. M. Viana) (2004-2005).

Membro do projeto temático CNPq *Teoria geométrica e ergódica de sistemas dinâmicos caóticos* (Coord. M. Viana) (dez 2000-dez 2001).

Membro do Programa de Núcleos de Excelência (PRONEX) do MCT para Sistemas Dinâmicos (Coord. C. Camacho e J. Palis) (1997 – 2002).

Coordenador responsável na PUC-Rio do Acordo de Cooperação internacional França - Brasil em Matemática, (2001–?).

Membro do projeto cooperação CNPq-ICCTI (Instituto de Cooperação Científica Internacional, Portugal), em Sistemas dinâmicos, (Coord. M. Viana) (1999-2000).

Programa de Conferências Departamento de Matemática PUC-Rio, Faperj, (fevereiro 98–fevereiro 99) e (agosto 02–agosto 03).

Programa de Conferências Departamento de Matemática PUC-Rio, CNPq, (janeiro 04 – dezembro 04).

Atividade Docente

Orientação

Doutorado

Carlos Carballo, *Propriedades genéricas de classes homoclínicas* (abril 2001). Tese publicada em Bol. S.B.M, **31**, 287-303, (2000), Ergodic Theory Dynam. Systems, **23**(2), 403–415, (2003) e. Bull. London Math. Soc., **35**(1), 85–91, (2003).

Mestrado

Bianca Santoro, tese *Colisão, colapso e explosão de classes homoclínicas,* (aceita em Nonlinearity).

Pós-Doutorado

Armando de Castro, bolsita Faperj, maio 99–março 00.
Vitor Araujo, bolsita CNPq, agosto 04–julho 05.

Participação em Bancas

Teses de Doutorado

2004 Instituto de Matemática Pura e Aplicada, orientador M. Viana, Carlos Matheus Silva Santos, *Contribuições à teoria ergódica de sistemas não hiperbólicos.*

Departamento de Matemática, PUC-Rio, orientador R. O. Ruggiero, José Barbosa Gomes, *Rigidez de superfícies cujos fluxos geodésicos preservam folheações de codimensão 1.*

2001 Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad Complutense de Madrid, orientador Francisco Romero, José Manuel Salazar Crespo, *Indice de punto fijo en hiperespacios e índice de Conley,*
Instituto de Matemática Pura e Aplicada, orientador M. Viana, Jairo Bochi, *Expoentes de Lyapunov nulos em sistemas conservativos.*

2000 Instituto de Matemática Pura e Aplicada, orientador M. Viana, Alexandre Baraviera, *Robust non uniform hyperbolicity for volume preserving maps,*

Departamento de Matemática, PUC-Rio, orientador R. O. Ruggiero, Wladimir Rosas *Sobre fluxos geodésicos expansivos: Conjunto de Pesin e E Teorema de Livsic.*

1999	Instituto de Matemática Pura e Aplicada, orientador M. Viana, Wanderley Horita, *Dimensão de Hausdorff de repulsores de não hiperbólicos de derivados difeomorfismos de Anosov.*
1998	Instituto de Matemática Pura e Aplicada, orientador J. Palis, Fernando José Sánchez-Salas, *Propriedades geometricas de atratores ergódicos,*
	Instituto de Matemática Pura e Aplicada, orientador M. Viana, Isabel Rios, *Desdobramentos de tangências homoclínicas internas de ferraduras: hiperbolicidade, dimensoões fracionárias e tangencias persistentes.*
	Instituto de Matemática Pura e Aplicada, orientador M. Viana, Augusto Armando de Castro *Backward inducing and exponential decay of correlations for partially hyperbolic attractors with mostly contracting central direction,*
	Instituto de Matemática Pura e Aplicada, orientador M. Viana, Jose F. Alves, *Medidas SRB para atratores multidimensionais.*
1996	Instituto de Matemática Pura e Aplicada, orientador J. Palis, Eduardo Colli, *Coexisteência de infinitos atratores estranhos.*
1995	Instituto de Matemática Pura e Aplicada, orientador R. Mañé, Heber Henrich, *Bifurcações Heteroclínicas de difeomorfismos de Anosov.*
1992	Instituto de Matemática Pura e Aplicada, orientador J. Palis, Carlos Martín Rivas, *Bifurcações Homoclínicas e Cascatas de Duplicação de Periodo em Qualquer Dimensão.*

Teses de Mestrado

2002	Departamento de Matemática, PUC-Rio, orientador M. Souza, Cristiane Aparecida Monteiro, *Análise de modelos inmunológicos de rede.*

Concursos Públicos

2004	Membro titular da *Comissão Julgadora do Concurso Público de Provas e Títulos para Professor Adjunto* do Departamento de Métodos Matemáticos (Setor Geometria Diferencial e Sistemas Dinâmicos), Instituto de Matemática, Universidade Federal do Rio de Janeiro.

Cursos ministrados

Depto. Matemática Aplicada a la Telecomunicación, Escuela Técnica Superior de Ingenieros de Telecomunicación, Univ. Politécnica de Madrid: *Matemática Aplicada II* (ano 84–85).

Instituto de Matemática Pura e Aplicada: *Equações diferenciais ordinarias,* (assistente), (março – junho 84), *Algebra linear,* (assistente), (março–junho 86).

Instituto de Matemática, Univ. Federal de Rio de Janeiro: *Algebra linear,* (agosto–novembro 91 e março–junho 92), *Equações diferenciais,* (março–junho 93), *Cálculo avana̧do,* (mestrado), (março–maio 92), *Métodos topológicos,* (mestrado), (março–maio 93).

Universidade do Norte Fluminense: *Introdução aos computadores,* (agosto–dezembro 93).

Departamento de Matemática, Pontifícia Universidade Católica, Rio de Janeiro: *Algebra II,* (agosto–dezembro 98), *Algebra linear I,* (agosto–dezembro 94, 99, 01, 02, 03 e 04, março–julho 96, 99, 01, 02, 03 e 04) (coordenador do curso nos semestres março–julho 01, 02, 03, e agosto–dezembro 00, 01, 02, 03 e 04), *Algebra linear II,* (agosto – dezembro 96), *Análise I,* (agosto–dezembro 97 e 04), *Cálculo a uma variável,* (março–julho 00), *Cálculo II,* (março–julho 95), *Cálculo III,* (março–julho 94), *Equações diferencias ordinárias,* (março–julho 97, 00 e 03)), *Introdução ao Análise,* (março–julho 96), *Introdução às Equações Diferenciais Parciais,* (agosto–dezembro 99), *Introdução às funções de variável complexas,* (março–julho 98 e 03), *Medida e Integração,* (março–julho 98), *Variaveis complexas,* (março–julho 99 e 01).

Iniciação científica

Processos estocás ticos e passeios aleatórios, (agosto 04–julho 05).

Introdução aos Sistemas Dinâmicos, (fevereiro 95–março 97 e agosto 02–dezembro03).

Introdução à criptografia, (com Renata M. da Rosa) (Agosto 97 – Fevereiro 99).

Material Didático na PUC

Responsável e elaborador do material didático do curso de Álgebra Linear 1, Mat 1200, disponível na página de internet (http://www.mat.puc-rio.br/disciplinas/MAT1200/).

Atividade administrativa na PUC

Departamento de Matemática

Coordenador do Colóquio do Departamento de Matemática (agosto 04–?).

Coordenador de Pós-Graduação do Departamento de Matemática, PUC-Rio (97 – 98) e (01-?).

Coordenador Administrativo de Pós-Graduação do Departamento de Matemática, PUC-Rio (01).

Membro eleito da Comissão Geral do Departamento de Matemática PUC-Rio (97, 99, 00, 01, 03, 05)

Membro da Comissão de Pos-Graduação e Pesquisa do Departamento de Matemática PUC-Rio (97, 98, 04).

Suplente da Comissão de Carreira Docente Departamento de Matemática PUC-Rio (01).

Centro Técnico Científico

Suplente Conselho Departamental do CTC (00).

Membro eleito do Conselho Departamental do CTC (01 e 02).

Membro eleito do Comissão de Carreira Docente do CTC (03–05).

Vários

Resenhador (Reviewer) do Mathematical Reviews da American Mathematical Society (47 resenhas publicadas).

Atividade como Referee: *Asterisque, Boletim Sociedade Brasileira de Matemática, Discrete and continuous dynamical systems, Duke Mathematical Journal, Ergodic Theory and Dynamical Systems, Inventiones Mathematicae, Journal of Dynamics and Stability of Systems, Nonlinearity, Proceedings of the American Mathematical Society, Publications Mathématiques Institut de Hautes Études Scientifiques.*

Consultor CAPES, CNPq, Faperj, Fapesp, Facepe, UFBa (Brasil), Fondecyt (Chile) Agencia Nacional de Evaluacion y Prospectiva (ANEP) do Ministerio de Ciencia y Tecnología (Espanha), Programa de Desarrollo Tecnológico (PDT) do Ministerio de Educación y Cultura de Uruguai, Proyectos Fondo Clemente Estable (2004) do Ministerio de Educación y Cultura de Uruguai.

Membro de Corpo editorial de *Matemática Universitaria* (desde janeiro 2001).

Rio de Janeiro, 20 de janeiro de 2005

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Collision, explosion and collapse of homoclinic classes

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Abstract

Homoclinic classes of generic C^1-diffeomorphisms are maximal transitive sets and pairwise disjoint. We here present a model explaining how two different homoclinic classes may intersect, failing to be disjoint. For that we construct a one-parameter family of diffeomorphisms $(g_s)_{s\in[-1,1]}$ with hyperbolic points P and Q having nontrivial homoclinic classes, such that, for $s<0$, the classes of P and Q are disjoint, for $s=0$, the classes collide and their intersection is a saddle-node, and, for $s>0$, after an explosion, the two classes are equal. Our constructions involve bifurcations through heterodimensional and saddle-node cycles.

Introduction

In this paper we study the collision of non-trivial homoclinic classes *via* saddle-node bifurcations and the dynamics before and after this collision. The main motivation of this paper comes from recent results about *maximal transitive sets*: for generic[1] C^1-diffeomorphisms, the homoclinic classes are either disjoint or equal ([Ar] and [CMP]). Our objective is to understand how two homoclinic classes may be non-disjoint and different as well as the dynamical consequences of this pathology.

Let us start by recalling some definitions. Given a diffeomorphism f, an f-invariant set Λ is *transitive* if there is an $x\in\Lambda$ whose forward orbit is dense in Λ, i.e., $\Lambda=\overline{\cup_{i\in\mathbb{N}}f^i(x)}$. A transitive set is *maximal* if it is a maximal element of the family of all transitive sets partially ordered by inclusion. Observe that any transitive set is contained in a maximal one. A transitive set Λ is *saturated* if it contains every transitive set Σ such that $\Lambda\cap\Sigma\neq\emptyset$. Clearly, every saturated transitive set is also maximal. The *homoclinic class* of a saddle P of f, denoted by $H(P,f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Every homoclinic class is a transitive set, not necessarily maximal nor saturated.

The problem of characterizing and describing (for a large class of systems) maximal and saturated transitive sets is a key problem in dynamics. In fact, these saturated transitive sets are the natural candidates for playing the role of the elementary pieces of dynamics (similar to the role of the basic sets in the Smale hyperbolic theory, [Sm]). Recently, [Ab] states that for generic C^1-diffeomorphisms f having finitely many different homoclinic classes the non-wandering set of f, $\Omega(f)$, is the disjoint union of such classes. Moreover, these classes verify a weak form of hyperbolicity (existence of a dominated splitting, see [BDP]) and are the maximal invariant sets of a fixed filtration (see Section 6.3) independent of the generic diffeomorphism in a neighborhood of f.

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[1] by a generic diffeomorphism we mean a diffeomorphism in a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$.

Consider a closed manifold M and denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the usual uniform topology. In [Ar], it is proved that homoclinic classes of generic diffeomorphisms are *maximal transitive sets.* [CMP] generalizes this result by proving that homoclinic classes of generic diffeomorphisms are saturated transitive sets. Thus homoclinic classes of generic diffeomorphisms are either equal or disjoint. We observe that there are locally generic diffeomorphisms having saturated transitive sets without periodic orbits (so which are not homoclinic classes), see [BD$_2$]. The goal of this paper is to give examples of homoclinic classes which are not saturated transitive sets, presenting an explanation of how this pathology arises. In fact, we exhibit homoclinic classes which are not contained in any saturated transitive set. For simplicity, we consider diffeomorphisms defined on three manifolds, but our constructions can be carried out to higher dimensions after straightforward modifications.

We construct a diffeomorphism f with saddles P and Q with *Morse index* (dimension of the unstable bundle) one and two such that their homoclinic classes are nonhyperbolic, nontrivial, and maximal transitive, and whose intersection is just a saddle-node. So these classes are not saturated transitive sets. In fact, as mentioned above, we will prove that they are not contained in any saturated transitive set.

Theorem A. *Let M be a 3-dimensional closed manifold. There exist an open set $W \subset M$ and a family of diffeomorphisms $(g_s)_{s\in[-1,1]}$, $g_s\colon M \to M$, such that, for every s, the diffeomorphism g_s has hyperbolic fixed points P and Q of Morse indices 1 and 2 such that the maximal invariant set of g_s in W, denoted by Λ_s, verifies the following:*

- *For every small $s < 0$, the set $\Lambda_s \cap \Omega(g_s)$ is the disjoint union of the homoclinic classes $H(P, g_s)$ and $H(Q, g_s)$, where $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and locally maximal.*
- *For $s = 0$, $\Lambda_0 = \Lambda_0 \cap \Omega(g_0) = H(P, g_0) \cup H(Q, g_0)$, where $H(P, g_0)$ and $H(Q, g_0)$ are locally maximal and $H(P, g_0) \cap H(Q, g_0) = \{S\}$, where S is a saddle-node fixed point.*
- *For every small $s > 0$, $\Lambda_s = \Lambda_0 \cap \Omega(g_0) = H(P, g_s) = H(Q, g_s)$.*

This result means that the homoclinic classes of P and Q *collide* at $s = 0$ and thereafter *explode* (the point P that does not belong to $H(Q, g_0)$ is in $H(Q, g_s)$ for every small positive s, and the same holds for the point Q and $H(P, g_s)$). Finally, the homoclinic classes also *collapse*: $H(Q, g_s) = H(P, g_s)$ for positive s.

In the previous theorem the open set W is a level of a *filtration,* (see Section 6.3). Theorem A now implies the following:

Theorem B. *Under the hypotheses of Theorem A, the homoclinic classes $H(P, g_0)$ and $H(Q, g_0)$ are not saturated and they are not contained in any saturated transitive set.*

Our construction involves *saddle-node bifurcations* and *heterodimensional cycles.* We introduce a codimension-two bifurcation, the *saddle-node heterodimensional cycles,* and study the *lateral homoclinic classes of a saddle-node.* Let us explain all that in details.

Consider a diffeomorphism f having two hyperbolic fixed points P and Q with Morse indices 1 and 2, respectively. Then, f has a *heterodimensional cycle associated to P and Q* if the 2-dimensional stable manifold of P and unstable manifold of Q, denoted by $W^s(P, f)$ and $W^u(Q, f)$, have a non-empty transverse intersection, and the 1-dimensional unstable manifold of P, $W^u(P, f)$, and stable one of Q, $W^s(Q,f)$, have a quasi-transverse intersection throughout the orbit of a point x_0, i.e., $T_{x_0}W^s(Q,f)+T_{x_0}W^u(P,f) = T_{x_0}W^s(Q,f)\oplus T_{x_0}W^u(P,f)$, thus $\dim(T_{x_0}W^s(Q,f)+T_{x_0}W^u(P,f)) = 2$. A heterodimensional cycle is depicted in Figure 1 in Section 1. Bifurcations through heterodimensional cycles have been systematically studied in the series of papers [D$_1$, D$_2$, DR$_1$, DR$_2$, DU, DR$_4$, DR$_5$].

A *saddle-node* S of a diffeomorphism f is a periodic point (we here assume to be fixed) such that the derivative of f at S has 1 as its only eigenvalue in the unitary circle. We consider saddle-nodes of *saddle-type* (i.e., the derivative of f at S simultaneously has eigenvalues inside and outside the unitary circle). Thus the tangent bundle of M at S has a Df-invariant splitting $E^{ss}\oplus E^c\oplus E^{uu}$, where E^{ss} (resp. E^{uu}) is the bundle spanned by the eigenvectors associated to the contracting (resp. expanding) eigenvalues, and E^c is the eigenspace associated to the eigenvalue 1 (in our context, all these spaces have dimension 1). By the theory of invariant manifolds, see [HPS], there exist the *strong stable* and *unstable* manifolds of the saddle-node, defined as the unique f-invariant manifolds tangent at S to E^{ss} and to E^{uu} and denoted by $W^{ss}(S,f)$ and $W^{uu}(S,f)$, respectively.

Motivated by the fact that (generic) saddle-nodes (of saddle type) simultaneously behave as points of index two and one (the stable and unstable manifolds of the saddle-node have both dimension 2), we introduce *saddle-node heterodimensional cycles.* A diffeomorphism f has a saddle-node heterodimensional cycle associated to a saddle-node S and the saddle P of Morse index one if the (two-dimensional) unstable manifold of S and stable manifold of P have nonempty transverse intersection and the (one-dimensional) invariant manifolds $W^{ss}(S,f)$ and $W^u(P,f)$ have a quasi-transverse intersection along the orbit of some point. A saddle-node heterodimensional cycle is depicted in Figure 4 in Section 4. One similarly defines saddle-node heterodimensional cycles associated to a (saddle-type) saddle-node S and a saddle Q of Morse index two.

Roughly speaking, in our construction we consider a diffeomorphism f simultaneously having two saddle-node heterodimensional cycles. We consider a two parameter family $(f_{t,s})_{t,s\in[-1,1]}$ of diffeomorphisms such that $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, one associated to a saddle-node S and a saddle P of Morse index one and other one associated to a saddle Q of index one and the saddle-node S. The parameter t describes the unfolding of the cycles (relative motion between compact parts of $W^u(P,f_{t,0})$ and $W^{ss}(S,f_{t,0})$ and of $W^s(Q,f_{t,0})$ and $W^{uu}(S,f_{t,0})$). The parameter s describes the unfolding of the saddle-node: for positive s there are two saddles S_s^+ and S_s^- of indices 2 and 1, colliding at $s=0$ to the saddle-node S and disappearing for negative s. We see that, fixed any small $\bar{t}>0$, for $s>0$ (before the collapse of the saddles), $H(P,f_{\bar{t},s}) = H(S_s^+,f_{\bar{t},s})$ and $H(Q,f_{\bar{t},s}) = H(S_s^-,f_{\bar{t},s})$ for all small positive s. Moreover, $H(P,f_{\bar{t},s})\cap H(Q,f_{\bar{t},s}) = \emptyset$. At the saddle-node bifurcation we have $H(P,f_{\bar{t},0})\cap H(Q,f_{\bar{t},0}) = \{S\}$. Finally, for $s<0$, after the disappearing of the saddles, $H(P,f_{\bar{t},s}) = H(Q,f_{\bar{t},s})$. See the results in Section 6. Theorem A follows by considering the arc $g_s = f_{\bar{t},-s}$. To deduce Theorem B from Theorem A, we consider a filtration having the open set W as a level and analyze the orbits of recurrent points of Λ_s.

In forthcoming papers, we will illustrate how this type of bifurcation naturally appear as secondary bifurcations in the unfolding of heterodimensional cycles and give a model for the collision, explosion, and collapse of (nontrivial) hyperbolic homoclinic classes, see [DR$_6$].

Let us say a few words about our constructions. As mentioned, our setting necessarily corresponds to a non-generic situation, so we focus our attention on an example (we have not done any effort for generality). We begin by presenting (in Section 1) a model for the unfolding of a heterodimensional cycle. This model (motivated by [D$_1$] and [BD$_1$]) allows us to give a rather transparent explanation of the dynamics in the unfolding of a heterodimensional cycle by reducing it to the study of the dynamics of an iterated system of functions defined on an interval, this is done in Section 2. Recall that the dynamics of a (linear) Smale horseshoe is given by two affine expanding maps of the interval (say $I = [0,1]$) whose domains of definition are two disjoint closed subintervals of I (say $[0,1/3]$ and $[1/3,1]$). The interval $(1/3,2/3)$ is the main gap of the horseshoe and corresponds to points in the basin of attraction of a sink. The affine model for heterodimensional cycles is a system of iterated functions with infinitely many maps F_i defined on subintervals I_i of I which are non-disjoint (the interior of the intervals I_i are pairwise disjoint, but I_i and I_{i+1} have a common extreme). Thus in this model there are no gaps and there are no escaping points.

In Section 3, we prove that, after unfolding the cycle, the dynamics of the model family is non-hyperbolic: the point of index 1 in the cycle belongs to the homoclinic class of the point of index 2 in the cycle. Here, using the one-dimensional reduction, we give a shorter and clearer proof of the results in [D$_1$]. Since our constructions rely heavily on this proof and there is not any written version of this approach, we have decided to include a short description of it.

In Section 4, we introduce the *lateral homoclinic classes of a saddle-node* S of a diffeomorphism f as above, $H^+(S,f)$ and $H^-(S,f)$, respectively defined as the closure of the transverse intersections $W^u(S,f) \pitchfork W^{ss}(S,f)$ and $W^s(S,f) \pitchfork W^{uu}(S,f)$. These lateral homoclinic classes essentially behave as the usual ones. We see that for arcs f_t unfolding at $t = 0$ the saddle-node heterodimensional cycle (associated to a saddle P of index one and the saddle-node S) one has $H(P,f_t) \subset H^+(S,f_t)$ for all small positive t. Moreover, under mild conditions, one also gets $H^+(S,f_t) = H(P,f_t)$ for all small $t > 0$. The inclusion $H(P,f_t) \subset H^+(S,f_t)$ follows adapting (in a rather straightforward way) the results for the model family in Section 3. For the inclusion $H^+(S,f_t) \subset H(P,f_t)$ we need new ingredients that we borrow from [D$_2$].

Using the results in Sections 3 and 4, we get a complete description of the homoclinic classes $H(P, f_{\bar{t},s})$ and $H(Q, f_{\bar{t},s})$ before the collapse of the saddles S^+_s and S^-_s to the saddle-node. Finally, to study $H(P, f_{\bar{t},s})$ and $H(Q, f_{\bar{t},s})$ after the collision, we introduce new systems of iterated functions and analyze their dynamics.

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1 Heterodimensional cycles: a model family

In this section, we construct a model one-parameter family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms unfolding a heterodimensional cycle. The study of the semi-local dynamics of f_t will be reduced to the analysis of a one-parameter family of endomorphisms with infinitely many discontinuities which describe the dynamics of f_t in the *central direction,* see Section 2.

Consider a diffeomorphism f with a heterodimensional cycle having the following dynamical configuration. In local coordinates in $\mathbb{R}^3$, the cycle is associated to saddle fixed points $Q = (0,0,0)$ and $P = (0,1,0)$ of indices 2 and 1, respectively, verifying the following conditions:

Partially hyperbolic (semi-local) dynamics of the cycle:

- In the cube $[-1,1] \times [-1,2] \times [-1,1]$ the diffeomorphism has the form

$$f(x,y,z) = (\lambda_s x, F(y), \lambda_u z),$$

where $F\colon [-1,2] \to (-1,2)$ is an increasing map with exactly two fixed points, a source at 0 and a sink at 1, and $0 < \lambda_s < d_m < 1 < d_M < \lambda_u$, where $0 < d_m < F'(x) < d_M$ for all $x \in [0,1]$.

- There is $\delta > 0$ such that F is linear in $[-\delta,\delta]$ and affine in $[1-\delta, 1+\delta]$. We denote by $\beta > 1$ and $0 < \lambda < 1$, the derivative of F at 0 and 1, respectively.

Observe that $[-1,1]\times\{(0,0)\} \subset W^s(Q)$, $\{0\}\times[0,1)\times[-1,1] \subset W^u(Q)$, $\{(0,0)\}\times[-1,1] \subset W^u(P)$, and $[-1,1]\times(0,1]\times\{0\} \subset W^s(Q)$. Thus the curve $\gamma = \{0\}\times(0,1)\times\{0\}$ (called *connexion*) is a normally hyperbolic curve contained in $W^u(Q) \pitchfork W^s(P)$.

Existence and unfolding of the cycle:

- **The cycle:** There exist $k_0 \in \mathbb{N}$ and a small neighborhood U of $(0,1,-1/2) \in W^u(P)$ such that the restriction of f^{k_0} to U is a translation,

  $$f^{k_0}(x,y,z) = (x - 1/2, y - 1, z + 1/2).$$

  Thus $f^{k_0}(0,1,-1/2) = (-1/2,0,0) \in W^u(Q)$ and $W^s(Q)$ and $W^u(P)$ meet throughout the orbit of the heteroclinic point $(-1/2,0,0)$. By construction, $(-1/2,0,0)$ is a quasi-transverse heteroclinic point.

- **The unfolding of the cycle:** Consider the family $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ of diffeomorphisms coinciding with f in $[-1,1]\times[-1,2]\times[-1,1]$ and such that the restriction of $f_t^{k_0}$ to U is of the form

  $$f_t^{k_0}(x,y,z) = (x - 1/2, y - 1 + t, z + 1/2) = f^{k_0}(x,y,z) + (0,t,0).$$

  So, for $t > 0$, $\{(-1/2,t)\}\times[-1,1] \subset W^u(P, f_t)$ and $x_t = (-1/2,t,0)$ is a transverse homoclinic point of P (for f_t). Similarly, $y_t = (-1/2,0,0)$ is a transverse homoclinic point of Q.

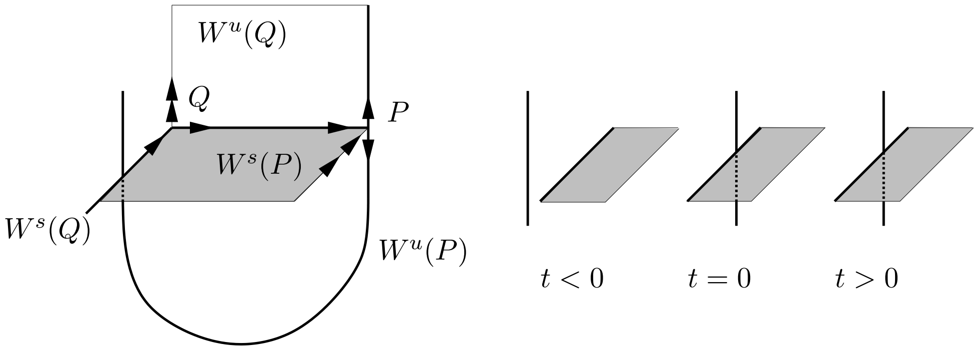

Figure 1: The model cycle and its unfolding

Consider a small *neighborhood of the heterodimensional cycle associated with* f_0, that is, an open set W containing the connexion $\gamma = \{0\}\times[0,1]\times\{0\}$ and the f_0-orbit of the heteroclinic point $(-1/2,0,0)$. For small t, let Λ_t be the *maximal* f_t*-invariant set in* W, $\Lambda_t = \cap_{n\in\mathbb{Z}} f_t^n(W)$. Consider also the forward and backward f_t-invariant sets in W, $\Lambda_t^+ = \cap_{n\geq 0} f_t^{-n}(W)$ and $\Lambda_t^- = \cap_{n\geq 0} f_t^n(W)$.

Fix a small positive ρ and consider the fundamental domains of F given by $D^+ = [\beta^{-1}\rho, \rho]$ and $D^- = [1-\rho, \lambda(1-\rho)]$ contained in the neighborhoods of 0 and 1 where F is affine. We can choose ρ such that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$ (for notational simplicity let us put $N = 1$). The map F^N is the *transition from* 0 *to* 1 [2]. Assume that that $F'(1) = \lambda$ and $F'(0) = \beta$ verify:

(T1) $F'(x) \geq \dfrac{1}{2}\,\dfrac{1-\lambda}{1-\beta^{-1}}$, for all $x \in D^+$,

(T2) $(1-\lambda) < \beta^{-1}$, and

(T3) $\dfrac{(1-\lambda)\,\lambda}{2\,(1-\beta^{-1})\,\beta} = \ell > 1$.

To get condition (T1) it is enough to consider F with small distortion. For conditions (T2) and (T3) it is enough to take β close enough to 1^+. In fact, later we will consider the case where Q is a saddle-node (saddle-node heterodimensional cycles) and $\beta = 1$, see Section 4. Our first result is:

Theorem 1.1. *For every $t > 0$ sufficiently small, $H(P, f_t) \subset H(Q, f_t)$ and $\Lambda_t \subset H(Q, f_t)$.*

This theorem was stated in [D$_1$]. Here we give a more conceptual prove of it, which enables us to introduce some technical tools to be used systematically later on. First, in Section 2 we will introduce the system of iterated functions associated to the cycle (this approach is motivated by [DR$_5$]). In Section 3, we deduce the theorem from the results in Section 2.

2 Expanding one-dimensional dynamics associated to the cycle

For each small $t > 0$, consider the *scaled fundamental domains* $D_t^{\pm}$ defined as follows: let $D_t^- = [1-t, \lambda(1-t)]$ and define k_t as the smaller $k \in \mathbb{N}$ with $F^{-k}(D_t^-) \subset [0, t]$. We define

$$D_t^+ = [a_t, b_t] = [\beta^{-1}(b_t), b_t] = F^{-k_t}(D_t^-), \quad \text{where} \quad \beta^{-2}t < a_t < b_t \leq t.$$

We next define an expanding map R_t, $R_t\colon D_t^+ \to D_t^+$, with discontinuities describing the central dynamics of the return map of f_t defined on $[-1,1] \times D_t^+ \times [-1,1]$. First, for each small $t > 0$ define the *transition map T_t from D_t^+ to D_t^-* by

$$T_t\colon D_t^+ \to D_t^-, \quad x \mapsto T_t(x) = F^{k_t}(x).$$

Lemma 2.1. *The map T_t verifies $T_t'(x) > \ell > 1$ for all $x \in D_t^+$, where ℓ is as in condition* **(T3)**.

---

[2] This transition plays a key role for determining the dynamics after unfolding the cycle and it is determined by the *Mather invariant* of F, [Ma]: in a neighborhood of 0, the map F is the time-one of the vector field $X(y) = (\log\beta)\, y\, \frac{\partial}{\partial y}$, whose flow is $y \mapsto \beta^t\, y$. Similarly, in a neighborhood of 1, F is the time-one of $Y(y) = (\log\lambda)\,(y-1)\,\frac{\partial}{\partial y}$. Consider, for y close to 0^+, an large n such that $F^n(x)$ is close to 1^- and write $DF^n(y)(X(y)) = \mu(y)\,Y(F^n(y))$. Using the local F-invariance of X and Y (near 0 and 1), one has that $\mu(x)$ does not depend on n and that $\mu(x) = \mu(F(x))$. The function μ is the Mather invariant of F which describes its distortion. For instance, if μ is identically 1, then F is exactly the exponential of a vector field.

Proof: Given $x \in D_t^+$ let $k_t = n_t(x) + 1 + m_t(x)$, where $F^{n_t(x)}(x) \in D^+$ and $F^i(x) \notin D^+$ for all $1 \le i < n_t(x)$. We claim that

$$\frac{1}{\beta\, t} \le \beta^{n_t(x)} \le \frac{\beta^2}{t} \quad \text{and} \quad \lambda\, t \le \lambda^{m_t(x)} \le \frac{t}{\lambda}. \tag{1}$$

For first inequalities just note $x \in D_t^+ \subset (\beta^{-2}\, t, t]$ and $\beta^{n_t(x)}\, x \in [\beta^{-1}, 1]$. The other ones follow analogously. As $T_t(x) = \lambda^{m_t(x)}\, F(\beta^{n_t(x)}\, x)$, hypotheses (T1) and (T3) and the estimates in (1) give

$$|T_t'(x)| = \lambda^{m_t(x)}\, |F'(x)|\, \beta^{n_t(x)} \ge (\lambda\, t) \left(\frac{1}{2} \frac{1-\lambda}{1-\beta^{-1}} \right) \left(\frac{1}{\beta\, t} \right) = \frac{1}{2} \frac{\lambda\,(1-\lambda)}{\beta\,(1-\beta^{-1})} = \ell > 1,$$

as we claimed. ☐

Since $T_t(x) \in D_t^- = [1-t, 1-\lambda t]$ for all $x \in D_t^+$, we can define the map G_t by

$$G_t \colon D_t^+ \to [0, t], \quad x \mapsto G_t(x) = T_t(x) + (t-1).$$

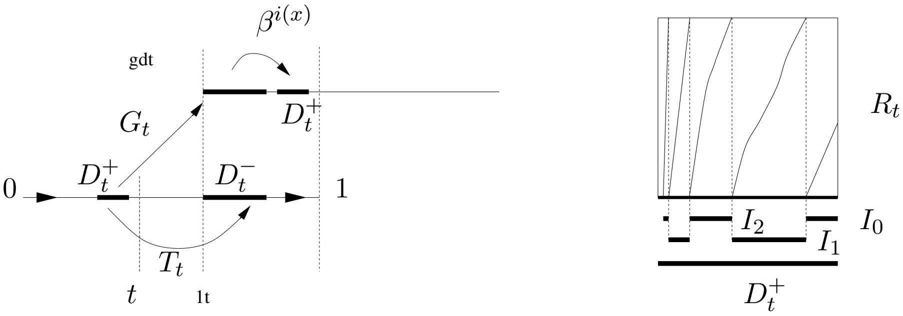

Figure 2: The expanding maps T_t, G_t and R_t

Remark 2.2. *The map G_t is monotone increasing and $G_t(D_t^+) = [0, t(1-\lambda)]$.*

Claim 2.3. *Let $(a_t, b_t] = \tilde{D}_t^+ \subset D_t^+$. Given $x \in \tilde{D}_t^+$ let $i(x) \in \mathbb{Z}$ be the minimum i with $\beta^i(G_t(x)) \in D_t^+$. Then there is $i_0 > 0$ (maximum with such property) such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: Recall first that $b_t \in (\beta^{-1}\, t, t]$. On the other hand, from $(1-\lambda) < \beta^{-1}$ (condition (T2)) and Remark 2.2, $G_t(\tilde{D}_t^+) = (0, t\,(1-\lambda)] \subset (0, \beta^{-1}\, t)$. Thus the right extreme of $G_t(\tilde{D}_t^+)$ is less than the left extreme of D_t^+, hence $i(x) > 0$ for all $x \in \tilde{D}_t^+$, ending the proof of the claim. ☐

Finally, the return map R_t is defined by

$$R_t \colon \tilde{D}_t^+ \to D_t^+, \quad R_t(x) = \beta^{i(x)}(G_t(x)) = \beta^{i(x)}\,(T_t(x) + (t-1)).$$

Next we study the dynamics of R_t: the map R_t is uniformly expanding and has (infinitely many) discontinuities where the lateral derivatives are well defined. These discontinuities will play a key role in our constructions. The definition of $i_0 \in \mathbb{N}$ in Claim 2.3 implies that $\beta^{-i_0}(a_t) \in G_t(D_t^+)$. For each $i \ge i_0$ define $d_i \in \tilde{D}_t^+$ by $G_t(d_i) = \beta^{-i}(a_t)$. By construction, the sequence $(d_i)_{i \ge i_0}$ corresponds to the discontinuities of R_t and verifies the following:

- $d_{i+1} < d_i$ and $d_i \to a_t$,
- Let $[d_{i+1}, d_i] = I_i$, $i > i_0$, and $I_{i_0} = [d_{i_0}, b_t]$. The map R_t is continuous and strictly increasing in the interior of each interval I_i. We continuously extend R_t to the whole I_i, obtaining a bi-valuated return map R_t with $R_t(d_i) = \{a_t, F(a_t) = b_t\}$ for all $i > i_0$. In particular, the restriction of R_t to any I_i, $i > i_0$, is onto. We let $R_t(b_t) = c_t \leq b_t$.

The main properties of R_t are summarized in the next lemma.

Lemma 2.4. *The restriction of R_t to each interval $I_i, i > i_0$, is onto and $R_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = d_i$ this means that the lateral derivatives of R_t at x are greater than ℓ). Moreover, $0 \in G_t(R_t(d_i))$ for all $i \geq i_0$.*

The expansiveness of R_t follows from Lemma 2.1 and Claim 2.3. Condition $0 \in G_t(R_t(d_i))$ follows from $a_t \in R_t(d_i)$ and $0 \in G_t(a_t)$.

Lemma 2.5. *Consider small $t > 0$ and an open subinterval J of $\tilde{D}_t^+$. Then there is $k \in \mathbb{N} \cup \{0\}$ such that $R_t^k(J)$ contains a discontinuity of R_t. In particular, there is $x \in J$ such that $G_t(R_t^k(x)) = 0$.*

Proof: If the interval J contains a discontinuity we are done. Otherwise, let $i > 0$ be such that the intervals J, $R_t(J), \ldots, R_t^i(J)$ do not contain discontinuities. Thus, for each $k \in \{0, \ldots, i\}$, there is $i_k \geq i_0$ such that $R_t^k(J) \subset I_{i_k}$. Lemma 2.4 implies that $|R_t^k(J)| \geq \ell^k |J|$, $\ell > 1$, for all $k \in \{0, \ldots, i\}$. Since the size of the intervals I_i is upper bounded, this inequality implies that there is a first $m \in \mathbb{N}$ such that $R_t^m(J)$ is not contained in any I_i, thus it intersects the set of discontinuities of R_t. □

3 The maximal invariant set: Proof of Theorem 1.1

Next proposition is the main technical result of this section. Heuristically, it means that the one-dimensional stable manifold of Q topologically behaves as a two-dimensional manifold.

Proposition 3.1. *For every small $t > 0$ and every two-disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^s(Q, f_t) \pitchfork \chi \neq \emptyset$. In particular, $W^s(P, f_t)$ is contained in the closure of $W^s(Q, f_t)$.*

Proof of the inclusion $H(P, f_t) \subset H(Q, f_t)$ in Theorem 1.1. By the definition of $H(P, f_t)$, it suffices to see that any $x \in W^s(P, f_t) \pitchfork W^u(P, f_t)$ is accumulated by homoclinic points of Q. By the configuration of the cycle, $W^u(P, f_t) \subset \text{closure}\,(W^u(Q, f_t))$, thus, given any $x \in H(P, f_t)$ and any $n > 0$, there is a disk Δ_n, contained in $W^u(Q, f_t)$ and in the ball of radius $1/n$ centered at x, whose interior meets transversely $W^s(P, f_t)$. By Proposition 3.1, $\Delta_n \pitchfork W^s(Q, f_t) \neq \emptyset$. Thus there is $y_n \in \Delta_n \cap H(Q, f_t)$. By construction, $y_n \to x$, proving the inclusion $H(P, f_t) \subset H(Q, f_t)$.

3.1 Proof of Proposition 3.1

We now go into the details of the proof of Proposition 3.1. We first introduce some definitions.

- A set $\Delta \subset [-1, 1] \times [-1, 2] \times [-1, 1]$ is a *vertical strip* if $\Delta = \{x_1\} \times [l_1, l_2] \times [r_1, r_2]$, where $l_1 < l_2$ and $r_1 < 0 < r_2$. The segment $\{x_1\} \times [l_1, l_2] \times \{0\}$ is the *basis* of Δ. The *width* and the *height* of Δ are $w(\Delta) = (l_2 - l_1)$ and $h(\Delta) = (r_2 - r_1)$. The strip Δ is *complete* if $r_2 = 1$ and $r_1 = -1$, *well located* if $[l_1, l_2]$ is contained in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.

- A subset $J \subset [-1,1]\times[-1,2]\times[-1,1]$ is a *vertical segment* if $J = \{x_1\}\times\{l_1\}\times[r_1,r_2]$, where $r_1 < 0 < r_2$. The point $(x,\ell_1,0)$ is the basis of J. The height of J is $h(J) = (r_2 - r_1)$. As above, the segment J is *complete* if $r_1 = -1$ and $r_2 = 1$, *well located* if l_1 is in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.

- A vertical segment J (resp. strip Δ) is *at the right of* Q if $l_1 \in (0,1]$.

Given an interval $\alpha \subset [-1,2]$, let $\Delta(\{x\}\times\alpha\times\{0\}) = \{x\}\times\alpha\times[-1,1]$ be the *unique* complete vertical strip with basis $\{x\}\times\alpha\times\{0\}$. Similarly, $J(x,y,0)$ is the unique complete vertical segment with basis $(x,y,0)$. The next algorithm associates to perfect segments and strips their successors:

Algorithm 3.2. *Let $\Delta = \Delta(\{x\}\times\alpha\times\{0\})$ be a perfect strip and define $\mathcal{G}_t(\Delta)$ as the perfect strip such that:*

- *the basis of $\mathcal{G}_t(\Delta)$ is of the form $(\{x'\}\times G_t(\alpha)\times\{0\})$, where $x' = \lambda_s^{k_t}\, x - 1/2$,*

- *$\mathcal{G}_t(\Delta)$ is contained in $f_t^{k_t}(\Delta)$ (where $F^{k_t}(D_t^+) = D_t^-$).*

Suppose now that α does not contain discontinuities, i.e. $\alpha \subset (d_{i+1}, d_i)$ for some i. Define $\mathcal{R}_t(\Delta)$ as the perfect strip such that:

- *the basis of $\mathcal{R}_t(\Delta)$ is of the form $(\{\hat{x}\}\times R_t(\alpha)\times\{0\})$, where $\hat{x} = \lambda_s^i(\lambda_s^{k_t}\, x - 1/2)$,*

- *$\mathcal{R}_t(\Delta)$ is contained in $f_t^{k_t+i}(\Delta)$.*

Similarly, to a perfect segment $J = J(x,y,0)$ we associate perfect segments $\mathcal{G}_t(J)$ and $\mathcal{R}_t(J)$ (provided $y \neq d_i$ for all i).

The strips $\mathcal{G}_t(\Delta)$ and $\mathcal{R}_t(\Delta)$ in Algorithm 3.2 are obtained as follows. Given a set A and a point $x \in A$, let $C(x,A)$ be the connected component of A containing x. Take a small neighborhood V of $f_0^{k_0}(\{0,1\}\times[-1,1]) \subset W^u(P,f_0)$, k_0 as in the definition of the cycle in Section 1, then

$$\begin{aligned}\mathcal{G}_t(\Delta) &= C(f_t^{k_t}(x,y,0), f_t^{k_t}(\Delta)\cap V)\cap[-1,1]^3),\\ \mathcal{R}_t(\Delta) &= C(f_t^{i}(x',y',0), f_t^{k_t+i}(\Delta)\cap V)\cap[-1,1]^3),\end{aligned}$$

where $(x,y,0)$ is any point in the basis of Δ, $x' = (\lambda_s^{k_t}\, x - 1/2)$, and $y' \in G_t(\alpha)$. The construction for the successors of segments is analogous.

Lemma 3.3. *The manifold $W^u(P,f_t)$ contains a perfect segment for all small $t > 0$.*

Proof: Consider the transverse homoclinic point $x_t = (-1/2, t, 0)$ of P. Recall that $t \geq b_t$ and $\beta^{-1}\, t \in D_t^+ = [a_t, b_t]$. Let us assume that $t > b_t$, and thus $\beta^{-1} t \in (a_t, b_t)$ (the case $t = b_t$ follows similarly, so it will be omitted). Consider the complete vertical segment $\mathcal{R}_t(J)$, where $J = J(-(\lambda_s^{-1}/2), \beta^{-1}t, 0) \subset W^u(P,f_t)$. If $R_t(\beta^{-1}t)$ belongs to the interior of D_t^+, then $\mathcal{R}_t(J) \subset W^u(P,f_t)$ is the announced segment. Otherwise, $R_t(\beta^{-1}t) = b_t$ and there is a homoclinic point of P of the form $(x', b_t, 0)$. Using the λ-lemma and the product structure of the cycle, one gets homoclinic points $(x_n, y_n, 0)$ of P and complete segments $J_n = J(x_n, y_n, 0) \subset W^u(P,f_t)$ such that $x_n \to x'$, $y_n \to b_t$, and y_n is increasing. Thus y_n is in the interior of D_t^+ for every big n and $J_n \subset W^u(P,f_t)$ is perfect. □

For clearness we first prove Proposition 3.1 in the following special case:

Proposition 3.4. *Let* $\chi \subset [-1,1] \times [-1,2] \times [-1,1] = \{x\} \times A$*, where* A *is a disk of* $\mathbb{R}^2$ *whose interior contains a point of the form* $(y,0)$ *with* $y \in (0,2)$*. Then* χ *intersects transversely* $W^s(Q,f_t)$*.*

We claim that is enough to prove the Proposition 3.4 for perfect strips:

Lemma 3.5. *Let* Δ *be a perfect strip. Then there is* $k \in \mathbb{N}$ *such that* $f_t^k(\Delta) \pitchfork W^s(Q,f_t) \neq \emptyset$*.*

Proof: Suppose that $\Delta = \Delta(\{x_0\} \times \alpha \times \{0\})$, α in the interior of D_t^+. By Lemma 2.5, there exist y_0 in the interior of α and $k \in \mathbb{N}$ such that $G_t(R_t^k(y_0)) = 0$. Thus the vertical strip $\mathcal{G}_t(\mathcal{R}_t^k(\Delta))$ (contained in the forward orbit of Δ) intersects transversely $[-1,1] \times \{(0,0)\} \subset W^s(Q,f_t)$. $\square$

Proof of Proposition 3.4: By Lemma 3.3, $W^u(P,f_t)$ contains a perfect vertical segment J. Since, by definition, χ meets transversely $W^s(P,f_t)$, the λ-lemma implies that forward orbit of χ contains a sequence of complete strips χ_n accumulating to J. Thus χ_n contains a perfect strip for all n large. Lemma 3.5 implies that χ_n (and thus χ) transversely meets $W^s(Q,f_t)$. $\square$

Proof of Proposition 3.1: We can assume that χ is transverse to $W^s_{loc}(P,f_t)$ and contained in $[-1,1] \times [-1,2] \times [-1,1]$. If χ contains a subset of the form $\{x\} \times A$, where A is an open subset of $\mathbb{R}^2$ containing a point $(0,y)$ with $y \in (0,2)$, Proposition 3.4 implies the result. For the general case, consider a point $(x_0,y_0,0)$, $y_0 \in (0,2)$, in the interior of $\chi \pitchfork W^s_{loc}(P,f_t)$ and for every big n the vertical strip

$$\Sigma_n = \{x_0\} \times [y_0 - 1/n, y_0 + 1/n] \times [-1/n, 1/n].$$

The strips Σ_n verify the hypotheses of Proposition 3.4, hence there is $(x_0,y_n,z_n) \in \Sigma_n \pitchfork W^s(Q,f_t)$ such that $H_n = [-1,1] \times \{(y_n,z_n)\} \subset W^s(Q,f_t)$. Since $(x_0,y_n,z_n) \to (x_0,y_0,0)$, it is immediate that H_n meets transversely χ for all large n, ending the proof of the proposition. $\square$

3.2 The maximal f_t-invariant set

To prove the second part of Theorem 1.1 (the inclusion $\Lambda_t \subset H(Q,f_t)$), let V_0 be the connected component of the neighborhood of the cycle W containing the heteroclinic point $(-1/2,0,0)$. There are two types of points of Λ_t: (a) those points whose orbit does not meet V_0 (i.e., the set $\gamma = \{0\} \times [0,1] \times \{0\} \subset W^s(P,f_t) \cap W^u(Q,f_t)$) and (b) those having an iterate in V_0.

We claim that every point of type (a) belongs to $H(Q,f_t)$: given any $(0,x,0)$, $x \in (0,1)$, consider the disk $\Delta_n = \{0\} \times [x - 1/n, x + 1/n] \times [-1/n, +1/n] \subset W^u(Q,f_t)$ satisfying the hypothesis of Proposition 3.1. Hence $\Delta_n \pitchfork W(Q,f_t) \neq \emptyset$ and thus $\Delta_n \cap H(Q,f_t) \neq \emptyset$. Since this holds for all $n \in \mathbb{N}$, $(0,x,0) \in H(Q,f_t)$.

For points $w \in \Lambda_t$ of type (b), after replacing w by some iterate of it, we can assume that $w \in V_0$. Consider the sequence $(n_i(w))_{i \in \mathcal{I}_t(w)}$ associated to w, where $\mathcal{I}_t(w) \subset \mathbb{Z}$ is an interval in $\mathbb{Z}$, inductively defined as follows: let $n_0(w) = 0$ and, assuming defined $n_j(w)$, $j \geq 0$, we define $n_{j+1}(w)$ as the first integer $k > (n_j(w)+1)$ such that $f_t^k(w) \in V_0$. If the forward orbit of w does not return to V_0 for every $k > (n_j(w)+1)$ then j is the right extreme of $\mathcal{I}_t(w)$. We argue analogously for negative j: assuming defined $n_j(x)$, $j \leq 0$, $n_{j-1}(w)$ is the first negative integer $k < (n_j(w) - 1)$ with $f_t^k(w) \in V_0$. If the backward orbit of w does not return to V_0 for every $k < (n_j(w)-1)$, then j is the left extreme of $\mathcal{I}_t(w)$.

Consider the subset $\mathcal{I}_t^+(\infty)$ (resp. $\mathcal{I}_t^-(\infty)$) of $\Lambda_t \cap V_0$ of points w such that $\mathcal{I}_t(w)$ is not upper (resp. lower) bounded. Let $\mathcal{I}_t^+(b)$ be the subset of $\Lambda_t \cap V_0$ of points w such that $\mathcal{I}_t(w)$ is upper bounded. The set $\mathcal{I}_t^-(b)$ is defined similarly. Let $\mathcal{I}_t^\pm(\infty) = \mathcal{I}_t^+(\infty) \cap \mathcal{I}_t^-(\infty)$ and $\mathcal{I}_t^\pm(b) = \mathcal{I}_t^+(b) \cap \mathcal{I}_t^-(b)$. We borrow from [DR$_2$, Lemma 4.1] the following lemma, whose proof is straightforward:

Lemma 3.6. *For every $t > 0$, $\mathcal{I}_t^+(b) \subset W^s(P, f_t) \cup W^s(Q, f_t)$ and $\mathcal{I}_t^-(b) \subset W^u(P, f_t) \cup W^u(Q, f_t)$.*

Next result immediately follows by observing that f_t (resp. f_t^{-1}) exponentially expands the vertical (resp. horizontal) segments:

Remark 3.7. *Let $w = (x, y, z) \in \mathcal{I}_t^+(\infty)$ (resp. $w \in \mathcal{I}_t^-(\infty)$). Then $\{(x, y)\} \times [z - \varepsilon, z + \varepsilon] \pitchfork W^s(P, f_t) \neq \emptyset$ (resp. $[x - \varepsilon, x + \varepsilon] \times \{(y, z)\} \pitchfork W^u(Q, f_t) \neq \emptyset$) for every $\varepsilon > 0$.*

To prove the inclusion $\Lambda_t \subset H(Q, f_t)$ in Theorem 1.1 we consider the following four cases.

Case (i): $w = (x, y, z) \in \mathcal{I}_t^-(b) \setminus \mathcal{I}_t^+(b) = \mathcal{I}_t^-(b) \cap \mathcal{I}_t^+(\infty)$**.**

By Remark 3.7, there is a sequence $w_n = (x, y, z_n) \in W^s(P, f_t)$ with $w_n \to w$. We claim that $w_n \in H(Q, f_t)$ for all large n. Thus $w \in H(Q, f_t)$. To prove the claim, note that the distances between the backward iterates of w_n and w exponentially decrease, thus $w_n \in \Lambda_t$. Moreover, since $w \in \mathcal{I}_t^-(b)$ we also have that $w_n \in \mathcal{I}_t^-(b)$. By Lemma 3.6, $w, w_n \in W^u(P, f_t) \cup W^u(Q, f_t)$. If $w_n \in W^u(P, f_t)$, then $w_n \in H(P, f_t) \subset H(Q, f_t)$ (recall the first part of Theorem 1.1 proved above) and we are done. Otherwise, $w_n \in W^u(Q, f_t)$ and for each k large, there is a small vertical strip $\Delta_k = \Delta_k(n)$ of diameter less than $1/k$, whose interior is contained in $W^u(Q, f_t)$ and contains w_n. Since $w_n \in W^s(P, f_t)$, $\Delta_k \pitchfork W^s(P, f_t)$. Thus, by Proposition 3.1, $W^s(Q, f_t)$ intersects transversely the interior of Δ_k. Hence, since the interior of Δ_k is contained in $W^u(Q, f_t)$, Δ_k contains a homoclinic point y_k of Q. From $\operatorname{diam}(\Delta_k) \to 0$, we get $y_k \to w_n$, which implies $w_n \in H(Q, f_t)$.

Case (ii): $w = (x, y, z) \notin \mathcal{I}_t^+(b) \cup \mathcal{I}_t^-(b)$**.**

We claim that w is accumulated by points $w_n \in \mathcal{I}_t^+(\infty) \cap \mathcal{I}_t^-(b)$, and the result follows from Case (i). To prove the claim observe that, by Remark 3.7, there is a sequence $w_n = (x_n, y, z) \in W^u(Q, f_t)$ with $w_n \to w$. Since the distances between the forward iterates of w_n and w exponentially decrease, it is immediate to check that $w_n \in \Lambda_t$. This also implies that $w_n \in \mathcal{I}_t^+(\infty)$. Finally, $w_n \in W^u(Q, f_t)$ implies $w_n \in \mathcal{I}_t^-(b)$, ending the proof of the claim.

Case (iii): $w = (x, y, z) \in \mathcal{I}_t^+(b) \setminus \mathcal{I}_t^-(b) = \mathcal{I}_t^+(b) \cap \mathcal{I}_t^-(\infty)$**.**

By Lemma 3.6, $w \in W^s(P, f_t) \cup W^s(Q, f_t)$ and, by replacing w by a forward iterate, we can assume that $w = (x, y, 0)$, $y \geq 0$. Remark 3.7 gives a sequence $w_n = (x_n, y, 0) \in W^u(Q, f_t)$ with $w_n \to w$. For each n, there is a vertical disk $\Delta_n \subset W^u(Q, f_t)$ centered at w_n, of diameter less than $1/n$. Clearly, Δ_n intersects transversely $W^s(P, f_t)$. Thus, by Proposition 3.1, $\Delta_n \pitchfork W^s(Q, f_t) \neq \emptyset$. As in the previous cases, this implies that $\Delta_n \cap H(Q, f_t) \neq \emptyset$ for all n large, thus $w \in H(P, f_t)$.

Case (iv): $w \in \mathcal{I}_t^{\pm}(b)$**.**

By Lemma 3.6, there are four possibilities: (1) $w \in W^s(Q, f_t) \cap W^u(Q, f_t)$, (2) $w \in W^s(P, f_t) \cap W^u(P, f_t)$, (3) $w \in W^s(P, f_t) \cap W^u(Q, f_t)$, and (4) $w \in W^u(P, f_t) \cap W^s(Q, f_t)$. Recall that the intersections above are transverse or quasi-transverse, depending on the case. Hence, in case (1), $w \in H(Q, f_t)$ and, in case (2), $w \in H(P, f_t) \subset H(Q, f_t)$. In case (3), the same proof of $\{0\} \times [0, 1] \times \{0\} \subset H(Q, f_t)$ implies that $w \in H(Q, f_t)$: just observe that for every disk $\Delta \subset W^u(Q, f_t)$ containing w, $W^s(P, f_t) \pitchfork \Delta \neq \emptyset$, thus $\Delta \cap H(Q, f_t) \neq \emptyset$. It still remains the case $w \in W^s(Q, f_t) \cap W^u(P, f_t)$. By replacing w by a forward iterate, we can assume that $w = (x, 0, 0)$, $x \in [-1, 1]$, and the following lemma and Cases (i) and (ii) easily imply Case (4):

Lemma 3.8. *Consider $w \in \Lambda_t$ of the form $w = (x, 0, 0) \in V_0 \cap (W^s(Q, f_t) \cap W^u(P, f_t))$. Then there is a sequence $w_n \to w$ with $w_n \in \mathcal{I}_t^+(\infty)$.*

Proof: For each $n \in \mathbb{N}$, consider the rectangle $R_n(x) = \{x\} \times [0, 1/n] \times [-1/n, 1/n]$.

Claim 3.9. *There exists $\kappa_n \in R_n(x) \cap \Lambda_t^+$ whose forward orbit returns to V_0 infinitely many times.*

Assuming the claim, we now finish the proof of the lemma. As in Remark 3.7, but now considering points in Λ_t^+, we have that the point $\kappa_n = (x_n, y_n, z_n)$ is accumulated by points $\kappa_n^m \in W^u(Q, f_t)$ of the form (x_n^m, y_n, z_n). Since the distances between the forward iterates of κ_n and κ_n^m decrease, the forward orbit of κ_n^m is contained in the neighbourhood W of the cycle and returns infinitely many times to V_0. On the other hand, since $\kappa_n^m \in W^u(Q, f_t)$, its backward orbit also is in W. Thus the whole orbit of κ_n^m is in W, so $\kappa_n^m \in \Lambda_t$ and $\kappa_n^m \in \mathcal{I}_t^+(\infty)$. By Cases (i) and (ii) above, $\kappa_n^m \in H(Q, f_t)$, thus $\kappa_n \in H(Q, f_t)$. Since $\kappa_n \to w$, $w \in H(Q, f_t)$, ending the proof of Case (iv).

To prove Claim 3.9, we need the following fact:

Fact 3.10. *Let $R = R_n(x)$, $1/n < t$. Then there is $i = i(R) \in \mathbb{N}$ such that, for every $j \geq i$, $f_t^j(R)$ contains a rectangle $\Gamma(R, j)$ of the form $\{a\} \times [0, 1/n] \times [-1/n, 1/n]$, $a \in [-1, 0]$.*

Proof: Let N_t be the smaller $i \in \mathbb{N}$ such that $F^i(1/n) \in (1 - t + 1/n, 1)$ and write

$$e = (1 - t + 1/n + g) = F^{N_t}(1/n), \quad g \in (0, t - 1/n).$$

By the definition of the unfolding of the cycle, for each $j \geq 0$, $f_t^{N_t+j}(R_n(x))$ contains the rectangle

$$\{\lambda_s^{N_t+j}(x) - 1/2\} \times [0, t + \lambda^j(g + 1/n - t)] \times [-1, 1] \supset \{\lambda_s^{N_t+j}(x) - 1/2\} \times [0, 1/n] \times [-1, 1],$$

since $t + \lambda^j(g + 1/n - t) \geq t + \lambda^j(1/n - t) \geq 1/n$. This finishes the proof of the fact. ☐

To prove Claim 3.9, consider $R_n(x) = R(0)$ and, using Fact 3.10, let $R(1) = \Gamma(R(0), i(R(0))$. Write $i_0 = i(R_0)$ and $R^1 = f_t^{-i_0}(R(1)) \subset R(0)$. Assume inductively defined numbers $i_0, i_1, \dots, i_{k-1}$ and rectangles $R(0), R(1), \dots, R(k)$ and $R^1, \dots, R^k$ as follows:

- $R(k) = \Gamma(R(k-1), i(R(k-1)))$ and $i_{k-1} = i(R(k-1))$, in particular, $R(k)$ satisfies the hypotheses of Fact 3.10,
- $R^k \subset R^{k-1} \subset \cdots \subset R^1 \subset R(0) = R_n(x)$ and $R^k = f_t^{-i_0 - \cdots - i_{k-1}}(R(k))$.

We define $i_k = i(R(k))$, $R(k+1) = \Gamma(R(k), i_k)$ and $R^{k+1} = f_t^{-i_0 - \cdots - i_k}(R(k+1))$, completing the inductive process. Now it suffices to take any point in the non-empty intersection $\cap_{k \in \mathbb{N}} R^k$. ☐

The proof of Theorem 1.1 is now complete.

4 Saddle-node heterodimensional cycles

In this section, we consider *saddle-node heterodimensional cycles.* For that, in the model heterodimensional cycle in Section 1, we replace the function F (defining the central dynamics) by a one-parameter family of C^2-maps $\Phi_s\colon [-1, 2] \to \mathbb{R}$ such that:

- For every s, the point 1 is an attracting hyperbolic point of Φ_s and Φ_s is affine in a neighborhood of 1 (independent of s). We denote by $0 < \lambda < 1$ the eigenvalue of Φ_s at 1.
- Locally in 0, the map Φ_s is of the form $\Phi_s(x) = x + x^2 - s$. Thus, for $s > 0$, Φ_s has two hyperbolic fixed points $\pm\sqrt{s}$ (an attractor and a repellor) collapsing at $s = 0$. Moreover, for every $s < 0$, Φ_s has no fixed points close to 0.

- Every Φ_s is strictly increasing and has no fixed points different from 1 and $\pm\sqrt{s}$.

In this way, as in Section 1, one has a two-parameter family of diffeomorphisms $f_{t,s}$, the parameters t and s describing the motion of the unstable manifold of P and the unfolding of the saddle-node, respectively. Note that $P = (0,1,0)$ and $S_s^{\pm} = (0, \pm\sqrt{s}, 0)$ $(s \geq 0)$ are fixed points of $f_{t,s}$.

We let $f_t = f_{t,0}$. For the saddle-node $S = (0,0,0)$ of f_t there are defined the stable and unstable manifolds (denoted $W^s(S, f_t)$ and $W^u(S, f_t)$) and the strong stable and unstable manifolds (denoted by $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$). Observe that $W^s(S, f_t)$ and $W^u(S, f_t)$ are two-manifolds with boundary and $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$ have both dimension one. Note that, by construction,

$$\{0\} \times [0,1) \times [-1,1] \subset W^u(S, f_t), \quad [-1,1] \times [-1,0] \times \{0\} \subset W^s(S, f_t),$$
$$[-1,1] \times \{(0,0)\} \subset W^{ss}(S, f_t), \quad \{(0,0)\} \times [-1,1] \subset W^{uu}(S, f_t).$$

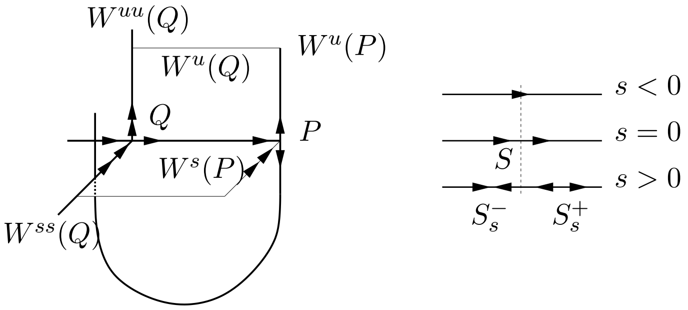

Figure 3: A saddle-node heterodimensional cycle

Keeping in mind these relations, we have that,

- for all t, $W^u(S, f_t)$ meets transversely $W^s(P, f_t)$ throughout the segment $\{0\} \times (0,1) \times \{0\}$,
- for $t = 0$, $W^u(P, f_0)$ meets quasi-transversely $W^{ss}(S, f_0)$ along the orbit of $(-1/2, 0, 0)$,
- for $t > 0$, the point $(-1/2, t, 0)$ is a transverse homoclinic point of P and $(-1/2, 0, 0)$ is a point of transverse intersection between $W^{ss}(S, f_t)$ and $W^u(S, f_t)$.

In this case, we say that the arc $f_t = f_{t,0}$ has a *saddle-node heterodimensional cycle* associated to P and S at $t = 0$. This cycle can be thought as a limit case of the heterodimensional cycles in Section 1, where the derivative of the point of index two Q is 1^+.

The two-fold behavior of the saddle-node S, as a point of index two and one simultaneously, leads us to consider, for small positive t, the *lateral homoclinic classes of* S defined by

$$H^+(S, f_t) = \overline{W^u(S, f_t) \pitchfork W^{ss}(S, f_t)} \quad \text{and} \quad H^-(S, f_t) = \overline{W^s(S, f_t) \pitchfork W^{uu}(S, f_t)}.$$

As in the case of the usual homoclinic classes, we have that:

Proposition 4.1. *For every small $t > 0$, $H^+(S, f_t)$ (resp. $H^-(S, f_t)$) is transitive and the periodic points of index two (resp. one) form a dense subset of it.*

Consider a neighborhood W of the saddle-node heterodimensional cycle defined as in Section 3 and denote by Υ_t the maximal invariant set of f_t in W.

Theorem 4.2. *For every small $t > 0$, one has that $H(P, f_t) \subset H^+(S, f_t)$ and $\Upsilon_t \subset H(S^+, f_t)$.*

The proof of Theorem 4.2 follows as the one of Theorem 1.1, the only difficulty being to redefine appropriately the one-dimensional dynamics associated to the cycle (recall Section 2). This will be briefly done in the next section. To get the inclusion $H^+(S, f_t) \subset H(P, f_t)$ we need the following distortion property for the saddle-node map $\Phi = \Phi_0$.

(SN) Let $K = \max\{|\Phi''(x)|/|\Phi'(x)|,\ x \in [0,1]\} > 0$. Then $\dfrac{4\,e^K\,(1-\lambda)}{\lambda^6} < \dfrac{1}{2}$, where $\lambda \in (2/3, 1)$.

Theorem 4.3. *Under the assumption (SN), $H^+(S, f_t) \subset H(P, f_t)$ holds for all small positive $t > 0$.*

To prove this theorem we need new ingredients that will be introduced in Section 4.3. Theorems 4.2 and 4.3 imply $H^+(S, f_t) = H(P, f_t)$ for all small $t > 0$.

4.1 One-dimensional dynamics for the saddle-node cycle

We now adapt the definitions of scaled fundamental domains, transitions and returns for saddle-node cycles. As in Section 2, for each $t > 0$, define the fundamental domains $D_t^- = [1-t, 1-\lambda\, t]$ and $D_t^+ = [a_t, b_t]$, $a_t = \Phi^{-1}(b_t)$, where D_t^+ is the first backward iterate of D_t^- by Φ contained in $[0, t]$. We have $\Phi^{k_t}(D_t^+) = D_t^-$, for some $k_t \in \mathbb{N}$. Observe that $|D_t^-| = t\,(1-\lambda)$ and, since $b_t \in (0, t]$, $|D_t^+| \le t^2$. For small $t > 0$, define the *transition* $\mathfrak{T}_t$ and the map $\mathfrak{G}_t$ by

$$\mathfrak{T}_t\colon D_t^+ \to D_t^-,\ \ x \mapsto \mathfrak{T}_t(x) = \Phi^{k_t}(x) \quad \text{and} \quad \mathfrak{G}_t\colon D_t^+ \to [0, t(1-\lambda)],\ \ x \mapsto \mathfrak{G}_t(x) = \mathfrak{T}_t(x) + (t-1).$$

Lemma 4.4. *The maps $\mathfrak{T}_t$ and $\mathfrak{G}_t$ are uniformly expanding for all small $t > 0$.*

Proof: It suffices to see that $(\Phi^{k_t})'(z) > 1$ for all $z \in D_t^+$. We use the following standard lemma (whose proof is omitted here):

Bounded Distortion Lemma 4.5. *Let $K > 0$ be as in condition (SN). Then, for every pair of points $z, y \in D_t^+$ and every small $t > 0$,*

$$e^{-K} \le \frac{(\Phi^{k_t})'(z)}{(\Phi^{k_t})'(y)} \le e^K.$$

The lemma now follows by the mean value theorem, taking $y \in D_t^+$ with

$$(\Phi^{k_t})'(y) = |D_t^-|/|D_t^+| \ge (1-\lambda)/t.$$

Thus, if t is small, $(\Phi^{k_t})'(z) \ge (e^{-K}\,(1-\lambda))/t > 1$, for all $z \in D_t^+$. □

As in Section 2, given $x \in (a_t, b_t] = \tilde{D}_t^+$, let $i(x) \in \mathbb{Z}$ be the first i with $\Phi^i(\mathfrak{G}_t(x)) \in D_t^+$. The *return map* $\mathfrak{R}_t$ is now defined by

$$\mathfrak{R}_t\colon \tilde{D}_t^+ \to D_t^+, \quad \mathfrak{R}_t(x) = \Phi^{i(x)}(\mathfrak{G}_t(x)) = \Phi^{i(x)}(\mathfrak{T}_t(x) + (t-1)).$$

Lemma 4.6. *There exists $i_0 > 0$ such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: To prove the lemma it is enough to see that $\mathfrak{G}_t(D_t^+) \subset (0, a_t)$. By definition, $\mathfrak{G}_t(D_t^+) = [0, (1-\lambda)\, t]$. Observe that, if t is small enough,

$$\Phi^2((1-\lambda)\, t) = \Phi((1-\lambda)\, t + (1-\lambda)^2\, t^2) = (1-\lambda)\, t + 2\,(1-\lambda)^2\, t^2 + \text{h.o.t} < t.$$

Thus, the right extreme $\Phi^2((1-\lambda)t)$ of $\Phi^2(\mathfrak{G}_t(D_t^+))$ is less than t. In particular, the right extreme of $\mathfrak{G}_t(D_t^+)$ is less than $\Phi^{-2}(t)$, and the the lemma follows from $D_t^+ \subset (\Phi^{-2}(t), t]$. ☐

As in the case of the return map R_t in Section 2, for each $i \geq i_0$, there is $\delta_i \in \tilde{D}_t^+$ with $\mathfrak{G}_t(\delta_i) = \Phi^{-i}(a_t)$. The points δ_i are the discontinuities of $\mathfrak{R}_t$. In this way, we get a decreasing sequence $(\delta_i)_{i \geq i_0}$ with $\delta_i \to a_t$, and intervals $J_i = [\delta_{i+1}, \delta_i]$, $i > i_0$, and $J_{i_0} = [\delta_{i_0}, b_t]$ such that $\mathfrak{R}_t$ is continuous and increasing in the interior of each J_i. Extending $\mathfrak{R}_t$ continuously to the whole J_i we get a bi-valuated map with $\mathfrak{R}_t(\delta_i) = \{a_t, b_t\}$ for all $i > i_0$.

Lemma 4.7. *The restriction of $\mathfrak{R}_t$ to each interval $J_i, i > i_0$, is onto. Moreover, there is $\ell > 1$ such that $\mathfrak{R}_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = \delta_i$ this means that the lateral derivatives of $\mathfrak{R}_t$ at x are greater than ℓ). Finally, $0 \in \mathfrak{G}_t(\mathfrak{R}_t(\delta_i))$ for all $i \geq i_0$.*

Proof: The lemma follows as Lemma 2.4 observing that $i_0 > 0$ (Lemma 4.6), $\mathfrak{G}_t$ is expanding (Lemma 4.4), and that the derivative of Φ in $(0, t]$ is bigger than one. ☐

Arguing as in Section 2, one gets the following lemma (corresponding to Lemma 2.5):

Lemma 4.8. *Given any subinterval I of D_t^+ there are $x \in I$ and $i \geq 0$ with $\mathfrak{G}_t(\mathfrak{R}_t^i(x)) = 0$.*

4.2 Lateral Homoclinic classes. Proof of Theorem 4.2

To prove Theorem 4.2 we proceed as in Section 3. After redefining vertical strips and segments and using Lemma 4.8, one gets that, for any small $t > 0$ and any disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^{ss}(S, f_t) \pitchfork \chi \neq \emptyset$ (recall Proposition 3.1). The inclusion $(H(P, f_t) \cup \Upsilon_t) \subset H^+(S, f_t)$ follows exactly as $(H(P, f_t) \cup \Lambda_t) \subset H(Q, f_t)$ in the case of heterodimensional cycles. So we omit the details of the proofs of these inclusions.

4.3 Proof of Theorem 4.3: the inclusion $H^+(S, f_t) \subset H(P, f_t)$

Consider the homoclinic point $x_t = (-1/2, t, 0)$ of P for f_t and the fundamental domains of Φ $\Delta_t^+(i) = \Phi^{-i}(\Delta_t^+(0))$, $i \geq 0$, where $\Delta_t^+(0) = [\Phi^{-1}(t), t]$. We now construct a family $\mathcal{H}_t$ of homoclinic points of P for f_t of the form $(x, y, 0)$ such that the set $\{y\colon\ (x, y, 0) \in \mathcal{H}_t\}$ is dense in $\Delta_t^+(0)$:

Proposition 4.9. *For every small $t > 0$ there are sequences of homoclinic points of P of the form $(b_{i_1,i_2,\dots,i_m,k}, x_{i_1,i_2,\dots,i_m,k}, 0)_{k \in \mathbb{N}^*}$, $b_{i_1,i_2,\dots,i_m,k} \in [-1, 0]$, $i_j \in \mathbb{N}^*$, such that*

- **(H1)** $x_{i_1,i_2,\dots,i_m,k} \in \bigcup_{i=0}^4 \Delta_t^+(i) = \Delta_t$,
- **(H2)** $x_{i_1,i_2,\dots,i_m,k} \to x_{i_1,i_2,\dots,i_m}$ *as* $k \to \infty$,
- **(H3)** $x_{i_1,i_2,\dots,i_m,0} < x_{i_1,i_2,\dots,(i_m-1)}$ *for every* $i_m \geq 1$,
- **(H4)** $\operatorname{diam}((x_{i_1,i_2,\dots,i_m,k})_k) \to 0$ *as* $m \to \infty$,
- **(H5)** (x_k) *is increasing and* $x_k \to t^-$ *as* $k \to \infty$,

(H6) $x_0 \in \Delta_t^+(1)$ *and* $x_0 \notin \Delta_t^+(0)$.

This proposition will be proved in Section 4.3.2. From the proposition one gets the following:

Corollary 4.10. *The set* $\mathcal{H}_t = \bigcup_{n,k\in\mathbb{N}^*}(x_{i_1,i_2,\dots,i_k,n})$ *contains a dense subset of* $\Delta_t^+(0)$.

Proof: The proof of is identical to [D$_2$, Lemma 4.1], but we repeat it here for completeness. Take any point x in the interior of $\Delta_t^+(0)$. If $x \in \mathcal{H}_t$ we are done. Otherwise, by (H5) and (H6), there is $i_1 > 1$ with $x_{(i_1-1)} < x < x_{i_1}$. Analogously, by (H2) and (H3), there is $i_2 > 1$ with $x_{i_1,(i_2-1)} < x < x_{i_1,i_2}$. Inductively, using (H2) and (H3) as above, we get a sequence $\{i_k\}$, $i_k > 1$, such that, for all k, $x_{i_1,\dots,(i_k-1)} < x < x_{i_1,\dots,i_k}$. Finally, from (H4), $\lim_{k\to\infty} x_{i_1,\dots,i_k} = x$. ☐

4.3.1 Proof of Theorem 4.3

The deduction of Theorem 4.3 from Corollary 4.10 follows as in [D$_2$, Section 5]. For completeness, we sketch here this proof. Consider any $w \in W^u(S,f_t) \pitchfork W^{ss}(S,f_t)$. By replacing w by some iterate of it, we can assume that $w = (x,0,0)$, $|x|$ small. We prove that, for every $\varepsilon > 0$, the square $S(\varepsilon) = (x-\varepsilon, x+\varepsilon)\times(0,\varepsilon)\times\{0\} \subset W^s(P,f_t)$ intersects transversely $W^u(P,f_t)$. This implies that $w \in H(P,f_t)$. The configuration of the cycle and the λ-lemma imply that there is $n(\varepsilon) > 0$ such that $f_t^{-n(\varepsilon)}(S(\varepsilon))$ contains a disk $S'(\varepsilon)$ of the form

$$S'(\varepsilon) = [-1,1]\times(\bar{y}-\xi,\bar{y}+\xi)\times\{\bar{z}\}, \quad \bar{y}\in(1-t,1),\ \bar{z}\in[-1,1] \text{ and small } \xi > 0.$$

Let $m \in \mathbb{N}$ be such that $\Phi^{-m}(\bar{y}) \in \Delta_t^+(0)$. Thus $f_t^{-m}(S'(\varepsilon))$ contains the horizontal strip

$$\hat{S}(\varepsilon) = [-1,1]\times(\Phi^{-m}(\bar{y}-\xi),\Phi^{-m}(\bar{y}+\xi))\times\{\lambda_u^{-m}\,\bar{z}\} \subset W^s(P,f_t).$$

Since $\Phi^{-m}(\bar{y})$ belongs to $\Delta_t^+(0)$, Corollary 4.10 implies that $\hat{S}(\varepsilon)$ meets $W^u(P,f_t)$. Thus $\hat{S}(\varepsilon)$ contains a homoclinic point of P and the same holds for $S(\varepsilon)$. ☐

4.3.2 Proof of Proposition 4.9: Sequences of homoclinic points:

Let κ_t be the first $k \in \mathbb{N}$ such that $\Phi^k(\Delta_t^+(0)) \subset [1-t,1]$, $\Delta_t^+(0) = [\Phi^{-1}(t),t]$. Since, for small $t > 0$, $|\Phi^{\kappa_t}(\Delta_t^+(0))| \le t\,(1-\lambda)$ and $|\Delta_t^+(0)| \ge \lambda\,t^2$, the Bounded Distortion Lemma 4.5 implies that

$$(\Phi^{\kappa_t})'(x) < \frac{1-\lambda}{\lambda\,t}\,e^K, \quad \text{for all } x\in\Delta_t^+(0). \tag{2}$$

Denote by δ_t^i the length of $\Delta_t^+(i)$. Since the derivative of Φ near 0 is close to 1 and strictly bigger than 1 in $(0,t]$, for small t, we have that

$$\delta_t^0 \ge \delta_t^i \ge \frac{9\,\delta_t^0}{10}, \quad i=1,\dots,4. \quad \text{In particular,} \quad \sum_{i=0}^{4}\delta_t^i \in [4\delta_t^0, 5\delta_t^0]. \tag{3}$$

Consider the interval $[1-\eta_t,1]$, $\eta_t = \delta_t^1 + \delta_t^0$, and let α_t be the first natural number α with

$$\Phi^{\kappa_t+\alpha}(\Delta_t^+(0)) \subset [1-\eta_t,1].$$

Observe that $|\Delta_t^+(0)| = \delta_t^0 < t^2$ and $\delta_t^1 < \delta_t^0$. Thus, for small $t > 0$, $\eta_t < 2\,\delta_t^0 < 2\,t^2 < \lambda\,t$. Since, $\Phi^{\kappa_t}(\Phi^{-1}(t)) \in [1-t, 1-\lambda\,t]$ and $(1-\lambda\,t) < (1-\eta_t)$, we get that $\alpha_t \ge 1$ for all small t. Observe also that $t^2 < \eta_t < 2\,t^2$, where the first inequality follows from (3) and $\delta_t^0 > 3t^2/4$ if t is small enough.

Lemma 4.11. *For every small $t > 0$ it holds $\lambda^{\alpha_t} \le (2\,t)/\lambda$.*

Proof: By definitions of κ_t and α_t, $\Phi^{\kappa_t}(\Delta_t^+(0)) = [1 - e_t^-, 1 - e_t^+]$, where $e_t^- \in [\lambda\, t, t]$, and $\Phi^{\alpha_t}(1 - e_t^-) \in [1 - \eta_t, 1 - \lambda\,\eta_t]$. Thus, since Φ is affine near 1, $\lambda^{\alpha_t}(e_t^-) \in (0, \eta_t]$. Hence, from $t^2 < \eta_t < 2\,t^2$ and $\lambda\, t \le e_t^- \le t$,

$$\lambda^{\alpha_t} \le \frac{\eta_t}{e_t^-} \le \frac{2\,t^2}{\lambda\, t} = \frac{2\,t}{\lambda},$$

ending the proof of the lemma. □

Next lemma is necessary for getting (H4) and along the inductive definition of $(x_{i_1,i_2,\dots,i_m,k})_k$.

Lemma 4.12. $L = \max\{(\Phi^{\kappa_t+\alpha_t+j})'(x);\ x \in \cup_{i=0}^{4}\Delta_t^+(i)$ *and* $j \ge 0\} < \frac{1}{2}$.

Proof: Since Φ is a contraction near 1, it is enough to compute the estimate when $j = 0$. We split the trajectory of a point $x \in \Delta_t^+(i)$ going from $\Delta_t^+(i)$ to $[1 - \eta_t, 1)$ as follows: **(i)** i iterates, $i \le 4$, for x going from $\Delta_t^+(i)$ to $\Delta_t^+(0)$; **(ii)** κ_t iterates for $\Phi^i(x)$ going from $\Delta_t^+(0)$ to $\Phi_t^{k_t}(\Delta_t^+(0))$; and **(iii)** α_t iterates for $\Phi^{\kappa_t+i}(x)$ going from $\Phi_t^{k_t}(\Delta_t^+(0))$ to $[1 - \eta_t, 1]$. This construction involves $(i + \kappa_t + \alpha_t)$ iterations of x by Φ, so we need to remove the last i iterations, corresponding to a contraction by λ^i. We claim that

$$L \le \underbrace{((2\,t+1)^4)}_{(\mathbf{a})}\ \underbrace{\frac{e^K\,(1-\lambda)}{\lambda\, t}\,\lambda^{\alpha_t}}_{(\mathbf{b})}\ \underbrace{\frac{1}{\lambda^4}}_{(\mathbf{c})}\,, \tag{4}$$

corresponding (**a**) to the expansion of the first i iterates by Φ (just observe that in $[0, t]$ the derivative of Φ is upper bounded by $(2\,t + 1)$ and that $i \le 4$), (**b**) to an upper bound of the derivative of $\Phi^{\kappa_t+\alpha_t}$, recall (2), and (**c**) to the i ($i \le 4$) negative iterates of Φ close to 1. By (SN), Lemma 4.11, and the fact that $(2\,t + 1)^4 < 2$ if $t > 0$ is small, we get

$$L \le ((2\,t+1)^4)\,\frac{e^K\,(1-\lambda)}{\lambda\, t}\,\frac{2\,t}{\lambda}\,\frac{1}{\lambda^4} = (2\,t+1)^4\,\frac{2\,(1-\lambda)\,e^K}{\lambda^6} \le 4\,e^K\,\frac{(1-\lambda)}{\lambda^6} < \frac{1}{2},$$

which ends the proof of the lemma. □

Construction of the sequences $(x_{i_1,i_2,\dots,i_m,k})$**.** To construct the sequences $(x_{i_1,i_2,\dots,i_m,k})$, we need the following algorithm about the creation of homoclinic points, which is a consequence of the definition of the unfolding of the heterodimensional cycle.

Algorithm 4.13. *Let $(x, y, 0)$, $x \in [-1, 1]$ and $y \in [0, t]$, be a homoclinic point of P (for f_t) such that $\{(x, y)\} \times [-1, 1] \subset W^u(P, f_t)$. Then, for every m with $\Phi^m(y) \in (1-t, 1)$, there is a homoclinic point of P of the form $(\bar{x}, \Phi^m(y) + t - 1, 0)$ such that $\{(\bar{x}, \Phi^m(y) + t - 1)\} \times [-1, 1] \subset W^u(P, f_t)$.*

Take the homoclinic point $(-1/2, t, 0)$ of P and the sequences $(y_i)_{i\in\mathbb{N}^*}$ and $(x_i)_{i\in\mathbb{N}^*}$ defined by

$$y_i = \Phi^{\kappa_t+\alpha_t+i}(t) \quad \text{and} \quad x_i = (t-1) + y_i, \quad y_i \to 1 \quad \text{and} \quad x_i \to t.$$

Observe that, for each $i \ge 0$, there is a homoclinic point $(b_i, x_i, 0)$ of P verifying Algorithm 4.13. Also, by the definitions of α_t and κ_t, $y_i \in [1 - \eta_t, 1]$ for all $i \ge 0$. Thus, since $\eta_t = \delta_t^0 + \delta_t^1$, one has

$$x_i \in [t - \eta_t, t] = [t - (\delta_t^1 + \delta_t^0), t] = \Delta_t^+(1) \cup \Delta_t^+(0). \tag{5}$$

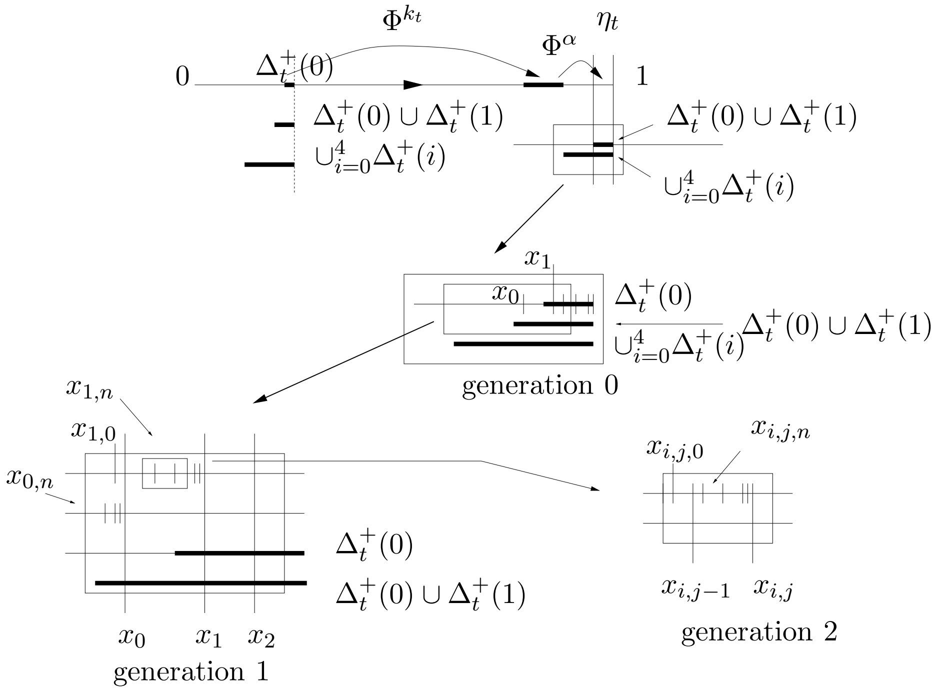

Figure 4: The sequences $(x_{i_1,i_2,\dots,i_m,n})_n$

Lemma 4.14. *The sequence $(x_i)_{i\in\mathbb{N}^*}$ verifies (H5) and (H6).*

Proof: Condition (H5) follows by definition. To get (H6), i.e.$x_0 \in (\Delta_t^+(1)\setminus\Delta_t^+(0))$, note that, by construction, $x_0\in[t-\eta_t, t-\lambda\,\eta_t]$ and $\Delta_t^+(1)=[t-\eta_t, t-\delta_t^0]$. Thus, by (5), it is enough to check that $\lambda\,\eta_t=\lambda\,(\delta_t^0+\delta_t^1)>\delta_t^0$. This inequality follows from $\delta_t^0>\delta_t^1$, (3) and (SN), observing that

$$\frac{\delta_t^0}{\eta_t}=\frac{\delta_t^0}{\delta_t^0+\delta_t^1}<\frac{\delta_t^0}{2\,\delta_t^1}<\frac{\delta_t^0}{2\,(9/10)\,\delta_t^0}=\frac{10}{18}<\frac{2}{3}<\lambda.$$

The proof of the lemma is now complete. □

We now proceed with the construction of the sequences in Proposition 4.9. For each $j\in\mathbb{N}^*$, define the sequences $(y_{j,i})_{i\in\mathbb{N}^*}$ and $(x_{j,i})_{i\in\mathbb{N}^*}$ as follows,

$$y_{j,i}=\Phi^{\kappa_t+\alpha_t+j}(x_i)\quad\text{and}\quad x_{j,i}=(t-1)+y_{j,i}.$$

We claim that $y_{j,i}\to y_j$ and, consequently, $x_{j,i}\to x_j$, as $i\to\infty$. For that just observe that $\lim_{i\to\infty}x_i=t$, thus, by continuity, $\lim_{i\to\infty}y_{j,i}=\lim_{i\to\infty}\Phi^{\kappa_t+\alpha_t+j}(x_i)=\Phi^{\kappa_t+\alpha_t+j}(t)=y_j$.

Lemma 4.15. *The points $(x_{j,i})_i$ belong to $\cup_{i=0}^4\Delta_t^+(i)$ for all $i,j\in\mathbb{N}^*$.*

Proof: Since, by construction, the sequences $(x_{j,i})_i$ are increasing, it is enough to see that $x_{0,0}\in\cup_{i=0}^4\Delta_t^+(i)$. Consider the diameter $d_0=(t-x_0)$ of $(x_i)_{i\in\mathbb{N}^*}$. By (5), $d_0<\delta_t^0+\delta_t^1$. Let $d_1=|x_0-x_{0,0}|$ be the diameter of $(x_{0,i}=\Phi^{\kappa_t+\alpha_t}(x_i)+(t-1))_i$, which is equal to the diameter of $(\Phi^{\kappa_t+\alpha_t}(x_i))_i$. Thus, since $(x_i)_i\subset\Delta_t^+(0)\cup\Delta_t^+(1)$, by Lemma 4.12, the diameter d_1 is bounded by

$$d_1\le L\,d_0<d_0/2<(\delta_t^0+\delta_t^1)/2<(2\,\delta_t^0)/2<\delta_t^0. \tag{6}$$

Since $x_0 \in \Delta_t^+(1)$ (Lemma 4.14), to prove that $x_{0,0} \in \cup_{i=0}^4 \Delta_t^+(i)$, it is enough to see that

$$x_{0,0} = x_0 - d_1 > x_0 - (\delta_t^2 + \delta_t^3 + \delta_t^4) \Longleftrightarrow d_1 < (\delta_t^2 + \delta_t^3 + \delta_t^4),$$

which immediately follows from $\delta_t^2 + \delta_t^3 + \delta_t^4 > \delta_t^0 > d_1$, the first inequality being consequence of $\delta_t^i > (9\,\delta_t^0)/(10)$, see (3), and the last from (6). This ends the proof of the lemma. □

Suppose now inductively defined sequences $(y_{i_i,i_2,\dots,i_m,i})_{i\in\mathbb{N}^*}$ and $(x_{i_i,i_2,\dots,i_m,i})_{i\in\mathbb{N}^*}$ by

$$y_{i_1,i_2,\dots,i_m,i} = \Phi^{\kappa_t+\alpha_t+i_1}(x_{i_2,\dots,i_m,i}) \quad \text{and} \quad x_{i_1,i_2,\dots,i_m,i} = (t-1) + y_{i_1,i_2,\dots,i_m,i},$$

satisfying conditions (H1), (H2), (H3) and

(H4b) Let d_m, $m \geq 0$, be the diameter of the sequence $(x\underbrace{0,\dots,0,i}_{m\,0's})_i$. Then $d_m \leq (d_{m-1})/2$.

Observe that (H2) and (H3) and are equivalent to (H2b) and (H3b) below, respectively,

(H2b) $(y_{i_1,i_2,\dots,i_m,i})_i \to y_{i_i,i_2,\dots,i_m}$ as $i \to \infty$,

(H3b) $y_{i_1,i_2,\dots,i_m,0} < y_{i_i,i_2,\dots,(i_m-1)}$ for all $i_m \geq 1$.

Notice that, for $m = 1$, (H1) follows from Lemma 4.15, (H2) (or (H2b)) from definition, and (H4b) from the estimates in (6). To check (H3b), $y_{i,0} < y_{i-1}$, for every $i \geq 1$, recall that, by Lemma 4.14, $x_0 < \Phi^{-1}(t) < t$, thus

$$y_{i-1} = \Phi^{\kappa_t+\alpha_t+i-1}(t) = \Phi^{\kappa_t+\alpha_t+i}(\Phi^{-1}(t)) > \Phi^{\kappa_t+\alpha_t+i}(x_0) = y_{i,0}.$$

For simplicity, we say that the sequences $(z_{i_i,i_2,\dots,i_m,i})_{i\in\mathbb{N}^*}$, $z = x, y$, are *of generation m*.

Lemma 4.16. *Property (H4b) implies (H4) in Proposition 4.9.*

Proof: By construction, the diameter of any sequence $(x_{i_1,\dots,i_m,k})$ of generation m is bounded by the diameter d_m of $(x\underbrace{0,\dots,0,i}_{m\,0's}))_{i\in\mathbb{N}^*}$. Thus, inductively, $d_m \leq (1/2)\,d_{m-1}$, so $d_m \to 0$. □

Keeping in mind Lemmas 4.14 and 4.16, to prove Proposition 4.9 it suffices to see that the sequences verify (H1), (H2b), (H3b) and (H4b). We argue inductively on the generation of the sequences and assume satisfied these conditions for sequences of generation less than or equal to m. To verify (H2b) for the sequences of generation $m+1$ note that, by induction, $(y_{i_1,i_2,\dots,i_m,i})_i \to y_{i_1,i_2,\dots,i_m}$. Thus, by continuity of Φ and by definition,

$$(y_{j,i_1,i_2,\dots,i_m,i})_i = (\Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m,i}))_i \to \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m}) = y_{j,i_1,i_2,\dots,i_m}.$$

To prove (H3b) observe that, by induction, $x_{i_1,i_2,\dots,i_m,0} < x_{i_1,i_2,\dots,i_m-1}$. Thus, since Φ is increasing,

$$y_{j,i_1,i_2,\dots,i_m,0} = \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m,0}) < \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m-1}) = y_{j,i_1,i_2,\dots,i_m-1}.$$

To check (H4b) observe that, by the induction hypotheses (H1), $x_{0,\dots,0,i} \in \cup_{i=0}^4 \Delta_t^+(i)$, Lemma 4.12 and the fact that the sequences $(y_{0,\dots,0,i})$ and $(x_{0,\dots,0,i})$ have the same diameter imply that

$$d_{m+1} = \mathrm{diam}(\underbrace{(y_{0,\dots,0,i})_i}_{(m+1)\,0's}) = \mathrm{diam}((\Phi^{\kappa_t+\alpha_t}(\underbrace{(x_{0,\dots,0,i})_i)}_{m\,0's}) = L\,\mathrm{diam}((x_{0,\dots,0,i}))_i) = L\,d_m \leq \frac{d_m}{2}.$$

Finally, to get (H1) for the generation $(m+1)$ it is enough to see that, for every m, $\sum_{i=0}^{m} d_m < 4\delta_t^0 < \sum_{i=0}^{4} \delta_t^i$, recall (3). By induction and Lemma 4.14, which implies that $d_0 \leq \delta_t^0 + \delta_t^1$, we have

$$\sum_{i=0}^{m} d_m \leq \sum_{i=0}^{m-1} (1/2)^i \, d_0 \leq \sum_{i=0}^{m-1} (1/2)^i \, (\delta_t^0 + \delta_t^1) \leq \frac{1}{1-2} \, (\delta_t^0 + \delta_t^1) < 2(\delta_t^0 + \delta_t^1) < 4\delta_t^0,$$

finishing the proof of our claim, and the construction of the sequences in Proposition 4.9.

5 Homoclinic classes before collapsing the saddles S_s^+ and S_s^-

We now return to the family $f_{t,s}$ in Section 4. Note that $S_s^+ = (0, \sqrt{s}, 0)$, $s > 0$, is a fixed point of index two of $f_{t,s}$ (any $t > 0$) and that $f_{\sqrt{s},s}$ has a heterodimensional cycle associated to S_s^+ and P:

- $W^u(S_s^+, f_{\sqrt{s},s})$ meets transversely $W^s(P, f_{\sqrt{s},s})$ throughout the segment $\{0\} \times (\sqrt{s}, 1) \times \{0\}$,
- $W^u(P, f_{\sqrt{s},s})$ meets quasi-transversely $W^s(S_s^+, f_{\sqrt{s},s})$ along the orbit of $(-1/2, \sqrt{s}, 0)$ (just observe that $[-1,1] \times \{(\sqrt{s}, 0)\} \subset W^s(S_s^+, f_{\sqrt{s},s})$ and that $(-1/2, \sqrt{s}, 0) \in W^u(P, f_{\sqrt{s},s})$).

In what follows we assume that the saddle-node arc Φ_s verifies condition (SN) in Section 4.

Theorem 5.1. *There exist a small $s_0 > 0$ and a strictly positive map τ defined on $(0, s_0)$ such that, for every $s \in (0, s_0)$ and $t \in (\sqrt{s}, \sqrt{s} + \tau(s))$,*

- *$H(P, f_{t,s}) = H(S_s^+, f_{t,s})$, and*
- *there is a neighborhood W_s of the cycle of $f_{\sqrt{s},s}$ (associated to P and S_s^+) such that the maximal invariant set $\Lambda_{t,s}$ of $f_{t,s}$ in W_s is contained to $H(S_s^+, f_{t,s})$.*

The the inclusion $H(P, f_{t,s}) \subset H(S_s, f_{t,s})$ and the second part of the theorem follow as in Theorem 1.1. So we just sketch these proofs. For a fixed $s > 0$ and $t > \sqrt{s}$, t close to $\sqrt{s}$, $t = \sqrt{s} + \tau$, consider the scaled fundamental domains $D_{t,s}^{\pm}$ of Φ_s,

$$D_{t,s}^- = [1 - (t - \sqrt{s}), 1 - \lambda\,(t - \sqrt{s})] = [1 - \tau, 1 - \lambda\,\tau]$$

and $D_{t,s}^+$ defined as the first backward iterate of $D_{t,s}^-$ in $[\sqrt{s}, \sqrt{s} + \tau]$. Let $\Phi_s^{k_{t,s}}(D_{t,s}^+) = D_{t,s}^-$, where $k_{t,s} \in \mathbb{N}$. These domains play the role of $D_t^{\pm}$ in Section 2. Observe that

$$\ell(t,s) = \frac{|D_{t,s}^-|}{|D_{t,s}^+|} \geq \frac{\tau\,(1-\lambda)}{(\sqrt{s}+\tau)^2 - s} = \frac{\tau\,(1-\lambda)}{\tau\,(2\sqrt{s}+\tau)} = \frac{1-\lambda}{2\,\sqrt{s}+\tau}.$$

By shrinking s, we can assume that $|\Phi_s''(x)|/|\Phi_s'(x)| < 2\,K$ for all $x \in [-1, 2]$ (K as in condition (SN)). Thus, there is $s_0 > 0$ and a map $\tau\colon (0, s_0) \to \mathbb{R}^+$ such that

$$\ell(t,s)\, e^{-2\,K} > 2, \quad \text{for all } s \in (0, s_0) \text{ and } t \in (\sqrt{s}, \sqrt{s} + \tau(s)). \tag{7}$$

Exactly as in Section 2, for $s \in (0, s_0)$ and $t \in (\sqrt{s}, \sqrt{s} + \tau(s))$, we define maps

$$\begin{aligned}
&T_{t,s}\colon D_{t,s}^+ \to D_{t,s}^-, \quad T_{t,s}(x) = \Phi_s^{k_{t,s}}(x),\\
&G_{t,s}\colon D_{t,s}^- \to [\sqrt{s}, \sqrt{s} + \tau\,(1-\lambda)], \quad G_{t,s}(x) = T_{t,s}(x) + (t-1),\\
&R_{t,s}\colon D_{t,s}^+ \to D_{t,s}^+, \quad R_{t,s}(x) = \Phi_s^{i(x)}(G_{t,s}(x)),
\end{aligned}$$

where, as in Section 2, $i(x)$ is the first forward iterate of $G_{t,s}(x)$ by Φ_s in $D^+_{t,s}$. As in Lemma 4.4, the Bounded Distortion Lemma 4.5 and equation (7) imply that $T_{t,s}$, $G_{t,s}$ and $R_{t,s}$ are uniformly expanding. The inclusion $(H(P, f_{t,s}) \cup \Lambda_{t,s}) \subset H(S^+_s, f_{t,s})$ now follows as in Theorem 1.1.

To get $H(S^+_s, f_{t,s}) \subset H(P, f_{t,s})$ recall first that Theorem 4.3 gives small $\bar{t} > 0$ with $H^+(S, f_{t,0}) \subset H(P, f_{t,0})$ for all $t \in (0, \bar{t})$. This inclusion follows by constructing the sequences of homoclinic points of P in Proposition 4.9. The proof Proposition 4.9 only involves distortion control of the saddle-node map in $[0, 1]$ and the contracting itineraries in Lemma 4.12. Clearly, these properties hold after replacing, for small positive s, the saddle-node S by the hyperbolic point S^+_s and considering the restriction of the saddle-node map to $[\sqrt{s}, 1]$. This ends the sketch of the proof of Theorem 5.1.

6 Collision, explosion and collapse of homoclinic classes

In this section we prove Theorems A and B. Consider a two parameter family of diffeomorphism $f_{t,s}$ locally defined as follows:

Partially hyperbolic local dynamics:

- In the set $C = [-1, 1]\times[-2, 2]\times[-1, 1]$, $f_{t,s}(x, y, z) = (\lambda^s x, \Psi_s(y), \lambda^u z)$, where $0 < \lambda^s < 1 < \lambda^u$ and $\Psi_s \colon [-2, 2] \to (-3, 2)$ is a strictly increasing C^2-map such that $\lambda^s < d_m < \Psi'_s(y) < d_M < \lambda^u$ for all $y \in [-2, 2]$ and small $|s|$.
- $\Psi_s(1) = 1$ and $\Psi_s(-1) = -1$ for all s and Ψ_s is affine and independent of s in $[-1-\delta, -1+\delta]$ and $[1-\delta, 1+\delta]$, for some small $\delta > 0$. Furthermore, $\Psi'_s(-1) = \beta > 1 > \Psi'_s(1) = \lambda > 0$.
- There is $\delta > 0$ such that the restriction of Ψ_s to $[-\delta, \delta]$ is of the form $\Psi_s(x) = x + x^2 - s$.
- For $s < 0$, Ψ_s has (exactly) two fixed points (± 1), for $s = 0$, Ψ_0 has three fixed points (± 1 and 0), and, for $s > 0$, Φ_s has four fixed points (± 1 and $\pm\sqrt{s}$). Let $P = (0, 1, 0)$, $Q = (0, -1, 0)$, $S = (0, 0, 0)$ and, for positive s, $S^{\pm}_s = (0, \pm\sqrt{s}, 0)$, the fixed points of $f_{t,s}$ in C.

Existence and unfolding of cycles:

- There are $k_0 \in \mathbb{N}$ and small neighborhoods of $(0, -1, -1/2)$, $(0, 0, -1/2)$ and $(0, 1, -1/2)$ such that, for each small $|s|$, in such neighborhoods, each $f^{k_0}_{t,s}$ is the translation $f^{k_0}_{t,s}(x, y, z) = (x - 1/2, y - 1 + t, z + 1/2)$, recall the definition of f_t in Section 1.

As in previous sections, we have:

- for $(t, s) = (0, 0)$, $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, associated to P and S, $W^u(P, f_{0,0})$ meets quasi-transversely $W^{ss}(S, f_{0,0})$, and to Q and S, $W^s(Q, f_{0,0})$ intersects $W^{uu}(S, f_{0,0})$, (note that $[-1, 1]\times\{(0, 0)\} \subset W^{ss}(S, f_{0,0})$ and $\{(0, 0)\}\times[-1, 1] \subset W^{uu}(S, f_{0,0})$).
- For small $|t|$ and $s < 0$, the homoclinic classes of P and Q are both nontrivial: notice that, for negative s, $[-1, 1] \times (-1, 2) \times \{0\} \subset W^s(P, f_{t,s})$ and $\{0\} \times (-1, 2) \times [-1, 1] \subset W^u(Q, f_{t,s})$, thus $(-1/2, t, 0)$ and $(-1/2, -1, 0)$ are homoclinic points of P and Q, respectively.
- $f_{\sqrt{s},s}$ has a pair of heterodimensional cycles associated to P and S^+_s and to Q and S^-_s (this is obtained exactly as in Section 5).

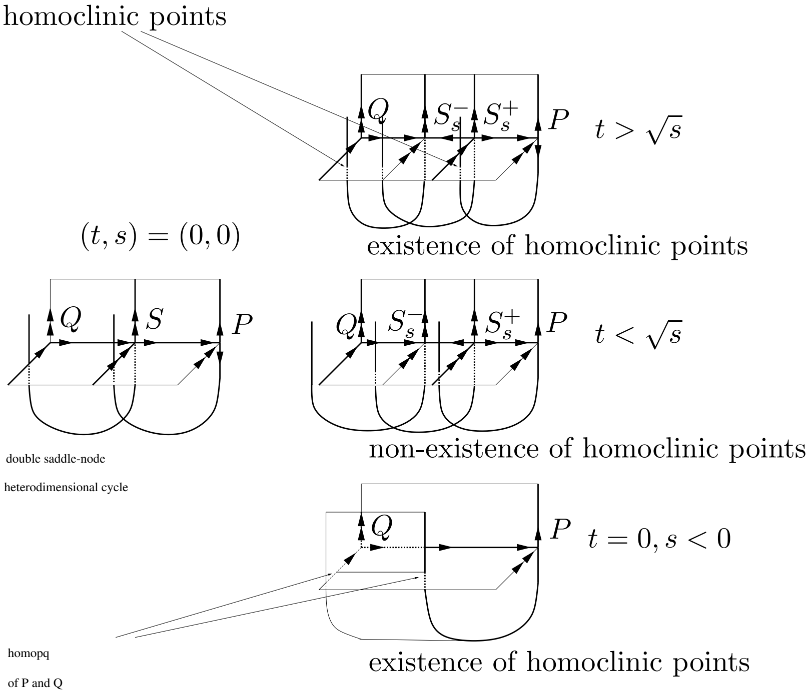

Figure 5: The two parameter family $f_{t,s}$

As in Section 4, we assume the following distortion property (similar to (SN)). Let K be an upper bound for $|\Psi_s''(x)|/|\Psi_s'(x)|$, for small $|s|$ and $x \in [-1, 1]$,

$$\textbf{(DS)}\ \max\left\{\frac{4\,e^K\,(1-\lambda)}{\lambda^6}, \frac{4\,e^K\,(1-\beta^{-1})}{\beta^{-6}}\right\} < \frac{1}{2}, \text{ where } 3/4 < \lambda < 1 < \beta < 4/3.$$

6.1 Dynamics before collapsing the saddles S_s^+ and S_s^-

Theorems 4.2 and 4.3 give a small $\bar{t}_0$ such that $H(P, f_{\bar{t},0}) = H^+(S, f_{\bar{t},0})$ and $H(Q, f_{\bar{t},0}) = H^-(S, f_{\bar{t},0})$ (for this inclusion consider $f_{\bar{t},0}^{-1}$) for all $\bar{t} \in (0, \bar{t}_0]$. These proofs only involve the following ingredients:

- The inclusions $H(P, f_{t_0,0}) \subset H^+(S, f_{t,0})$ and $H(Q, f_{t,0}) \subset H^-(S, f_{t,0})$ are obtained considering the ratio between the lengths of the scaled fundamental domains at the hyperbolic point and at the saddle-node and using that such a ratio is arbitrarily large.
- The inclusion $H^+(S, f_{t,0}) \subset H(P, f_{t_0,0})$ (resp. $H^-(S, f_{t,0}) \subset H(Q, f_{t_0,0})$) is obtained by constructing sequences of homoclinic points of P (resp. Q) verifying Proposition 4.9. The proof of such a proposition only involves distortion control of the saddle-node map in $[0, 1]$ (resp. $[-1, 0]$) and contracting itineraries (Lemma 4.12).

These properties hold after replacing, for small $s > 0$, the saddle-node S by the saddle S_s^+ (considering the restriction of Ψ_s to $[\sqrt{s}, 1]$) and S by the saddle S_s^- (considering the restriction of Ψ_s to $[-1, -\sqrt{s}]$). In this way, we get:

Theorem 6.1. *If $t > 0$ is small, there is a small $s(t) > 0$ such that $H(P, f_{t,s}) = H(S_s^+, f_{t,s})$ and $H(Q, f_{t,s}) = H(S_s^-, f_{t,s})$ for all $s \in (0, s(t))$.*

6.2 Dynamics after collapsing the saddles S_s^+ and S_s^-

Theorem 6.2. *For every small $t > 0$ there is $s(t) < 0$ such that $H(P, f_{t,s}) = H(Q, f_{t,s})$ for all $s \in [s(t), 0)$.*

This theorem follows adapting the constructions in Theorem 1.1. We prove the inclusion $H(P, f_{t,s}) \subset H(Q, f_{t,s})$ ($H(Q, f_{t,s}) \subset H(P, f_{t,s})$ follows by taking $f_{t,s}^{-1}$). As in Sections 2 and 4.1, define transitions $\mathcal{T}_{t,s}$ and returns $\mathcal{R}_{t,s}$ as follows: take the fundamental domain $D_t^- = [1-4\,t, 1-4\,\lambda\,t]$ of Ψ_s and let $k_{t,s}$ be the first $k \in \mathbb{N}$ with $-2\,t \in \Psi_s^{-k}(D_t^-)$. Let $D_{t,s}^+ = \Psi_s^{-k_{t,s}}(D_t^-)$ and define

$$\mathcal{T}_{t,s} \colon D_{t,s}^+ \to D_t^-,\ \ \mathcal{T}_{t,s}(x) = \Psi_s^{k_{t,s}}(x), \qquad \mathcal{G}_{t,s} \colon D_{t,s}^+ \to [-3\,t, -2\,t],\ \ \mathcal{G}_{t,s}(x) = \mathcal{T}_{t,s}(x) + (t-1).$$

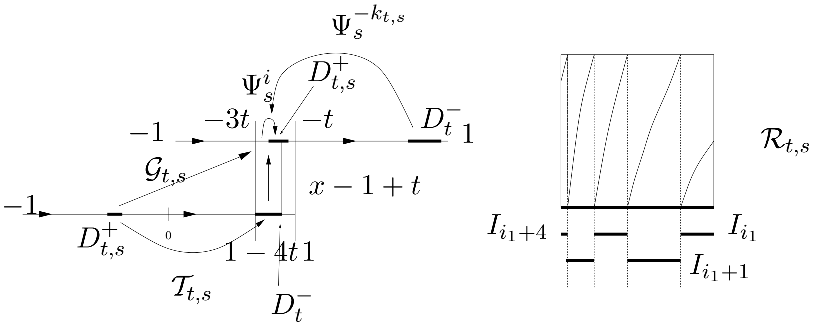

Figure 6: The expanding maps $\mathcal{T}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$

By definition, $\mathcal{G}_{t,s}(D_{t,s}^+) \subset [-3\,t, t\,(1-4\,\lambda)] \subset [-3\,t, -2\,t]$ (recall that $\lambda > 3/4$). Since, $|D_{t,s}^+| \leq 9\,t^2 + s$, the Bounded Distortion Lemma 4.5 and $|D_t^-| = 4\,\lambda\,(1-t)$ immediately give

$$\mathcal{T}_{t,s}'(x) \geq (e^{-2\,K})\,\frac{|D_t^-|}{|D_{t,s}^+|} \geq (e^{-2\,K})\,\frac{4\,\lambda(1-t)}{9\,t^2 + s}.$$

This inequality immediately implies the following:

Lemma 6.3. *For every small $t > 0$ there is $s(t) < 0$ such that $\mathcal{T}_{t,s}$ and $\mathcal{G}_{t,s}$ are 63-expanding for all $s \in [s(t), 0)$.*

Since, by definition, the right extreme of $\mathcal{G}_{t,s}(D_{t,s}^+)$ is less than the right extreme of $D_{t,s}^+$, for each $x \in D_{t,s}^+$ there is a first $i(x) \geq 0$ with $\Psi_s^{i(x)}(\mathcal{G}_{t,s}(x)) \in D_{t,s}^+$. The return map $\mathcal{R}_{t,s}$ is defined by

$$\mathcal{R}_{t,s} \colon D_{t,s}^+ \to D_{t,s}^+, \quad \mathcal{R}_{t,s}(x) = \Psi_s^{i(x)} \mathcal{G}_{t,s}(x).$$

Lemma 6.4. *The map $\mathcal{R}_{t,s}$ is 3-expanding for all small $t > 0$ and $s \in [s(t), 0)$.*

Proof: Note that the expansion for $\mathcal{R}_{t,s}$ does not follow immediately from the expansion of $\mathcal{T}_{t,s}$: the $i(x)$ iterates by Ψ_s at the left of 0 introduce a contraction. By Lemma 6.3, it is enough to see that this contraction is at most $1/21$. Recall that $\mathcal{G}_{t,s}(D^+_{t,s}) \subset [-3\,t, -2\,t]$ and observe that $0 \leq i(x) \leq i_0$, where $\Psi^{i_0}(-3\,t) \in D^+_{t,s}$. Write $D^+_{t,s}(j) = \Psi_s^{-j}(D^+_{t,s})$ and note that, for every $x \in D^+_{t,s}$, $|D^+_{t,s}(i(x))| \leq |D^+_{t,s}(i_0)| < 10\,t^2 + s$. For each $i = i(x)$ there is $z_i \in D^+_{t,s}(i)$ such that

$$(\Psi^i_s)'(z_i) = \frac{|D^+_{t,s}|}{|D^+_{t,s}(i)|} \geq \frac{|D^+_{t,s}|}{|D^+_{t,s}(i_0)|} \geq \frac{|D^+_{t,s}|}{10\,t^2+s}.$$

Using the arguments in the Bounded Distortion Lemma 4.5, we have that, for every $x \in D^+_{t,s}(i)$,

$$(\Psi^i_s)'(x) \geq \frac{e^{-\,K\,L_t}\,|D^+_{t,s}|}{10\,t^2+s},$$

where $L_t = 2\,t$ is the length of the interval $[-3\,t, -t]$ where the iterations by Ψ_s are considered. Finally, since $|D^+_{t,s}| > (t^2)/2$,

$$(\Psi^{i(x)}_s)'(x) \geq \frac{e^{-K\,2\,t}\,|D^+_{t,s}|}{10\,t^2+s} > \frac{e^{-K\,2\,t}\,t^2}{20\,t^2+2s} > \frac{1}{21},$$

for every small t and $s \in [s(t), 0)$, ending the proof of the lemma. $\square$

Let $D^+_{t,s} = [e^-_{t,s}, e^+_{t,s}]$. Note that there are $i_1, i_2 \in \mathbb{N}$ such that $i(x) \in [i_1, i_2]$ for all $x \in D^+_{t,s}$ and, for each $i \in [i_1, i_2 - 1]$, there is $d_i \in D^+_{t,s}$ with $\Psi_s(\mathcal{G}_{t,s}(d_i)) = e^-_{t,s}$. As in Section 2, $d_{i_2-1} < d_{i_2-2} < \cdots < d_{i_1}$ and the d_i are the discontinuities of $\mathcal{R}_{t,s}$. Moreover, $\mathcal{G}_{t,s}((d_{i+1}, d_i)) = \operatorname{int}(D^+_{t,s})$ and $\mathcal{G}_{t,s}$ is increasing in (d_{i+1}, d_i). Write $I_{i_1} = [d_{i_1}, e^+_{t,s}]$, $I_i = [d_i, d_{i-1}]$, $i_1 < i \leq i_2 - 2$, and $I_{i_2} = [e^-_{t,s}, d_{i_2-1}]$. We now continuously extend $\mathcal{R}_{t,s}$ to the closed intervals I_i (so $\mathcal{R}_{t,s}$ is bivaluated at any d_i).

Lemma 6.5. *Given any subinterval J of $D^+_{t,s}$, there is $m \geq 0$ such that $\mathcal{R}^m_{t,s}(J) = D^+_{t,s}$.*

Proof: The proof follows as in Lemma 2.5. It is enough to see $\mathcal{R}^m_{t,s}(J)$ contains an interval I_i, $i_1 < i < i_2$, for some $m \in \mathbb{N}^*$. Write $J = J_0$. If $\mathcal{R}_{t,s}(J_0)$ contains two discontinuities we are done. Otherwise, $\mathcal{R}_{t,s}(J_0) \subset I_i \cup I_{i-1}$ for some i. Write $J_1^- = \mathcal{R}_{t,s}(J_0) \cap I_{i-1}$ and $J_1^+ = \mathcal{R}_{t,s}(J_0) \cap I_i$, and let J_1 be the biggest $J_1^\pm$. By Lemma 6.4, $|J_1| > (3/2)\,|J_0|$. Inductively, one gets intervals J_i contained in the orbit of J_0 such that either J_{i+1} contains two discontinuities or $|J_{i+1}| \geq (3/2)^i |J_0|$. Since the size of $D^+_{t,s}$ is finite, this ends the proof of the lemma. $\square$

Lemma 6.5 is the main step to prove Theorem 6.2, whose proof follows arguing as in the proof of Theorem 1.1 after proving the following:

Proposition 6.6. *Let $\chi = \{x\} \times A \subset [-1,1] \times [-1,2] \times [-1,1]$, where $x \in [-1,1]$ and A is a disk of $\mathbb{R}^2$ whose interior contains a point $(y,0)$ with $y \in (-1,2)$. Then $\chi \pitchfork W^s(Q, f_{t,s}) \neq \emptyset$.*

This result corresponds to Proposition 3.4. After proving it, Theorem 6.2 is deduced as follows. As in Section 3, Proposition 6.6 implies that $W^s(Q, f_{t,s})$ meets transversely every two-disk χ with $W^s(P, f_{t,s}) \pitchfork \chi \neq \emptyset$. Now, arguing as in Section 3.1, we get $H(P, f_{t,s}) \subset H(Q, f_{t,s})$. We now prove Proposition 6.6. The first step is the next lemma, corresponding to Lemma 3.3 in the proof of Propositions 3.1 and 3.4 (its proof follows exactly as Lemma 3.3, so it will be omitted).

Lemma 6.7. *There are $x \in [-1,1]$ and $y \in int(D^+_{t,s})$ such that $\{(x,y)\} \times [-1,1] \subset W^u(P, f_{t,s})$.*

Lemma 6.8. *For every $x \in [-1,1]$, $W^s(Q, f_{t,s})$ meets transversely $\{x\} \times D^+_{t,s} \times [-1,1]$.*

Proof: Observe that, by construction, the point $(0, -t, -1/2)$ belongs to $W^s(Q, f_{t,s}) \cap W^u(Q, f_{t,s})$. Define $j > 0$ by $\Psi_s^{-j}(-t) \in D^+_{t,s}$. It is now immediate that

$$H = [-1,1] \times \{(\Psi_s^{-j}(0), -\lambda_u^{-j}(1/2))\} \subset W^s(Q, f_{t,s}) \quad \text{and} \quad H \pitchfork (\{x\} \times D^+_{t,s} \times [-1,1]) \neq \emptyset,$$

ending the proof of the lemma. □

We are now ready to prove Proposition 6.6. By the λ-lemma and Lemma 6.7, the forward orbit of the disk χ contains a strip Δ of the form $\{x\} \times [a,b] \times [-1,1]$, where $x \in [-1,1]$, and $\alpha_0 = [a,b] \subset D^+_{t,s}$. Using the map $\mathcal{R}_{t,s}$ and arguing as in Section 3.1, we inductively define strips $\Delta_k = \{x_k\} \times \alpha_k \times [-1,1]$, $x_k \in [-1,1]$ and $\alpha_k \subset D^+_{t,s}$, such that $\Delta_{k+1} \subset f^{n_k}_{t,s}(\Delta_k)$ and $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$. By Lemma 6.5, there is a first $k \in \mathbb{N}$ such that $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$ contains $D^+_{t,s}$. By Lemma 6.8, $W^s(Q, f_{t,s}) \pitchfork \Delta_{k+1} \neq \emptyset$, thus $W^s(Q, f_{t,s}) \pitchfork \Delta \neq \emptyset$, ending the proof of the proposition.

6.3 End of the proof of Theorem A

We now construct a one-parameter family of diffeomorphisms (g_s) satisfying Theorem A. For that consider the arc $f_{t,s}$ defined as in the beginning of Section 6. We fix small $\bar{t} > 0$ and consider the arc $g_s = f_{\bar{t},-s}$. The results in the previous section imply that

- for every $s < 0$, $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and disjoint (Theorem 6.1),
- $s = 0$, $H(P, g_0) = H^+(S, g_0)$ and $H(Q, f_0) = H^-(S, g_0)$, (Theorems 4.2 and 4.3),
- for every $s > 0$, $H(P, g_s) = H(Q, g_s)$, (Theorem 6.2).

To finish the proof of Theorem A we need to see that $\{S\} \in H(P, g_s) \cap H(Q, g_s)$ and to describe the maximal invariant set of g_s in the neighborhood W of the cycle. We assume that W is a *level of a fitration* of $f_{0,0}$ (thus, by continuity and compacity, it is also a level of a filtration for $f_{t,s}$ for every small $|s|$ and $|t|$): there are compact sets M_2 and M_1, $M_1 \subset \text{int}(M_2)$, such that $M_2 \setminus M_1 = W$ and $f_{0,0}(M_i) \subset \text{int}(M_i)$, $i = 1, 2$. Hence, if $x \in W$ and $f^i_{t,s}(x) \notin W$ for some i, then x is wandering: suppose, for instance, that $f^{i_0}_{t,s}(x) \in \text{int}(M_1)$, where $i_0 > 0$. Then, there is a neighborhood $U_x \subset W$ of x with $f^{i_0}_{t,s}(U_x) \subset M_1$. By the definition of the filtration, $f^{i_0+j}_{t,s}(U_x) \subset M_1$ for all $j \geq 0$. Thus $f^{i_0+j}_{t,s}(U_x) \cap U_x = \emptyset$ for all $j \geq 0$. By shrinking U_x, we have that $f^j_{t,s}(U_x) \cap U_x = \emptyset$ for all $j > 0$, and x is wandering.

Using the definition of the arc $f_{t,s}$, it is immediate to check the following:

Remark 6.9. *Let $\Lambda_{t,s}$ be the maximal invariant set of $f_{t,s}$ in W. Then, for $s > 0$ small,*

- *Every point $(x, y, s) \in C = [-1,1] \times [-2,2] \times [-1,1]$ with $y \in (-\sqrt{s}, \sqrt{s})$ (resp, $y \in [-2,-1)$ or $y \in (1,2)$) is wandering.*
- *Consider $w = (x,y,z) \in C \cap \Lambda_{t,s}$ and, for $i \in \mathbb{Z}$, let $w_i = g^i_s(w) = f^i_{\bar{t},-s}(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in [\sqrt{s}, 1]$ and $y_j \in [-2, \sqrt{s})$ for some $j > 0$. Then, for every $n \geq j$ with $w_n \in C$, $y_n \in [-2, \sqrt{s})$.*

Consider now $s = 0$.

- *Every point $(x, y, s) \in C$ with $y \in [-2, -1)$ or $y \in (1, 2)$ is wandering.*
- *Consider $w = (x, y, z) \in C \cap \Lambda_{t,0}$ and, for $i \in \mathbb{Z}$, let $w_i = g_0^i(w) = f_{\bar{t},0}^i(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in (0, 1]$ and $y_j \in [-2, 0)$ for some $j > 0$. Then, for every $n \geq j$ with $w_n \in C$, $y_n \in [-2, 0)$.*

For $s \geq 0$ let $\Lambda^+_{\bar{t},s}$ (resp. $\Lambda^-_{\bar{t},s}$) be the set of points $w \in \Lambda_{\bar{t},s} \cap \Omega(f_{\bar{t},s})$ such that $y_i \in [\sqrt{s}, 1]$ (resp. $y_i \in [-1, \sqrt{s}]$) for all $i \in \mathbb{Z}$ with $w_i = (x_i, y_i, z_i) \in C$. Remark 6.9 implies that, for every small $s \geq 0$,

$$\Lambda_{\bar{t},s} \cap \Omega(f_{\bar{t},s}) = \Lambda^+_{\bar{t},s} \cup \Lambda^-_{\bar{t},s}.$$

Arguing as in Section 3, one gets $H(P, f_{\bar{t},s}) = \Lambda^+_{\bar{t},s}$ and $H(Q, f_{\bar{t},s}) = \Lambda^-_{\bar{t},s}$. Observe that, for positive s, one needs to exclude the segment $\{0\} \times (-\sqrt{s}, \sqrt{s}) \times \{0\} \subset \Lambda_{\bar{t},s}$ consisting of wandering points.

Finally, for the saddle-node parameter $s = 0$, it is immediate that $\Lambda^+_{\bar{t},0} \cap \Lambda^-_{\bar{t},0} = \{S\}$.

For parameters $s < 0$ the result follows similarly (but now the situation is much more simple).

6.4 Proof of Theorem B

Clearly, the homoclinic classes $H(P, f_{\bar{t},0})$ and $H(Q, f_{\bar{t},0})$ are not saturated. We claim that there is not any transitive saturated set Σ containing $H(P, f_{\bar{t},0})$. Otherwise, the set Σ must also contain $H(Q, f_{\bar{t},0})$. Thus Σ contains the whole $\Lambda_{\bar{t},0}$. Using the filtration, one has $\Sigma = \Lambda_{\bar{t},0}$. But this set is not transitive: Remark 6.9 implies that there is no orbit going from a small neighborhood U_Q of Q to a small neighborhood U_P of P and thereafter returning to U_Q. This contradiction ends the proof of the theorem.

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Heterodimensional tangencies

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Abstract

We consider C^1-diffeomorphisms f defined on three dimensional manifolds having a pair of saddles P_f and Q_f (of unstable indices one and two) whose homoclinic classes coincide persistently. We prove that if the two-dimensional stable manifold of P_f and the two-dimensional unstable manifold of Q_f have some non-transverse intersection (a heterodimensional tangency) the unfolding of such a tangency leads to diffeomorphisms h such that the homoclinic class of Q_h (the continuation of Q_f for h) is robustly non-dominated. This leads to the phenomena of (C^1-locally generic) coexistence of infinitely many sinks or sources and, in some relevant cases, to the coexistence of infinitely many minimal Cantor sets. We give examples where the previous dynamical configuration occurs, providing a natural transition from partially hyperbolic to robustly non-dominated dynamics.

Keywords: dominated splitting, heterodimensional cycle, heterodimensional tangency, homoclinic class, index of a saddle, robustly transitive set.
MSC 2000: 37C05, 37C20, 37C25, 37C29, 37D30.

1 Introduction

The main motivation of this paper comes from the so-called *Newhouse C^2-generic coexistence phenomenon*: given any surface M, there are a non-empty open set $\mathcal{U} \subset \mathrm{Diff}^2(M)$ and a residual subset $\mathcal{R}_{\mathcal{U}}$ of it, such that every diffeomorphism $f \in \mathcal{R}_{\mathcal{U}}$ has infinitely many sinks, see [22, 23]. On one hand, these results rely on the construction of hyperbolic sets having persistent homoclinic tangencies. On the other hand, the construction of these hyperbolic sets with persistent tangencies is based on the creation of thick horseshoes in the unfolding of homoclinic tangencies. Shortly, [23] claims that *any* homoclinic tangency leads to the C^2-Newhouse coexistence phenomenon. We will discuss briefly the Newhouse's construction, after stating our results, in the end of this introduction.

We point out that the ingredients of the Newhouse's construction are typically C^2 (bounded distortion and fractal dimensions). Moreover, the results in [37] are an indication that those results cannot be adapted to the C^1-topology. In fact, the extension of Newhouse's construction to the

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C^1-topology is closely related to the Smale's Conjecture on the C^1-density of hyperbolic surface diffeomorphisms (Axiom A and no-cycles condition)[1].

On the other hand, in dimension greater than or equal to three and in the C^1-topology, there are open sets of diffeomorphisms satisfying the Newhouse generic coexistence phenomenon: existence of an open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks, see [6, 7][2]. These examples rely on the construction of persistently *non-dominated homoclinic classes* (see Definitions 1.1 and 1.7). There is the following dichotomy for homoclinic classes of C^1-generic diffeomorphisms: either a homoclinic class has a *dominated splitting* or it is contained in the closude of an infinite set of sinks or sources, see [8]. Therefore, to get C^1-open sets of diffeomorphisms in which the Newhouse phenomenon can be found, it is enough to construct homoclinic classes being persistently non-dominated.

An important step of the construction of non-dominated homoclinic classes in [6, 7] is the generation of persistent *heterodimensional cycles* (see Definition 1.8) associated to saddles having different *indices* (dimension of the unstable bundle) and non-real multipliers (we observe that heterodimensional cycles can only appear in dimension strictly greater than two). We point out that heterodimensional cycles may be persistent in the C^1-topology (this is used in [6, 7]). In contra-position, to get persistent homoclinic tangencies one needs to consider C^2-diffeomorphisms. This is probably the main reason because the C^1-coexistence phenomenon may occur in dimension three (and higher dimensions). The C^1-persistence of heterodimensional cycles involves a blender-like argument, which does not require distortion estimates and roughly plays the role of the thick horseshoes in the Newhouse C^2-construction (we will briefly comment this point in Section 1.2, for a discussion about blenders see [10, Chapter 6]).

In summary, homoclinic tangencies is the natural context for the C^2-Newhouse coexistence phenomenon. On the other hand, the C^1-Newhouse coexistence phenomenon is closely related to the generation of non-dominated homoclinic classes and its natural context are manifolds of dimension three (or greater). Thus a natural question is how to get such a non-dominated dynamical configuration. This paper provides a natural scenario for the C^1-Newhouse coexistence phenomenon (and strong forms of it) in dimension three: the unfolding of *heterodimensional tangencies,* (see Definition 1.2). Roughly, we first consider non-hyperbolic homoclinic classes (persistently) containing a pair of saddles P of index one and Q of index two. Then we study the dynamics at non-transverse intersections (heterodimensional tangencies) between the two-dimensional stable manifold $W^s(P)$ of P and the two-dimensional unstable manifold $W^u(Q)$ of Q. We see that the unfolding of this tangency leads to robustly non-dominated homoclinic classes, therefore (by [8]) to the C^1-Newhouse coexistence phenomenon.

In a somewhat different context, our results also (partially) answer the following question: *how do partially hyperbolic sets/diffeomorphisms bifurcate?* This question is motivated by the series of papers by Newhouse, Palis, and Takens studying bifurcations of hyperbolic sets (of C^2-surface diffeomorphisms) via homoclinic tangencies (see [26, 27, 23, 28, 29, 30]): a hyperbolic set Λ of saddle-type (the homoclinic class of a saddle) losses its hyperbolicity via the creation of a

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[1] We recall that the density of hyperbolic dynamics does not hold in the C^2-topology in any dimension greater than or equal to two, (see [22] and the extensions to higher dimensions in [34, 31]) and in the C^1-topology in dimensions higher than or equal to three (see, for instance, the pioneering papers [2], for dimension $n \geq 3$, and [35], for dimension three).

[2] The results in [6, 7] are stated for three dimensional manifolds, but they can be extended to higher dimensions after straightforward modifications.

homoclinic (or heteroclinic) tangency between the invariant manifolds of some saddle(s) of Λ. The unfolding of this tangency leads to a string of bifurcations and interesting dynamical phenomena: period doubling bifurcations ([38]), persistence of tangencies, persistence of coexistence of infinitely many sinks or sources ([23]), and persistence of strange attractors ([20]), among others. See [30] for a comprehensive study of homoclinic bifurcations and [25] for a survey on recent results on this subject.

The dynamical configuration (heterodimensional tangencies) we study here is reminiscent of the one of homoclinic tangencies. Instead bifurcations of hyperbolic homoclinic classes as in the papers above, we consider partially hyperbolic ones. Typically, these non-hyperbolic classes contain saddles of indices one and two. We analyze what happens at heterodimensional tangencies relating these saddles with different indices. In many relevant cases, this kind of bifurcation provides the transition from *partially hyperbolic to non-dominated dynamics.*

1.1 Definitions and statement of results

In this section, we state precisely our results. We begin by recalling the definition of homoclinic class and stating some results about them.

1.1.1 Homoclinic classes and heterodimensional tangencies

Homoclinic classes were introduced by Newhouse in [21] as an abstraction of the basic sets in the Smale theory (see [36]). They are an essential ingredient in the structure of the dynamics of diffeomorphisms and in many relevant cases correspond to the *elementary pieces* of dynamics (see [10, Chapter 10.4]).

Definition 1.1 (Homoclinic class). *Let P be a hyperbolic periodic saddle of a diffeomorphism f. The* homoclinic class *of P, denoted by $H(P,f)$, is the closure of the transverse intersections of the invariant manifolds (stable and unstable ones) of the orbit $\mathcal{O}(P)$ of P,*

$$H(P,f) = \overline{W^s(\mathcal{O}(P),f) \pitchfork W^u(\mathcal{O}(P),f)}.$$

The homoclinic class of a hyperbolic periodic saddle P can be also (equivalently) defined as the closure of the set of (hyperbolic) saddles Q *homoclinically related* to P: the stable manifold of the orbit of Q transversely meets the unstable one of the orbit of P and vice-versa. This implies that two saddles P and Q which are homoclinically related have the same index. Moreover, the saddles of $H(P,f)$ having the same index as P form a dense subset of the whole class $H(P,f)$. Recall that the *index* of a saddle is the dimension of its unstable bundle. Finally, any homoclinic class $H(P,f)$ is f-invariant and *transitive* (i.e., there is some $x \in H(P,f)$ whose orbit is dense in $H(P,f)$). In fact, the points of $H(P,f)$ whose forward orbits are dense in the whole class form a residual subset of $H(P,f)$. For these properties about homoclinic classes see, for instance, [24, Section 3].

On the other hand, a homoclinic class $H(P,f)$ may fail to be uniformly hyperbolic: it may contain in a robust way hyperbolic saddles having indices different as the one of P (see, for instance, the constructions in [13, 14, 15] and the examples in [5, 11]). Finally, a homoclinic class may also fail to be locally maximal: for instance, the class can be contained with some persistence in the closure of an infinite set of sinks or sources, (see [6]). In fact, the homoclinic classes we obtain here after unfolding a heterodimensional tangency are not neither hyperbolic nor locally maximal.

For C^1-*generic* diffeomorphisms f (i.e., diffeomorphisms in some residual subset of $\mathrm{Diff}^1(M)$) the homoclinic classes of the periodic points of f form a partition (i.e., a pair of homoclinic classes either coincide or they are disjoint) of a dense subset of the limit set of f (this follows from [32, 3, 12]). Moreover, [1, Lemma 2.1] (which follows from [3, 12] using an argument of genericity) claims the following:

Consider a diffeomorphism f and any pair of (hyperbolic) saddles P_f and Q_f of f of different indices. Then there are a C^1-open neighbourhood $\mathcal{U}_f$ of f and a residual subset $\mathcal{R}_{\mathcal{U}_f}$ of $\mathcal{U}_f$ with the following property: let P_g and Q_g the continuations of the saddles P_f and Q_f for g close to f, then

- *either $H(P_g, g) = H(Q_g, g)$, for all $g \in \mathcal{R}_{\mathcal{U}_f}$;*
- *or $H(P_g, g) \cap H(Q_g, g) = \emptyset$, for all $g \in \mathcal{R}_{\mathcal{U}_f}$.*

In the first case, we say that the saddles P_f and Q_f are persistently linked *(in $\mathcal{U}_f$).*

Remark that any homoclinic class containing two different orbits is necessarily infinite. Thus if P_f and Q_f have different indices and are persistently linked then $H(P_f, f) = H(Q_f, f)$ is infinite.

In what follows, we consider saddles (of different indices) which are persistently linked and heterodimensional tangencies associated to them as follows:

Definition 1.2 (Heterodimensional tangency). *Let f be a diffeomorphism and P_f and Q_f two saddles of f with index(P_f) < index(Q_f) and such that $H(P_f, f) = H(Q_f, f)$. We say that f has a* heterodimensional tangency *at y_f associated to P_f and Q_f if $W^s(P_f, f)$ and $W^u(Q_f, f)$ have some non-transverse intersection at some point y_f.*

Note that in the previous definition $\dim(W^s(P_f, f)) + \dim(W^u(Q_f, f)) > \dim(M)$. We observe that (for us) the interesting case in Definition 1.2 occurs when the saddles P_f and Q_f are persistently linked.

Finally, let us recall a definition:

Definition 1.3 (Minimal set). *Let f be a diffeomorphism. A closed f-invariant set Λ_f is* minimal *if the orbit of any point $x \in \Lambda_f$ is dense in the whole Λ_f.*

1.1.2 Statement of results

Before stating our main result let us stand some notation. From now on M is a closed manifold of dimension three and $\mathrm{Diff}^1(M)$ denotes the space of C^1-diffeomorphisms endowed with the usual uniform topology. Given a hyperbolic periodic point P_f of a diffeomorphism f we denote by $\pi(P_f)$ the period of P_f and by P_g the continuation of P_f for g close to f.

Theorem 1. *Let $\mathcal{L}_{P,Q}(M)$ be an open set of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{L}_{P,Q}(M)$, there are saddles P_f and Q_f (depending continuously on f) with different indices which are persistently linked in $\mathcal{L}_{P,Q}(M)$. Assume that there is $f_0 \in \mathcal{L}_{P,Q}(M)$ with a heterodimensional tangency associated to P_{f_0} and Q_{f_0}. Then there are a C^1-open set $\mathcal{V}_0$ and a residual subset $\mathcal{R}_{\mathcal{V}_0}$ of $\mathcal{V}_0$ such that f_0 belongs to the closure of $\mathcal{V}_0$ and for every $h \in \mathcal{R}_{\mathcal{V}_0}$ it holds*

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\{\text{sinks and sources of } h\}.$$

In particular, any $h \in \mathcal{R}_{\mathcal{V}_0}$ has either infinitely many sinks or sources.

Moreover, if the jacobians of P_{f_0} and Q_{f_0} verify

$$(\star) \qquad \mathrm{Jac}[f_0^{\pi(P_{f_0})}(P_{f_0})] > 1 > \mathrm{Jac}[f_0^{\pi(Q_{f_0})}(Q_{f_0})] \quad \textit{or} \quad \mathrm{Jac}[f_0^{\pi(P_{f_0})}(P_{f_0})] < 1 < \mathrm{Jac}[f_0^{\pi(Q_{f_0})}(Q_{f_0})],$$

then the sets $\mathcal{V}_0$ and $\mathcal{R}_{\mathcal{V}_0}$ can be taken such that every $h \in \mathcal{R}_{\mathcal{V}_0}$ also satisfies

$$H(P_h, h) = H(Q_h, h) \subset \text{closure} \left(\bigcup \text{minimal Cantor set of } h \right).$$

In particular, any $h \in \mathcal{R}_{\mathcal{V}_0}$ has infinitely many non-trivial minimal Cantor sets.

Remark 1.4. *By* [7, Corollary C]*, if in Theorem 1 Condition ($\star$) holds one can take the locally residual set $\mathcal{R}_{\mathcal{V}_0}$ such that, for every $h \in \mathcal{R}_{\mathcal{V}_0}$, the homoclinic class $H(P_h, h) = H(Q_h, h)$ is (simultaneously) contained in the closure of the union of infinitely many*

- *disjoint non-trivial homoclinic classes of h,*
- *disjoint non-trivial hyperbolic attractors and repellors of h, and*
- *disjoint non-trivial partially hyperbolic attractors and repellors of h.*

Theorem 1 follows from the following somewhat more general result (Proposition 1.5 below). Assume that the saddles P_f and Q_f have indices one and two, respectively. Consider the subset $\mathcal{L}^c_{P,Q}(M)$ of $\mathcal{L}_{P,Q}(M)$ consisting of diffeomorphisms $f \in \mathcal{L}_{P,Q}(M)$ such that the transverse intersection between $W^s(P_f, f)$ and $W^u(Q_f, f)$ contains a closed curve diffeomorphic to a circle. Note that, by definition, the set $\mathcal{L}^c_{P,Q}(M)$ is open.

Proposition 1.5. *There is a residual subset $\mathcal{R}_{\mathcal{L}}(M)$ of $\mathcal{L}^c_{P,Q}(M)$ of diffeomorphisms g such that*

$$H(P_g, g) = H(Q_g, g) \subset \text{closure}\{\text{sinks and sources of } h\}.$$

Moreover, if P_f and Q_f verify Condition ($\star$) in Theorem 1 then the residual subset $\mathcal{R}_{\mathcal{L}}(M)$ can be taken such that

$$H(P_g, g) = H(Q_g, g) \subset \text{closure} \left(\bigcup \text{minimal Cantor set of } g \right),$$

for all $g \in \mathcal{R}_{\mathcal{L}}(M)$.

To derive Theorem 1 from this proposition it is enough to see that after perturbing a diffeomorphism f with a heterodimensional tangency, associated to saddles P_f and Q_f persistently linked, one can get a diffeomorphism g such that the corresponding invariant manifolds $W^s(P_g, g)$ and $W^u(Q_g, g)$ have a transverse intersection diffeomorphic to a circle.

A context where heterodimensional tangencies arise in a natural way is the one of *robustly transitive sets* we will describe in the next paragraphs. Given $f \in \mathrm{Diff}^1(M)$ and a subset K of M, denote by $\Lambda_f(K) = \cap_{i \in \mathbb{Z}} f^i(K)$ the maximal f-invariant set of f in K.

Definition 1.6 (Robustly transitive set). *Consider $f \in \mathrm{Diff}^1(M)$. An f-invariant compact set Λ_f is* robustly transitive *if there are neighbourhoods U of Λ_f in the ambient manifold M and $\mathcal{V}_f$ of f in* $\mathrm{Diff}^1(M)$ *such that $\Lambda_f = \Lambda_f(U)$ and, for every $g \in \mathcal{V}_f$, the set $\Lambda_g(U)$ is transitive and contained in the interior of U.*

By [8], every robustly transitive set has a dominated splitting (see Definition 1.7). Moreover, robustly transitive sets are partially hyperbolic in dimension three, [17], and uniformly hyperbolic in dimension two, [19].

Robustly transitive sets of C^1-generic diffeomorphisms are *relative homoclinic classes* (see [6]), meaning the following. Given an open set U of M let $\mathcal{T}(U)$ be the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive. By [6], there is a residual subset $\mathcal{H}(U)$ of $\mathcal{T}(U)$ such that, for every $g \in \mathcal{H}(U)$ and every saddle $P_g \in \Lambda_g(U)$, the set $\Lambda_g(U)$ is the subset of the homoclinic class $H(P_g, g)$ of P_g of points whose whole orbit are contained in U. The fact that robustly transitive sets are (C^1-generically) relative homoclinic classes and the results in [3, 12] will allow us to apply Theorem 1 to robustly transitive sets of non-hyperbolic type. Let us explain briefly this point.

We next consider robustly transitive sets of non-hyperbolic type of three dimensional diffeomorphisms. First, let $\mathcal{N}(U)$ be the subset of $\mathcal{T}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ is not uniformly hyperbolic. Consider now the subset $\mathcal{J}(U)$ of $\mathcal{T}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ contains (hyperbolic) saddles of indices one and two. This implies that $\mathcal{J}(U) \subset \mathcal{N}(U)$. By definition of robustly transitive set, the set $\mathcal{J}(U)$ is open. Moreover, by [9, Theorem A], the set $\mathcal{J}(U)$ is dense in $\mathcal{N}(U)$.

Take now $f \in \mathcal{J}(U)$, by definition, $\Lambda_f(U)$ contains saddles P_f and Q_f of indices one and two. By the previous results, for C^1-generic f, $\Lambda_f(U)$ is (simultaneously) the relative homoclinic class in U of P_f and Q_f. Moreover, these relative homoclinic classes coincide (this is just a relative version of [12]). Therefore, the saddles P_f and Q_f are persistently linked. In other words, for generic $f \in \mathcal{J}(U)$, there are saddles P_f and Q_f such that $f \in \mathcal{L}_{P,Q}(M)$.

Corollary 2. *Let M be a three dimensional closed manifold, U an open set of M, and $f \in \mathcal{J}(U)$ a diffeomorphism with a heterodimensional tangency associated to saddles P_f and $Q_f \in \Lambda_f(U)$. Then there are an open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ whose closure contains f and a residual subset $\mathcal{R}_\mathcal{U}$ of $\mathcal{U}$ such that for every $h \in \mathcal{R}_\mathcal{U}$*

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\{\text{sinks and sources of } h\}.$$

In particular, any $h \in \mathcal{R}_\mathcal{U}$ has either infinitely many sinks or sources.

Moreover, if the jacobians of P_f and Q_f verify Condition ($\star$) in Theorem 1 then the sets $\mathcal{U}$ and $\mathcal{R}_\mathcal{U}$ can be taken such that every $h \in \mathcal{R}_\mathcal{U}$ also satisfies

$$H(P_h, h) = H(Q_h, h) \subset \text{closure}\left(\bigcup \text{minimal Cantor set of } h\right).$$

In particular, any $h \in \mathcal{R}_\mathcal{U}$ has infinitely many non-trivial minimal Cantor sets.

Let us observe that, in, general, in the corollary the point corresponding to the heterodimensional tangency does not belong to U. Moreover, the orbits of the sinks and the sources, as well as the minimal sets, are not contained in the set U. This situation is similar to the one of homoclinic tangencies associated to hyperbolic sets in the series of papers mentioned above (the tangency does not belong to the hyperbolic set). It is interesting to compare this situation to the one in [33] where the tangency occurs *inside* a horseshoe.

Corollary 2 provides a context where heterodimensional tangencies occur in a natural way, exemplifying the transition from *partially hyperbolic to non-dominated dynamics.* In Section 4, we construct examples of robustly transitive sets $\Lambda_f(U)$ with heterodimensional tangencies. In these examples the robustly transitive sets have partially hyperbolic splittings with three bundles, $T_{\Lambda_f(U)} = E^s \oplus E^c \oplus E^u$, where E^s and E^u are uniformly hyperbolic (note that this is the non-hyperbolic situation closest to the hyperbolic one).

These examples also motivated by the series of papers [26, 27, 28, 29] where the boundary of the set of uniformly hyperbolic diffeomorphisms is crossed via a homoclinic or heteroclinic tangency. Here the situation is similar: the boundary of partially hyperbolic systems is crossed via a heterodimensional tangency. In [26, 27, 28, 29], after crossing the frontier of hyperbolicity one also returns (in fact, infinitely many times) to uniformly hyperbolic regions. The situation here is rather different: after crossing the frontier of partial hyperbolicity one remains in a domain where the dynamics is non-dominated.

1.2 Ingredients of the proofs

We begin by recalling some definitions and results that will play a key role in this paper:

Definition 1.7 (Dominated splitting and partial hyperbolicity). *An f-invariant compact set Λ has a* dominated splitting *if the tangent bundle $T_\Lambda M$ over Λ splits into two Df-invariant bundles E and F, $T_\Lambda M = E \oplus F$, whose fibers $E(x)$ and $F(x)$ have constant dimensions, and there exists an integer $\ell \geq 1$ such that for every point $x \in \Lambda$ and every pair of unit vectors $u \in E(x)$ and $v \in F(x)$ it holds*

$$\|Df^\ell(x)\, u\| \leq 2\, \|Df^\ell(x)\, v\|.$$

If the set Λ has a dominated splitting $T_\Lambda M = E \oplus F$ such that one of the bundles is uniformly hyperbolic we say that Λ is partially hyperbolic.

Definition 1.8 (Heterodimensional cycle). *Let P_f and Q_f be two saddles of different indices of a diffeomorphism f. The diffeomorphism f has a* heterodimensional cycle *associated to P_f and Q_f if*

$$W^s(P_f, f) \cap W^u(Q_f, f) \neq \emptyset \quad \text{and} \quad W^u(P_f, f) \cap W^s(Q_f, f) \neq \emptyset.$$

Remark 1.9. *Assume that the indices of the saddles Q_f and P_f in the definition of heterodimensional cycle are two and one, respectively. Then, after an arbitrarily small perturbation, one can assume that $W^s(P_f, f) \cap W^u(Q_f, f)$ contains transverse intersections and that $W^u(P_f, f)$ and $W^s(Q_f, f)$ meet quasi-transversely along the orbit of some point x_f, i.e., $x_f \in W^u(P_f, f) \cap W^s(Q_f, f)$ and $T_{x_f} W^u(P_f, f) \cap T_{x_f} W^s(Q_f, f) = \{\bar{0}\}$.*

Following [7], we introduce the notions of *wild homoclinic class* and *wild set of diffeomorphisms.* The homoclinic class $H(Q_f, f)$ of a saddle Q_f of $f \in \mathrm{Diff}^1(M)$ is *wild* if there is a neighbourhood $\mathcal{V}_f$ of f in $\mathrm{Diff}^1(M)$ such that, for every $g \in \mathcal{V}_f$, the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting. In this case, we say that the homoclinic class $H(Q_f, f)$ belongs to $W(f)$ (the *set of wild homoclinic classes of f*).

Denote by $\mathcal{W}(M)$ the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms having some wild homoclinic class (i.e., $f \in \mathcal{W}(M)$ if $W(f) \neq \emptyset$). We let $\mathcal{W}_{[1,2,3]}(M)$ the subset of $\mathcal{W}(M)$ of diffeomorphisms f having a wild homoclinic class $H(Q_f, f) \in W(f)$ and a C^1-neighbourhood $\mathcal{V}_f$ such that:

W1) for every $g \in \mathcal{V}_f$, the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting and contains saddles R_g and T_g (R_g and T_g continuously depend on g) having different indices;

W2) either R_g or T_g is homoclinically related to Q_g;

W3) the jacobians of R_g and T_g verify

$$\mathrm{Jac}[g^{\pi(R_g)}(R_g)] > 1 > \mathrm{Jac}[g^{\pi(Q_g)}(Q_g)] \quad \text{or} \quad \mathrm{Jac}[g^{\pi(R_g)}(R_g)] < 1 < \mathrm{Jac}[g^{\pi(Q_g)}(Q_g)].$$

We say that a homoclinic class $H(Q_f, f)$ verifying (W1)–(W3) belongs to $W_{[1,2,3]}(f)$.

Theorem 1.10.

- **([8, Corollary 0.3])** *There is a residual subset $\mathcal{R}(M)$ of $\mathcal{W}(M)$ of diffeomorphisms f such that any homoclinic class $H(P_f, f)$ of f in $W(f)$ is contained in the closure of the (infinite) set of sinks and sources of f;*

- **([7, Theorem B, Heuristic principle, and Corollary C])** *There is a residual subset $\mathcal{R}_{[1,2,3]}(M)$ of $\mathcal{W}_{[1,2,3]}(M)$ of diffeomorphisms f such that any homoclinic class $H(P_f, f) \in W_{[1,2,3]}(f)$ is simultaneously contained in the closures of the (infinite) set of sinks of f, of the (infinite) set of sources of f; and of the (infinite) set of minimal Cantor sets of f*[3].

We are now ready to sketch the proof of our results. In order to simplify the exposition, we prove first our main result in the robustly transitive case (Corollary 2). This allows us to present the main arguments in the proofs while avoiding technical difficulties which are typical in the case of homoclinic classes.

Consider a diffeomorphism f having a robustly transitive set $\Lambda_f(U)$. Assume that f has a heterodimensional tangency associated to saddles P_f and Q_f of $\Lambda_f(U)$, say of indices one and two, respectively. After an arbitrarily small perturbation, we can assume that this tangency is parabolic (see Definition 2.1). Next, after perturbing f, we get a diffeomorphism g arbitrarily close to f such that the transverse intersection between $W^s(P_g, g)$ and $W^u(Q_g, g)$ contains a small contractible circle σ_g (the circle σ_g is contained in a small disk of $W^s(P_g, g)$). Clearly, this property is open. The key step is to see that, after a new perturbation of g, we can assume that the circles σ_g are (robustly) contained in the homoclinic class of Q_g: $\sigma_h \subset H(Q_h, h)$, for every h in an open set whose closure contains g. This is done in Proposition 2.2. Finally, the property of containing small contractible circles prevents the homoclinic class of Q_h of having a dominated splitting, see Proposition 2.9. Corollary 2 now follows from Theorem 1.10.

We now explain the steps of the proof of Proposition 2.2. We first obtain (after a perturbation) a saddle-node S in $\Lambda_f(U)$ (see Lemma 2.3). Unfolding this saddle-node, we get two saddles A_g and B_g of indices one and two, respectively, in $\Lambda_g(U)$. Using the robust transitivity of $\Lambda_g(U)$ and the Connecting Lemma in [18], we get a partially hyperbolic heterodimensional cycle associated to A_g and B_g, (see Lemma 2.4 and Definition 2.5). Finally, by unfolding this heterodimensional cycle, we see that *the (one-dimensional) stable manifold of B_g transversely meets any two-disk Δ transverse to $W^s(A_g, g)$,* see Lemma 2.6 and Figure 1. Following [10, Lemma 6.6], we call this property *distinctive intersection property* of the heterodimensional cycle (although the formulations are not exactly the same). The distinctive intersection property is the key argument in the blender-like construction mentioned above.

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[3] These minimal Cantor sets are maximal ones (they contain any invariant set intersecting them) and (clearly) aperiodic (without periodic orbits).

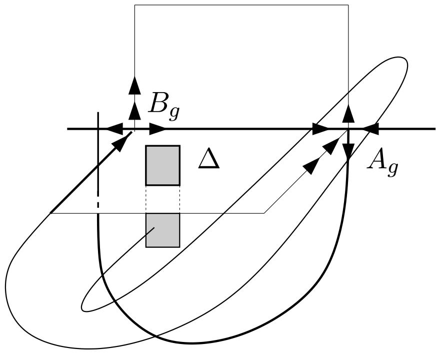


Figure 1: The distinctive intersection property

Using the distinctive property and the fact that B_g is homoclinically related to Q_g and A_g is homoclinically related to P_g, we get that the circles σ_g are (robustly) contained in the homoclinic class of Q_g. Proposition 2.2 now implies that the class $H(Q_g, g)$ does not admit any dominated splitting.

The proof for homoclinic classes follows along the same ideas (see Section 3). The main difference (also the main difficulty) is that we need to check that the saddles A_g and B_g above, obtained after perturbing the dynamics, belong to the homoclinic class of Q_g. We observe that this property follows immediately in the robustly transitive case. To prove this fact for homoclinic classes we use some arguments in [1] and results in [12, 4] about homoclinic classes of C^1-generic diffeomorphisms.

We finish this introduction by comparing our C^1-construction and the Newhouse's construction of C^2-diffeomorphisms having infinitely many sinks. On one hand, the main step of the Newhouse's construction is to see that the unfolding of a homoclinic tangency generates two one-dimensional Cantor sets presenting persistent intersections. These Cantor sets are obtained as the intersections of the stable and unstable laminations of a hyperbolic set Λ and a curve γ. Denote these sets by K^s and K^u, respectively. The curve γ corresponds to the tangencies between extensions of the stable and unstable laminations. Thus each intersection between these Cantor sets corresponds to a tangency between the leaves of the stable and unstable manifolds of a hyperbolic set. See Figure 2. Finally, each tangency leads to the creation of a sink. The construction of these (so-called *thick*) Cantor sets relies on genuinely C^2-properties (bounded distortion).

On the other hand, our construction (as well as the ones in [8, 7]) contains implicitly the persistence of tangencies between the stable and unstable manifolds of points of the homoclinic class. The existence of such tangencies essentially follows from the distinctive intersection property above and the fact that the homoclinic class contains circles. Roughly, the distinctive property means that the one-dimensional stable manifold of any saddle (of the homoclinic class) of index two topologically behaves as a two-dimensional manifold. Similarly for the one dimensional unstable manifolds of saddles of index one. So one can think of the saddles of the homoclinic class as saddles having simultaneously indices one and two. The distinctive property also allows us to avoid considering persistent intersections of Cantor sets, as in the Newhouse's construction. Moreover, in our context, the distinctive property is also the key for proving that the homoclinic classes contain circles. Finally, the distinctive property also guarantees that our construction can be done in the

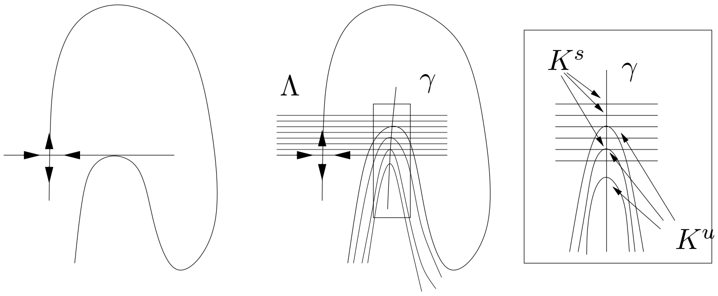


Figure 2: Persistent tangencies

C^1-case.

2 Homoclinic classes without dominated splittings

Recall that $\mathcal{T}(U)$ is the (open) subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive and that $\mathcal{J}(U)$ is the (open) subset of $\mathcal{T}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ contains saddles of indices one and two.

Definition 2.1 (Parabolic heterodimensional tangency). *A diffeomorphism $f \in \mathcal{J}(U)$ has a* parabolic heterodimensional tangency *at the point y_f, associated to the saddles P_f of index 1 and Q_f of index 2 of the set $\Lambda_f(U)$, if there are compact two-disks $K^s_f \subset W^s(P_f, f)$ and $K^u_f \subset W^u(Q_f, f)$ containing y_f in their interiors and local coordinates around $y_f = (0,0,0)$ such that, in these coordinates,*

$$K^s_f = \{z = 0\} \quad \text{and} \quad K^u_f = \{z = x^2 + y^2\}.$$

Thence $W^s(P_f, f)$ and $W^u(Q_f, f)$ have a non-transverse intersection at the heteroclinic point y_f.

For g close to f (g in a small neighbourhood $\mathcal{U}_f \subset \mathcal{J}(U)$ of f), we consider continuations $K^s_g \subset W^s(P_g, g)$ and $K^u_g \subset W^u(Q_g, g)$ of K^s_f and K^u_f. Let $\mathcal{U}^+_f$ be the (open) subset of $\mathcal{U}_f$ of diffeomorphisms g such that $\sigma_g = K^s_g \cap K^u_g$ is a transverse intersection diffeomorphic to a circle. Clearly, f is in the closure of $\mathcal{U}^+_f$.

The main result in this section is the following:

Proposition 2.2. *Let $g \in \mathcal{U}^+_f$. Then there is an open set $\mathcal{V}_g \subset \mathcal{U}^+_f$ whose closure contains g such that, for every $h \in \mathcal{V}_g$, the circle $\sigma_h = K^s_h \cap K^u_h \subset W^s(P_h, h) \cap W^u(Q_h, h)$ is contained in the homoclinic class $H(Q_h, h)$ of Q_h.*

Note that the curve σ_h is contained in a disk of $W^s(P_h, h)$, thus, for some big k, $h^k(\sigma_h)$ is contained in the local stable manifold of P_h.

2.1 Robust transitivity and heterodimensional cycles

In order to prove Proposition 2.2, we begin by collecting some results about the occurrence of saddle-node bifurcations (Lemma 2.3) and (partially hyperbolic) heterodimensional cycles (see Lemma 2.4

and Definition 2.5) inside robustly transitive sets. We also study the dynamics arising from partially hyperbolic heterodimensional cycles (Lemma 2.6).

Consider the subset $\mathcal{S}(U)$ of $\mathcal{J}(U)$ of diffeomorphisms f such that $\Lambda_f(U)$ contains a *periodic saddle-node* (i.e., a periodic point having an unique multiplier in the unitary circle which is equal to one).

Lemma 2.3. ([9, page 203]) *The set $\mathcal{S}(U)$ is dense in $\mathcal{J}(U)$.*

Consider $f \in \mathcal{J}(U)$ and a pair of saddles A_f and B_f of f in $\Lambda_f(U)$. Let $\mathcal{J}_{A,B}(U)$ be an open subset of $\mathcal{J}(U)$ such that for every $g \in \mathcal{J}_{A,B}(U)$ the continuations A_g of A_f and B_g of B_f are defined and belong to $\Lambda_g(U)$.

Next result follows from the Connecting Lemma in [18] and the robust transitivity of $\Lambda_f(U)$:

Lemma 2.4. ([17, Lemma 3.3])

1. *Suppose that the saddles A_f and B_f of f have the same index. Then there is an open and dense subset $\mathcal{H}_{A,B}(U)$ of $\mathcal{J}_{A,B}(U)$ such that for every $g \in \mathcal{H}_{A,B}(U)$ the saddles A_g and B_g are homoclinically related.*

2. *Suppose that the saddles A_f and B_f of f verify* $\operatorname{index}(A_f) < \operatorname{index}(B_f)$. *Then there is a dense subset $\mathcal{C}_{A,B}(U)$ of $\mathcal{J}_{A,B}(U)$ such that every $g \in \mathcal{C}_{A,B}(U)$ has a heterodimensional cycle associated to A_g and B_g as follows:*
    - *$W^s(A_g, g) \cap W^u(B_g, g)$ contains transverse intersections;*
    - *there is $x_g \in W^u(A_g, g) \cap W^s(B_g, g)$ such that $x_g \in \Lambda_g(U)$ and*

    $$T_{x_g} W^u(A_g, g) \oplus T_{x_g} W^s(B_g, g) = T_{x_g} W^u(A_g, g) + T_{x_g} W^s(B_g, g),$$

    that is, $W^u(A_g, g)$ and $W^s(B_g, g)$ meet quasi-transversely at the point y_g.

Definition 2.5 (Partially hyperbolic heterodimensional cycle). *A heterodimensional cycle associated to saddles A_f and B_f, of indices one and two and the same period π, is* partially hyperbolic *if there are a π-periodic connected component Γ_f of the transverse intersection between $W^s(A_f, f)$ and $W^u(B_f, f)$ and a heteroclinic point $x_f \in W^u(A_f, f) \cap W^s(B_f, f)$ such that the set Σ_f obtained as the closure of the orbits of the saddles A_f and B_f and the heteroclinic intersections Γ_f and x_f has a dominated splitting $E^s \oplus E^c \oplus E^u$ having one-dimensional bundles and such that the central bundle E^c is tangent to Γ_f.*

We say that the saddles A_f and B_f and the heteroclinic intersections Γ_f and x_f are the elements of the cycle.

Given a partially hyperbolic heterodimensional cycle (with elements A_f, B_f, Γ_f and x_f), let $\lambda_c(A_f)$ and $\lambda_c(B_f)$ be the eigenvalues of $Df^\pi(A_f)$ and $Df^\pi(B_f)$ corresponding to the central direction E^c. In the partially hyperbolic heterodimensional cycles we consider, the curve Γ_f has extremes A_f and B_f. We call this curve *connection of the cycle* and denote by ℓ_f its length. Finally, we also assume that the map $\log |Df^\pi_{|\Gamma_f}|$ is Lipschitz with Lipschitz constant of L_f (we will explain this point in Remark 2.7). Denote by

$$\Delta_{[B_f, A_f]} = (\mathrm{e}^{-\ell_f L_f}\, \lambda_c(B_f)^k)\, \frac{1 - \lambda_c(A_f)}{1 - \lambda_c(B_f)^{-1}}, \quad \text{where} \quad k = \max\{i \colon \lambda_c(B_f)^{-i} > (1 - \lambda_c(A_f))\}.$$

We say that the cycle is $[B_f, A_f]$-*expanding* if $\Delta_{[B_f, A_f]} > 1$.

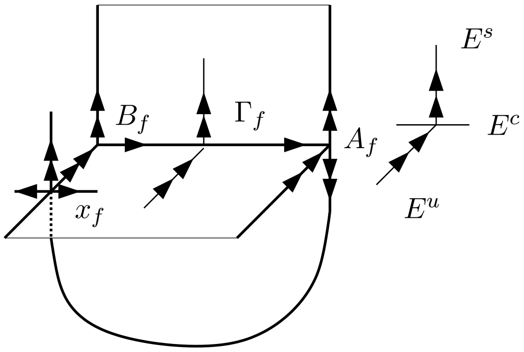


Figure 3: A partially hyperbolic heterodimensional cycle

Lemma 2.6 (Distinctive intersection property, [13]). *Let f be a diffeomorphism having a $[B_f, A_f]$-expanding partially hyperbolic heterodimensional cycle associated to saddles A_f and B_f in a robustly transitive set. Then there is a C^1-open set $\mathcal{U}$ such that $f \in \overline{\mathcal{U}}$ with the following property: for every $g \in \mathcal{U}$ and every two-disk Δ^{cu} intersecting transversely $W^s(A_g, g)$, one has that $W^s(B_g, g)$ transversely meets Δ^{cu}.*

Remark 2.7. Lemma 2.6 follows using the arguments in the proof of [13, Theorem 1.1, pages 302-3]. Let us briefly explain how Lemma 2.6 follows in our setting.

First, [13] considers cycles corresponding to first bifurcations. This assumption, in particular, implies that the cycle is *untwisted,* meaning the following. The heteroclinic point x_f simultaneously belongs to the manifolds with boundary

$$W^u([A_f, B_f]) = W^u(A_f, f) \cup W^u(B_f, f), \qquad W^s([A_f, B_f]) = W^s(A_f, f) \cup W^s(B_f, f).$$

The cycle is *untwisted* if there are neighbourhoods $D^u(x_f)$ and $D^s(x_f)$ of x_f in $W^u([A_f, B_f])$ and $W^s([A_f, B_f])$ such that $D^u(x_f) \cap D^s(x_f) = \{x_f\}$.

Observe that, a priori, the cycle associated to A_f and B_f given by Lemma 2.4 may be twisted. Using the transitivity and the fact that the homoclinic class of A_f is non-trivial, after a perturbation (again using the Connecting Lemma) one gets g arbitrarily close to f with a untwisted (partially hyperbolic) heterodimensional cycle with elements A_g, B_g, Γ_g and y_g, where A_g, B_g and Γ_g are the continuations of the elements A_f, B_f and Γ_f of the initial cycle. The heteroclinic point y_g may involve a longer part of the unstable manifold of P_g. The configuration leading to the untwisted cycle is depicted in Figure 4.

The previous construction implies that if the initial cycle is $[B_f, A_f]$-expanding, one can take the new cycle being $[B_g, A_g]$-expanding (note that this definition only involves the restriction to the connection and the eigenvalues, which continuously depend on the diffeomorphism).

Second, the proof in [13] only involves the restriction of f to the connection Γ_f and the central eigenvalues $\lambda_c(A_f)$ and $\lambda_c(B_f)$. Thus one can repeat the arguments in [13] considering only the dynamics in a neighbourhood of cycle (i.e., in an open set containing the orbits of the elements of the cycle).

Finally, [13] considers C^∞-diffeomorphisms, but using the blender construction in [5] one can carry the same construction to the C^1-case.

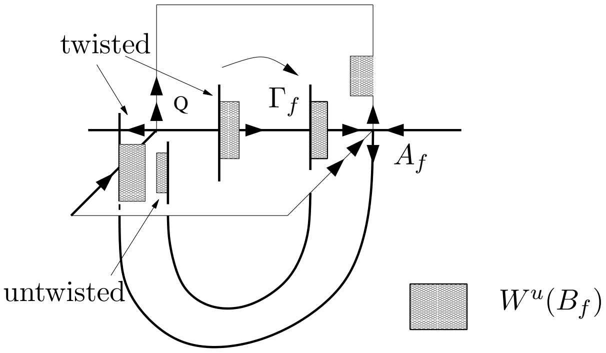


Figure 4: Twisted and untwisted intersections

For an expository explanation of the proof of Lemma 2.6 in the C^1-case, we refer to [10, Chapter 6.1]. We observe that Lemma 2.6 also holds for partially hyperbolic cycles that do not verify the $[B_f, A_f]$-expanding assumption, see [15]. We quote here the result in [13] because we will use it explicitly it in Section 4.

2.2 Proof of Proposition 2.2

We are now ready to prove the proposition. The proof consists of the following steps. Let $g \in \mathcal{U}_f^+$.

Step 1: By Lemma 2.3, there is h_1 arbitrarily close to g such that $\Lambda_{h_1}(U)$ contains a periodic saddle-node S_{h_1} of period $\pi = \pi(S_{h_1})$. Let λ_s, λ_c, and λ_u be the eigenvalues of $Dh_1^{\pi}(S_{h_1})$, where $|\lambda_s| \le |\lambda_c| = 1 \le |\lambda_u|$. Note that $|\lambda_s| < 1$: otherwise, we can perturb the saddle-node to get a diffeomorphism h_2 (arbitrarily close to h_1) with a source inside U, contradicting the robust transitivity of $\Lambda_{h_1}(U)$. Similarly, $|\lambda_u| > 1$. Therefore, $\lambda_c = 1$ and the multipliers of S_{h_1} are all real.

Step 2: By unfolding the saddle-node S_{h_1}, we get h_2 arbitrarily close to h_1 with two saddles A_{h_2} and B_{h_2}, of period π and indices one and two, contained in $\Lambda_{h_2}(U)$. Moreover, there is a π-periodic segment Υ_{h_2} bounded by A_{h_2} and B_{h_2} and contained in $\Lambda_{h_2}(U)$ and in the transverse intersection between $W^s(A_{h_2})$ and $W^u(B_{h_2})$. Furthermore, the restriction of h_2 to the curve Υ_{h_2} is partially hyperbolic with three bundles (corresponding to the stable, central, and unstable directions of the saddle-node). See Figure 5.

Step 3: Using the second part of Lemma 2.4, we get h_3 arbitrarily close to h_2 with a heterodimensional cycle associated to A_{h_3} and B_{h_3}. This cycle is partially hyperbolic: it is enough to take $\Gamma_{h_3} = \Upsilon_{h_3}$ and the heteroclinic point x_{h_3} given by Lemma 2.4. Moreover, the dynamics of $h_3^{\pi(\Upsilon_{h_3})}$ in Υ_{h_3} is close to the identity (the curve is obtained by perturbing a saddle-node). Thus we can take the heterodimensional cycle being $[B_{h_3}, A_{h_3}]$-expanding. Let us briefly explain this point. First, we can take the diffeomorphism being C^2 around the cycle. Moreover, $h_3^{\pi(\Upsilon_{h_3})}$ is close to the identity and the length ℓ_{h_3} of Υ_{h_3} can be taken arbitrarily small. Thus we can assume that

$$e^{-\ell_{h_3} L_{h_3}} > 1/2.$$

On the other hand, we can take $\lambda_c(A_{h_3})$ and $\lambda_c(B_{h_3})$ close to one in such a way

$$2\,(1 - \lambda_c(A_{h_3})) > \lambda_c(B_{h_3})^{-k} > (1 - \lambda_c(A_{h_3})),$$

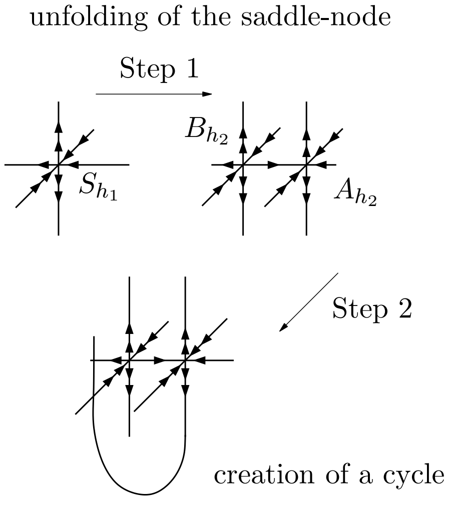


Figure 5: Creation of a partially hyperbolic heterodimensional cycle

for some $k \geq 1$. Hence,

$$\lambda_c(B_{h_3})^k > \frac{1}{2\,(1-\lambda_c(A_{h_3}))}.$$

Therefore, putting together all the previous inequalities,

$$\Delta_{[B_{h_3},A_{h_3}]} = (\mathrm{e}^{-\ell_{h_3}\, L_{h_3}}\, \lambda_c(B_{h_3})^k)\, \frac{1-\lambda_c(A_{h_3})}{1-\lambda_c(B_{h_3})^{-1}} > \frac{1}{4\,(1-\lambda_c(B_{h_3})^{-1})}.$$

Hence if $\lambda_c(B_{h_3})$ is close enough to 1, $(1-\lambda_c(B_{h_3})^{-1}) < 1/8$ and thus $\Delta_{[B_{h_3},A_{h_3}]} > 2$.

Step 4: Applying Lemma 2.6 to h_3, we get an open set $\mathcal{U}$, contained in $\mathcal{U}_f^+$ and arbitrarily close to the initial g, such that, for every $h \in \mathcal{U}$, $W^s(B_h, h)$ transversely meets any two disk transverse to $W^s(A_h, h)$.

Last step: Consider now the sets $\mathcal{H}_{A,P}(U)$ and $\mathcal{H}_{B,Q}(U)$ in Lemma 2.4. Note that the saddles A_h and P_h have same index. The same holds for the saddles B_h and Q_h. Thus the first part of Lemma 2.4 implies that the set

$$\mathcal{U}_{[A,P],[B,Q]}(U) = \mathcal{U} \cap \mathcal{H}_{A,P}(U) \cap \mathcal{H}_{B,Q}(U).$$

is open and dense in $\mathcal{U}$.

Claim 2.8. *For every $h \in \mathcal{U}_{[A,P],[B,Q]}(U)$ it holds $\sigma_h \subset H(Q_h, h)$.*

This claim immediately implies Proposition 2.2: just note that the open set $\mathcal{U}$ can be taken arbitrarily close to the initial $g \in \mathcal{U}_f^+$ and that $\mathcal{U}_{[A,P],[B,Q]}(U)$ is open and dense in $\mathcal{U}$.

Proof of the claim: Consider $h \in \mathcal{U}_{[A,P],[B,Q]}(U)$ and any point $x \in \sigma_h$. We will see that *$W^s(Q_h, h)$ transversely meets any two-disk $W^u(x)$ contained in $W^u(Q_h, h)$ and containing x in its interior.* Since $W^u(x) \subset W^u(Q_h, h)$, we have that the disk $W^u(x)$ contains a transverse homoclinic point of Q_h. Thus $W^u(x) \cap H(Q_h, h) \neq \emptyset$. As $W^u(x)$ can be taken arbitrarily small and $H(Q_h, h)$ is closed, one has that $x \in H(Q_h, h)$, ending the proof of the claim.

To prove the assertion recall that, by construction, the circle σ_h is contained in the transverse intersection between $W^s(P_h, h)$ and $W^u(Q_h, h)$. Thus $W^s(P_h, h)$ transversely meets $W^u(x)$. As A_h is homoclinically related to P_h, $W^s(A_h, h)$ transversely meets $W^u(x)$. Now, by Step 4, $W^s(B_h, h)$ transversely meets $W^u(x)$. Finally, as Q_h and B_h are homoclinically related, $W^s(Q_h, h)$ transversely meets $W^u(x)$, which implies our assertion. ☐

The proof of Proposition 2.2 is now complete.

2.3 Non-dominated homoclinic classes

For each $g \in \mathcal{U}_f^+$ consider the set $\mathcal{V}_g$ given by Proposition 2.2 (i.e., g is in the closure of $\mathcal{V}_g$ and, for every $h \in \mathcal{V}_g$, the homoclinic class $H(Q_h, h)$ contains a small contractible circle σ_h). Let

$$\mathcal{N} = \cup_{g \in \mathcal{U}_f^+} \mathcal{V}_g.$$

Note that, by construction, the set $\mathcal{N}$ is open and dense in $\mathcal{U}_f^+$.

Proposition 2.9. *For every $g \in \mathcal{N}$ the homoclinic class $H(Q_g, g)$ does not admit any dominated splitting.*

Proof: For the properties about dominated splittings we use here see, for instance, [10, Appendix B.1]. We need the following lemma (the ideas in its proof are borrowed from [7, Lemma 3.8]).

Lemma 2.10. *Let f be a diffeomorphism defined on a three dimensional manifold. Suppose that there are saddles P and Q of f, of indices 1 and 2, such that $P \in H(Q, f)$ and $H(Q, f)$ has a dominated splitting $E \oplus F$.*

- *If $\dim(E) = 1$ then P has real multipliers λ_{ss}, λ_c, and λ_u, with $|\lambda_{ss}| < |\lambda_c| < 1 < |\lambda_u|$.*
- *If $\dim(F) = 1$ then Q has real multipliers β_s, β_c and β_{uu}, with $|\beta_s| < 1 < |\beta_c| < |\beta_{uu}|$.*

Proof: We prove the lemma when $\dim(E) = 1$. The case $\dim(F) = 1$ follows similarly taking f^{-1}.

First, as $\dim(E^s(P)) = 2$, the domination of $E \oplus F$ implies that $E(P) \subset E^s(P)$. Let $H(P) = F(P) \cap E^s(P)$. This bundle is one-dimensional and Df-invariant. Thus $E^s(P) = E(P) \oplus H(P)$, where the splitting is invariant and dominated.

The proof now is by contradiction. Assume that P has a pair of non-real multipliers, say λ and $\bar{\lambda}$. Since $\dim(E^u(P)) = 1$, these multipliers are contracting. In such a case, $E^s(P)$ does not admit any Df-invariant splitting, contradicting the existence of $E^s(P) = E(P) \oplus H(P)$.

We have that the three multipliers of P are real. Finally, if two multipliers of P have the same modulus (again such multipliers are necessarily contracting) then $E^s(P)$ does not admit any dominated splitting, contradicting the existence of $E^s(P) = E(P) \oplus H(P)$ above. This completes the proof of the lemma. ☐

The proof of the proposition now follows by contradiction. Consider $g \in \mathcal{N}$ and suppose that $H(Q_g, g)$ has a dominated splitting $T_{H(Q_g,g)}M = E \oplus F$. Suppose, for example, that $\dim(E) = 1$ (the case $\dim(F) = 1$ follows analogously considering g^{-1}). Lemma 2.10 implies that the eigenvalues of $Dg^{\pi(P_g)}(P_g)$ are λ_{ss}, λ_c, and λ_u, where $|\lambda_{ss}| < |\lambda_c| < 1 < |\lambda_u|$. The domination now implies that $E(P) = E^{ss}(P)$.

For each $x \in H(Q_g, g)$ consider the fibers $E(x)$ and $F(x)$ and the angle $\alpha(x)$ between $E(x)$ and $F(x)$. The domination of $E \oplus F$ implies that these fibers depend continuously on x and that there is of some $\alpha_0 > 0$ such that $\alpha(x) > \alpha_0$ for all $x \in H(Q_g, g)$.

Consider now the small contractible curve $\sigma_g \subset W^s(P_g, g) \cap W^u(Q_g, g)$ with $\sigma_g \subset H(Q_g, g)$ given by Proposition 2.2. Then there is a sequence $(i_k)_{k\in\mathbb{Z}}$ such that $g^{i_k}(\sigma_g) \to P_g$ as $i_k \to \infty$ and $g^{i_k}(\sigma_g) \to Q_g$ as $i_k \to -\infty$, where the limits mean convergence in the Hausdorff metric. Since $H(Q_g, g)$ is closed and g-invariant, this implies that $P_g \in H(Q_g, g)$.

Let $\Sigma_g = \cup_{i_k} g^{i_k}(\sigma)$. For each $x \in \Sigma_g$ consider $H(x) = T_x\Sigma_g$. By definition, the family $\mathcal{H} = \{H(x)\}_{x\in\Sigma_g}$ is Dg-invariant. The domination of $E \oplus F$ and the invariance of the family $\mathcal{H}$ imply that either $H(x) \subset E(x)$ for all $x \in \Sigma_g$, or $H(x) \subset F(x)$ for all $x \in \Sigma_g$.

Since the curves $g^{i_k}(\sigma)$ (big i_k) are arbitrarily small contractible circles arbitrarily close to P_g (contained in a small local stable manifold of P), for each big i_k there are points $z, w \in g^{i_k}(\sigma_g)$ such that: (a) $H(z)$ is *parallel* to $E^{ss}(P) = E(P)$ (we write $H(z) \simeq E(P)$) and (b) $H(w)$ is *parallel* to $E^c(P)$, where $E^c(P)$ is the eigenspace associated to the contracting multiplier λ_c of P (we write $H(z) \simeq E^c(P)$). Note that $E^c(P) \subset F(P)$. These possibilities are depicted in Figure 6

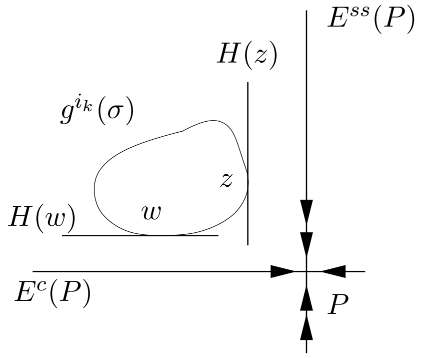


Figure 6: The contradiction

We are not ready to get a contradiction. First, if $H(x) \subset E(x)$ for all $x \in \Sigma_g$ (thus $H(x) = E(x)$) then (b) above gives the contradiction: there is w arbitrarily close to P such that $E(P) = E^{ss}(P)$ and $E(w) = H(w) \simeq E^c(P)$, hence $E(P)$ and $E(w)$ are not close.

Finally, if $H(x) \subset F(x)$ then (a) above gives the contradiction: there is z arbitrarily close to P such that the angle between $F(z)$ and $E(z)$ is arbitrarily close to zero, contradicting the fact that $\alpha(z) > \alpha_0 > 0$ for all $z \in H(P_g, g)$. This completes the proof of the proposition. □

2.4 Consequences: proof of Corollary 2

We are now ready to finish the proof of Corollary 2. Consider $f \in \mathcal{J}(U)$ having a heterodimensional tangency associated to saddles P_f (of index one) and Q_f (of index two) of a robustly transitive set $\Lambda_f(U)$. Proposition 2.9 gives an open set $\mathcal{N}$ whose closure contains f such that, for every $g \in \mathcal{N}$, the homoclinic class of Q_g does not admit any dominated splitting. By the first part of

Theorem 1.10, there is a residual subset $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{N}$ of diffeomorphisms h such that the homoclinic class of $H(Q_h, h)$ is contained in the closure of the (infinite) set of sinks and sources of h.

Moreover, if the homoclinic class $H(Q_f, f)$ contains saddles verifying (W1)–(W3) (i.e., saddles R_f and T_f having different indices and such that $\mathrm{Jac}[f^{\pi(R_f)}(R_f)] > 1 > \mathrm{Jac}[f^{\pi(T_f)}(T_f)]$) the second part of Theorem 1.10 guarantees that the open set $\mathcal{N}$ and the residual subset $\mathcal{R}_{\mathcal{N}}$ of $\mathcal{N}$ above can be chosen such that for every $h \in \mathcal{R}_{\mathcal{N}}$ the homoclinic class of $H(Q_h, h)$ is simultaneously contained in the closure of the (infinite) set of sinks of h, in the closure of the (infinite) set of sources of h, and in the closure of the (infinite) set of minimal Cantor sets of h.

3 Proof of Theorem 1

Theorem 1 essentially follows from arguments in the robustly transitive case. The main difference concerns Steps 2 and 3 in Section 2.2. More precisely, in the robustly transitive case, after a perturbation, one has a *the robustly transitive set* $\Lambda_f(U)$ *containing a saddle-node* S_f. This saddle-node is obtained by collapsing two saddles A_g and B_g of different indices of the robustly transitive set $\Lambda_g(U)$. Thus, by definition of robustly transitive set, the resulting saddle-node S_f necessarily belongs to $\Lambda_f(U)$. Using the saddles A_g and B_g, we obtain a partially hyperbolic heterodimensional cycle. The unfolding of such a cycle leads to diffeomorphisms h such that the homoclinic class of the saddle B_h contains the small contractible circle $\sigma_h \subset W^s(P_h, h) \cap W^u(Q_h, h)$. This fact implies that the homoclinic class of B_h (which coincides with the one of Q_h) does not admit any dominated splitting (Proposition 2.9).

In the case of homoclinic classes, one can also obtain a saddle-node S_f as well as the two collapsing saddles A_g and B_g. The main difficulty (and the main difference with the robustly transitive case) is that, a priori, the saddles A_g and B_g may have nothing to do with the homoclinic classes of initial saddles P_g and Q_g being persistently linked. In fact, we will see that one can take the saddles A_g and B_g homoclinically related to P_g and Q_g. This step involves the notion of *chain recurrence class* and the fact that, C^1*-generically, any chain recurrence class containing a periodic saddle* P *is the homoclinic class of* P (see [4, Remarque 1.10]). After proving this step, the proof follows exactly as in the robustly transitive case. Let us sketch this construction (whose main ingredients can be found in [1]) more precisely.

Given a saddle R_f of a diffeomorphism f, let $\mathrm{Per}_{\mathbb{R}}(H(R_f, f))$ be the subset of saddles T_f of $H(R_f, f)$ such that

- T_f and R_f are homoclinically related (thus the saddles T_f and R_f have the same index);
- the eigenvalues of $Df^{\pi(T_f)}(T_f)$ are real, different in modulus, and of multiplicity one, recall that $\pi(T_f)$ is the period of T_f.

Lemma 3.1. ([1, Proposition 2.6] following [8, Lemma 4.16]) *There is a residual subset* $\mathcal{R}_0$ *of* $\mathrm{Diff}^1(M)$ *of diffeomorphisms* f *such that* $\mathrm{Per}_{\mathbb{R}}(H(R_f, f))$ *is dense in* $H(R_f, f)$*, for every non-trivial homoclinic class* $H(R_f, f)$ *of* f*.*

Let $\mathcal{L}_{P,Q}(M)$, P_f, Q_f, and f_0 as in Theorem 1 ($\mathcal{L}_{P,Q}(M)$ is open and for all $f \in \mathcal{L}_{P,Q}(M)$ the saddles P_f and Q_f are persistently linked, and f_0 has a heterodimensional tangency associated to P_{f_0} and Q_{f_0}). Let $\mathcal{G}$ be a residual subset of $\mathcal{L}_{P,Q}(M)$ such that $H(P_f, f) = H(Q_f, f)$ for all $f \in \mathcal{G}$.

Assume that the indices of P_f and Q_f are one and two, respectively. By Lemma 3.1, there is an open neighbourhood $\mathcal{U}_0$ of f_0 and a residual subset $\mathcal{R}_{\mathcal{U}_0}$ of $\mathcal{U}_0$ such that, for all $h \in \mathcal{R}_{\mathcal{U}_0}$, there are saddles $P'_h \in \mathrm{Per}_{\mathbb{R}}(H(P_h, h))$ and $Q'_h \in \mathrm{Per}_{\mathbb{R}}(H(Q_h, h))$. Let us suppose that the saddles P'_h and Q'_h are defined in the whole $\mathcal{U}_0$. This implies that

$$H(P'_h, h) = H(P_h, h) = H(Q_h, h) = H(Q'_h, h), \quad \text{for all } h \in \mathcal{G}_{\mathcal{U}_0} = \mathcal{R}_{\mathcal{U}_0} \cap \mathcal{G}. \tag{1}$$

Using the Connecting Lemma in [18], the transitivity of the homoclinic classes, and the fact that the unstable manifold of Q'_h and the stable manifold of P'_h are both two-dimensional, one has (exactly as in Lemma 2.4) the following:

Lemma 3.2. *There are an open and dense subset $\mathcal{T}_0$ of $\mathcal{U}_0$ and a dense subset $\mathcal{D}_0$ of $\mathcal{U}_0$ such that*

1. *$W^s(P'_h, h)$ and $W^u(Q'_h, h)$ have some transverse intersection for all $h \in \mathcal{T}_0$;*
2. *$W^u(P'_h, h) \cap W^s(Q'_h, h) \neq \emptyset$ for all $h \in \mathcal{D}_0$.*

Therefore, there is a dense subset of $\mathcal{U}_0$ of diffeomorphisms h having a heterodimensional cycle associated to P'_h and Q'_h.

Exactly as in the robustly transitive case, unfolding this heterodimensional cycle (associated to saddles with real multipliers) one obtains a saddle-node S_g. This saddle-node is originated by the collapse of two saddles A_f and B_f of indices one and two (exactly as in Step 2). Arguing exactly as in the proof of [1, Theorem 3.2][4], one gets that there is an open set $\mathcal{U}_1$, whose closure contains g, such that for every $f \in \mathcal{U}_1$ the saddles A_f and B_f are defined and verify

$$W^u(Q'_f, f) \cap W^s(A_f, f) \neq \emptyset \quad \text{and} \quad W^u(A_f, f) \cap W^s(P'_f, f) \neq \emptyset, \tag{2}$$

and

$$W^s(P'_f, f) \cap W^u(B_f, f) \neq \emptyset \quad \text{and} \quad W^s(B_f, f) \cap W^u(Q'_f, f) \neq \emptyset. \tag{3}$$

We now need the notion of chain recurrence class. A point x is *chain recurrent* if for any $\varepsilon > 0$, there are closed ε-pseudo orbits containing x. The set of chain recurrent points of f, denoted by $\mathcal{R}(f)$, splits into f-invariant compact sets called *chain recurrence classes*: two points $x, y \in \mathcal{R}(f)$ are in the same class if, for any $\varepsilon > 0$, there are ε-pseudo orbits going from x to y and from y to x. [4, Remarque 1.10] states that, for C^1-generic f, any chain recurrence class containing a periodic point P is the homoclinic class of P.

Consider now the residual subset $\mathcal{G}_{\mathcal{U}_1} = \mathcal{G}_{\mathcal{U}_0} \cap \mathcal{U}_1$ of $\mathcal{U}_1$.

Lemma 3.3. *The saddles Q'_f, P'_f, A_f, and B_f are in the same chain recurrence class for every $f \in \mathcal{G}_{\mathcal{U}_1}$.*

Proof: We first prove that the saddles Q'_f, A_f, and P'_f are in the same chain recurrence class. This assertion will follow from equation (2) and the fact that $H(P'_f, f) = H(Q'_f, f)$ for every $f \in \mathcal{G}_{\mathcal{U}_1}$ (see (1)). Let us prove this claim.

---

[4] In fact the situation in [1] is much more subtle: the saddle P'_f has index k and the saddle Q'_f has index $k + m$, $m \geq 2$, and one gets a saddle A_f of intermediate index ℓ, $m < \ell < m + k$ such that $W^u(Q'_f, f) \cap W^s(A_f, f) \neq \emptyset$ and $W^u(A_f, f) \cap W^s(P'_f, f) \neq \emptyset$.

Taking any point $x \in W^u(Q'_f, f) \cap W^s(A_f, f)$, we get a finite segment of orbit $f^{-m_1}(x)$, $f^{-m_1+1}(x), \dots, f^{n_1}(x)$, $m_1, n_1 \in \mathbb{N}$, such that $f^{-m_1}(x)$ is arbitrarily close to Q'_f and $f^{n_1}(x)$ is arbitrarily close to A_f. Analogously, taking $y \in W^u(A_f, f) \cap W^s(P'_f, f)$, we get a segment of orbit $f^{-m_2}(y)$, $f^{-m_2+1}(y), \dots, f^{n_2}(y)$, $m_2, n_2 \in \mathbb{N}$, such that $f^{-m_2}(y)$ is arbitrarily close to A_f and $f^{n_2}(y)$ is arbitrarily close to P'_f. Finally, as $H(P'_f, f) = H(Q'_f, f)$ and this set is transitive, there is a segment of orbit $z, f(z), \dots, f^{n_3}(z)$, $n_3 \in \mathbb{N}$, starting arbitrarily close to P'_f and ending arbitrarily close to Q'_f. The announced pseudo-orbit containing the points Q'_f, A_f, and P'_f is obtained concatenating the three segments of orbits above. This shows that the saddles Q'_f, A_f, and P'_f are in the same chain recurrence class.

Similarly, from equation (3) and $H(P'_f, f) = H(Q'_f, f)$, one has that the saddles Q'_f, B_f, and P'_f are in the same chain recurrence class. Thus, for all $f \in \mathcal{G}_{\mathcal{U}_1}$, the saddles Q'_f, A_f, B_f, and P'_f are in the same chain recurrence class. This completes the proof of the lemma. $\square$

Since (C^1-generically) any chain recurrence class containing a saddle is the homoclinic class of such a saddle, [4, Remarque 1.10], we get a residual subset $\mathcal{R}_{\mathcal{U}_1}$ of $\mathcal{U}_1$ of diffeomorphisms f such that

$$H(P'_f, f) = H(A_f, f) = H(B_f, f) = H(Q'_f, f).$$

Applying now Lemma 3.2 to the saddles A_f and B_f (which are arbitrarily close to a saddle-node) we get a heterodimensional cycle associated to A_f and B_f. Exactly as in Step 3, this cycle can be taken partially hyperbolic and $[B_f, A_f]$-expanding. The proof of the result now follows exactly as in the robustly transitive case noting that

$$H(P_f, f) = H(P'_f, f) = H(A_f, f) = H(B_f, f) = H(Q'_f, f) = H(Q_f, f).$$

This completes the sketch of the proof of Theorem 1.

4 Examples of diffeomorphisms with heterodimensional tangencies

In this section, we construct diffeomorphisms with heterodimensional tangencies obtained as *bifurcations* of partially hyperbolic sets. These examples illustrate a natural transition from partially hyperbolic to non-dominated dynamics. The examples are motivated by the construction of robustly transitive sets (which are non-hyperbolic homoclinic classes) arising from Morse-Smale systems via bifurcations through heterodimensional cycles in [13]. Our construction is similar to the one in [24, page 89-90] for diffeomorphisms with homoclinic tangencies. Let us recall the main steps of the construction in [24], .

Consider a diffeomorphism f having a two-dimensional Plykin attractor Σ in a disk D of $\mathbb{R}^2$. Take a fixed saddle p of f in Σ and note that Σ is the homoclinic class of p. Multiply now f by a linear expansion $g(x) = \lambda\, x$ on the line $\mathbb{R}$. We take λ big enough in such a way $D \times \{0\}$ is normally hyperbolic (any expansion of the derivative of g in the disk D is strictly upper bounded by λ). Note that the point $S = (p, 0)$ is a saddle of index two of the product diffeomorphism $\phi = (f, g)$ and that $\Lambda = \Sigma \times \{0\} \subset D \times \{0\}$ is a hyperbolic set of ϕ which is the homoclinic class of the saddle $S = (p, 0)$. Finally, the unstable bundle of Λ is two-dimensional.

We can now select points z_1 in the strong unstable manifold of S (tangent to $\{0, 0\} \times \mathbb{R}$) and z_2 in the stable manifold of S and perform a semi-local perturbation of ϕ preserving the hyperbolic set Λ as follows: there is an arc $(\phi_t)_{t\in[0,1]}$, with $\phi_0 = \phi$, such that for every $t \in [0, 1)$ the set Λ is a hyperbolic set of ϕ_t equal to the homoclinic class of S, and, for $t = 1$, ϕ_1 has a homoclinic tangency associated to S at the point z_2. The perturbation is depicted in Figure 7. A key condition here (that we will use later) is that Λ does not contain points in $z > 0$.

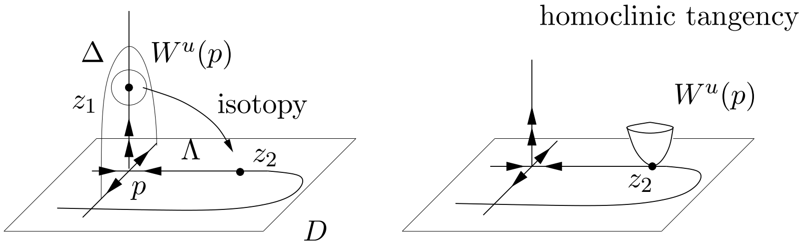


Figure 7: Creation of a homoclinic tangency

We now sketch our construction and explain how the arguments above can be used in our setting. For the sake of simplicity we use local coordinates in $\mathbb{R}^3$, but our construction clearly work in the broader setting of diffeomorphisms bifurcating from Morse-Smale diffeomorphisms via expanding partially hyperbolic heterodimensional cycles.

Next construction is analogous to the one in [16, Section 2]. We begin with a diffeomorphism f having a heterodimensional cycle related to saddles $Q = (0, 0, 0)$ and $P = (0, 4, 0)$, of indices 2 and 1, such that dynamics around the cycle is the following. In the cube $R = [-1, 1] \times [-1, 5] \times [-1, 1]$,

$$f(x, y, z) = (\lambda_s\, x, F(y), \lambda_u\, z), \qquad 0 < \lambda_s < 1 < \lambda_u,$$

where F is a strictly increasing function such that $F'(0) = \beta > 1$ and $F'(4) = \lambda < 1$, $0 < \lambda < 1$, and $\lambda_s < F'(x) < \lambda_u$ for every $x \in [0, 4]$. By construction, the restriction of f to the cube R has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with three non-trivial bundles given by the coordinate directions. Moreover,

$$\begin{array}{lcl} \{0\} \times [-1, 4) \times [-1, 1] \subset W^u(Q) & \text{and} & [-1, 1] \times \{(0, 0)\} \subset W^s(Q), \\ \{(0, 4)\} \times [-1, 1] \subset W^u(P) & \text{and} & [-1, 1] \times (0, 5] \times \{0\} \subset W^s(P). \end{array}$$

Thus $W^s(P)$ and $W^u(Q)$ meet transversely along the curve $\{0\} \times (0, 4) \times \{0\}$.

Take the points $A = (0, 4, -\frac{1}{2}) \in W^u(P)$ and $B = (-\frac{1}{2}, 0, 0) \in W^s(Q)$. We assume that there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(A) = B$ and $f^j(A) \notin R$ for all $0 < j < k_0$. Thus $W^u(P) \cap W^s(Q) \neq \emptyset$ and f has a heterodimensional cycle associated to P and Q.

Take a small compact neighbourhood U_A of A such that $U_A, \dots, f^{k_0}(U_A)$ are pairwise disjoint, $U_A, f^{k_0}(U_A) \subset R$, and $f^j(U_A) \cap R = \emptyset$, for all $0 < j < k_0$. We assume that transition f^{k_0} from U_A to $f^{k_0}(U_A)$ is of the form

$$f^{k_0}(x, y, z) = (x - \frac{1}{2}, y - 4, z + \frac{1}{2}), \quad \text{for every } (x, y, z) \in U_A.$$

Thus $W^u(P)$ and $W^s(Q)$ meet *quasi-transversely* throughout the orbit of the heteroclinic point B:

$$T_B W^u(P) + T_B W^s(Q) = T_B W^u(P) \oplus T_B W^s(Q) = \mathbb{XZ}.$$

We can now extend f to a diffeomorphism defined in a three dimensional closed manifold M in the boundary of the Morse-Smale systems. Moreover, we can choose the restriction F of f to $\{0\} \times [0,4] \times \{0\}$ in such a way f has a $[Q,P]$-expanding partially hyperbolic heterodimensional cycle whose elements are P, Q, $\Gamma_f = \{0\} \times [0,4] \times \{0\}$, and $x_f = B$.

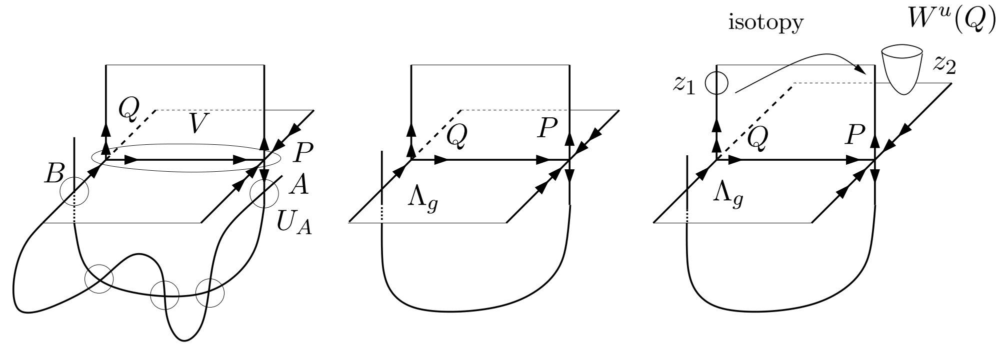
We select a small neighbourhood of the cycle

$$W = V \cup U_A \cup f(U_A) \cup f^2(U_A) \cup \cdots \cup f^{k_0-1}(U_A) \cup f^{k_0}(U_A),$$

where V is a small neighbourhood of the curve $\{0\} \times [0,4] \times \{0\}$ contained in the cube R such that $f^{-1}(U_A) \subset V$ and $f^{k_0+1}(U_A) \subset V$. This configuration is depicted in Figure 8

Given g close to f, let Λ_g be the maximal invariant set of g in W. By construction, the set Λ_g is partially hyperbolic (with a splitting with three one-dimensional bundles). Moreover, if we take g such that A is a transverse homoclinic point of Q (i.e., after unfolding the cycle), this maximal invariant set Λ_g is robustly transitive and equal to the non-hyperbolic homoclinic class of Q (note that by definition $P \in \Lambda_g$). Thus the set $\Lambda_g = H(Q,g)$ verifies the hypotheses of Theorem 1. For details see the proof of [13, Theorem 3] (see also [16, Theorem 2.1]).

We now take h close to g such that $\Lambda_h = H(Q,h)$ is robustly transitive. We can now use the arguments above to get a heterodimensional tangency associated to P and Q. A key point here is that the set Λ_h does not intersect the regions $z > 0$ and $x < 0$. We now exactly repeat the procedure in [24, page 89-90] taking, for instance, $z_1 = (0,0,1/2) \in W^u(Q,h)$ and $z_2 = (-1/2,1/2,0) \in W^s(P,h)$. As above, there is an arc $(h_t)_{t\in[0,1]}$, with $h_0 = h$, such that for every $t \in [0,1)$ the set $\Lambda_{h_t} = H(Q,h_t)$ is robustly transitive (we do not modify the dynamics around the cycle) and for $t = 1$, h_1 has a heterodimensional tangency associated to Q and P at the point z_2. See Figure 8. This completes the sketch of our construction.



heterodimensional cycle

unfolding of the cycle

heterodimensional tangency

Figure 8: Creation of a heterodimensional tangency

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The end of domination

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Let M be a closed manifold and $\mathrm{Diff}^1(M)$ be the space of C^1-diffeomorphisms defined on M endowed with the usual topology. A property of diffeomorphisms is said *C^1-generic* (or shortly, generic) if it holds for diffeomorphisms in a residual subset of $\mathrm{Diff}^1(M)$. In this note we adopt the generic point of view[1]

The *homoclinic class* of a hyperbolic saddle point p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of the invariant manifolds of p. On the one hand, every homoclinic class is f-invariant, *transitive* (there is a dense orbit in the whole class) and the periodic points with the same *Morse index* (dimension of the unstable bundle) as p are dense in $H(p, f)$. On the other hand, in general, homoclinic classes fail to be locally maximal and hyperbolic. For instance, there exist nontrivial homoclinic class accumulated by sinks and sources, thus failing to be locally maximal. In fact, there is a dichotomy for C^1-generic homoclinic classes: *either a homoclinic class has some weak hyperbolicity (existence of a* dominated splitting*) or else it is accumulated by infinitely many sinks or sources,* see [5].

One of the aims of this note is to discuss the dynamics of homoclinic classes without *domination.* Let us recall the definition of *dominated splitting.* Consider an f-invariant set Λ. A Df-invariant splitting over Λ, $T_\Lambda M = E \oplus F$, is dominated if the dimension of the fibers $E(x)$ and $F(x)$ are independent of $x \in \Lambda$ and there is $\ell \in \mathbb{N}$ such that

$$\|Df^{\ell}|_{E(x)}\| \cdot \|Df^{-\ell}|_{F(f^{\ell}(x))}\| < \frac{1}{2}$$

for every $x \in \Lambda$. This means that the derivative of f expands the vector in F unifromly more than the vectors in E.

An important feature of the nontrivial homoclinic classes $H(p, f)$ without domination is that there is a perturbation of the diffeomorphism f throughout the orbit of some periodic point $x \in H(p, f)$ which is an homothety, see [5, Proposition 2.1]. For the non-dominated homoclinic classes considered in this

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[1] This approach allows us to use appropriate perturbation lemmas: e.g., the Pugh's closing lemma [17], the ergodic closing lemma [14], the Hayashi's connecting lemma [13], and the Franks' lemma [11] on dynamical perturbations of derivatives.

note (the so called *wild homoclinic classes,* see definition below) this homothety is in fact the identity. This property enables us to find (essentially) any prescribed dynamics after perturbing the initial diffeomorphism f.

A second objective comes from an attemp to understand the *elementary pieces of dynamics* of generic diffeomorphisms. In the hyperbolic setting, Smale's decomposition theorem [18] claims that these elementary pieces are the homoclinic classes, which are then basic sets (hyperbolic and locally maximal sets). This is also the case of generic diffeomorphisms of surfaces having finitely many sinks or sources, see [14].

In the nonhyperbolic setting and in higher dimensions, the first problem is to find a good candidate for the role of an elementary piece of dynamics. In the Smale theory, these elementary pieces are *dynamically undecomposable* (this leads to the notion of transitivity), *maximal* (an elementary piece is not contained in any greater one), and *isolated* (two different pieces are dynamically independent).

The notion of transitivity is fundamental in any attempt to define the concept of elementary piece of dynamics. There are two natural strategies for generalizing the notion of basic set. The notions of transitivity and maximality lead to the concept of *maximal transitive set,* and transitivity together with isolation leads to the concept of *robust transitive set* introduced in [8] [2]. On the other hand, from the mere constructive point of view, the homoclinic relation associates to each hyperbolic saddle p (in a natural way) a transitive set containing it: its homoclinic class. Thus there are three natural candidates for the role of elementary piece of dynamics: maximal transitive sets, robustly transitive sets and homoclinic classes. Recently, in [1] is proved that (in any dimension) for generic diffeomorphisms having finitely many homoclinic classes (the so-called *tame* diffeomorphisms) these three approaches essentially coincide [3].

In this note we focus our attention on the notions of maximal transitive sets and homoclinic classes, as well as on diffeomorphisms with infinitely many homoclinic classes (*wild diffeomorphisms*).

First, let us observe that the intersection of two maximal transitive sets may be non-empty. This is an obstruction for considering these sets as elementary

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[2] A transitive set Λ is *maximal* if it is a maximal element of the family of all compact transitive sets of f ordered by inclusion. An f-invariant set Λ is robustly transitive for f if there are neighbourhoods U of Λ in M and $\mathcal{N}$ of f in $\mathrm{Diff}^1(M)$ such that, for every $g \in \mathcal{N}$, the set $\Lambda_g = \cap_{i \in \mathbb{Z}} g^i(\bar{U})$ is transitive and $\Lambda_f = \Lambda$.

[3] More precisely, the results in [1] assert that there is a residual subset $\mathcal{T}$ of $\mathrm{Diff}^1(M)$ (the set of *tame diffeomorphisms*) such that, for every $f \in \mathcal{T}$, the number of homoclinic classes is well defined and locally constant in $\mathcal{T}$. Moreover, for a fixed $f \in \mathcal{T}$, there are a neighbourhood $\mathcal{N}$ of f, a *filtration* $M_1 \subset M_2 \subset \cdots \subset M_k$, $f(M_i) \subset \mathrm{int} M_i$, and hyperbolic points $p_1(f), p_2(f), \dots p_k(f)$ whose continuations are defined for all $g \in \mathcal{N}$, such that for each $g \in \mathcal{N} \cap \mathcal{T}$ the maximal invariant set of g in $M_{i+1} \setminus M_i$ is the homoclinic class of $p_{i+1}(g)$. Moreover, $H(p_{i+1}(g))$ admits a dominated splitting.

This result generalizes to higher dimensions the Mañé's generic dichotomy in [14] between hyperbolicity (existence of a spectral decomposition of the limit set) and coexistence of infinitely many sinks or sources for surface diffeomorphisms. Here the dichotomy is between the existence of a spectral decomposition of the limit set whose pieces are weakly hyperbolic and have some topological stability and the coexistence of infinitely many homoclinic classes.

pieces of dynamics. To avoid this problem, one considers the stronger notion of *saturated transitive set,* defined as a transitive set containing any transitive set intersecting it. We observe that there are maximal transitive sets which are not contained in any saturated one, see [10]. The link between homoclinic classes and maximal or saturated transitive sets is stated in recent results by Arnaud and Carballo-Morales-Pacifico:

Theorem 1 (Arnaud, [2]) *Generically in* $\mathrm{Diff}^1(M)$*, every transitive set containing a periodic orbit* p *is contained in the homoclinic class of* p*. In particular, generically, homoclinic classes are maximal transitive sets.*

Theorem 2 (Carballo, Morales, Pacifico, [7]) *Generically in* $\mathrm{Diff}^1(M)$*, the homoclinic classes are saturated transitive sets. In particular, generically, any two homoclinic classes are either equal or disjoint.*

These two results lead naturally to the question of the *existence of saturated transitive sets which are not homoclinic classes with some persistence.* We give a positive answer to this question, showing the existence of persistent saturated transitive sets without periodic orbits. We show that the appearance of such sets is connected with the existence of homoclinic classes which do not admit any dominated splitting.

1 Non-dominated homoclinic classes

In this section we construct homoclinic classes which do not admit any dominated splitting exhibiting pathological dynamics. For simplicity, from now on, we will restrict our attention to diffeomorphisms of 3 manifolds.

Consider a diffeomorphism f having two saddles p and q of Morse indices 1 and 2 involved in a *heterodimensional cycle.* This means that the two dimensional stable manifold of p and unstable manifold of q meet transversely and the one dimensional unstable manifold of p and unstable manifold of q meet quasi-transversely along the orbit of a point x (i.e., $T_xW^u(p,f)+T_xW^s(q,f) = T_xW^u(p,f)\oplus T_xW^s(q,f)$). Under some assumptions on the geometry of the transverse intersection $W^s(p,f)\pitchfork W^u(q,f)$ (including the assumption that the eigenvalues of the derivatives of f at p and q are all real and different in modulus) one gets an open set $\mathcal{W}$ of $\mathrm{Diff}^1(M)$ whose closure contains f such that the homoclinic classes of p and q are equal for every diffeomorphism g nearby f, see [9]. The proof of this result only involves orbits in a neighbourhood of the cycle (the orbits of the transverse intersection $W^s(p,f)\pitchfork W^u(q,f)$ and of the quasi-transverse intersection $W^u(p,f)\cap W^s(q,f)$).

The next step consists in considering points p' and q' homoclinically related to p and q, respectively, such that p' has a non-real contracting eigenvalue and q' has a non-real expanding eigenvalue. Then, by shrinking the open set $\mathcal{W}$ above, one has that, for every $g\in\mathcal{W}$, the homoclinic classes of p' and q' are equal.

We claim that the transitive set $\Lambda_g = H(p', g) = H(q', g)$ does not admit any dominated splitting. Otherwise, $T_{\Lambda_g}(M) = E \oplus F$, where F dominates E. Suppose, for instance, that E has dimension 2. Then $E(p') = E^s(p')$ and $E(q') \cap E^u(q')$ is an one dimensional g-invariant bundle, which contradicts the fact that the derivative of g at q has non-real expanding eigenvalues (which in turn implies that $E^u(q')$ does not contain nontrivial invariant sub-bundles).

Theorem 3 (Bonatti, D., [4]) *Let $\mathcal{W}$ be an open set as above. Then there is a residual subset $\mathcal{R}$ (in the C^1-topology) of $\mathcal{W}$ of diffeomorphisms g such that $H(p', g) = H(q', g)$ is contained in the closure of the (infinite) set consisting of sinks and sources of g.*

Moreover, if the jacobian of p is greater than one and the jacobian of q is less than one, then $H(p', g) = H(q', g)$ is simultaneously contained in the closure of the set of sinks and in the closure of the set of sources for every $g \in \mathcal{R}$.

This result shows the occurrence of the Newhouse phenomenon (coexistence of infinitely many sinks or sources) in the C^1-topology in dimension bigger than or equal to three. Recall that for surface diffeomorphism this phenomenon is associated to the unfolding of homoclinic tangencies of C^2-diffeomorphisms, [15, 16]. Theorem 3 shows how the Newhouse phenomenon is related with the failure of domination of a nontrivial homoclinic class. In fact, the previous result illustrates a much more general principle:

Theorem 4 (Bonatti, D., Pujals [5]) *There is a residual subset of diffeomorphisms $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every diffeomorphism $f \in \mathcal{R}$ and every saddle p of f, the homoclinic class $H(p, f)$ verifies one of the following properties:*

- *either $H(p, f)$ is contained in the closure of infinitely many sources or sinks,*
- *or $H(p, f)$ admits a dominated splitting.*

If the dimension of the ambient manifold is two, the dominated splittings in the theorem are hyperbolic, and if the dimension of the ambient manifold is three the splittings are (at least) *partially hyperbolic* (at least one bundle of the dominated splitting is hyperbolic). In higher dimensions, there are generic homoclinic classes having a dominated splitting without any hyperbolic direction, see [6].

Our objective now is to describe another stronger form of the Newhouse phenomenon in homoclinic classes that do not admit any dominated splitting.

2 Wild dynamics

The construction in [4] motivates the following definition. A homoclinic class Λ_f of a diffeomorphism f (defined on a 3-manifold) is *wild* if it contains saddles p_f, q_f and r_f and there is a neighbourhood $\mathcal{V}$ of f where the continuations p_g, q_g and r_g of p_f, q_f and r_f are defined such that, for every $g \in \mathcal{V}$:

1. the saddles p_g and r_g have Morse index one and are homoclinically related, the saddle p_g has a pair of contracting non-real eigenvalues, and the saddle q_g has Morse index two and a pair of expanding non-real eigenvalues,

2. $q_g \in H(p_g, g)$,

3. $\mathrm{Jac}_g\mathrm{per}_{(p_g)}(p_g) > 1 > \mathrm{Jac}_g\mathrm{per}_{(r_g)}(r_g)$, where $\mathrm{Jac}_g(r)$ denotes the Jacobian of g at r.

Denote by $\mathcal{W}(M)$ the set of diffeomorphisms of $\mathrm{Diff}^1(M)$ having a wild homoclinic class. By definition, $\mathcal{W}(M)$ is open in $\mathrm{Diff}^1(M)$. Theorem 3 guarantees the existence of wild homoclinic classes.

Before stating the next theorem, let us recall some definitions. An f-invariant set Σ is *minimal* if the orbit of any point of Σ is dense in Σ (i.e., Σ has no proper f-invariant sets). Observe that every infinite minimal set Σ does not contain periodic orbits. An f-invariant set Σ is *uniquely ergodic* if it supports only one f-invariant probability measure (which is necessarily ergodic). Observe that there are minimal sets where the restriction of f is not uniquely ergodic (see [12]). Finally, a compact set Σ is *Lyapunov stable* for f if for every neighbourhood U of Σ there is a neighbourhood V of Σ such that $f^i(V) \subset U$ for all $i \geq 0$. Observe that every Lyapunov stable transitive set Λ is saturated: any transitive set intersecting Λ is contained in any neighbourhood of Λ (so it is contained in Λ).

Next theorem gives a sample of some pathological dynamics associated to the appearance of a wild homoclinic class:

Theorem 5 (Bonatti, D., [3]) *There is a residual subset $\mathcal{R}(M)$ of the set $\mathcal{W}(M)$ of diffeomorphisms with wild homoclinic classes consisting of diffeomorphisms f having simultaneously infinitely many:*

a) *Sinks and sources.*

b) *Saddles with* trivial homoclinic classes *(i.e., the classes consist only of the periodic saddle itself).*

c) *Non-trivial uniformly hyperbolic attractors and repellers.*

d) *Non-trivial partially hyperbolic attractors and repellers.*

e) *Non-trivial (non-hyperbolic) homoclinic classes Λ_i containing points of different indices. Moreover, infinitely many of these homoclinic classes are wild ones.*

f) *Maximal transitive Cantor set $\mathcal{F}_i$ where f is minimal. Moreover, the sets $\mathcal{F}_i$ are simultaneously Lyapunov stable for f and f^{-1} and uniquely ergodic.*

Moreover, the wild homoclinic classes Λ_i and the minimal transitive sets $\mathcal{F}_i$ can be chosen such that,

g) *each wild homoclinic class Λ_i is contained in the closure of the union of infinitely many maximal transitive sets $\mathcal{F}_j$ with minimal dynamics,*

h) *each wild maximal transitive sets $\mathcal{F}_i$ with minimal dynamics is contained in the closure of the union of infinitely many wild homoclinic classes Λ_j.*

The main naive idea in the proof of this theorem is that, after perturbing the dynamics, one gets an open set where the dynamics (up to the period) is the identity. This allows us, after a new perturbation, to get new wild homoclinic classes. Finally, the existence of such a new wild homoclinic class allows us to repeat the previous procedure infinitely many times. The main technical step of the proof of the theorem is the following:

Proposition 2.1 *Let p_f be a point of Morse index one such that, for every g in a neighbourhood $\mathcal{N}$ of f, the homoclinic class $H(p_g, g)$ is wild. Then, for every $g \in \mathcal{N}$ and every $\varepsilon > 0$, there are a saddle x_g homoclinically related to p_g whose orbit is ε-dense in $H(p_g, g)$ and a neighbourhood V_g of x_g such that*

- *V_g, $g(V_g), \dots, g^{n-1}(V_g)$ are pairwise disjoint, where n is the period of x_g,*
- *the derivative of g^n at x_g is ε-close to the identity.*

This proposition is a fundamental technical step in the proofs in [5]. Using this result one perturbs the derivative of g^n at x_g to get the identity. A lemma by Franks in [11] enables us to perform dynamically such a perturbation of the derivative, getting a diffeomorphism h arbitrarily close to g whose restriction to V is the identity.

Let us briefly explain how, using the previous result, one constructs a transitive set with minimal dynamics. The next step is, after perturbation, to get a new diffeomorphism f_1 and disk $D_1 \subset V_g$ such that D_1, $f_1(D_1), \dots, f_1^{n_1-1}(D_1)$ are pairwise disjoint, $f_1^{n_1}(D_1)$ is contained in the interior of D_1, and $f_1^{n_1}$ has a wild homoclinic class in D_1. Arguing inductively, one constructs a nested sequence of open sets $\mathcal{U}_k$ and of periodic disks D_k of period n_k, $n_k \to \infty$ and $n_{k+1} = n_k\, m_k$, $m_k > 1$, such that

- $\mathcal{U}_{k+1}$ is dense in $\mathcal{U}_k$,
- for every $f \in \mathcal{U}_k$, D_k, $f(D_k), \dots, f^{n_k-1}(D_k)$ are pairwise disjoint, $f^{n_k}(D_k)$ is contained in the interior of D_k, and f^{n_k} has a wild homoclinic class in D_k,
- the diameters of the disks D_k go to zero as $k \to \infty$.

It is now immediate that, for each f in the locally residual set $\mathcal{R} = \cap_{k\in\mathbb{N}}\mathcal{U}_k$, the set

$$\Upsilon_f = \bigcap_{k\in\mathbb{N}} (\cup_{i=0}^{n_k-1} f^i(D_k))$$

is a minimal Cantor set which is Lyapunov stable.

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How do hyperbolic homoclinic classes collide at heterodimensional cycles?

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Abstract

We present a model illustrating heterodimensional cycles (i.e., cycles associated to saddles having different indices) as a mechanism leading to the *collision* of hyperbolic homoclinic classes (of points of different indices) and thereafter to the *persistence of (heterodimensional) cycles.* The collisions are associated to secondary (saddle-node) bifurcations appearing in the unfolding of the initial cycle.

1 Introduction

1.1 Context

We study partially hyperbolic dynamics obtained when uniformly hyperbolic diffeomorphisms loss their stability by the creation of *heterodimensional cycles,* that is, cycles associated to hyperbolic saddles having different *indices* (dimension of the unstable bundle). We analyze the phenomena of collision of hyperbolic homoclinic classes and persistence of cycles at such bifurcations.

We consider one-parameter families of diffeomorphisms $(f_t)_{t\in[-1,1]}$ starting in a hyperbolic region (Morse-Smale systems) and crossing its boundary at a bifurcation (say $t=0$) corresponding to a cycle associated to a pair of saddles having different indices (i.e., a heterodimensional cycle). The goal is to describe the dynamical features of the diffeomorphisms f_t having some persistence or prevalence (in terms of the parameter t) immediately after crossing the boundary of the hyperbolicity. Clearly, the resulting dynamics depends on the type of bifurcation responsible for the breakdown of the hyperbolicity (here a heterodimensional cycle). Typically, one considers codimension one bifurcations: the bifurcating diffeomorphism f_0 belongs to a (local) submanifold Σ which is contained in the boundary of the hyperbolic systems and has codimension one in the space of diffeomorphisms. Moreover, the arc $(f_t)_{t\in[-1,1]}$ transversely intersects the submanifold Σ at f_0.

Newhouse-Palis conjectured in [NP$_2$] that the generic way of crossing the boundary of a hyperbolic region is via one of the following two types of bifurcations: (i) loss of hyperbolicity of some periodic orbit (saddle-node, period doubling, and Hopf bifurcations), and (ii) loss of transversality of the intersection between the invariant manifolds of hyperbolic periodic points (creation of cycles). This conjecture holds for surface diffeomorphisms (in this case, the cycle is due to a

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homoclinic or heteroclinic tangency) in the boundary of the Morse-Smale systems, [NP_1]. The first sort of bifurcation is local (the dynamics on a finite orbit) while the second one is semi-global and involves the whole orbit of a non-periodic point (a non-transverse intersection between invariant manifolds). An interesting dynamical feature rises when there is an interplay between these two types of bifurcations. For instance, generic bifurcations via homoclinic tangencies lead to the creation of infinitely many saddle-node, period doubling and Hopf bifurcations immediately after their unfolding, see [YA] and [PT_3, Chapter 3]. For the converse, in general, some semi-global hypotheses on the dynamics are necessary (this is the case, for instance, of the saddle-node cycles, [NPT]). This paper illustrates such an interplay between local (saddle-node) and semi-global (creation of cycles) bifurcations in the context of heterodimensional cycles.

We now focus on the partially hyperbolic (with one-dimensional central direction) case and say a few words about the dynamics after the bifurcation. We first note that in our setting after unfolding the cycle (and for all small $t > 0$) new transverse homoclinic points of P and Q (we omit the dependence of the saddles on the parameter t) are created. Moreover, there are no homoclinic tangencies and all the cycles created throughout the unfolding are heterodimensional. In fact, in the unfolding process infinitely many heterodimensional cycles are created and the parameters corresponding to cycles accumulate the first bifurcation $t = 0$. Finally, by the Smale's homoclinic theorem, there are transitive hyperbolic Cantor sets Λ_P and Λ_Q, containing P and Q and contained in the homoclinic classes of P and Q. In particular, the sets Λ_P and Λ_Q have different indices (dimension of the unstable bundle). In some cases the homoclinic classes of P and Q are both nonhyperbolic and contain properly the sets Λ_P and Λ_Q. But there are examples of arcs $(f_t)_{t\in[-1,1]}$ such that there are parameter intervals corresponding to hyperbolic diffeomorphisms f_t for which these two classes are simultaneously hyperbolic. A goal of this paper is to describe how these diffeomorphisms can bifurcate. So three crucial steps in this context are to determine whether:

(i) the homoclinic classes of P and Q are hyperbolic (so they can be taken equal to Λ_P and Λ_Q);

(ii) the resulting non-wandering set of f_t has elements disjoint from these homoclinic classes; and

(iii) the diffeomorphism f_t exhibits a secondary cycle, associated either to saddles in the homoclinic classes of P and Q or to new saddles, disjoint from the homoclinic classes of P and Q, created throughout the unfolding.

Heuristically, several previous results (see [$\mathrm{Dí}_1$, $\mathrm{Dí}_2$, DRo_1, DRo_3]) suggest that the behavior after the bifurcation is essentially determined by the dynamics of the bifurcating diffeomorphism f_0 in the central direction (we will precise the meaning of this sentence later on). Taking bifurcating diffeomorphisms f_0 with an appropriate central dynamics one gets arcs $(f_t)_{t\in[-1,1]}$ as above such that there are parameter intervals H_n, $H_n \to 0^+$, such that f_t is hyperbolic for all $t \in H_n$. In particular, for every $t \in H_n$ the homoclinic classes of P and Q are hyperbolic and disjoint, see [DRo_1]. But, for the same arc $(f_t)_{t\in[-1,1]}$, there are also other parameter intervals B_n (corresponding to nonhyperbolic diffeomorphisms) for which the homoclinic classes of P and Q coincide, see [DRo_2]. So a natural problem is to understand the transition from hyperbolic to nonhyperbolic dynamics (i.e., the secondary bifurcations of $(f_t)_{t\in[-1,1]}$). This is one of the goals of this article.

We construct arcs $(f_t)_{t\in[-1,1]}$ as before such that there is a sequence of secondary bifurcations t_n, $t_n \to 0^+$, such that for each t_n the hyperbolic homoclinic classes of P and Q collide via a saddle-node bifurcation. This situation exemplifies the interplay between semi-global (heterodimensional cycles) and local (saddle-node) bifurcations, where the (local) saddle-node bifurcations bring substantial global changes of the dynamics. For instance, we see that these collisions lead to *persistence of cycles* and *intermingled homoclinic classes* immediately after each saddle-node.

The phenomenon of persistence of (heterodimensional) cycles here is, in some sense, similar to the one of persistence of tangencies in the setting of homoclinic tangencies, [Ne_1, Ne_2, Mo], and critical saddle-node horseshoes, [DU]: it is also associated to the creation of thick Cantor sets (see Definition 6.1 and the Gap Lemma 7.1, and for an overview [PT_3, Chapter 6.1]). But the underlying dynamical reasons of the creation of these thick Cantor sets are rather different. In the case of homoclinic tangencies, it is related to the critical behavior of the dynamics (proximity to the quadratic family after a renormalization), while in our context the dynamics after the bifurcation is partially hyperbolic and non-critical: the invariant manifolds of the saddles in the cycle do not have folding points and the angles between these manifolds in the intersection points are uniformly bounded away from zero. Thus the mechanism leading to arbitrarily thick hyperbolic Cantor sets in this paper is new. Roughly, the thick Cantor sets are dynamically defined (in the sense in [PT_3, Chapter 4.1]), associated to a system of (infinitely many) iterated functions (whose generators are given by the central dynamics) without criticalities but having discontinuities.

1.2 Homoclinic classes and heterodimensional cycles

1.2.1 Homoclinic classes and their collisions

The *homoclinic class* of a (hyperbolic) saddle P of a diffeomorphism f, denoted by $H(P, f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Homoclinic classes are f-invariant, *transitive* (i.e., there exists a dense orbit), and the subset of the periodic points with the same index of P is dense in $H(P, f)$. An important property of homoclinic classes is their lower semi-continuous dependence on the diffeomorphisms, this property follows from the continuity of the invariant manifolds of hyperbolic saddles and of their transverse intersections on compact parts. Homoclinic classes may fail to be hyperbolic: for example, a homoclinic class may contain saddles of different indices. In fact, this sort of phenomenon typically occurs in the contexts of heterodimensional cycles, see [$\mathrm{Dí}_1$, $\mathrm{Dí}_2$, DRo_2], and (nonhyperbolic) robustly transitive sets and diffeomorphisms, see [BD, BDPR].

Homoclinic classes are natural elementary pieces of dynamics (as in the Axiom A and tame cases, see [Sm, Ab]) playing a key role in the structure of the dynamics. Recently, [Ar, CMP] proved that, for diffeomorphisms f in a residual set of the space of C^1-diffeomorphisms, $\mathrm{Diff}^1(M)$, any two homoclinic classes of f are either disjoint or coincide. Thus, by the C^1 closing lemma [Pu], the homoclinic classes generically define a partition of (a dense part of) the non-wandering set.

Motivated by these results, one aims to understand how homoclinic classes may change from the metric point of view, that is, to understand the discontinuities of the map $(P_f, f) \to H(P_f, f)$, where P_f is a hyperbolic saddle depending continuously on f. By a *(metric) bifurcation of a homoclinic class* $H(P_f, f)$ we mean a discontinuity of the previous map. A well studied way in which homoclinic classes bifurcate is via the creation of cycles. We study (simultaneous) bifurcations of a pair of homoclinic classes via their *collision* and see how this sort of bifurcation can appear related to the unfolding of heterodimensional cycles.

Given a one-parameter family of diffeomorphisms $(f_t)_{t\in[a,c]}$, by a *collision of a pair of homoclinic classes,* say $H(P, f_t)$ and $H(Q, f_t)$, at $t = b$, we mean that $H(P, f_t) \cap H(Q, f_t) = \emptyset$ for every $t \in [a, b)$, and $H(P, f_t) \cap H(Q, f_t) \neq \emptyset$ for $t = b$. In the case studied here, the classes are both nontrivial and hyperbolic before the collision, and they are different for the collision parameter $t = b$. For $t = b$, the intersection of the classes is the orbit of a saddle-node, which is the responsible of the loss of hyperbolicity of the classes. This collision leads to the (simultaneous) explosions of the colliding homoclinic classes as well as to a string of bifurcations (persistence of heterodimensional cycles).

The simplest case of collision of homoclinic classes occurs when two isolated saddles of different indices collide via a saddle-node bifurcation. A much more interesting dynamics rises when the saddle-nodes are non-isolated. In dimension two, the main examples are the *Derived from Anosov (DA)* bifurcations, [Sm], where a repeller and a non-trivial hyperbolic attractor (the homoclinic class of a saddle) collide, and the *saddle-node horseshoes,* [Ze, Ta, Cr], where a horseshoe (the homoclinic class of a saddle) and a sink collide. The DA bifurcations are isolated. Saddle-node horseshoe bifurcations may be isolated (as in [CK]) or lead to a string of bifurcations (including Hénon-like attractors, persistence of tangencies, and variation of entropy, as in [DRoV, Co, DU, DRi]), depending on semi-global aspects of the dynamics. For a survey on this subject see [DRiV].

In the collisions in the previous paragraph, one of the classes is trivial (a sink or a repeller). In our construction the two colliding homoclinic classes are hyperbolic (before the collision) and non-trivial (for that we need to consider an ambient manifold of dimension 3 or higher). Collisions of two non-trivial and nonhyperbolic homoclinic classes were studied in [DS], where the collision leads to (using the terminology in [DS]) the *collapse* of the classes in the sequel of the bifurcation: the two classes explode and each class absorbs the other one, that is, they coincide.

1.2.2 Dynamics at heterodimensional cycles

The *index of a hyperbolic transitive set* Λ of a diffeomorphism f is the dimension of its unstable bundle. A diffeomorphism f has a heterodimensional cycle if there are (transitive) hyperbolic sets Λ and Σ having different indices such that

$$W^s(\Lambda) \cap W^u(\Sigma) \neq \emptyset \neq W^u(\Lambda) \cap W^s(\Sigma).$$

This sort of cycles only can occur in dimension bigger than or equal to three. We consider the simplest case (in fact, codimension-one case): there are saddles $P \in \Lambda$ and $Q \in \Sigma$ such that

- $\operatorname{index}(Q) = \operatorname{index}(P) + 1$;
- $W^u(Q) \cap W^s(P)$ is non-empty and transverse; and
- $W^s(Q) \cap W^u(P)$ is non-empty and quasi-transverse (i.e., for every heteroclinic point $x \in W^s(Q) \cap W^u(P)$ it holds $T_xW^s(Q) \oplus T_xW^u(P) = T_xW^s(Q) + T_xW^u(P)$ and the sum has codimension one in T_xM, where M is the ambient manifold).

In many cases, including bifurcations from Morse-Smale systems, heterodimensional cycles lead to persistent nonhyperbolic dynamics. Given an arc $(f_t)_{t\in[-1,1]}$ unfolding a cycle, say at $t = 0$ and for positive t, let $H(t) = \{s \in [0,t]\colon f_t \text{ is hyperbolic}\}$ and $B(t) = \{s \in [0,t]\colon H(P, f_s) = H(Q, f_s)\}$. For $t \in B(t)$, we say that the homoclinic classes of P and Q are *intermingled.* There is an open class of families of $(f_t)_{t\in[-1,1]}$ such that $B(t) = (0,t]$ for all small t, see [Dí$_1$, Dí$_2$]. There are also cases where $B(t) \neq (0,t]$, but $B(t)$ contains intervals and verifies $\liminf |B(t)|/t > 0$, ($|\cdot|$ denotes the measure), see [DRo$_2$]. In these cases we say (following [PT$_3$, Chapter 7]) that the dynamics is *robustly* and *prevalently nonhyperbolic* after the bifurcation, respectively.

In some cases both sets $H(t)$ and $B(t)$ contain intervals and satisfy $\liminf |B(t)|/t > 0$ and $\liminf |H(t)|/t > 0$ (using the terminology above hyperbolicity and nonhyperbolicity are both prevalent). Moreover, the arc can be constructed with *density of hyperbolicity arbitrarily close to* 1 *at the bifurcation*: given any $\varepsilon > 0$, there is an open set of arcs (as above) such that $\liminf |H(t)|/t > 1 - \varepsilon$. Following [$\mathrm{PT}_3$] we say that the phenomena of nonhyperbolic dynamics (intermingled classes) and hyperbolic dynamics are both *not totally prevalent* after the bifurcation.

The hyperbolic parameters $t \in H(t)$ in the construction in [DRo_1] are such that the exploding non-wandering set of f_t is the (disjoint) union of the homoclinic classes of P and Q. Thus a natural question is to describe how these hyperbolic diffeomorphisms bifurcate. Two possibilities are the creation of new heterodimensional cycles and the loss of hyperbolicity of a saddle (for instance, and typically, via a saddle-node bifurcation). We construct examples where the unfolding of the cycle leads to a sequence of collisions of (non-trivial) hyperbolic homoclinic classes via saddle-node bifurcations: There is sequence of parameter intervals $(h_n^-, h_n^+) \subset H(t)$, $h_n^\pm \to 0^+$, such that each h_n^- is a first (secondary) bifurcation such that there is

1. a collision of the hyperbolic homoclinic classes of P and Q at h_n^-;
2. persistence of heterodimensional cycles in an interval (c_n, h_n^-).

By *persistence of heterodimensional cycles* in a parameter interval (t_0, t_1) we mean that there are two hyperbolic (transitive) sets of different indices Λ_t and Σ_t of f_t (depending continuously on t) such that, for any $t \in (t_0, t_1)$, the diffeomorphism f_t has a cycle associated to Λ_t and Σ_t.

Our construction illustrates the importance of secondary bifurcations in the unfolding of cycles. Such bifurcations also play a key role in the homoclinic setting, think for instance of the secondary tangencies associated to *thick hyperbolic sets,* responsible for the persistence of tangencies and the coexistence of infinitely many sinks, see [Ne_2, Ne_3, PT_3].

1.3 Main result and ingredients of the proof

Let f be a diffeomorphism with a cycle associated to hyperbolic saddles P and Q. A neighbourhood of the cycle is an open set W containing the elements of the cycle: the orbits of the saddles P and Q and the intersections $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$.

Given an arc $(f_t)_{t \in [-1,1]}$ unfolding a cycle at $t = 0$ (for positive t) and a neighbourhood W of the cycle, the *relative nonwandering set* of f_t in W is the intersection of the nonwandering set $\Omega(f_t)$ of f_t and the maximal invariant set $\Lambda_t(W) = \cap_{\mathbb{Z}} f_t^i(W)$ of f_t in W.

Theorem 1. *There are an arc $(f_t)_{t \in [-1,1]}$ unfolding a heterodimensional cycle at $t = 0$ (for positive t) associated to saddles P and Q, of indices r and $r+1$, a neighbourhood W of the cycle, $n_0 \in \mathbb{N}$, and a sequence of scaled parameter intervals C_n and H_n, $n \geq n_0$,*

$$C_n = [a_n, \bar{t}_n) = \lambda^n \,[1 + \mu^-, 1 + \mu^\star) \quad \textit{and} \quad H_n = [\bar{t}_n, b_n] = \lambda^n \,[1 + \mu^\star, 1 + \mu^+],$$

for some $0 < \lambda < 1$ and $\mu^- < \mu^\star < \mu^+$, such that:

(1) Hyperbolicity and secondary collision bifurcations: *for every $t \in H_n$, the relative nonwandering set of f_t in the neighbourhood W of the cycle is the union of the nontrivial homoclinic classes $H(P, f_t)$ and $H(Q, f_t)$ of P and Q. Moreover,*

- **(1a) prevalent hyperbolicity:** *for every $t \in (H_n \setminus \{\bar{t}_n\})$, $H(P, f_t)$ and $H(Q, f_t)$ are disjoint and hyperbolic;*

(1b) collisions of homoclinic classes via saddle-node bifurcations: *for every* $\bar{t}_n$, $H(P, f_{\bar{t}_n}) \cap H(Q, f_{\bar{t}_n})$ *is the orbit of a saddle-node. Moreover, every compact invariant set contained in each homoclinic class disjoint from the saddle-node is hyperbolic.*

(2) Persistence of cycles: *the sets* C_n *are intervals of persistence of heterodimensional cycles. More precisely, for every* $t \in C_n$*, the diffeomorphism* f_t *has a cycle associated to a hyperbolic set of index* r *(contained in the homoclinic class of* P*) and a hyperbolic set of index* $r+1$*.*

A key ingredient in our construction is the one-dimensional limit dynamics associated to (partially hyperbolic) heterodimensional cycles introduced in [DRo$_3$]. We will detail this construction (also used in a particular case in [DS]) in Section 3.2. This limit dynamics plays a role similar to the quadratic family in the context of homoclinic tangencies (see, for instance, [PT$_3$, Chapter 6]) or to the limit dynamics associated to saddle-node cycles (see [NPT, DRoV]), for describing the dynamics after the bifurcation. This limit dynamics is non-critical, has discontinuities, and is given by a one-parameter family of systems of iterated functions generated by:

(i) two affine maps, corresponding to the linearization of the dynamics in the weak unstable direction of Q and the weak stable direction of P,

(ii) a transition map given by the iterates from a (scaled) fundamental domain of Q to a (scaled) fundamental domain of P,

(iii) a translation corresponding to the rescaling of the parameter t unfolding the cycle.

The series of works [PT$_1$, PT$_2$, Mo, PY, MPV] show that the frequency of hyperbolicity of the dynamics after unfolding a homoclinic tangency is mainly determined by the fractal dimension of the hyperbolic set in the cycle (limit capacity and Hausdorff dimension). For heterodimensional cycles, the role of fractal dimensions is quite enigmatic and until now unexplored. In fact, the dynamics after the bifurcation is essentially determined by the limit dynamics above. Here we describe the creation of *thick one dimensional Cantor sets* (as in [Ne$_2$, Ne$_3$] the sets are dynamically defined but here they are defined by non-critical maps) associated to collisions of hyperbolic homoclinic classes and study how they lead to persistence of heterodimensional cycles.

Let us say a few words about our construction. Denote by $(\varphi_\mu)_{\mu \in [-\delta, \delta]}$ the (non-critical) limit family. As in the case of saddle-node cycles, there are sequences $t_n^\pm$ of parameters, $t_n^\pm \to 0$, and reparametrizations $\varrho_n \colon [-\delta, \delta] \to [t_n^-, t_n^+]$ with nice distortion properties such that the dynamics of $f_{\varrho_n(\mu)}$ is the product of φ_μ (the central part) by a hyperbolic saddle dynamics. Every system φ_μ has two hyperbolic sets Λ'_μ and Σ'_μ (depending continuously on μ) with non-empty intersection. Such an intersection is assured by a condition on the product of their thickness (see the Gap Lemma 7.1). The sets Λ'_μ and Σ'_μ are the projections along the stable and unstable directions of two hyperbolic sets $\Lambda_{\varrho_n(\mu)}$ and $\Sigma_{\varrho_n(\mu)}$ of $f_{\varrho_n(\mu)}$ having different indices. In this way, the intersections between Λ'_μ and Σ'_μ correspond to (heterodimensional) cycles associated to $\Lambda_{\varrho_n(\mu)}$ and $\Sigma_{\varrho_n(\mu)}$.

Some comments about our constructions arise naturally. First, by [Ar, CMP], the collision of homoclinic classes is a non-generic phenomenon (at least in the C^1-topology) and we do not know the flexibility of our construction, roughly a product dynamics where we fix a one-dimensional dynamics and construct a cycle whose central limit dynamics is the fixed one. For instance, we do not know whether our result holds for arcs of diffeomorphisms in a neighbourhood of the arcs constructed in this paper. We point out that the product structure of the example allows us to estimate the thickness of Cantor sets created in the unfolding of the cycle, concluding the persistence of cycles thereafter. But, in general, estimates of the thickness in dimension bigger than two are highly complicated and depend on the regularity of the stable and unstable foliations, see [PV]. Probably, the ideas sketched in [MPV] for studying homoclinic tangencies in higher dimensions could be an ingredient for extending our results to non-product dynamics.

Finally, related to our results, in the context of vector fields of three manifolds, in [BMP] is constructed a vector field with two non-trivial homoclinic classes whose intersection is the closure of the unstable manifold of a hyperbolic singularity. See also [MP], where it is proved that a singular hyperbolic set (see [MPP] for the precise definition) containing a unique singularity is either transitive or the union of two homoclinic classes with non-empty intersection.

This paper is organized as follows. In Section 2, we describe a model arc unfolding a heterodimensional cycle. In Section 3, we introduce a one-parameter system of iterated (one-dimensional) functions describing the central dynamics in the sequel of the bifurcation (limit dynamics). After introducing the basic terminology, we give a sketch of the proof of the theorem in Section 3.3. In Section 4, we determine the central dynamics by considering saddle-node arcs. In Section 5, we construct two isolated hyperbolic sets (of different indices) and describe their collision via a saddle-node bifurcation. In Section 6, we construct two one-dimensional dynamically defined hyperbolic Cantor sets (contracting and expanding), associated to the system of iterated functions, the expanding one with large thickness. Finally, we see that these Cantor sets correspond to two hyperbolic sets of diffeomorphisms of the initial arc (see Sections 6.1 and 6.3) and prove, in Section 7, that these hyperbolic sets have heterodimensional cycles.

2 The model one-parameter family

For notational simplicity, we will carry our constructions in dimension 3, the extension to higher dimensions is straightforward. We consider a one-parameter family of (local) diffeomorphisms $(f_t)_{t\in[-\tau,\tau]}$ defined on the cube $R = [-1,1] \times [-1,5] \times [1,1]$ of $\mathbb{R}^3$. Using local coordinates, this construction can be carried to any manifold of dimension three.

The definition of the arc is done in three steps: (a) semi-local partially hyperbolic dynamics, (b) existence and unfolding of the cycle, and (c) global hypotheses: existence of a filtration and invariant foliations. The filtration will enable us to control the dynamics of the diffeomorphisms f_t after the bifurcation and the invariant foliations will allow us to reduce the study of the dynamics to a one-dimensional family of iterated functions.

The heterodimensional cycle is related to hyperbolic fixed saddles $Q = (0,0,0)$ and $P = (0,4,0)$ of indices 2 and 1, respectively, and the semi-local dynamics satisfies the following properties.

(a) Partially hyperbolic product dynamics. In the cube $R = [-1,1] \times [-1,5] \times [-1,1]$,for every $t \in [-\tau,\tau]$ it holds $f_t = f$, where

$$f(x,y,z) = (\lambda_s\, x, F(y), \lambda_u\, z), \qquad 0 < \lambda_s < 1 < \lambda_u,$$

where F is a strictly increasing function to be fixed in Section 4 such that:

local affinity: $F(y) = \beta\, y$, if $y \in [-1, \beta^{-1}]$, and $F(y) = \lambda\,(y-4)+4$, if $y \in [3,5]$, where $0 < \lambda < 1$ and $\beta = \lambda^{-1}$,

domination: $\lambda_s < F'(y) < \lambda_u$, for every $y \in [-1,5]$.

The last condition implies that $\lambda_s < \lambda < \beta < \lambda_u$ and that the restriction of f to R has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with three non-trivial bundles given by the coordinate directions. Observe also that, by definition of the diffeomorphism f,

$$\begin{array}{lcl} \{0\} \times [-1,4) \times [-1,1] \subset W^u(Q) & \text{and} & [-1,1] \times \{0\} \times \{0\} \subset W^s(Q), \\ \{0\} \times \{4\} \times [-1,1] \subset W^u(P) & \text{and} & [-1,1] \times (0,5] \times \{0\} \subset W^s(P). \end{array}$$

Thus the invariant manifolds $W^s(P)$ and $W^u(Q)$ meet transversely along the curve $\gamma = \{0\} \times (0,4) \times \{0\}$, named *connection.*

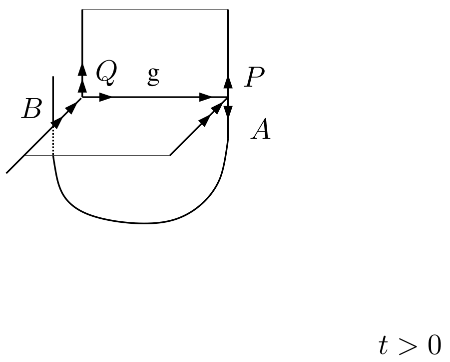

Figure 1: The model cycle and its unfolding

(b) Existence and unfolding of the heterodimensional cycle. Take the points $A = (0,4,-\frac{1}{2}) \in W^u(P)$ and $B = (-\frac{1}{2},0,0) \in W^s(Q)$. We assume that there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(A) = B$ and $f^j(A) \notin R$ for all $0 < j < k_0$. Thus $W^u(P) \cap W^s(Q) \neq \emptyset$ (i.e., there is a heterodimensional cycle associated to P and Q). Take a small compact neighbourhood U_A of A such that $U_A, \dots, f^{k_0}(U_A)$ are pairwise disjoint, $U_A, f^{k_0}(U_A) \subset R$, and $f^j(U_A) \cap R = \emptyset$, for all $0 < j < k_0$. We assume that the dynamics of f^{k_0} from U_A to $f^{k_0}(U_A)$ is of the form

affine return: $f^{k_0}(x,y,z) = (x + \frac{1}{2}, y - 4, z + \frac{1}{2})$, for every $(x,y,z) \in U_A$.

This implies that, for some $\delta > 0$, $\{(-1/2,0)\} \times (-\delta,\delta) \subset W^u(P)$, thus $W^u(P)$ and $W^s(Q)$ meet *quasi-transversely* throughout the orbit of the heteroclinic point A (i.e., $T_AW^u(P) + T_AW^s(Q) = T_AW^u(P) \oplus T_AW^s(Q) = \mathbb{XZ}$ and this sum has dimension two). Hence, since $W^s(P) \pitchfork W^u(Q) \neq \emptyset$, f has a (codimension one) heterodimensional cycle associated to P and Q.

We now define the arc arc $(f_t)_{t \in [-\tau,\tau]}$, small $\tau > 0$, unfolding the cycle above at $t = 0$. For all $(x,y,z) \in R$, $f_t(x,y,z) = f(x,y,z)$ and the restriction of $f_t^{k_0}$ to U_A is the composition of f^{k_0} and a t-translation parallel to the $\mathbb{Y}$ direction:

$$f_t^{k_0}(x,y,z) = f^{k_0}(x,y,z) + (0,t,0) = (x + \frac{1}{2}, y - 4 + t, z + \frac{1}{2}). \tag{1}$$

By construction, for each $t > 0$ and $s \in (-\delta,\delta)$, one has that $(-1/2,t,s) \in W^u(P,f_t)$. As $[-1,1] \times (0,5) \times \{0\} \subset W^s(P,f_t)$ one has that $(\frac{1}{2},t,0) = f_t^{k_0}(0,4,-\frac{1}{2}) = f_t^{k_0}(A)$ is a transverse homoclinic point of P. Similarly, the orbit of B consists of transverse homoclinic point of Q. Thus, for $t > 0$ the homoclinic classes of P and Q are both non-trivial.

(c) Global dynamics: filtrations and invariant foliations We now assume some global properties of the bifurcating diffeomorphism f that allow us to control the dynamics of f in the ambient manifold. First, we say that a pair (M_1, M_2) is a *filtration* of f if M_1 and M_2 are compact manifolds with boundary of the same dimension as M such that $f(M_i) \subset \operatorname{int}(M_i)$, $i = 1, 2$.

(c$_1$) Relative Morse-Smale dynamics. The set $W = R \cup (\cup_{j \in \{1,\dots,k_0-1\}} f^j(U_A))$ is a *filtrating neighborhood of the cycle:* there is a filtration (M_1, M_2) of f such that $W = M_2 \setminus \operatorname{int}(M_1)$. This implies that if $x \in W$ and $f(x) \notin W$ then $f^n(x) \notin W$, for every $n \in \mathbb{N}$. Similarly, if $x \in W$ and $f^{-1}(x) \notin W$ then $f^{-n}(x) \notin W$, for all $n \in \mathbb{N}$. It follows that, for every small $|t|$, the set W is also a filtrating neighbourhood of f_t. Hence, for all small $|t|$,

$$\Omega(f_t) \cap W = \Omega(f_t|_W) \subset \bigcap_{n \in \mathbb{Z}} f_t^n(W), \qquad H(P, f_t) \cup H(Q, f_t) \subset \bigcap_{n \in \mathbb{Z}} f_t^n(W).$$

The semi-local dynamics of f_t implies that, for $t < 0$, the restriction of f_t to W is Morse-Smale (in particular, the homoclinic classes of P and Q are both trivial). Thus at $t = 0$ there is a Ω-*explosion*: for positive t, the classes of $H(P, f_t)$ and $H(Q, f_t)$ are both non-trivial.

(c$_2$) Invariant foliations. Given $X = (x_0, y_0, z_0) \in R$, define its *strong stable, strong unstable* and *central leaves,* denoted by $F^s(X), F^u(X)$, and $F^c(X)$, by

$$\begin{aligned} F^s(X) &= \{(x, y, z) \in R : \; y = y_0, z = z_0\}, \\ F^u(X) &= \{(x, y, z) \in R : \; x = x_0, y = y_0\}, \\ F^c(X) &= \{(x, y, z) \in R : \; x = x_0, z = z_0\}. \end{aligned}$$

This defines three foliations (strong stable, strong unstable, and central) on R. For each t, we extend them, via f_t, to the whole set W. By construction, these foliations (denoted by $\mathcal{F}_t^\sigma$, $\sigma = s, u, c$) are f_t-invariant in the following sense: if $X \in R$ and $f_t(X) \in R$ then $f_t(F^s(X)) \subset F_t^s(f_t(X))$, $f_t(F^u(X)) \supset F^u(f_t(X))$ and $f_t(F^c(X)) \cap F^c(f_t(X))$ is a neighbourhood of $f_t(X)$ in $F^c(f_t(X))$; if $X \in R$, $f_t(X) \notin R$ and $f_t^{k_0}(X) \in R$ (i.e., $X \in U_A$)) then $f_t(F_t^\sigma(X)) \subset F_t^\sigma(f_t(X))$, $\sigma = s, u, c$.

Let $\Lambda_t = \cap_{i \in \mathbb{Z}} f_t^i(W)$. The dynamics of f_t in Λ_t has two hyperbolic directions (parallel to the axis $\mathbb{X}$ and $\mathbb{Z}$). Observe that if $X \in R$ then $Df_t(X)$ is uniformly contracting in the $\mathbb{X}$-direction and uniformly expanding in the $\mathbb{Z}$-direction, and if $X \in U_A$ then $Df_t^{k_0}(X)$ is the identity. These remarks and the partial hyperbolicity of f_t in R imply that the set Λ_t is partially hyperbolic: the $\mathbb{X}$ and $\mathbb{Z}$-directions are hyperbolic (contracting and expanding) and dominate the (central) $\mathbb{Y}$-direction. This means that the dynamics of f_t is essentially determined by its central dynamics. The study of such central dynamics is done in the next section.

3 Normalized central dynamics, selection of parameters, and outline of the construction

In this section we introduce the key ideas in the proof of the theorem and sketch its proof. We first present the main terminology in our constructions: itineraries (Section 3.1) and normalized central dynamics (Section 3.2). Using these tools we outline the proof in Section 3.3.

3.1 Itineraries

Take fundamental domains $D^+ = [\beta^{-1}, 1]$ and $D^- = [3, 4-\lambda]$ of F and assume that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$. Consider the reference cubes $\Delta^\pm = [-1,1] \times D^\pm \times [-1,1]$, small $t > 0$, and $X \in \Lambda_t \cap \Delta^+$. There are two possibilities concerning its positive iterates: either there is $n > 1$ such that $f_t^n(X) \in \Delta^+$ or not. In the first case, the smallest $n > 1$ with $f_t^n(X) \in \Delta^+$ is the *forward return time of* X, denoted by $r^+(X)$. In the second case, $X \in W^s(P) \cup W^s(Q)$. Similarly, for negative iterates of X, either there is $n > 1$ with $f_t^{-n}(X) \in \Delta^+$ or not. In the first case, the *backward return time of* X, $r^-(X)$, is defined as above. In the second case, $X \in W^u(P) \cup W^u(Q)$. These facts follow from the existence of the filtration and the geometry of the cycle. For a proof see [DRo$_1$, Lemma 4.1]. As, by the definition of the f_t, the invariant manifolds of P and Q meet transversely or quasi-transversely, according to the case, the previous results may be read as follows:

Lemma 3.1. *Given any point $X \in \Lambda_t \cap \Delta^+$ there are the following possibilities:*

- *X has infinitely many forward or backward returns to Δ^+,*
- *X is a transverse homoclinic point of P or Q,*
- *$X \in W^u(P) \cap W^s(Q)$ (heteroclinic cycle point) and $W^u(P, f_t)$ and $W^s(Q, f_t)$ meet quasi-transversely at X, and*
- *$X \in W^s(P) \cap W^u(Q)$ (heteroclinic connection point) and $W^s(P, f_t)$ and $W^u(Q, f_t)$ meet transversely at X.*

Given $X = (x, y, z)$ let $X_i = (x_i, y_i, z_i) = f_t^i(X)$. Assume that X has a first forward return $r^+(X) = \bar{q} + k_0 + \bar{p} + N$ to Δ^+, where k_0 is as in the definition of the cycle, $F^N(D^+) = D^-$, and

$$\begin{array}{l}
X_N \in \Delta^-, \\
X_N, \dots, X_{\bar{p}-1+N}, X_{\bar{p}+N} \in R \quad \text{and} \quad X_{\bar{p}+N} \in U_A, \\
X_{k_0+\bar{p}+N} \in f_t^{k_0}(U_A) \subset R, \\
X_{i+k_0+\bar{p}+N} \in (R \setminus \Delta^+) \quad \text{for all } i = 0, 1, \dots, \bar{q}-1, \text{ and} \\
X_{\bar{q}+k_0+\bar{p}+N} \in \Delta^+.
\end{array}$$

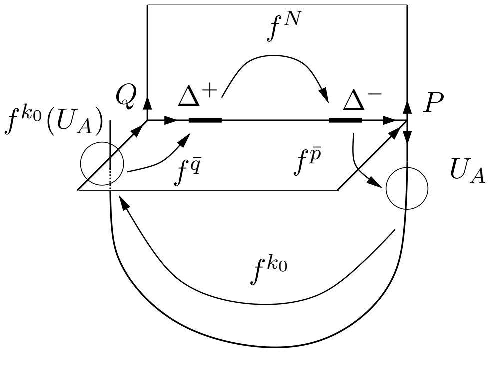

Figure 2: Itineraries

3.2 Choice of parameters, normalized dynamics, and iterated functions

For special parameters, we can estimate $\bar{p}$ and $\bar{q}$, controlling the returns in the central direction.

Lemma 3.2. *Consider the sequences of parameters* $t_n = \lambda^n$ *and of scaled parameter intervals* $B_n = [t_n(1-\delta), t_n(1+\delta)]$, *small* $\delta > 0$. *Then for every point* $X \in \Delta^+$ *having a positive return to* Δ^+ *it holds* $\bar{p}, \bar{q} \geq (n-1)$. *Moreover, if* $t = t_n(1+\mu)$, $\mu > 0$, $\bar{p}, \bar{q} \geq n$.

Proof: Fix $t = t_n(1+\mu)$, $\mu \in (-\delta, \delta)$, and $X = (x, y, z)$. By definition of f_t and since F is affine in $[3, 5]$, $y_{\bar{p}+N} = \lambda^{\bar{p}}(y_N - 4) + 4$. Since X returns to Δ^+, the unfolding condition (1) implies that

$$0 < y_{k_0+\bar{p}+N} = \lambda^{\bar{p}}(y_N - 4) + 4 + (t - 4) = \lambda^{\bar{p}}(y_N - 4) + t.$$

Finally, as $y_N \in D^- = [3, 4-\lambda]$, $-1 \leq (y_N - 4) \leq -\lambda$, it follows

$$-t = -t_n(1+\mu) = -\lambda^n(1+\mu) < \lambda^{\bar{p}}(y_N - 4) \leq -\lambda^{\bar{p}}\lambda = -\lambda^{\bar{p}+1}, \qquad \lambda^n(1+\mu) > \lambda^{\bar{p}+1}.$$

So, if μ is small, $\bar{p} \geq (n-1)$. Moreover, if $\mu > 0$ one has $\bar{p} \geq n$.

Finally, the estimate for $\bar{q}$ follows similarly noting that $y_{k_0+\bar{p}+N} < t$. □

Fix $t = t_n(1+\mu) \in \lambda^n(1-\delta, 1+\delta)$. Using Lemma 3.2, we write

$$r^+(X) = q(X) + n + k_0 + p(X) + n + N, \qquad q(X), p(X) \geq -1,$$

and we say that X has a *return* $r^+(X)$ *of type* $(q(X), p(X))$. Since $\beta^n = t_n^{-1}$ and $\lambda^n = t_n$, the y-coordinate of $X_{r^+(X)}$ is

$$\begin{aligned} y_{r^+(X)} &= \beta^{q(X)+n}\left[(t_n(1+\mu) - 4) + (\lambda^{n+p(X)}(F^N(y) - 4) + 4\right] = \\ &= \beta^{q(X)+n}\left[t_n(1+\mu) + (\lambda^{n+p(X)}(F^N(y) - 4))\right] = \\ &= \beta^{q(X)}\left[1 + \mu + \lambda^{p(X)}(F^N(y) - 4)\right]. \end{aligned} \tag{2}$$

Equation (2) leads to the following one-parameter family of maps of the interval describing the central dynamics of returns to Δ^+:

$$\Phi_\mu^{q,p} \colon D_\mu^{q,p} \to D^+, \qquad \Phi_\mu^{q,p}(y) = \beta^q\left[1 + \mu + \lambda^p(F^N(y) - 4)\right], \quad \mu \in [-\delta, \delta], \quad p, q \geq -1, \tag{3}$$

where $D_\mu^{q,p}$ is the maximal subset of D^+ consisting of points y with $\Phi_\mu^{q,p}(y) \in D^+$ (see Figure 3). We call this family the *normalized central dynamics of* f_t. Next remark justifies this name and implies that $\Phi_\mu^{q,p}$ gives the y-coordinate of any (q, p)-return of f_t, $t = t_n(1+\mu)$, *for any* n.

Remark 3.3. (Central dynamics). *Let* $t = \lambda^n(1+\mu) \in B_n$.

- *By the monotonicity of* F, $D_\mu^{q,p}$ *is either empty or a closed interval.*
- *Suppose that* $X = (x_0, y_0, z_0) \in \Delta^+$ *has a* (q, p)*-return. Then* $y_0 \in D_\mu^{q,p}$ *and the* y*-coordinate of* $f_t^{q+n+k_0+p+n+N}(X)$ *is* $\Phi_\mu^{q,p}(y_0)$. *Conversely, given any* $y_0 \in D_\mu^{q,p}$, *there is at least one point of the form* $X = (x, y_0, z) \in \Delta^+$ *with a* (q, p)*-return.*
- **(consecutive returns)** *Suppose that* $X_0 = (x_0, y_0, z_0) \in \Delta^+$ *has a* (q_0, p_0)*-return* X^1 *and* X^1 *has a* (q_1, p_1)*-return* X^2. *Then* X^2 *is the* second return *of* X_0 *whose* y*-coordinate is* $\Phi_\mu^{q_1,p_1} \circ \Phi_\mu^{q_0,p_0}(y_0)$.

The dynamics of the maps $\Phi_\mu^{q,p}$ depend *only* on F^N, λ, and μ. In the next section, we will choose F^N and λ accordingly. We first consider returns of type $(0, 0)$, given by $F^N(y) - 3 + \mu$.

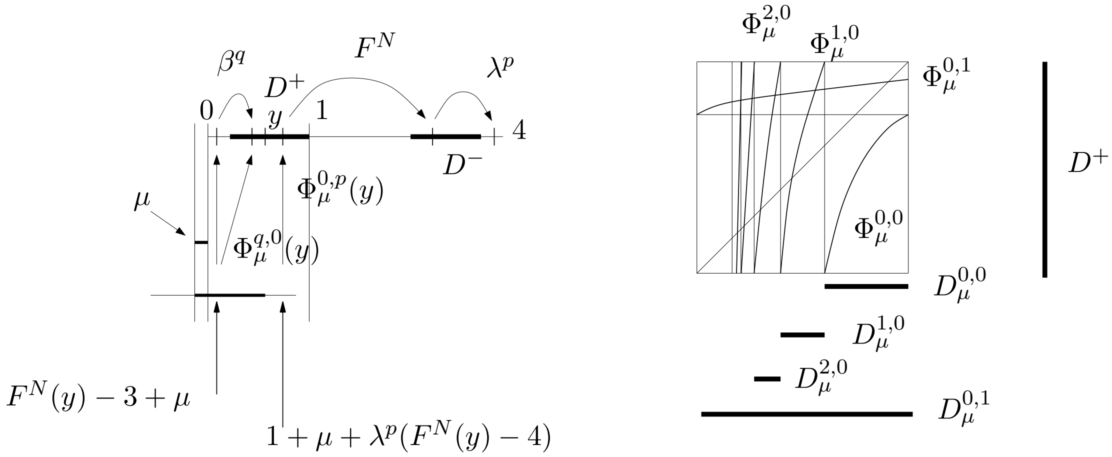

Figure 3: The maps $\Phi_\mu^{q,p}$

3.3 Outline of the constructions. Sketch of the proof

In this section, we present the main steps of our constructions and sketch the proof of the theorem.

Selection of the central dynamics: the map F. To define F, we consider the auxiliary saddle-node arc g_α

$$g_\alpha(y) = y - \varepsilon(y - 1/2)^2 + \alpha\,(1 - y),$$

with a saddle-node at the point 1/2 for the parameter $\alpha = 0$ (here $\varepsilon > 0$ is small). The saddle-node is unfolded for negative α: for $\alpha > 0$, the map g_α has two hyperbolic fixed points close to 1/2, collapsing to 1/2 for $\alpha = 0$ and disappearing thereafter. We select an interval L containing the saddle-node 1/2 in the reference fundamental domain $D^+ = [\beta^{-1}, 1]$. To each parameter α (close to 0), we associate a map F_α defining a possible central dynamics of f: in the cube R, $f(x, y, z) = (\lambda_s\, x, F_\alpha(y), \lambda_u\, z)$. The key property of F_α, see condition (F3) in Section 4.1, is

$$F_\alpha^N : L \to D^- = [3, 4 - \lambda], \quad F_\alpha^N(y) = g_\alpha(y) + 3. \tag{4}$$

We select α_0 and let $F = F_{\alpha_0}$. We will explain latter on this choice, which is done for guaranteeing the creation of one dimensional Cantor sets with large thickness after the unfolding the saddle-node (see the last paragraph in this section). This is a key step for obtaining the persistence of cycles.

Dynamics of the returns. Fixed α_0, so selected the central dynamics F and defined the arc, we consider the new saddle-node arc $h_\mu(y) = g_{\alpha_0}(y) + \mu$. Then, for each parameter of the form $t = t_n\,(1+\mu)$ (any n), the map h_μ defines the central y-coordinate for the points in $[-1, 1] \times L \times [-1, 1]$ having a return of type $(0, 0)$: by equations (2) and (4) and Remark 3.3 it holds:

$$\begin{aligned} y_{r^+(X)} &= \beta^{0+n}\left[(t_n\,(1+\mu) - 4) + (\lambda^{n+0}\,(F^N(y) - 4) + 4)\right] = \\ &= 1 + \mu + (F^N(y) - 4) = \mu + F^N(y) - 3 = g_\alpha(y) + \mu = h_\mu(y). \end{aligned}$$

Secondary (rescaled) saddle-node bifurcations. The arc (h_μ) has a saddle-node for the parameter $\mu^\star$ (close to 0) at some point $s_{\mu^\star}$ (close to 1/2). This saddle-node is also unfolded for negative μ. The construction of the arc (f_t) and the comments above imply that, for every $t_n^\star = t_n\,(1+\mu^\star)$, the diffeomorphism $f_{t_n^\star}$ has a saddle-node at some point $S_{t_n^\star}$ in Δ^+ whose y-coordinate is $s_{\mu^\star}$. For every $t_n^\star$ the saddle-node is *essentially* the same, the only difference being that the periods p_n of the saddle-nodes $S_{t_n^\star}$ increase as n, but the local central dynamics of the $f_{t_n^\star}^{p_n}$ at the $S_{t_n^\star}$ are always the same (and given by $h_{\mu^\star}$).

Hyperbolicity before the unfolding of the saddle-node. By Remark 3.3, for $t = t_n\,(1+\mu)$ small μ, the system $(\Phi_\mu^{p,q})$ determines the central dynamics of the returns to the reference cube Δ^+. We prove that there is μ^+, $\mu^+ > \mu^\star$, such that the system $(\Phi_\mu^{p,q})$ is *hyperbolic* for all $\mu \in (\mu^\star, \mu^+]$. For the precise meaning of this assertion see Section 5.1.

For $\mu > \mu^\star$, the map $h_\mu = \Phi_\mu^{0,0}$ has two fixed points s_μ^- (repelling) and s_μ^+ (contracting) collapsing to the saddle-node $s_{\mu^\star}$ at $\mu^\star$. As above, for $t = t_n(1+\mu)$, these points correspond to saddles S_t^- and S_t^+ of f_t, of indices 2 and 1, whose y-coordinate are s_μ^- and s_μ^+.

Using Remark 3.3, the product structure of the dynamics, and noting that the diffeomorphisms f_t uniformly contract in the $\mathbb{X}$-direction and uniformly expand in the $\mathbb{Z}$-direction, we translate the *hyperbolicity* of $(\Phi_\mu^{p,q})$ to f_t for every $t = t_n\,(1+\mu)$ with $\mu \in (\mu^\star, \mu^+]$ (any n). The next step is to prove, using the system $(\Phi_\mu^{p,q})$, that for such parameters one has that

$$H(P, f_t) = H(S_t^+, f_t) \qquad \text{and} \qquad H(Q, f_t) = H(S_t^-, f_t),$$

these classes are both hyperbolic, and their union is the relative non-wandering set in the filtrating neighbourhood W. Finally, we see that these hyperbolic classes collide for $t_n^\star$.

Persistence of cycles at the collisions. To get the intervals of persistence of cycles (after the unfolding of the saddle-nodes) we need to choose carefully the parameter α_0 in the definition of the map F. In fact, in the previous steps the choice of α_0 is essentially irrelevant.

For $\mu < \mu^\star$ and close to $\mu^\star$, the system $(\Phi_\mu^{p,q})$ defines (using an inducing map method) a family of (uniformly) expanding maps from the subset $[\beta^{-1} + \varepsilon(\mu), 1]$ of D^+ to the whole D^+, where $\varepsilon(\mu)$ is small. The choice of α_0 is such that $\varepsilon(\mu) \simeq \ell(\alpha_0)$, where $\ell(\alpha_0)$ is very small.

We consider an invariant Cantor set Σ_μ of these functions in $[\beta^{-1}+\epsilon(\mu), 1]$ (roughly, the maximal invariant set). Since $\epsilon(\mu)$ is very small (here the choice of α_0 is crucial), we get that the set Σ_μ has large thickness (roughly of order of $1/\ell(\alpha_0)$, see Proposition 6.6). Essentially, the *thickness* of a Cantor set is the quotient length(bridge)/length(gap), where in each step of the construction of the Cantor set a *gap* is a removed open interval and its two *bridges* are the adjacent resulting connected components, see Definition 6.1. Roughly, here the gaps of Σ_μ are the pre-images of $[\beta^{-1}, \beta^{-1}+\epsilon(\mu)]$ and the corresponding bridges pre-images of $[\beta^{-1}+\epsilon(\mu), 1]$, using bounded distortion properties, we see that these quotients are of order of $(1-\beta^{-1}-\epsilon(\mu))/\epsilon(\mu)$. As $\varepsilon(\mu)$ is very small, this ratio (and thus the thickness of the Cantor set) is very big. Finally, Remark 3.3 implies that Σ_μ is the projection in the central direction of a hyperbolic set Γ_t, $t = t_n\,(1+\mu)$, of index two.

We also consider a (hyperbolic) part Υ_t of the homoclinic class of P having thickness uniformly bounded from zero for all t, see Section 6.1.

The Gap Lemma (Lemma 7.1) asserts that two Cantor sets whose convex hulls are non-disjoint and whose product of their thickness is larger than one have non-empty intersection. The previous construction (taking $\ell(\alpha_0)$ small) implies that Cantor sets defined as projections of Γ_t and Υ_t verify the hypotheses of the Gap Lemma. The product structure gives now the announced persistence of heterodimensional cycles (associated to the sets Γ_t and Υ_t).

4 Definition of the map F. Saddle-node arcs

In this section we define the map F giving the central dynamics of the diffeomorphisms f_t.

4.1 A saddle-node family

For small positive ε (further restrictions on the size of ε will appear throughout the text) consider the saddle-node arc (g_α), small $|\alpha|$,

$$g_\alpha(y) = y - \varepsilon(y - 1/2)^2 + \alpha\,(1 - y).$$

The arc g_α unfolds, at $\alpha = 0$, a saddle-node at $1/2$ as follows. For $\alpha > 0$, the map g_α has two hyperbolic points. These points collapse to the saddle-node at $\alpha = 0$ and disappear for negative α.

For α and ε small, let d_α be the critical point of g'_α, $d_\alpha = 1/2 + (1-\alpha)/(2\,\varepsilon) > 1$. Since the maps g_α are increasing in $(-\infty, 1]$, $g_0(1/2) = 1/2 > \beta^{-1}$, and $g_0(\beta^{-1}) < \beta^{-1}$ (small ε), for every α close to 0 there is a unique $c_\alpha \in (\beta^{-1}, 1/2)$ with $g_\alpha(c_\alpha) = \beta^{-1}$. For each α, write $D^+ = [\beta^{-1}, c_\alpha] \cup [c_\alpha, 1]$.

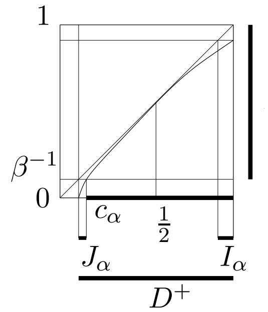

Figure 4: The saddle-node arc g_α

We take $\lambda = \beta^{-1} = \varepsilon/4$ and, for small α, choose a C^2-map $F_\alpha\colon [-1, 5] \to \mathbb{R}$ verifying conditions (F1)–(F5) below:

- **(F1)** F_α is strictly increasing and $F_\alpha^N(D^+) = D^- = [3, 4 - \lambda]$,
- **(F2)** $F_\alpha(y) = \beta\,y$ in $[-1, \beta^{-1}]$, $F_\alpha(y) = \lambda\,(y - 4) + 4$ in $[3, 5]$,
- **(F3)** the restriction of F_α^N to the interval $[c_\alpha, 1]$ is equal to $g_\alpha + 3$,
- **(F4)** $(F_\alpha^N)'(y) > 1$ for all $y \in [\beta^{-1}, c_\alpha)$, and
- **(F5)** $\lambda\,(F_\alpha^N)'(\beta^{-1}) = \beta\,(F_\alpha^N)'(1)$.

We will take the map F giving the central dynamics, see Section 2, equal to some F_α, for a convenient α chosen in Proposition 6.6.

For the existence of F_α, note first that (F1) and (F3) are compatible. On the one hand, by definition, $g_\alpha(1) = 1 - \varepsilon/4 = 1 - \lambda$, so $F_\alpha^N(1) = 4 - \lambda = g_\alpha(1) + 3$. On the other hand, and that $g_\alpha(c_\alpha) + 3 = \beta^{-1} + 3 > 3$. So there is space for extending F_α^N to the subinterval $[\beta^{-1}, c_\alpha]$ of D^+ in such a way $F_\alpha^N([\beta^{-1}, c_\alpha]) = [3, 3 + \beta^{-1}]$.

By continuity, it suffices to check (F4) for $\alpha = 0$. Since $F_0^N([\beta^{-1}, c_0]) = [3, 3+\beta^{-1}]$, it is enough to see that $c_0 - \beta^{-1} < \beta^{-1}$, that is, $c_0 < 2\,\beta^{-1}$. Since g_0 is strictly increasing in $[0,1]$, $c_0 > 0$, and $2\,\beta^{-1} < 1$, this condition is equivalent to

$$g_0(c_0) = \beta^{-1} < g_0(2\,\beta^{-1}). \tag{5}$$

Since $\beta^{-1} = \varepsilon/4$,

$$g_0(2\,\beta^{-1}) = g_0(\varepsilon/2) = \varepsilon/2 - \varepsilon\,(\varepsilon/2 - 1/2)^2 = \varepsilon(1/2 - 1/4\,(\varepsilon - 1)^2).$$

Thus, $g_0(2\,\beta^{-1}) > \beta^{-1}$ is equivalent to

$$\varepsilon(1/2 - 1/4(\varepsilon-1)^2) > \varepsilon/4, \quad \text{or} \quad 1/4 > 1/4(\varepsilon-1)^2, \quad \text{or} \quad 1 > (\varepsilon-1)^2,$$

which is automatically verified for small positive ε.

Finally, to check condition (F5), observe that by (F3), $(F_\alpha^N)'(1) = g_\alpha'(1)$, so (F5) is equivalent to $(F_\alpha^N)'(\beta^{-1}) = 16\,(1 - \varepsilon - \alpha)/\varepsilon^2$, which is clearly compatible with the previous conditions.

4.2 Estimates on the derivative of the transition map

For α close to 0, consider the fundamental domains of g_α and the constants

$$\begin{aligned} &J_\alpha = [c_\alpha, g_\alpha^{-1}(c_\alpha)] \quad \text{and} \quad I_\alpha = [g_\alpha(1), 1] = [1-\lambda, 1],\\ &d_m(\alpha) = \min\{g_\alpha'(y),\, y \in [c_\alpha, 1]\} \quad \text{and} \quad d_M(\alpha) = \max\{g_\alpha'(y),\, y \in [c_\alpha, 1]\}. \end{aligned}$$

Note that for $\alpha < 0$, given any $w \in (g_\alpha^{-1}(c_\alpha), 1]$ there is a first $n_\alpha(w) \in \mathbb{N}$ such that $g_\alpha^{n_\alpha(w)}(w) \in J_\alpha$.

Proposition 4.1. *Fix $\varepsilon \in (0, \frac{1}{10})$. There is $\alpha_\varepsilon > 0$ such that*

$$(g_\alpha^{n_\alpha(w)})'(w) > 1/2, \qquad \textit{for all } \alpha \in [-\alpha_\varepsilon, 0) \textit{ and every } w \in (g_\alpha^{-1}(c_\alpha), 1].$$

The main step in the proof of Proposition 4.1 is the next estimate of the derivative of $g_\alpha^{n_\alpha(w)}$:

Lemma 4.2. *There is $\alpha_\epsilon > 0$ such that, for any $\alpha \in [-\alpha_\varepsilon, 0)$ and any $w \in I_\alpha = [g_\alpha(1), 1]$, it holds*

$$(g_\alpha^{n_\alpha(w)})'(w) \geq \frac{d_m(\alpha)}{(d_M(\alpha))^2}\,\frac{|J_\alpha|}{|I_\alpha|} > \frac{1}{2}.$$

Proof of the lemma: To prove the second inequality it is enough to check it for $\alpha = 0$. The general case follows from the continuous dependence of $d_m(\alpha)$, $d_M(\alpha)$, $|J_\alpha|$, and $|I_\alpha|$ on the parameter α. Since g_0' is decreasing in $(-\infty, d_0)$, I_0 and J_0 are contained in $(-\infty, d_0)$, and $c_0 > \beta^{-1}$,

$$\begin{aligned} d_m(0) &= g_0'(1) = (1-\varepsilon),\\ d_M(0) &= g_0'(c_0) < g_0'(\beta^{-1}) = g_0'(\varepsilon/4) = 1 - 2\,\varepsilon\,(\varepsilon/4 - 1/2) = (1 + \varepsilon - \varepsilon^2/2) < (1+\varepsilon). \end{aligned}$$

By definition, $|I_0| = \varepsilon/4$. It remains to estimate $|J_0|$. By equation (5), $g_0(\varepsilon/2) = g_0(2\,\beta^{-1}) > \beta^{-1}$, thus, since g_0 is increasing in $(-\infty, d_0)$, $\varepsilon/2 > g_0^{-1}(\beta^{-1}) = c_0$. There are two cases, either $\varepsilon/2 > g_0^{-1}(c_0)$ or $\varepsilon/2 \in [c_0, g_0^{-1}(c_0)]$. In the first case, since g_0' is decreasing in $(-\infty, d_0)$, $\varepsilon/2, c_0, g_0^{-1}(c_0) < 1/2$, and $g_0'(1/2) = 1$,

$$|J_0| \geq \varepsilon/2 - g_0(\varepsilon/2) = \varepsilon/2 - \varepsilon/2 + \varepsilon\,(\varepsilon/2 - 1/2)^2 = (\varepsilon/4)\,(1-\varepsilon)^2.$$

In the second case, write $[c_0, g_0^{-1}(c_0)] = g^{-1}([\beta^{-1}, c_0])$ and observe that

$$|J_0| = \frac{c_0 - \beta^{-1}}{g_0'(\xi)}, \quad \text{for some } \xi \in [\beta^{-1}, c_0].$$

Thus, since (by hypothesis) $g_0^{-1}(\beta^{-1}) < \varepsilon/2 < g_0^{-1}(c_0)$ and the derivative of g_0 is decreasing,

$$|J_0| \geq \frac{\varepsilon/2 - g_0(\varepsilon/2)}{g_0'(\beta^{-1})} > \frac{(\varepsilon/4)\,(1-\varepsilon)^2}{1+\varepsilon}.$$

In both cases one has that

$$|J_0| \geq \frac{\varepsilon\,(1-\varepsilon)^2}{4\,(1+\varepsilon)}.$$

Putting all the previous inequalities together and noting that $\varepsilon < 1/10$, we get the second inequality:

$$\frac{|J_0|}{|I_0|}\,\frac{d_m(0)}{(d_M(0))^2} \geq \frac{\varepsilon\,(1-\varepsilon)^2}{4\,(1+\varepsilon)\,(\varepsilon/4)}\,\frac{(1-\varepsilon)}{(1+\varepsilon)^2} \geq \frac{(1-\varepsilon)^3}{(1+\varepsilon)^3} > \frac{1}{2}.$$

We now prove the first inequality in the lemma. Given $\alpha < 0$, let N_α be the smallest natural number such that $g_\alpha^{N_\alpha}(I_\alpha) \cap J_\alpha \neq \emptyset$. By the choice of N_α, since g_α' is decreasing, and J_α and $g_\alpha^{N_\alpha}(I_\alpha)$ are at the left of the point where the derivative of g_α is 1 (if α is small),

$$|J_\alpha| \geq |g_\alpha^{N_\alpha}(I_\alpha)| \geq |g_\alpha^{-1}(J_\alpha)| \geq \frac{|J_\alpha|}{d_M(\alpha)}.$$

Therefore, taking $z_\alpha \in I_\alpha$ such that $(g_\alpha^{N_\alpha})'(z_\alpha) = |g_\alpha^{N_\alpha}(I_\alpha)|/|I_\alpha|$, one has

$$(g_\alpha^{N_\alpha})'(z_\alpha) \geq \frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}. \tag{6}$$

Define $b_\alpha \in I_\alpha$ by $g_\alpha^{N_\alpha}(b_\alpha) = g_\alpha^{-1}(c_\alpha)$. For a given $w \in I_\alpha$ there are three possibilities:

- $w \in [g_\alpha(1), z_\alpha]$. Note that $n_\alpha(w) = N_\alpha$ or $n_\alpha(w) = N_\alpha + 1$. Observe that if $n_\alpha(w) = N_\alpha + 1$ then $0 < g_\alpha'(g_\alpha^{N_\alpha}(w)) < 1$. Thus using the monotonicity of g_α', one gets

$$(g_\alpha^{n_\alpha(w)})'(w) \geq (g_\alpha^{N_\alpha})'(w) \geq (g_\alpha^{N_\alpha})'(z_\alpha) \geq \frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}.$$

- $w \in [b_\alpha, 1]$. In this case, $n_\alpha(w) = N_\alpha + 1$. Since $g_\alpha(w) < z_\alpha$, from the monotonicity of g_α', $(g_\alpha^{N_\alpha})'(g_\alpha(w)) \geq (g_\alpha^{N_\alpha})'(z_\alpha)$. Thus from equation (6) and $g_\alpha'(w) \geq d_m(\alpha)$,

$$(g_\alpha^{N_\alpha+1})'(w) = (g_\alpha^{N_\alpha})'(g_\alpha(w))\,g_\alpha'(w) \geq (g_\alpha^{N_\alpha})'(z_\alpha)\,g_\alpha'(w) \geq \frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}\,d_m(\alpha).$$

- $w \in [z_\alpha, b_\alpha]$ (if $z_\alpha < b_\alpha$). In this case $n_\alpha(w) = N_\alpha$. Arguing as in the previous cases,

$$(g_\alpha^{N_\alpha})'(w) \geq (g_\alpha^{N_\alpha})'(1) = \frac{(g_\alpha^{N_\alpha+1})'(1)}{(g_\alpha)'(g_\alpha^{N_\alpha}(1))} \geq \frac{1}{d_M(\alpha)}\,(g_\alpha^{N_\alpha+1})'(1) \geq \frac{1}{d_M(\alpha)}\frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}\,d_m(\alpha).$$

The proof of the inequality (and of the lemma) now follows noting that $d_m(\alpha) < 1 < d_M(\alpha)$. ☐

Proof of the proposition: Let s_α be the (unique) point with $g'_\alpha(s_\alpha) = 1$. Take any $w \in (g_\alpha^{-1}(c_\alpha), g_\alpha(1)]$. First, if $w \leq s_\alpha$ the result is obvious: all the derivatives throughout the orbit of w are bigger or equal than 1. If $w \geq s_\alpha$ take $k \geq 0$ with $g_\alpha^{-k}(w) = \bar{w} \in I_\alpha$. The result follows from the case $w \in I_\alpha$ (Lemma 4.2) noting that

$$(g_\alpha^{n_\alpha(w)})'(w) = (g_\alpha^{n_\alpha(w)+k})'(\bar{w})\,(g_\alpha^{-k})'(w), \quad n_\alpha(w) + k = n_\alpha(\bar{w}), \quad \text{and} \quad (g_\alpha^{-k})'(w) \geq 1.$$

This completes the proof of Proposition 4.1. ☐

5 Hyperbolic dynamics

In this section, we prove the parts of the theorem concerning hyperbolic dynamics and collisions of homoclinic classes. In Section 5.1, we state a hyperbolicity result for the family $(\Phi_\mu^{p,q})$, for parameters $\mu \in (\mu^\star, \mu^+]$, where $\mu^\star$ is the saddle-node parameter of the family (h_μ) in Section 3.3. In Section 5.2, using Remark 3.3 and the product structure, we will translate the hyperbolicity of the system $(\Phi_\mu^{p,q})$ to the diffeomorphisms f_t, $t \in t_n\,(1+\mu^\star, 1+\mu^+]$. Finally, in Section 5.3, we state the collision of homoclinic classes, related to the saddle-node bifurcations $\mu^\star$ of $h_\mu = \Phi_\mu^{0,0}$. Since we will follow closely [DRo$_1$, Section 4] we will omit most of the details of the proofs.

5.1 Hyperbolicity of the normalized family

Fix small $\mu^\star < 0$ and let $\alpha^\star = \alpha(\mu^\star) > 0$ be the saddle-node parameter of the family $g_\alpha + \mu^\star$. Define now the auxiliary saddle-node arc

$$h_\mu = h_\mu^{\alpha^\star} = g_{\alpha^\star} + \mu$$

having a saddle-node close to 1/2 for the parameter $\mu^\star$. For small $\mu > \mu^\star$, the map h_μ has (exactly) a pair of (hyperbolic) fixed points in D^+, s_μ^- and s_μ^+, $s_\mu^- < s_\mu^+$, where s_μ^+ is contracting and s_μ^- is expanding. These points collapse to the saddle-node $s_{\mu^\star}$. For small $\mu \geq \bar{\mu}$, consider the partition of D^+ given by the intervals

$$L_u(\mu) = [\beta^{-1}, s_\mu^-], \quad L_c(\mu) = (s_\mu^-, s_\mu^+), \quad \text{and} \quad L_s(\mu) = [s_\mu^+, 1].$$

As in Section 4.1, let $h_{\mu^\star}(c_{\mu^\star}) = \beta^{-1}$. Take the system of functions $\Phi_\mu^{q,p}$, defined on D^+, induced by $F = F_{\alpha^\star}$ associated to $h_{\mu^\star}$, i.e., the restriction of F^N to $[c_{\mu^\star}, 1]$ is $h_{\mu^\star} + 3$, see condition (F3) in Section 4. Equation (3) now implies that, for all $y \in [c_{\mu^\star}, 1]$ and μ close to $\mu^\star$, one has

$$\Phi_\mu^{0,0} = \beta^0(1 + \mu + \lambda^0(F^N - 4)) = (1 + \mu + (g_{\alpha^\star} + 3) - 4) = g_{\alpha^\star} + \mu = h_\mu.$$

Thus, for $\mu > \mu^\star$, $s_\mu^\pm$ are (hyperbolic) fixed points of $\Phi_\mu^{0,0}$ in $[c_{\mu^\star}, 1]$.

We need to state two definitions. We say that y is a *ω-limit point of* $\{\Phi_\mu^{p,q}\}$ if there are a point x_0 and sequences of non-negative numbers (p_n) and (q_n) such that $x_n \to y$, where $x_{n+1} = \Phi_\mu^{p_n,q_n}(x_n)$. The *(forward) limit set of* $\{\Phi_\mu^{p,q}\}$, $L^+(\Phi_\mu^{p,q})$, is the closure of all the ω-limit points of $\{\Phi_\mu^{p,q}\}$.

Next lemmas mean that there is μ^+, $0 > \mu^+ > \mu^\star$, such that, for every $\mu \in H = [\mu^\star, \mu^+]$, the forward limit set $L^+(\Phi_\mu^{q,p})$ is contained in $L_s(\mu) \cup L_u(\mu)$. Moreover, if $\mu \in (\mu^\star, \mu^+]$ this set is hyperbolic (the part contained in $L_s(\mu)$ is contracting and the part in $L_u(\mu)$ is expanding). By Remark 3.3, these properties have a direct translation (in terms of hyperbolicity) to the arc of diffeomorphisms $(f_t)_t$ (defined in terms of F above) for $t \in [t_n\,(1+\mu^\star), t_n(1+\mu^+)]$.

Lemma 5.1. *There is μ^+, $0 > \mu^+ > \mu^\star$, such that, for every $\mu \in H = [\mu^\star, \mu^+]$, it holds:*

- $\Phi_\mu^{0,p}(L_s(\mu)) \subset L_s(\mu)$ *for all $p \geq 0$. If $\mu \neq \mu^\star$, these applications are uniformly contracting.*
- *All the returns of points of $L_s(\mu)$ to D^+ are of the form $(0,p)$ with $p \geq 0$, i.e., there are no returns of type (q,p) with $q > 0$.*

Proof: To prove the first item, recall that, by definition of F, $F^N(D^+) = D^-$, so $F^N(1) = (4-\lambda)$. The definition of $\Phi_\mu^{p,q}$, see (3), now implies that $\Phi_\mu^{0,0}([s_\mu^+, 1]) = [s_\mu^+, 1-\lambda+\mu] \subset [s_\mu^+, 1]$ and that $\Phi_\mu^{0,p+1}([s_\mu^+, 1])$ is at the right of $\Phi_\mu^{0,p}([s_\mu^+, 1])$ for all $p \geq 0$. As $\beta^{-1} = \lambda = \varepsilon/4$, small ε, it holds

$$(1-\lambda) + \mu > \beta^{-1}, \qquad \text{for every small } |\mu|. \tag{7}$$

Observe that $F^N(y) \in [3, 4-\lambda]$, thus $(F^N(y) - 4) \in (-1, -\lambda)$, for all if $y \in D^+$. Hence

$$\Phi_\mu^{0,p}(y) \in (1+\mu-\lambda^p, 1+\mu-\lambda^{p+1}) \subset (\beta^{-1}, 1) \subset D^+. \tag{8}$$

Thus, $\Phi_\mu^{0,p}([s_\mu^+, 1]) \subset [s_\mu^+, 1)$ for all $p \geq 0$, proving the inclusions. The hyperbolicity follows from the definition of the maps $\Phi_\mu^{0,p}$ and $0 < h'_\mu < 1$ in $(s_\mu^+, 1]$, $\mu \in (\mu^\star, \mu^+]$.

The second part of the lemma follows directly from $\Phi_\mu^{0,p}(L_s(\mu)) \subset (\beta^{-1}, 1)$ (equation (8)), hence $\Phi_\mu^{q,p}(y) = \beta^q\, \Phi_\mu^{0,p}(y) > \beta\, \beta^{-1} = 1$ if $y \in L_s(\mu)$ and $q \geq 1$. □

The proof of the next lemma follows arguing as in the previous one, so it will be omitted.

Lemma 5.2. *For every $\mu \in H$, $\mu \neq \mu^\star$, it holds*

- $\Phi_\mu^{0,0}(L_c(\mu)) = L_c(\mu)$ *and the restriction of $\Phi_\mu^{0,0}$ to $L_c(\mu)$ is strictly increasing.*
- $\Phi_\mu^{0,p}(L_c(\mu)) \subset L_s(\mu)$ *for all $p \geq 1$.*
- *All the returns of points of $L_c(\mu)$ to D^+ are of the form $(0,p)$ with $p \geq 0$. Moreover, the returns of $L_c(\mu)$ to $L_c(\mu)$ are of the form $(0,0)$.*

Lemma 5.3. *For every $\mu \in H$ it holds*

- $\Phi_\mu^{0,p}(L_u(\mu)) \subset L_s(\mu)$ *for every $p \geq 1$.*
- *All the returns of points of $L_u(\mu)$ to D^+ are of the form $(0,p)$ or $(q,0)$, with $p, q \geq 0$.*
- *If $\mu \neq \mu^\star$, the restriction of $\Phi_\mu^{q,0}$ to $L_u(\mu)$ is uniformly expanding for all $q \geq 0$.*

Proof: For the first claim just note that, if $p \geq 1$, then $\Phi_\mu^{0,p}(L_u(\mu))$ is at the right of $\Phi_\mu^{0,0}(L_s(\mu)) \subset L_s(\mu)$. The second assertion follows from the first one and the proof of Lemma 5.1. The expansion of $\Phi_\mu^{0,0}$ in $D_\mu^{0,0} \cap L_u(\mu)$ ($D_\mu^{0,0}$ is the interval of definition of $\Phi_\mu^{0,0}$) follows noting that $D_\mu^{0,0} \cap L_u(\mu) = [c_\mu, s_\mu^-]$ and using the expansivity of $g_{\alpha(\bar\mu)}$ in $[c_\mu, s_\mu^-]$, where (as above) c_μ is defined by $h_\mu(c_\mu) = \beta^{-1}$. The expansivity of $\Phi_\mu^{q,0}$ in $D_\mu^{q,0} \subset [\beta^{-1}, c_\mu)$ follows from its definition and condition (F4). □

5.2 Prevalent hyperbolicity for the arc (f_t)

In this section we translate the hyperbolicity of the system $(\Phi_\mu^{p;q})$, $\mu \in (\mu^\star, \mu^+]$, to the diffeomorphisms f_t for $t \in (t_n\,(1+\mu^\star), t_n\,(1+\mu^+)]$. Remark 3.3 allows us to perform this translation.

To each point $X_0 = (x_0, y_0, z_0) \in \Delta^+ \cap \Lambda_t$ (Λ_t is the maximal invariant set of f_t in W) we associate a sequence $i(X_0) = (i_k(X_0))_{k \in E(X_0)}$ of symbols s, c and u, as follows. We let $i_0(X_0) = j$ if $y_0 \in L_j(\mu)$. Let $X_1 = (x_1, y_1, z_1)$ and $X_{-1} = (x_{-1}, y_{-1}, z_{-1})$ be the first forward and backward returns of X_0 to Δ^+ by f_t (if they are defined); let $i_{\pm 1}(X_0) = j$ if $y_{\pm 1} \in L_j(\mu)$. If possible, we inductively define first forward and backward returns $X_{\pm(n+1)}$ of $X_{\pm n}$ and let $i_{n+1}(X_0) = i_1(X_n) = i_n(X_1)$ and $i_{-(n+1)}(X_0) = i_{-1}(X_{-n}) = i_{-n}(X_{-1})$. The set $E(X_0) \subset \mathbb{Z}$ is the maximal interval where these indices are defined. The sequence $i(X_0) = (i_k(X_0))_{k \in E(X_0)}$ is the *itinerary of* X.

Using the cycle configuration, it is easy to see that every $X \in \Lambda_t$, $X \neq P, Q$, has some (backward or forward) iterate in Δ^+. So we can associate to each $X \in \Lambda_t \setminus \{P, Q\}$ the sequence $i(Y)$, where Y is some iterate of X in Δ^+. This sequence is well defined, up to a shift, for points whose returns do not have central coordinates equal to β^{-1} or 1 (in fact, for the parameters that we select, we will see that this situation corresponds to wandering points). Define now the sets:

$$\begin{aligned}
\Omega_s(t) &= \{X \in \Omega(f_t)\colon E(X) \neq \emptyset \text{ and } i_k(X) = s \text{ for all } k \in E(X)\} \cup \{P\},\\
\Omega_u(t) &= \{X \in \Omega(f_t)\colon E(X) \neq \emptyset \text{ and } i_k(X) = u \text{ for all } k \in E(X)\} \cup \{Q\},\\
\Omega_c(t) &= \{X \in \Omega(f_t)\colon E(X) \neq \emptyset \text{ and } i_k(X) = c \text{ for all } k \in E(X)\}.
\end{aligned}$$

For each n, write $H_n = \{t = t_n\,(1+\mu),\ \mu \in H\}$, $H = [\mu^\star, \mu^+]$ as in Lemma 5.1, and fix $t = t_n(1+\mu) \in H_n$. Take $X = (x, y, z) \in \Delta^+$ with a first return $X_1 = (x_1, y_1, z_1)$ to Δ^+ of type (q, p). By Remark 3.3, $y_1 = \Phi_\mu^{q,p}(y)$. This fact and Lemmas 5.1, 5.2 and 5.3 imply that *the itinerary of a non-wandering point of* Λ_t *is constant (equal to s, c or u).* Moreover:

- If $i_k(X) = s$ for all $k \in E(X)$ then all the returns are of type $(0, p)$, $p \geq 0$. Thus, if $\mu \neq \mu^\star$, the returns are uniformly contracting in the $\mathbb{Y}$-direction.
- If $i_k(X) = u$ for all $k \in E(X)$ then all the returns are of type $(q, 0)$, $q \geq 0$. Thus, if $\mu \neq \mu^\star$, the returns are uniformly expanding in the $\mathbb{Y}$-direction.
- If $i_k(X) = c$ for all $k \in E(X)$ then all the returns are of type $(0, 0)$.

Actually, the last possibility does not hold for a non-wandering point: just note that $\Phi_\mu^{0,0}$ restricted to $L_c(\mu)$ is a monotone map without fixed points. Similar arguments imply that a point having a return with central coordinate equal to β^{-1} or 1 is wandering. These facts are the ingredients of the proof of the next lemma (for details see [DRo$_1$, Section 4]) which implies (1a) in Theorem 1.

Lemma 5.4. ([DRo$_1$, Steps B-G]) *For every* $t \in H_n$,

- $\Omega(f_t) \cap W = \Omega_s(t) \cup \Omega_u(t)$. *In particular,* $\Omega_c(t) = \emptyset$.
- $\Omega_s(t) = H(P, f_t)$ *and* $\Omega_u(t) = H(Q, f_t)$.
- *If* $t \neq \lambda^n\,(1+\mu^\star)$, *the homoclinic classes of* P *and* Q *are hyperbolic.*
- *There are no cycles associated to the homoclinic classes of* P *and* Q.

5.3 Sequence of saddle-node bifurcations

For $t = \lambda^n\,(1+\mu)$, $\mu \in (\mu^\star, \mu^+]$, let $S_t^\pm$ be the periodic point of f_t in Δ^+ whose central coordinate is $s_\mu^\pm$. The points S_t^+ and S_t^- are hyperbolic (of indices 1 and 2, respectively) with period $n+k_0+n+N$. By, Lemma 5.4, $S_t^+ \in H(P, f_t)$ and $S_t^- \in H(Q, f_t)$. For $t_n^\star = t_n\,(1+\mu^\star)$, one has a saddle-node $S_{t_n^\star}^+ = S_{t_n^\star}^- = S_{t_n^\star}$ whose central coordinate is $s_{\mu^\star}$. Next lemma implies (1b) in Theorem 1:

Lemma 5.5. *The intersection $H(P, f_{t_n^\star}) \cap H(Q, f_{t_n^\star})$ is the orbit of the saddle-node $S_{t_n^\star}$. Moreover, every compact $f_{t_n^\star}$-invariant subset of $H(P, f_{t_n^\star}) \cup H(Q, f_{t_n^\star})$ disjoint from $S_{t_n^\star}$ is hyperbolic.*

This lemma follows as Lemma 5.4 using Remark 3.3 and Lemmas 5.1, 5.2, and 5.3. First, note that $L_c(\mu^\star) = \emptyset$. For $p \geq 0$, the map $\Phi_{\mu^\star}^{0,p}$ is uniformly contracting in any compact subset of $L_s(\mu^\star)$ that does not contain $s_{\mu^\star}$. For $q \geq 0$, the map $\Phi_{\mu^\star}^{q,0}$ is uniformly expanding in any compact subset of $L_u(\mu^\star)$ disjoint from $s_{\mu^\star}$. These properties can be translated to $f_{t_n^\star}$. Finally, the fact that $H(P, f_{t_n^\star}) \cap H(Q, f_{t_n^\star})$ is the orbit of $S_{t_n^\star}$ follows from $s_{\mu^\star} = L_s(\mu^\star) \cap L_u(\mu^\star)$, Lemmas 5.1, 5.2, and 5.3, and that $s_{\mu^\star}$ is a saddle-node of $\Phi_{\mu^\star}^{0,0}$.

6 Dynamically defined Cantor sets

In this section we construct dynamically defined Cantor sets for the family $(\Phi_\mu^{p,q})$ having arbitrarily big thickness. In this step, the selection of the parameter α in the definition of F (so in the definition of $(\Phi_\mu^{p,q})$) is crucial. Therefore, for the sake of clearness, we write explicitly the two parameters μ and α involved in our constructions. In this way, we consider a two parameter family of maps $\Phi_{\mu,\alpha}^{q,p}$ defined as $\Phi_\mu^{q,p}$ just taking the central map F equal to F_α. Similarly, we consider the two-parameter family of diffeomorphisms $f_{t,\alpha}$.

In what follows, we systematically use *dynamically defined Cantor sets* and their *thickness.* For further information on this topic we refer to [PT$_3$, Chapter 4].

Definition 6.1. *A* presentation *of a Cantor set $\Upsilon \subset \mathbb{R}$ is an enumeration $\mathcal{G} = \{G_n\}$ of the* bounded gaps *of Υ (i.e., the bounded connected components of $\mathbb{R} \setminus \Upsilon$). The* thickness of the presentation $\mathcal{G}$, *$\tau(\Upsilon, \mathcal{G})$, is defined as follows. For each n and each point g in the boundary of a gap G_n, let B_g be the connected component of $I \setminus (G_1 \cup \cdots \cup G_n)$ containing g (here I is the convex hull of Υ). The interval B_g is the* left *or* right bridge *of G_n, according to its relative position. Then*

$$\tau(\Upsilon, \mathcal{G}) = \inf_{n\in\mathbb{N}, g\in\partial G_n} \tau(\Upsilon, \mathcal{G}, g), \quad \tau(\Upsilon, \mathcal{G}, g) = |B_g|/|G_n|.$$

Finally, the thickness *of Υ is $\tau(\Upsilon) = \sup \tau(\Upsilon, \mathcal{G})$, where the supremum is taken over all the presentations of Υ. Thus, for any presentation $\mathcal{G}$ of Υ, $\tau(\Upsilon, \mathcal{G})$ is a lower bound of $\tau(\Upsilon)$.*

We define the *central localization* of an $f_{t,\alpha}$-invariant set Λ in the neighbourhood W by

$$\Lambda^c = \{y_0 \in D^+ : \text{ there is } X \in \Lambda \text{ of the form } (x, y_0, z)\}. \tag{9}$$

In this section, for appropriate t and α, we build two (transitive) hyperbolic sets $\Lambda_{t,\alpha}$ and $\Omega_{t,\alpha}$ of $f_{t,\alpha}$, $\Lambda_{t,\alpha}$ of index one contained in the homoclinic class of P and $\Omega_{t,\alpha}$ of index two, such that the product of the thickness of their central localizations is bigger than one. This fact (and Lemma 7.1) is the key for getting a t-parameter interval with persistence of cycles, (part (2) of the theorem).

6.1 Construction of the hyperbolic set $\Lambda_{t,\alpha}$ of index one

Lemma 6.2. *There are parameters μ_0 and $\alpha_0 > 0$ and strictly positive constants d^c and τ^c such that, for every $t \in B_n = [t_n\,(1-\mu_0), t_n\,(1+\mu_0)]$, $t_n = \lambda^n$, and $\alpha \in [-\alpha_0, \alpha_0]$, there exists a (transitive) hyperbolic set $\Lambda_{t,\alpha}$ of $f_{t,\alpha}$ of index one, contained in the homoclinic class of P, such that the diameter and the thickness of $(\Lambda_{t,\alpha})^c$ are lower bounded by d^c and τ^c, respectively.*

Proof: The proof of this lemma has two parts. We first construct a dynamically defined Cantor set for the system $\Phi^{0,p}_{\mu,\alpha}$, for (μ,α) close to $(0,0)$. We next see, using Remark 3.3 that such a set is the central localization of a hyperbolic set of $f_{t_n\,(1+\mu),\alpha}$ in the homoclinic class of P.

We claim that there is p_1 such that, for all $p \geq p_1$, **(i)** $\Phi^{0,p}_{\mu,\alpha}$ is contracting, **(ii)** $D^{0,p}_{\mu,\alpha} = D^+$, and **(iii)** $\Phi^{0,p}_{\mu,\alpha}(D^+)$ is in the interior of D^+. To get the contraction property (i), just take p_1 big enough in order that the factor λ^{p_1} compensates any expansion introduced by F^N_α, any small $|\alpha|$, see the definition of $\Phi^{0,p}_{\mu,\alpha}$ in (3). Equation (8) implies that $\Phi^{0,p}_{\mu,\alpha}(D^+) \subset \mathrm{int}(D^+)$, for $p \geq 1$, which (by continuity) implies (ii) and (iii), for small $|\mu|$ and $|\alpha|$ and $p \geq 1$.

Define $K^{p_1,p_1+1}_{\mu,\alpha}$ as the maximal invariant set in D^+ of $\Phi^{0,p_1}_{\mu,\alpha}$ and $\Phi^{0,p_1+1}_{\mu,\alpha}$, that is,

$$K^{p_1,p_1+1}_{\mu,\alpha} = \bigcap_{j\geq 0} D^+_{\mu,\alpha}(j), \quad \text{where} \quad D^+_{\mu,\alpha}(0) = D^+ \quad \text{and} \quad D^+_{\mu,\alpha}(i+1) = \bigcup_{p=p_1}^{p_1+1} \Phi^{0,p}_{\mu,\alpha}(D^+(i)).$$

By construction, $K^{p_1,p_1+1}_{\mu,\alpha}$ is a hyperbolic dynamically defined Cantor set, which is transitive and has a dense subset of periodic points. Let $d^c(0,0) > 0$ and $\tau^c(0,0) > 0$ be the diameter and the thickness of $K^{p_1,p_1+1}_{0,0}$. As the thickness and the diameters of $K^{p_1,p_1+1}_{\mu,\alpha}$ depend continuously on $\Phi^{0,p_1}_{\mu,\alpha}$ and $\Phi^{0,p_1+1}_{\mu,\alpha}$, thus on μ and α, there are small μ_0 and α_0 such that, for all $(\mu,\alpha) \in [-\mu_0,\mu_0]\times[-\alpha_0,\alpha_0]$, it holds

$$\mathrm{diam}(K^{p_1,p_1+1}_{\mu,\alpha}) > d^c(0,0)/2 = d^c, \quad \text{and} \quad \tau(K^{p_1,p_1+1}_{\mu,\alpha}) > \tau^c(0,0)/2 = \tau^c.$$

To prove the lemma, it suffices to find a set $\Lambda_{t,\alpha}$ (contained in $H(P,f_{t,\alpha})$) whose central localization is $K^{p_1,p_1+1}_{\mu,\alpha}$. Let us explain this in details.

Fix an interval $B_n = [t_n\,(1-\mu_0), t_n\,(1+\mu_0)]$. By Remark 3.3, for each $t(\mu) = t_n\,(1+\mu) \in B_n$, the central coordinate of a return of type $(0,p)$ by $f_{t(\mu),\alpha}$ is given by $\Phi^{0,p}_{\mu,\alpha}$. Define $\Lambda_{t(\mu),\alpha}$ as the $f_{t(\mu),\alpha}$-invariant set of points with infinitely many consecutive returns to $\Delta^+ = [-1,1]\times D^+\times[-1,1]$ of types $(0,p_1)$ or $(0,p_1+1)$. By construction, the central localization of $\Lambda_{t(\mu),\alpha}$ is $K^{p_1,p_1+1}_{\mu,\alpha}$. Since the directions parallel to the axis $\mathbb{X}$ and $\mathbb{Z}$ are contracting and expanding and, by the previous step, the $\mathbb{Y}$-direction is contracting for the restriction of $f_{t(\mu),\alpha}$ to $\Lambda_{t(\mu),\alpha}$, this set is hyperbolic of index one. The claims for the diameter and the thickness now are obvious.

It remains to prove that $\Lambda_{t(\mu),\alpha}$ is a transitive subset of the homoclinic class of P. As a consequence of the contraction in the $\mathbb{X}$-direction and the expansion in the $\mathbb{Z}$-direction and the $f_{t,\alpha}$-invariance of the planes $\mathbb{XZ}$, the set $\Lambda_{t(\mu),\alpha}$ intersects every plane parallel to the $\mathbb{XZ}$-plane in at most one point. Thus there is a one-to-one correspondence between $\Lambda_{t(\mu),\alpha}\cap\Delta^+$ and $(\Lambda_{t(\mu),\alpha})^c$, so the dynamics (via $f_{t(\mu),\alpha}$) of the returns of the points of $\Lambda_{t(\mu),\alpha}\cap\Delta^+$ to Δ^+ and the dynamics of $(\Lambda_{t(\mu),\alpha})^c = K^{p_1,p_1+1}_{\mu,\alpha}$ by the system $\Phi^{0,p_1}_{\mu,\alpha}$ and $\Phi^{0,p_1+1}_{\mu,\alpha}$ are conjugate. Hence, the periodic points are dense in $\Lambda_{t(\mu),\alpha}$ and this set is transitive.

To see that any periodic point $A \in \Lambda_{t(\mu),\alpha}$ is homoclinically related to P note that, by construction, the orbit of the unstable manifold of A meets the rectangle $[-1,1]\times D^+\times\{0\} \subset W^s(P,f_{t(\mu),\alpha})$. Thus $W^u(A,f_{t(\mu),\alpha}) \pitchfork W^s(P,f_{t(\mu),\alpha}) \neq \emptyset$. To see that $W^u(P,f_{t(\mu),\alpha}) \pitchfork W^s(A,f_{t(\mu),\alpha}) \neq \emptyset$ note that, by construction and considering negative iterations by $f_{t(\mu),\alpha}$, the stable manifold of A contains a disk of the form $[-1,1]\times D^+\times\{z_0\}$ and that $W^u(P,f_{t(\mu),\alpha})$ intersects any disk of this form (any $z_0\in[-1,1]$). This completes the proof of the lemma. $\square$

6.2 Construction of the thick hyperbolic set $\Omega_{t,\alpha}$ of index two

As in Section 6.1, we first construct a dynamically defined Cantor set $\Sigma_{\mu,\alpha}$ associated to the family $\Phi^{q,0}_{\mu,\alpha}$, $q\geq 0$, with large thickness (Proposition 6.6). This set corresponds to the central localization of a transitive hyperbolic set of index two (the announced set $\Omega_{t,\alpha}$).

Since $\Phi^{0,0}_{\mu,\alpha}=F^N_\alpha-3+\mu$, see (3), by condition (F1) it holds $\Phi^{0,0}_{\mu,\alpha}(\beta^{-1})=\mu$. So, for small $\mu<0$, there are $c_{\mu,\alpha}$ (the continuation of c_α in Section 4) and $\ell_{\mu,\alpha}\in D^+$, $\beta^{-1}<\ell_{\mu,\alpha}<c_{\mu,\alpha}$, with

$$\Phi^{0,0}_{\mu,\alpha}(\ell_{\mu,\alpha})=0 \quad\text{and}\quad \Phi^{0,0}_{\mu,\alpha}(c_{\mu,\alpha})=\beta^{-1}.$$

Lemma 6.3. (Bounded distortion lemma) *There is a constant L, independent of small $|\alpha|$ and $|\mu|$, such that*

$$L^{-1}\,\frac{|I|}{|J|}\leq\frac{|(\Phi^{0,0}_{\mu,\alpha})^k(I)|}{|(\Phi^{0,0}_{\mu,\alpha})^k(J)|}\leq L\,\frac{|I|}{|J|},$$

for every pair of intervals I and J in the same fundamental domain of $\Phi^{0,0}_{\mu,\alpha}$ in $[c_{\mu,\alpha},1]$ with $(\Phi^{0,0})^j_{\mu,\alpha}(I\cup J)\subset[c_{\mu,\alpha},1]$ for all $j=1,\dots,k-1$.

The proof of this lemma is quite standard and uses that the map $\Phi^{0,0}_{\mu,\alpha}$ is C^2 and monotone in $[c_{\mu,\alpha},1]$, which follows from condition (F3) and equation (3). For details see [PT$_3$, pages 58-59].

Define, for $\mu<0$, the interval $G_0(\mu,\alpha)=(\beta^{-1},\ell_{\mu,\alpha})$. Since $|\mu|=|(\Phi^{0,0}_{\mu,\alpha})'(\xi)|\,|G_0(\mu,\alpha)|$ for some $\xi\in[\beta^{-1},\ell_{\mu,\alpha}]\subset[\beta^{-1},1]$, there is a constant K (independent of small α and μ) such that

$$\frac{|\mu|}{K}<|G_0(\mu,\alpha)|=(\ell_{\mu,\alpha}-\beta^{-1})<K\,|\mu|. \tag{10}$$

We take small $\mu<0$ with

$$K\,|\mu|<<\min\{d^c,\tau^c\},\qquad\text{where } d^c \text{ and } \tau^c \text{ are as in Lemma 6.2.}$$

6.2.1 The induced map

Fix small $\mu<0$ and let $\alpha(\mu)$ be the parameter corresponding to the saddle-node bifurcation of the family $(g_\alpha+\mu)_\alpha$. Take $\alpha<\alpha(\mu)$ and note that $\Phi^{0,0}_{\mu,\alpha}=g_\alpha+\mu$ has no fixed points in $[c_{\mu,\alpha},1]$. Define

$$I_0(\mu,\alpha)=[\ell_{\mu,\alpha},c_{\mu,\alpha}] \quad\text{and}\quad H_0(\mu,\alpha)=[\beta^{-1},c_{\mu,\alpha}]=\{\beta^{-1}\}\cup G_0(\mu,\alpha)\cup I_0(\mu,\alpha).$$

Since $H_0(\mu,\alpha)$ is a fundamental domain of $\Phi^{0,0}_{\mu,\alpha}$ and the graph of $\Phi^{0,0}_{\mu,\alpha}$ is under the diagonal, there is a first $k_0\in\mathbb{N}$ such that $(\Phi^{0,0}_{\mu,\alpha})^{k_0}(1)\in H_0(\mu,\alpha)$. For each $k\in\{1,\dots,k_0\}$, let

$$L_k(\mu,\alpha)=(\Phi^{0,0}_{\mu,\alpha})^{-k}(L_0(\mu,\alpha))\cap[\beta^{-1},1],\qquad L=G,H,I. \tag{11}$$

Note that $I_{k_0}(\mu,\alpha) = \emptyset$ if $1 \in G_{k_0}(\mu,\alpha)$ and $\Phi^{0,0}_{\mu,\alpha}((\ell_{\mu,\alpha}, c_{\mu,\alpha})) = (0, \beta^{-1})$. Take the following partition $J_i(\mu,\alpha)$ of $(\ell_{\mu,\alpha}, c_{\mu,\alpha}]$ by intervals accumulating to $\ell_{\mu,\alpha}$ whose interiors are pairwise disjoint:

$$(\ell_{\mu,\alpha}, c_{\mu,\alpha}] = \bigcup_{i\geq 1} J_i(\mu,\alpha), \quad \text{where} \quad J_i(\mu,\alpha) = (\Phi^{0,0}_{\mu,\alpha})^{-1}(\beta^{-i}\,([\beta^{-1}, 1]). \tag{12}$$

Using this partition, we define the following induced map (depicted in Figure 5)

$$\Psi_{\mu,\alpha}\colon (\ell_{\mu,\alpha}, c_{\mu,\alpha}] \to [\beta^{-1}, 1], \quad x \in J_i(\mu,\alpha) \mapsto \Psi_{\mu,\alpha}(x) = \beta^i\,(\Phi^{0,0}_{\mu,\alpha}(x)) = \Phi^{i,0}_{\mu,\alpha}(x). \tag{13}$$

The map $\Psi_{\mu,\alpha}$ is bi-valuated in boundary of $J_i(\mu,\alpha)$ and maps the interior of $J_i(\mu,\alpha)$ onto $(\beta^{-1}, 1)$.

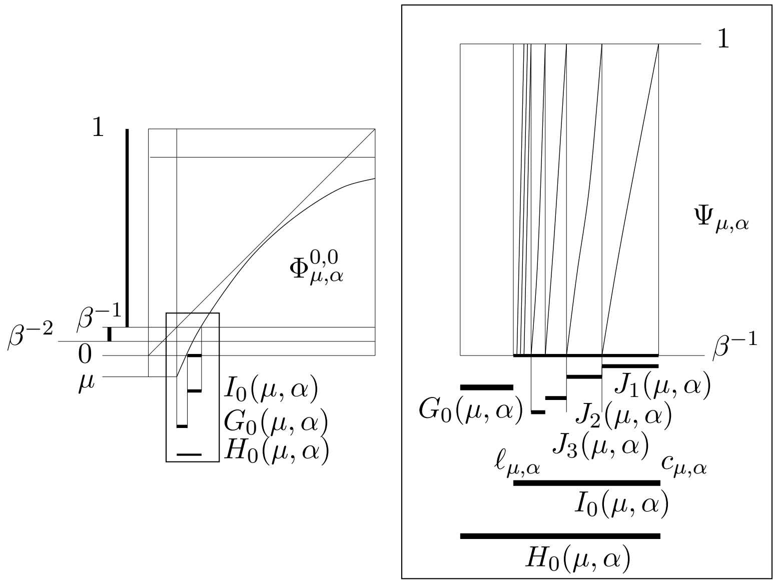

Figure 5: The induced map $\Psi_{\mu,\alpha}$

Define the map $\Gamma_{\mu,\alpha}$ (which is bi-valuated at infinitely many points) as follows (see Figure 6):

$$\Gamma_{\mu,\alpha}\colon I(\mu,\alpha) = \bigcup_{i=0}^{k_0} I_i(\mu,\alpha) \to [\beta^{-1}, 1], \quad x \in I_i(\mu,\alpha) \mapsto \Gamma_{\mu,\alpha}(x) = \Psi_{\mu,\alpha} \circ (\Phi^{0,0}_{\mu,\alpha})^i(x). \tag{14}$$

Remark 6.4. *Let m^+ and $m^- > 0$ be upper and lower bounds of the derivative of $\Phi^{0,0}_{\mu,\alpha}$ in $[\beta^{-1}, 1]$ independent of small μ and α. Then $\beta^i\, m^- \leq \Psi'_{\mu,\alpha}(x) \leq \beta^i\, m^+$ for all $x \in J_i(\mu,\alpha)$.*

Lemma 6.5. *The map $\Gamma_{\mu,\alpha}$ verifies the following:*

- *its lateral derivatives are defined at every point in the interior of $I(\mu,\alpha)$;*
- *it is uniformly expanding (i.e., there is $\rho > 1$ such that every lateral derivative is strictly bigger than ρ); and*

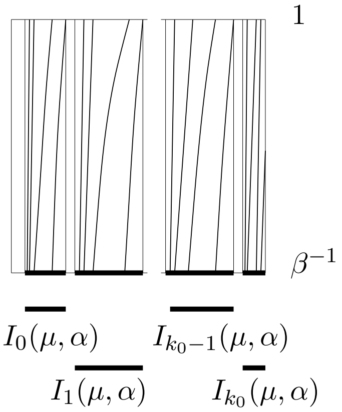

Figure 6: The map $\Gamma_{\mu,\alpha}$

- $\Gamma'_{\mu,\alpha}(x) \to +\infty$ *as* $x \to (\Phi^{0,0}_{\mu,\alpha})^{-i}(\ell_{\mu,\alpha})$*,* $i = 0, \ldots, k_0$.

Proof: The first and the third items follow immediately from the definition of $\Gamma_{\mu,\alpha}$. For the second assertion, we begin by estimating the derivative of $\Psi_{\mu,\alpha}$ in $I_0(\mu,\alpha)$. Take any $x \in I_0(\mu,\alpha)$, $x \in J_k(\mu,\alpha)$ for some $k \geq 1$, by definition,

$$\Psi_{\mu,\alpha}(x) = \beta^k\,(\Phi^{0,0}_{\mu,\alpha}(x)) = \beta^k\,(F^N_\alpha(x) + 3 + \mu) = \beta^k\,(g_\alpha(x) + \mu).$$

Since $J_k(\mu,\alpha) \subset [\beta^{-1}, c_{\mu,\alpha}]$, condition (F4) implies that $\Psi'_{\mu,\alpha}(y) > \beta$.

Suppose now that $x \in I_j(\mu,\alpha)$, $j \geq 1$, then $x, (\Phi^{0,0}_{\mu,\alpha})(x), \ldots, (\Phi^{0,0}_{\mu,\alpha})^{j-1}(x)$ belong to $[c_{\mu,\alpha}, 1]$, thus $\Phi^{0,0}_{\mu,\alpha}$ coincides with $g_\alpha + \mu$ along the first j iterates of x. Therefore,

$$\Gamma_{\mu,\alpha}(x) = \Psi_{\mu,\alpha} \circ (g_\alpha + \mu)^j(x).$$

We claim that $((g_\alpha + \mu)^{j-1})'((g_\alpha + \mu)(x)) > 1/2$, which implies the result: Since $(g_\alpha + \mu)^j(x) \in I_0(\mu,\alpha)$, $\Psi'_{\mu,\alpha}((g_\alpha + \mu)^j(x)) > \beta$, now it is enough to note that $\beta/4 > 1$ and $g'_\alpha(x) > 1/2$ for all x.

To prove the claim, note that, fixed small $\mu < 0$, $(g_\alpha + \mu)_\alpha$ is a saddle-node arc bifurcating at $\alpha(\mu) = \varepsilon\,(-1 + \sqrt{1 - 4\mu})$ close to 0. Repeating the proof of Proposition 4.1 and noting that, with the notation of the proposition, $j - 1 = n_\alpha((g_\alpha + \mu)(x))$, one gets, for small $\alpha < \alpha(\mu)$,

$$((g_\alpha + \mu)^{j-1})'((g_\alpha + \mu)(x)) > 1/2,$$

ending the proof of the claim and of the lemma. □

We are now ready to construct a dynamically defined Cantor set $\Sigma_{\mu,\alpha}$, associated to $\Gamma_{\mu,\alpha}$, with large thickness and small gaps, whose central localization is a hyperbolic set $\Omega_{\mu,\alpha}$ of index 2. The key step in this construction is Proposition 6.6, which together with Lemma 6.2 and (Gap) Lemma 7.1 are the ingredients of the persistence of cycles in the theorem, see Section 7.

Archetypally, the set $\Sigma_{\mu,\alpha}$ consists of the points in D^+ whose orbits by $\Gamma_{\mu,\alpha}$ do not meet $G_0(\mu,\alpha)$. In general, this set fails to be dynamically defined (and thus we do not know how to calculate its thickness). To bypass this difficulty, we consider a dynamically defined subset of it. This is done by constructing a Markov partition for $\Gamma_{\mu,\alpha}$ and defining $\Sigma_{\mu,\alpha}$ as the maximal invariant set associated to such a partition (this corresponds to exhibit a good presentation of $\Sigma_{\mu,\alpha}$).

Recall that, fixed small $\mu^\star < 0$, $\alpha(\mu^\star)$ is the parameter corresponding to the saddle-node bifurcation of the family $(g_\alpha + \mu^\star)_\alpha$. Thus the arc $(g_{\alpha(\mu^\star)} + \mu)_\mu$ unfolds a saddle-node at $\mu^\star$.

Proposition 6.6. *There are constants τ_0, $\tau_1 > 0$, and $0 < \eta \leq 1$, independent of small $\mu^\star < 0$, such that, for all $\mu < \mu^\star$, the map $\Gamma_{\mu,\alpha(\mu^\star)}$ has a dynamically defined Cantor set $\Sigma_{\mu,\alpha(\mu^\star)}$ such that:*

- **(a)** *$\tau(\Sigma_{\mu,\alpha(\mu^\star)}) \geq \tau_0/|\mu|^\eta$, where τ denotes the thickness of the Cantor set,*
- **(b)** *every (bounded) gap of the Cantor set $\Sigma_{\mu,\alpha(\mu^\star)}$ has length less than $\tau_1\,|\mu|^\eta$, and*
- **(c)** *the diameter of $\Sigma_{\mu,\alpha(\mu^\star)}$ is bigger than $(1-\beta^{-1})/2$.*

Notational remark: *From now on, the parameter α is fixed and equal to $\alpha(\mu^\star)$, the saddle-node parameter of the arc $(g_\alpha + \mu^\star)_\alpha$, for some small negative $\mu^\star$. Thus for notational simplicity, the dependence on α of the maps and the associated points and sets will be omitted.*

For $\mu < \mu^\star$ and close to μ, to construct the set Σ_μ we consider a Markov partition. There are two cases, according to the position of $G_{k_0}(\mu)$.

6.2.2 Construction of the Markov partition. Case 1: $1 \in G_{k_0}(\mu)$ (i.e., $I_{k_0}(\mu) = \emptyset$)

Let $q_\mu \in I_{k_0-1}(\mu)$ be the closest point to 1 with $\Gamma_\mu(q_\mu) = q_\mu$ (the existence of q_μ is guaranteed by $\Gamma_\mu(I_{k_0-1}(\mu)) = [0,1]$). By construction, there is a sequence $x_n \in (\ell_\mu, 1)$ of fixed points of Γ_μ converging to ℓ_μ. In particular, there is a fixed point p_μ of Γ_μ arbitrarily close to ℓ_μ satisfying

$$\Gamma_\mu(p_\mu) = p_\mu, \quad p_\mu - \ell_\mu < |\mu|. \tag{15}$$

Lemma 6.7. *There are constants K_1, K_2, and K_3, independent of small $\mu^\star < 0$, such that, for all $\mu < \mu^\star$ close to $\mu^\star$,*

1. *$|G_{k_0}(\mu)| < K_1\,|G_0(\mu)| < K_2\,|\mu|$,*
2. *$(1-q_\mu) \leq \dfrac{K_1\,\rho}{\rho-1}\,|G_0(\mu)| \leq \dfrac{K_2\,\rho}{\rho-1}\,|\mu| \leq K_3\,|\mu|$, where $\rho > 1$ is as in Lemma 6.5.*

Proof: The inequalities in item (1) follow from Proposition 4.1, the first one, and equation (10), the second one. To prove item (2), let $G_{k_0}(\mu) = (1-v_\mu, 1)$. By construction, considering the branch of Γ_μ in $I_{k_0-1}(\mu)$, one has $\Gamma_\mu([q_\mu, 1-v_\mu]) = [q_\mu, 1]$. Since Γ_μ has no discontinuities in $[q_\mu, 1-v_\mu]$, by Lemma 6.5, there is $x \in (q_\mu, 1-v_\mu)$ such that

$$(1-q_\mu) = \Gamma'_\mu(x)(1-v_\mu-q_\mu) \geq \rho(1-v_\mu-q_\mu).$$

Since $v_\mu = |G_{k_0}(\mu)|$, using the first part of the lemma, one has that

$$(1-q_\mu) \leq \frac{\rho}{\rho-1}\,v_\mu \leq \frac{\rho}{\rho-1}\,K_1\,|G_0(\mu)| \leq \frac{\rho}{\rho-1}\,K_2\,|\mu| \leq K_3\,|\mu|,$$

ending the proof of the inequalities in the second item of the lemma. □

Let $A_0(\mu) = (\beta^{-1}, p_\mu)$ and $A_{k_0}(\mu) = (q_\mu, 1)$. Equations (10) and (15) and Lemma 6.7 imply:

Lemma 6.8. *There is a constant $K_4 > 0$, independent of small $\mu^\star$, such that*

$$\max\{|A_0(\mu)|, |A_{k_0}(\mu)|\} \leq K_4\,|\mu|, \qquad \textit{for all } \mu < \mu^\star \textit{ close to } \mu^\star.$$

We now define the intervals corresponding to the *initial* gaps and bridges of the Cantor set Σ_μ:

- For $j \in \{1, \dots, (k_0-1)\}$, $A_j(\mu) = (a_j, b_j)$ is an interval containing $G_j(\mu)$ such that $\Gamma_\mu(a_j) = q_\mu$ and $\Gamma_\mu(b_j) = p_\mu$. More precisely, a_j is the biggest point in $I_{j-1}(\mu)$ with $\Gamma_\mu(a_j) = q_\mu$ and $b_j \in I_j(\mu)$ is chosen close to the right extreme of $G_j(\mu)$. A priori, there are infinitely many possibilities for the choice of b_j, we will explicit this choice below. In Figure 7 are depicted the gaps $A_0(\mu)$ and $A_1(\mu)$.
- For $j = 0, \dots, k_0 - 1$, $B_j(\mu) = [b_j, a_{j+1}]$, where $a_{k_0} = q_\mu$ and $b_0 = p_\mu$.

Remark 6.9. *By construction, $(\Phi_\mu^{0,0})^k(B_k(\mu)) = B_0(\mu)$.*

In $[p_\mu, c_\mu]$, the map $\Psi_\mu = \Gamma_\mu$ has finitely many discontinuities, say r. The number r increases as p_μ approaches to ℓ_μ, so r can be taken arbitrarily large after an appropriate choice of p_μ. We choose the points b_j above such that the restriction of Γ_μ to each $B_j(\mu)$ also has r discontinuity points $d_j^1 < d_j^2 < \cdots < d_j^r$, where Γ_μ is bi-valuated and takes the values β^{-1} and 1. Thus the restriction of Γ_μ to each $B_j(\mu)$ has $(r+1)$ (injective) branches, where

- $\Gamma_\mu([b_j, d_j^1]) = [p_\mu, 1]$,
- $\Gamma_\mu([d_j^i, d_j^{i+1}]) = [\beta^{-1}, 1]$, for $i = 1, \dots, r-1$, and
- $\Gamma_\mu([d_j^r, a_{j+1}]) = [\beta^{-1}, q_\mu]$.

To get a Markov partition of Γ_μ, we refine the bridges $B_j(\mu)$ by adding new gaps, see Figure 7.

- For each $j \in \{0, \dots, k_0-1\}$ and $i \in \{1, \dots, r\}$, $A_j^i(\mu)$ is the smallest interval (a_j^i, b_j^i) containing the discontinuity d_j^i such that $\Gamma_\mu(a_j^i) = q_\mu$ and $\Gamma_\mu(b_j^i) = p_\mu$.
- The intervals $B_j^0(\mu)$, $B_j^1(\mu), \dots, B_j^r(\mu)$ are the connected components of $B_j(\mu) \setminus \cup_{i=1}^r A_j^i(\mu)$, ordered in such a way $B_j^i(\mu)$ is at the left of $B_j^{i+1}(\mu)$. By construction, $\Gamma_\mu(B_j^i(\mu)) = [p_\mu, q_\mu]$. Note that $B_j^i(\mu) = [b_j^i, a_j^{i+1}]$ and that $b_0^0 = p_\mu$.

The previous construction and Lemma 6.5 imply the following:

Proposition 6.10. *Let $\mathcal{B}_\mu = \{B_j^i(\mu)\}_{j=0,\dots,(k_0-1)}^{i=1,\dots,r}$, $\mathbb{B}_\mu = \cup_{i,j} B_j^i(\mu)$, and Σ_μ be the maximal invariant set of Γ_μ in $\mathbb{B}_\mu$. The set Σ_μ is a (expanding) dynamically defined Cantor set associated to the Markov partition $\mathcal{B}_\mu$.*

Throughout this section, in our estimates of the thickness, we use repeatedly the next result:

Lemma 6.11. (Bounded distortion property, [PT$_3$, Theorem 1 page 58]) *Let Σ be a dynamically defined Cantor set of a $C^{1+\varepsilon}$-expanding map Γ. Given any $\delta > 0$, there is $\kappa(\delta) > 0$ such that: for every pair of points x and y and $n \geq 1$ such that $|\Gamma^n(x) - \Gamma^n(y)| \leq \delta$ and the interval of extremes $\Gamma^i(x)$ and $\Gamma^i(y)$ is contained in the domain of definition of Γ, for all $i = 0, 1, \dots, (n-1)$, it holds*

$$|\log|(\Gamma^n)'(x)| - \log|(\Gamma^n)'(y)|| \leq \kappa(\delta).$$

Moreover, $\kappa(\delta)$ converges to 0 as $\delta \to 0$.

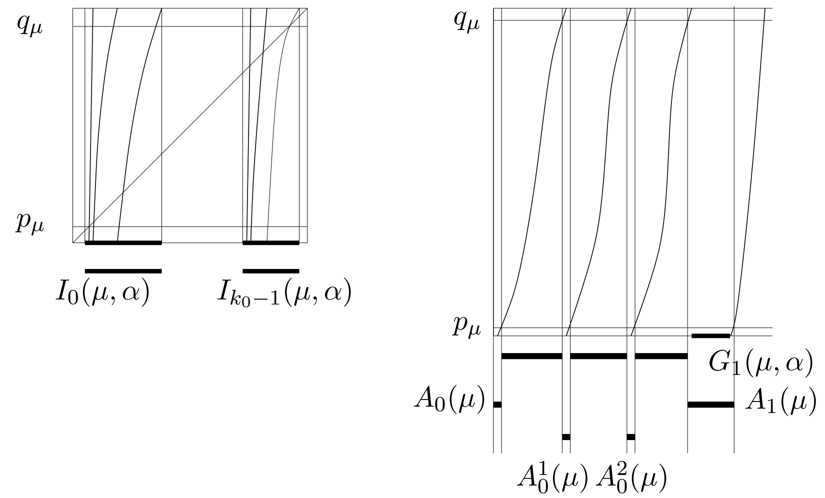

Figure 7: Initial gaps and bridges

We apply this lemma to the set Σ_μ associated to the expanding map Γ_μ (restricted to the Markov partition $\mathbb{B}_\mu$). In our context, it is enough to fix $\delta = (1-\beta^{-1})$, which is an upper bound for the length of any interval of the partition, and let $\kappa_\mu = \kappa_\mu(\delta)$, the constant given by the lemma. Using the definition of Γ_μ, see (13) and (14), and that it is C^2, one gets

$$\kappa_\mu \leq \frac{\max |\Psi''_\mu(x)|}{(\min |\Psi'_\mu(y)|)^2}\,(1-\beta^{-1})\,\frac{1}{\rho-1},$$

where $\rho > 1$ is as in Lemma 6.5. Thus $\kappa_\mu = \kappa$ can be taken independent of small μ. This inequality (whose proof we omit here) is obtained adapting straightforwardly the proof of Lemma 6.11 to the C^2-case (see [PT$_3$, page 59]).

To estimate the thickness of Σ_μ we choose the following presentation $\mathcal{A}_\mu$ of it:

Presentation of $\mathcal{A}_\mu$: *The generation* $(0,1)$ *of gaps is* $A_1(\mu), A_2(\mu), \ldots, A_{k_0}(\mu)$. *The generation* $(0,2)$ *of gaps is defined by*

$$A_0^1(\mu), A_0^2(\mu), \ldots, A_0^r(\mu), A_1^1(\mu), A_1^2(\mu), \ldots, A_1^r(\mu), \ldots, A_{k_0}^1(\mu), A_{k_0}^2(\mu), \ldots, A_{k_0}^r(\mu).$$

Inductively, for each $i = 1$ *or* 2, *we define the* $(k+1,i)$*-generation of gaps as the pre-images by* Γ_μ *of the gaps of the* (k,i)*-generation, enumerated according to the ordering in* $\mathbb{R}$. *Moreover, any gap of generation* $(k+1,2)$ *is posterior to any gap of generation* $(k+1,1)$.

Take a gap G of (n,i)-generation and a bridge B of G. Then the number n and any pair of points $x \in G$ and $y \in B$ verify the hypotheses of Lemma 6.11. Moreover, $\Gamma_\mu^n(G)$ is a gap of $(0,i)$-generation and $\Gamma_\mu^n(B)$ is a bridge of it. Using Lemma 6.11 one gets (this is a standard argument)

$$e^{-\kappa}\frac{|B|}{|G|} \leq \frac{|\Gamma_\mu^n(B)|}{|\Gamma_\mu^n(G)|} \leq e^{\kappa}\frac{|B|}{|G|}. \tag{16}$$

Remark 6.12. *Equation* (16) *implies that if* $\tau > 0$ *is such that* $|B|/|G| > \tau$, *for every gap* G *of generation* $(0,1)$ *or* $(0,2)$ *and every bridge* B *of* G, *then the thickness of* Σ_μ *is bigger than* $\tau\, e^{-\kappa}$.

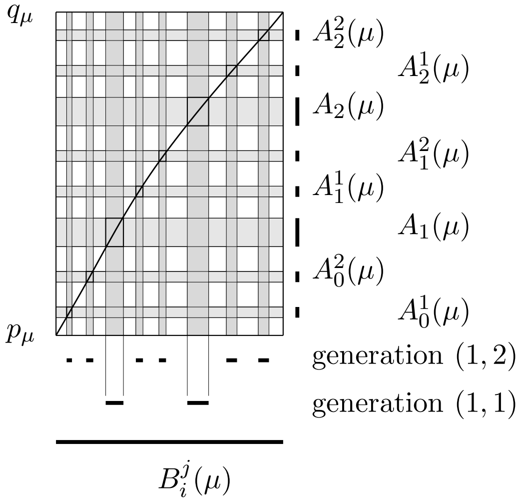

Figure 8: New generation of gaps

By Remark 6.12, to get a lower bound for $\tau(\Sigma_\mu)$ (i.e. to prove Proposition 6.6 in Case 1) it suffices to estimate the ratios between bridges and gaps of generations $(0,1)$ and $(0,2)$. This is done in the next lemma.

Lemma 6.13. *There is K_0, independent of small $\mu^\star < 0$, such that, for every gap G of generation $(0,1)$ or $(0,2)$ and any bridge B of G, it holds $|B|/|G| > K_0/|\mu|$, for every $\mu < \mu^\star$ close to $\mu^\star$.*

Proof: We first estimate $|B_0(\mu)|/|A_1(\mu)|$ and $|B_1(\mu)|/|A_1(\mu)|$. Note that $B_0(\mu)$ is the left bridge of $A_1(\mu)$ and $B_1(\mu)$ is contained in the right bridge of $A_1(\mu)$. To estimate the length of $A_1(\mu) = (a_1, b_1)$, write $G_1(\mu) = (c_\mu, g_1^r) \subset A_1(\mu)$ (see equation (11)). Then

$$|A_1(\mu)| = b_1 - a_1 = (b_1 - g_1^r) + |G_1(\mu)| + (c_\mu - a_1).$$

As c_μ and a_1 are in the same branch of Γ_μ, $\Gamma_\mu(c_\mu) = 1$ and $\Gamma_\mu(a_1) = q_\mu$, Lemmas 6.5 and 6.7 imply

$$(c_\mu - a_1) \le \frac{1 - q_\mu}{\rho} \le \frac{K_3\,|\mu|}{\rho}. \tag{17}$$

Also, since by construction, $\Gamma_\mu(G_1(\mu)) = G_0(\mu)$, from Lemmas 6.5 and 6.7,

$$|G_1(\mu)| \le \frac{|G_0(\mu)|}{\rho} \le \frac{K_2\,|\mu|}{K_1\,\rho} = \frac{K_5\,|\mu|}{\rho}.$$

By construction, $\Phi_\mu^{0,0}(g_1^r) = \ell_\mu$ and $\Phi_\mu^{0,0}(b_1) = p_\mu$, Proposition 4.1 and equation (15) imply that

$$b_1 - g_1^r \le 2\,(p_\mu - \ell_\mu) \le 2\,|\mu|.$$

Putting these inequalities together, one gets

$$|A_1(\mu)| \le \frac{K_3 + K_5 + 2\,\rho}{\rho}\,|\mu| = K_6\,|\mu|. \tag{18}$$

To estimate $B_0(\mu) = [p_\mu, a_1]$ write

$$|B_0(a_\mu)| = (c_\mu - \ell_\mu) - (p_\mu - \ell_\mu) - (c_\mu - a_1).$$

Note first that the size of $[\ell_\mu, c_\mu]$ is uniformly bounded away from zero (independent of small $\mu^\star$). By (15) and (17), $(p_\mu - \ell_\mu) + (c_\mu - a_1) \leq (1 + (K_3/\rho))\,|\mu|$. Thus there is a constant K_7 such that

$$|B_0(a_\mu)| > K_7 > 0.$$

Finally, by Remark 6.9 and Lemma 6.3, the previous inequality implies that

$$|B_1(\mu)| > K_8 > 0.$$

All the estimates above lead to

$$\min\left\{\frac{|B_1(\mu)|}{|A_1(\mu)|}, \frac{|B_0(\mu)|}{|A_1(\mu)|},\right\} \geq \frac{K_9}{|\mu|}. \tag{19}$$

Similarly, using Lemma 6.3, Remark 6.9, and equation (19), one gets

$$\min\left\{\frac{|B_i(\mu)|}{|A_i(\mu)|}, \frac{|B_{i-1}(\mu)|}{|A_i(\mu)|},\right\} \geq \frac{L\,K_9}{|\mu|} = \frac{K_{10}}{|\mu|}. \tag{20}$$

Observing that, for each i, $B_{i-1}(\mu)$ is the left bridge of $A_i(\mu)$ and $B_i(\mu)$ is contained in the right bridge of $A_i(\mu)$, this ends the estimates for the generation $(0,1)$.

We now estimate the ratio between bridges and gaps of generation $(0,2)$. In this step, each interval $B_j(\mu)$ is subdivided into $(r+1)$ new bridges $B_j^i(\mu)$, $i = 0,\ldots,r$, obtained from $B_j(\mu)$ by removing the intervals $A_j^i(\mu)$ around the discontinuities d_j^i, $i = 1,\ldots,r$, where r is the number of of discontinuities of Ψ_μ in $[p_\mu, c_\mu]$. As before, bearing in mind Lemma 6.3, we first estimate these ratios for the bridges $B_0^i(\mu)$.

Observe that, by definition, $B_0^i(\mu) = [b_0^i, a_0^{i+1}] \subset J_{r+1-i}(\mu)$ (recall equation (12)) and

$$\Gamma_\mu((d_0^i, b_0^i)) = (\beta^{-1}, p_\mu) \quad \text{and} \quad \Gamma_\mu((a_0^i, d_0^i)) = (q_\mu, 1).$$

Recall that $|(\beta^{-1}, p_\mu)| < (K+1)\,|\mu|$ (see (10) and (15)) and $|(q_\mu, 1)| < K_3\,|\mu|$ (Lemma 6.7). Note that $(a_0^i, d_0^i) \cup B_0^i(\mu) \subset J_{r+1-i}(\mu)$ and that $(d_0^i, b_0^i) \subset J_{r+1-i-1}(\mu)$. Remark 6.4 now implies that

$$|A_0^i(\mu)| = (b_0^i - a_0^i) = (b_0^i - d_0^i) + (d_0^i - a_0^i) \leq \frac{(K+1+K_3)\,\mu}{\beta^{r-i}\,m^-} = \frac{K_{11}\,|\mu|}{\beta^{r-i}\,m^-}. \tag{21}$$

Analogously, as $\Psi_\mu(B_0^i(\mu)) = [p_\mu, q_\mu]$, and the size of $[p_\mu, q_\mu]$ is uniformly lower bounded, one has

$$|B_0^i(\mu)| \geq \frac{K_{12}}{\beta^{r-i+1}\,m^+}. \tag{22}$$

Now, from inequalities (21) and (22), it follows that, for $i = 1,\ldots,r$,

$$\min\left\{\frac{|B_0^{i-1}(\mu)|}{|A_0^i(\mu)|}, \frac{|B_0^i(\mu)|}{|A_0^i(\mu)|},\right\} \geq \frac{K_{13}}{|\mu|}. \tag{23}$$

Arguing exactly as in the gaps of generation $(0,1)$, by using Lemma 6.3, for all $i = 1, \dots, r$ and $j = 0, \dots, k_0$, one gets

$$\min\left\{\frac{|B_j^{i-1}(\mu)|}{|A_j^i(\mu)|}, \frac{|B_j^i(\mu)|}{|A_j^i(\mu)|},\right\} \geq \frac{K_{14}}{|\mu|}. \tag{24}$$

Since, for each i, $B_j^{i-1}(\mu)$ is the left bridge of $A_j^i(\mu)$ and $B_j^i(\mu)$ is contained in the right bridge of $A_j^i(\mu)$, this ends the estimates for the generation $(0,2)$ and the proof of the lemma. $\square$

Part (b) of Proposition 6.6 follows from Lemma 6.5, noting that the previous estimates show that the lengths of the gaps of $(0,1)$ and $(0,2)$-generations are (at most) of order of $|\mu|$. Finally, to get (c), just note that, by Lemmas 6.7 and 6.8,

$$\text{diameter}(\Sigma_\mu) = q_\mu - p_\mu = (1-\beta^{-1}) - (1-q_\mu) - (p_\mu - \beta^{-1}) \geq (1-\beta^{-1}) - K_3\,|\mu| - K_4\,|\mu|.$$

The proof of Proposition 6.6 in Case 1 is now complete.

6.2.3 Construction of the Markov partition. Case 2. $1 \notin G_{k_0}(\mu)$ (i.e., $1 \in I_{k_0}(\mu)$)

Let $\tau > 2/\tau^c$ (τ^c as in Lemma 6.2). We will modify the Markov partition in Case 1 in order to construct a dynamically defined Cantor set Σ_μ (associated to such a partition) such that the ratios bridges/gaps of the *zero generation* are (at least) of order of $1/|\mu|^\eta$, for some $\eta \in (0,1)$. By Remark 6.12 and Proposition 6.6 the thickness of the set Σ_μ is larger than τ. We consider two cases, according to the size of $I_{k_0}(\mu)$. Let

$$\xi = 1/|\mu^\star|^{3/5}. \tag{25}$$

Case 2.1. $|I_{k_0}(\mu)| \leq \xi\,|\mu|$. We modify the construction of Case 1 replacing $G_{k_0}(\mu)$ by

$$G'_{k_0}(\mu) = G_{k_0}(\mu) \cup I_{k_0}(\mu)$$

and defining p_μ and q_μ exactly as before (i.e., we add the relatively small interval $I_{k_0}(\mu)$ to the relatively big gap $G_{k_0}(\mu)$). The estimate for $p_\mu - \ell_\mu$ (equation (15)) is exactly the same and the one for $(1-q_\mu)$ follows as in Lemma 6.7, but in this case one gets

$$(1-q_\mu) \leq (K_3 + \xi)\,|\mu|.$$

We consider the dynamically defined Cantor set Σ_μ associated to the map Γ_μ in $[p_\mu, q_\mu]$ (the Markov partition and the gaps of generations $(0,1)$ and $(0,2)$ are defined as before). In this way, the Cantor set Σ_μ verifies Proposition 6.6 (taking η equal to $\frac{2}{5}$).

Case 2.2. $|I_{k_0}(\mu)| > \xi\,|\mu|$. We split $I_{k_0}(\mu)$ into two intervals $I^\ell_{k_0}(\mu)$ and $I^r_{k_0}(\mu)$ with disjoint interiors, where $I^\ell_{k_0}(\mu)$ is the domain of all (infinitely many) surjective branches of the restriction of Γ_μ to $I_{k_0}(\mu)$ and $I^r_{k_0}(\mu)$ is its complement (corresponding to the only non surjective branch of Γ_μ). See Figure 9. Write $\Gamma_\mu(I^r_{k_0}(\mu)) = [\beta^{-1}, r]$, where $r < 1$. By Lemma 6.5,

$$|I^r_{k_0}(\mu)| \leq \frac{(r-\beta^{-1})}{\rho}. \tag{26}$$

There are two sub-cases, according to the ratios between the sizes of $I^r_{k_0}(\mu)$ and $I^\ell_{k_0}(\mu)$: either $|I^r_{k_0}(\mu)|/|I^\ell_{k_0}(\mu)| \leq \xi\,|\mu|$ (Case 2.2.1) or $|I^r_{k_0}(\mu)|/|I^\ell_{k_0}(\mu)| > \xi\,|\mu|$ (Case 2.2.2).

Case 2.2.1. $|I^r_{k_0}(\mu)|/|I^\ell_{k_0}(\mu)| \le \xi\,|\mu|$**.** In this case $I^r_{k_0}(\mu)$ is considered as an extremal gap and plays a role similar to the gap $G_{k_0}(\mu)$ in Case 1. Let q'_μ be the fixed point of Γ_μ in $I^\ell_{k_0}(\mu)$ closest to 1. Consider the dynamically defined Cantor set Σ_μ in $[p_\mu, q'_\mu]$ associated to Γ_μ and the Markov partition obtained by modifying the one in Case 1 as follows:

- We consider k_0 gaps $A_i(\mu)$ ($i = 1, \ldots, k_0$) of generation $(0,1)$ (instead of $k_0 - 1$ as in Case 1), where each gap $A_i(\mu)$ contains $G_i(\mu)$ and is defined as in Case 1 replacing q_μ by q'_μ.
- The gaps of generation $(0,2)$ are obtained as the ones in Case 1, but now adding the discontinuities of Γ_μ in $I^\ell_{k_0}(\mu)$.

By Remark 6.12, it is enough to estimate the ratios bridge/gap for the generations $(0,1)$ and $(0,2)$. We first consider the generation $(0,1)$. Since $I^r_{k_0}(\mu)$ is (at most) of order of $\xi\,|\mu| \simeq |\mu|^{2/5}$, arguing as in (2) in Lemma 6.7, it follows that $|1-q'_\mu|$ is of the same order as $|I^r_{k_0}(\mu)|$, hence $|1-q'_\mu|$ is at most of order of $|\mu|^{2/5}$. This implies that the ratios bridge/gap of generation $(0,1)$, excluded $|I^\ell_{k_0}(\mu)|/|A_{k_0}(\mu)|$, are of order of (at least) $1/|\mu|^{2/5}$. As in Case 1, to estimate $|I^\ell_{k_0}(\mu)|/|A_{k_0}(\mu)|$ it suffices to estimate $|I^\ell_{k_0}(\mu)|/|G_{k_0}(\mu)|$ (these quotients are of the same order). Note that

$$\xi\,|\mu| \le |I_{k_0}(\mu)| = |I^r_{k_0}(\mu)| + |I^\ell_{k_0}(\mu)| = |I^\ell_{k_0}(\mu)| \left(\frac{|I^r_{k_0}(\mu)|}{|I^\ell_{k_0}(\mu)|} + 1 \right) \le |I^\ell_{k_0}(\mu)|\,(\xi\,|\mu| + 1).$$

Thus,

$$|I^\ell_{k_0}(\mu)| \ge \frac{\xi\,|\mu|}{1 + \xi\,|\mu|}.$$

Arguing as in Lemma 6.7, one gets $|G_{k_0}(\mu)| < K_2\,|\mu|$. Finally, recalling the definition of ξ in (25),

$$\frac{|I^\ell_{k_0}(\mu)|}{|G_{k_0}(\mu)|} \ge \frac{\xi}{K_2\,(1 + \xi\,|\mu|)} = K'_2\,\frac{1}{|\mu|^{3/5}} \ge K'_2\,\frac{1}{|\mu|^{2/5}},$$

ending the estimates for the generation $(0,1)$.

The estimates for the ratios bridge/gap of generation $(0,2)$ follow as in Case 1, but now these ratios are of order of (at least) $1/|\mu|^{2/5}$. The previous calculations and the choice of ξ in (25) imply that, for small $\mu^\star$ and $\mu < \mu^\star$ close to $\mu^\star$, the ratios bridge/gap of generations $(0,1)$ and $(0,2)$ are at least of order of $1/|\mu|^{2/5}$. Thus, by Remark 6.12, the set Σ_μ verifies Proposition 6.6 taking $\eta = 2/5$. Finally, the lower bound for the diameter of Σ_μ is obtained similarly replacing q_μ by q'_μ.

Case 2.2.2. $|I^r_{k_0}(\mu)|/|I^\ell_{k_0}(\mu)| > \xi\,|\mu|$**.** We claim that, in this case, $\Gamma_\mu(I^r_{k_0}(\mu))$ contains (at least) two surjective branches of Γ_μ. Note first that we can choose the fixed point p_μ of Γ_μ arbitrarily close to ℓ_μ such that there is $s_\mu > p_\mu$ such that $[p_\mu, s_\mu]$ contains two consecutive branches of Γ_μ and $(s_\mu - \ell_\mu) < |\mu|$. Since $\Gamma_\mu(I^r_{k_0}(\mu)) = [\beta^{-1}, r]$ and $G_0(\mu) = (\beta^{-1}, \ell_\mu)$, to get $(p_\mu, s_\mu) \subset \Gamma_\mu(I^r_{k_0}(\mu))$ it is enough to see that $(\beta^{-1}, s_\mu) \subset \Gamma_\mu(I^r_{k_0}(\mu))$. By equation (10),

$$(s_\mu - \beta^{-1}) \le |G_0(\mu)| + |\mu| \le K\,|\mu| + |\mu| < |\Gamma_\mu(I^r_{k_0}(\mu))|.$$

We argue by contradiction, if the last inequality does not hold, then, by Lemma 6.5,

$$|I^r_{k_0}(\mu)| \le \frac{(1+K)\,|\mu|}{\rho}.$$

Thus, by hypothesis,

$$\xi\,|\mu| < |I_{k_0}(\mu)| = |I^r_{k_0}(\mu)| + |I^\ell_{k_0}(\mu)| = |I^r_{k_0}(\mu)|\left(1+\frac{|I^\ell_{k_0}(\mu)|}{|I^r_{k_0}(\mu)|}\right) \le \frac{(K+1)\,|\mu|}{\rho}\left(1+\frac{1}{\xi\,|\mu|}\right).$$

Hence,

$$\xi \le \frac{(K+1)}{\rho}\left(1+\frac{1}{\xi\,|\mu|}\right),$$

which, by the definition of ξ in (25), is not verified for μ close to small $\mu^\star$. This proves the claim.

By the claim, there is $h \ge 3$ such that $I^r_{k_0}(\mu)$ is of the form

$$I^r_{k_0}(\mu) = \bigcup_{j=1}^{h}(G_{k_0+j}(\mu)\cup I_{k_0+j}(\mu)), \qquad \text{where}$$

- the interval $I_{k_0+h}(\mu)$ may be empty,
- $\Gamma_\mu(G_{k_0+j}(\mu)) = G_{j-1}(\mu)$ and $\Gamma_\mu(I_{k_0+j}(\mu)) = I_{j-1}(\mu)$, if $j \neq h$,
- if $j = h$ and $I_{k_0+h}(\mu) \neq \emptyset$, then $\Gamma_\mu(G_{k_0+h}(\mu)) = G_{h-1}(\mu)$ and $\Gamma_\mu(I_{k_0+h}(\mu)) \subset I_{h-1}(\mu)$,
- if $j = h$ and $I_{k_0+h}(\mu) = \emptyset$, then $\Gamma_\mu(G_{k_0+h}(\mu)) \subset G_{h-1}(\mu)$.

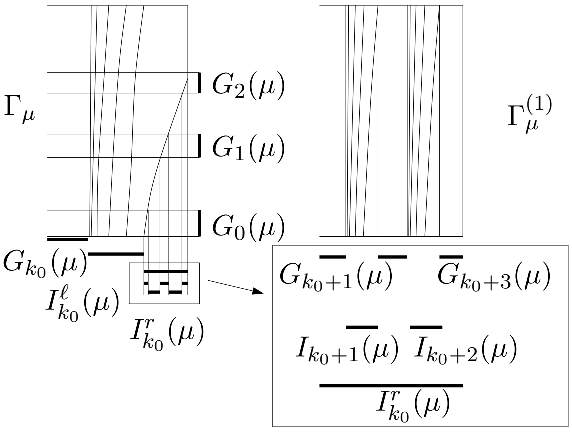

Figure 9: New gaps and the map $\Gamma^{(1)}_\mu$

Consider now the map (see Figure 9)

$$\Gamma^{(1)}_\mu\colon [\beta^{-1},1]\setminus(\cup_{j=0}^{k_0+h}G_j(\mu)) \to [\beta^{-1},1], \qquad \begin{cases} \Gamma^{(1)}_\mu(y) = \Gamma_\mu(y), & \text{if } y \notin I^r_{k_0}(\mu), \\ \Gamma^{(1)}_\mu(y) = \Gamma_\mu\circ\Gamma_\mu(y), & \text{if } y \in I^r_{k_0}(\mu). \end{cases}$$

Remark 6.14. *By construction and Lemma 6.5, the map* $\Gamma^{(1)}_\mu$ *is* ρ*-expanding. Moreover, its restriction to* $I_{k_0+1}(\mu),\ldots,I_{k_0+h}(\mu)$ *is* ρ^2*-expanding.*

As in the case of the map Γ_μ, we consider two cases according to the position of 1.

Case (a): the interval $I_{k_0+h}(\mu)$ is empty, or, equivalently, $1 \in G_{k_0+h}(\mu)$.

Case (b): $I_{k_0+h}(\mu) \neq \emptyset$.

Case (a). In this case, we argue as in Case 1 (Section 6.2.2) with the following modifications.

- Take the fixed points p_μ (the same as above) and q^1_μ (the fixed point of $\Gamma^{(1)}_\mu$ closest to 1). The gaps of generation $(0,1)$ are the intervals $A_0(\mu),\ldots,A_{k_0+h}(\mu)$, each of them containing the corresponding $G_j(\mu)$ and whose extremes are mapped by $\Gamma^{(1)}_\mu$ into p_μ and q^1_μ.
- To define the gaps of generation $(0,2)$, consider the discontinuities d^j_i, $d^j_i \in I_i(\mu)$, of $\Gamma^{(1)}_\mu$ and, for each d^j_i, the interval $A^j_i(\mu)$ containing d^j_i and mapped onto $[\beta^{-1},p_\mu)\cup(q^1_\mu,1]$ by $\Gamma^{(1)}_\mu$.

As in Case 1, the key step is to estimate the ratios bridges/gaps of generations $(0,1)$ and $(0,2)$. As before, it is enough to estimate the ratios $|I_j(\mu)|/|G_j(\mu)|$ and $|I_{j-1}(\mu)|/|G_j(\mu)|$ instead of the ratios of the bridges and gaps (for notational simplicity, we let $I^\ell_{k_0}(\mu)=I_{k_0}(\mu)$). First, as in Case 1, one has $|1-q^1_\mu|<(K_{15}\,|\mu|)/\rho^2)$, where K_{15} is independent of small μ. We have the following:

- For $j=1,\ldots,k_0-1$, the estimates for the quotients
$$\frac{|I_j(\mu)|}{|G_j(\mu)|} \quad\text{and}\quad \frac{|I_{j-1}(\mu)|}{|G_j(\mu)|}$$
are the similar to the ones made in Case 1. This also holds for $|I_{k_0-1}(\mu)|/|G_{k_0}(\mu)|$
- Since the *old* interval $I_{k_0}(\mu)$ was subdivided, we need to estimate the ratios
$$\frac{|I_{k_0}(\mu)|}{|G_{k_0}(\mu)|} \quad\text{and}\quad \frac{|I_{k_0}(\mu)|}{|G_{k_0+1}(\mu)|},$$
considering the *new* $I_{k_0}(\mu)=I^\ell_{k_0}(\mu)$. This is done using Claim 6.15 below.
- The ratios $|I_{k_0+j}(\mu)|/|G_{k_0+j}(\mu)|$ and $|I_{k_0+j}(\mu)|/|G_{k_0+j+1}(\mu)|$, $j\geq 1$, are calculated exactly as the ratios of gaps of generation $(1,1)$ in Case 1.

Claim 6.15. $|I^r_{k_0}(\mu)|<K_{16}\,|I^\ell_{k_0}(\mu)|$, *$K_{16}$ is independent of μ close to $\mu^\star$.*

The claim follows immediately from equation (16), taking $B=I^\ell_{k_0}(\mu)$ and $A=I^r_{k_0}(\mu)$ and noting that $I^\ell_{k_0}(\mu)\cup I^\ell_{k_0}(\mu)$ is contained in one interval (branch) of definition of Γ_μ, see Lemma 6.11.

By Lemma 6.7, $|G_{k_0}(\mu)|<K_2\,|\mu|$, by the claim and hypothesis (Case 2.2), one has
$$\frac{\xi}{K_2}=\frac{\xi\,|\mu|}{K_2\,|\mu|}\leq\frac{|I_{k_0}(\mu)|}{|G_{k_0}(\mu)|}=\frac{|I^\ell_{k_0}(\mu)|+|I^r_{k_0}(\mu)|}{|G_{k_0}(\mu)|}\leq\frac{(K_{16}+1)\,|I^\ell_{k_0}(\mu)|}{|G_{k_0}(\mu)|}.$$
Thus
$$\frac{|I^\ell_{k_0}(\mu)|}{|G_{k_0}(\mu)|}>\frac{\xi}{K_2\,(1+K_{16})}.$$

We now estimate $|I^\ell_{k_0}(\mu)|/|G_{k_0+1}(\mu)|$. From the definition of $G_{k_0+1}(\mu)$, inequality (10), and Lemma 6.5,
$$|G_{k_0+1}(\mu)|\leq\frac{|G_0(\mu)|}{\rho}\leq\frac{K\,|\mu|}{\rho}.$$

By hypothesis (Case 2.2) and the claim,

$$\xi\,|\mu| \leq |I_{k_0}(\mu)| = |I^{\ell}_{k_0}(\mu)| + |I^{r}_{k_0}(\mu)| \leq (1+K_{16})\,|I^{\ell}_{k_0}(\mu)|.$$

Putting these inequalities together we have,

$$\frac{|I^{\ell}_{k_0}(\mu)|}{|G_{k_0+1}(\mu)|} > \frac{\xi\,\rho}{(1+K_{16})\,K}.$$

The estimates for the ratios bridge/gap of generation $(0,2)$ follow as in Case 1, using Lemma 6.11 and noting that the sizes of $[\beta^{-1}, p_\mu]$ and $[q^1_\mu, 1]$ are of order of (at most) $|\mu|$. Arguing as in the previous case, this implies that all the ratios $|I|/|G|$ of generations $(0,1)$ and $(0,2)$ are of order of (at least) $1/|\mu|^{2/5}$. This ends the estimate of the thickness of Σ_μ in Case (a).

Case (b). In this the situation similar to Case 2, now considering the map $\Gamma^{(1)}_\mu$ and $1 \in I_{k_0+h}(\mu) = I_{k_1}(\mu)$. We proceed exactly as before, if we fall in Cases (similar to) 1, 2.1, 2.2, 2.2.1 or 2.2.2(a) then the construction goes as above ends and the proof of Proposition 6.6 is finished.

If we are in Case 2.2.2(b), we define a new map $\Gamma^{(2)}_\mu$, obtained from $\Gamma^{(1)}_\mu$ exactly as $\Gamma^{(1)}_\mu$ from Γ_μ. There is a new interval $I_{k_2}(\mu)$ containing the point 1. Since the definition of $I_{k_2}(\mu)$ involves two backwards iterations by Γ_μ, by Lemma 6.5, $|I_{k_2}(\mu)| \leq 1/\rho^2$.

Lemma 6.11 and equation (16) guarantee that all the constants in Case 2.2.2(b) for $\Gamma^{(1)}_\mu$ are the same as the ones obtained in the first inductive step. We now proceed inductively: in each step n either the construction is finished (i.e., we fall in *new* Cases 1, 2.1, 2.2, 2.2.1 or 2.2.2(a)) or we define a new map $\Gamma^{(n)}_\mu$ and an interval $I_{k_n}(\mu)$ of size less than $1/\rho^n$ containing the point 1. Therefore, there is a first n such that $|I_{k_n}(\mu)| < 1/\rho^n < \xi\,|\mu|$, so we are in a situation similar to Case 2.1 and the recurrent procedure ends.

The proof of item (a) (lower bound for the thickness) of Proposition 6.6 is now complete. By construction, all the gaps of Σ_μ (in all sub-cases) are at most of order of $|\mu|^{2/5}$, implying (b). The claim on the diameter follows noting that the convex hull of Σ_μ always contains $[p_\mu, q_\mu]$.

6.3 The thick hyperbolic unstable set Ω_t

Consider $t \in B_n$, $t = \lambda^n(1+\mu)$, where $\mu < \mu^\star$ and close to $\mu^\star$. Recall that α is fixed and equal to $\alpha(\mu^\star)$ and the transition map F associated to the arc f_t is $F_{\alpha(\mu^\star)}$. The (transitive) hyperbolic set Ω_t of index two is the (minimal) f_t-invariant set such that:

- **(a)** The central localization of Ω_t (see (9) for the definition) is $(\Omega_t)^c = \Sigma_\mu$.
- **(b)** Every $X = (x,y,z) \in \Omega_t \cap \Delta^+$ has *consecutive* returns (by f_t) to Δ^+ such that the central dynamics of the composition of the returns is $\Gamma^{(n)}_\mu$ (n is the last step in the inductive process of the construction of Σ_μ),

  $$f^r_t(X) = (x', y', z') \quad \text{and} \quad y' = \Gamma^n_\mu(y),$$

  where r is the number of iterations for obtaining such a final return.

Let us explain a bit more condition (b) and why the set Ω_t is hyperbolic (of index two). Let us assume, for simplicity, that $\Sigma_\mu = (\Omega_t)^c$ is obtained in the first step of the process (i.e., associated to Γ_μ). Note that $y \in \cup_{j=0}^{k_0} I_j(\mu)$, see (11). First, if $y \in I_0(\mu)$ then $y \in J_i(\mu)$ and thus it has a return of type $(i, 0)$, recall the definitions of the intervals $J_i(\mu)$ in (12) and of return of type (p, q) in Section 3.2 (just before equation (2)). Using Remark 3.3 one has

$$f_t^{i+n+k_0+0+n+N}(x, y, z) = (x', y', z'), \qquad y' = \Psi_\mu(y) = \Phi_\mu^{i,0}(y) = \Gamma_\mu(y).$$

By Lemma 6.5, $Df_t^{i+n+k_0+n+N}(X)$ expands in the central direction (at least) by a factor $\rho > 1$.

Second, if $y \in I_k(\mu)$ then $(\Phi_\mu^{0,0})^k$ belongs to some subinterval $J_i(\mu)$ of $I_0(\mu)$. Therefore, using Remark 3.3, the point $X = (x, y, z) \in \Delta^+$ has k consecutive returns to Δ^+ of type $(0, 0)$ (each one corresponding to $(0 + n + k_0 + 0 + n + N)$ iterations of f_t) followed by a return of type $(i, 0)$ (corresponding to $(i + n + k_0 + 0 + n + N)$ iterations). Thus

$$f_t^{(i+n+k_0+n+N)+k\,(n+k_0+n+N)}(x, y, z) = (x', y', z'), \qquad y' = \Gamma_\mu(y).$$

By Lemma 6.5, the central direction is (at least) ρ-expanding, $\rho > 1$. As the number of iterations involved in those returns is upper bounded (the bound depending on the localization of p_μ and the number k_0 of initial bridges of the Markov partition), one gets the hyperbolicity of the set Ω_t.

If the set Σ_μ is obtained in the step n of the process (associated to $\Gamma_\mu^{(n)}$), one needs to consider other types of returns (defined similarly). The hyperbolicity follows from the fact that $\Gamma_\mu^{(n)}$ is ρ-expanding and that the returns involve a uniformly upper bounded number of iterates of f_t.

7 Linked sets. Persistence of heterodimensional cycles

We prove that, for $\mu^- < \mu^\star$ close to $\mu^\star$, for every $t \in C_n = [\lambda^n\,(1+\mu^-), \lambda^n(1+\mu^\star))$, the hyperbolic sets $\Lambda_t = \Lambda_{t,\alpha(\mu^\star)}$ in Lemma 6.2 (of index one and contained in the homoclinic class of P) and Ω_t in Section 6.3 (of index two) are heteroclinically related. This will end the proof of the theorem.

Lemma 7.1. (Gap Lemma, [Ne$_2$] and [PT$_3$, page 63]) *Let K_1 and K_2 be Cantor sets of $\mathbb{R}$ with thickness τ_1 and τ_2, respectively. If $\tau_1 \cdot \tau_2 > 1$ then one of the following alternatives occur: K_1 is contained in a gap of K_2; K_2 is contained in a gap of K_1; $K_1 \cap K_2 \neq \emptyset$.*

Write $t = \lambda^n(1 + \mu)$ and recall that $\Lambda_\mu = (\Lambda_t)^c$ and $\Sigma_\mu = (\Omega_t)^c$. By Lemma 6.2 and Proposition 6.6, the product of the thickness of Λ_μ and Σ_μ is bigger than one. Moreover, by construction, the convex hull of Λ_μ and Σ_μ are non-disjoint. Finally, Lemma 6.2 and Proposition 6.6 imply that each Cantor set is not contained in a gap of the other one (recall the conditions on the size of the gaps of Σ_μ and on the diameters of these two sets). Thus, by Lemma 7.1, $\Sigma_\mu \cap \Lambda_\mu \neq \emptyset$.

By construction, it is immediate that $W^s(\Lambda_t) \cap W^u(\Omega_t) \neq \emptyset$. To see that $W^u(\Lambda_t) \cap W^s(\Omega_t) \neq \emptyset$ take any $y_0 \in \Lambda_\mu \cap \Sigma_\mu$. By Remark 3.3, there are points $X_1 = (x_1, y_0, z_1) \in \Lambda_t$ and $X_2 = (x_2, y_0, z_2) \in \Omega_t$. By construction,

$$\{(x_1, y_0)\} \times [-1, 1] \subset W^u(X_1) \quad \text{and} \quad [-1, 1] \times \{(y_0, z_2)\} \subset W^s(X_2).$$

So $(x_1, y_0, z_2) \in W^u(\Lambda_t) \cap W^s(\Omega_t)$ and there is a (heterodimensional) cycle related to Λ_t and Ω_t.

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Collision, explosion and collapse of homoclinic classes

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Abstract

Homoclinic classes of generic C^1-diffeomorphisms are maximal transitive sets and pairwise disjoint. We here present a model explaining how two different homoclinic classes may intersect, failing to be disjoint. For that we construct a one-parameter family of diffeomorphisms $(g_s)_{s\in[-1,1]}$ with hyperbolic points P and Q having nontrivial homoclinic classes, such that, for $s<0$, the classes of P and Q are disjoint, for $s=0$, the classes collide and their intersection is a saddle-node, and, for $s>0$, after an explosion, the two classes are equal. Our constructions involve bifurcations through heterodimensional and saddle-node cycles.

Introduction

In this paper we study the collision of non-trivial homoclinic classes *via* saddle-node bifurcations and the dynamics before and after this collision. The main motivation of this paper comes from recent results about *maximal transitive sets*: for generic[1] C^1-diffeomorphisms, the homoclinic classes are either disjoint or equal ([Ar] and [CMP]). Our objective is to understand how two homoclinic classes may be non-disjoint and different as well as the dynamical consequences of this pathology.

Let us start by recalling some definitions. Given a diffeomorphism f, an f-invariant set Λ is *transitive* if there is an $x\in\Lambda$ whose forward orbit is dense in Λ, i.e., $\Lambda=\overline{\cup_{i\in\mathbb{N}}f^i(x)}$. A transitive set is *maximal* if it is a maximal element of the family of all transitive sets partially ordered by inclusion. Observe that any transitive set is contained in a maximal one. A transitive set Λ is *saturated* if it contains every transitive set Σ such that $\Lambda\cap\Sigma\neq\emptyset$. Clearly, every saturated transitive set is also maximal. The *homoclinic class* of a saddle P of f, denoted by $H(P,f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Every homoclinic class is a transitive set, not necessarily maximal nor saturated.

The problem of characterizing and describing (for a large class of systems) maximal and saturated transitive sets is a key problem in dynamics. In fact, these saturated transitive sets are the natural candidates for playing the role of the elementary pieces of dynamics (similar to the role of the basic sets in the Smale hyperbolic theory, [Sm]). Recently, [Ab] states that for generic C^1-diffeomorphisms f having finitely many different homoclinic classes the non-wandering set of f, $\Omega(f)$, is the disjoint union of such classes. Moreover, these classes verify a weak form of hyperbolicity (existence of a dominated splitting, see [BDP]) and are the maximal invariant sets of a fixed filtration (see Section 6.3) independent of the generic diffeomorphism in a neighborhood of f.

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[1] by a generic diffeomorphism we mean a diffeomorphism in a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$.

Consider a closed manifold M and denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the usual uniform topology. In [Ar], it is proved that homoclinic classes of generic diffeomorphisms are *maximal transitive sets.* [CMP] generalizes this result by proving that homoclinic classes of generic diffeomorphisms are saturated transitive sets. Thus homoclinic classes of generic diffeomorphisms are either equal or disjoint. We observe that there are locally generic diffeomorphisms having saturated transitive sets without periodic orbits (so which are not homoclinic classes), see [BD$_2$]. The goal of this paper is to give examples of homoclinic classes which are not saturated transitive sets, presenting an explanation of how this pathology arises. In fact, we exhibit homoclinic classes which are not contained in any saturated transitive set. For simplicity, we consider diffeomorphisms defined on three manifolds, but our constructions can be carried out to higher dimensions after straightforward modifications.

We construct a diffeomorphism f with saddles P and Q with *Morse index* (dimension of the unstable bundle) one and two such that their homoclinic classes are nonhyperbolic, nontrivial, and maximal transitive, and whose intersection is just a saddle-node. So these classes are not saturated transitive sets. In fact, as mentioned above, we will prove that they are not contained in any saturated transitive set.

Theorem A. *Let M be a 3-dimensional closed manifold. There exist an open set $W \subset M$ and a family of diffeomorphisms $(g_s)_{s\in[-1,1]}$, $g_s\colon M \to M$, such that, for every s, the diffeomorphism g_s has hyperbolic fixed points P and Q of Morse indices 1 and 2 such that the maximal invariant set of g_s in W, denoted by Λ_s, verifies the following:*

- *For every small $s < 0$, the set $\Lambda_s \cap \Omega(g_s)$ is the disjoint union of the homoclinic classes $H(P, g_s)$ and $H(Q, g_s)$, where $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and locally maximal.*
- *For $s = 0$, $\Lambda_0 = \Lambda_0 \cap \Omega(g_0) = H(P, g_0) \cup H(Q, g_0)$, where $H(P, g_0)$ and $H(Q, g_0)$ are locally maximal and $H(P, g_0) \cap H(Q, g_0) = \{S\}$, where S is a saddle-node fixed point.*
- *For every small $s > 0$, $\Lambda_s = \Lambda_0 \cap \Omega(g_0) = H(P, g_s) = H(Q, g_s)$.*

This result means that the homoclinic classes of P and Q *collide* at $s = 0$ and thereafter *explode* (the point P that does not belong to $H(Q, g_0)$ is in $H(Q, g_s)$ for every small positive s, and the same holds for the point Q and $H(P, g_s)$). Finally, the homoclinic classes also *collapse*: $H(Q, g_s) = H(P, g_s)$ for positive s.

In the previous theorem the open set W is a level of a *filtration,* (see Section 6.3). Theorem A now implies the following:

Theorem B. *Under the hypotheses of Theorem A, the homoclinic classes $H(P, g_0)$ and $H(Q, g_0)$ are not saturated and they are not contained in any saturated transitive set.*

Our construction involves *saddle-node bifurcations* and *heterodimensional cycles.* We introduce a codimension-two bifurcation, the *saddle-node heterodimensional cycles,* and study the *lateral homoclinic classes of a saddle-node.* Let us explain all that in details.

Consider a diffeomorphism f having two hyperbolic fixed points P and Q with Morse indices 1 and 2, respectively. Then, f has a *heterodimensional cycle associated to P and Q* if the 2-dimensional stable manifold of P and unstable manifold of Q, denoted by $W^s(P, f)$ and $W^u(Q, f)$, have a non-empty transverse intersection, and the 1-dimensional unstable manifold of P, $W^u(P, f)$, and stable one of Q, $W^s(Q,f)$, have a quasi-transverse intersection throughout the orbit of a point x_0, i.e., $T_{x_0}W^s(Q,f)+T_{x_0}W^u(P,f)=T_{x_0}W^s(Q,f)\oplus T_{x_0}W^u(P,f)$, thus $\dim(T_{x_0}W^s(Q,f)+T_{x_0}W^u(P,f))=2$. A heterodimensional cycle is depicted in Figure 1 in Section 1. Bifurcations through heterodimensional cycles have been systematically studied in the series of papers [D_1, D_2, DR_1, DR_2, DU, DR_4, DR_5].

A *saddle-node* S of a diffeomorphism f is a periodic point (we here assume to be fixed) such that the derivative of f at S has 1 as its only eigenvalue in the unitary circle. We consider saddle-nodes of *saddle-type* (i.e., the derivative of f at S simultaneously has eigenvalues inside and outside the unitary circle). Thus the tangent bundle of M at S has a Df-invariant splitting $E^{ss}\oplus E^c\oplus E^{uu}$, where E^{ss} (resp. E^{uu}) is the bundle spanned by the eigenvectors associated to the contracting (resp. expanding) eigenvalues, and E^c is the eigenspace associated to the eigenvalue 1 (in our context, all these spaces have dimension 1). By the theory of invariant manifolds, see [HPS], there exist the *strong stable* and *unstable* manifolds of the saddle-node, defined as the unique f-invariant manifolds tangent at S to E^{ss} and to E^{uu} and denoted by $W^{ss}(S,f)$ and $W^{uu}(S,f)$, respectively.

Motivated by the fact that (generic) saddle-nodes (of saddle type) simultaneously behave as points of index two and one (the stable and unstable manifolds of the saddle-node have both dimension 2), we introduce *saddle-node heterodimensional cycles.* A diffeomorphism f has a saddle-node heterodimensional cycle associated to a saddle-node S and the saddle P of Morse index one if the (two-dimensional) unstable manifold of S and stable manifold of P have nonempty transverse intersection and the (one-dimensional) invariant manifolds $W^{ss}(S,f)$ and $W^u(P,f)$ have a quasi-transverse intersection along the orbit of some point. A saddle-node heterodimensional cycle is depicted in Figure 4 in Section 4. One similarly defines saddle-node heterodimensional cycles associated to a (saddle-type) saddle-node S and a saddle Q of Morse index two.

Roughly speaking, in our construction we consider a diffeomorphism f simultaneously having two saddle-node heterodimensional cycles. We consider a two parameter family $(f_{t,s})_{t,s\in[-1,1]}$ of diffeomorphisms such that $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, one associated to a saddle-node S and a saddle P of Morse index one and other one associated to a saddle Q of index one and the saddle-node S. The parameter t describes the unfolding of the cycles (relative motion between compact parts of $W^u(P,f_{t,0})$ and $W^{ss}(S,f_{t,0})$ and of $W^s(Q,f_{t,0})$ and $W^{uu}(S,f_{t,0})$). The parameter s describes the unfolding of the saddle-node: for positive s there are two saddles S_s^+ and S_s^- of indices 2 and 1, colliding at $s=0$ to the saddle-node S and disappearing for negative s. We see that, fixed any small $\bar t>0$, for $s>0$ (before the collapse of the saddles), $H(P,f_{\bar t,s})=H(S_s^+,f_{\bar t,s})$ and $H(Q,f_{\bar t,s})=H(S_s^-,f_{\bar t,s})$ for all small positive s. Moreover, $H(P,f_{\bar t,s})\cap H(Q,f_{\bar t,s})=\emptyset$. At the saddle-node bifurcation we have $H(P,f_{\bar t,0})\cap H(Q,f_{\bar t,0})=\{S\}$. Finally, for $s<0$, after the disappearing of the saddles, $H(P,f_{\bar t,s})=H(Q,f_{\bar t,s})$. See the results in Section 6. Theorem A follows by considering the arc $g_s=f_{\bar t,-s}$. To deduce Theorem B from Theorem A, we consider a filtration having the open set W as a level and analyze the orbits of recurrent points of Λ_s.

In forthcoming papers, we will illustrate how this type of bifurcation naturally appear as secondary bifurcations in the unfolding of heterodimensional cycles and give a model for the collision, explosion, and collapse of (nontrivial) hyperbolic homoclinic classes, see [DR_6].

Let us say a few words about our constructions. As mentioned, our setting necessarily corresponds to a non-generic situation, so we focus our attention on an example (we have not done any effort for generality). We begin by presenting (in Section 1) a model for the unfolding of a heterodimensional cycle. This model (motivated by [D_1] and [BD_1]) allows us to give a rather transparent explanation of the dynamics in the unfolding of a heterodimensional cycle by reducing it to the study of the dynamics of an iterated system of functions defined on an interval, this is done in Section 2. Recall that the dynamics of a (linear) Smale horseshoe is given by two affine expanding maps of the interval (say $I = [0, 1]$) whose domains of definition are two disjoint closed subintervals of I (say $[0, 1/3]$ and $[1/3, 1]$). The interval $(1/3, 2/3)$ is the main gap of the horseshoe and corresponds to points in the basin of attraction of a sink. The affine model for heterodimensional cycles is a system of iterated functions with infinitely many maps F_i defined on subintervals I_i of I which are non-disjoint (the interior of the intervals I_i are pairwise disjoint, but I_i and I_{i+1} have a common extreme). Thus in this model there are no gaps and there are no escaping points.

In Section 3, we prove that, after unfolding the cycle, the dynamics of the model family is non-hyperbolic: the point of index 1 in the cycle belongs to the homoclinic class of the point of index 2 in the cycle. Here, using the one-dimensional reduction, we give a shorter and clearer proof of the results in [D_1]. Since our constructions rely heavily on this proof and there is not any written version of this approach, we have decided to include a short description of it.

In Section 4, we introduce the *lateral homoclinic classes of a saddle-node* S of a diffeomorphism f as above, $H^+(S, f)$ and $H^-(S, f)$, respectively defined as the closure of the transverse intersections $W^u(S, f) \pitchfork W^{ss}(S, f)$ and $W^s(S, f) \pitchfork W^{uu}(S, f)$. These lateral homoclinic classes essentially behave as the usual ones. We see that for arcs f_t unfolding at $t = 0$ the saddle-node heterodimensional cycle (associated to a saddle P of index one and the saddle-node S) one has $H(P, f_t) \subset H^+(S, f_t)$ for all small positive t. Moreover, under mild conditions, one also gets $H^+(S, f_t) = H(P, f_t)$ for all small $t > 0$. The inclusion $H(P, f_t) \subset H^+(S, f_t)$ follows adapting (in a rather straightforward way) the results for the model family in Section 3. For the inclusion $H^+(S, f_t) \subset H(P, f_t)$ we need new ingredients that we borrow from [D_2].

Using the results in Sections 3 and 4, we get a complete description of the homoclinic classes $H(P, f_{\bar{t},s})$ and $H(Q, f_{\bar{t},s})$ before the collapse of the saddles S_s^+ and S_s^- to the saddle-node. Finally, to study $H(P, f_{\bar{t},s})$ and $H(Q, f_{\bar{t},s})$ after the collision, we introduce new systems of iterated functions and analyze their dynamics.

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1 Heterodimensional cycles: a model family

In this section, we construct a model one-parameter family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms unfolding a heterodimensional cycle. The study of the semi-local dynamics of f_t will be reduced to the analysis of a one-parameter family of endomorphisms with infinitely many discontinuities which describe the dynamics of f_t in the *central direction,* see Section 2.

Consider a diffeomorphism f with a heterodimensional cycle having the following dynamical configuration. In local coordinates in $\mathbb{R}^3$, the cycle is associated to saddle fixed points $Q = (0, 0, 0)$ and $P = (0, 1, 0)$ of indices 2 and 1, respectively, verifying the following conditions:

Partially hyperbolic (semi-local) dynamics of the cycle:

- In the cube $[-1, 1] \times [-1, 2] \times [-1, 1]$ the diffeomorphism has the form

$$f(x, y, z) = (\lambda_s x, F(y), \lambda_u z),$$

where $F\colon [-1,2] \to (-1,2)$ is an increasing map with exactly two fixed points, a source at 0 and a sink at 1, and $0 < \lambda_s < d_m < 1 < d_M < \lambda_u$, where $0 < d_m < F'(x) < d_M$ for all $x \in [0,1]$.

- There is $\delta > 0$ such that F is linear in $[-\delta, \delta]$ and affine in $[1-\delta, 1+\delta]$. We denote by $\beta > 1$ and $0 < \lambda < 1$, the derivative of F at 0 and 1, respectively.

Observe that $[-1,1] \times \{(0,0)\} \subset W^s(Q)$, $\{0\} \times [0,1) \times [-1,1] \subset W^u(Q)$, $\{(0,0)\} \times [-1,1] \subset W^u(P)$, and $[-1,1] \times (0,1] \times \{0\} \subset W^s(Q)$. Thus the curve $\gamma = \{0\} \times (0,1) \times \{0\}$ (called *connexion*) is a normally hyperbolic curve contained in $W^u(Q) \pitchfork W^s(P)$.

Existence and unfolding of the cycle:

- **The cycle:** There exist $k_0 \in \mathbb{N}$ and a small neighborhood U of $(0,1,-1/2) \in W^u(P)$ such that the restriction of f^{k_0} to U is a translation,

  $$f^{k_0}(x,y,z) = (x - 1/2, y - 1, z + 1/2).$$

  Thus $f^{k_0}(0,1,-1/2) = (-1/2,0,0) \in W^u(Q)$ and $W^s(Q)$ and $W^u(P)$ meet throughout the orbit of the heteroclinic point $(-1/2,0,0)$. By construction, $(-1/2,0,0)$ is a quasi-transverse heteroclinic point.

- **The unfolding of the cycle:** Consider the family $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ of diffeomorphisms coinciding with f in $[-1,1] \times [-1,2] \times [-1,1]$ and such that the restriction of $f_t^{k_0}$ to U is of the form

  $$f_t^{k_0}(x,y,z) = (x - 1/2, y - 1 + t, z + 1/2) = f^{k_0}(x,y,z) + (0,t,0).$$

  So, for $t > 0$, $\{(-1/2,t)\} \times [-1,1] \subset W^u(P, f_t)$ and $x_t = (-1/2,t,0)$ is a transverse homoclinic point of P (for f_t). Similarly, $y_t = (-1/2,0,0)$ is a transverse homoclinic point of Q.

Figure 1: The model cycle and its unfolding

Consider a small *neighborhood of the heterodimensional cycle associated with f_0*, that is, an open set W containing the connexion $\gamma = \{0\} \times [0,1] \times \{0\}$ and the f_0-orbit of the heteroclinic point $(-1/2,0,0)$. For small t, let Λ_t be the *maximal f_t-invariant set in W*, $\Lambda_t = \cap_{n\in\mathbb{Z}} f_t^n(W)$. Consider also the forward and backward f_t-invariant sets in W, $\Lambda_t^+ = \cap_{n\geq 0} f_t^{-n}(W)$ and $\Lambda_t^- = \cap_{n\geq 0} f_t^n(W)$.

Fix a small positive ρ and consider the fundamental domains of F given by $D^+ = [\beta^{-1}\rho, \rho]$ and $D^- = [1-\rho, \lambda(1-\rho)]$ contained in the neighborhoods of 0 and 1 where F is affine. We can choose ρ such that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$ (for notational simplicity let us put $N = 1$). The map F^N is the *transition from* 0 *to* 1 [2]. Assume that that $F'(1) = \lambda$ and $F'(0) = \beta$ verify:

(T1) $F'(x) \geq \dfrac{1}{2}\,\dfrac{1-\lambda}{1-\beta^{-1}}$, for all $x \in D^+$,

(T2) $(1-\lambda) < \beta^{-1}$, and

(T3) $\dfrac{(1-\lambda)\,\lambda}{2\,(1-\beta^{-1})\,\beta} = \ell > 1.$

To get condition (T1) it is enough to consider F with small distortion. For conditions (T2) and (T3) it is enough to take β close enough to 1^+. In fact, later we will consider the case where Q is a saddle-node (saddle-node heterodimensional cycles) and $\beta = 1$, see Section 4. Our first result is:

Theorem 1.1. *For every $t > 0$ sufficiently small, $H(P, f_t) \subset H(Q, f_t)$ and $\Lambda_t \subset H(Q, f_t)$.*

This theorem was stated in [D$_1$]. Here we give a more conceptual prove of it, which enables us to introduce some technical tools to be used systematically later on. First, in Section 2 we will introduce the system of iterated functions associated to the cycle (this approach is motivated by [DR$_5$]). In Section 3, we deduce the theorem from the results in Section 2.

2 Expanding one-dimensional dynamics associated to the cycle

For each small $t > 0$, consider the *scaled fundamental domains* $D_t^{\pm}$ defined as follows: let $D_t^- = [1-t, \lambda(1-t)]$ and define k_t as the smaller $k \in \mathbb{N}$ with $F^{-k}(D_t^-) \subset [0, t]$. We define

$$D_t^+ = [a_t, b_t] = [\beta^{-1}(b_t), b_t] = F^{-k_t}(D_t^-), \quad \text{where} \quad \beta^{-2}t < a_t < b_t \leq t.$$

We next define an expanding map R_t, $R_t \colon D_t^+ \to D_t^+$, with discontinuities describing the central dynamics of the return map of f_t defined on $[-1,1] \times D_t^+ \times [-1,1]$. First, for each small $t > 0$ define the *transition map T_t from D_t^+ to D_t^-* by

$$T_t \colon D_t^+ \to D_t^-, \quad x \mapsto T_t(x) = F^{k_t}(x).$$

Lemma 2.1. *The map T_t verifies $T_t'(x) > \ell > 1$ for all $x \in D_t^+$, where ℓ is as in condition* **(T3)**.

---

[2] This transition plays a key role for determining the dynamics after unfolding the cycle and it is determined by the *Mather invariant* of F, [Ma]: in a neighborhood of 0, the map F is the time-one of the vector field $X(y) = (\log\beta)\, y\, \frac{\partial}{\partial y}$, whose flow is $y \mapsto \beta^t\, y$. Similarly, in a neighborhood of 1, F is the time-one of $Y(y) = (\log\lambda)\,(y-1)\,\frac{\partial}{\partial y}$. Consider, for y close to 0^+, an large n such that $F^n(x)$ is close to 1^- and write $DF^n(y)(X(y)) = \mu(y)\, Y(F^n(y))$. Using the local F-invariance of X and Y (near 0 and 1), one has that $\mu(x)$ does not depend on n and that $\mu(x) = \mu(F(x))$. The function μ is the Mather invariant of F which describes its distortion. For instance, if μ is identically 1, then F is exactly the exponential of a vector field.

Proof: Given $x \in D_t^+$ let $k_t = n_t(x) + 1 + m_t(x)$, where $F^{n_t(x)}(x) \in D^+$ and $F^i(x) \notin D^+$ for all $1 \leq i < n_t(x)$. We claim that

$$\frac{1}{\beta\, t} \leq \beta^{n_t(x)} \leq \frac{\beta^2}{t} \quad \text{and} \quad \lambda\, t \leq \lambda^{m_t(x)} \leq \frac{t}{\lambda}. \tag{1}$$

For first inequalities just note $x \in D_t^+ \subset (\beta^{-2}\, t, t]$ and $\beta^{n_t(x)}\, x \in [\beta^{-1}, 1]$. The other ones follow analogously. As $T_t(x) = \lambda^{m_t(x)}\, F(\beta^{n_t(x)}\, x)$, hypotheses (T1) and (T3) and the estimates in (1) give

$$|T_t'(x)| = \lambda^{m_t(x)}\, |F'(x)|\, \beta^{n_t(x)} \geq (\lambda\, t) \left(\frac{1}{2}\, \frac{1-\lambda}{1-\beta^{-1}}\right) \left(\frac{1}{\beta\, t}\right) = \frac{1}{2}\, \frac{\lambda\,(1-\lambda)}{\beta\,(1-\beta^{-1})} = \ell > 1,$$

as we claimed. □

Since $T_t(x) \in D_t^- = [1-t, 1-\lambda t]$ for all $x \in D_t^+$, we can define the map G_t by

$$G_t \colon D_t^+ \to [0, t], \quad x \mapsto G_t(x) = T_t(x) + (t-1).$$

Figure 2: The expanding maps T_t, G_t and R_t

Remark 2.2. *The map G_t is monotone increasing and $G_t(D_t^+) = [0, t(1-\lambda)]$.*

Claim 2.3. *Let $(a_t, b_t] = \tilde{D}_t^+ \subset D_t^+$. Given $x \in \tilde{D}_t^+$ let $i(x) \in \mathbb{Z}$ be the minimum i with $\beta^i(G_t(x)) \in D_t^+$. Then there is $i_0 > 0$ (maximum with such property) such that $i(x) \geq i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: Recall first that $b_t \in (\beta^{-1}\, t, t]$. On the other hand, from $(1-\lambda) < \beta^{-1}$ (condition (T2)) and Remark 2.2, $G_t(\tilde{D}_t^+) = (0, t\,(1-\lambda)] \subset (0, \beta^{-1}\, t)$. Thus the right extreme of $G_t(\tilde{D}_t^+)$ is less than the left extreme of D_t^+, hence $i(x) > 0$ for all $x \in \tilde{D}_t^+$, ending the proof of the claim. □

Finally, the return map R_t is defined by

$$R_t \colon \tilde{D}_t^+ \to D_t^+, \quad R_t(x) = \beta^{i(x)}(G_t(x)) = \beta^{i(x)}\,(T_t(x) + (t-1)).$$

Next we study the dynamics of R_t: the map R_t is uniformly expanding and has (infinitely many) discontinuities where the lateral derivatives are well defined. These discontinuities will play a key role in our constructions. The definition of $i_0 \in \mathbb{N}$ in Claim 2.3 implies that $\beta^{-i_0}(a_t) \in G_t(D_t^+)$. For each $i \geq i_0$ define $d_i \in \tilde{D}_t^+$ by $G_t(d_i) = \beta^{-i}(a_t)$. By construction, the sequence $(d_i)_{i \geq i_0}$ corresponds to the discontinuities of R_t and verifies the following:

- $d_{i+1} < d_i$ and $d_i \to a_t$,
- Let $[d_{i+1}, d_i] = I_i$, $i > i_0$, and $I_{i_0} = [d_{i_0}, b_t]$. The map R_t is continuous and strictly increasing in the interior of each interval I_i. We continuously extend R_t to the whole I_i, obtaining a bi-valuated return map R_t with $R_t(d_i) = \{a_t, F(a_t) = b_t\}$ for all $i > i_0$. In particular, the restriction of R_t to any I_i, $i > i_0$, is onto. We let $R_t(b_t) = c_t \leq b_t$.

The main properties of R_t are summarized in the next lemma.

Lemma 2.4. *The restriction of R_t to each interval $I_i, i > i_0$, is onto and $R_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = d_i$ this means that the lateral derivatives of R_t at x are greater than ℓ). Moreover, $0 \in G_t(R_t(d_i))$ for all $i \geq i_0$.*

The expansiveness of R_t follows from Lemma 2.1 and Claim 2.3. Condition $0 \in G_t(R_t(d_i))$ follows from $a_t \in R_t(d_i)$ and $0 \in G_t(a_t)$.

Lemma 2.5. *Consider small $t > 0$ and an open subinterval J of $\tilde{D}_t^+$. Then there is $k \in \mathbb{N} \cup \{0\}$ such that $R_t^k(J)$ contains a discontinuity of R_t. In particular, there is $x \in J$ such that $G_t(R_t^k(x)) = 0$.*

Proof: If the interval J contains a discontinuity we are done. Otherwise, let $i > 0$ be such that the intervals J, $R_t(J), \ldots, R_t^i(J)$ do not contain discontinuities. Thus, for each $k \in \{0, \ldots, i\}$, there is $i_k \geq i_0$ such that $R_t^k(J) \subset I_{i_k}$. Lemma 2.4 implies that $|R_t^k(J)| \geq \ell^k |J|$, $\ell > 1$, for all $k \in \{0, \ldots, i\}$. Since the size of the intervals I_i is upper bounded, this inequality implies that there is a first $m \in \mathbb{N}$ such that $R_t^m(J)$ is not contained in any I_i, thus it intersects the set of discontinuities of R_t. $\square$

3 The maximal invariant set: Proof of Theorem 1.1

Next proposition is the main technical result of this section. Heuristically, it means that the one-dimensional stable manifold of Q topologically behaves as a two-dimensional manifold.

Proposition 3.1. *For every small $t > 0$ and every two-disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^s(Q, f_t) \pitchfork \chi \neq \emptyset$. In particular, $W^s(P, f_t)$ is contained in the closure of $W^s(Q, f_t)$.*

Proof of the inclusion $H(P, f_t) \subset H(Q, f_t)$ in Theorem 1.1. By the definition of $H(P, f_t)$, it suffices to see that any $x \in W^s(P, f_t) \pitchfork W^u(P, f_t)$ is accumulated by homoclinic points of Q. By the configuration of the cycle, $W^u(P, f_t) \subset \text{closure}\,(W^u(Q, f_t))$, thus, given any $x \in H(P, f_t)$ and any $n > 0$, there is a disk Δ_n, contained in $W^u(Q, f_t)$ and in the ball of radius $1/n$ centered at x, whose interior meets transversely $W^s(P, f_t)$. By Proposition 3.1, $\Delta_n \pitchfork W^s(Q, f_t) \neq \emptyset$. Thus there is $y_n \in \Delta_n \cap H(Q, f_t)$. By construction, $y_n \to x$, proving the inclusion $H(P, f_t) \subset H(Q, f_t)$.

3.1 Proof of Proposition 3.1

We now go into the details of the proof of Proposition 3.1. We first introduce some definitions.

- A set $\Delta \subset [-1, 1] \times [-1, 2] \times [-1, 1]$ is a *vertical strip* if $\Delta = \{x_1\} \times [l_1, l_2] \times [r_1, r_2]$, where $l_1 < l_2$ and $r_1 < 0 < r_2$. The segment $\{x_1\} \times [l_1, l_2] \times \{0\}$ is the *basis* of Δ. The *width* and the *height* of Δ are $w(\Delta) = (l_2 - l_1)$ and $h(\Delta) = (r_2 - r_1)$. The strip Δ is *complete* if $r_2 = 1$ and $r_1 = -1$, *well located* if $[l_1, l_2]$ is contained in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.

- A subset $J \subset [-1,1] \times [-1,2] \times [-1,1]$ is a *vertical segment* if $J = \{x_1\} \times \{l_1\} \times [r_1, r_2]$, where $r_1 < 0 < r_2$. The point $(x, \ell_1, 0)$ is the basis of J. The height of J is $h(J) = (r_2 - r_1)$. As above, the segment J is *complete* if $r_1 = -1$ and $r_2 = 1$, *well located* if l_1 is in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.
- A vertical segment J (resp. strip Δ) is *at the right of* Q if $l_1 \in (0,1]$.

Given an interval $\alpha \subset [-1,2]$, let $\Delta(\{x\} \times \alpha \times \{0\}) = \{x\} \times \alpha \times [-1,1]$ be the *unique* complete vertical strip with basis $\{x\} \times \alpha \times \{0\}$. Similarly, $J(x,y,0)$ is the unique complete vertical segment with basis $(x,y,0)$. The next algorithm associates to perfect segments and strips their successors:

Algorithm 3.2. *Let $\Delta = \Delta(\{x\} \times \alpha \times \{0\})$ be a perfect strip and define $\mathcal{G}_t(\Delta)$ as the perfect strip such that:*

- *the basis of $\mathcal{G}_t(\Delta)$ is of the form $(\{x'\} \times G_t(\alpha) \times \{0\})$, where $x' = \lambda_s^{k_t} x - 1/2$,*
- *$\mathcal{G}_t(\Delta)$ is contained in $f_t^{k_t}(\Delta)$ (where $F^{k_t}(D_t^+) = D_t^-$).*

Suppose now that α does not contain discontinuities, i.e. $\alpha \subset (d_{i+1}, d_i)$ for some i. Define $\mathcal{R}_t(\Delta)$ as the perfect strip such that:

- *the basis of $\mathcal{R}_t(\Delta)$ is of the form $(\{\hat{x}\} \times R_t(\alpha) \times \{0\})$, where $\hat{x} = \lambda_s^i(\lambda_s^{k_t} x - 1/2)$,*
- *$\mathcal{R}_t(\Delta)$ is contained in $f_t^{k_t+i}(\Delta)$.*

Similarly, to a perfect segment $J = J(x,y,0)$ we associate perfect segments $\mathcal{G}_t(J)$ and $\mathcal{R}_t(J)$ (provided $y \neq d_i$ for all i).

The strips $\mathcal{G}_t(\Delta)$ and $\mathcal{R}_t(\Delta)$ in Algorithm 3.2 are obtained as follows. Given a set A and a point $x \in A$, let $C(x, A)$ be the connected component of A containing x. Take a small neighborhood V of $f_0^{k_0}(\{0,1\} \times [-1,1]) \subset W^u(P, f_0)$, k_0 as in the definition of the cycle in Section 1, then

$$\begin{aligned} \mathcal{G}_t(\Delta) &= C(f_t^{k_t}(x,y,0), f_t^{k_t}(\Delta) \cap V) \cap [-1,1]^3), \\ \mathcal{R}_t(\Delta) &= C(f_t^{i}(x',y',0), f_t^{k_t+i}(\Delta) \cap V) \cap [-1,1]^3), \end{aligned}$$

where $(x,y,0)$ is any point in the basis of Δ, $x' = (\lambda_s^{k_t} x - 1/2)$, and $y' \in G_t(\alpha)$. The construction for the successors of segments is analogous.

Lemma 3.3. *The manifold $W^u(P, f_t)$ contains a perfect segment for all small $t > 0$.*

Proof: Consider the transverse homoclinic point $x_t = (-1/2, t, 0)$ of P. Recall that $t \geq b_t$ and $\beta^{-1} t \in D_t^+ = [a_t, b_t]$. Let us assume that $t > b_t$, and thus $\beta^{-1} t \in (a_t, b_t)$ (the case $t = b_t$ follows similarly, so it will be omitted). Consider the complete vertical segment $\mathcal{R}_t(J)$, where $J = J(-(\lambda_s^{-1}/2), \beta^{-1} t, 0) \subset W^u(P, f_t)$. If $R_t(\beta^{-1} t)$ belongs to the interior of D_t^+, then $\mathcal{R}_t(J) \subset W^u(P, f_t)$ is the announced segment. Otherwise, $R_t(\beta^{-1} t) = b_t$ and there is a homoclinic point of P of the form $(x', b_t, 0)$. Using the λ-lemma and the product structure of the cycle, one gets homoclinic points $(x_n, y_n, 0)$ of P and complete segments $J_n = J(x_n, y_n, 0) \subset W^u(P, f_t)$ such that $x_n \to x'$, $y_n \to b_t$, and y_n is increasing. Thus y_n is in the interior of D_t^+ for every big n and $J_n \subset W^u(P, f_t)$ is perfect. □

For clearness we first prove Proposition 3.1 in the following special case:

Proposition 3.4. *Let* $\chi \subset [-1,1]\times[-1,2]\times[-1,1] = \{x\}\times A$*, where* A *is a disk of* $\mathbb{R}^2$ *whose interior contains a point of the form* $(y,0)$ *with* $y\in(0,2)$*. Then* χ *intersects transversely* $W^s(Q,f_t)$*.*

We claim that is enough to prove the Proposition 3.4 for perfect strips:

Lemma 3.5. *Let* Δ *be a perfect strip. Then there is* $k\in\mathbb{N}$ *such that* $f_t^k(\Delta)\pitchfork W^s(Q,f_t)\neq\emptyset$*.*

Proof: Suppose that $\Delta=\Delta(\{x_0\}\times\alpha\times\{0\})$, α in the interior of D_t^+. By Lemma 2.5, there exist y_0 in the interior of α and $k\in\mathbb{N}$ such that $G_t(R_t^k(y_0))=0$. Thus the vertical strip $\mathcal{G}_t(\mathcal{R}_t^k(\Delta))$ (contained in the forward orbit of Δ) intersects transversely $[-1,1]\times\{(0,0)\}\subset W^s(Q,f_t)$. $\square$

Proof of Proposition 3.4: By Lemma 3.3, $W^u(P,f_t)$ contains a perfect vertical segment J. Since, by definition, χ meets transversely $W^s(P,f_t)$, the λ-lemma implies that forward orbit of χ contains a sequence of complete strips χ_n accumulating to J. Thus χ_n contains a perfect strip for all n large. Lemma 3.5 implies that χ_n (and thus χ) transversely meets $W^s(Q,f_t)$. $\square$

Proof of Proposition 3.1: We can assume that χ is transverse to $W^s_{loc}(P,f_t)$ and contained in $[-1,1]\times[-1,2]\times[-1,1]$. If χ contains a subset of the form $\{x\}\times A$, where A is an open subset of $\mathbb{R}^2$ containing a point $(0,y)$ with $y\in(0,2)$, Proposition 3.4 implies the result. For the general case, consider a point $(x_0,y_0,0)$, $y_0\in(0,2)$, in the interior of $\chi\pitchfork W^s_{loc}(P,f_t)$ and for every big n the vertical strip

$$\Sigma_n=\{x_0\}\times[y_0-1/n,y_0+1/n]\times[-1/n,1/n].$$

The strips Σ_n verify the hypotheses of Proposition 3.4, hence there is $(x_0,y_n,z_n)\in\Sigma_n\pitchfork W^s(Q,f_t)$ such that $H_n=[-1,1]\times\{(y_n,z_n)\}\subset W^s(Q,f_t)$. Since $(x_0,y_n,z_n)\to(x_0,y_0,0)$, it is immediate that H_n meets transversely χ for all large n, ending the proof of the proposition. $\square$

3.2 The maximal f_t-invariant set

To prove the second part of Theorem 1.1 (the inclusion $\Lambda_t\subset H(Q,f_t)$), let V_0 be the connected component of the neighborhood of the cycle W containing the heteroclinic point $(-1/2,0,0)$. There are two types of points of Λ_t: (a) those points whose orbit does not meet V_0 (i.e., the set $\gamma=\{0\}\times[0,1]\times\{0\}\subset W^s(P,f_t)\cap W^u(Q,f_t)$) and (b) those having an iterate in V_0.

We claim that every point of type (a) belongs to $H(Q,f_t)$: given any $(0,x,0)$, $x\in(0,1)$, consider the disk $\Delta_n=\{0\}\times[x-1/n,x+1/n]\times[-1/n,+1/n]\subset W^u(Q,f_t)$ satisfying the hypothesis of Proposition 3.1. Hence $\Delta_n\pitchfork W(Q,f_t)\neq\emptyset$ and thus $\Delta_n\cap H(Q,f_t)\neq\emptyset$. Since this holds for all $n\in\mathbb{N}$, $(0,x,0)\in H(Q,f_t)$.

For points $w\in\Lambda_t$ of type (b), after replacing w by some iterate of it, we can assume that $w\in V_0$. Consider the sequence $(n_i(w))_{i\in\mathcal{I}_t(w)}$ associated to w, where $\mathcal{I}_t(w)\subset\mathbb{Z}$ is an interval in $\mathbb{Z}$, inductively defined as follows: let $n_0(w)=0$ and, assuming defined $n_j(w)$, $j\geq0$, we define $n_{j+1}(w)$ as the first integer $k>(n_j(w)+1)$ such that $f_t^k(w)\in V_0$. If the forward orbit of w does not return to V_0 for every $k>(n_j(w)+1)$ then j is the right extreme of $\mathcal{I}_t(w)$. We argue analogously for negative j: assuming defined $n_j(x)$, $j\leq0$, $n_{j-1}(w)$ is the first negative integer $k<(n_j(w)-1)$ with $f_t^k(w)\in V_0$. If the backward orbit of w does not return to V_0 for every $k<(n_j(w)-1)$, then j is the left extreme of $\mathcal{I}_t(w)$.

Consider the subset $\mathcal{I}_t^+(\infty)$ (resp. $\mathcal{I}_t^-(\infty)$) of $\Lambda_t\cap V_0$ of points w such that $\mathcal{I}_t(w)$ is not upper (resp. lower) bounded. Let $\mathcal{I}_t^+(b)$ be the subset of $\Lambda_t\cap V_0$ of points w such that $\mathcal{I}_t(w)$ is upper bounded. The set $\mathcal{I}_t^-(b)$ is defined similarly. Let $\mathcal{I}_t^\pm(\infty)=\mathcal{I}_t^+(\infty)\cap\mathcal{I}_t^-(\infty)$ and $\mathcal{I}_t^\pm(b)=\mathcal{I}_t^+(b)\cap\mathcal{I}_t^-(b)$. We borrow from [DR$_2$, Lemma 4.1] the following lemma, whose proof is straightforward:

Lemma 3.6. *For every $t > 0$, $\mathcal{I}_t^+(b) \subset W^s(P, f_t) \cup W^s(Q, f_t)$ and $\mathcal{I}_t^-(b) \subset W^u(P, f_t) \cup W^u(Q, f_t)$.*

Next result immediately follows by observing that f_t (resp. f_t^{-1}) exponentially expands the vertical (resp. horizontal) segments:

Remark 3.7. *Let $w = (x, y, z) \in \mathcal{I}_t^+(\infty)$ (resp. $w \in \mathcal{I}_t^-(\infty)$). Then $\{(x, y)\} \times [z - \varepsilon, z + \varepsilon] \pitchfork W^s(P, f_t) \neq \emptyset$ (resp. $[x - \varepsilon, x + \varepsilon] \times \{(y, z)\} \pitchfork W^u(Q, f_t) \neq \emptyset$) for every $\varepsilon > 0$.*

To prove the inclusion $\Lambda_t \subset H(Q, f_t)$ in Theorem 1.1 we consider the following four cases.

Case (i): $w = (x, y, z) \in \mathcal{I}_t^-(b) \setminus \mathcal{I}_t^+(b) = \mathcal{I}_t^-(b) \cap \mathcal{I}_t^+(\infty)$.

By Remark 3.7, there is a sequence $w_n = (x, y, z_n) \in W^s(P, f_t)$ with $w_n \to w$. We claim that $w_n \in H(Q, f_t)$ for all large n. Thus $w \in H(Q, f_t)$. To prove the claim, note that the distances between the backward iterates of w_n and w exponentially decrease, thus $w_n \in \Lambda_t$. Moreover, since $w \in \mathcal{I}_t^-(b)$ we also have that $w_n \in \mathcal{I}_t^-(b)$. By Lemma 3.6, $w, w_n \in W^u(P, f_t) \cup W^u(Q, f_t)$. If $w_n \in W^u(P, f_t)$, then $w_n \in H(P, f_t) \subset H(Q, f_t)$ (recall the first part of Theorem 1.1 proved above) and we are done. Otherwise, $w_n \in W^u(Q, f_t)$ and for each k large, there is a small vertical strip $\Delta_k = \Delta_k(n)$ of diameter less than $1/k$, whose interior is contained in $W^u(Q, f_t)$ and contains w_n. Since $w_n \in W^s(P, f_t)$, $\Delta_k \pitchfork W^s(P, f_t)$. Thus, by Proposition 3.1, $W^s(Q, f_t)$ intersects transversely the interior of Δ_k. Hence, since the interior of Δ_k is contained in $W^u(Q, f_t)$, Δ_k contains a homoclinic point y_k of Q. From $\operatorname{diam}(\Delta_k) \to 0$, we get $y_k \to w_n$, which implies $w_n \in H(Q, f_t)$.

Case (ii): $w = (x, y, z) \notin \mathcal{I}_t^+(b) \cup \mathcal{I}_t^-(b)$.

We claim that w is accumulated by points $w_n \in \mathcal{I}_t^+(\infty) \cap \mathcal{I}_t^-(b)$, and the result follows from Case (i). To prove the claim observe that, by Remark 3.7, there is a sequence $w_n = (x_n, y, z) \in W^u(Q, f_t)$ with $w_n \to w$. Since the distances between the forward iterates of w_n and w exponentially decrease, it is immediate to check that $w_n \in \Lambda_t$. This also implies that $w_n \in \mathcal{I}_t^+(\infty)$. Finally, $w_n \in W^u(Q, f_t)$ implies $w_n \in \mathcal{I}_t^-(b)$, ending the proof of the claim.

Case (iii): $w = (x, y, z) \in \mathcal{I}_t^+(b) \setminus \mathcal{I}_t^-(b) = \mathcal{I}_t^+(b) \cap \mathcal{I}_t^-(\infty)$.

By Lemma 3.6, $w \in W^s(P, f_t) \cup W^s(Q, f_t)$ and, by replacing w by a forward iterate, we can assume that $w = (x, y, 0)$, $y \geq 0$. Remark 3.7 gives a sequence $w_n = (x_n, y, 0) \in W^u(Q, f_t)$ with $w_n \to w$. For each n, there is a vertical disk $\Delta_n \subset W^u(Q, f_t)$ centered at w_n, of diameter less than $1/n$. Clearly, Δ_n intersects transversely $W^s(P, f_t)$. Thus, by Proposition 3.1, $\Delta_n \pitchfork W^s(Q, f_t) \neq \emptyset$. As in the previous cases, this implies that $\Delta_n \cap H(Q, f_t) \neq \emptyset$ for all n large, thus $w \in H(P, f_t)$.

Case (iv): $w \in \mathcal{I}_t^\pm(b)$.

By Lemma 3.6, there are four possibilities: (1) $w \in W^s(Q, f_t) \cap W^u(Q, f_t)$, (2) $w \in W^s(P, f_t) \cap W^u(P, f_t)$, (3) $w \in W^s(P, f_t) \cap W^u(Q, f_t)$, and (4) $w \in W^u(P, f_t) \cap W^s(Q, f_t)$. Recall that the intersections above are transverse or quasi-transverse, depending on the case. Hence, in case (1), $w \in H(Q, f_t)$ and, in case (2), $w \in H(P, f_t) \subset H(Q, f_t)$. In case (3), the same proof of $\{0\} \times [0, 1] \times \{0\} \subset H(Q, f_t)$ implies that $w \in H(Q, f_t)$: just observe that for every disk $\Delta \subset W^u(Q, f_t)$ containing w, $W^s(P, f_t) \pitchfork \Delta \neq \emptyset$, thus $\Delta \cap H(Q, f_t) \neq \emptyset$. It still remains the case $w \in W^s(Q, f_t) \cap W^u(P, f_t)$. By replacing w by a forward iterate, we can assume that $w = (x, 0, 0)$, $x \in [-1, 1]$, and the following lemma and Cases (i) and (ii) easily imply Case (4):

Lemma 3.8. *Consider $w \in \Lambda_t$ of the form $w = (x, 0, 0) \in V_0 \cap (W^s(Q, f_t) \cap W^u(P, f_t))$. Then there is a sequence $w_n \to w$ with $w_n \in \mathcal{I}_t^+(\infty)$.*

Proof: For each $n \in \mathbb{N}$, consider the rectangle $R_n(x) = \{x\} \times [0, 1/n] \times [-1/n, 1/n]$.

Claim 3.9. *There exists* $\kappa_n \in R_n(x) \cap \Lambda_t^+$ *whose forward orbit returns to* V_0 *infinitely many times.*

Assuming the claim, we now finish the proof of the lemma. As in Remark 3.7, but now considering points in Λ_t^+, we have that the point $\kappa_n = (x_n, y_n, z_n)$ is accumulated by points $\kappa_n^m \in W^u(Q, f_t)$ of the form (x_n^m, y_n, z_n). Since the distances between the forward iterates of κ_n and κ_n^m decrease, the forward orbit of κ_n^m is contained in the neighbourhood W of the cycle and returns infinitely many times to V_0. On the other hand, since $\kappa_n^m \in W^u(Q, f_t)$, its backward orbit also is in W. Thus the whole orbit of κ_n^m is in W, so $\kappa_n^m \in \Lambda_t$ and $\kappa_n^m \in \mathcal{I}_t^+(\infty)$. By Cases (i) and (ii) above, $\kappa_n^m \in H(Q, f_t)$, thus $\kappa_n \in H(Q, f_t)$. Since $\kappa_n \to w$, $w \in H(Q, f_t)$, ending the proof of Case (iv).

To prove Claim 3.9, we need the following fact:

Fact 3.10. *Let* $R = R_n(x)$, $1/n < t$. *Then there is* $i = i(R) \in \mathbb{N}$ *such that, for every* $j \geq i$, $f_t^j(R)$ *contains a rectangle* $\Gamma(R, j)$ *of the form* $\{a\} \times [0, 1/n] \times [-1/n, 1/n]$, $a \in [-1, 0]$.

Proof: Let N_t be the smaller $i \in \mathbb{N}$ such that $F^i(1/n) \in (1 - t + 1/n, 1)$ and write

$$e = (1 - t + 1/n + g) = F^{N_t}(1/n), \quad g \in (0, t - 1/n).$$

By the definition of the unfolding of the cycle, for each $j \geq 0$, $f_t^{N_t+j}(R_n(x))$ contains the rectangle

$$\{\lambda_s^{N_t+j}(x) - 1/2\} \times [0, t + \lambda^j(g + 1/n - t)] \times [-1, 1] \supset \{\lambda_s^{N_t+j}(x) - 1/2\} \times [0, 1/n] \times [-1, 1],$$

since $t + \lambda^j(g + 1/n - t) \geq t + \lambda^j(1/n - t) \geq 1/n$. This finishes the proof of the fact. $\square$

To prove Claim 3.9, consider $R_n(x) = R(0)$ and, using Fact 3.10, let $R(1) = \Gamma(R(0), i(R(0))$. Write $i_0 = i(R_0)$ and $R^1 = f_t^{-i_0}(R(1)) \subset R(0)$. Assume inductively defined numbers $i_0, i_1, \dots, i_{k-1}$ and rectangles $R(0), R(1), \dots, R(k)$ and $R^1, \dots, R^k$ as follows:

- $R(k) = \Gamma(R(k-1), i(R(k-1)))$ and $i_{k-1} = i(R(k-1))$, in particular, $R(k)$ satisfies the hypotheses of Fact 3.10,
- $R^k \subset R^{k-1} \subset \cdots \subset R^1 \subset R(0) = R_n(x)$ and $R^k = f_t^{-i_0 - \cdots - i_{k-1}}(R(k))$.

We define $i_k = i(R(k))$, $R(k+1) = \Gamma(R(k), i_k)$ and $R^{k+1} = f_t^{-i_0 - \cdots - i_k}(R(k+1))$, completing the inductive process. Now it suffices to take any point in the non-empty intersection $\cap_{k \in \mathbb{N}} R^k$. $\square$

The proof of Theorem 1.1 is now complete.

4 Saddle-node heterodimensional cycles

In this section, we consider *saddle-node heterodimensional cycles.* For that, in the model heterodimensional cycle in Section 1, we replace the function F (defining the central dynamics) by a one-parameter family of C^2-maps $\Phi_s \colon [-1, 2] \to \mathbb{R}$ such that:

- For every s, the point 1 is an attracting hyperbolic point of Φ_s and Φ_s is affine in a neighborhood of 1 (independent of s). We denote by $0 < \lambda < 1$ the eigenvalue of Φ_s at 1.
- Locally in 0, the map Φ_s is of the form $\Phi_s(x) = x + x^2 - s$. Thus, for $s > 0$, Φ_s has two hyperbolic fixed points $\pm\sqrt{s}$ (an attractor and a repellor) collapsing at $s = 0$. Moreover, for every $s < 0$, Φ_s has no fixed points close to 0.

- Every Φ_s is strictly increasing and has no fixed points different from 1 and $\pm\sqrt{s}$.

In this way, as in Section 1, one has a two-parameter family of diffeomorphisms $f_{t,s}$, the parameters t and s describing the motion of the unstable manifold of P and the unfolding of the saddle-node, respectively. Note that $P = (0, 1, 0)$ and $S_s^{\pm} = (0, \pm\sqrt{s}, 0)$ $(s \geq 0)$ are fixed points of $f_{t,s}$.

We let $f_t = f_{t,0}$. For the saddle-node $S = (0, 0, 0)$ of f_t there are defined the stable and unstable manifolds (denoted $W^s(S, f_t)$ and $W^u(S, f_t)$) and the strong stable and unstable manifolds (denoted by $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$). Observe that $W^s(S, f_t)$ and $W^u(S, f_t)$ are two-manifolds with boundary and $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$ have both dimension one. Note that, by construction,

$$\begin{array}{ll} \{0\} \times [0, 1) \times [-1, 1] \subset W^u(S, f_t), & [-1, 1] \times [-1, 0] \times \{0\} \subset W^s(S, f_t), \\ [-1, 1] \times \{(0, 0)\} \subset W^{ss}(S, f_t), & \{(0, 0)\} \times [-1, 1] \subset W^{uu}(S, f_t). \end{array}$$

Figure 3: A saddle-node heterodimensional cycle

Keeping in mind these relations, we have that,

- for all t, $W^u(S, f_t)$ meets transversely $W^s(P, f_t)$ throughout the segment $\{0\} \times (0, 1) \times \{0\}$,
- for $t = 0$, $W^u(P, f_0)$ meets quasi-transversely $W^{ss}(S, f_0)$ along the orbit of $(-1/2, 0, 0)$,
- for $t > 0$, the point $(-1/2, t, 0)$ is a transverse homoclinic point of P and $(-1/2, 0, 0)$ is a point of transverse intersection between $W^{ss}(S, f_t)$ and $W^u(S, f_t)$.

In this case, we say that the arc $f_t = f_{t,0}$ has a *saddle-node heterodimensional cycle* associated to P and S at $t = 0$. This cycle can be thought as a limit case of the heterodimensional cycles in Section 1, where the derivative of the point of index two Q is 1^+.

The two-fold behavior of the saddle-node S, as a point of index two and one simultaneously, leads us to consider, for small positive t, the *lateral homoclinic classes of* S defined by

$$H^+(S, f_t) = \overline{W^u(S, f_t) \pitchfork W^{ss}(S, f_t)} \quad \text{and} \quad H^-(S, f_t) = \overline{W^s(S, f_t) \pitchfork W^{uu}(S, f_t)}.$$

As in the case of the usual homoclinic classes, we have that:

Proposition 4.1. *For every small* $t > 0$, $H^+(S, f_t)$ *(resp.* $H^-(S, f_t)$*) is transitive and the periodic points of index two (resp. one) form a dense subset of it.*

Consider a neighborhood W of the saddle-node heterodimensional cycle defined as in Section 3 and denote by Υ_t the maximal invariant set of f_t in W.

Theorem 4.2. *For every small $t > 0$, one has that $H(P, f_t) \subset H^+(S, f_t)$ and $\Upsilon_t \subset H(S^+, f_t)$.*

The proof of Theorem 4.2 follows as the one of Theorem 1.1, the only difficulty being to redefine appropriately the one-dimensional dynamics associated to the cycle (recall Section 2). This will be briefly done in the next section. To get the inclusion $H^+(S, f_t) \subset H(P, f_t)$ we need the following distortion property for the saddle-node map $\Phi = \Phi_0$.

(SN) Let $K = \max\{|\Phi''(x)|/|\Phi'(x)|,\ x \in [0,1]\} > 0$. Then $\dfrac{4\,e^K\,(1-\lambda)}{\lambda^6} < \dfrac{1}{2}$, where $\lambda \in (2/3, 1)$.

Theorem 4.3. *Under the assumption (SN), $H^+(S, f_t) \subset H(P, f_t)$ holds for all small positive $t > 0$.*

To prove this theorem we need new ingredients that will be introduced in Section 4.3. Theorems 4.2 and 4.3 imply $H^+(S, f_t) = H(P, f_t)$ for all small $t > 0$.

4.1 One-dimensional dynamics for the saddle-node cycle

We now adapt the definitions of scaled fundamental domains, transitions and returns for saddle-node cycles. As in Section 2, for each $t > 0$, define the fundamental domains $D_t^- = [1-t, 1-\lambda\, t]$ and $D_t^+ = [a_t, b_t]$, $a_t = \Phi^{-1}(b_t)$, where D_t^+ is the first backward iterate of D_t^- by Φ contained in $[0,t]$. We have $\Phi^{k_t}(D_t^+) = D_t^-$, for some $k_t \in \mathbb{N}$. Observe that $|D_t^-| = t\,(1-\lambda)$ and, since $b_t \in (0,t]$, $|D_t^+| \le t^2$. For small $t > 0$, define the *transition* $\mathfrak{T}_t$ and the map $\mathfrak{G}_t$ by

$$\mathfrak{T}_t\colon D_t^+ \to D_t^-,\ \ x \mapsto \mathfrak{T}_t(x) = \Phi^{k_t}(x) \quad \text{and} \quad \mathfrak{G}_t\colon D_t^+ \to [0, t(1-\lambda)],\ \ x \mapsto \mathfrak{G}_t(x) = \mathfrak{T}_t(x) + (t-1).$$

Lemma 4.4. *The maps $\mathfrak{T}_t$ and $\mathfrak{G}_t$ are uniformly expanding for all small $t > 0$.*

Proof: It suffices to see that $(\Phi^{k_t})'(z) > 1$ for all $z \in D_t^+$. We use the following standard lemma (whose proof is omitted here):

Bounded Distortion Lemma 4.5. *Let $K > 0$ be as in condition (SN). Then, for every pair of points $z, y \in D_t^+$ and every small $t > 0$,*

$$e^{-K} \le \frac{(\Phi^{k_t})'(z)}{(\Phi^{k_t})'(y)} \le e^K.$$

The lemma now follows by the mean value theorem, taking $y \in D_t^+$ with

$$(\Phi^{k_t})'(y) = |D_t^-|/|D_t^+| \ge (1-\lambda)/t.$$

Thus, if t is small, $(\Phi^{k_t})'(z) \ge (e^{-K}\,(1-\lambda))/t > 1$, for all $z \in D_t^+$. □

As in Section 2, given $x \in (a_t, b_t] = \tilde{D}_t^+$, let $i(x) \in \mathbb{Z}$ be the first i with $\Phi^i(\mathfrak{G}_t(x)) \in D_t^+$. The *return map* $\mathfrak{R}_t$ is now defined by

$$\mathfrak{R}_t\colon \tilde{D}_t^+ \to D_t^+, \quad \mathfrak{R}_t(x) = \Phi^{i(x)}(\mathfrak{G}_t(x)) = \Phi^{i(x)}(\mathfrak{T}_t(x) + (t-1)).$$

Lemma 4.6. *There exists $i_0 > 0$ such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: To prove the lemma it is enough to see that $\mathfrak{G}_t(D_t^+) \subset (0, a_t)$. By definition, $\mathfrak{G}_t(D_t^+) = [0, (1-\lambda)\, t]$. Observe that, if t is small enough,

$$\Phi^2((1-\lambda)\, t) = \Phi((1-\lambda)\, t + (1-\lambda)^2\, t^2) = (1-\lambda)\, t + 2\,(1-\lambda)^2\, t^2 + \text{h.o.t} < t.$$

Thus, the right extreme $\Phi^2((1-\lambda)t)$ of $\Phi^2(\mathfrak{G}_t(D_t^+))$ is less than t. In particular, the right extreme of $\mathfrak{G}_t(D_t^+)$ is less than $\Phi^{-2}(t)$, and the the lemma follows from $D_t^+ \subset (\Phi^{-2}(t), t]$. □

As in the case of the return map R_t in Section 2, for each $i \geq i_0$, there is $\delta_i \in \tilde{D}_t^+$ with $\mathfrak{G}_t(\delta_i) = \Phi^{-i}(a_t)$. The points δ_i are the discontinuities of $\mathfrak{R}_t$. In this way, we get a decreasing sequence $(\delta_i)_{i \geq i_0}$ with $\delta_i \to a_t$, and intervals $J_i = [\delta_{i+1}, \delta_i]$, $i > i_0$, and $J_{i_0} = [\delta_{i_0}, b_t]$ such that $\mathfrak{R}_t$ is continuous and increasing in the interior of each J_i. Extending $\mathfrak{R}_t$ continuously to the whole J_i we get a bi-valuated map with $\mathfrak{R}_t(\delta_i) = \{a_t, b_t\}$ for all $i > i_0$.

Lemma 4.7. *The restriction of $\mathfrak{R}_t$ to each interval $J_i, i > i_0$, is onto. Moreover, there is $\ell > 1$ such that $\mathfrak{R}_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = \delta_i$ this means that the lateral derivatives of $\mathfrak{R}_t$ at x are greater than ℓ). Finally, $0 \in \mathfrak{G}_t(\mathfrak{R}_t(\delta_i))$ for all $i \geq i_0$.*

Proof: The lemma follows as Lemma 2.4 observing that $i_0 > 0$ (Lemma 4.6), $\mathfrak{G}_t$ is expanding (Lemma 4.4), and that the derivative of Φ in $(0, t]$ is bigger than one. □

Arguing as in Section 2, one gets the following lemma (corresponding to Lemma 2.5):

Lemma 4.8. *Given any subinterval I of D_t^+ there are $x \in I$ and $i \geq 0$ with $\mathfrak{G}_t(\mathfrak{R}_t^i(x)) = 0$.*

4.2 Lateral Homoclinic classes. Proof of Theorem 4.2

To prove Theorem 4.2 we proceed as in Section 3. After redefining vertical strips and segments and using Lemma 4.8, one gets that, for any small $t > 0$ and any disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^{ss}(S, f_t) \pitchfork \chi \neq \emptyset$ (recall Proposition 3.1). The inclusion $(H(P, f_t) \cup \Upsilon_t) \subset H^+(S, f_t)$ follows exactly as $(H(P, f_t) \cup \Lambda_t) \subset H(Q, f_t)$ in the case of heterodimensional cycles. So we omit the details of the proofs of these inclusions.

4.3 Proof of Theorem 4.3: the inclusion $H^+(S, f_t) \subset H(P, f_t)$

Consider the homoclinic point $x_t = (-1/2, t, 0)$ of P for f_t and the fundamental domains of Φ $\Delta_t^+(i) = \Phi^{-i}(\Delta_t^+(0))$, $i \geq 0$, where $\Delta_t^+(0) = [\Phi^{-1}(t), t]$. We now construct a family $\mathcal{H}_t$ of homoclinic points of P for f_t of the form $(x, y, 0)$ such that the set $\{y\colon\ (x, y, 0) \in \mathcal{H}_t\}$ is dense in $\Delta_t^+(0)$:

Proposition 4.9. *For every small $t > 0$ there are sequences of homoclinic points of P of the form $(b_{i_1,i_2,\dots,i_m,k}, x_{i_1,i_2,\dots,i_m,k}, 0)_{k \in \mathbb{N}^*}$, $b_{i_1,i_2,\dots,i_m,k} \in [-1, 0]$, $i_j \in \mathbb{N}^*$, such that*

- **(H1)** $x_{i_1,i_2,\dots,i_m,k} \in \bigcup_{i=0}^4 \Delta_t^+(i) = \Delta_t$,
- **(H2)** $x_{i_1,i_2,\dots,i_m,k} \to x_{i_1,i_2,\dots,i_m}$ *as* $k \to \infty$,
- **(H3)** $x_{i_1,i_2,\dots,i_m,0} < x_{i_1,i_2,\dots,(i_m-1)}$ *for every* $i_m \geq 1$,
- **(H4)** $\operatorname{diam}((x_{i_1,i_2,\dots,i_m,k})_k) \to 0$ *as* $m \to \infty$,
- **(H5)** (x_k) *is increasing and* $x_k \to t^-$ *as* $k \to \infty$,

(H6) *$x_0 \in \Delta_t^+(1)$ and $x_0 \notin \Delta_t^+(0)$.*

This proposition will be proved in Section 4.3.2. From the proposition one gets the following:

Corollary 4.10. *The set $\mathcal{H}_t = \bigcup_{n,k\in\mathbb{N}^*}(x_{i_1,i_2,\dots,i_k,n})$ contains a dense subset of $\Delta_t^+(0)$.*

Proof: The proof of is identical to [D_2, Lemma 4.1], but we repeat it here for completeness. Take any point x in the interior of $\Delta_t^+(0)$. If $x \in \mathcal{H}_t$ we are done. Otherwise, by (H5) and (H6), there is $i_1 > 1$ with $x_{(i_1-1)} < x < x_{i_1}$. Analogously, by (H2) and (H3), there is $i_2 > 1$ with $x_{i_1,(i_2-1)} < x < x_{i_1,i_2}$. Inductively, using (H2) and (H3) as above, we get a sequence $\{i_k\}$, $i_k > 1$, such that, for all k, $x_{i_1,\dots,(i_k-1)} < x < x_{i_1,\dots,i_k}$. Finally, from (H4), $\lim_{k\to\infty} x_{i_1,\dots,i_k} = x$. $\square$

4.3.1 Proof of Theorem 4.3

The deduction of Theorem 4.3 from Corollary 4.10 follows as in [D_2, Section 5]. For completeness, we sketch here this proof. Consider any $w \in W^u(S,f_t) \pitchfork W^{ss}(S,f_t)$. By replacing w by some iterate of it, we can assume that $w = (x,0,0)$, $|x|$ small. We prove that, for every $\varepsilon > 0$, the square $S(\varepsilon) = (x-\varepsilon, x+\varepsilon)\times(0,\varepsilon)\times\{0\} \subset W^s(P,f_t)$ intersects transversely $W^u(P,f_t)$. This implies that $w \in H(P,f_t)$. The configuration of the cycle and the λ-lemma imply that there is $n(\varepsilon) > 0$ such that $f_t^{-n(\varepsilon)}(S(\varepsilon))$ contains a disk $S'(\varepsilon)$ of the form

$$S'(\varepsilon) = [-1,1]\times(\bar y - \xi, \bar y + \xi)\times\{\bar z\}, \quad \bar y \in (1-t,1),\ \bar z \in [-1,1] \text{ and small } \xi > 0.$$

Let $m \in \mathbb{N}$ be such that $\Phi^{-m}(\bar y) \in \Delta_t^+(0)$. Thus $f_t^{-m}(S'(\varepsilon))$ contains the horizontal strip

$$\hat S(\varepsilon) = [-1,1]\times(\Phi^{-m}(\bar y - \xi), \Phi^{-m}(\bar y + \xi))\times\{\lambda_u^{-m}\,\bar z\} \subset W^s(P,f_t).$$

Since $\Phi^{-m}(\bar y)$ belongs to $\Delta_t^+(0)$, Corollary 4.10 implies that $\hat S(\varepsilon)$ meets $W^u(P,f_t)$. Thus $\hat S(\varepsilon)$ contains a homoclinic point of P and the same holds for $S(\varepsilon)$. $\square$

4.3.2 Proof of Proposition 4.9: Sequences of homoclinic points:

Let κ_t be the first $k \in \mathbb{N}$ such that $\Phi^k(\Delta_t^+(0)) \subset [1-t,1]$, $\Delta_t^+(0) = [\Phi^{-1}(t), t]$. Since, for small $t > 0$, $|\Phi^{k_t}(\Delta_t^+(0))| \le t\,(1-\lambda)$ and $|\Delta_t^+(0)| \ge \lambda\, t^2$, the Bounded Distortion Lemma 4.5 implies that

$$(\Phi^{\kappa_t})'(x) < \frac{1-\lambda}{\lambda\, t}\, e^K, \quad \text{for all } x \in \Delta_t^+(0). \tag{2}$$

Denote by δ_t^i the length of $\Delta_t^+(i)$. Since the derivative of Φ near 0 is close to 1 and strictly bigger than 1 in $(0,t]$, for small t, we have that

$$\delta_t^0 \ge \delta_t^i \ge \frac{9\,\delta_t^0}{10}, \quad i = 1,\dots,4. \quad \text{In particular,} \quad \sum_{i=0}^4 \delta_t^i \in [4\delta_t^0, 5\delta_t^0]. \tag{3}$$

Consider the interval $[1-\eta_t, 1]$, $\eta_t = \delta_t^1 + \delta_t^0$, and let α_t be the first natural number α with

$$\Phi^{\kappa_t+\alpha}(\Delta_t^+(0)) \subset [1-\eta_t, 1].$$

Observe that $|\Delta_t^+(0)| = \delta_t^0 < t^2$ and $\delta_t^1 < \delta_t^0$. Thus, for small $t > 0$, $\eta_t < 2\,\delta_t^0 < 2\,t^2 < \lambda\, t$. Since, $\Phi^{\kappa_t}(\Phi^{-1}(t)) \in [1-t, 1-\lambda\, t]$ and $(1-\lambda\, t) < (1-\eta_t)$, we get that $\alpha_t \ge 1$ for all small t. Observe also that $t^2 < \eta_t < 2\,t^2$, where the first inequality follows from (3) and $\delta_t^0 > 3t^2/4$ if t is small enough.

Lemma 4.11. *For every small $t > 0$ it holds $\lambda^{\alpha_t} \leq (2\,t)/\lambda$.*

Proof: By definitions of κ_t and α_t, $\Phi^{\kappa_t}(\Delta_t^+(0)) = [1 - e_t^-, 1 - e_t^+]$, where $e_t^- \in [\lambda\, t, t]$, and $\Phi^{\alpha_t}(1 - e_t^-) \in [1 - \eta_t, 1 - \lambda\, \eta_t]$. Thus, since Φ is affine near 1, $\lambda^{\alpha_t}(e_t^-) \in (0, \eta_t]$. Hence, from $t^2 < \eta_t < 2\,t^2$ and $\lambda\, t \leq e_t^- \leq t$,

$$\lambda^{\alpha_t} \leq \frac{\eta_t}{e_t^-} \leq \frac{2\,t^2}{\lambda\, t} = \frac{2\,t}{\lambda},$$

ending the proof of the lemma. □

Next lemma is necessary for getting (H4) and along the inductive definition of $(x_{i_1,i_2,\dots,i_m,k})_k$.

Lemma 4.12. $L = \max\{(\Phi^{\kappa_t+\alpha_t+j})'(x);\ x \in \cup_{i=0}^4 \Delta_t^+(i)$ *and* $j \geq 0\} < \frac{1}{2}$.

Proof: Since Φ is a contraction near 1, it is enough to compute the estimate when $j = 0$. We split the trajectory of a point $x \in \Delta_t^+(i)$ going from $\Delta_t^+(i)$ to $[1 - \eta_t, 1)$ as follows: **(i)** i iterates, $i \leq 4$, for x going from $\Delta_t^+(i)$ to $\Delta_t^+(0)$; **(ii)** κ_t iterates for $\Phi^i(x)$ going from $\Delta_t^+(0)$ to $\Phi_t^{k_t}(\Delta_t^+(0))$; and **(iii)** α_t iterates for $\Phi^{\kappa_t+i}(x)$ going from $\Phi_t^{k_t}(\Delta_t^+(0))$ to $[1 - \eta_t, 1]$. This construction involves $(i + \kappa_t + \alpha_t)$ iterations of x by Φ, so we need to remove the last i iterations, corresponding to a contraction by λ^i. We claim that

$$L \leq \underbrace{((2\,t+1)^4)}_{(\mathbf{a})}\, \underbrace{\frac{e^K\,(1-\lambda)}{\lambda\, t}\,\lambda^{\alpha_t}}_{(\mathbf{b})}\, \underbrace{\frac{1}{\lambda^4}}_{(\mathbf{c})}, \tag{4}$$

corresponding (**a**) to the expansion of the first i iterates by Φ (just observe that in $[0, t]$ the derivative of Φ is upper bounded by $(2\,t + 1)$ and that $i \leq 4$), (**b**) to an upper bound of the derivative of $\Phi^{\kappa_t+\alpha_t}$, recall (2), and (**c**) to the i ($i \leq 4$) negative iterates of Φ close to 1. By (SN), Lemma 4.11, and the fact that $(2\,t+1)^4 < 2$ if $t > 0$ is small, we get

$$L \leq ((2\,t+1)^4)\,\frac{e^K\,(1-\lambda)}{\lambda\, t}\,\frac{2\,t}{\lambda}\,\frac{1}{\lambda^4} = (2\,t+1)^4\,\frac{2\,(1-\lambda)\,e^K}{\lambda^6} \leq 4\,e^K\,\frac{(1-\lambda)}{\lambda^6} < \frac{1}{2},$$

which ends the proof of the lemma. □

Construction of the sequences $(x_{i_1,i_2,\dots,i_m,k})$**.** To construct the sequences $(x_{i_1,i_2,\dots,i_m,k})$, we need the following algorithm about the creation of homoclinic points, which is a consequence of the definition of the unfolding of the heterodimensional cycle.

Algorithm 4.13. *Let $(x, y, 0)$, $x \in [-1, 1]$ and $y \in [0, t]$, be a homoclinic point of P (for f_t) such that $\{(x, y)\} \times [-1, 1] \subset W^u(P, f_t)$. Then, for every m with $\Phi^m(y) \in (1-t, 1)$, there is a homoclinic point of P of the form $(\bar{x}, \Phi^m(y) + t - 1, 0)$ such that $\{(\bar{x}, \Phi^m(y) + t - 1)\} \times [-1, 1] \subset W^u(P, f_t)$.*

Take the homoclinic point $(-1/2, t, 0)$ of P and the sequences $(y_i)_{i\in\mathbb{N}^*}$ and $(x_i)_{i\in\mathbb{N}^*}$ defined by

$$y_i = \Phi^{\kappa_t+\alpha_t+i}(t) \quad \text{and} \quad x_i = (t-1) + y_i, \quad y_i \to 1 \quad \text{and} \quad x_i \to t.$$

Observe that, for each $i \geq 0$, there is a homoclinic point $(b_i, x_i, 0)$ of P verifying Algorithm 4.13. Also, by the definitions of α_t and κ_t, $y_i \in [1 - \eta_t, 1]$ for all $i \geq 0$. Thus, since $\eta_t = \delta_t^0 + \delta_t^1$, one has

$$x_i \in [t - \eta_t, t] = [t - (\delta_t^1 + \delta_t^0), t] = \Delta_t^+(1) \cup \Delta_t^+(0). \tag{5}$$

Figure 4: The sequences $(x_{i_1,i_2,\dots,i_m,n})_n$

Lemma 4.14. *The sequence $(x_i)_{i\in\mathbb{N}^*}$ verifies (H5) and (H6).*

Proof: Condition (H5) follows by definition. To get (H6), i.e.$x_0 \in (\Delta_t^+(1)\setminus\Delta_t^+(0))$, note that, by construction, $x_0 \in [t-\eta_t, t-\lambda\,\eta_t]$ and $\Delta_t^+(1) = [t-\eta_t, t-\delta_t^0]$. Thus, by (5), it is enough to check that $\lambda\,\eta_t = \lambda\,(\delta_t^0+\delta_t^1) > \delta_t^0$. This inequality follows from $\delta_t^0 > \delta_t^1$, (3) and (SN), observing that

$$\frac{\delta_t^0}{\eta_t} = \frac{\delta_t^0}{\delta_t^0+\delta_t^1} < \frac{\delta_t^0}{2\,\delta_t^1} < \frac{\delta_t^0}{2\,(9/10)\,\delta_t^0} = \frac{10}{18} < \frac{2}{3} < \lambda.$$

The proof of the lemma is now complete. □

We now proceed with the construction of the sequences in Proposition 4.9. For each $j \in \mathbb{N}^*$, define the sequences $(y_{j,i})_{i\in\mathbb{N}^*}$ and $(x_{j,i})_{i\in\mathbb{N}^*}$ as follows,

$$y_{j,i} = \Phi^{\kappa_t+\alpha_t+j}(x_i) \quad \text{and} \quad x_{j,i} = (t-1) + y_{j,i}.$$

We claim that $y_{j,i} \to y_j$ and, consequently, $x_{j,i} \to x_j$, as $i \to \infty$. For that just observe that $\lim_{i\to\infty} x_i = t$, thus, by continuity, $\lim_{i\to\infty} y_{j,i} = \lim_{i\to\infty}\Phi^{\kappa_t+\alpha_t+j}(x_i) = \Phi^{\kappa_t+\alpha_t+j}(t) = y_j$.

Lemma 4.15. *The points $(x_{j,i})_i$ belong to $\cup_{i=0}^4\Delta_t^+(i)$ for all $i, j \in \mathbb{N}^*$.*

Proof: Since, by construction, the sequences $(x_{j,i})_i$ are increasing, it is enough to see that $x_{0,0} \in \cup_{i=0}^4\Delta_t^+(i)$. Consider the diameter $d_0 = (t-x_0)$ of $(x_i)_{i\in\mathbb{N}^*}$. By (5), $d_0 < \delta_t^0+\delta_t^1$. Let $d_1 = |x_0 - x_{0,0}|$ be the diameter of $(x_{0,i} = \Phi^{\kappa_t+\alpha_t}(x_i) + (t-1))_i$, which is equal to the diameter of $(\Phi^{\kappa_t+\alpha_t}(x_i))_i$. Thus, since $(x_i)_i \subset \Delta_t^+(0)\cup\Delta_t^+(1)$, by Lemma 4.12, the diameter d_1 is bounded by

$$d_1 \le L\,d_0 < d_0/2 < (\delta_t^0+\delta_t^1)/2 < (2\,\delta_t^0)/2 < \delta_t^0. \tag{6}$$

Since $x_0 \in \Delta_t^+(1)$ (Lemma 4.14), to prove that $x_{0,0} \in \cup_{i=0}^4 \Delta_t^+(i)$, it is enough to see that

$$x_{0,0} = x_0 - d_1 > x_0 - (\delta_t^2 + \delta_t^3 + \delta_t^4) \Longleftrightarrow d_1 < (\delta_t^2 + \delta_t^3 + \delta_t^4),$$

which immediately follows from $\delta_t^2 + \delta_t^3 + \delta_t^4 > \delta_t^0 > d_1$, the first inequality being consequence of $\delta_t^i > (9\,\delta_t^0)/(10)$, see (3), and the last from (6). This ends the proof of the lemma. ☐

Suppose now inductively defined sequences $(y_{i_i,i_2,\dots,i_m,i})_{i\in\mathbb{N}^*}$ and $(x_{i_i,i_2,\dots,i_m,i})_{i\in\mathbb{N}^*}$ by

$$y_{i_1,i_2,\dots,i_m,i} = \Phi^{\kappa_t+\alpha_t+i_1}(x_{i_2,\dots,i_m,i}) \quad \text{and} \quad x_{i_1,i_2,\dots,i_m,i} = (t-1) + y_{i_1,i_2,\dots,i_m,i},$$

satisfying conditions (H1), (H2), (H3) and

(H4b) Let d_m, $m \geq 0$, be the diameter of the sequence $(x_{\underbrace{0,\dots,0,i}_{m\,0's}})_i$. Then $d_m \leq (d_{m-1})/2$.

Observe that (H2) and (H3) and are equivalent to (H2b) and (H3b) below, respectively,

(H2b) $(y_{i_1,i_2,\dots,i_m,i})_i \to y_{i_i,i_2,\dots,i_m}$ as $i \to \infty$,

(H3b) $y_{i_1,i_2,\dots,i_m,0} < y_{i_i,i_2,\dots,(i_m-1)}$ for all $i_m \geq 1$.

Notice that, for $m = 1$, (H1) follows from Lemma 4.15, (H2) (or (H2b)) from definition, and (H4b) from the estimates in (6). To check (H3b), $y_{i,0} < y_{i-1}$, for every $i \geq 1$, recall that, by Lemma 4.14, $x_0 < \Phi^{-1}(t) < t$, thus

$$y_{i-1} = \Phi^{\kappa_t+\alpha_t+i-1}(t) = \Phi^{\kappa_t+\alpha_t+i}(\Phi^{-1}(t)) > \Phi^{\kappa_t+\alpha_t+i}(x_0) = y_{i,0}.$$

For simplicity, we say that the sequences $(z_{i_i,i_2,\dots,i_m,i})_{i\in\mathbb{N}^*}$, $z = x, y$, are *of generation m.*

Lemma 4.16. *Property (H4b) implies (H4) in Proposition 4.9.*

Proof: By construction, the diameter of any sequence $(x_{i_1,\dots,i_m,k})$ of generation m is bounded by the diameter d_m of $(x_{\underbrace{0,\dots,0,i}_{m\,0's}}))_{i\in\mathbb{N}^*}$. Thus, inductively, $d_m \leq (1/2)\,d_{m-1}$, so $d_m \to 0$. ☐

Keeping in mind Lemmas 4.14 and 4.16, to prove Proposition 4.9 it suffices to see that the sequences verify (H1), (H2b), (H3b) and (H4b). We argue inductively on the generation of the sequences and assume satisfied these conditions for sequences of generation less than or equal to m. To verify (H2b) for the sequences of generation $m+1$ note that, by induction, $(y_{i_1,i_2,\dots,i_m,i})_i \to y_{i_1,i_2,\dots,i_m}$. Thus, by continuity of Φ and by definition,

$$(y_{j,i_1,i_2,\dots,i_m,i})_i = (\Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m,i}))_i \to \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m}) = y_{j,i_1,i_2,\dots,i_m}.$$

To prove (H3b) observe that, by induction, $x_{i_1,i_2,\dots,i_m,0} < x_{i_1,i_2,\dots,i_m-1}$. Thus, since Φ is increasing,

$$y_{j,i_1,i_2,\dots,i_m,0} = \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m,0}) < \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\dots,i_m-1}) = y_{j,i_1,i_2,\dots,i_m-1}.$$

To check (H4b) observe that, by the induction hypotheses (H1), $x_{0,\dots,0,i} \in \cup_{i=0}^4 \Delta_t^+(i)$, Lemma 4.12 and the fact that the sequences $(y_{0,\dots,0,i})$ and $(x_{0,\dots,0,i})$ have the same diameter imply that

$$d_{m+1} = \operatorname{diam}\underbrace{((y_{0,\dots,0,i})_i)}_{(m+1)\,0's} = \operatorname{diam}((\Phi^{\kappa_t+\alpha_t}\underbrace{((x_{0,\dots,0,i})_i))}_{m\,0's} = L\operatorname{diam}((x_{0,\dots,0,i}))_i) = L\,d_m \leq \frac{d_m}{2}.$$

Finally, to get (H1) for the generation $(m+1)$ it is enough to see that, for every m, $\sum_{i=0}^{m} d_m < 4\delta_t^0 < \sum_{i=0}^{4} \delta_t^i$, recall (3). By induction and Lemma 4.14, which implies that $d_0 \le \delta_t^0 + \delta_t^1$, we have

$$\sum_{i=0}^{m} d_m \le \sum_{i=0}^{m-1} (1/2)^i\, d_0 \le \sum_{i=0}^{m-1} (1/2)^i\, (\delta_t^0 + \delta_t^1) \le \frac{1}{1-2}\, (\delta_t^0 + \delta_t^1) < 2(\delta_t^0 + \delta_t^1) < 4\delta_t^0,$$

finishing the proof of our claim, and the construction of the sequences in Proposition 4.9.

5 Homoclinic classes before collapsing the saddles S_s^+ and S_s^-

We now return to the family $f_{t,s}$ in Section 4. Note that $S_s^+ = (0, \sqrt{s}, 0)$, $s > 0$, is a fixed point of index two of $f_{t,s}$ (any $t > 0$) and that $f_{\sqrt{s},s}$ has a heterodimensional cycle associated to S_s^+ and P:

- $W^u(S_s^+, f_{\sqrt{s},s})$ meets transversely $W^s(P, f_{\sqrt{s},s})$ throughout the segment $\{0\} \times (\sqrt{s}, 1) \times \{0\}$,
- $W^u(P, f_{\sqrt{s},s})$ meets quasi-transversely $W^s(S_s^+, f_{\sqrt{s},s})$ along the orbit of $(-1/2, \sqrt{s}, 0)$ (just observe that $[-1,1] \times \{(\sqrt{s}, 0)\} \subset W^s(S_s^+, f_{\sqrt{s},s})$ and that $(-1/2, \sqrt{s}, 0) \in W^u(P, f_{\sqrt{s},s})$).

In what follows we assume that the saddle-node arc Φ_s verifies condition (SN) in Section 4.

Theorem 5.1. *There exist a small $s_0 > 0$ and a strictly positive map τ defined on $(0, s_0)$ such that, for every $s \in (0, s_0)$ and $t \in (\sqrt{s}, \sqrt{s} + \tau(s))$,*

- *$H(P, f_{t,s}) = H(S_s^+, f_{t,s})$, and*
- *there is a neighborhood W_s of the cycle of $f_{\sqrt{s},s}$ (associated to P and S_s^+) such that the maximal invariant set $\Lambda_{t,s}$ of $f_{t,s}$ in W_s is contained to $H(S_s^+, f_{t,s})$.*

The the inclusion $H(P, f_{t,s}) \subset H(S_s, f_{t,s})$ and the second part of the theorem follow as in Theorem 1.1. So we just sketch these proofs. For a fixed $s > 0$ and $t > \sqrt{s}$, t close to $\sqrt{s}$, $t = \sqrt{s} + \tau$, consider the scaled fundamental domains $D_{t,s}^{\pm}$ of Φ_s,

$$D_{t,s}^- = [1 - (t - \sqrt{s}), 1 - \lambda\,(t - \sqrt{s})] = [1 - \tau, 1 - \lambda\,\tau]$$

and $D_{t,s}^+$ defined as the first backward iterate of $D_{t,s}^-$ in $[\sqrt{s}, \sqrt{s} + \tau]$. Let $\Phi_s^{k_{t,s}}(D_{t,s}^+) = D_{t,s}^-$, where $k_{t,s} \in \mathbb{N}$. These domains play the role of $D_t^{\pm}$ in Section 2. Observe that

$$\ell(t,s) = \frac{|D_{t,s}^-|}{|D_{t,s}^+|} \ge \frac{\tau\,(1-\lambda)}{(\sqrt{s} + \tau)^2 - s} = \frac{\tau\,(1-\lambda)}{\tau\,(2\sqrt{s} + \tau)} = \frac{1-\lambda}{2\,\sqrt{s} + \tau}.$$

By shrinking s, we can assume that $|\Phi_s''(x)|/|\Phi_s'(x)| < 2\,K$ for all $x \in [-1, 2]$ (K as in condition (SN)). Thus, there is $s_0 > 0$ and a map $\tau\colon (0, s_0) \to \mathbb{R}^+$ such that

$$\ell(t,s)\, e^{-2\,K} > 2, \quad \text{for all } s \in (0, s_0) \text{ and } t \in (\sqrt{s}, \sqrt{s} + \tau(s)). \tag{7}$$

Exactly as in Section 2, for $s \in (0, s_0)$ and $t \in (\sqrt{s}, \sqrt{s} + \tau(s))$, we define maps

$$\begin{aligned}
&T_{t,s}\colon D_{t,s}^+ \to D_{t,s}^-, \quad T_{t,s}(x) = \Phi_s^{k_{t,s}}(x),\\
&G_{t,s}\colon D_{t,s}^- \to [\sqrt{s}, \sqrt{s} + \tau\,(1-\lambda)], \quad G_{t,s}(x) = T_{t,s}(x) + (t-1),\\
&R_{t,s}\colon D_{t,s}^+ \to D_{t,s}^+, \quad R_{t,s}(x) = \Phi_s^{i(x)}(G_{t,s}(x)),
\end{aligned}$$

where, as in Section 2, $i(x)$ is the first forward iterate of $G_{t,s}(x)$ by Φ_s in $D^+_{t,s}$. As in Lemma 4.4, the Bounded Distortion Lemma 4.5 and equation (7) imply that $T_{t,s}$, $G_{t,s}$ and $R_{t,s}$ are uniformly expanding. The inclusion $(H(P, f_{t,s}) \cup \Lambda_{t,s}) \subset H(S^+_s, f_{t,s})$ now follows as in Theorem 1.1.

To get $H(S^+_s, f_{t,s}) \subset H(P, f_{t,s})$ recall first that Theorem 4.3 gives small $\bar{t} > 0$ with $H^+(S, f_{t,0}) \subset H(P, f_{t,0})$ for all $t \in (0, \bar{t})$. This inclusion follows by constructing the sequences of homoclinic points of P in Proposition 4.9. The proof Proposition 4.9 only involves distortion control of the saddle-node map in $[0, 1]$ and the contracting itineraries in Lemma 4.12. Clearly, these properties hold after replacing, for small positive s, the saddle-node S by the hyperbolic point S^+_s and considering the restriction of the saddle-node map to $[\sqrt{s}, 1]$. This ends the sketch of the proof of Theorem 5.1.

6 Collision, explosion and collapse of homoclinic classes

In this section we prove Theorems A and B. Consider a two parameter family of diffeomorphism $f_{t,s}$ locally defined as follows:

Partially hyperbolic local dynamics:

- In the set $C = [-1, 1] \times [-2, 2] \times [-1, 1]$, $f_{t,s}(x, y, z) = (\lambda^s x, \Psi_s(y), \lambda^u z)$, where $0 < \lambda^s < 1 < \lambda^u$ and $\Psi_s \colon [-2, 2] \to (-3, 2)$ is a strictly increasing C^2-map such that $\lambda^s < d_m < \Psi'_s(y) < d_M < \lambda^u$ for all $y \in [-2, 2]$ and small $|s|$.
- $\Psi_s(1) = 1$ and $\Psi_s(-1) = -1$ for all s and Ψ_s is affine and independent of s in $[-1 - \delta, -1 + \delta]$ and $[1 - \delta, 1 + \delta]$, for some small $\delta > 0$. Furthermore, $\Psi'_s(-1) = \beta > 1 > \Psi'_s(1) = \lambda > 0$.
- There is $\delta > 0$ such that the restriction of Ψ_s to $[-\delta, \delta]$ is of the form $\Psi_s(x) = x + x^2 - s$.
- For $s < 0$, Ψ_s has (exactly) two fixed points (± 1), for $s = 0$, Ψ_0 has three fixed points (± 1 and 0), and, for $s > 0$, Φ_s has four fixed points (± 1 and $\pm\sqrt{s}$). Let $P = (0, 1, 0)$, $Q = (0, -1, 0)$, $S = (0, 0, 0)$ and, for positive s, $S^\pm_s = (0, \pm\sqrt{s}, 0)$, the fixed points of $f_{t,s}$ in C.

Existence and unfolding of cycles:

- There are $k_0 \in \mathbb{N}$ and small neighborhoods of $(0, -1, -1/2)$, $(0, 0, -1/2)$ and $(0, 1, -1/2)$ such that, for each small $|s|$, in such neighborhoods, each $f^{k_0}_{t,s}$ is the translation $f^{k_0}_{t,s}(x, y, z) = (x - 1/2, y - 1 + t, z + 1/2)$, recall the definition of f_t in Section 1.

As in previous sections, we have:

- for $(t, s) = (0, 0)$, $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, associated to P and S, $W^u(P, f_{0,0})$ meets quasi-transversely $W^{ss}(S, f_{0,0})$, and to Q and S, $W^s(Q, f_{0,0})$ intersects $W^{uu}(S, f_{0,0})$, (note that $[-1, 1] \times \{(0, 0)\} \subset W^{ss}(S, f_{0,0})$ and $\{(0, 0)\} \times [-1, 1] \subset W^{uu}(S, f_{0,0})$).
- For small $|t|$ and $s < 0$, the homoclinic classes of P and Q are both nontrivial: notice that, for negative s, $[-1, 1] \times (-1, 2) \times \{0\} \subset W^s(P, f_{t,s})$ and $\{0\} \times (-1, 2) \times [-1, 1] \subset W^u(Q, f_{t,s})$, thus $(-1/2, t, 0)$ and $(-1/2, -1, 0)$ are homoclinic points of P and Q, respectively.
- $f_{\sqrt{s},s}$ has a pair of heterodimensional cycles associated to P and S^+_s and to Q and S^-_s (this is obtained exactly as in Section 5).

Figure 5: The two parameter family $f_{t,s}$

As in Section 4, we assume the following distortion property (similar to (SN)). Let K be an upper bound for $|\Psi''_s(x)|/|\Psi'_s(x)|$, for small $|s|$ and $x \in [-1,1]$,

(DS) $\max\left\{\dfrac{4\,e^K\,(1-\lambda)}{\lambda^6}, \dfrac{4\,e^K\,(1-\beta^{-1})}{\beta^{-6}}\right\} < \dfrac{1}{2}$, where $3/4 < \lambda < 1 < \beta < 4/3$.

6.1 Dynamics before collapsing the saddles S_s^+ and S_s^-

Theorems 4.2 and 4.3 give a small $\bar{t}_0$ such that $H(P, f_{\bar{t},0}) = H^+(S, f_{\bar{t},0})$ and $H(Q, f_{\bar{t},0}) = H^-(S, f_{\bar{t},0})$ (for this inclusion consider $f_{\bar{t},0}^{-1}$) for all $\bar{t} \in (0, \bar{t}_0]$. These proofs only involve the following ingredients:

- The inclusions $H(P, f_{t_0,0}) \subset H^+(S, f_{t,0})$ and $H(Q, f_{t,0}) \subset H^-(S, f_{t,0})$ are obtained considering the ratio between the lengths of the scaled fundamental domains at the hyperbolic point and at the saddle-node and using that such a ratio is arbitrarily large.
- The inclusion $H^+(S, f_{t,0}) \subset H(P, f_{t_0,0})$ (resp. $H^-(S, f_{t,0}) \subset H(Q, f_{t_0,0})$) is obtained by constructing sequences of homoclinic points of P (resp. Q) verifying Proposition 4.9. The proof of such a proposition only involves distortion control of the saddle-node map in $[0,1]$ (resp. $[-1,0]$) and contracting itineraries (Lemma 4.12).

These properties hold after replacing, for small $s > 0$, the saddle-node S by the saddle S_s^+ (considering the restriction of Ψ_s to $[\sqrt{s}, 1]$) and S by the saddle S_s^- (considering the restriction of Ψ_s to $[-1, -\sqrt{s}]$). In this way, we get:

Theorem 6.1. *If $t > 0$ is small, there is a small $s(t) > 0$ such that $H(P, f_{t,s}) = H(S_s^+, f_{t,s})$ and $H(Q, f_{t,s}) = H(S_s^-, f_{t,s})$ for all $s \in (0, s(t))$.*

6.2 Dynamics after collapsing the saddles S_s^+ and S_s^-

Theorem 6.2. *For every small $t > 0$ there is $s(t) < 0$ such that $H(P, f_{t,s}) = H(Q, f_{t,s})$ for all $s \in [s(t), 0)$.*

This theorem follows adapting the constructions in Theorem 1.1. We prove the inclusion $H(P, f_{t,s}) \subset H(Q, f_{t,s})$ ($H(Q, f_{t,s}) \subset H(P, f_{t,s})$ follows by taking $f_{t,s}^{-1}$). As in Sections 2 and 4.1, define transitions $\mathcal{T}_{t,s}$ and returns $\mathcal{R}_{t,s}$ as follows: take the fundamental domain $D_t^- = [1-4\,t, 1-4\,\lambda\,t]$ of Ψ_s and let $k_{t,s}$ be the first $k \in \mathbb{N}$ with $-2\,t \in \Psi_s^{-k}(D_t^-)$. Let $D_{t,s}^+ = \Psi_s^{-k_{t,s}}(D_t^-)$ and define

$$\mathcal{T}_{t,s} \colon D_{t,s}^+ \to D_t^-,\ \ \mathcal{T}_{t,s}(x) = \Psi_s^{k_{t,s}}(x), \qquad \mathcal{G}_{t,s} \colon D_{t,s}^+ \to [-3\,t, -2\,t],\ \ \mathcal{G}_{t,s}(x) = \mathcal{T}_{t,s}(x) + (t-1).$$

Figure 6: The expanding maps $\mathcal{T}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$

By definition, $\mathcal{G}_{t,s}(D_{t,s}^+) \subset [-3\,t, t\,(1-4\,\lambda)] \subset [-3\,t, -2\,t]$ (recall that $\lambda > 3/4$). Since, $|D_{t,s}^+| \leq 9\,t^2 + s$, the Bounded Distortion Lemma 4.5 and $|D_t^-| = 4\,\lambda\,(1-t)$ immediately give

$$\mathcal{T}_{t,s}'(x) \geq (e^{-2\,K})\,\frac{|D_t^-|}{|D_{t,s}^+|} \geq (e^{-2\,K})\,\frac{4\,\lambda(1-t)}{9\,t^2+s}.$$

This inequality immediately implies the following:

Lemma 6.3. *For every small $t > 0$ there is $s(t) < 0$ such that $\mathcal{T}_{t,s}$ and $\mathcal{G}_{t,s}$ are 63-expanding for all $s \in [s(t), 0)$.*

Since, by definition, the right extreme of $\mathcal{G}_{t,s}(D_{t,s}^+)$ is less than the right extreme of $D_{t,s}^+$, for each $x \in D_{t,s}^+$ there is a first $i(x) \geq 0$ with $\Psi_s^{i(x)}(\mathcal{G}_{t,s}(x)) \in D_{t,s}^+$. The return map $\mathcal{R}_{t,s}$ is defined by

$$\mathcal{R}_{t,s} \colon D_{t,s}^+ \to D_{t,s}^+, \quad \mathcal{R}_{t,s}(x) = \Psi_s^{i(x)} \mathcal{G}_{t,s}(x).$$

Lemma 6.4. *The map $\mathcal{R}_{t,s}$ is 3-expanding for all small $t > 0$ and $s \in [s(t), 0)$.*

Proof: Note that the expansion for $\mathcal{R}_{t,s}$ does not follow immediately from the expansion of $\mathcal{T}_{t,s}$: the $i(x)$ iterates by Ψ_s at the left of 0 introduce a contraction. By Lemma 6.3, it is enough to see that this contraction is at most 1/21. Recall that $\mathcal{G}_{t,s}(D^+_{t,s}) \subset [-3\,t, -2\,t]$ and observe that $0 \le i(x) \le i_0$, where $\Psi^{i_0}(-3\,t) \in D^+_{t,s}$. Write $D^+_{t,s}(j) = \Psi_s^{-j}(D^+_{t,s})$ and note that, for every $x \in D^+_{t,s}$, $|D^+_{t,s}(i(x))| \le |D^+_{t,s}(i_0)| < 10\,t^2 + s$. For each $i = i(x)$ there is $z_i \in D^+_{t,s}(i)$ such that

$$(\Psi^i_s)'(z_i) = \frac{|D^+_{t,s}|}{|D^+_{t,s}(i)|} \ge \frac{|D^+_{t,s}|}{|D^+_{t,s}(i_0)|} \ge \frac{|D^+_{t,s}|}{10\,t^2 + s}.$$

Using the arguments in the Bounded Distortion Lemma 4.5, we have that, for every $x \in D^+_{t,s}(i)$,

$$(\Psi^i_s)'(x) \ge \frac{e^{-K\,L_t}\,|D^+_{t,s}|}{10\,t^2 + s},$$

where $L_t = 2\,t$ is the length of the interval $[-3\,t, -t]$ where the iterations by Ψ_s are considered. Finally, since $|D^+_{t,s}| > (t^2)/2$,

$$(\Psi^{i(x)}_s)'(x) \ge \frac{e^{-K\,2\,t}\,|D^+_{t,s}|}{10\,t^2 + s} > \frac{e^{-K\,2\,t}\,t^2}{20\,t^2 + 2s} > \frac{1}{21},$$

for every small t and $s \in [s(t), 0)$, ending the proof of the lemma. □

Let $D^+_{t,s} = [e^-_{t,s}, e^+_{t,s}]$. Note that there are $i_1, i_2 \in \mathbb{N}$ such that $i(x) \in [i_1, i_2]$ for all $x \in D^+_{t,s}$ and, for each $i \in [i_1, i_2 - 1]$, there is $d_i \in D^+_{t,s}$ with $\Psi_s(\mathcal{G}_{t,s}(d_i)) = e^-_{t,s}$. As in Section 2, $d_{i_2-1} < d_{i_2-2} < \cdots < d_{i_1}$ and the d_i are the discontinuities of $\mathcal{R}_{t,s}$. Moreover, $\mathcal{G}_{t,s}((d_{i+1}, d_i)) = \operatorname{int}(D^+_{t,s})$ and $\mathcal{G}_{t,s}$ is increasing in (d_{i+1}, d_i). Write $I_{i_1} = [d_{i_1}, e^+_{t,s}]$, $I_i = [d_i, d_{i-1}]$, $i_1 < i \le i_2 - 2$, and $I_{i_2} = [e^-_{t,s}, d_{i_2-1}]$. We now continuously extend $\mathcal{R}_{t,s}$ to the closed intervals I_i (so $\mathcal{R}_{t,s}$ is bivaluated at any d_i).

Lemma 6.5. *Given any subinterval J of $D^+_{t,s}$, there is $m \ge 0$ such that $\mathcal{R}^m_{t,s}(J) = D^+_{t,s}$.*

Proof: The proof follows as in Lemma 2.5. It is enough to see $\mathcal{R}^m_{t,s}(J)$ contains an interval I_i, $i_1 < i < i_2$, for some $m \in \mathbb{N}^*$. Write $J = J_0$. If $\mathcal{R}_{t,s}(J_0)$ contains two discontinuities we are done. Otherwise, $\mathcal{R}_{t,s}(J_0) \subset I_i \cup I_{i-1}$ for some i. Write $J_1^- = \mathcal{R}_{t,s}(J_0) \cap I_{i-1}$ and $J_1^+ = \mathcal{R}_{t,s}(J_0) \cap I_i$, and let J_1 be the biggest $J_1^{\pm}$. By Lemma 6.4, $|J_1| > (3/2)\,|J_0|$. Inductively, one gets intervals J_i contained in the orbit of J_0 such that either J_{i+1} contains two discontinuities or $|J_{i+1}| \ge (3/2)^i |J_0|$. Since the size of $D^+_{t,s}$ is finite, this ends the proof of the lemma. □

Lemma 6.5 is the main step to prove Theorem 6.2, whose proof follows arguing as in the proof of Theorem 1.1 after proving the following:

Proposition 6.6. *Let $\chi = \{x\} \times A \subset [-1,1] \times [-1,2] \times [-1,1]$, where $x \in [-1,1]$ and A is a disk of $\mathbb{R}^2$ whose interior contains a point $(y,0)$ with $y \in (-1,2)$. Then $\chi \pitchfork W^s(Q, f_{t,s}) \neq \emptyset$.*

This result corresponds to Proposition 3.4. After proving it, Theorem 6.2 is deduced as follows. As in Section 3, Proposition 6.6 implies that $W^s(Q, f_{t,s})$ meets transversely every two-disk χ with $W^s(P, f_{t,s}) \pitchfork \chi \neq \emptyset$. Now, arguing as in Section 3.1, we get $H(P, f_{t,s}) \subset H(Q, f_{t,s})$. We now prove Proposition 6.6. The first step is the next lemma, corresponding to Lemma 3.3 in the proof of Propositions 3.1 and 3.4 (its proof follows exactly as Lemma 3.3, so it will be omitted).

Lemma 6.7. *There are $x \in [-1,1]$ and $y \in int(D^+_{t,s})$ such that $\{(x,y)\} \times [-1,1] \subset W^u(P, f_{t,s})$.*

Lemma 6.8. *For every $x \in [-1,1]$, $W^s(Q, f_{t,s})$ meets transversely $\{x\} \times D^+_{t,s} \times [-1,1]$.*

Proof: Observe that, by construction, the point $(0, -t, -1/2)$ belongs to $W^s(Q, f_{t,s}) \cap W^u(Q, f_{t,s})$. Define $j > 0$ by $\Psi_s^{-j}(-t) \in D^+_{t,s}$. It is now immediate that

$$H = [-1,1] \times \{(\Psi_s^{-j}(0), -\lambda_u^{-j}\,(1/2))\} \subset W^s(Q, f_{t,s}) \quad \text{and} \quad H \pitchfork (\{x\} \times D^+_{t,s} \times [-1,1]) \neq \emptyset,$$

ending the proof of the lemma. □

We are now ready to prove Proposition 6.6. By the λ-lemma and Lemma 6.7, the forward orbit of the disk χ contains a strip Δ of the form $\{x\} \times [a,b] \times [-1,1]$, where $x \in [-1,1]$, and $\alpha_0 = [a,b] \subset D^+_{t,s}$. Using the map $\mathcal{R}_{t,s}$ and arguing as in Section 3.1, we inductively define strips $\Delta_k = \{x_k\} \times \alpha_k \times [-1,1]$, $x_k \in [-1,1]$ and $\alpha_k \subset D^+_{t,s}$, such that $\Delta_{k+1} \subset f^{n_k}_{t,s}(\Delta_k)$ and $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$. By Lemma 6.5, there is a first $k \in \mathbb{N}$ such that $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$ contains $D^+_{t,s}$. By Lemma 6.8, $W^s(Q, f_{t,s}) \pitchfork \Delta_{k+1} \neq \emptyset$, thus $W^s(Q, f_{t,s}) \pitchfork \Delta \neq \emptyset$, ending the proof of the proposition.

6.3 End of the proof of Theorem A

We now construct a one-parameter family of diffeomorphisms (g_s) satisfying Theorem A. For that consider the arc $f_{t,s}$ defined as in the beginning of Section 6. We fix small $\bar{t} > 0$ and consider the arc $g_s = f_{\bar{t},-s}$. The results in the previous section imply that

- for every $s < 0$, $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and disjoint (Theorem 6.1),
- $s = 0$, $H(P, g_0) = H^+(S, g_0)$ and $H(Q, f_0) = H^-(S, g_0)$, (Theorems 4.2 and 4.3),
- for every $s > 0$, $H(P, g_s) = H(Q, g_s)$, (Theorem 6.2).

To finish the proof of Theorem A we need to see that $\{S\} \in H(P, g_s) \cap H(Q, g_s)$ and to describe the maximal invariant set of g_s in the neighborhood W of the cycle. We assume that W is a *level of a fitration* of $f_{0,0}$ (thus, by continuity and compacity, it is also a level of a filtration for $f_{t,s}$ for every small $|s|$ and $|t|$): there are compact sets M_2 and M_1, $M_1 \subset \mathrm{int}(M_2)$, such that $M_2 \setminus M_1 = W$ and $f_{0,0}(M_i) \subset \mathrm{int}(M_i)$, $i = 1, 2$. Hence, if $x \in W$ and $f^i_{t,s}(x) \notin W$ for some i, then x is wandering: suppose, for instance, that $f^{i_0}_{t,s}(x) \in \mathrm{int}(M_1)$, where $i_0 > 0$. Then, there is a neighborhood $U_x \subset W$ of x with $f^{i_0}_{t,s}(U_x) \subset M_1$. By the definition of the filtration, $f^{i_0+j}_{t,s}(U_x) \subset M_1$ for all $j \geq 0$. Thus $f^{i_0+j}_{t,s}(U_x) \cap U_x = \emptyset$ for all $j \geq 0$. By shrinking U_x, we have that $f^j_{t,s}(U_x) \cap U_x = \emptyset$ for all $j > 0$, and x is wandering.

Using the definition of the arc $f_{t,s}$, it is immediate to check the following:

Remark 6.9. *Let $\Lambda_{t,s}$ be the maximal invariant set of $f_{t,s}$ in W. Then, for $s > 0$ small,*

- *Every point $(x, y, s) \in C = [-1,1] \times [-2,2] \times [-1,1]$ with $y \in (-\sqrt{s}, \sqrt{s})$ (resp, $y \in [-2,-1)$ or $y \in (1,2)$) is wandering.*
- *Consider $w = (x, y, z) \in C \cap \Lambda_{t,s}$ and, for $i \in \mathbb{Z}$, let $w_i = g^i_s(w) = f^i_{\bar{t},-s}(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in [\sqrt{s}, 1]$ and $y_j \in [-2, \sqrt{s})$ for some $j > 0$. Then, for every $n \geq j$ with $w_n \in C$, $y_n \in [-2, \sqrt{s})$.*

Consider now $s = 0$.

- *Every point* $(x, y, s) \in C$ *with* $y \in [-2, -1)$ *or* $y \in (1, 2)$ *is wandering.*
- *Consider* $w = (x, y, z) \in C \cap \Lambda_{t,0}$ *and, for* $i \in \mathbb{Z}$, *let* $w_i = g_0^i(w) = f_{\bar{t},0}^i(w)$. *If* $w_i \in C$ *we let* $w_i = (x_i, y_i, z_i)$. *Suppose that* $y_i \in (0, 1]$ *and* $y_j \in [-2, 0)$ *for some* $j > 0$. *Then, for every* $n \geq j$ *with* $w_n \in C$, $y_n \in [-2, 0)$.

For $s \geq 0$ let $\Lambda_{\bar{t},s}^+$ (resp. $\Lambda_{\bar{t},s}^-$) be the set of points $w \in \Lambda_{\bar{t},s} \cap \Omega(f_{\bar{t},s})$ such that $y_i \in [\sqrt{s}, 1]$ (resp. $y_i \in [-1, \sqrt{s}]$) for all $i \in \mathbb{Z}$ with $w_i = (x_i, y_i, z_i) \in C$. Remark 6.9 implies that, for every small $s \geq 0$,

$$\Lambda_{\bar{t},s} \cap \Omega(f_{\bar{t},s}) = \Lambda_{\bar{t},s}^+ \cup \Lambda_{\bar{t},s}^-.$$

Arguing as in Section 3, one gets $H(P, f_{\bar{t},s}) = \Lambda_{\bar{t},s}^+$ and $H(Q, f_{\bar{t},s}) = \Lambda_{\bar{t},s}^-$. Observe that, for positive s, one needs to exclude the segment $\{0\} \times (-\sqrt{s}, \sqrt{s}) \times \{0\} \subset \Lambda_{\bar{t},s}$ consisting of wandering points.

Finally, for the saddle-node parameter $s = 0$, it is immediate that $\Lambda_{\bar{t},0}^+ \cap \Lambda_{\bar{t},0}^- = \{S\}$.

For parameters $s < 0$ the result follows similarly (but now the situation is much more simple).

6.4 Proof of Theorem B

Clearly, the homoclinic classes $H(P, f_{\bar{t},0})$ and $H(Q, f_{\bar{t},0})$ are not saturated. We claim that there is not any transitive saturated set Σ containing $H(P, f_{\bar{t},0})$. Otherwise, the set Σ must also contain $H(Q, f_{\bar{t},0})$. Thus Σ contains the whole $\Lambda_{\bar{t},0}$. Using the filtration, one has $\Sigma = \Lambda_{\bar{t},0}$. But this set is not transitive: Remark 6.9 implies that there is no orbit going from a small neighborhood U_Q of Q to a small neighborhood U_P of P and thereafter returning to U_Q. This contradiction ends the proof of the theorem.

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Pseudo-orbit shadowing in the C^1 topology

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April 6, 2005

To Carlos Gutierrez and Marco Antonio Teixeira
for their 60th birthday

Abstract

We prove that the shadowing property does not hold for diffeomorphisms in an open and dense subset of the set of C^1-robustly non-hyperbolic transitive diffeomorphisms (i.e., diffeomorphisms with a C^1-neighbourhood consisting of non-hyperbolic transitive diffeomorphisms).

1 Introduction

Numerical simulations of dynamical systems generally produce approximate orbits (or *pseudo-orbits*) rather than actual orbits of the system. In order for such simulations to be relevant, it is important to ensure that sufficiently precise computations produce pseudo-orbits which are followed – or *shadowed* – by true orbits. This property – called the *pseudo-orbit shadowing property* – is therefore of fundamental importance to applications of dynamics.

Moreover, shadowing of pseudo-orbits plays a central role in the general theory of dynamical systems. Every basic set of every hyperbolic dynamical system (in this paper all dynamical systems are given by iterations of diffeomorphisms) displays the shadowing property, as proved by Anosov and Bowen [An, Bo]. The shadowing property in turn is fundamental to the stability of hyperbolic diffeomorphisms. Indeed, shadowing and stability/hyperbolicity are so closely intertwined that it is generally believed that shadowing and hyperbolicity are in some sense equivalent. This equivalence is of course not strictly true: there are examples of systems which are not hyperbolic and nevertheless do exhibit the shadowing property ([YY], for instance, mentions several such examples which are very fragile). But it is reasonable to expect that, for *most* systems in some sense, the shadowing property and hyperbolicity are indeed equivalent. The goal of this paper is to show that, if *most systems* is taken to mean C^1*-generic* or C^1*-open and dense* diffeomorphisms, then the equivalence between shadowing and hyperbolicity does hold in some important contexts.

Let us begin by setting our context and recalling some standard definitions:

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Throughout the paper M denotes a compact boundaryless manifold whose dimension d is greater than or equal to 3. The space of C^1 diffeomorphisms on M, endowed with the usual C^1 topology, is denoted by $\mathrm{Diff}^1(M)$. Given $\delta > 0$ and $f \in \mathrm{Diff}^1(M)$, a sequence $(x_n)_{n\in\mathbb{N}}$ in M is a *δ-pseudo-orbit of f* if the distance between $f(x_n)$ and x_{n+1} is less than δ for all n. Given $\varepsilon > 0$, the pseudo-orbit (x_n) is *ε-shadowed* by a true orbit $(f^n(x))_{n\in\mathbb{N}}$ of f if the distance between x_n and $f^n(x)$ is less than ε for all n.

An invariant set Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ has the *shadowing property* if for every $\varepsilon > 0$ there is some $\delta > 0$ such that every δ-pseudo-orbit of points in Λ is ε-shadowed by some orbit of f in Λ. A diffeomorphism $f \in \mathrm{Diff}^1(M)$ has the shadowing property or *is shadowable* if M has the shadowing property for f. So for a set or diffeomorphism *not* to have the shadowing property means that there is some $\varepsilon > 0$ such that for every $\delta > 0$ there is some δ-pseudo-orbit which is not ε-shadowed by any actual orbit. Finally, given an open set $U \subset M$, then f is *nonshadowable in U* if there is some $\varepsilon > 0$ such that for every $\delta > 0$ there is some δ-pseudo-orbit *in U* which is not ε-shadowed by any actual orbit *in all of M*.

Given $f \in \mathrm{Diff}^1(M)$, an f-invariant compact set Λ is *transitive* if there is some $x \in \Lambda$ such that the ω-limit set $\omega_f(x)$ of x coincides with Λ. The diffeomorphism f is *transitive* if M is a transitive set for f; the diffeomorphism f is *C^1-robustly transitive* if there is a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(M)$ such that every $g \in \mathcal{U}_f$ is transitive. We denote by $\mathcal{T}$ the subset of $\mathrm{Diff}^1(M)$ of C^1-robustly transitive diffeomorphisms. We also consider the open subset $\mathcal{RNT}$ of $\mathcal{T}$ of *C^1-robustly non-hyperbolic transitive diffeomorphisms* of M, that is diffeomorphisms $f \in \mathcal{T}$ having a neighbourhood $\mathcal{V}_f$ consisting of non-hyperbolic diffeomorphisms.

A transitive set Λ is an *attractor* if there is an open neighborhood V of Λ in M such that the closure of $f(V)$ is contained in V and $\bigcap_{n\in\mathbb{N}} f^n(V) = \Lambda$; such a neighborhood V is called an *attracting block* of Λ. The *basin* of Λ is the set $\mathcal{B}(\Lambda) \equiv \{x \in M : \omega(x) \subset \Lambda\}$. It is easy to verify that if V is an attracting block of Λ then $\mathcal{B}(\Lambda) = \bigcup_{n\in\mathbb{N}} f^{-n}(V)$; in particular the basin $\mathcal{B}(\Lambda)$ is an open subset of M. An attractor Λ is *C^1-robust* if there are an attracting block V of Λ and a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{U}$ then $\Lambda_g \equiv \bigcap_{n\in\mathbb{N}} g^n(V)$ is an attractor of g; in this case Λ_g is said to be the *continuation of Λ relative to g*.

A transitive set Λ is *isolated* if there is an open neighborhood V of Λ, called an *isolating block* of Λ, such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$. An isolated transitive set Λ is *C^1-robust* if there are an isolating block V of Λ and a neighborhood $\mathcal{U}$ of f such that given $g \in \mathcal{U}$, then the continuation $\Lambda_g \equiv \bigcap_{n\in\mathbb{Z}} g^n(V)$ of Λ is also a transitive set.

Finally, let us recall some terminology concerning residual sets and genericity. Given an open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$, a subset $\mathcal{R}$ of $\mathcal{U}$ is *residual in $\mathcal{U}$* if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of $\mathcal{U}$; in this case $\mathcal{R}$ is dense in $\mathcal{U}$. A set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ is said to be *residual* if it is residual in all of $\mathrm{Diff}^1(M)$. A property (P) is *residual* or *generic* in $\mathcal{U}$ if (P) holds for all diffeomorphisms which belong to some residual subset of $\mathcal{U}$; the property (P) is said to be *residual* or *generic* if it is residual in all of $\mathrm{Diff}^1(M)$.

1.1 Statement of the results

Motivated by previous definitions in [Ki, Yo], Yuan and Yorke introduced in [YY] a strong type of non-shadowing property called *absolute non-shadowability*. Let us recall their definition. Fix a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a point $x \in M$. Consider the set $\Omega_{x,\delta}$ of δ-pseudo-orbits starting at x. Now, given an element x_n of a δ-pseudo-orbit starting at x, the successor x_{n+1} of x_n in the pseudo-orbit can be any one of the points in the open ball $B_\delta(f(x_n))$. Let us assume the uniform probability distribution on $B_\delta(f(x_n))$. In this way, the set $\Omega_{x,\delta}$ constitutes a Markov chain having each δ-pseudo-orbit as a sample sequence. The point x is said to be *absolutely non-shadowable* if there is some $\varepsilon > 0$ such that, for all $\delta > 0$, almost every δ-pseudo trajectory $(x_i)_{i=0}^{\infty}$ (i.e., a set of full measure of the space of δ-pseudo trajectories) is ε-shadowed by no true orbit.

For the sake of concision, we will say that an attractor Λ is *absolutely non-shadowable* if every point in the basin $\mathcal{B}(\Lambda)$ of Λ is absolutely non-shadowable. In [YY] the intersection of the following three conditions is shown to be sufficient for an attractor Λ of a map f to be absolutely non-shadowable:

y1) *dimension variability*: the attractor Λ contains saddles P and Q having different *indices* (i.e., dimension of the unstable manifold), say $\operatorname{index}(P) < \operatorname{index}(Q)$;

y2) the attractor Λ is the closure of the unstable manifold of the saddle P of lower index; and

y3) there is a dominated splitting $E \oplus F$ (see Definition 2.7) over Λ such that the dimension of F coincides with $\operatorname{index}(P)$.

We observe that, a priori, these properties are quite fragile: in general, they can be destroyed after arbitrarily small perturbations of the map. Yorke and Yuan [YY], however, construct some examples for which the hypotheses (y1), (y2), and (y3) hold robustly: these examples include some non-invertible maps on the two-dimensional torus (following [KKGOY]) and some diffeomorphisms on $\mathbb{S}^3 \times \mathbb{S}^1$ (following [Sm] and modifying [KKGOY]). It follows that these examples are absolutely non-shadowable in a robust way.

Our first result follows from verifying that the three hypotheses from [YY] – and hence the absolute non-shadowability condition – hold for generic robustly transitive non-hyperbolic diffeomorphisms (in this case, the attractor is the whole manifold):

Theorem 1. *Generically in the set $\mathcal{RNT}$ of robustly non-hyperbolic and transitive diffeomorphisms, the whole ambient manifold is absolutely non-shadowable.*

At this point we note that, as far as the authors know, all of the known robustly non-hyperbolic transitive diffeomorphisms (see [Sh, Mñ$_1$, Ca, BD$_1$, BV]) satisfy the aforementioned conditions (y1) and (y3) given in [YY]. Condition (y2) always holds for generic robustly transitive diffeomorphisms (this is a consequence of the Connecting Lemma of Hayashi, [Ha], see Lemma 2.5). Indeed, it seems possible that *all* robustly non-hyperbolic transitive diffeomorphisms – not just C^1-generic ones – are absolutely non-shadowable. Unfortunately, we do not know how to prove this at this point.

We can, however, prove that C^1*-open and densely* every robustly non-hyperbolic transitive diffeomorphism does not have the shadowing property.

Theorem 2. *There is an open and dense subset $\mathcal{N}$ of the set $\mathcal{RNT}$ of robustly non-hyperbolic transitive diffeomorphisms consisting of non-shadowable diffeomorphisms.*

This theorem was proved in [BDT] for diffeomorphisms of dimension three under the additional hypothesis on the global dynamics of the existence of a dominated splitting having three non-trivial directions. Naively speaking, the proof of Theorem 2 consists in showing that this global hypothesis can be obtained locally. This fact and an argument we import from [DR] about invariant manifolds of saddles involved in partially hyperbolic heterodimensional cycles allow us to get an analogue of condition (y2) in [YY]. Our proof also involves the transitions associated to heterodimensional cycles introduced in [BDPR] following [BDP] (see the sketch of the proof at the end of this introduction).

Theorem 2 is in fact a corollary of the following more general result (whose proof is essentially the same as above, with the addition of certain generic arguments):

Theorem 3. *Given an isolated transitive set Λ of a generic diffeomorphism f, then either:*

a) Λ is a hyperbolic set; or

b) there are a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and arbitrarily small isolating blocks V of Λ such that every $g \in \mathcal{U}$ is non-shadowable in the neighborhood V.

This result easily yields the following corollary, of which Theorem 2 is a special case:

Corollary 1.1. *Let Λ be a C^1-robust attractor of a diffeomorphism f, with continuations Λ_g defined for all diffeomorphisms g in some open neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f. Assume that Λ_g is non-hyperbolic for every $g \in \mathcal{U}$. Then there is an open and dense subset $\mathcal{U}_0$ of $\mathcal{U}$ such that for every $g \in \mathcal{U}_0$ there is an arbitrarily small isolating block V_g of Λ_g in which g is non-shadowable.*

From Theorem 3, which is a “semilocal” result, we immediately obtain not just Theorem 2 but also an interesting “global” result. It was explained in [Ab1] that generic diffeomorphisms come in one of two types: *tame* diffeomorphisms, which have a finite number of homoclinic classes and whose nonwandering sets admit partitions into a finite number of disjoint transitive sets; and *wild* diffeomorphisms, which have an infinite number of (disjoint and different) homoclinic classes and whose nonwandering sets admit no such partitions.

Theorem 4. *There is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ is tame, then the following two conditions are equivalent:*

a) f is hyperbolic (i.e., Axiom A without cycles)

b) f is shadowable

The result above motivates the following conjecture:

Conjecture 1. *Generically hyperbolicity and shadowability are equivalent. That is, there is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that $f \in \mathcal{R}$ is shadowable if and only if it is Axiom A without cycles.*

In view of Theorem 4 above, which settles the conjecture in the tame case, Conjecture 1 above is reduced to the following:

Conjecture 2. *Every wild generic diffeomorphism is non-shadowable.*

We remark that this last conjecture at first seems easy to prove, by constructing pseudo-orbits which “jump” from one homoclinic class to the other, but in fact this naive idea does not work by itself, so that this is a much more subtle problem than might appear at first sight.

1.2 Further comments

We also stress that while our results above show that "typically" (i.e. C^1-generically or C^1-open and densely, at least in some contexts) hyperbolicity and shadowability are equivalent, it may very well be that in a statistical sense shadowability is quite abundant: that is, it may be that many nonhyperbolic systems exhibit the pseudo-orbit property for most (but not all) pseudo-orbits. In the appendix to [KH] Katok and Mendoza have shown that given a $C^{1+\alpha}$-diffemorphism f which preserves a hyperbolic measure μ, then "most" pseudo-orbits near the support of μ are shadowed by actual orbits. The work in progress [ABC] shows that C^1-generic transitive difeomorphisms exhibit hyperbolic measures whose supports coincide with the whole ambient manifold. If Katok's techniques (which rely on a weak form of Pesin theory) can be adapted to the C^1-generic context, then this would imply that generic (including generic nonhyperbolic) transitive diffeomorphisms do have the shadowing property, *modulo sets of arbitrarily small measure.*

It is worthwhile to compare the previous results with [PP], which claims that shadowing is generic in the C^0-topology. See also the papers [AHK, Mz, Od].

We close this part of the introduction with some comments concerning shadowing and hyperbolicity. Different notions of weak shadowing have been introduced in recent years (for the various types of shadowing and their properties we refer the reader to the books [Pa, Pi]). One of them is the *first weak shadowing property, (f.w.s.p),* see [CP, Sa_2], meaning that, for every $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo trajectory is contained in an ε-neighbourhood of a true orbit (in our context this definition automatically is verified: the diffeomorphisms are transitive and thus the ε-neighbourhood of a dense orbit is the whole manifold). [Sa_2] states that the C^1-interior of the surfaces diffeomorphisms verifying the f.s.w.p are structurally stable (so hyperbolic).

For results in the same spirit of [Sa_2], relating shadowing and hyperbolicity, see [Sa_1] and [PRS]. We observe that the existence of robustly transitive diffeomorphisms, see [Sh, $\mathrm{Mñ}_1$, Ca, BD_1, BV], in dimensions greater than or equal to three shows that this result does not hold in higher dimensions. In fact, the main difference between dimension two and higher dimensions arises from the dimension variability, which is forbidden in dimension two (in the context of diffeomorphisms). Finally, recently [Cr, Corollary 1.10] states that there is a generic subset of C^1-diffeomorphisms satisfying a stronger formulation of the f.w.s.p. formulated above.

1.3 Ingredients of the proofs

Let us describe the broad outlines of the proofs of our main results. These proofs have two main ingredients: the properties of the dominated splittings of homoclinic classes and robustly transitive diffeomorphisms obtained in [DPU, BDP, BDPR] and the analysis of the dynamics arising from partially hyperbolic heterodimensional cycles, see [DR].

Theorem 1, as was previously mentioned, follows from checking that conditions (y1), (y2), and (y3) of [YY] hold for any generic non-hyperbolic robustly transitive diffeomorphism. Condition (y1), *dimension variability,* follows from combining the Connecting Lemma of Hayashi [Ha] (which extends the previous work of Mañé [$\mathrm{Mñ}_2$] on the stability conjecture) with some standard C^1-generic arguments. Not being hyperbolic implies that non-hyperbolic periodic points may be created via small C^1-perturbations. This implies that saddles having different indices may be created. Since hyperbolic saddles are persistent, it follows that the existence of points of different indices is an open and dense property in $\mathcal{RNT}$.

Condition (y2), *existence of a periodic point (of appropriate index) whose unstable manifold is* *dense in the manifold,* is guaranteed generically by combining Pugh's General Density Theorem [Pu] with a result from [BD_2] which in turn is a consequence of the aforementioned Connecting Lemma. Indeed, this combination proves that generically the unstable manifold of *every* periodic point in the attractor is dense in the manifold (see Lemma 2.5).

Condition (y3), *existence of a dominated splitting* $E \oplus F$ *such that the dimension of* F *is the index of a saddle in the manifold whose index is not maximal,* is the toughest of the three to check. The existence of a dominated splitting is a straightforward consequence of the results in [BDP]. Showing that there is a dominated splitting with an appropriate dimension, however, requires more subtlety. We employ an argument by contradiction using techniques from [BDPR], a paper which extends some of the ideas in [BDP].

For the sake of clarity we first prove Theorem 2 and later (in Subsection 3.3.4) show how this proof can be modified in order to obtain the more general Theorem 3. In fact, the proof of Theorem 3 only differs from that of Theorem 2 in that it requires some extra, "soft", generic arguments; the core of both results are the arguments found in the proof of Theorem 2.

The proof of Theorem 2 relies on the C^1-generic machinery generated by the Connecting Lemma and on the analysis of the dynamics arising from the unfolding of partially hyperbolic heterodimensional cycles. In our context, the creation of a heterodimensional cycle follows immediately from the Connecting Lemma and the robust transitivity (see Remark 2.4). But here it is essential to have some local domination around the cycle: we need to create partially hyperbolic heterodimensional cycles. To get these cycles we use the machinery developed in [BDP, BDPR] on dominated splittings of homoclinic classes.

Heterodimensional cycles, introduced by Newhouse-Palis in [NP] in the seventies and recently developed in the series of papers of the second author and Rocha, see for instance [D_1, D_2, DR], consist of cycles between two (or more) hyperbolic periodic saddles which have different indices. That is, there are two saddles P and Q, with $\mathrm{index}(P) \neq \mathrm{index}(Q)$, such that the stable manifold of P intersects the unstable manifold of Q and vice-versa.

In case the indices of P and Q are consecutive integers, however, the use of the *blenders* studied in [BD_2, DR] does allow the creation of C^1-robust *quasi-heterodimensional cycles*: here the invariant manifolds of P and Q are not required to intersect each other, but rather to (C^1-robustly) accumulate (in a dominated way) on each other, see Theorem 3.9. This is the main step to construct arbitrarily fine pseudo-orbits which cannot be shadowed by actual orbits. That is, the presence of a dominated quasi-heterodimensional cycle is an obstruction to the shadowing property, and so the presence of a robust dominated quasi-heterodimensional cycle is a robust obstruction to the shadowing property.

Let us now explain how such a partially hyperbolic heterodimensioanl cycle is created. As in the proof of Theorem 1, the lack of hyperbolicity of the attractor generates (modulo C^1-small perturbations) *dimension variability,* that is, two periodic saddles of different indices. The arguments from [BDPR] show that the indices of the periodic points form a non-trivial interval in $\mathbb{N}$; in particular, the manifold contains two periodic points P and Q with consecutive indices. Moreover, using [BDP] (see also Lemma 3.4) we see that this cycle can be associated to saddles P and Q whose multipliers are all real, positive and have multiplicity one. Unfolding this cycle we obtain a partially hyperbolic saddle-node (Proposition 3.5). Unfolding this saddle-node one gets two new saddles of different indices bounding a segment where the dynamics is locally partially hyperbolic. Finally, the Connecting Lemma allows us to create a partially hyperbolic heterodimensional cycle associated to these saddles.

Finally, we note that the proof of [YY, Theorem 2.2] shows that their conditions (y1), (y2), and (y3) essentially imply the existence of a quasi-heterodimensuonal cycle (actually, this type of quasi-cycle condition is considered in [CK] in the context of homeomorphisms). The main point is how to obtain such quasi-cycles. So, despite the substantial differences in terms of vocabulary and technique between this paper and [YY], we in fact rely on the same type of mechanism, which can be summed up by the phrase *dimensional variability, local domination and transitivity imply no shadowing.*

This paper is organized as follows. In Section 2, we recall the C^1-generic machinery that allows us to create cycles after perturbations and state the main facts about dominated splittings of robustly transitive diffeomorphisms and homoclinic classes. Using these facts, we prove Theorem 1. In Section 3, we prove Theorems 2 and 3 (in Section 3.3.3 and 3.3.4, respectively). In Section 3.1, we explain how partially hyperbolic heterodimensional cycles are produced in the robustly transitive setting. This construction relies on the creation of partially hyperbolic saddle-nodes. In Section 3.2, we prove that the diffeomorphisms having partially hyperbolic heterodimensional cycles are dense in $\mathcal{RNT}$. Finally, in Section 3.3, we explain how the constructions in [DR] implies the persistence of dominated quasi-cycles in the set of robustly transitive diffeomorphisms. Using this fact, we construct non-shadowable pseudo-orbits.

2 Generic non-hyperbolic robustly transitive diffeomorphisms

In this section we prove Theorem 1. Recall that $\mathcal{RNT}$ denotes the set of C^1-robustly non-hyperbolic transitive diffeomorphisms of M, that is, diffeomorphisms f having a neighbourhood $\mathcal{U}_f$ in $\mathrm{Diff}^1(M)$ such that every $g \in \mathcal{U}_f$ is non-hyperbolic and transitive. Denote by $\mathcal{Y}$ the subset of $\mathcal{RNT}$ consisting of diffeomorphisms f such that:

(y1) there are (hyperbolic) saddles P_f and Q_f of f such that $\mathrm{index}(P_f, f) < \mathrm{index}(Q_f, f)$;

(y2) the ambient manifold M is the closure of the unstable manifold of the saddle P_f;

(y3) there is a (non-trivial) dominated splitting $E \oplus F$ of f (see Definition 2.7) defined on the whole manifold such that the dimension of F is equal to the dimension of the unstable bundle of P_f.

Denote by $\mathcal{G}$ the set of diffeomorphisms $f \in \mathcal{RNT}$ such that either f or f^{-1} belongs to $\mathcal{Y}$.

Proposition 2.1. *The set $\mathcal{G}$ contains a residual subset of $\mathcal{RNT}$.*

For proving this proposition we need some preliminary results and definitions.

2.1 Perturbation lemmas

We need the following perturbation lemma about the creation of heteroclinic intersections:

Lemma 2.2 (Connecting Lemma, [Ha]). *Let f be a C^1 diffeomorphism and Λ a transitive set of f containing a pair of hyperbolic saddles A_f and B_f. Then there is g arbitrarily C^1-close to f such that $W^u(A_g, g) \cap W^s(B_g, g) \neq \emptyset$, where A_g and B_g are the continuations of the saddles A_f and B_f of f for g. This lemma holds when $A_f = B_f$.*

We state two consequences from Lemma 2.2 we use repeatedly throughout the paper. Next remark is an immediate consequence from the lemma in the case $A_f = B_f$:

Remark 2.3. *Let f be a C^1 diffeomorphism and Λ a transitive set of f containing saddle A_f. Then there is g arbitrarily C^1-close to f such that $W^u(A_g, g) \pitchfork W^s(A_g, g) \neq \emptyset$.*

Remark 2.4. *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f having a transitive set Λ_f containing a pair of (hyperbolic) saddles A_f and B_f depending continuously on f and having different indices (there is no hypotheses on the variation of the transitive set Λ_f). There is a dense subset $\mathcal{C}$ of $\mathcal{U}$ of diffeomorphisms g having a heterodimensional cycle associated to A_g and B_g.*

Proof: Suppose that the $p = \mathrm{index}(A_f, f) < \mathrm{index}(B_f, f) = q$. Applying Lemma 2.2 to $W^s(A_f, f)$ and $W^u(B_f, f)$, we get g close to f such that $W^s(A_g, g) \cap W^u(B_g, g) \neq \emptyset$. As the sum of the dimensions of these manifolds is greater than the dimension n of the ambient manifold M (this sum is $(n-p)+q > n-q+q = n$), after a new perturbation we can assume that $W^s(A_g, g) \pitchfork W^u(A_g, g) \neq \emptyset$. This implies that the set $\mathcal{C}_0$ of diffeomorphisms $g \in \mathcal{U}$ such that $W^s(A_g, g) \pitchfork W^u(B_g, g) \neq \emptyset$ is open and dense in $\mathcal{U}$.

To finish the proof of the remark just take $g \in \mathcal{C}_0$ and apply Lemma 2.2 to $W^u(A_f, f)$ and $W^s(B_f, f)$, obtaining $h \in \mathcal{C}_0$ close to g with $W^u(A_h, h) \cap W^s(B_h, h) \neq \emptyset$. As $h \in \mathcal{C}_0$, $W^s(A_h, h) \cap W^u(B_h, h) \neq \emptyset$, thus h has a heterodimensional cycle associated to A_h and B_h. $\square$

Next Lemma 2.5 is a reformulation of [BD$_2$, Théorème 1.4] to the context of robustly transitive diffeomorphisms and it is obtained from Lemma 2.2 using a standard argument of genericity. Let us recall that the *homoclinic class* of a saddle P_f of a diffeomorphism f, denoted by $H(P_f, f)$, is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P_f. A homoclinic class can be equivalentely defined as the closure of the set of saddles Q *homoclinically related* to P_f (that is, the stable manifold of the orbit of P_f transversely meets the unstable manifold of the orbit of Q and vice-versa). Note that two saddles homoclinicially related have the same index. This implies that the periodic points of the same index as P_f are dense in the class $H(P_f, f)$. Finally, a homoclinic class is always transitive (for the proof of these properties see, for instance, [Ne]).

Lemma 2.5. *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(M)$ of transitive diffeomorphisms f such that every f has a saddle A_f depending continuously on $f \in \mathcal{U}$. There is a residual subset $\mathcal{J}_A$ of $\mathcal{U}$ such that, for all $f \in \mathcal{J}_A$, the homoclinic class of A_f is the whole ambient manifold M. In particular, the stable and unstable manifolds of A_f are both dense in M.*

Given a hyperbolic saddle A_f of a diffeomorphism f, denote by Σ_{A_f} the subset of the homoclinic class $H(A_f, f)$ of A_f of saddles of the same index as A_f and homoclinically related to A_f. The set Σ_{A_f} always is dense in $H(A_f, f)$. The previous lemma and an argument of transversality implies the following.

Remark 2.6. *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(M)$ of transitive diffeomorphisms f such that every f has a saddle A_f depending continuously on $f \in \mathcal{U}$. There is a residual subset $\mathcal{L}$ of $\mathcal{U}$ such that, for all $f \in \mathcal{L}$, the set Σ_{A_f} is dense in M.*

2.2 Dominated splittings

In this section we state the main facts about dominated splittings of robust transitive diffeomorphisms and prove Proposition 2.1. For properties of dominated splittings see, for example, [BDV, Appendix B].

Definitions and properties 2.7. *Consider a diffeomorphism f and an f-invariant set Λ.*

- **Dominated splitting:** *A Df-invariant splitting $E \oplus F$ of TM over Λ is* dominated *if the fibers of the bundles have constant dimension and there are a metric $||\cdot||$ and a natural number $n \in \mathbb{N}$ such that*
$$||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2} \quad \textit{for all } x \in \Lambda.$$
*In this case, we say that the splitting is n-*dominated.

- **Extension to the closure,** [BDV, Section B.1.1]: *Every n-dominated splitting defined over a set Λ can be extended to an n-dominated splitting defined over the closure of Λ.*

- **Extension and persistence of dominated splittings,** [BDV, Section B.1.1]**:** *Suppose that Λ has a dominated splitting. Then this splitting can be extended in a dominated way to the maximal invariant set of f in a neigbourhood of Λ. Moreover, every n-dominated splitting persists under C^1 perturbations: for any $\varepsilon > 0$ there are neighbourhoods U of Λ and $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that for every $g \in \mathcal{U}$ the maximal invariant set in the closure of U has an (n-ε)-dominated splitting*[1] *having the same dimensions as the initial dominated splitting over Λ.*

- **Finest dominated splitting:** *The splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is* dominated *if, for all $i = 1, \ldots, (k-1)$, the splitting $(\oplus_{j=1}^{i} E_j) \oplus (\oplus_{j=i+1}^{k} E_j)$ is dominated. The splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is the* finest dominated splitting *of Λ if any bundle of the splitting is* indecomposable, *that is no E_i admits any dominated splitting.*

- **Clustering property of dominated splittings,** [BDV, Proposition B.2]**:** *Let $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ be the finest dominated splitting of Λ and $E \oplus F$ a dominated splitting over Λ, then there is i such that $E = \oplus_{j=1}^{i} E_j$ and $F = \oplus_{j=i+1}^{k} E_j$.*

- **Volume hyperbolicity:** *The dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is* volume hyperbolic *if there is n such that the derivative of f^n uniformly contracts the (induced) volume in E_1 and uniformly expands the (induced) volume in E_k.*

Theorem 2.8. [BDP, Theorems 2 and 4] *Every robustly transitive diffeomorphism f has a dominated splitting. Moreover, the finest dominated splitting of f (which is well and uniquely defined) is volume hyperbolic.*

Given $f \in \mathcal{RNT}$, denote by $s^-(f)$ and $s^+(f)$ the minimum and the maximum of the dimensions of the stable bundles of the (hyperbolic) saddles of f (if f has no hyperbolic saddles, these numbers are not defined). Consider also the finest dominated splitting $E_1(f) \oplus \cdots \oplus E_{m(f)}(f)$ of f defined on the whole M. Denote by $d_i(f)$ the dimension of the bundle $E_i(f)$.

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[1] meaning $||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2} + \varepsilon$ for all x.

Theorem 2.9. ([BDPR, Theorem A and Lemma 4.1]) *There is an open and dense subset $\mathcal{I}$ of $\mathcal{RNT}$ consisting of diffeomorphisms f such that*

- *the numbers $s^+(f)$ and $s^-(f)$ are well defined, locally constant, and $s^-(f) < s^+(f)$,*
- *for every $i \in [s^-(f), s^+(f)] \cap \mathbb{N}$ there is a (hyperbolic) saddle of f of index i,*
- *the number $m(f)$ of bundles of the finest dominated splitting and the dimensions $d_i(f)$ of the bundles of the splitting are locally constant.*

The last assertion in Theorem 2.9 follows from the fact that any dominated splitting of f defined on the whole M has a continuation for any g close to f (see, for instance, [BDP, Lemma 1.4]) and the uniqueness of a dominated splitting fixed the number of sub-bundles, their dimensions, and the ordering of such dimensions, (see [BDPR, Lemma 4.1]). In fact, this assertion follows essentially from the clustering property in Definition 2.7.

We now pick $f \in \mathcal{I}$ and a small neighbourhood $\mathcal{U}_f$ of it where the numbers above are constant in $\mathcal{U}_f$, so we omit the dependence on g of $s^\pm(g)$, $m(g)$ and $d_i(g)$ on $g \in \mathcal{U}_f$. Finally, write $r_j = d_1 + \cdots + d_j$, $j = 1, \dots, m$.

Let $s^+ - s^- = k$ and consider saddles $Q_f^0, Q_f^1, \dots, Q_f^k$ such that the dimensions of the stable bundles are $s^-, (s^- + 1), \dots, s^+ = s^- + k$. Since the continuations of these saddles are defined on a neighbourhood of f, we also omit the dependence on f of them.

Lemma 2.10. *Let $f \in \mathcal{I}$ and $\mathcal{U}_f$ be a neighbourhood of f as above (i.e., $s^\pm(g)$, $m(g)$, and $d_i(g)$ are constant in $\mathcal{U}_f$). There is an open and dense subset $\mathcal{V}_f$ of $\mathcal{U}_f$ such that for every $g \in \mathcal{V}_f$ there is $i \in [s^-, s^+]$ such that there is g-invariant splitting $E \oplus F$ such that* $\dim(F)$ *is the index of Q_g^i.*

We postpone the proof of the lemma and prove Proposition 2.1.

Proof of Proposition 2.1: Lemma 2.10 implies that the diffeomorphisms g such that either g of g^{-1} verifies (simultaneously) (y1) and (y3) form an open and dense subset of $\mathcal{I}$. To see how this assertion follows note that if $i \in [s^-, s^+)$ then it is enough to take in (y1) the saddles $P_g = Q_i^g$ and $Q_g = Q_{i+1}^g$ and in (y3) the dominated splitting $E \oplus F$ given by Lemma 2.10. If $i = s^+$ consider g^{-1}, the saddles $Q_{s^+}^g$ and $Q_{s^+-1}^g$, and the dominated splitting $F \oplus E$ of g^{-1} (we need to reverse the ordering of the bundles to get a dominated splitting of g^{-1}). It is now enough to take in (y1) the saddles $P_{g^{-1}} = Q_{s^+}^g$ and $Q_{g^{-1}} = Q_{s^+-1}^g$ and in (y3) the dominated splitting $F \oplus E$. Noting that the index of $P_{g^{-1}}$ (for g^{-1}) is

$$n - \operatorname{index}(Q_g^{s^+}, g) = n - \dim(F) = \dim(E)$$

we get that $F \oplus E$ verifies the dimension condition (y3).

Finally, the density condition (y2) follows from Lemma 2.5, which assures that (locally) generically the invariant manifolds $W^s(Q_g^i, g)$ and $W^u(Q_g^i, g)$ are both dense in M. This completes the proof of the proposition. $\square$

Proof of the lemma: As the properties in Lemma 2.10 are open, it suffices to prove the density of the diffeomorphisms verifying the lemma. By Lemma 2.5, there is residual subset $\mathcal{J}_f$ of $\mathcal{U}_f$ such that every $g \in \mathcal{J}_f$ has a saddle P_g whose homoclinic class $H(P_g, g)$ is the whole manifold M. Thus the finest dominated splittings of Df over M and over $H(P_g, g)$ coincide. Moreover, by Remark 2.6, we also can assume the set Σ_{P_g} is dense in M. Since every dominated splitting defined on a set can be extended (in a dominated way) to its closure (recall Definition 2.7), the restriction of the finest dominated splitting of TM to Σ_{P_g} consists also of indecomposable sub-bundles. In other words, $T_{\Sigma_g}M = E_1(g) \oplus \cdots \oplus E_m(g)$ is a dominated splitting consisting of indecomposable sub-bundles.

For a given $g \in \mathcal{J}_f$ there are two possibilities:

Case (a): There are $j < m$ and $i = 0, \ldots, (k-1)$ such that $s^- + i = r_j$.

Case (b): for every j and every $i = 0, \ldots, (k-1)$, one has $r_j \neq s^- + i$.

In Case (a) the proof is trivial: just take the saddle Q^i and the g-invariant dominated splitting $E \oplus F$, $E = E_1 \oplus \cdots \oplus E_j$ and $F = E_{j+1} \oplus \cdots \oplus E_m$. By construction, $E^s(Q^i) = E$ and $E^u(Q^i) = F$, thus $\mathrm{index}(Q_g, g) = \dim(F)$.

Note that $s^+ < r_m = \dim(M)$ (otherwise, g has a sink, which prevents the transitivity of g). In Case (b) we have that for every diffeomorphism close to g:

$$\text{there is } 0 \leq j \leq (m-1) \text{ such that } r_j < s^- < \cdots < s^+ < r_{j+1} \text{ (here we let } r_0 = 0). \tag{2.1}$$

We now need the following results we export from [BDP, Propositions 2.4 and 2.5] (we use here the formulation in [BDPR] which is more convenient in our context) and [Fr].

Lemma 2.11. [BDPR, Lemma 5.3] . *Let P_f be a saddle of a diffeomorphism f whose homoclinic class $H(P_f, f)$ is non-trivial. Let $E_1 \oplus \cdots \oplus E_m$ be the restriction to Σ_{P_f} of the finest dominated splitting of $T_{H(P_f,f)}M$ of Df. Then, for every $\varepsilon > 0$, there are a dense subset Σ_ε of Σ_{P_f} and an ε-perturbation A_ε of Df preserving the splitting $E_1 \oplus \cdots \oplus E_m$ such that, for every saddle $R \in \Sigma_\varepsilon$, the restriction of the linear map*

$$M_{A_\varepsilon}(R) = A_\varepsilon(f^{n(R)-1}(R)) \circ \cdots \circ A_\varepsilon(f(R)) \circ A_\varepsilon(R)$$

to each bundle $E_i(R)$ is a homothety, where $n(R)$ is the period of the saddle R.

Moreover, if there are $i \in \{1, \ldots, m\}$ and $Q \in \Sigma_{P_f}$ such that the modulus of the Jacobian of the restriction of $f^{n(Q)}$ to $E_i(Q)$ is one then $R \in \Sigma_\varepsilon$ can be taken such that the restriction of $M_{A_\varepsilon}(R)$ to $E_i(R)$ is identity.

Lemma 2.12. ([Fr], [Mñ$_2$]) *Consider a $\mathcal{C}^1$-diffeomorphism f and an f-invariant finite set Σ. Let A be an ε-perturbation of the derivative Df of f along Σ (i.e., the linear maps $A(x)$ and $Df(x)$ are ε-close for all $x \in \Sigma$). Then for every neighbourhood U of Σ there is a diffeomorphism g C^1-ε-close to f such that*

- *$g(x) = f(x)$ if $x \in \Sigma$ or if $x \notin U$,*
- *$Dg(x) = A(x)$ for all $x \in \Sigma$.*

We are now ready to finish the proof of the lemma. Consider the bundle E_{j+1}, by Lemma 2.11, there are a saddle R of g and an arbitrarily small perturbation A of the derivative of g along the orbit of R such that the restriction of $M_A(R)$ to E_{j+1} is a homothety. After a new perturbation, we can take the ratio of this homothety different from 1. Applying Lemma 2.12, taking Σ equal to the orbit of R and considering the perturbation A, we get h close to g such that R is a periodic orbit of h with $Dh^{n(R)}(R) = M_A(R)$.

Let $T_R M = E^s(R) \oplus E^u(R)$ and note that for every vector $v \notin E^u$ one has $Dh^m(v) \to E^s(R)$ as $m \to \infty$ (similarly, if $v \notin E^s$ then $Dh^{-m}(v) \to E^u(R)$). The invariance of the bundles $E_i(R)$ and the fact that the restrictions of $Dh^{n(R)}$ to the bundles $E_i(R)$ are homotheties imply that either $E_i(R) \subset E^s(R)$ or $E_i(R) \subset E^u(R)$.

If the homothety $M_A(R)|_{E_{j+1}(R)}$ is a contraction then $E_{j+1}(R)$ is a stable bundle of R. Thus, by the domination, every $E_i(Q)$, $i = 1, \ldots, (j+1)$, is also contracting. Hence $E_1(R) \oplus \cdots \oplus E_{j+1}(R) \subset E^s(R)$. Therefore the dimension of the stable bundle of R is at least r_{j+1}. Thus $r_{j+1} \leq s^+$, contradicting (2.1).

Similarly, if the homothety $M_A(R)|_{E_{j+1}(R)}$ is an expansion then $E_{j+1}(R)$ is a unstable bundle of R. Thus, by the domination, every $E_i(Q)$, $i = (j+1), \ldots, m$, is also expanding. Hence $E_{j+1}(R) \oplus \cdots \oplus E_m(R) \subset E^u(R)$. Therefore $E^s(R) \subset E_1(R) \oplus \cdots \oplus E_j(R)$. Thus $\dim(E^s(R)) \leq r_j$. Hence $s^- \leq r_j$, contradicting (2.1). This ends the proof of the lemma. □

3 Partially hyperbolic heterodimensional cycles

In this section we prove Theorems 2 and 3.

3.1 Creation of partially hyperbolic heterodimensional cycles

Denote by $\mathcal{P}$ the subset of $\mathcal{RNT}$ of diffeomorphisms g satisfying the following conditions:

(p1) There are saddles A_g and B_g of the same period k of indices m_s and $m_s + 1$, respectively, and an open curve γ_g (called *connection*) whose extremes are A_g and B_g such that $\gamma_g \subset W^s(A_g) \pitchfork W^u(B_g)$ and $g^k(\gamma_g) = \gamma_g$.

(p2) There is a small neigbourhood U of the orbit of the closure of γ_g, $\overline{\gamma_g} = \{A_g, B_g\} \cup \gamma_g$, and a g-invariant dominated splitting defined of U

$$E^s_1(g) \oplus \cdots \oplus E^s_{m_s}(g) \oplus E^c(g) \oplus E^u_1(g) \oplus \cdots \oplus E^u_{m_u}(g)$$

such that the bundles $E^s_i(g)$ $(i = 1, \ldots, m_s)$, $E^c(g)$, and $E^u_j(g)$ $(j = 1, \ldots, m_u)$ are one dimensional and

$$\begin{aligned}
&E^s(A_g) = E^s_1(g) \oplus \cdots \oplus E^s_{m_s}(g) \oplus E^c(g),\\
&E^u(B_g) = E^c(g) \oplus E^u_1(g) \oplus \cdots \oplus E^u_{m_u}(g),\\
&T_{\gamma_g} M = E^c(g).
\end{aligned}$$

Theorem 3.1. *The set $\mathcal{P}$ is open and dense in $\mathcal{RNT}$.*

First note that the set $\mathcal{P}$ is open (transverse intersections and dominated splittings are persistent). Thus the point is the density of $\mathcal{P}$. The proof of the theorem has three steps:

1. Consider the open subset $\mathcal{O}$ of $\mathcal{RNT}$ consisting of diffeomorphisms f having a pair of (hyperbolic) periodic saddles P_f and Q_f such that:

   (o1) $\operatorname{index}(P_f, f) + 1 = \operatorname{index}(Q_f, f)$,

(o2) all the eigenvalues of $Df^{n(P_f)}(P_f)$ and $Df^{n(Q_f)}(Q_f)$ ($n(R_f)$ is the period of R_f) are real, have multiplicity one, and their moduli are different.

We prove that set $\mathcal{O}$ is dense in $\mathcal{RNT}$ (Proposition 3.2).

2. Lemma 3.5 claims that every f in $\mathcal{O}$ can be C^1-approximated by some g in $\mathcal{RNT}$ having a saddle-node S_g such that all the eigenvalues of $Dg^{n(S_g)}(S_g)$ are different and positive and have multiplicity one. Thus the splitting given by the sum of the eigenspaces of S_g is dominated.
3. Theorem 3.1 follows by unfolding the saddle-node S_g to get a pair of hyperbolic saddles A_g and B_g as in the theorem, considering the segment $\gamma_g = (A_g, B_g)$ tangent to the central direction of the saddle-node, and using that a dominated splitting is persistent (recall Definition 2.7).

Proposition 3.2. *The set $\mathcal{O}$ is open and dense in $\mathcal{RNT}$.*

Proof: Since the conditions in the definition of the set $\mathcal{O}$ are open, it is enough to prove the density of the set $\mathcal{O}$ in $\mathcal{RNT}$.

Lemma 3.3. *There is an open and dense subset $\mathcal{K}$ of $\mathcal{RNT}$ of diffeomorphisms f with the following property: there are $p \in \mathbb{N}$ and saddles P_f and Q_f of f of indices p and $p+1$ such that $H(P_f, f)$ and $H(Q_f, f)$ are non-trivial.*

Proof: Since the existence of a pair of saddles P_f and Q_f of f of indices p and $p+1$ such that $H(P_f, f)$ and $H(Q_f, f)$ are both non-trivial is an open property (this follows from the persistence of hyperbolic saddles and of transverse intersections of invariant manifolds), it suffices to prove the density of the set $\mathcal{K}$ in $\mathcal{RNT}$.

Consider the open and dense subset $\mathcal{I}$ of $\mathcal{RNT}$ of diffeomorphisms f such that the indices of the saddles of f form an interval in $\mathbb{N}$ given by Theorem 2.9. By the definition of the set $\mathcal{RNT}$, we can assume that f has saddles P_f and Q_f of indices p and $p+1$ for some p. Consider a small neighbourhood $\mathcal{U}$ of f where the continuations of these saddles are defined and the residual subset $\mathcal{J} = \mathcal{J}_P \cap \mathcal{J}_Q$ of $\mathcal{U}$, where $\mathcal{J}_P$ and $\mathcal{J}_Q$ are the residual subsets of $\mathcal{U}$ given by Lemma 2.5, i.e., $H(R_g, g) = M$ if $g \in \mathcal{J}_R$. Then every $g \in \mathcal{J}$ verifies Lemma 3.3. □

We need the following result:

Lemma 3.4 ([BDP]). *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(M)$ such that every f in $\mathcal{U}$ has a saddle P_f whose homoclinic class is non-trivial (here the saddle P_f depends continuously on f).*

There is a residual subset $\mathcal{D}$ of $\mathcal{U}$ of diffeomorphisms g such that the homoclinic class of P_g contains a saddle R_g of the same index as P_g such that every eigenvalue of $Dg^{n(R_g)}(R_g)$ is real and positive and has multiplicity one.

This lemma is just a reformulation of the results in [BDP], where Lemma 3.4 is formulated for *periodic linear systems (cocyles)* $(\Sigma, f, \mathcal{E}, A)$ *with transitions* (here f is a diffeomorphisms, Σ a set of periodic points of f, $\mathcal{E}$ an Euclidean bundle over Σ, and $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ so that for each $x \in \Sigma$ the map $A(x)$ is a linear isomorphism $A(x) \colon \mathcal{E}_x \to \mathcal{E}_{f(x)}$). See [BDP, Section 1] for the precise definition.

[BDP, Lemma 4.16] asserts that if $(\Sigma, f, \mathcal{E}, A)$ is a periodic linear system with transitions then, for every $\varepsilon > 0$, there is a *diagonalizable* ε-perturbation A' of A defined on a dense subset of Σ' of Σ. By a diagonalizable system we mean that if x is a periodic orbit of period k then all the eigenvalues of the linear map $M_A(x) = A(f^{k-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x) \colon \mathcal{E}_x \to \mathcal{E}_x$ are real and positive and have multiplicity one. Lemma 3.4 now follows from the next two remarks:

- Let $H(P_f, f)$ be the homoclinic class of a saddle P_f of index k. Then the derivative of f induces a periodic linear system with transitions in Σ_{P_f} (recall that $\Sigma_{P_f} \subset H(P_f, f)$ is the set of saddles of index k homoclinically related to P_f). See [BDP, Lemma 1.9].
- By the previous item, we can apply [BDP, Lemma 4.16] to the dense subset Σ_{P_f} of $H(P_f, f)$ to obtain a dense subset Σ' of Σ_{P_f} such that for each $R_f \in \Sigma'$ there is an ε-perturbation A of Df along the finite orbit of R_f such that every eigenvalue of $M_A(R_f) = A(f^{n(R_f)-1}(R_f)) \circ \cdots \circ A(f(R_f)) \circ A(R_f)$ is real and positive and has multiplicity one. Using Lemma 2.12, we can perform the previous perturbation dynamically: there is g close to f such that $R_f = R_g$ is a periodic point of g of period $n(R_g) = n(R_f)$ with $Dg^{n(R_g)}(R_g) = M_A(R_f)$. Finally, using that R_f and P_f are homoclinically related, one has that the same holds for P_g and $R_g = R_f$. Note that in our case the fact that R_g is in the homoclinic class of P_g immediately follows from the robust transitivity.

Lemmas 3.3 and 3.4 imply immediately Proposition 3.2. □

Let $\mathcal{D}$ be the subset of $\mathcal{RNT}$ of diffeomorphisms g having a saddle-node S_g such that:

(d1) every eigenvalue λ of $Df^{n(S_g)}(S_g)$ is real and positive, and has multiplicity one,

(d2) 1 is an eigenvalue of $Df^{n(S_g)}(S_g)$,

(d3) there is a pair of eigenvalues λ and β of $Df^{n(S_g)}(S_g)$ with $0 < \lambda < 1 < \beta$.

Proposition 3.5. *The set $\mathcal{D}$ is dense in $\mathcal{RNT}$.*

Proof: For a given $f \in \mathcal{O}$ (the open and dense subset of $\mathcal{RNT}$ in Proposition 3.2), let A_f and B_f be the saddles in the definition of $\mathcal{O}$ satisfying (o1) and (o2). Let $0 < \lambda_1 < \cdots < \lambda_{p+1} < 1 < \lambda_{p+2} < \cdots < \lambda_n$ be the eigenvalues of $Df^{n(P_f)}(P_f)$ and consider the dominated splitting $T_{P_f}M = E_1(P_f) \oplus \cdots \oplus E_n(P_f)$ given by the one-dimensional eigenspaces of $Df^{n(P_f)}(P_f)$. Similarly, $0 < \beta_1 < \cdots < \beta_p < 1 < \beta_{p+1} < \cdots < \beta_n$ are the eigenvalues of $Df^{n(Q_f)}(Q_f)$ and $T_{Q_f}M = E_1(Q_f) \oplus \cdots \oplus E_n(Q_f)$ is the splitting given by the eigenspaces of $Df^{n(Q_f)}(Q_f)$. By Remark 2.4, after an arbitrarily small C^1-perturbation, we can assume that f has heterodimensional cycle associated to A_f and B_f. This cycle has a dominated structure in the saddles exactly as in [BDPR, Theorem 3.1] (in fact, here we have even more: all the bundles of the dominated splittings of the saddles in the cycle are one-dimensional). In fact, [BDPR, Theorem 3.1] can be read as follows (see also the proof of [BDPR, Corollary 3.6]):

Lemma 3.6. *Let $f \in \mathcal{O}$ such that the saddles A_f and B_f has a heterodimensional cycle. Then there are linear maps T_1 and T_2 preserving the dominated splittings of A_f and B_f*

$$T_1(E_i(P_f)) = E_i(Q_f) \quad \text{and} \quad T_2(E_i(Q_f)) = E_i(P_f), \qquad \text{for all } i = 1, \ldots, n,$$

such that for every m and n large enough there is a diffeomorphism g close to f having a saddle S_g of period $n(S_g) \geq n \cdot n(P_f) + m \cdot n(Q_f)$ such that $Dg^{n(S_g)}(S_g)$ is arbitrarily close[2] *to*

$$T_1 \circ (Df^{n(P_f)}(P_f))^n \circ T_2 \circ (Df^{n(Q_f)}(Q_f))^m.$$

---

[2] In this statement there is an abuse of notation: we can take the saddle S_g nearby Q_g in such a way that a continuation of the dominated splitting of Q_g is defined in S_g and thus the previous expression makes sense.

As a consequence, the central eigenvalue of $Dg^{n(S_g)}(S_g)$ (the $(p+1)$-th eigenvalue λ_c) is arbitrarily close to

$$C \cdot \lambda_{p+1}^n \cdot \beta_{p+1}^m,$$

for some constant C independent of n and m.

The last equation in Lemma 3.6 means that we can select large n and m such that $C_1 < \log(\lambda_c) < C_2$, for some constants C_1 and C_2 independent of n and m. Taking large n and m, this implies that S_g can be chosen such that its $(p+1)$-th Lyapunov exponent is arbitrarily close to zero. Thus, after a new perturbation, we can assume that $\lambda_c = 1$, thus S_g is a saddle node. For details, see the proof of [BDPR, Corollary 3.6], specially the equation in [BDPR, page 203] and observe that we are considering cycles whose saddles only have positive eigenvalues. This completes the proof of Proposition 3.5. $\square$

Remark 3.7. *Proposition 3.5 implies a weaker version of Theorem 2: there is a dense subset of $\mathcal{RNT}$ of non-shadowable diffeomorphisms. It is enough to take $f \in \mathcal{D}$ having a partially hyperbolic saddle-node S with one-dimensional center-stable $W^{cs}(S)$ and center-unstable $W^{cu}(S)$ manifolds (these manifods are tangent to the central direction and have the saddle-node as an extreme). Consider $x \in W^{cs}(S)$ and a one-jump pseudo-orbit (z_n) of the form $z_n = f^n(x)$ for $n = 0, \dots, k$, $z_{k+1} = y$ for some $y \in W^{cu}(S)$, and $z_{k+1+m} = f^m(y)$. Fixed any $\varepsilon > 0$, one gets an ε-pseudo orbit by taking big k and y close to the saddle-node. Due to the partial hyperbolicity of the saddle-node, this pseudo-orbit can not be shadowed.*

3.1.1 End of the proof of Theorem 3.1

Given $g \in \mathcal{D}$ consider the saddle-node S_g satisfying conditions (d1)–(d3). Let $0 < \lambda_1 < \cdots < \lambda_{m_s} < 1 = \lambda_c < \beta_1 < \cdots < \beta_{m_u}$ be the eigenvalues of $Dg^{n(S_g)}(S_g)$. Consider the dominated splitting along the orbit $O(S_g)$ of S_g given by

$$T_{O(S_g)}M = (E_1^s(g) \oplus \cdots \oplus E_{m_s}^s(g)) \oplus E^c(g) \oplus (E_1^u(g) \oplus \cdots \oplus E_{m_u}^u(g)),$$

where $E_i^s(g)$, $E^c(g)$, and $E_i^u(g)$ are the (one-dimensional) eigenspaces associated to λ_i, 1, and β_i. Note that $E^s(g) = E_1^s(g) \oplus \cdots \oplus E_{m_s}^s(g)$ is the stable bundle of S_f, $E^u(g) = E_1^u(g) \oplus \cdots \oplus E_{m_u}^u(g)$ is the unstable bundle of S_g, and $E^c(g)$ is the central direction.

Using the dominated splitting of the saddle-node S_g and the fact that a dominated splitting admits an extension (see, Definition 2.7), we can perform a local perturbation of g throughout the finite orbit of S_g to get a diffeomorphism f verifying (p1) and (p2): it suffices to obtain the saddles A_f and B_f in a normally hyperbolic local central manifold of S_g in such a way γ_f is a center stable manifold of A_f and a center unstable manifold of B_f tangent to the central direction (see Figure 1). This ends the proof of the theorem. $\square$

3.2 Density of partially hyperbolic heterodimensional cycles in $\mathcal{RNT}$

Denote by $\mathcal{H}$ the subset of $\mathcal{P}$ such that the saddles A_f and B_f verifying (p1) and (p2) are related by a heterodimensional cycle as follows:

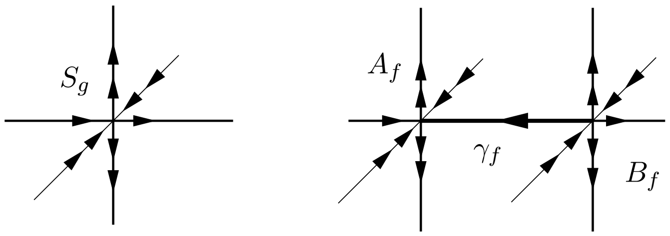

Figure 1: Unfolding a saddle-node

(p3) The unstable manifold of A_f and the stable one of B_f meet throughout the orbit of a heteroclinic point x_f:

$$W^u(A_f)\cap W^s(B_f)\supset \bigcup_{i\in\mathbb{Z}} f^i(x_f).$$

Considering some forward iterate of x_f, we can assume that the point x_f belongs to open set U where is defined the dominated splitting in the definition of $\mathcal{P}$. Moreover, the intersection between $W^u(A_f)$ and $W^s(B_f)$ at x_f is quasi-transverse, meaning that

$$T_{x_f}W^u(A_f)+T_{x_f}W^s(B_f)=T_{x_f}W^u(A_f)\oplus T_{x_f}W^s(B_f)$$

and

$$T_{x_f}M=T_{x_f}W^u(A_f)\oplus T_{x_f}W^s(B_f)\oplus E^c(f)(x_f).$$

Lemma 3.8. *The set $\mathcal{H}$ is dense in $\mathcal{RNT}$.*

Proof: By Theorem 3.1, it is enough to prove the density of $\mathcal{H}$ in $\mathcal{P}$. Let $f\in\mathcal{P}$, since f is transitive and $\mathcal{P}$ is open, by Lemma 2.2, there is $h\in\mathcal{P}$ arbitrarily C^1-close to f such that $W^u(A_h)\cap W^s(B_h)\neq\emptyset$. Thus, since $W^s(A_h)\pitchfork W^u(B_h)\neq\emptyset$, h has a heterodimensional cycle associated to A_h and B_h. By a transversality argument, we can assume (after a new perturbation if necessary) that this intersection occurs along the orbit of a point x_h and that $T_{x_h}W^u(A_h)$ and $T_{x_h}W^s(B_h)$ are in general position. A new transversality argument assures that we can assume that $T_{x_h}W^u(A_h)\oplus T_{x_h}W^s(A_h)$ and $E^c(h)(x_h)$ are in general position, thus $T_{h_f}M=T_{h_f}W^u(A_h)\oplus T_{x_h}W^s(B_h)\oplus E^c(h)(x_h)$. $\square$

3.3 Unfolding partially hyperbolic heterodimensional cycles

Consider $f\in\mathcal{H}$, the saddles A_f and B_f (say of indices p and $p+1$), the connection γ_f, and the heteroclinic point x_f in the definitions of the sets $\mathcal{P}$ and $\mathcal{H}$. Take a small neigbourhood W of the cycle as follows: consider the open set U containing the orbits of A_f, B_f, and γ_f where the dominated splitting $E^s(f)\oplus E^c(f)\oplus E^u(f)$ in (p2) is defined and a neighbourhood U_0 of x_f, then

$$W=U\cup(\cup_{i=-N}^{N}f^i(U_0)),$$

where $N\in\mathbb{N}$ is such that $f^i(x_f)\in U$ for all $|i|\geq N$ (in fact, without loss of generality we can assume that $x_f\in U$ and its whole forward orbit is contained in U).

Next theorem is a reformulation (in fact, a weaker version stated in a more suitable form for our goals) of the results in [DR] on bifurcations of partially hyperbolic heterodimensional cycles.

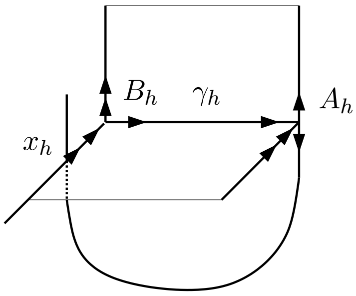

Figure 2: A partially hyperbolic heterodimensional cycle

Theorem 3.9 ([DR]). *Given any* $f \in \mathcal{H}$ *there is an open set* $\mathcal{B}$ *in* $\mathrm{Diff}^1(M)$ *whose closure contains* f *such that, for every* $g \in \mathcal{B}$*, the unstable manifold* $W^u(A_g, g)$ *of* A_g *accumulates the local stable manifold* $W^s_{loc}(B_g, g)$ *of* B_g *as follows: there are* $r > 0$ *and sequences* $(x_n)_n$ *of transverse homoclinic points of* A_g*,* $x_n \in W^u(A_g, g) \pitchfork W^s(A_g, g)$*, and* $(\Delta^u(x_n))_n$ *of disks* $\Delta^u(x_n) \subset W^u(A_g, g)$ *(of the same dimension as* $W^u(A_g, g)$*) centered at* x_n *such that:*

- (x_n) *converges to some point* $x_\infty \in W^s_{loc}(B_g, g)$*,* $x_\infty \neq B_g$*,*
- *the radius of each disk* $\Delta^u(x_n)$ *is greater than* r*,*
- *the tangent direction of* $\Delta^u(x_n)$ *is* ϵ*-close to* $E^u(g)$ *(the number* ε *can be taken arbitrarily small by reducing the sizes of* $\mathcal{B}$ *and* U*),*
- $\Delta^u(x_n) \cap W^s_{loc}(B_g, g) = \emptyset$*,*
- $\Delta^u(x_n) = g^{k_n}(\delta^u(n))$*, where* $k_n > 0$*,* $\delta^u(n)$ *is a disk contained in* $W^u_{loc}(A_g) \subset U$ *such that* $\delta(n), g(\delta(n)), \dots, g^{k_n}(\delta(n))$ *are contained in the neighbourhood* W *of the cycle.*

[DR, Theorem A] states that if f has a (partially hyperbolic) heterodimensional cycle as the ones above, then there is a C^1-open set $\mathcal{U}_f$ of diffeomorphisms, whose closure contains f, such that for every $g \in \mathcal{U}_f$ the homoclinic classes of A_g and B_g coincide. The main step to prove this theorem is to see that the there is an open set $\mathcal{V}_f$ of diffeomorphisms, $f \in \overline{\mathcal{V}_f}$, such that for every $g \in \mathcal{V}_f$ the closure of $W^u(A_g, g)$ (of dimension p) contains the whole $W^u(B_g, g)$ (of dimension $(p+1)$). The proof of this result involves a blender argument (see [BD$_1$, DR] and also [BDV, Chapter 6.2] for an expository explanation of blenders). The proof of this fact consists in showing the existence of a cube $C \subset U$ around a fundamental domain of the curve γ_g (the continuation of the connection γ_f for g close to f) endowed with a cone-field structure $\mathcal{C}^u$,$\mathcal{C}^{cu}$, $\mathcal{C}^{cs}$, and $\mathcal{C}^s$, corresponding to the bundles $E^u(g)$, $E^c(g) \oplus E^u(g)$, $E^c(g) \oplus E^s(g)$, and $E^s(g)$, such that every $(p+1)$-dimensional strip S, tangent to the cone-field $\mathcal{C}^{cs}$ and crossing the two sides of the cube C parallel to $\mathcal{C}^{cu}$, transversely intersects a disk Δ^u ($\Delta^u \pitchfork S \neq \emptyset$) contained $W^u(A_g, g) \cap C$ and tangent to $\mathcal{C}^u$. Moreover, this construction only involves the semi-global dynamics of g in the neighbourhood W of the cycle. The previous statement essentially corresponds to [DR, Proposition 3.6].

We next detail the construction of the points x_n and the disks $\Delta^u(x_n)$ in Theorem 3.9.

3.3.1 Itineraries in the neighbourhood of the cycle

To define exactly the disks in Theorem 3.9, we need to choose carefully the neighbourhood of the cycle. For the sake of simplicity, suppose that A_f and B_f are fixed points We assume that $W^u_{loc}(B_f, f)$ and $W^s_{loc}(A_f, f)$ are contained in the neighbourhood U of $\overline{\gamma_f}$. There is a first k_0 such that $f^{k_0}(W^u_{loc}(A_f, f)) \cap W^s_{loc}(B_f, f) \neq \emptyset$ and this intersection is just the heteroclinic point x_f in condition (p3) in the definition of $\mathcal{H}$ (we assume that x_f is in the interior of $f^{k_0}(W^u_{loc}(A_f, f))$). We take a small neighbourhood U_0 of x_f such that $U_0 \subset U$ and, for every g close to f, $g^{-k_0}(U_0) \subset U$ and the compact sets

$$K_0(g) = \overline{U}, \quad K_1(g) = g^{-1}(\overline{U_0}), \quad K_2(g) = g^{-2}(\overline{U_0}), \quad \dots, \quad K_{k_0-1}(g) = g^{-k_0+1}(\overline{U_0})$$

are pairwise disjoint. There is $\varepsilon_1 > 0$ such that

$$d(K_i(g), K_j(g)) > \varepsilon_1 \quad \text{for every } g \text{ close to } f \text{ and } i \neq j. \tag{3.2}$$

We now consider the following neighbourhood W of the cycle,

$$W = U \cup f^{-1}(U_0) \cup f^{-2}(U_0) \cup \dots \cup f^{-k_0+1}(U_0).$$

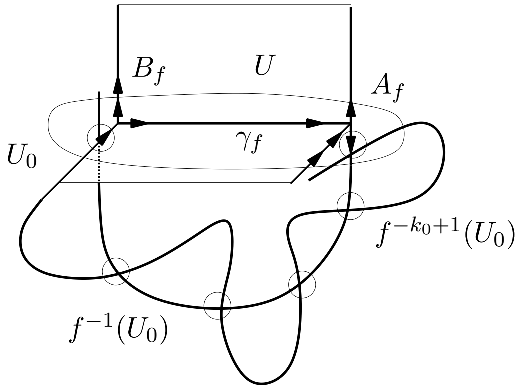

Figure 3: Neighbourhood of the cycle

Next step is to unfold the cycle associated to A_f and B_f, considering g close to f such that there is a continuation x_g of the heteroclinic point x_f which is a transverse homoclinic point of A_g (see Figure 2). Take a point $x \in W^u_{loc}(A_g) \subset U$ and a segment of its orbit $x, g(x), \dots, g^m(x)$ in W such that $g^m(x) \in U$. We split this segment of orbit as follows: there are $m_1(x), \dots, m_\ell(x)$ such that

- $x, g(x), \dots, g^{m_1(x)}(x) \in U$,
- $g^{m_1(x)+1}(x) \in g^{-k_0+1}(U_0), \quad \dots, \quad g^{m_1(x)+k_0-1}(x) \in g^{-1}(U_0), \quad g^{m_1(x)+k_0}(x) \in U_0 \subset U$,
- $g^j(x) \in U$ for all $j \in [m_1(x) + k_0, m_2(x)]$.

Inductively, for $i = 2, \dots, \ell$,

- $g^{m_i(x)+1}(x) \in g^{-k_0+1}(U_0), \quad \ldots, \quad g^{m_i(x)+k_0-1}(x) \in g^{-1}(U_0), \quad g^{m_i(x)+k_0}(x) \in U_0 \subset U$,
- $g^j(x) \in U$ for all $j \in [m_i(x)+k_0, m_{i+1}]$ if $i < \ell$, if $i = \ell$ it holds $g^j(x) \in U$ for all $j \in [m_\ell(x)+k_0, m]$.

Take points $x, y \in W^u_{loc}(A_g, g)$ and $m > 0$ such that the segments of orbits $x, g(x), \ldots, g^m(x)$ and $y, g(y), \ldots, g^m(y)$ are contained in W but not in U, and $g^m(x), g^m(y) \in U$. Consider the numbers $m_1(x), \ldots, m_{\ell(x)}(x)$ and $m_1(y), \ldots, m_{\ell(y)}(y)$ defined as above. The choice of x and y (i.e., the segments of orbits are not contained in U) implies that $\ell(x), \ell(y) \geq 1$ and $m_1(x) \neq m \neq m_1(y)$.

Lemma 3.10. *Consider x and y as above. Suppose that there is j such that $m_j(x) \neq m_j(y)$. Then $d(g^k(x), g^k(y)) > \varepsilon_1$ for some $k \in \{0, \ldots, m\}$, where ε_1 is as in equation (3.2).*

Proof: Take a first j with $m_j(x) \neq m_j(y)$ and suppose that $m_j(x) > m_j(y)$. By definition,

$$g^{m_j(x)+1}(x) \in g^{-k_0+1}(U_0) \subset K_{k_0-1}(g).$$

Since $m_{j-1}(y) = m_{j-1}(x) < m_j(x)+1 \leq m_j(y)$ (if $j = 1$ we let $m_0(y) = m_0(x) = 0$),

$$g^{m_j(x)+1}(y) \in U \subset K_0(g).$$

The lemma follows from equation (3.2). □

3.3.2 The disks $\Delta^u(x_n)$

We are now ready to explain the construction of the transverse homoclinic points x_n and the disks $\Delta^u(x_n)$. The homoclinic points x_n in Theorem 3.9 have their full orbit contained in the neighbourhood W of the cycle. One can take these points in U_0 and for each x_n consider its first backward iterate $\bar{x}_n$ in $W^u_{loc}(A_g, g)$, $\bar{x}_n = g^{-k_n}(x_n) \in W^u_{loc}(A_g, g)$, and the numbers $m_1(\bar{x}_n), \ldots, m_\ell(\bar{x}_n)$ corresponding to the segment of orbit of the first k_n iterates of $\bar{x}_n$. The set $\delta^u(n)$ is the set of points $z \in W^u_{loc}(A_g, g)$ such that

- $g^i(z) \in W$ for all $i = 0, \ldots, k_n$,
- $\ell(z) = \ell(\bar{x}_n)$ and $m_i(\bar{x}_n) = m_i(z)$ for all $i = 1, \ldots, \ell(\bar{x}_n)$.

Then

$$\Delta^u(x_n) = g^{k_n}(\delta^u(n)).$$

3.3.3 Proof of Theorem 2

For notational simplicity, we will explain the proof of Theorem 2 when A_g and B_g are fixed points (for the general case it suffices to take a power of g). Next remark is an immediate consequence from the hyperbolicity of A_g and B_g. Define $W^{s,u}_{loc}(C_g, g)$, $C = A, B$, as the connected component of $W^{s,u}(C_g, g) \cap D_\delta(C_g)$ containing C_g, here $D_\delta(C_g)$ is the ball of radius small δ centered at C_g.

Remark 3.11. *Let $C = A, B$. There is $\varepsilon_0 > 0$ such that:*

- *if $y \in D_\delta(C_g)$ and $y \notin W^s_{loc}(C_g, g)$ then there is $n \geq 0$ such that $d(g^n(y), W^s_{loc}(C_g, g)) > \varepsilon_0$;*

- *if $y \in D_\delta(C_g)$ and $y \notin W^u_{loc}(B_g, g)$ then there is $n \geq 0$ such that $d(g^{-n}(y), W^u_{loc}(C_g, g)) > \varepsilon_0$.*

Take x_n as in Theorem 3.9 close to x_∞ and the one-jump pseudo-orbit $(z_n)_n$ defined by:

$$z_k = g^k(x_n), \quad k \leq 0, \quad z_k = g^k(x_\infty), \quad k \geq 0.$$

Lemma 3.12. *The pseudo-orbit $(z_n)_n$ can not be shadowed by a true orbit.*

Proof: The proof is by contradiction, suppose that there is a true orbit $(g^k(y))_k$ that ε-shadows $(z_k)_k$ for some small ε. Remark 3.11 immediately implies that (if ε is small enough)

$$y \in W^s_{loc}(B_g, g) \quad \text{and} \quad g^{-k_n}(y) \in W^u_{loc}(A_g, g). \tag{3.3}$$

Therefore, $y \in W^s(B_g, g) \cap W^u(A_g, g)$.

Remark 3.13. *The fact $y \in W^s(B_g, g) \cap W^u(A_g, g)$ implies that if g is Kupka-Smale, then the pseudo-orbit $(z_n)_n$ can not be shadowed. The genericity of the Kupka-Smale diffeomorphisms implies a weaker version of Theorem 2: there is a residual subset of $\mathcal{RNT}$ of non-shadowable diffeomorphisms.*

We are now ready to finish the proof of Lemma 3.12, which is by contradiction. Fix $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ and take a homoclinic point x_n at distance less than ε from x_∞. Then $(z_k)_k$ is an ε-pseudo-orbit. Assume that the g-orbit of y δ-shadows z_n if $\delta < \min\{\varepsilon_0, \varepsilon_1\}$. By equation (3.3) and the definition of k_n,

$$g^{-k_n}(y) = \bar{y} \in W^u_{loc}(A_g, g) \quad \text{and} \quad g^{-k_n}(x_n) = \bar{x}_n \in W^u_{loc}(A_g, g).$$

Consider the segments of orbits of $\bar{y}$ and $\bar{x}_n$ corresponding to the first k_n iterates of $\bar{y}$ and $\bar{x}_n$ and the numbers $m_1(\bar{x}_n), \ldots, m_{\ell(\bar{x}_n)}(\bar{x}_n)$ and $m_1(\bar{y}), \ldots, m_{\ell(\bar{y})}(\bar{y})$ defined as above. Lemma 3.10 implies that $\ell = \ell(\bar{x}) = \ell(\bar{y})$ and $m_i(\bar{x}) = m_i(\bar{y})$ for all $i = 0, \ldots, \ell$ (otherwise, $d(f^j(y), z_j) > \varepsilon_1 > \varepsilon$ for some j). Therefore, by the definition of $\delta^u(n)$, $\bar{y} \in \delta^u(n)$. Thus $g^{k_n(\bar{y})} = y \in \Delta^u(x_n)$. Since $y \in W^s_{loc}(B_g, g)$ and $\Delta^u(x_n) \cap W^s_{loc}(B_g, g) = \emptyset$, we obtain a contradiction. $\square$

3.3.4 Proof of Theorem 3

The proof of this theorem involves the ingredients used to prove Theorem 2 plus some generic machinery. We now explain how the proof of Theorem 2 may be modified so as to yield Theorem 3.

Let U be an open set in M and f be a generic diffeomorphism[3] such that the set $\Lambda_f = \Lambda_f(U) \equiv \cap_{i\in\mathbb{Z}} f^i(\overline{U}) \subset U$ is transitive and non-hyperbolic. (If Λ_f were hyperbolic, one would of course have the shaowing property in a sufficiently small neighborhood of Λ_f.)

- By Theorem B of [Ab2] there is a neighborhood $\mathcal{V}$ of f in $\mathrm{Diff}^1(M)$, an arbitrarily small neighborhood (which we still call U) of Λ_f in M, and a residual subset $\mathcal{R}$ of $\mathcal{V}$ such that if $g \in \mathcal{R}$ then the set $\Lambda_g(U) = \cap_{i\in\mathbb{Z}} g^i(\overline{U}) \subset U$ is transitive and non-hyperbolic[4].

---

[3] When we say that f is a "generic diffeomorphism", we mean that f simultaneously satisfies all of the generic properties listed in the remainder of the proof. So we are doing the proof backwards: a more formal proof would begin with "Let $\mathcal{R}$ be the residual subset of $\mathrm{Diff}^1(M)$ obtained by intersecting the residual sets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$ of diffeomorphisms having respectively the following properties...".

[4] Theorem B of [Ab2] actually deals with attracting sets, but the same arguments yield the result in the case of locally maximal transitive sets

- Now, the transitive set $\Lambda_g(U)$ of the generic diffeomorphism g is a *relative homoclinic class* in U of some periodic point $P_g \in U$. By the *relative homoclinic class* $H(P_g, g, U)$ in U we mean the set of points of the homoclinic class $H(P_g, g)$ whose full orbits are contained in U. This result is stated in [BD$_2$, Proposition 2.4] for robustly transitive sets, but the arguments can be carried out in the same way for locally maximal transitive sets of generic diffeomorphisms.

- We also have that for generic g the set $\Lambda_g(U)$ contains two saddles P_g and Q_g of consecutive indices and having only real positive multipliers of multiplicity one. This is just a reformulation of [BDPR, Theorem A] for the generic case (in fact, the proof is the same as in the robust case). Theorem 2.9 formulates this result in the special case of robustly transitive diffeomorphisms.

- Now, using the Connecting Lemma (Lemma 2.2), we create a cycle associated to these saddles. Unfolding the cycle and using Lemma 3.6, we get $h \in \mathcal{V}$ arbitrarily close to g having a saddle-node in $\Lambda_h(U)$ (a priori h might not belong to $\mathcal{R}$).

- Exactly as in the proof of Theorem 2, we unfold the saddle-node to obtain (after an arbitrarily small C^1 perturbation) $\ell \in \mathcal{V}$ having two saddles (with different indices) A_ℓ and B_ℓ bounding a (normally hyperbolic) periodic segment γ_ℓ such that its (finite) orbit is contained in $\Lambda_\ell(U)$. Now, since this configuration is robust under C^1 perturbations, we can after another small perturbation assume that $\ell \in \mathcal{R}$.

- We now use the transitivity of $\Lambda_\ell(U)$ and the Connecting Lemma to get a dominated heterodimensional cycle associated to A_h and B_h. We then proceed exactly as in the proof of Theorem 2.

- Once this is done, we have achieved the following: arbitrarily near the generic diffeomorphism f we have created an open subset $\mathcal{W}$ of $\mathrm{Diff}^1(M)$ such that every $g \in \mathcal{W}$ exhibits arbitrarily fine pseudo-orbits *inside* U (because the pseudo-orbits occur arbitrarily near the dominated heterodimensional cycle, which is contained in U) which cannot be shadowed by actual orbits. Simple generic arguments (see for instance the proof of Theorem A in [Ab2]) now allow us to show that given a generic diffeomorphism f with a non-hyperbolic locally maximal transitive set Λ, then f is *contained* in some neighborhood $\mathcal{W}$ as above, that is, in a neighborhood consisting of diffeomorphisms g which exhibit non-shadowable pseudo-orbits arbitrarily near Λ.

This completes the sketch of the proof of Theorem 3.

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Collisions of Hyperbolic Homoclinic Classes

ictp 2004, trieste, italy

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(I) HETERODIMENSIONAL CYCLES

$f\colon M \to M$, dim $M \geq 3$.

P hyperbolic saddle of f, index of P,
ind(P) = dimension of the unstable bundle.

P, Q hyperbolic periodic saddles of f

dimension variability: $\boxed{\text{ind}(P) < \text{ind}(Q)}$

$\boxed{\text{heterodimensional cycle}}$ (P and Q)

$$\underbrace{W^s(P) \cap W^u(Q)}_{\textbf{(a)}} \neq \emptyset \neq \underbrace{W^u(P) \cap W^s(Q)}_{\textbf{(b)}}$$

Generic situation:

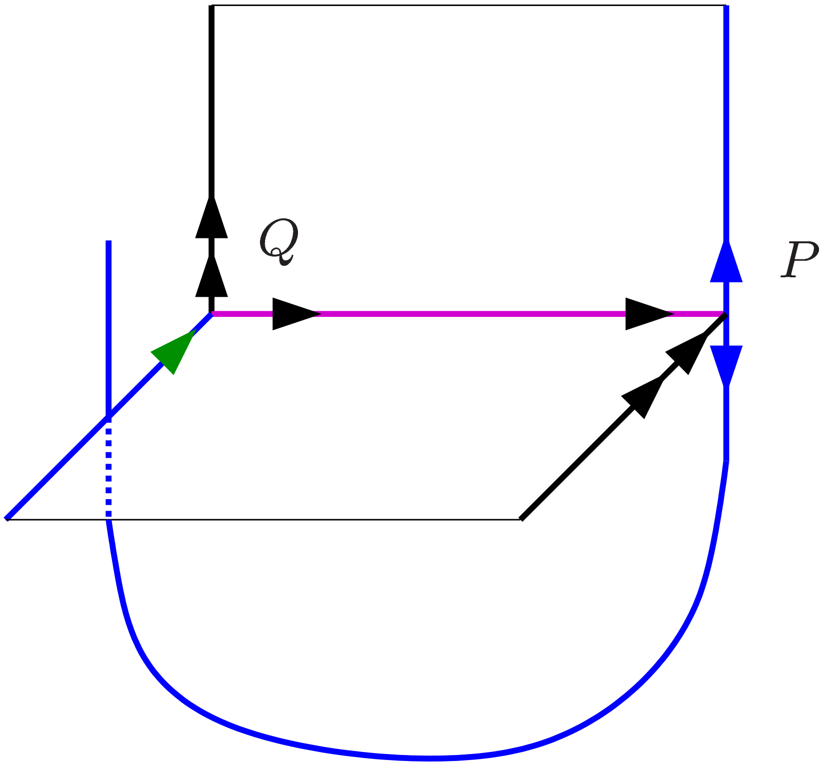
(a) transverse intersection;

(b) quasi-transverse:
for all $x \in W^s(Q) \cap W^u(P)$ heteroclinic point:

$$\boxed{T_x W^u(P) + T_x W^s(Q) = T_x W^u(P) \oplus T_x W^s(Q)}$$

Codimension one-case: $\boxed{\text{ind}(P) + 1 = \text{ind}(Q)}$

(for simplicity: $\dim(M) = 3$, $\text{ind}(Q) = 2$).



Homoclinic class of P:

$$\boxed{H(P(f) = \overline{W^s(P) \pitchfork W^u(P)}}$$

- transitive (dense orbit)
- density of periodic points (of the same index as P) (homoclinically related)
- (in general) it may contain points of different indices (non-hyperbolicity)

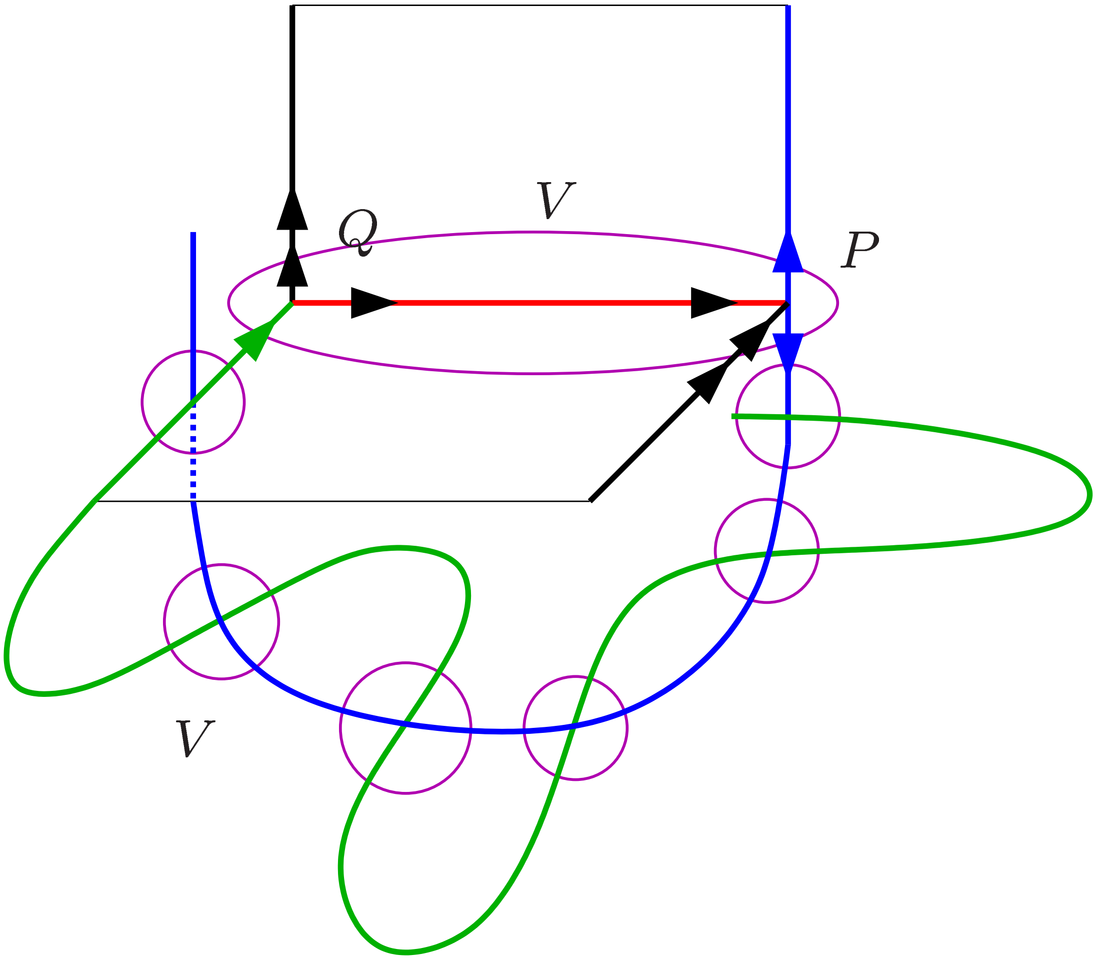
Goal:

Dynamics of nearby diffeomorphisms in a neighborhood of the cycle after the unfolding

explosion of the homoclinic classes P and Q in the cycle neighborhood of the cycle:

(small) open set V containing all elements of the cycle, orbits of the:

- saddles P and Q,
- heteroclinic points of P and Q:
  $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$



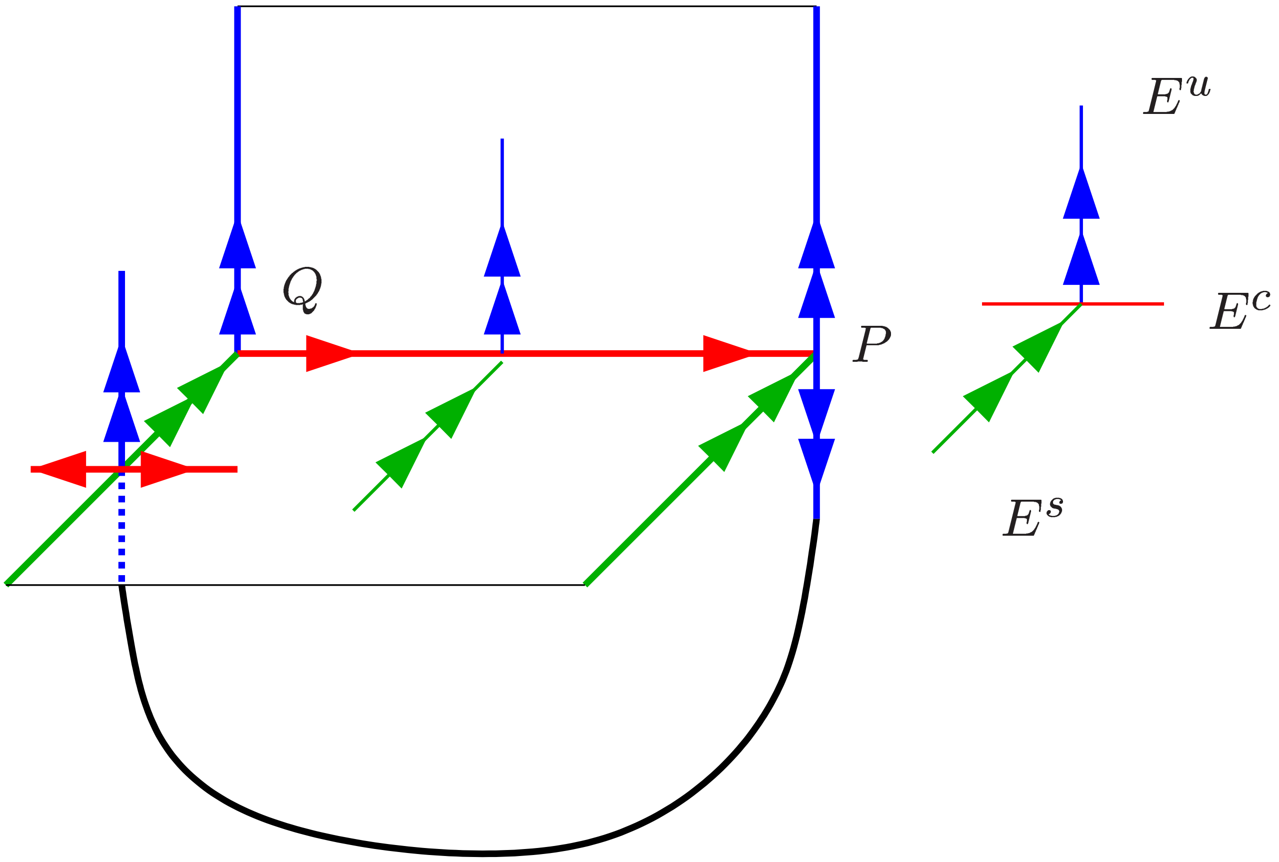
Partially hyperbolic case: g close to f,

$$\Lambda_g = \bigcap_{\mathbb{Z}} g^i(V)$$

Λ_g maximal inv. set of g in V

$$T(\Lambda_g) = E^s \oplus E^c \oplus E^u$$

E^s, E^u unif. hyper. (stable and unstable),
E^c one-dimensional.



Strategy (Newhouse-Palis-Takens):

- Consider $(f_t)_{t=[-1,1]}$:

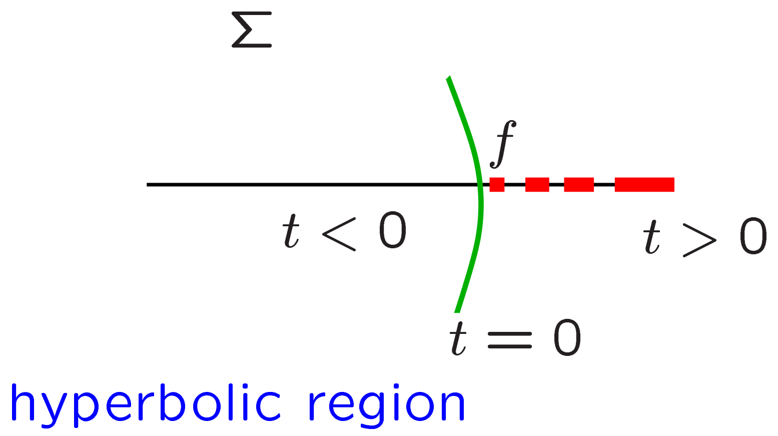
– starting in a hyperbolic region (f.i., Morse-Smale relative to neigh. of the cycle)

– crossing transversely its boundary Σ for $t = 0$ at $f_0 = f$

(here: the component of the boundary containing f has codimension one)

- dynamics for small $t > 0$ (?)

- Interesting dynamics: those displayed by diffeomorphism f_t for a positive measure set of parameters $t > 0$.



(II) DYNAMICS ARISING FROM HETERODIMENSIONAL CYCLES

$\mathcal{P}$ a dynamical property:

$$\mathbb{P} = \{t > 0 \colon f_t \text{ satisfies } \mathcal{P}\}$$

Prevalent: $\liminf_{t\to 0+} \frac{|\mathbb{P}\cap[0,t]|}{t} > 0.$

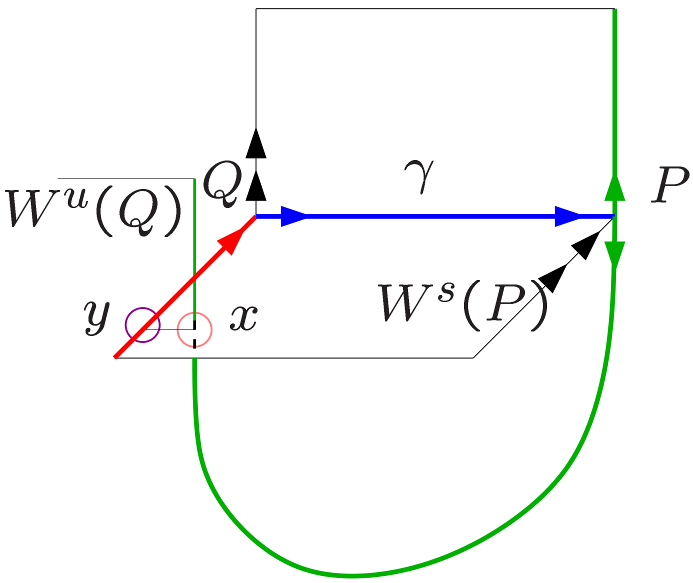
Absolutely Prev.: $\liminf_{t\to 0+} \frac{|\mathbb{P}\cap[0,t]|}{t} = 1.$

Persistent: $|\mathbb{P}\cap[0,t]| > 0$ for all $t > 0.$

(Some) typical features after unfolding a partially hyperbolic heterodimensional cycle:

(1) intermingled homoclinic classes:

$$\boxed{H(P, f_t) \cap H(Q, f_t) \neq \emptyset}$$



Typically: $H(P, f_t) = H(Q, f_t)$, $\gamma \subset H(P, f_t)$

(Carballo et al.) C^1-Generically:
$H(P, f_t) \cap H(Q, f_t) \neq \emptyset \implies H(P, f_t) = H(Q, f_t)$

(2) hyperbolicity: $\Lambda_{f_t} = H(P, f_t) \cup H(Q, f_t)$
the classes are hyperbolic (indices 1 and 2)

(3) collisions of homoclinic classes (secondary saddle-node biffurcations)

(4) persistence of heterodimensional cycles
(thick one-dimensional Cantor sets)

Thm A (D-Rocha):

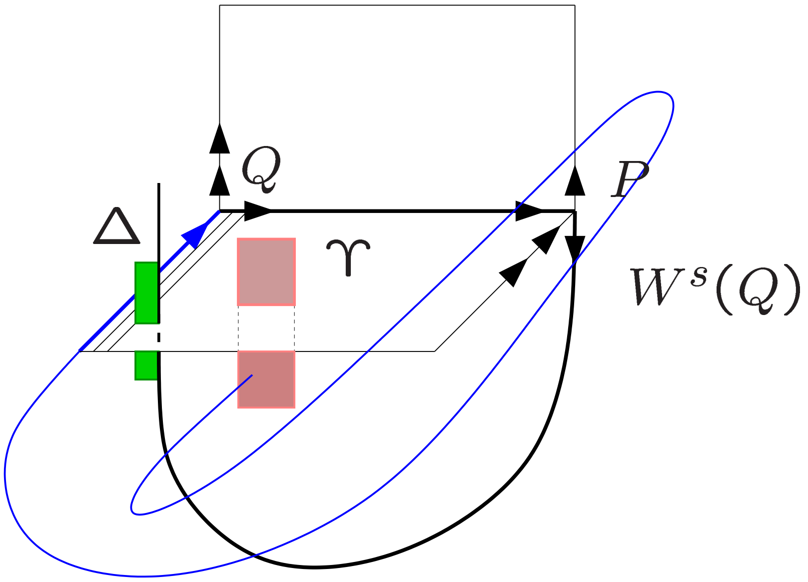
For every $(f_t)_{t\in[-1,1]}$ unfolding a partial. hyperbol. heterodimen. cycle there is a sequence of parameter intervals B_n, $B_n \to 0^+$, such that $\forall t \in B_n$:

A) $H(P, f_t) = H(Q, f_t)$,
Λ_{f_t} is transitive (containing P and Q).

B) (first bifurc.) prevalent non-hyperb.

$$\liminf_{t\to 0} \frac{|B \cap [-t,t]|}{2\,t} > 0; \qquad B = \cup B_n.$$

Main ingredient: Blender-like construction
$W^u(Q)$ intersects *every* 2-disk Υ crossing transversely $W^s(P)$.



Given an arc $(f_t)_{t\in[-1,1]}$ unfolding a cycle at $t=0$:

$$\boxed{H(s)=\{t\in[0,s]: f_s \text{ hyperbolic }\}}$$

hyperbolic: Axiom A $\oplus$ no-cycles

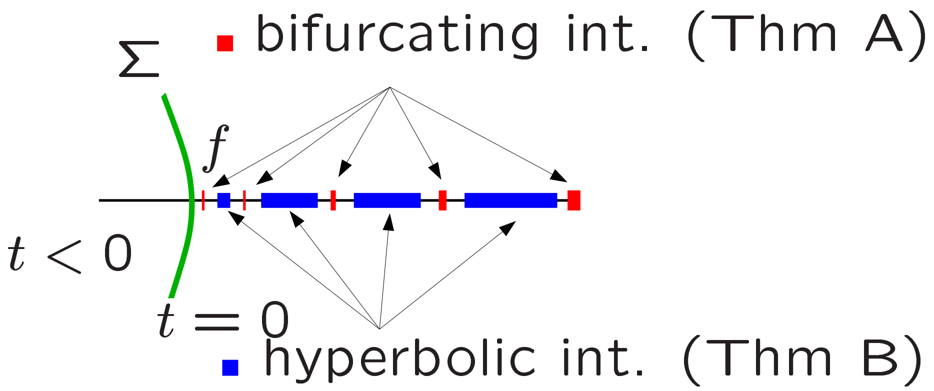
Thm B (D-Rocha):

For every $\varepsilon>0$ there is an open set of $\mathcal{A}_\epsilon$ of arcs $(f_t)_{t\in[-1,1]}$ unfolding a partially hyperbolic heterodimensional cycle such that:

$$\liminf_{t\to 0}\frac{|H(t)\cap[0,t]|}{t}>1-\varepsilon$$

(by the previous theorem this limit is <1)

hyperbolicity is prevalent but not absolute

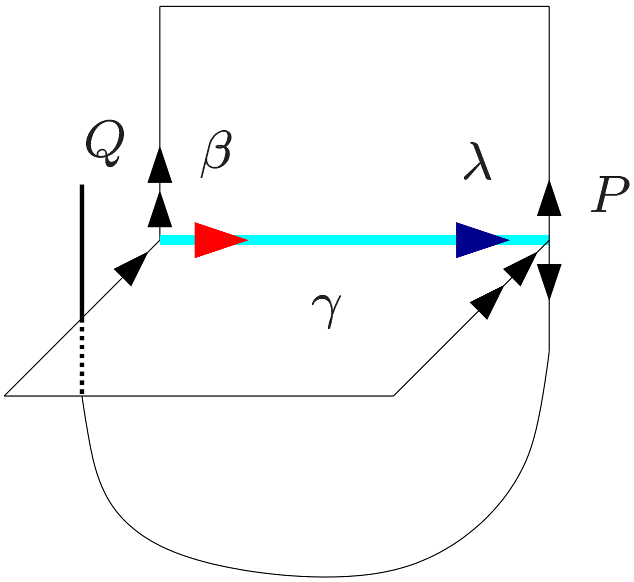


(III) LIMIT DYNAMICS

Dynamics after the bifurcation *depends* on the central dynamics of the bifurc. diffeo.

Ingredients

central eigenvalues of P and Q (corresponding to the central direction E^c)
transition from Q to P throughout the curve $\gamma = W^s(P) \pitchfork W^u(Q)$ (restriction f/γ)



Mather invariant:

f/γ close to 0:

time-one of $\boxed{X(y) = (\log\beta)\, y\, \frac{\partial}{\partial y}}$

flow: $\boxed{y \mapsto \beta^t\, y}$

f/γ close to 8:

time-one of $\boxed{Y(y) = (\log\lambda)\,(y-8)\, \frac{\partial}{\partial y}}$

flow: $\boxed{y \mapsto \beta^t\, y}$

y near 0^+, big n with $f^n(y)$ near 8^-:

$$\boxed{\;Df^n(y)(X(y)) = \mu(y)\, Y(f^n(y))}$$

local f-invariance (X, Y near $0, 8$),

- independent of n,
- $\mu(y) = \mu(f(y))$.

μ measures the distortion.

$\mu(y)$: Mather invariant of f/γ

Heuristic principle

- small distortion:

intermingled homoclinic classes,
for $t > 0$: $H(P, f_t) = H(Q, f_t)$. (D).

- For Thm B one needs a **BIG** distortion

(as $\varepsilon \to 0$, distortion $\to \infty$).

- Thm. A: dynamics in *intermediate* cases.

Strategy

- projection to the central direction

curve γ

- fix a reference fundam. domain D^+ of γ,

returns to D^+.

Limit dynamics (asumptions)

$\gamma = [0,8]$, $D^+ = [\beta^{-1}, 1]$, $D^- = [7, 8-\lambda]$,

- f linear in $[0,1]$, $\boxed{x \mapsto \beta x}$
- f affine in $[7,8]$, $\boxed{x \mapsto \lambda(x-8)+8}$
- $f^N(D^+) = D^-$, $\boxed{F = f^N \text{ transition}}$

Resulting dynamics

- returns to D^+ by iterations of f_t
- Quotient dynamics: After identifying the stable and unstable directions

(Quotient) Family of returns to D^+:

a one-parameter family of system of iterated functions (∞-many maps) generated by:

- two affine maps (dynamics at Q and P),

$$\boxed{x \mapsto \beta\, x} \qquad \boxed{x \mapsto \lambda\,(x-8)+8}$$

- a transition $\boxed{\text{the map } F}$
- a translation induced by the parameter t after reparametrization

- **Step 1:** Transition
- **Step 2:** Affine iterates

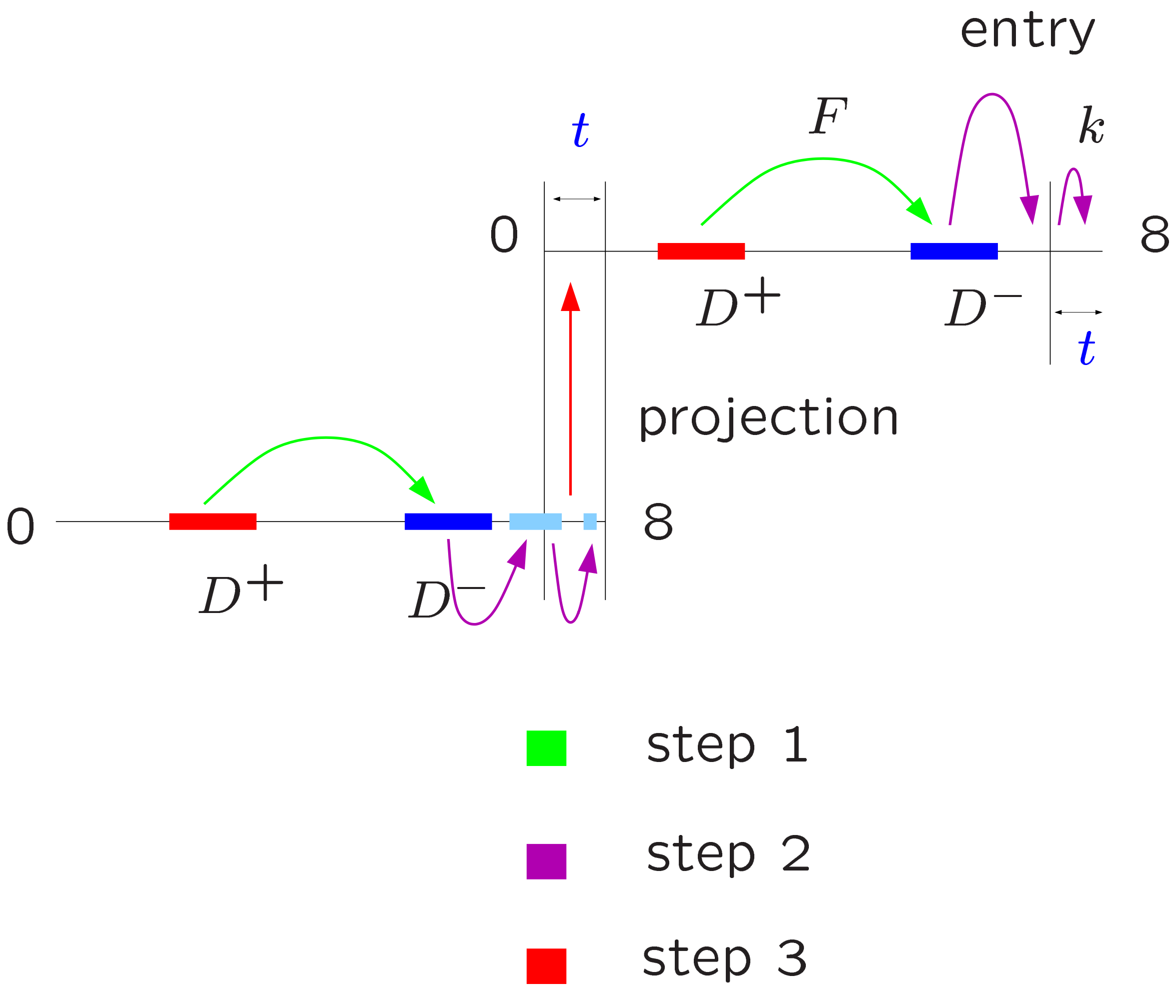
Part 1: approx. to $P = 8$ at distance t (entry iterates $\lambda^{\text{entry}} \simeq t$),
Part 2: further iterates ($k \geq 0$).

Steps 1–2: $\boxed{\lambda^{\text{entry}+k}(F(x)-8)+8}$

- **Step 3:** effect of the parameter t (acting as a translation) and projection

Steps 1–3: $\boxed{\boxed{(\lambda^{\text{entry}+k}(F(x)-8)+8)}\ +t-8}$

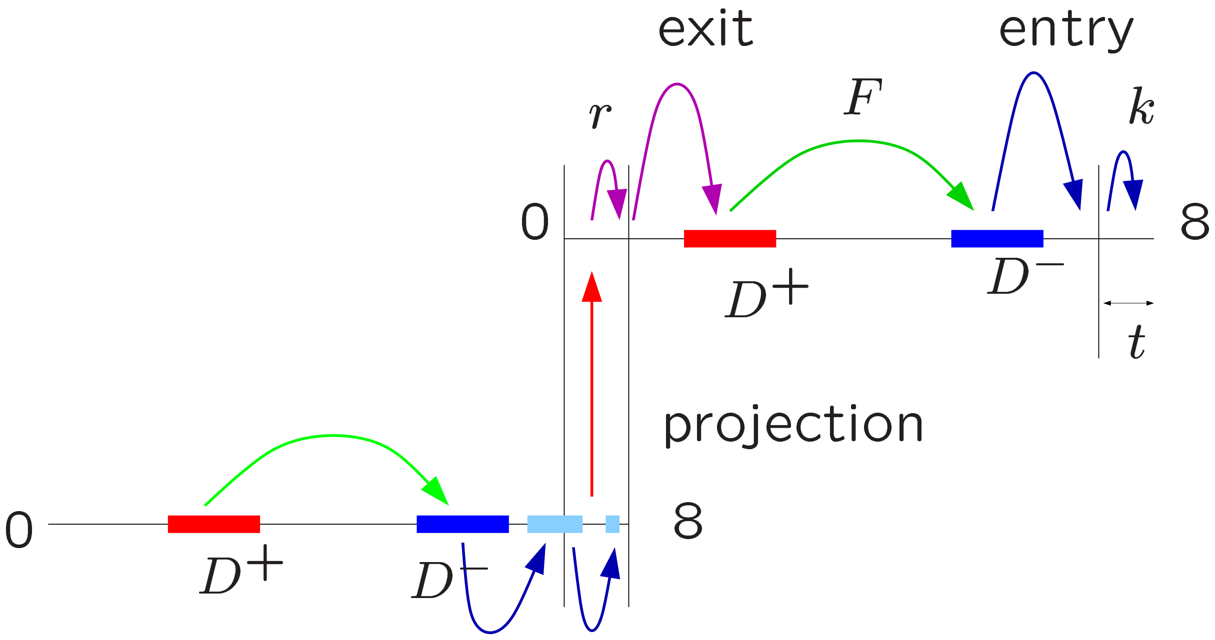
Steps 1–3: $\boxed{\lambda^{\text{entry}+k}(F(x)-8)}\ +t$



- **Step 4:** Linear iterates

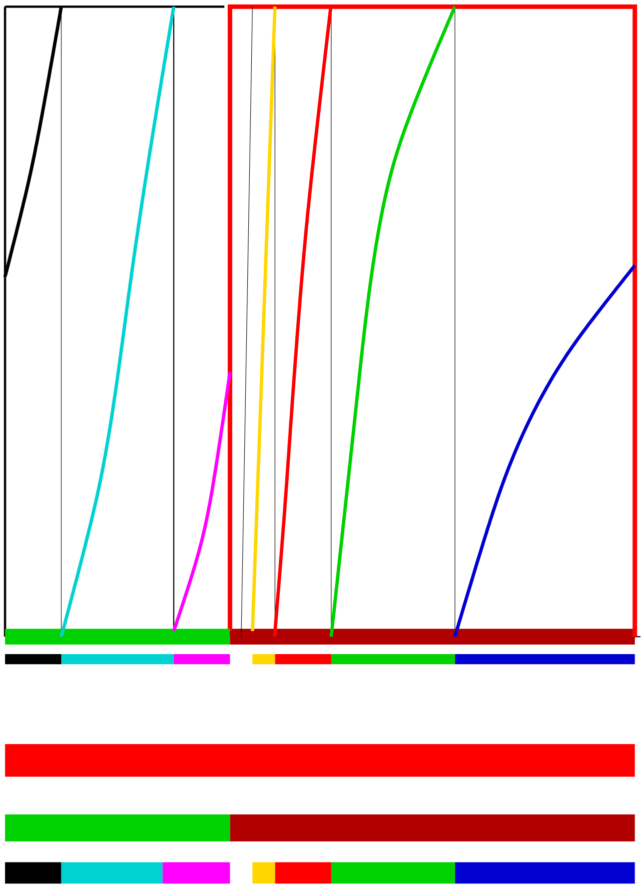
Part 1: Separation at distance t from $Q = 0$

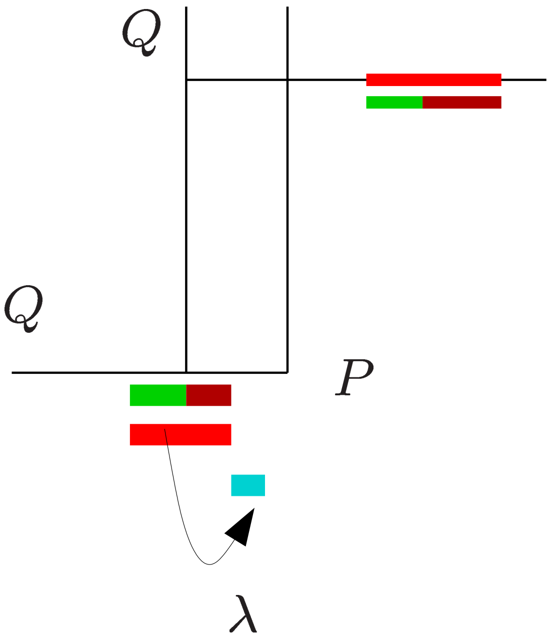
Part 2: Return to D^+, entry iterates $\beta^{\text{entry}} \simeq 1/t$.

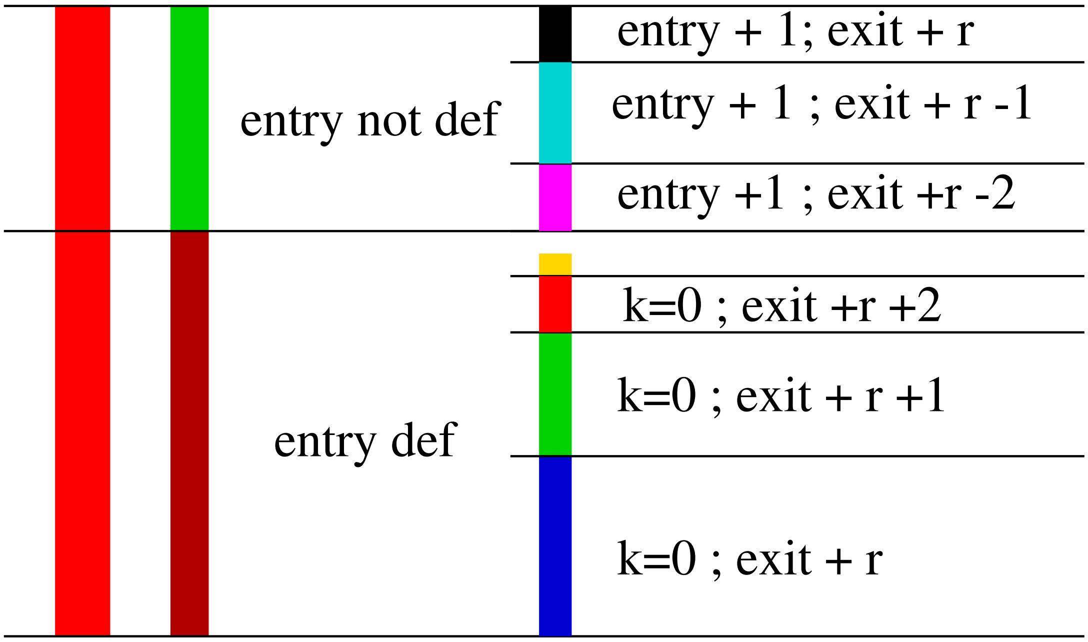


Steps 1–4:

$$\boxed{\beta^{\text{exit}+r}\ \boxed{(\ \lambda^{\text{entry}+k}(F(x)-8)+t\ )}}$$




Q
Q
P
λ


entry not def
entry + 1; exit + r
entry + 1 ; exit + r -1
entry +1 ; exit +r -2
entry def
k=0 ; exit +r +2
k=0 ; exit + r +1
k=0 ; exit + r

SIMPLIFIED CASE: $\boxed{\lambda^{\text{entry}} = t}$ $\boxed{\beta^{\text{exit}} = 1/t}$

Returns for the parameter t (of type k, r):

$$\boxed{\beta^r(\lambda^k(F(x) - 8)) + 1}$$

Re-scaling effect: $\boxed{s = t - \mu,\ \text{small}\ \mu > 0}$

One-param. family iterated functions $R_\mu^{k,r}$

$$\boxed{R_\mu^{k,r}(x) = \beta^r(\ \lambda^k(F(x) - 8)) + 1 + \beta^r \mu}$$

Remark k is free $r = r(n, x)$

Typically $k \to \infty$ $r(n, x) = 0$

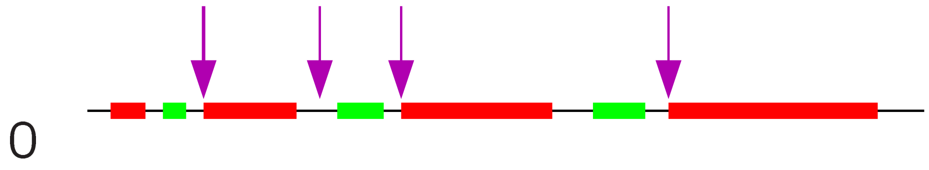
$k = 0$, there are x with $r(n, x)$ arbitrarily big

To get $H(P, f_t) = H(Q, f_t)$:

$R_\mu^{0,r}$, $R_\mu^{1,r}$ uniformly expanding

(IV) COLLISIONS OF HOMOCLINIC CLASSES

transition from hyperbolic to bifurcating intervals (Theorems A and B)



🟥 hyperbolicity

🟩 non-hyperbolicity

↓ secondary bifurcations

Secondary bifurcations

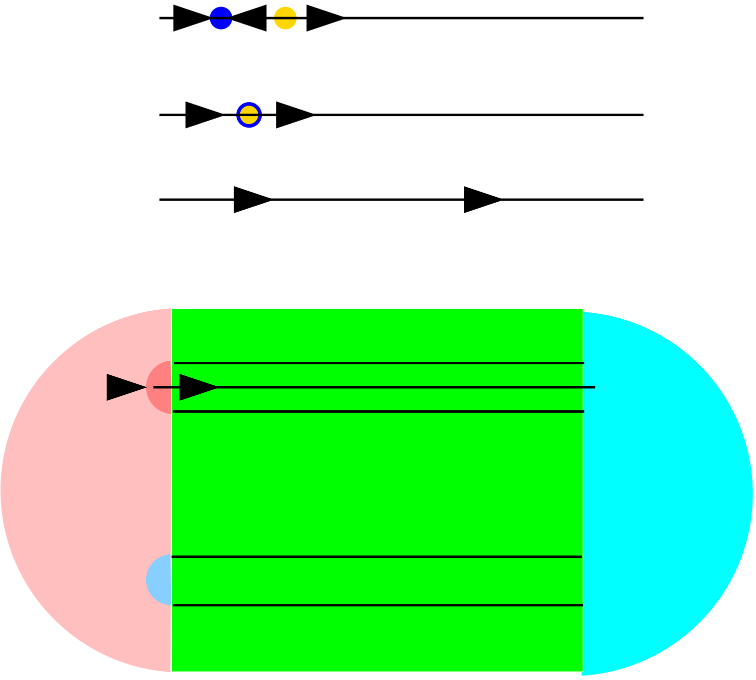
- new heterodimensional cycles
- saddle-node bifurcations

(loss of hyperbolicity of some saddle)

- others ?

Interesting: the saddle-node is involved in the dynamics

- saddle-node cycles and horseshoes



- derived from Anosov

- collisions of homoclinic classes

dimension two: derived from Anosov, saddle-node horeseshoes, always a class is trivial

Goal collisions of non-trivial hyperbolic homoclinic class. at heterodimensional cycles

$\boxed{\text{Model (product structure)}}$ (D. Rocha)

$(f_t)_{t\in[-1,1]}$ unfolds a (partially hyperbolic) heterodimensional cycle at $t = 0$ such that:

$\exists\ (t_n), (t_n^+), (t_n^-)$, $t_n, t_n^\pm \to 0^+$, with

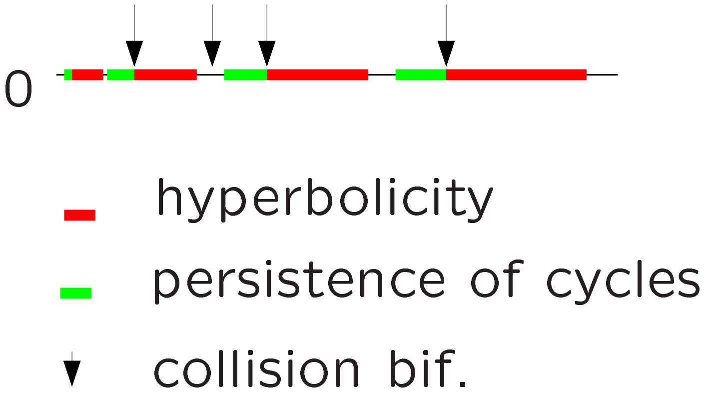
- f_t is hyperbolic if $t \in (t_n, t_n^+)$, (in particular $H(P, f_t)$ and $H(Q, f_t)$ are hyperbolic)

- for $t = t_n$ there is a collision of the homoclinic classes of P and Q:
  $H(P, f_{t_n}) \cap H(Q, f_{t_n}) \neq \emptyset$
  (the orbit of a saddle-node)

- there is persistence of heterodimensional cycles in (t_n^-, t_n^+)

Persistence of (heterodimensional) cycles (in the open interval I): there is a dense subset of I such that every f_t has a heterodimensional cycle.

- the intervals (t_n^-, t_n) and (t_n, t_n^+) are both prevalent are the bifurcation



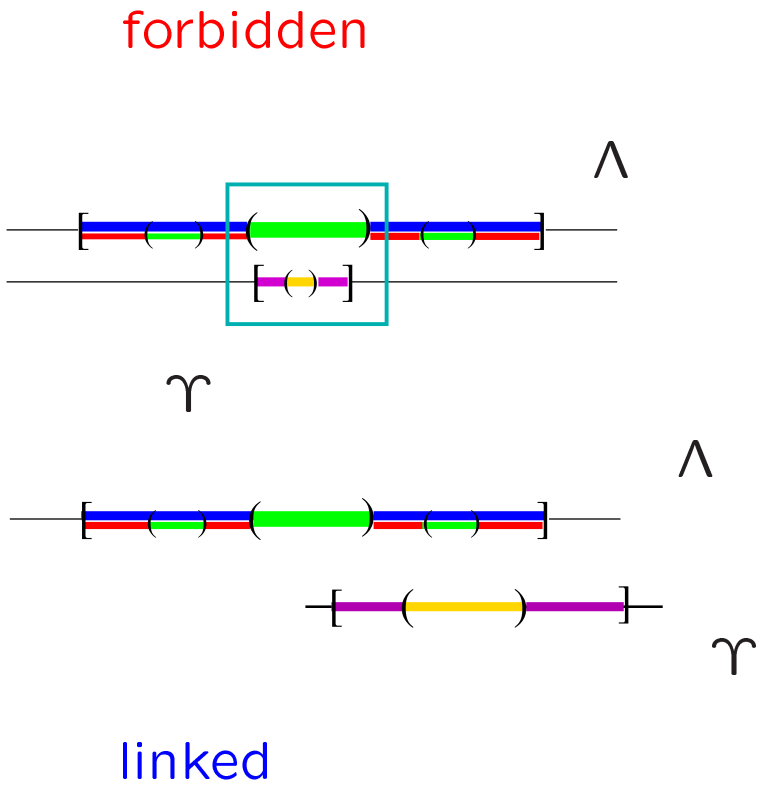
Ingredients

- construct F such that $R_0^{0,0}$ unfolds a saddle-node

- creation of arbitrarily thick hyperbolic sets Υ (of index 2) immediately before the unfolding of the saddle-node (the unfolding is for decreasing t):
Υ is dynamically defined, associated to the system of iterated functions
(non-critical dynamics)

- the homoclinic class of P contains hyperbolic sets Λ with thicknes uniformly bounded from zero

- Gap Lemma:
$\tau(\Lambda) \cdot \tau(\Upsilon) > 1$ and each set is not contained in a gap of the other one, then $\Lambda \cap \Upsilon \neq \emptyset$



- the product dynamics gives a cycle associated to Υ and Λ

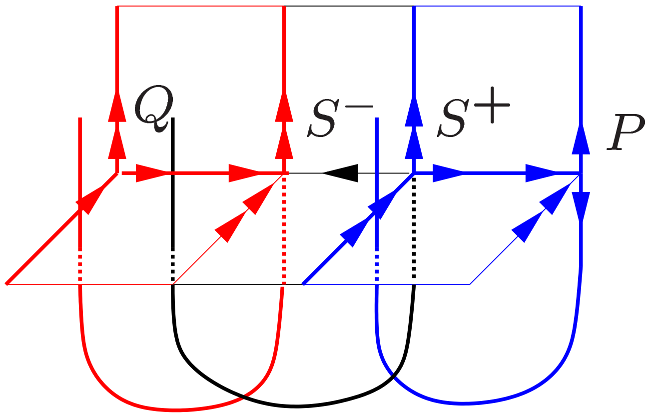
$\boxed{\text{Remark}}$ (D. Santoro)

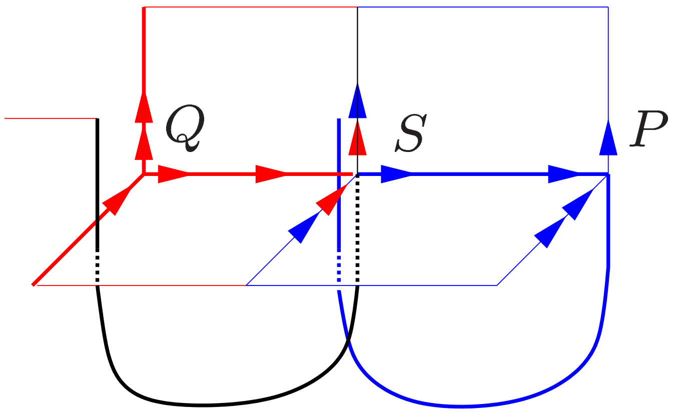
There is a model family for the collision of partially hyperbolic (non-hyp.) homoclinic classes $H(P)$ and $H(Q)$ leading to the collapse of the classes:

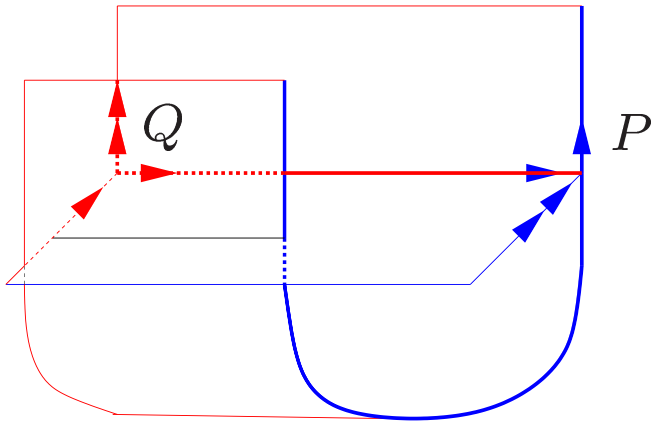
$$\boxed{H(P) = H(Q) \text{ after the collision}}$$

Methods for the collapse $\neq$.

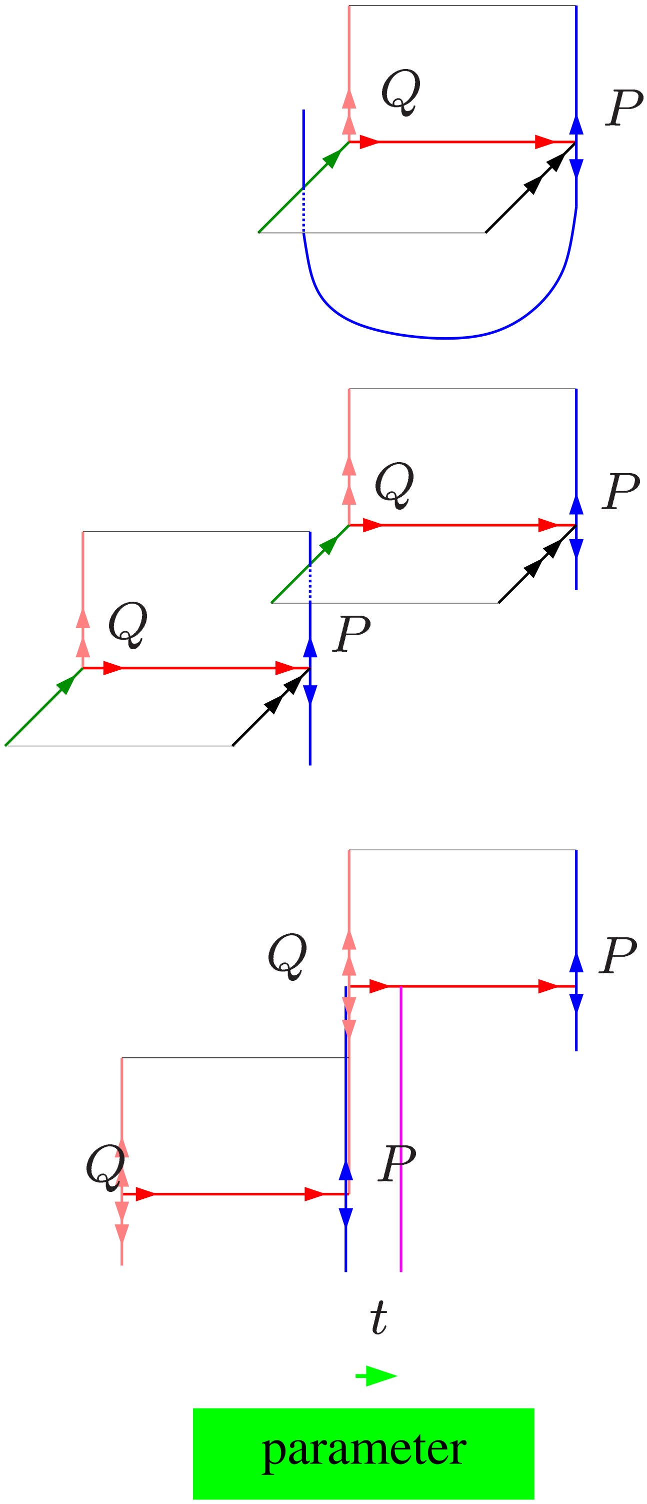
collision of non-hyperbolic homoclinic classes

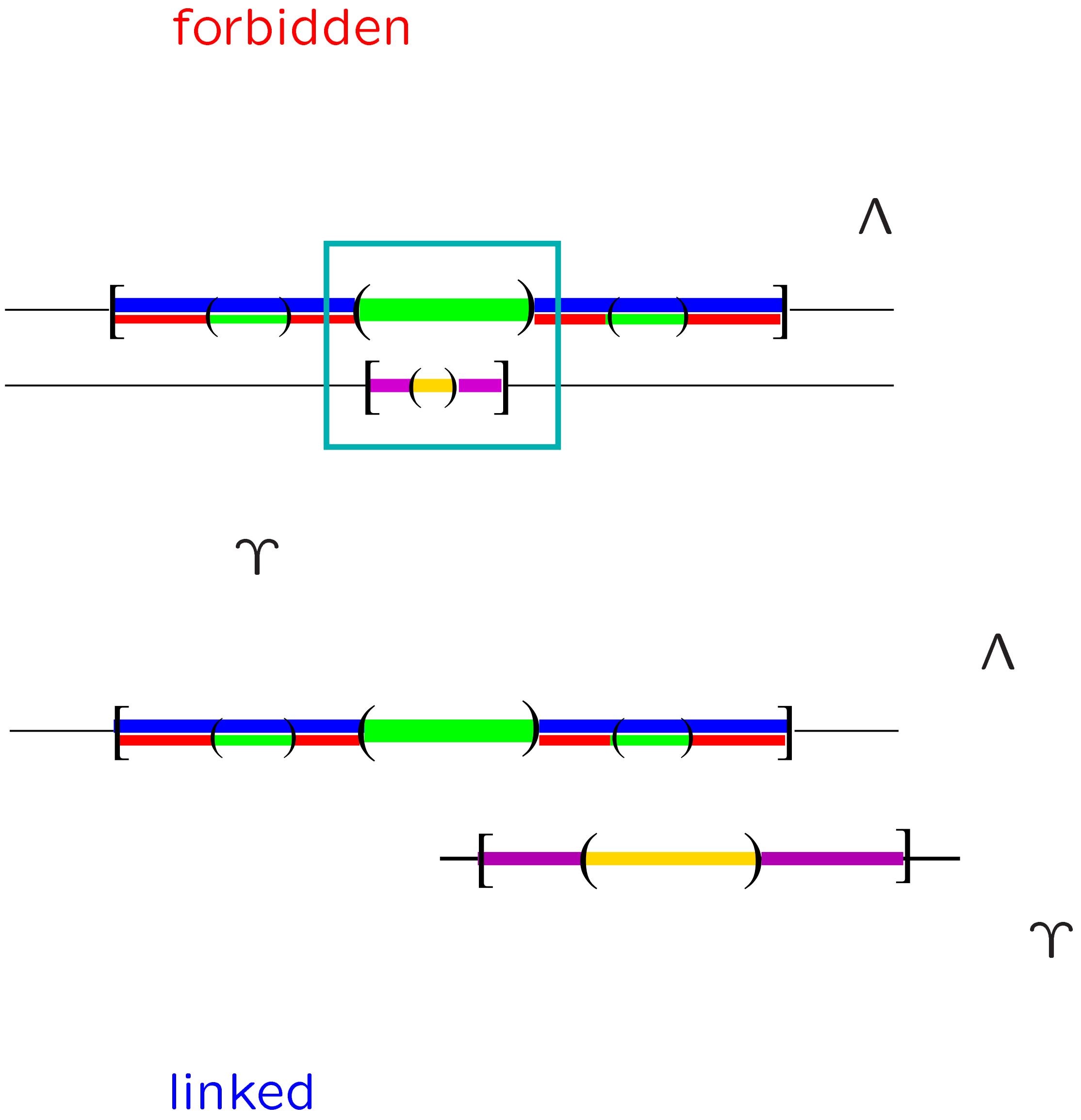






model cycle (one-dim reduction)

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Minimality of strong stable and unstable foliations for partially hyperbolic diffeomorphisms.

Christian Bonatti, Lorenzo J. Díaz and Raúl Ures [*]

March 14, 2001

Abstract

We give a topological criterion for the minimality of the strong unstable (or stable) foliation of robustly transitive partially hyperbolic diffeomorphisms.

As a consequence we prove that, for 3−manifolds, there is an open and dense subset of robustly transitive diffeomorphisms (far from homoclinic tangencies) such that either the strong stable or the strong unstable foliation is robustly minimal.

We also give a topological condition (existence of a central periodic compact leaf) guaranteeing (for an open and dense subset) the simultaneous minimality of the two strong foliations.

1 Introduction

Robustly transitive diffeomorphisms are one of the archetypes of chaotic dynamics: they are *transitive* (i.e., there are points whose forward orbits are dense in the whole ambient manifold) and the transitivity persists after C^1−perturbations. The first known examples are the Anosov (hyperbolic) ones. More recently non-hyperbolic robustly transitive diffeomorphisms were constructed using one of the following three methods: bifurcations from hyperbolic diffeomorphisms (see the *derived from Anosov* examples in [Mñ$_1$, C, BV]), skew products [S, BD$_1$], and perturbations of the time-one of Anosov flows, [BD$_1$].

For volume preserving diffeomorphisms the prototype should be the *stably ergodic* diffeomorphisms, that is, volume preserving diffeomorphisms whose volume preserving perturbations are ergodic (with respect to the volume). These two theories (robust transitivity and stable ergodicity) have been systematically developed in recent years and run in parallel. As a general rule, examples can be translated from one theory to the other (see for instance the example in [SW]), and many problems are common to both theories (see the survey [BPSW] on stable ergodicity).

Robustly transitive diffeomorphisms always satisfy some weak form of hyperbolicity: they are uniformly hyperbolic in dimension two [Mñ$_2$], partially hyperbolic in dimension three, [DPU], and they admit a dominated splitting in higher dimensions, [BDP].

In this paper we deal with robustly transitive diffeomorphisms defined on three-manifolds which are partially hyperbolic (but not Anosov). This means that the tangent bundle of the ambient manifold can be split into 2 or 3 invariant sub-bundles as follows:

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[*] partially supported by CNPq, FAPERJ, FINEP, PRONEX (Brazil), CONICYT (Uruguay), Ministère des Affaires Étrangères, CNRS and Université de Bourgogne (France)

- $TM = E^s \oplus E^c$, where E^s is one-dimensional and uniformly hyperbolic and E^c is two-dimensional, *undecomposable* (i.e., it does not admit any dominated splitting) and not uniformly hyperbolic,
- $TM = E^c \oplus E^u$, where E^u is one-dimensional and uniformly hyperbolic (expanding) and E^c is two-dimensional, undecomposable and not uniformly hyperbolic,
- $TM = E^s \oplus E^c \oplus E^u$, where the three bundles are one-dimensional, E^s and E^u are uniformly hyperbolic (contracting and expanding, respectively) and E^c is not uniformly hyperbolic.

In the first two cases it is possible to perturb the initial diffeomorphism to get a new diffeomorphism exhibiting a *homoclinic tangency* (i.e., there is a hyperbolic periodic point whose invariant manifolds are non-transverse), see [DPU, Corollary G]. In this paper we deal with the third situation, so-called *strongly partially hyperbolic* (see a more precise definition below), where the diffeomorphisms are *far from homoclinic tangencies.* For instance, the 3−dimensional examples in [Mñ$_1$, BD$_1$, SW] have this property.

In the strongly partially hyperbolic context, due to the uniform hyperbolicity of the bundles E^s and E^u, there exist one-dimensional foliations tangent to E^s and E^u, called strong stable $\mathcal{F}^s$ and strong unstable $\mathcal{F}^u$ foliations, respectively, see [BP, HPS]. However, we do not know if in general the central bundle is (uniquely) integrable and so the central foliation is not a priori defined, in fact this is an important open problem in this area.

Some key problems in the study of (robustly) transitive diffeomorphisms are the following:

- *Characterization of robust transitivity or stable ergodicity.*
- *Which* 3−*manifolds can support robustly transitive diffeomorphisms* ? Let us recall that Anosov diffeomorphisms in dimension 3 are always conjugate to a finite quotient of a linear Anosov map on the torus T^3, see [N, Fr, Mn]. We conjecture that *there are no robustly transitive diffeomorphisms on the sphere* S^3. [DPU, Theorem H] verifies this conjecture in a more restrictive context.
- *State the ergodic properties of these systems, in particular, the existence and finiteness of SRB measures* (for results on this subject see [BV, ABV, Do]).

In this paper we will begin the topological description of the strong stable or unstable foliation, motivated by the two following ideas:

- For describing the global dynamics of strongly partially hyperbolic diffeomorphisms one often attempts to propagate topological or measurable local properties along the leaves of the strong stable or unstable foliations to the whole manifold.
- The support of SRB-measures of strongly partially hyperbolic diffeomorphisms is saturated for the strong unstable foliation. For instance, [BV] uses the hypothesis of the density of the unstable leaves to prove the uniqueness of the SRB-measure.

A foliation is *minimal* if all its leaves are dense in the ambient manifold.

Problem 1.1. *Are the strong stable and strong unstable foliations* minimal *for an open and dense subset of the set of robustly transitive, strongly partially hyperbolic diffeomorphisms ?*

Before stating our results, let us recall that this problem has a long history in the uniformly hyperbolic context:

- The stable and unstable foliations of transitive Anosov diffeomorphisms are always dense.
- For transitive Anosov flows on $3-$manifolds, the strong stable and unstable foliations are both minimal, except for a very specific case: the suspension of an Anosov diffeomorphism of T^2 (see [P]). This study begun with the geodesic flows of surface having constant negative curvature, where the strong stable and unstable foliations correspond with the horocyclic flows and are uniquely ergodic (see [Fu]). Then the property of unique ergodicity was generalized to the strong stable or unstable foliation of Anosov flows (see [BoMa]).

In this paper, assuming that the central bundle admits an invariant orientation, we prove that *the minimality of at least one of the two strong foliations is an open and dense property* (see Theorem 1). Moreover, if the central bundle is uniquely integrable, we also prove that *the minimality of both strong foliations is an open and dense property among the diffeomorphisms having a periodic compact central leaf* (see Theorem 3).

Let us now state our results precisely.

Statement of results

Let $f\colon M \to M$ be a diffeomorphism of a compact manifold. Let us recall that an f_*-invariant splitting $E \oplus F$ of TM is *dominated* if the fibers of the bundles have constant dimension and there are a Riemannian metric $||\cdot||$ and a constant $\lambda < 1$ such that $||f_*(x)_{|E}|| \cdot ||f_*^{-1}(f(x))_{|F}|| < \lambda$, for all $x \in M$.

An f_*-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ with three bundles is called dominated if the splittings $E^{cs} \oplus E^u$ and $E^s \oplus E^{cu}$ are both dominated, where $E^{cs} = E^s \oplus E^c$ and $E^{cu} = E^c \oplus E^u$.

Definition 1.2. *A diffeomorphism f of a compact $3-$manifold M is* strongly partially hyperbolic *if there is a f_*-invariant* dominated *splitting $TM = E^s \oplus E^c \oplus E^u$ with three one-dimensional bundles such that E^s and E^u are respectively uniformly contracting and expanding.*

Given a compact closed manifold M of dimension 3 the set of robustly transitive diffeomorphisms of M is, by definition, open in $\mathrm{Diff}^1(M)$. As dominated splittings persist by C^1-perturbations, the set of strongly partially hyperbolic diffeomorphisms is open. The property of the central bundle not being hyperbolic is not open. However, Mañé's result [Mñ$_2$] implies that every diffeomorphism which is robustly transitive and strongly partially hyperbolic but not Anosov can be C^1-approximated by one having hyperbolic periodic points of different *indices* (dimension of the stable manifold). This property is clearly open.

We now denote by $\mathcal{T}(M)$ the open set of robustly transitive and strongly partially hyperbolic diffeomorphisms having hyperbolic periodic points of different indices (of index 1 and 2). We endow $\mathcal{T}(M)$ with the C^1-topology. Define by $\mathcal{T}^+(M)$ the subset of $\mathcal{T}(M)$ of diffeomorphisms f such that the bundles E^s, E^u and E^c are orientable and f_* preserves these orientations. This set is open in $\mathcal{T}(M)$.

A property of diffeomorphisms is called *robust* if whenever it holds for a diffeomorphism it also holds for a C^1-neighborhood of that diffeomorphism.

Denote by $\mathcal{O}^s(M)$ (resp. $\mathcal{O}^u(M)$) the subset of $\mathcal{T}(M)$ of diffeomorphisms f whose strong stable (resp. unstable) foliation is robustly minimal. In other words, $\mathcal{O}^s(M)$ (resp. $\mathcal{O}^u(M)$) is the interior of the subset of $\mathcal{T}(M)$ of diffeomorphisms f with a minimal strong stable (resp. unstable) foliation.

Similarly, let $\mathcal{O}^{i+}(M) = \mathcal{O}^i(M) \cap \mathcal{T}^+(M)$, $i = s, u$.

Theorem 1. *The open subset* $\mathcal{O}^+(M) = \mathcal{O}^{s+}(M) \cup \mathcal{O}^{u+}(M)$ *of diffeomorphisms with a robustly minimal strong stable or unstable foliation is dense in* $\mathcal{T}^+(M)$.

The condition on the existence of an f_*-invariant orientation on the stable and unstable bundles is purely technical and can easily be dropped, see Proposition 7.1 in Section 7. The hypothesis on the invariant orientation of the central bundle of f is much more delicate and subtle. Of course, this difficulty can be bypassed if either its lift to the orientation covering remains transitive or if the central bundle is orientable (f does not preserve this orientation) and f^2 is transitive. The robust transitivity of f^2 is guaranteed if f has some periodic point of odd period (see Lemma 7.3). The robust transitivity of the lift of f to the orientation covering is guaranteed if f has a hyperbolic periodic point p of period k such that $f_*^k(p)$ reverses the orientation of $E^c(p)$ (see Remark 7.5).

Another way to guarantee the transitivity of the lifts is to assume the *accessibility* hypothesis. More precisely, a point y is *accessible* from a point x if there is a path going from x to y consisting of segments contained in strong stable or strong unstable leaves. Accessibility is an equivalence relation. A diffeomorphism f has the *accessibility property* if there is only one accessibility class. Similarly, f has the *robust accessibility property* if every diffeomorphism g C^1-close to f has the accessibility property.

Theorem 2. *The open set of diffeomorphisms with a robustly minimal strong stable or unstable foliation is dense in the set* $\mathcal{T}_a(M)$ *of diffeomorphisms* $f \in \mathcal{T}(M)$ *having the robust accessibility property.*

The accessibility assumption does not seem to be too restrictive: Pugh and Shub conjecture that *the accessibility property is valid for an open and dense subset of* $\mathcal{T}(M)$, [PS], and recently Dolgopyat and Wilkinson, [DW][1], have announced the proof of this conjecture. For a partial answer for this conjecture see [N̦T̈]. Assuming this result our theorems can be re-formulated as follows:

The set $\mathcal{O}(M) = \mathcal{O}^s(M) \cup \mathcal{O}^u(M)$ *of diffeomorphisms with a robustly minimal strong stable or unstable foliation is dense in* $\mathcal{T}(M)$.

The previous results only give the robust minimality of one of the strong foliations. We now introduce a subset of $\mathcal{T}(M)$ for which we have been able to prove the minimality of both foliations. Consider the subset $\mathcal{T}_0(M)$ of $\mathcal{T}(M)$ consisting of the diffeomorphisms verifying the following two extra geometrical assumptions:

- *coherence,* that is, unique integrability of the bundles E^c, $E^c \oplus E^s$ and $E^c \oplus E^u$,
- existence of a compact and periodic central leaf.

From [HPS] the coherence and the existence of a periodic compact leaf are conditions which are open in the set of partially hyperbolic systems, so that $\mathcal{T}_0(M)$ is a C^1-open subset of $\mathcal{T}(M)$. In the partially hyperbolic setting the coherence assumption is quite common and natural, and has often been used in the context of stably ergodic systems (see for instance [PS, Do]).

As above we consider the subset $\mathcal{T}_0^+(M) = \mathcal{T}_0(M) \cap \mathcal{T}^+(M)$.

Theorem 3. *The open set* $\mathcal{O}^s(M) \cap \mathcal{O}^u(M)$ *of diffeomorphisms whose strong stable and unstable foliations are both robustly minimal is dense in* $\mathcal{T}_0^+(M)$.

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[1] The definition of partial hyperbolicity in these papers is a bit more restrictive than the one we use here: in their definition, for any pair of points x and y and every unitary vectors $u \in E$ and $v \in F$ one has $|f_*(x)(u)|/|f_*(y)(v)| < k$ for some $k < 1$, and in our definition we only consider the case $x = y$.

Again, we can substitute the hypothesis on the orientations by the robust accessibility property (see Section 7.2).

Finally, Theorem 1 remains valid in any dimension for robustly transitive diffeomorphisms whose central bundle has dimension 1 and is uniquely integrable.

Theorem 4. *Let M be a compact manifold of dimension $n \geq 3$. Denote by $\mathcal{T}_1(M) \subset \mathrm{Diff}^1(M)$ the set of robustly transitive diffeomorphisms whose central bundle has dimension one, is uniquely integrable, and has an $f-$invariant orientation.*

Then $\mathcal{O}^u(M) \cup \mathcal{O}^s(M)$ contains a dense open subset of $\mathcal{T}_1(M)$, where $\mathcal{O}^u(M) \cup \mathcal{O}^s(M)$ is the set of diffeomorphisms $f \in \mathcal{T}_1(M)$ having a robustly minimal strong unstable or stable foliation.

The theorems above follow from a result on the existence of complete transverse sections adapted to the dynamics.

Definition 1.3. *Let f be a strongly partially hyperbolic diffeomorphism of a compact $3-$manifold M.*

- *A compact surface with boundary T transverse to the strong unstable foliation of f is a $u-$*section for f *if $f(T)$ is contained in the interior of T and $\omega(T) = \cap_{n\geq 0} f^n(T)$ is the union of finitely many segments and circles tangent to the central direction.*
- *A $u-$section T is* complete *if its interior intersects each strong unstable leaf $F^u(x)$ transversely, $x \in M$.*

*We define $s-$*section *and* complete $s-$section for f *by replacing f by f^{-1} in the definition above.*

By the $f-$invariance, every complete $u-$section is tangent to the center stable bundle $E^c \oplus E^s$.

The proof of our results follows from the following heuristic principle: generically a complete $u-$section contains finitely many periodic points; we will show that the closure of any leaf of $\mathcal{F}^u$ contains the leaf through one of these periodic points. So the minimality of the foliation will follows from the density of these finitely many unstable manifolds.

Denote by $\mathcal{U}(M)$ (resp. $\mathcal{S}(M)$) the subset of $\mathcal{T}(M)$ of diffeomorphisms admitting complete $u-$sections and $s-$sections, respectively.

Theorem 5.

1. *The sets $\mathcal{S}(M)$ and $\mathcal{U}(M)$ are open in $\mathcal{T}(M)$.*
2. *The union $\mathcal{S}(M) \cup \mathcal{U}(M)$ is dense in $\mathcal{T}^+(M)$.*
3. *Both $\mathcal{S}(M)$ and $\mathcal{U}(M)$ are dense in $\mathcal{T}_0^+(M)$.*
4. *There is an open and dense subset $\mathcal{M}^u(M)$ of $\mathcal{U}(M)$ such that the strong stable foliation is minimal for all f in $\mathcal{M}^u(M)$. Similarly, there is an open and dense subset $\mathcal{M}^s(M)$ of $\mathcal{S}(M)$ such that the strong unstable foliation is $f-$minimal for all f in $\mathcal{M}^s(M)$.*

Theorems 1 and 3 are both direct consequences of Theorem 5, and the different items of this theorem correspond to the steps of the proofs of Theorems 1 and 3.

Let us return to the known examples of three-dimensional robustly transitive diffeomorphisms with a strongly partially hyperbolic structure, [Mñ$_1$, BD$_1$, SW]. The examples in [BD$_1$, SW] have closed periodic orbits and thus the two strong foliations are minimal. On the other hand, in the derived from Anosov examples in [Mñ1] our results only ensure the minimality of one of the strong foliations. In fact, our proof shows that if the initial Anosov diffeomorphism has a one-dimensional stable (resp. unstable) bundle then the strong stable (resp. unstable) foliation of the derived map is minimal. This leads us to the following question:

Problem 1.4. *Is having (simultaneously) minimal strong stable and unstable foliations a dense property in* $\mathcal{T}(M)$*?*

Concerning $3-$dimensional robustly transitive diffeomorphisms, we can now consider those admitting only a splitting with two bundles (see the constructions in [BV]), for instance, having a splitting of the form $E^u \oplus E^c$, where E^u is one-dimensional and uniformly hyperbolic (expanding) and E^c is two-dimensional, undecomposable and non-hyperbolic. The hyperbolicity of E^u ensures the existence of the strong stable foliation.

Problem 1.5. *Consider the robustly transitive diffeomorphisms of a compact* $3-$*manifold having a splitting* $E^c \oplus E^u$ *where* $\dim(E^c) = 2$*. Is the robust minimality of* $\mathcal{F}^u$ *a dense property for these diffeomorphisms?*

In higher dimensions the panorama of robustly transitive diffeomorphisms seems to be much more open. For instance, the existence of a hyperbolic direction (and therefore of an invariant foliation) is not guaranteed (see examples in [BV]). So it should be important to understand which geometrical object would replace the invariant foliations of the partially hyperbolic theory.

This paper is organized as follows:

In Section 2 we prove that there is an open and dense subset of $\mathcal{T}(M)$ of diffeomorphisms such that both stable and unstable manifolds of a periodic point are dense.

In Section 3 we prove that the existence of a complete s or $u-$section is an open property.

In Section 4 we show that the existence of a complete $u-$section generically implies the robust minimality of the unstable foliation.

In Section 5 we show that every Kupka-Smale diffeomorphism of $\mathcal{T}^+(M)$ has a complete s or $u-$section. The first step of this section is to prove the existence of invariant manifolds tangent to the central bundle of the periodic points, this property will substitute the unique integrability of the central bundle. In the second step we get a complete section by considering the union of the strong stable or strong unstable leaves through the points in these central curves.

In Section 6 (under the hypotheses of Theorem 3) we see that the local stable and unstable manifolds of any compact periodic leaf of the central foliation are complete u and $s-$sections.

In Section 7 we explain how to extend our results to the non-orientable case, assuming the accessibility property (Theorem 2).

In Section 8 we extend our results to higher dimensions (Theorem 4).

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2 Robust density of invariant manifolds of hyperbolic periodic points

Given a diffeomorphism f and a hyperbolic periodic orbit P_f of f we say that the stable manifold of P_f, $W^s(P_f)$, is *robustly dense* in M if there is a C^1-neighborhood of f consisting of diffeomorphisms g such that $W^s(P_g)$ is dense in M, where P_g is the continuation of P_f for the diffeomorphism g. We define in the same way the robust density of the stable manifold of the orbit of P_f, and of the unstable manifold of P_f or its orbit.

In this section we show that the invariant manifolds of periodic orbits of generic diffeomorphisms of $\mathcal{T}(M)$ are robustly dense in M. This property is just a remark (see Proposition 2.1) when the invariant manifold is $2-$dimensional. For the one-dimensional manifolds the key idea is to use the Hayashi's Connecting Lemma, [H], to get *heterodimensional cycles.* After unfolding this cycle one gets that the closure of this one-dimensional manifold contains in a robust way the $2-$dimensional stable manifold of a saddle of different index (see [DR] and Section 2.1). Then, using Proposition 2.1, this one-dimensional invariant manifold is robustly dense in the whole M.

Proposition 2.1. *For every diffeomorphism $f \in \mathcal{T}(M)$ and every hyperbolic periodic point P of index one of f, the unstable manifold of the orbit of P is robustly dense in M. Similarly, the stable manifold of the orbit of any hyperbolic periodic point of f of index two is robustly dense in M.*

Proof: Consider $f \in \mathcal{T}(M)$ and any hyperbolic periodic point P of f of index one. Consider any point $z \in M$ whose forward orbit is dense in M. Then there is some big i such that $w = f^i(z)$ is close enough to P so that the strong stable leaf $F^s(w)$ of w intersects transversely the interior of $W^u(P)$ at some point y. Now, using that f exponentially contracts the leaves of the strong stable foliation of f, one has that the $\omega-$limit sets of y, w and z are equal. In particular, one has $\omega(y) = \omega(w) = \omega(z) = M$. So the orbit of $W^u(P)$ is dense in M, that is, the stable manifold of the orbit of P is dense in M. As the set $\mathcal{T}(M)$ is open and the density holds for any $f \in \mathcal{T}(M)$, the stable manifold of the orbit of P is robustly dense in M, ending the proof of the lemma. $\square$

The arguments in the proof of the previous proposition give somewhat more:

Scholium 2.2.

- *Every $f-$invariant surface immersed in M transversely to $\mathcal{F}^s_f$ or $\mathcal{F}^u_f$ is dense in M.*
- *Proposition 2.1 and the item above hold in any dimension for robustly transitive diffeomorphisms having a partial hyperbolic structure $E^s \oplus E^c$ (E^s is a uniformly contracting bundle). In that case, every $\dim(E^c)-$dimensional unstable manifold of a periodic orbit is dense in M. A similar result holds for the stable manifolds.*

The main result of this section is:

Proposition 2.3. *Let $\mathcal{V}$ be an open subset of $\mathcal{T}(M)$ such that, for every f in $\mathcal{V}$, there is a hyperbolic periodic point P_f of index one depending continuously on $f \in \mathcal{V}$.*

Then there is an open and dense subset $\mathcal{V}_1$ of $\mathcal{V}$ such that, for every $f \in \mathcal{V}_1$, the stable manifold of the orbit of P_f is dense in M.

Proof: By definition of $\mathcal{T}(M)$ any $f \in \mathcal{T}(M)$ has some periodic point Q_f of index 2. Up to cover $\mathcal{V}$ by open subsets, we can assume that Q_f depends continuously on $f \in \mathcal{V}$.

By the robust transitivity of $f \in \mathcal{V}$ we have that P_f and Q_f belong to the same transitive set (the ambient manifold M). Now [BD$_2$, Proposition 1.1] implies that there is a dense subset $\mathcal{W} \subset \mathcal{V}$ such that, for every $f \in \mathcal{W}$, one has that $W^u(P_f) \cap W^s(Q_f) \neq \emptyset$ and $W^s(P_f) \cap W^u(Q_f) \neq \emptyset$ (in fact, this part of the proposition is a direct consequence of Hayashi's Connecting Lemma).

Observe that, by the strong partial hyperbolicity of f, every point in $W^u(P_f) \cap W^s(Q_f)$ corresponds to a transverse intersection of these $2-$dimensional invariant manifolds, and each point in $W^s(P_f) \cap W^u(Q_f)$ is a quasi-transverse intersection of these $1-$dimensional invariant manifolds. So P_f and Q_f belong to a generic heterodimensional cycle of codimension one as in [DR]. Moreover, the strong partial hyperbolicity of f implies that this cycle is far from homoclinic tangencies.

Now the fact in the proof of [BDPR, Lemma 7.2] (which is just a reformulation of [DR, Proposition 3.6(b)]) implies that any $f \in \mathcal{W}$ belongs to the closure of an open set $\mathcal{V}_f$ consisting of diffeomorphisms g such that the stable manifold of P_g meets any $2-$disk intersecting $W^s(Q_g)$ and transverse to the strong stable foliation. In particular, the closure of $W^s(P_g)$ contains $W^s(Q_g)$, which is dense in M (recall Proposition 2.1). Thus $W^s(P_g)$ is dense in M for every $g \in \mathcal{V}_f$. In the next subsection we will state more precisely the result on heterodimensional cycles which is the key of this argument (see Proposition 2.6).

Finally, by construction, the set $\mathcal{V}_1 = \bigcup_{f \in \mathcal{W}} \mathcal{V}_f$ is the announced dense open subset of $\mathcal{V}$. The proof of the proposition is now complete. $\square$

Remark 2.4. *Let P be a hyperbolic periodic point (of any index) of $f \in \mathcal{T}(M)$ such that the stable manifold $W^s(P)$ of its orbit is dense in M. Then given any surface Σ transverse to $\mathcal{F}^s_f$ one has that $W^s(P) \cap \Sigma$ is dense in Σ. This follows from the fact that $W^s(P)$ is saturated by the leaves of the strong stable foliation.*

Corollary 2.5. *Under the hypotheses of Proposition 2.3, the homoclinic class of P_f is equal to M for all $f \in \mathcal{V}_1$.*

As a consequence, there is an open and dense subset of $\mathcal{T}(M)$ of diffeomorphisms whose hyperbolic periodic points of index 1 (or index 2) are dense in M.

2.1 Heterodimensional cycles: Proposition 2.6

We say that a diffeomorphism f has a *codimension one heterodimensional cycle* if there are hyperbolic periodic points P_f and Q_f of indices $k+1$ and k, respectively, such that $W^s(P_f)$ and $W^u(Q_f)$ have nonempty transverse intersections and $W^u(P_f)$ and $W^s(Q_f)$ have a quasi-transverse intersection (i.e., there is a point x of $W^u(P_f) \cap W^s(Q_f)$ such that $T_x W^u(P_f) + T_x W^s(Q_f)$ has dimension $\dim(M) - 1$, where M is the ambient manifold). This heterodimensional cycle is *far from homoclinic tangencies* if every g in a C^1-neighbourhood of f has no homoclinic tangencies associated neither to P_g nor Q_g.

Proposition 2.6. *Let f be a C^1-diffeomorphism having a codimension one heterodimensional cycle associated to the hyperbolic periodic points P_f and Q_f, where the index of P_f is bigger than the index of Q_f. Suppose that the cycle is far from homoclinic tangencies. Then there is a C^1-open set $\mathcal{U}_f$ whose closure contains f such that*

$$W^s(P_g) \subset \text{closure}(W^s(Q_g)) \quad \textit{and} \quad W^u(Q_g) \subset \text{closure}(W^u(P_g))$$

for every diffeomorphism $g \in \mathcal{U}_f$.

This kind of phenomenon, an invariant manifold of dimension $k+1$ being contained in the closure of the corresponding invariant manifold of dimension k, was first exhibited in [Dí] for the unfolding of some heterodimensional cycles (in the context of parametrized families of diffeomorphisms). In [BD$_1$] using the *blenders* (a kind of skew horseshoe) was shown that the unfolding of some heterodimensional cycles is a key mechanism for obtaining robust transitivity. In fact, the blenders give a mechanism guaranteeing that (for instance) the closure of a stable manifold of dimension k contains a stable manifold of dimension strictly bigger than k in a robust way.

Later, in [DR] was introduced the *generalized blenders* (hyperbolic sets with skew Markov partitions) and proved that the previous result always holds in the context of bifurcations of heterodimensional cycles in parametrized families of diffeomorphisms (i.e., the closure of a stable manifold of the point of index k in the cycle contains the stable manifold of the point of index $k+1$ of the cycle in a robust way).

Proof: Proposition 2.6, is obtained as follows: The fact in the proof of [BDPR, Lemma 7.2] (which is just a reformulation of [DR, Proposition 3.6(b)]) ensures that f belongs to the closure of an open set $\mathcal{V}_f$ of diffeomorphisms g such that the stable manifold of Q_g transversely meets any $(\dim(M)-k)-$disk intersecting $W^s(P_g)$ and transverse to a strong stable foliation of $W^s(P_g)$. In particular, the closure of $W^s(Q_g)$ contains $W^s(P_g)$.

To proof the proposition observe that P_g has a transverse homoclinic point and that the stable manifold of P_g is contained in the closure of the stable manifold of Q_g for all $g \in \mathcal{V}_f$. The Connecting Lemma now implies that there is a dense subset of $\mathcal{V}_f$ consisting of diffeomorphisms g having a heterodimensional cycle associated to P_g and Q_g. By hypothesis, these cycles are far from tangencies. In this way one gets a sequence g_n, $g_n \in \mathcal{V}_f$, converging to f and such that every g_n has a heterodimensional cycle far from homoclinic tangencies.

Arguing as above, but now applying the previous construction to g_n^{-1}, for each n we get an open set $\mathcal{V}_n \subset \mathcal{V}_f$ whose closure contains g_n such that the closure of $W^u(P_g)$ contains $W^u(Q_g)$ for all $g \in \mathcal{V}_n$. Finally, it is enough to take $\mathcal{U}_f = \cup_n \mathcal{V}_n$. □

3 Proof of item 1 of Theorem 5: the existence of a complete section is an open property

The aim of this section is to prove the following proposition (corresponding to item 1 of Theorem 5).

Proposition 3.1. *The sets $\mathcal{S}(M)$ and $\mathcal{U}(M)$ of diffeomorphism admitting complete s and $u-$sections, respectively, are open in $\mathcal{T}(M)$.*

The first step of the proof is a general lemma about transverse sections of $1-$dimensional foliations. Consider a foliation $\mathcal{G}$ of dimension one, a submanifold (with boundary) T of M is a *complete section* if it cuts transversely every leaf of $\mathcal{G}$.

Lemma 3.2. *Consider a foliation $\mathcal{G}$ of dimension one having a complete section T. Then there is $K_T = K > 0$ such that every segment γ of length bigger than K contained in a leaf of $\mathcal{G}$ intersects transversely the interior of T. As a consequence, the interior of T contains a compact subset T_0 which also is a complete section.*

Proof: The proof is by contradiction. Suppose that the result is false. Then there is a sequence of segments γ_n of length n contained in leaves of $\mathcal{G}$ such that $\gamma_n \cap T = \emptyset$ for all n. Let x_n be the middle point of γ_n. Taking a subsequence, if necessary, we can assume that (x_n) converges to some point x. Since T is a complete section, there is a segment γ containing x and contained in a leaf of $\mathcal{G}$ that intersects transversely T in its interior. This implies that γ_n also intersects transversely T for all n big enough. This leads to a contradiction and ends the first part of the lemma.

To prove the second part of the lemma consider a finite compact covering C_i of M such that for each i there is a continuous function $\phi_i\colon C_i \to \operatorname{int}(T)$ which associates to each point $x \in C_i$ a point of intersection of $\operatorname{int}(T)$ and the segment of length K centered at x and contained in a leaf of $\mathcal{G}$. Then the compact set T_0 is the union of the compact sets $\phi_i(C_i)$. This completes the proof of the lemma. $\square$

We are now ready to prove Proposition 3.1:

Proof: We prove, for instance, that the set $\mathcal{U}(M)$ is open. Let $f \in \mathcal{T}(M)$ and T_f be a complete $u-$section. By Lemma 3.2 above, there are $K > 0$ and a compact subset T'_f of the interior of T_f such that every unstable segment of length K intersects transversely T'_f.

Consider now a compact neighbourhood T^*_f of T'_f in the interior of T_f such that $f(T^*_f) \subset \operatorname{int}(T^*_f)$. One can choose T^*_f containing $f(T_f)$. By the normal hyperbolicity of the transverse section T^*_f, for every g sufficiently C^1-close to f, there is defined a continuation T^*_g of T^*_f which is a $g-$invariant compact submanifold with boundary, i.e., $g(T^*_g) \subset \operatorname{int}(T^*_g)$. For g close enough to f denote by T'_g the projection of T'_f into T^*_g along the leaves of $\mathcal{F}^u_f$. Thus every segment of length $2K$ contained in a leaf of $\mathcal{F}^u_f$ meets transversely T'_g.

Using now the continuous dependence of the foliations $\mathcal{F}^u_g$ on g, for every g sufficiently C^1-close to f, we have that every segment of length $3K$ of $\mathcal{F}^u_g$ intersects transversely T^*_g. So T^*_g is a complete section for g.

It remains to see that T^*_g is a $u-$section. This follows from the normal hyperbolicity of $\omega(T_f) = \cap_{n\geq 0} f^n(T_f)$. More precisely, since T_f is a complete $u-$section, $\omega(T_f)$ consists of normally hyperbolic circles and central segments (not necessarily pairwise disjoint) whose extremes are periodic points of f (not necessarily hyperbolic). By the choice of T^*_f, $\omega(T_f) = \omega(T^*_f)$. Observe now that the circles of $\omega(T^*_f)$ admit a continuous continuation for every g close to f. On the other hand, the central segments of $\omega(T^*_f)$ only have (in general) a lower semi-continuous continuation. Such a semi-continuity is due to the possible disappearance of the non-hyperbolic extremities of the segments.

Take $\omega(T^*_g) = \cap_{n\geq 0} g^n(T_g)$. By the previous comment, for every g sufficiently close to f, the set $\omega(T^*_g)$ is the union of circles and central segments: the circles being continuations of the circle components of $\omega(T^*_f)$ and the segments being the lower semi-continuous continuation of the central segments of $\omega(T^*_f)$. This ends the proof of the proposition. $\square$

Remark 3.3.

1. *The sections T^*_g in the proof of Proposition 3.1 are complete $u-$sections depending continuously on g in a neighborhood of f.*

2. *Observe that the constants $K_{T^*_g}$ in Lemma 3.2 can be taken independent of the diffeomorphism g if the neighbourhood of f is small enough.*

4 Proof of item 4 of Theorem 5

In this section we prove the following proposition (corresponding to item 4 of Theorem 5).

Proposition 4.1. *There is a dense open subset $\mathcal{U}_m(M)$ (resp. $\mathcal{S}_m(M)$) of $\mathcal{U}(M)$ (resp. $\mathcal{S}(M)$) consisting of diffeomorphisms having a minimal strong unstable (resp. stable) foliation.*

We prove this proposition for the set $\mathcal{U}(M)$, the proof for $\mathcal{S}(M)$ is the same.

4.1 Perfect sections

A complete $u-$section Σ of $\mathcal{F}^u_f$ is called *perfect* if it is the (disjoint) union of finitely many local stable manifolds of hyperbolic periodic points of f of index two. In particular, every perfect $u-$section is a complete $u-$section.

Let $\mathcal{U}_p(M)$ be the subset of $\mathcal{U}(M)$ of diffeomorphism having a perfect $u-$section.

Proposition 4.2. *The set $\mathcal{U}_p(M)$ is a dense open subset of $\mathcal{U}(M)$.*

Proof: We only need to prove that the set $\mathcal{U}_p(M)$ is dense in $\mathcal{U}(M)$: in fact, Proposition 3.1 and item 1 of Remark 3.3 ensure that if f admits a perfect $u-$section then every g close to f also has a perfect $u-$section (i.e., to have perfect sections is an open property).

We now prove the density of $\mathcal{U}_p(M)$ in $\mathcal{U}(M)$. Take any $f \in \mathcal{U}(M)$ and let T_f be a complete $u-$section for f. Observe first that, after a perturbation of f, we can assume that f is Kupka-Smale (i.e., density and hyperbolicity of the periodic points and general position of their invariant manifolds). Hence its dynamics is Morse-Smale in restriction to $\omega(T_f)$. Thus given any point z of T_f there are two possibilities: either it belongs to the stable manifold of a (hyperbolic) periodic point of index two, or it belongs to the stable manifold of a point of index one.

Consider now all the periodic points $Q_1, \ldots, Q_m$ of index one of $\omega(T_f)$. Take a constant $K = K_{T_f} > 0$ as in Lemma 3.2 and for each $i \in \{1, \ldots, m\}$ consider a segment γ_i of length K contained in the strong unstable leaf of Q_i and such that γ_i does not contain Q_i. By Lemma 3.2, every curve γ_i intersects T_f at some point z_i (which is, by construction, different from Q_i). Now, after an arbitrarily small perturbation of f preserving the set $\omega(T_f)$ and its Morse-Smale dynamics, we can assume that the points z_i do not belong to the (one dimensional) stable manifolds of the points $Q_1, \ldots, Q_m$.

The $\lambda-$lemma now implies that the orbit of any strong unstable leaf of the new diffeomorphism f intersects the stable manifold of some point of index two in $\omega(T_f)$.

To end the proof of the proposition we need the following lemma:

Lemma 4.3. *Let Σ be a $u-$section of $\mathcal{F}^u_f$ intersecting the forward orbit of any leaf of $\mathcal{F}^u_f$. Then Σ is a complete $u-$section.*

Proof: By hypothesis, given any $x \in M$ there is $n = n_x$ such that the unstable leaf through $f^{n-1}(x)$ intersects Σ, so the unstable leaf of $f^n(x)$ meets $f(\Sigma) \subset \text{int}(\Sigma)$. By transversality, there is a finite covering of M by open sets U_i such that for each i there is n_i such that the unstable leaf through $f^{n_i}(x)$ intersects the interior of Σ for every $x \in U_i$.

Take $n = \sup(n_i)$. Then $f^{-n}(\Sigma)$ is a complete $u-$section. By Lemma 3.2 there is K such that any unstable segment of length K intersects the interior of $f^{-n}(\Sigma)$. Let $\lambda \geq \sup_{x \in M} \|f_*(x)\|$ and take any unstable segment γ of length bigger than $\lambda^n \cdot K$. Then $f^{-n}(\gamma)$ is a unstable segment of length larger than K, and so $f^{-n}(\gamma)$ intersects the interior of $f^{-n}(\Sigma)$. Thus γ intersects the interior of Σ, and so Σ is a $u-$complete section, ending the proof of the lemma. $\square$

To end the proof of the proposition consider the periodic points $P_1, \ldots, P_k$ of index 2 of $\omega(T_f)$ and the union Σ of the local stable manifolds of these points. Recall that (by the comment before the

Lemma 4.3) the forward orbit of any strong unstable leaf of f intersects Σ. Lemma 4.3 now implies that Σ is a complete $u-$section. To see that this section is perfect just observe that, by construction, $\omega(\Sigma) = \{P_1, \dots, P_k\}$. The proof of Proposition 4.2 is now complete. $\square$

4.2 Minimality of the strong unstable foliation

In this section we end the proof of Proposition 4.1 by proving that the strong unstable foliation of any diffeomorphism f in a dense open subset $\mathcal{U}_m(M)$ of $\mathcal{U}_p(M)$ is minimal. Since, by Proposition 4.2, $\mathcal{U}_p(M)$ is an open and dense subset of $\mathcal{U}(M)$, this implies Proposition 4.1.

We begin with the following lemma:

Lemma 4.4. *Let $f \in \mathcal{U}_p(M)$ and T_f be a perfect $u-$section for f with $\omega(T_f) = \{P_1, \dots, P_m\}$. Then the closure of any leaf of the strong unstable foliation $\mathcal{F}^u_f$ contains the closure of the unstable manifold of some $P_i \in \omega(T_f)$.*

Proof: Observe that Lemma 4.3 implies that $f^n(T_f)$ is a perfect $u-$section for every $n \in \mathbb{Z}$. Moreover, for every $n > 0$ sufficiently large, the set $f^n(T_f)$ is contained in the union of arbitrarily small local stable manifolds of the points P_i. This means that each unstable leaf passes arbitrarily close to some of the P_i. The lemma now follows easily from this fact. $\square$

Lemma 4.5. *Let f be a diffeomorphism of a compact manifold M and $\mathcal{G}$ an $f-$invariant foliation such that the orbit of any leaf of $\mathcal{G}$ is dense in M. Then the leaf of $\mathcal{G}$ of any periodic point p of f is dense in M.*

Proof: Consider a periodic point p of f (of period k) and let C_0 be the closure of the leaf of $\mathcal{G}$ containing p. We need to prove that $C_0 = M$. Let $C_i = f^i(C_0)$. By hypothesis, $M = C_0 \cup C_1 \cup \dots \cup C_k$. As a finite union of compact sets with empty interiors has empty interior, there is i such that C_i has non-empty interior. Thus every C_j also has non-empty interior. Observe that, by construction, each set C_i is saturated for $\mathcal{G}$.

We now claim that given any $j \in \{1, \dots, k\}$ either the interiors of C_0 and C_j are disjoint or $C_0 = C_j$. To prove this claim it suffices to observe that if $\operatorname{int}(C_0) \cap C_j \neq \emptyset$ then the leaf through $f^j(p)$ intersects C_0 and so (since the set C_0 is saturated by the foliation $\mathcal{G}$) the set C_j is contained in C_0. This implies that $C_j \subset C_0$. As a consequence, $C_0 \cap \operatorname{int}(C_j) \neq \emptyset$. The inclusion $C_0 \subset C_j$ follows analogously, ending the proof of our claim.

If all the sets C_i are equal to C_0 we have $C_0 = M$ and we are done. Otherwise, the boundary ∂C_0 of C_0 is a non-empty set saturated for $\mathcal{G}$. Hence, by hypothesis, the $f-$orbit of ∂C_0 is dense in M. Thus there is j such that $f^j(\partial C_0) \cap \operatorname{int}(C_0) \neq \emptyset$. In other words, $\partial C_j \cap \operatorname{int}(C_0) \neq \emptyset$. By the claim $C_0 = C_j$ and so $\partial C_0 \cap \operatorname{int}(C_0) \neq \emptyset$, which is a contradiction. This ends the proof of the lemma. $\square$

We can now finish the proof of Proposition 4.1.

Proof: Recall that, as $\mathcal{U}_p(M)$ is a dense open subset of $\mathcal{U}(M)$ (Proposition 4.2), it is enough to exhibit a dense open subset of $\mathcal{U}_p(M)$ with minimal strong unstable foliations.

Consider any $f \in \mathcal{U}_p(M)$ and let T_f be a perfect $u-$section for f. Let $P_{1,f}, \dots, P_{n,f}$ be the periodic points of index 2 whose union of local stable manifolds is T_f. Take a small open neighbourhood $\mathcal{U}_f$ of f such that, for any $g \in \mathcal{U}_f$, the union T_g of the local stable manifolds of the continuations $P_{1,g}, \dots, P_{n,g}$ is a perfect $u-$section for g.

Proposition 2.3 implies that there is an open and dense subset $\mathcal{V}_f$ of $\mathcal{U}_f$ of diffeomorphisms g such that the unstable manifold of the orbit of any point $P_{i,g}$ is dense in M.

Lemma 4.4 now implies that, for any $g \in \mathcal{V}_f$, the closure of any leaf of $\mathcal{F}^u_g$ contains a leaf through one of the $P_{i,g}$. In particular, the $g-$orbit of any leaf of $\mathcal{F}^u_f$ is dense in M. By Lemma 4.5 the unstable leaf through each $P_{i,g}$ is dense. Finally, applying again Lemma 4.4, we get that every unstable leaf of $\mathcal{F}^u_g$ is dense in M, for all $g \in V_f$. This ends the proof of Proposition 4.1. $\square$

5 Generic robust minimality of one of the two strong foliations: Proof of item 2 of Theorem 5

The goal of this section is to prove item 2 of Theorem 5. Recall that the Kupka-Smale diffeomorphisms form a residual subset of $\mathrm{Diff}^1(M)$. By Corollary 2.5, the set of diffeomorphisms of $\mathcal{T}(M)$ whose hyperbolic periodic points are dense in M is open and dense in $\mathcal{T}(M)$. Let now $\mathcal{T}^+_K(M)$ be the residual subset of $\mathcal{T}^+(M)$ consisting of Kupka-Smale diffeomorphisms having periodic points dense in M. Recall that if $f \in \mathcal{T}^+(M)$ then the bundles E^s, E^u and E^c are orientable and f preserves these orientations.

Item 2 of Theorem 5 claims the density in $\mathcal{T}^+(M)$ of the diffeomorphisms admitting complete u or $s-$sections, this fact follows immediately from the next proposition:

Proposition 5.1. *Every diffeomorphism $f \in \mathcal{T}^+_K(M)$ has a complete u or $s-$section.*

The steps of the proof of Proposition 5.1 are the following:

- Construction of invariant central curves of periodic points (Section 5.1 and Lemma 5.2).
- Analysis of the topology and the dynamics of the invariant central curves and classification of the periodic points according to this analysis (Section 5.2 and Proposition 5.6).
- Using the invariant central curves one constructs $u-$sections (or $s-$sections according to the case) by considering the union of the local stable leaves of the points in this central curve (Section 5.3 and Proposition 5.8).

5.1 Invariant central curves for periodic points

In this section we prove the following lemma:

Lemma 5.2. *Let $f \in \mathcal{T}^+_K(M)$ and p be a periodic point of f (of period k). Then there is a periodic curve (of period k) $\gamma_p\colon [-1, +\infty[\to M$ starting at p (i.e., $\gamma_p(0) = p$), tangent to E^c and positively oriented (according to the orientation of E^c) such that either γ_p contains a periodic circle or it is an injective immersion of infinite length.*

Observe that this lemma is trivial if the central bundle E^c is uniquely integrable. For a generalization of this lemma without the orientation assumption see Remark 5.5.

In what follows we call the curves tangent to the central bundle E^c *central curves.*

We start the proof of Lemma 5.2 with a local argument guaranteeing the existence of an f^k-invariant local central curves tangent to E^c.

Lemma 5.3. *Let p be a hyperbolic fixed point of index 2. Then there is an $f-$invariant curve γ tangent to E^c and embedded in $W^s_{loc}(p)$, $\gamma\colon [-1, 1] \to M$, such that $\gamma(0) = p$ and $f(\gamma) \subset \mathrm{int}(\gamma)$.*

Observe that this result does not follow directly from the existence of weak stable manifolds: we need a curve tangent to the central bundle at any point.

Proof: Observe first that $W^s(p)$ is tangent at any point to the center stable bundle $E^c \oplus E^s$, hence E^s and E^c define invariant line-fields in $W^s_{loc}(p)$. In fact, E^s is tangent (in $W^s(p)$) to the local strong stable foliation of the restriction of f to $W^s_{loc}(p)$.

Consider now a rectangle R contained in $W^s_{loc}(p)$ and centered at p such that $f(R)$ is contained in the interior of R and whose boundary is the union of four segments ℓ, r, u and d, where ℓ and r are contained in leaves of the local strong stable foliation and u and d are transverse to the strong stable foliation. Moreover, oriented local stable leaves go from d to u and oriented local central leaves go from ℓ to r.

Choosing the rectangle R sufficiently thin in the central direction, we can assume that any curve in R tangent to E^c and containing p does not intersect $u \cup d$. Thus, by the Cauchy's theorem of existence of solutions of continuous differential equations, every local solution through p (i.e., a curve tangent to E^c containing p) can be extended to a maximal solution reaching both ℓ and r.

By transversality, every maximal solution in R intersects any local leaf of $\mathcal{F}^s_f$ in R in exactly one point. Since the stable leaves are oriented, for any family (γ_σ) of maximal solutions starting at p in R and any local stable leaf we can consider the supremum of the intersections of such a local leaf with the solutions γ_σ. We now define the supremum γ of the family of solutions (γ_σ) taking the set of the suprema in each leaf.

It is easy to verify that the supremum of two maximal solutions is also a maximal solution. From this one deduces that the supremum of an arbitrary family of maximal solutions (γ_σ) can be obtained as the limit of an increasing sequence of maximal solutions. Then, by the compacity of the set of maximal solutions (Ascoli-Arzelá), the supremum is a maximal solution.

Denote by γ the supremum of the family of all the maximal solutions in R. We claim that γ is $f-$invariant (this claim implies the lemma).

To prove the claim observe that, since f preserves the orientations of E^s and E^c and these bundles are f_*-invariant, the image of γ by f is the supremum of all maximal solutions in $f(R) \subset R$. Notice that $\gamma \cap f(R)$ is a maximal solution in $f(R)$, so it is below $f(\gamma)$. Let γ_1 be an extension of $f(\gamma)$ to a maximal solution in R. By definition of γ, the curve γ_1 is below γ. Hence the curves γ and γ_1 coincide over $f(R)$. We have shown that $f(\gamma) = \gamma \cap f(R)$, ending the proof of the claim (and thus of the lemma). □

Let us observe that there are similar results for periodic points of index 2 and for periodic points of index 1.

Lemma 5.4. *Let p be a hyperbolic periodic point of period k and index 2 of f and $\gamma\colon [0,1] \to M$ be an f^k-invariant central curve starting at p (i.e., $\gamma(0) = p$) and contained in $W^s(p)$. Let $\Upsilon_p = \cup_{n\in\mathbb{N}} f^{-nk}(\gamma)$. Then there are are two possibilities:*

- *either the length of Υ_p is infinite*
- *or the closure of Υ_p is a compact central segment whose extremes are p and q, where q is a hyperbolic periodic point of index 1 whose period is a divisor of k.*

There is a similar result for points p of index 1, in this case if the length of Υ_p is finite then the extreme q has index 2.

Proof: Assume that the length of Υ_p is finite (otherwise we are done). By construction, the extreme q of Υ_p different from p is a periodic point whose period divide k. Analyzing the dynamics of f^k in this segment, since it is tangent to E^c, we have that (except the point p) this curve is contained in the unstable manifold of q. Thus $E^c \oplus E^u$ is the unstable bundle of q. Thus q has index 1, ending the proof of the lemma. $\square$

We are now ready to end the proof of Lemma 5.2. We repeat the procedure in Lemma 5.4 starting at the point $p = p_0$ and argue inductively. Observe that the existence of the curves γ satisfying Lemma 5.4 follows from Lemma 5.2. At each inductive step we obtain an f^k−invariant curve having either infinite length (and in this case we are done) or a point of different index and period bounded by k (the period of the initial periodic point).

In this way we get a sequence of points p_0, q_0, p_1, $q_1, \dots, p_k, q_k$, where the p_i have index two and the q_i index one, q_i is a extreme of Υ_{p_i}, and p_{i+1} is a extreme of Υ_{q_i}. Since the number of periodic points of f of period less than k is bounded, at some stage of the construction we get a curve of infinite length (and we are done) or a first j such that $p_i = p_j$ or $q_i = q_j$ for some $i < j$. In this last case and assuming (for instance) that $p_i = p_j$, we have that the curve $\Upsilon = \cup_{t=0}^{j-1}(\Upsilon_{p_t} \cup \Upsilon_{q_t})$ contains a circle. More precisely, observing that, by construction, $\Upsilon_{q_{j-1}}$ is tangent to p_j and that the curves Υ_r follows the orientation of E^c we have that the curve $\cup_{t=i}^{j-1}(\Upsilon_{p_t} \cup \Upsilon_{q_i})$ is a circle. The proof of Lemma 5.2 is now complete. $\square$

Let us make two remarks about the proof of Lemma 5.2 that we will use in the non-orientable case:

Remark 5.5. (Extensions of Lemma 5.2)

Non-transitive case: *The conclusions in Lemma 5.2 also hold (and the proof is exactly the same) for Kupka-Smale diffeomorphisms f with a partially hyperbolic splitting $TM = E^s \oplus E^c \oplus E^u$ defined in the whole manifold such that the bundles E^s, E^u and E^c are orientable and f preserve these orientations.*

Non-transitive and non-orientable case: *Suppose now that E^c is orientable and f preserves this orientation, but either some of the bundles E^s or E^u is non-orientable or the bundles E^s and E^u are orientable and f does not preserve some of these orientations. In this case Lemma 5.2 can be stated in the following way:*
Given any hyperbolic periodic point p of f of period k there is a periodic curve (of period less than $4\,k$) $\gamma_p\colon [-1, +\infty[\to M$ starting at p (i.e., $\gamma_p(0) = p$), tangent to E^c and positively oriented (according to the orientation of E^c) such that either γ_p contains a periodic circle or it is an injective immersion of infinite length.

Proof: It is enough to consider a lift $\tilde{f}$ of f to a covering of M corresponding to the possible local orientations of E^s and E^u. This lift may be not transitive, but it is Kupka-Smale and strongly partially hyperbolic (for completeness see comments in Section 7). Thus we can apply the first item of this remark. $\square$.

5.2 Topology and the dynamics of central manifolds: classification of the periodic points

We now classify the periodic points p of $f \in \mathcal{T}_K^+(M)$ in three (a priori non disjoint) classes. Observe that, since E^c is not necessarily uniquely integrable, to a periodic point p it is possible to associate (a priori) several central curves γ_p satisfying Lemma 5.2.

By construction, the period of any periodic point in γ_p divide the period of p. Thus, since the diffeomorphism f is Kupka-Smale, the curve γ_p only contains finitely many periodic points. Hence, if γ_p is an injective immersion, the last periodic point of γ_p (running γ_p in the positive orientation) is well defined. Denote such a point by $q(\gamma_p)$.

1. Let Γ_1 be the set of periodic points p of f such that there is an f^k-invariant central curve γ_p containing a periodic closed curve (k is the period of p).
2. Let Γ_2 be the set of periodic points p of f such that there is an f^k-invariant central curve γ_p being an injective immersion and whose last periodic point $q(\gamma_p)$ has index 2 (so $q(\gamma_p)$ is an attractor for the restriction of f^k to γ_p).
3. Let Γ_3 be the set of periodic points p of f such that there is an f^k-invariant central curve γ_p which is an injective immersion and whose last periodic point $q(\gamma_p)$ has index 1 (so $q(\gamma_p)$ is an repellor for the restriction of f^k to γ_p).

As f preserves the orientation of E^c each set Γ_i is $f-$invariant. Moreover, the union of the sets Γ_i is the set of periodic points of f which (by hypothesis) is dense in M. Hence the closure of some Γ_i has non-empty interior. Thus, by the transitivity of f, such a Γ_i is dense in M. Proposition 5.1 now follows directly from Proposition 5.6 below:

Proposition 5.6. *Let $f \in \mathcal{T}_K^+(M)$. The following holds:*

1. *If Γ_2 is dense in M then there is a complete $u-$section for f.*
2. *If Γ_3 is dense in M then there is a complete $s-$section for f.*
3. *If Γ_1 is dense in M then then there are (simultaneously) complete u and $s-$sections for f.*

Proof: We will prove that if the union Γ of Γ_1 and Γ_2 is dense in M then there exists a complete $u-$section for f. This will be done in the next section.

5.3 Construction of $u-$sections

We say that a compact central curve J is *$c-$contracting* if it is a circle or a simple segment such that $f^k(J)$ is contained in the interior of J for some $k > 0$. The *local stable manifold of a central curve J*, denoted by $W^s_{loc}(J)$, is the union of the local stable leaves of the points in J.

Lemma 5.7. *Let J be a $c-$contracting central curve. Then the local stable manifold of the orbit of J is a $u-$section.*

Proof: Remark that the only thing to verify is that this local stable manifold is a surface tangent to $E^s \oplus E^c$. This fact follows as in the proof of the local stable manifold theorem. □

Proposition 5.8. *Suppose that $\Gamma = \Gamma_1 \cup \Gamma_2$ is dense in M. Then given any point z there are a $c-$contracting central curve J and a neighbourhood U_z of z such that the local unstable leaf of y intersects $W^s_{loc}(J)$ for every $y \in U_z$.*

This proposition implies immediately Proposition 5.6: by compacity, there are finitely many points $z_1, \dots, z_n$ such that the neighbourhoods $U_{z_1}, \dots, U_{z_n}$ cover M, then the union of their corresponding $u-$sections is a complete $u-$section.

We will use the following remark which follows immediately from the continuity of the central bundle E^c:

Remark 5.9. *There is $\delta > 0$ such that every central curve of length less than δ has no auto-intersections. In particular, the length of any circle contained in a central curve is at least δ.*

Lemma 5.10. *Suppose that the set Γ is dense in M. Then for every $z \in M$ there are a neighbourhood U_z of z, a periodic point $p \in \Gamma$, and a segment I of length δ (δ as Remark 5.9) contained in a curve γ_p given by Lemma 5.2 such that the union of the local strong stable manifolds of the points in the interior of I intersects the local unstable leaf of every point in U_z.*

Proof: Consider the local unstable leaf $F^u_{loc}(z)$ of z and the topological surface Δ obtained considering the union of the local stable leaves of the points in $F^u_{loc}(z)$. The set $(U_z \setminus \Delta)$ has two connected components, say U_z^ℓ and U_z^r, where the points in U_z^ℓ are at the left of the points U_z^r (following the orientation of the central leaves). By the density of Γ in M, there is a hyperbolic periodic point $p \in \Gamma \cap U_z^\ell$. If U_z is sufficiently small (diameter less than $\delta/2$) then the segment I of length δ in γ_p starting at p meets Δ, ending the proof of the lemma. □

Proof of Proposition 5.8: To prove the proposition we now need to modify (by extending or shrinking) the segment I given by Lemma 5.10 in order to get the announced $c-$contracting curve J.

Recall that I is contained in the curve γ_p, $\gamma_p \colon]-1, +\infty[\to M$, parametrized following the positive orientation of E^c. Let $I = \gamma_p([0, s])$ and k the period of p.

To construct J we need to distinguish four cases, according to the indices of the first and the last periodic points of I:

1. *the point p has index 2 and the last periodic point $\gamma_p(t)$ in I has index 2:* Then $s \geq t$ and, for every $\varepsilon > 0$ small enough, the curve $J = \gamma_p([-\varepsilon, s + \varepsilon])$ contains the same periodic points as $\gamma_p([0, t])$ and $f^k(J)$ is contained in the interior of J. In this case J is a $c-$contracting curve.

2. *the point p has index 1 and the last periodic point $\gamma_p(t)$ in I has index 2:* Then $s \geq t$ and, for every $\varepsilon > 0$ small enough, the curve $J = \gamma_p([\varepsilon, s + \varepsilon])$ contains every periodic point of $\gamma_p([0, s])$ except the point p, and $f^k(J)$ is contained in the interior of J. In this case J is a $c-$contracting curve.
   Moreover, the union of the local strong stable manifolds of the points in J intersects transversely the local strong unstable manifold of every point close to z.

3. *the point p has index 2 and the last periodic point $\gamma_p(t)$ in I has index 1:* As $p \in \Gamma$, there is a periodic point $q = \gamma_p(r)$, $r > s$, of index 2. Then, for every $\varepsilon > 0$ small enough, the curve $J' = \gamma_p([-\varepsilon, r + \varepsilon])$ is such that $f^k(J')$ is contained in the interior of J'.
   Observe that J' may exhibit auto-intersections and there are the following possibilities: J' is either a circle or $c-$contracting curve (in these cases we take $J = J'$) or the union of a circle C and a segment H having one extremity in the circle (where $C \cap H$ is a periodic point). In this last case we need to shrink J'.

4. *the point p has index* 1 *and the last periodic point $\gamma_p(t)$ in I has index* 1*:* Since $p \in \Gamma$, there is a periodic point $q = \gamma_p(r)$, $r > s$, of index 2. Then, for every $\varepsilon > 0$ small enough, the curve $J' = \gamma_p([\varepsilon, r + \varepsilon])$ is such that $f^k(J')$ is contained in the interior of J'.
   Moreover, if ε is small enough, the union of the local strong stable manifolds of the points in J' meets transversely the local strong unstable manifold of every point close to z.
   As above, J' is either a circle or a $c-$contracting curve (in this case, $J = J'$) or the union of a circle C and a segment H. In this last case we will shrink J'.

It remains to consider the case where $J' = C \cup H$ (C is a circle, H is a segment, and $H \cap C$ is a periodic point), corresponding to items 3 and 4 above. In such a case we will replace J' by a smaller curve J being a circle or a $c-$contracting curve.

Let w be the point of J' whose local strong stable leaf intersects the local unstable leaf of z. There are the following possibilities for the position of w in J':

- $w \in C$: In this case we let $C = J$ (a circle).
- $w \notin C$ (i.e., w is in the interior of H): By construction of J', there are two possibilities:
    - either the point w belongs to the stable manifold of some periodic point q of index 2: In this case let J be the intersection of the local stable manifold of q and J'. Observe that J is a $c-$contracting curve.
    - or w is a periodic point of index 1: In this case, by construction of J', there are two periodic points q_1 and q_2 of index 2 in the curve J', $q_1 \in H$ and q_2 at the right of w, such that the sub-interval $[q_1, q_2]$ of J' joining q_1 and q_2 contains w. Then we take J as the intersection of J' and the local stable manifold of $[q_1, q_2]$. The resulting interval is a $c-$contracting curve.

The comments above ends the proof of Proposition 5.8. □

The proof of Proposition 5.6 is now complete. □

6 Proof of Theorem 3 (item 3 of Theorem 5)

First let us recall that, for $f \in \mathcal{T}_0(M)$ the bundle E^c is uniquely integrable, so there is a central foliation $\mathcal{F}^c_f$ tangent to E^c. Moreover, for $f \in \mathcal{T}_0^+(M)$ the foliations $\mathcal{F}^s_f$, $\mathcal{F}^c_f$ and $\mathcal{F}^u_f$ are all orientable and the diffeomorphism f preserves these orientations. Finally, by Propositions 3.1 and 4.1, to prove Theorem 3 it is enough to construct a pair of complete (u and $s-$) sections for a dense subset of $\mathcal{T}_0^+(M)$.

Let γ be a periodic compact leaf of the central foliation. For simplicity, we will assume that γ is fixed, the periodic case follows similarly. The curve γ is a normally hyperbolic compact manifold, so that one can speak of the stable and unstable manifolds, $W^s(\gamma)$ and $W^u(\gamma)$, of γ. These manifolds are the union of the stable and unstable leaves (respectively) through the points in γ. As the bundles E^s and E^u are 1$-$dimensional and oriented, the manifolds $W^s(\gamma)$ and $W^u(\gamma)$ are 2$-$dimensional cylinders. Then the set $(W^s(\gamma) \setminus \gamma)$ has two connected components, denoted by $W^s_-(\gamma)$ and $W^s_+(\gamma)$, following the convention that the oriented leaves of $\mathcal{F}^s_f$ go from $W^s_-(\gamma)$ to $W^s_+(\gamma)$. Both $W^s_-(\gamma)$ and $W^s_+(\gamma)$ are $f-$invariant so, by Scholium 2.2, they are both dense in M.

Similarly, given any point x we let $F^c_-(x)$ and $F^c_+(x)$ be the two connected components of $(F^c(x) \setminus x)$, where the points in $F^c_-(x)$ are at the left of the points of $F^c_+(x)$ following the orientation of the central manifold.

We now construct a complete $u-$section for f. The construction of the complete $s-$section is obtained analogously.

Proposition 6.1. *Every local stable manifold of γ is a complete $u-$section for f.*

Proof: One of the difficulties of the proof of the proposition arises from the fact that $W^s(\gamma)$ is not a priori saturated by the central foliation (recall that it is saturated for $\mathcal{F}^s_f$). The next lemma allows us to bypass this difficulty:

Lemma 6.2. *Consider the component $W^s_+(\gamma)$ of $W^s(\gamma)$. There is $i \in \{-,+\}$ such that, for all $x \in W^s_+(\gamma)$, the component $F^c_i(x)$ is contained in $W^s_+(\gamma)$.*

There is a similar result for the component $W^s_-(\gamma)$.

Proof: The fact that the lemma holds for x in a small neighborhood U of γ in W^s_+ is a general fact for foliations on surfaces in the neighborhood of a compact leaf. Now, observing that $W^s_+(\gamma) = \bigcup_{i\in\mathbb{Z}} f^i(U)$, we extend this property to the whole $W^s_+(\gamma)$. $\square$

End of the proof Proposition 6.1: Given $f \in \mathcal{T}^+_0(M)$ consider any leaf F^u of the strong unstable foliation and an (arbitrarily small) segment L^u of it. For each point $x \in L^u$ take a small segment of the central foliation centered at x, say $L^c(x) \subset F^c(x)$. As, by hypothesis, f is dynamically coherent the union of these central segments is a disk Σ tangent to $E^c \oplus E^u$, thus transverse to the strong stable foliation. Denote by Σ^- and Σ^+ the two connected components of $(\Sigma \setminus L^u)$, where the central leaves go from Σ^- to Σ^+.

Notice that $W^s_+(\gamma)$ remains dense after removing from it the local stable manifold of γ. Let $W^s_{+,0}(\gamma)$ be this reduced separatrix. By construction, there is $\delta > 0$ such that, for each $x \in W^s_{+,0}(\gamma)$, the stable segment of length δ centered at x is contained in $W^s_+(\gamma)$. Now, using the density of $W^s_{+,0}(\gamma)$, we get that $W^s_+(\gamma) \cap \Sigma$ is dense in Σ.

We fix a pair of points $x^- \in L^c_-(x) \subset \Sigma^-$ and $x^+ \in L^c_+(x) \subset \Sigma^+$. Since $W^s_+(\gamma) \cap \Sigma$ is dense in Σ there are intersections $z^\pm$ of $W^s_+(\gamma) \cap \Sigma^\pm$ arbitrarily close to the points $x^\pm$.

Observe that by definition (orientation criterion) one has that

- $F^c_+(z^-)$ intersects L^u,
- $F^c_-(z^+)$ intersects L^u.

Let $i \in \{+,-\}$ be such that the component $F^c_i(w)$ of $(F^c(w) \setminus w)$ is contained in $W^s_+(\gamma)$ for all $w \in W^s_+(\gamma)$, recall Lemma 6.2. By the previous comment it follows that if $i = +$ then $F^c_+(z^-)$ intersects L^u. Otherwise, $i = -$, $F^c_-(z^+)$ intersects L^u. In both cases we have that $W^s_+(\gamma)$ intersects (transversely) L^u, ending the proof of the proposition. $\square$

7 Extensions of the theorems for non-orientable bundles

The proof of Proposition 5.1 in the general case allows many possibilities: each bundle E^s, E^u and E^c may be orientable or not, and the diffeomorphism f may preserve or not these orientations. We split $\mathcal{T}(M)$ into the following subsets:

- $\mathcal{T}^{c+}(M)$ is the subset of $\mathcal{T}(M)$ of diffeomorphisms f such that E^c is orientable and f preserves this orientation. This set is open and closed in $\mathcal{T}(M)$ and contains $\mathcal{T}^+(M)$.

- $\mathcal{T}^{odd}(M)$ is the subset of $\mathcal{T}(M)$ of diffeomorphisms f such that E^c is orientable, f reverses this orientation, and f has a hyperbolic periodic of odd period. This set is open in $\mathcal{T}(M)$.

- $\mathcal{T}^{-}(M)$ is the set of diffeomorphisms f in $\mathcal{T}(M)$ having a hyperbolic periodic orbit p (of period k) such that $f_*^k(p)$ reverses the orientations of $E^c(p)$. Observe that this set is open.

- $\mathcal{T}^{\pm}(M)$ is the interior of $\big(\mathcal{T}(M) \setminus \big(\mathcal{T}^{c+}(M) \cup \mathcal{T}^{-}(M)\big)\big)$, that is, the set of diffeomorphisms $f \in \mathcal{T}(M)$ such that E^c does not admit any $f-$invariant orientation, but for every periodic point q of f and every g close enough to f, $g_*^r(q)$ preserves the orientations of $E^c(q)$ (where r is the period of q).

As in the previous sections we denote by $\mathcal{T}_K^{c+}(M)$, $\mathcal{T}_K^{-}(M)$ and $\mathcal{T}_K^{\pm}(M)$ the subsets of $\mathcal{T}^{c+}(M)$, $\mathcal{T}^{-}(M)$ and $\mathcal{T}^{\pm}(M)$, respectively, consisting of Kupka-Smale diffeomorphisms whose periodic points are dense in M.

By definition, the union $\mathcal{T}^{\pm}(M) \cup \mathcal{T}^{c+}(M) \cup \mathcal{T}^{-}(M)$ is a dense open subset of $\mathcal{T}(M)$. Recall that $\mathcal{T}_a(M)$ is the subset of $\mathcal{T}(M)$ of diffeomorphisms having the robust accessibility property.

The next result generalizes Proposition 5.1 and implies Theorem 2:

Proposition 7.1. *The set $\mathcal{S}(M) \cup \mathcal{U}(M)$ contains a dense open subset of $\mathcal{T}^{c+}(M) \cup \mathcal{T}^{odd}(M) \cup \mathcal{T}^{-}(M) \cup \mathcal{T}_a(M)$.*

In the next subsections we will prove this result.

7.1 Existence of central invariant curves in the non-orientable case. Proof of Proposition 5.1 for $f \in \mathcal{T}^{c+}(M)$

We begin this section by doing some general comments about coverings of M, the lift of transitive diffeomorphisms to these coverings, and their u and $s-$sections.

Lifts of transitive diffeomorphisms:
Given $f \in \mathcal{T}(M)$ let $\pi\colon \tilde{M} \to M$ be the covering of 8 leaves corresponding to the different orientations of the stable, unstable and central bundles of f. The diffeomorphism f acts in a natural way on these orientations, so that f has a lift $\tilde{f}$ defined on $\tilde{M}$. Observe that diffeomorphism $\tilde{f}$ is strongly partially hyperbolic, its stable, unstable and central bundles are orientable, and $\tilde{f}$ preserves these orientations. However, $\tilde{f}$ may fail to be transitive. In fact, this is the main difficulty that appears in the non-orientable case. For diffeomorphisms in $\mathcal{T}^{c+}(M) \cup \mathcal{T}^{-}(M) \cup \mathcal{T}^{odd}(M)$ this difficulty may be bypassed. For diffeomorphism in $\mathcal{T}^{\pm}(M)$ we will use the accessibility property (in fact, this is the only point in the paper where this condition is used).

Consider also the covering $\pi_c\colon M_c \to M$ of 2 leaves corresponding to the two orientations of the central bundle. Denote by f_c the corresponding lift of f to M_c. As above, f_c is strongly partially hyperbolic and it is not necessarily transitive.

Finally, observe that f is Kupka-Smale if and only if f_c is Kupka-Smale. Moreover, the periodic points of f are dense in M if and only the periodic points of f_c are dense in M_c. Similar results hold replacing f_c and M_c by $\tilde{f}$ and $\tilde{M}$.

Projections and lifts of sections:
First, observe that $\tilde{M}$ is a covering of M_c and $\tilde{f}$ is a lift of f_c.

Observe that the projection by π_c of any complete s or $u-$section for f_c (in M_c) onto M is a complete s or $u-$section for f. Conversely, the lift of any complete $u-$section for f (in M) to M_c is a complete $u-$section for f_c.

By the comments above and Remark 5.5, Lemma 5.2 can be now applied to the lift $\tilde{f}$ of f, obtaining the following:

Remark 7.2. *Let p be a periodic point of f (of period k), p_c a lift of p in M_c, and $\tilde{p}$ a lift of p_c in $\tilde{M}$. Denote by $\tilde{\gamma}$ the $\tilde{f}^{\tilde{k}}-$invariant curve starting at $\tilde{p}$ ($\tilde{k}$ is the period of $\tilde{p}$) given by Remark 5.9. Let γ_c be the projection of $\tilde{\gamma}$ onto M_c. Then γ_c is $f_c^{\tilde{k}}-$invariant curve positively oriented. Observe that the period of p_c is $\tilde{k}$, $\tilde{k}/2$ or $\tilde{k}/4$.*

Moreover, if the bundle E^c is orientable and f preserves this orientation, the projection γ of $\tilde{\gamma}$ is a $f^{\tilde{k}}-$invariant curve positively oriented.

7.1.1 Proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^{c+}(M)$

Consider any $f \in \mathcal{T}_K^{c+}(M)$, that is the bundle E^c is orientable and f preserves this orientation. Then the sets Γ_1, Γ_2 and Γ_3 defined exactly as in Section 5 are $f-$invariant and their union is dense in M. As in the case $f \in \mathcal{T}_K^{+}(M)$, the closure of some Γ_i has non-empty interior. Thus, by the transitivity of f, such a closure is the whole M. From this fact we now deduce that any $f \in \mathcal{T}_K^{c+}(M)$ has a complete u or $s-$section, ending the proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^{c+}(M)$.

7.1.2 Proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^{odd}(M)$

As is the previous case, for $f \in \mathcal{T}_K^{odd}(M)$ the orientability of E^c allows us to define the sets Γ_i, whose union is dense in M. These sets are not necessarily invariant, but they are f^2-invariant. Thus if f^2 is transitive the proof of the previous case works in this one. Next lemma ensures that f^2 is transitive for generic diffeomorphisms f of $\mathcal{T}_K^{odd}(M)$, implying Proposition 7.1 for $f \in \mathcal{T}_K^{odd}(M)$.

Lemma 7.3. *Let $f \in \mathcal{T}_K^{odd}(M)$. Suppose that f has a periodic point p of odd period such that its homoclinic class is the whole M. Then the set of periodic points of odd period of f is dense in M and f^2 is transitive.*

Proof: Just observe that the homoclinic class of p is the closure of a union of hyperbolic sets containing p and having a Markov partition. Now given any point q there are periodic orbits passing arbitrarily close to p and q. If all these orbits have even periods then we can consider new orbits whose itineraries are obtained by adding the itinerary of p (i.e., an odd number of iterates). This completes the proof of the first part of the lemma.

Consider now a point x whose forward orbit is dense in M and a point q of odd period k. Then there is a sequence n_i such that $f^{n_i}(x)$ converges to q. Observe that $f^{n_i+k}(x)$ also converges to q. Then either n_i or $n_i + k$ contains a subsequence of even numbers. Thus the closure of the orbit of x by f^2 contains all the periodic points of odd period and so it is dense in M. This completes the proof of the lemma. □

7.1.3 Lift of homoclinic classes. Proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^{-}(M)$

Let us state a general result about the lift of homoclinic classes¿ Given a hyperbolic periodic point denote by $H(p, f)$ its homoclinic class (the closure of the transverse intersections of its invariant manifolds).

Lemma 7.4. *Let p be a hyperbolic periodic point of f and p_1 and p_2 the lifts of p to M_c. Then*

$$\pi_c(H(p_1,f_c)) = \pi_c(H(p_2,f_c)) = H(p,f),$$

the homoclinic classes of p_1 and p_2 are interchanged by the automorphism of the covering π_c, and

$$H(p_1,f_c) \cup H(p_2,f_c) = \pi_c^{-1}(H(p,f)).$$

Moreover, if the points p_1 and p_2 are in the same f_c-orbit or they are homoclinically related then $H(p_1,f_c) = H(p_2,f_c) = \pi_c^{-1}(H(p,f))$.

Proof: Observe that the invariant manifolds of p_i projects onto the (corresponding) invariant manifold of p and that $W^i(p_1) \cup W^i(p_2) = \pi_c^{-1}(W^i(p))$, $i = s$ or u.

Consider a homoclinic point x of p, $x \in W^s(p) \cap W^u(p)$, and let x_1 and x_2 be the two lifts of x to M_c. Up to changing the indices of these points, we can assume that $x_1 \in W^s(p_1)$. Then, using the automorphism of the covering, we get that $x_2 \in W^s(p_2)$. Similarly, if $x_1 \in W^u(p_2)$ then $x_2 \in W^s(p_1)$, and if $x_2 \in W^u(p_2)$ then $x_1 \in W^s(p_1)$.

Suppose first that $x_1 \in W^u(p_2)$ and $x_2 \in W^u(p_1)$. Therefore $x_1 \in W^s(p_1) \cap W^u(p_2)$ and $x_2 \in W^u(p_1) \cap W^s(p_2)$. By the partial hyperbolicity of $\tilde{f}$, these intersections are transverse. Thus p_1 and p_2 are homoclinically related and their homoclinic classes are equal. Now it is easy to check that $H(p_i,f_c) = \pi_c^{-1}(H(p,f)$.

For the remaining case, i.e., for every homoclinic point $x \in W^s(p) \cap W^u(p)$ one has that the lifts x_1 and x_2 are such that (up to change of indices) $x_i \in W^s(p_i) \cap W^u(p_i)$, $i = 1, 2$. In this case, to get $H(p_i,f_c) = \pi_c^{-1}(H(p,f)$, it is enough to consider the closure of these transverse intersections of $W^s(p_i)$ and $W^u(p_i)$. This ends the proof of the lemma. □

Remark 7.5. *Let p be a hyperbolic periodic point of f of period k.*

- *If f^k reverses the orientations of $E^c(p)$ then the lifts p_1 of p_2 of p in M_c are in the same f_c-orbit. Thus, by Lemma 7.4, the lift of the homoclinic class of p to M_c is a homoclinic class of f_c.*
- *Otherwise, if f^k preserves the orientations of $E^c(p)$, Lemma 7.4 ensures that the lift of the homoclinic class of p is the union of two homoclinic classes (corresponding to the homoclinic classes of the two lifts p_1 and p_2 of p), which (a priori) may be different.*

We are now ready to prove Proposition 7.1 for diffeomorphisms f in $\mathcal{T}_K^-(M)$. Following [BD$_2$, Théorème B], there is a residual subset $\mathcal{R}_K^-(M)$ of $\mathcal{T}_K^-(M)$ of diffeomorphisms such that the homoclinic class of any periodic point of f is the whole M. By definition of $\mathcal{T}_K^-(M)$, there is a periodic point p of f such that $f_*^k(p)$ (k the period of p) reverses the orientation of $E^c(p)$. The first part of Remark 7.5 implies that the homoclinic class of p_c is the whole M_c, where p_c is any lift of p in M_c. In particular, f_c is transitive.

Observing that f_c preserves the orientation of the central bundle E^c, the proof follows (in M_c) as for $f \in \mathcal{T}_K^+(M)$. This ends the proof of Proposition 7.1 for diffeomorphisms in $\mathcal{T}^-(M)$.

7.1.4 End of the proof of Proposition 7.1: the case $f \in \mathcal{T}_K^{\pm}(M)$

In view of the two previous subsections to end the proof of Proposition 7.1 it remains to consider the diffeomorphisms f in $\mathcal{T}_K^{\pm}(M)$. As in previous cases, there is a residual subset $\mathcal{R}_K^{\pm}(M)$ of $\mathcal{T}_K^{\pm}(M)$ of diffeomorphisms such that every homoclinic class is equal to M. By Lemma 7.4, if there is some periodic point p of f having a homoclinic class whose lift to M_c is a unique homoclinic class then such a class is M_c. In this case we can construct a complete section as in the previous section. Thus it remains to consider the case where *the lift to M_c of any homoclinic class of f is the union of two different homoclinic classes.* To deal with this situation we need the following lemma:

Proposition 7.6. *Let $f \in \mathcal{T}(M)$ be a diffeomorphism such that:*

- *There is a hyperbolic periodic point p of f whose homoclinic class is equal to M and*
- *f satisfies the accessibility property.*

Then the lift of the homoclinic class of p to M_c is the whole M. In particular, the lift f_c of f is transitive.

Arguing as in the previous cases we have that this proposition implies Proposition 7.1 for $f \in \mathcal{T}_K^{\pm}(M)$.

Proof: The proof is by contradiction: Let p_1 and p_2 be the two lifts of p to M_c. Assume that $H(p_i, f_c) \neq M_c$, $i = 1, 2$. By Lemma 7.4, the orbits of p_1 and p_2 are disjoint and these two points are not homoclinically related. Moreover, again by Lemma 7.4, $H(p_1, f_c) \cup H(p_2, f_c) = M_c$ and these homoclinic classes are interchanged by the automorphism of the covering π_c. Thus these two classes have both non-empty interior and the union of their interiors is dense in M_c.

We claim that if the intersection A of these two interiors is non-empty then A is dense in M_c. In fact, we have the following:

- the projection of A by π_c in M is $f-$invariant, hence it is dense in M (due to transitivity of f),
- the set A is invariant by the the automorphism of the covering.

These two facts imply that the homoclinic classes of p_1 and p_2 are both M_c, ending the proof of the proposition when A is non-empty.

It remains to consider the case in which these two interiors are disjoint ($A = \emptyset$). In this case the boundaries of the homoclinic classes of p_1 and p_2 are equal and f_c-invariant. Denote by ∂ such a boundary.

Lemma 7.7. *The set ∂ is saturated by $\mathcal{F}_f^s$ and $\mathcal{F}_f^u$.*

Before proving this lemma, let us observe that it contradicts the accessibility hypothesis, ending the proof of Proposition 7.6.

Proof: By continuity of the foliations $\mathcal{F}_f^s$ and $\mathcal{F}_f^u$, it is enough to see that the interiors of the homoclinic classes of p_1 and p_2 are both saturated by $\mathcal{F}_f^s$ and $\mathcal{F}_f^u$.

As the periodic orbits homoclinically related to p_1 are dense in $H(p_1, f_c)$, there is a periodic orbit q homoclinically related to p_1 in the interior of $H(p_1, f_c)$. Observe that $H(p_1, f_c)$ is f_c-invariant, thus the invariant manifolds of q are contained in the interior of $H(p_1, f_c)$. Thus, since $H(p_1, f_c)$ is closed,

$$\operatorname{closure}(W^s(q)) \cup \operatorname{closure}(W^u(q)) \subset H(p_1, f_c).$$

To see the converse inclusion, observe that, since q is homoclinically related to p_1, the closure of each invariant manifold of q contains the corresponding invariant manifold of p_1, which, on its turn, contains a dense subset of $H(p_1, f_c)$. Thus the invariant manifolds of q are both dense in $H(p_1, f_c)$. Hence

$$H(p_1, f_c) \subset \text{closure}(W^s(q)),\ \text{closure}(W^u(q)).$$

Thus

$$\text{closure}(W^s(q)) = \text{closure}(W^u(q)) = H(p_1, f_c).$$

As the closure of saturated set is saturated, and $W^s(q)$ and $W^u(q)$ are saturated sets, this ends the proof of the lemma. ☐

Now the proof of Proposition 7.6 is complete. ☐

7.2 Theorem 3 in the non-orientable case

We have the following generalization of Theorem 3.

Theorem 6. *Consider the subset $\mathcal{T}_0^a(M)$ of $\mathcal{T}_0(M)$ of diffeomorphisms satisfying the robust accessibility property. Then the open set $\mathcal{O}^s(M) \cap \mathcal{O}^u(M)$ of diffeomorphisms whose strong stable and unstable foliations are both robustly minimal is dense in $\mathcal{T}_0^a(M)$.*

The proof follows arguing exactly as in the previous subsections, where the orientation hypothesis was replaced by the robust accessibility property.

As in section 7.1.4, if $f \in \mathcal{T}_0(M)$ has a unique accessibility class, then its admits a transitive lift $\tilde{f}$ on some orientations covering.

8 Generalization to higher dimensions. Proof of Theorem 4

Let M be a compact manifold of any dimension. Let us recall the $\mathcal{T}_1(M)$ denotes the set of robustly transitive diffeomorphisms f of M whose central bundle has dimension 1, is uniquely integrable and admits an $f-$invariant orientation.

Proposition 8.1. *Every Kupka-Smale diffeomorphism of $\mathcal{T}_1(M)$ whose periodic points are dense in M has a complete s or $u-$section.*

The proof of this proposition is identical to the proof of Proposition 5.1, observing that the unique integrability of the central bundle substitutes trivially Lemma 5.2. As the Kupka-Smale diffeomorphisms whose periodic points are dense in M is dense in $\mathcal{T}_1(M)$, this completes the proof of Theorem 4.

As in the previous cases, we can state Theorem 4 in the non-orientable case for diffeomorphisms with the robust accessibility property.

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Critical saddle-node horseshoes: bifurcations and entropy

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Abstract

In this paper we study one-parameter families $(f_\mu)_{\mu\in[-1,1]}$ of two dimensional diffeomorphisms unfolding critical saddle-node horseshoes (say at $\mu = 0$) such that f_μ is hyperbolic for negative μ. We describe the dynamics at some isolated secondary bifurcations which appear in the sequel of the unfolding of the initial saddle-node bifurcation.

We construct two classes of open sets of such arcs. For the first class, we exhibit a collection of parameter intervals I_n, $I_n \subset (0,1]$, converging to the saddle-node parameter, $I_n \to 0$, such that the topological entropy of f_μ is a constant h_n in I_n and h_n is an increasing sequence. So, for parameters in I_n, the topological entropy is upper bounded by the entropy of the initial saddle-node diffeomorphisms. This illustrates the following intuitive principle: a critical cycle of an attracting saddle-node horseshoe is a destroying dynamics bifurcation. In the second class, the entropy of f_μ does not depend monotonely on the parameter μ.

Finally, when the saddle-node horseshoe is not an attractor, we prove that the entropy may increase after the bifurcation.

1 Introduction

1.1 Context

The goal of this paper is to study the *destruction of non-trivial hyperbolic sets* via the loss of hyperbolicity of some periodic point of this set. We consider the case of the so-called *saddle-node horseshoes* introduced by Zeeman in [Z], constructed as follows: consider a two dimensional compact disk D and a diffeomorphism $f\colon D \to \mathrm{int}(D)$ such that the limit set of f in D is the usual Smale's horseshoe Λ (conjugate to a complete shift of two symbols) and a sink S. Then consider the collapse of the sink S and the accessible saddle P of the horseshoe via a saddle-node bifurcation in such a way that at the bifurcation the limit set of the resulting diffeomorphism in D is a non-hyperbolic attractor Λ' conjugate to the initial horseshoe Λ and whose basin of attraction contains the whole disk D. We say that Λ' is a *saddle-node horseshoe.* This bifurcation is depicted in Figure 1.

An important feature of this kind of bifurcations is that the dynamics in the sequel of the bifurcation depends simultaneously on the local dynamics in a neighbourhood of the saddle-node and on some global aspects of the bifurcating diffeomorphism which have no relation with the initial dynamics of the horseshoe Λ. Let us illustrate this fact, let P be the saddle-node of the saddle-node horseshoe Λ', then there is an interval (say) I of the unstable manifold of P such that $f_0(I)$ is

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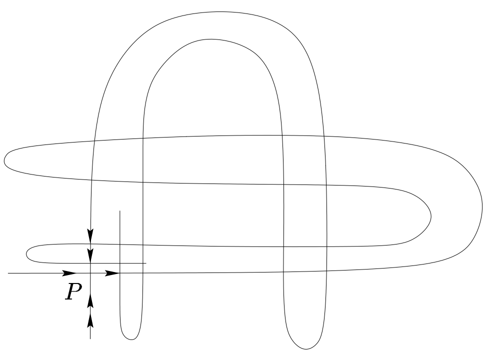

Figure 1: A critical saddle-node horseshoe

contained in the (local) stable manifold of P. So the dynamics of f_0 on I does not affect the limit set of f_0. In the sequel of the bifurcation and after an arbitrarily big number of iterations, points of I returns close to I and are involved in the creation of new limit points which are far from the initial horseshoe and that have no relations with the initial dynamics in the horseshoe (although these points may belong to hyperbolic sets containing the *remainders* of the initial horseshoe).

Consider an one-parameter family of diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ unfolding the previous attracting saddle-node horseshoe (say at $\mu = 0$). Then, after the unfolding of the saddle-node, new bifurcations appear: some periodic points of Λ lose their hyperbolicity and disappear, while homoclinic tangencies associated to the continuations of some hyperbolic periodic points of the initial horseshoe Λ are created, (see [NPT] and [DRoV]). Thus all the complexity of the homoclinic bifurcations appear in this bifurcation (as, for instance, co-existence of infinitely many sinks or sources, [N$_2$], existence of strange attractors, [MV], persistence of tangencies, [N$_2$], and cascades of period doubling bifurcations, [YA]). Conversely, Mora has observed that *critical saddle-node bifurcations* do appear as a consequence of the unfolding of homoclinic tangencies, see [PT$_3$, page 150].

In view of the previous paragraph, secondary bifurcations play a key role in the unfolding of *critical saddle-node horseshoes* (we will give a precise definition later). We analyze some of these secondary bifurcations and study the variation of the topological entropy in the sequel of the bifurcation. This problem is closely related to the one of the creation/destruction of periodic points and hyperbolic sets in the sequel of a bifurcation (see [KKY] for the case of homoclinic tangencies). Here we give a complete description of the dynamics of the limit set Λ_μ of f_μ in D by constructing a Markov partition for Λ_μ for a large set of parameters μ. We also discuss the behavior of the topological entropy in function of the type of saddle-node horseshoe (roughly speaking, if the saddle-node horseshoe is an attractor or not).

As far as we know, the first examples of saddle-node cycles appeared associated to the van der Pol equation as pointed out in [AW]. These cycles were also analyzed in [AL] and [Si]. The systematic study of saddle-node cycles was initiated in [AS, NPT] from the point of view of structural stability. We here use the tools introduced by Newhouse-Palis-Takens in [NPT] to study the local analysis of saddle-node cycles: *adapted vector fields* and *transition maps*, see Section 3.2.

As we have mentioned, saddle-node horseshoes were introduced in [Z] and currently there is a series of works that bring to light part of the richness of phenomena involved in the unfolding of this kind of bifurcation. In the example in [Z] the saddle-node has homoclinic transverse orbits (i.e., the strong stable manifold and the unstable manifold of the saddle-node exhibit transverse intersections, see definitions in Section 1.2). Let us observe that [LS] also considered this type of cycles from the point of view of the occurrence of hyperbolic dynamics following the bifurcation and the existence of non-trivial hyperbolic sets. Latter, [T] studied saddle-node cycles and saddle-node horseshoes as a model for intermittancy.

More recently, in [DRoV] were studied *critical saddle-node cycles* (see definition Section 1.2) from two points of view: occurrence of hyperbolic dynamics in the sequel of the bifurcation and existence of strange attractors. They proved that generic one parameter families of diffeomorphisms unfolding a critical saddle-node cycle always display strange attractors in a *prevalent* way (i. e., the set of parameters corresponding to diffeomorphisms with Hénon-like attractor have positive density at the bifurcation). Moreover, in the case of *one-cycles* (the unstable manifold of the saddle-node is contained in the stable manifold), these strange attractors may have a *global character,* that is, there is an invariant annulus containing the unstable manifold of the saddle-node which is contained in the basin of attraction of the strange attractor. This result was generalized in [Co] in the case of critical saddle-node horseshoes (in this case the strange attractor is the maximal invariant set in the disk D which is contained in its basin of attraction).

Finally, [DU] analyzes the phenomenon of *persistence of tangencies* in the sequel of the unfolding of a critical saddle-node horseshoe and proves that this persistence is essentially independent of the Hausdorff dimension of the saddle-node horseshoe (it is worthy to compare this result with the ones in the case of homoclinic tangencies, where there is a strong relation between frequency of hyperbolicity and fractal dimensions of the hyperbolic set involved in the homoclinic tangency, see the series of papers [N_1], [PT_1], [PT_2], [PY_1], [R] and [M]).

Non-critical saddle-node horseshoes (see definition in the next section) were studied in [CK], where the authors show that a noncritical saddle-node horseshoe may appear as an isolated bifurcation, as it occurs in the case of the derived from Anosov diffeomorphisms, see [W].

In the setting of saddle-node horseshoes (both critical and noncritical) it is known that a wide class of these bifurcation may appear as a first bifurcation, see [LS, Cr].

Finally, for a discussion on global saddle-node bifurcations we refer the reader to [AAIS, Chapter 3.4], and for a survey on saddle-node cycles to [DRiV].

We consider here the destruction of hyperbolic sets via the loss of hyperbolicity of some periodic point of it. Other main mechanism of destruction of hyperbolic sets may be obtained considering *tangencies inside horseshoes*: naively, the periodic points of the hyperbolic set remain hyperbolic but the angles between the stable and unstable manifolds of some periodic points of the horseshoe decreases converging to zero. So, some transverse homoclinic point of the hyperbolic set becomes a non-transverse homoclinic one, obtaining a homoclinic tangency before any hyperbolic point loses its hyperbolicity. This kind of bifurcations were studied in [As], [R] and [K]. See also [BS] and [Ya] for the same type of problems in the context of Hénon maps, and [BDV] and [E] in the case where the hyperbolic set is the whole manifold (Anosov diffeomorphisms) and the tangency is cubic.

1.2 Definitions and Statement of results

Before stating our results let us recall some definitions. A *saddle-node* of a diffeomorphism f is a periodic point P of f such that the derivative of Df^k at P, k is the period of P, has an eigenvalue equal to 1 and all the other eigenvalues outside the unitary circle. Here we consider two dimensional *contracting* saddle-nodes (i.e., all the eigenvalues are in the unitary disk). In such a case, the stable

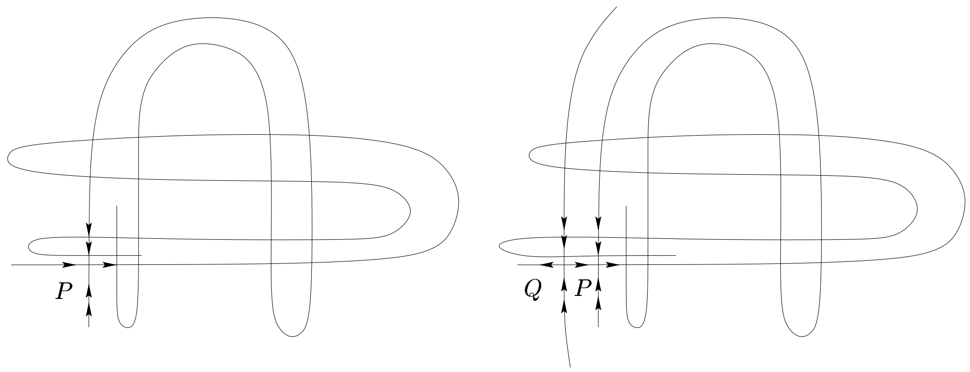

Figure 2: Critical and non critical saddle-node horseshoes

manifold of P has dimension two and its unstable manifold is one-dimensional. We say that the saddle-node is involved in a *saddle-node cycle* if its stable and unstable manifolds have nonempty intersection. This cycle is a *one saddle-node cycle* if the unstable manifold of the saddle-node is contained in the stable one.

In the stable manifold of P it is defined the *strong stable foliation,* the only f-invariant foliation having one dimensional leaves such that the leaf of the saddle-node is tangent to the strong stable bundle (the bundle associated to the contracting eigenvalue of the saddle-node), see [HPS]. The cycle is *critical* if the unstable manifold is tangent to some leaf of this strong stable foliation.

Consider a dissipative saddle-node P, its *homoclinic class,* denoted by $H(P)$, is the closure of the transverse intersections of its unstable and strong stable manifolds. If this homoclinic class is nontrivial we say that it is a *saddle-node horseshoe.* This homoclinic class $H(P)$ satisfies the properties of density of periodic points and transitivity of the usual homoclinic classes. Analogously, the saddle-node P is homoclinically related to a hyperbolic saddle Q if $W^u(Q) \pitchfork W^{ss}(P)$ and $W^u(P) \pitchfork W^s(Q)$ are both nonempty sets. We have that every saddle Q homoclinically related to P belongs to $H(P)$.

We consider here saddle-node horseshoes for which all the periodic points are hyperbolic (except for the saddle-node). A *saddle-node horseshoe cycle* is a saddle-node cycle such that the homoclinic class of the saddle-node is nontrivial. We define *critical* and *noncritical saddle-node horseshoe cycles* in the natural way.

Following the terminology introduced for saddle-node cycles, a *one saddle-node horseshoe cycle* is a saddle-node horseshoe cycle such that the stable manifold of the saddle-node horseshoe Λ contains its unstable manifold (i.e., Λ is an attractor). A *two saddle-node horseshoe cycle* is a saddle-node horseshoe cycle such that there is a saddle Q with $Q \notin \Lambda$, $W^u(P) \pitchfork W^s(Q) \neq \emptyset$ and $W^s(Q) \cap W^{ss}(P) \neq \emptyset$ (P is the saddle-node of Λ). In Figure 2 are depicted different types of saddle-node horseshoe cycles. We will see that, as in the case of one and two saddle-node cycles, the dynamics arising in the sequel of the unfolding of one and a two saddle-node horseshoe cycles may be rather different.

We are now ready to state our results. Let M be a two-dimensional manifold and D a compact disk of M. Consider the set $\mathcal{O}(D)$ of one-parameter families of $\mathcal{C}^\infty$-diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ defined on M unfolding *generically* (see Section 3.1) a critical one saddle-node horseshoe cycle such that:

- Each diffeomorphism f_μ maps D into its interior.
- For $\mu < 0$ the limit set of f_μ in D is the union of a sink S_μ and a hyperbolic saddle set Λ_μ conjugate to a complete shift of two symbols.
- For $\mu = 0$, there is a *saddle-node point* P_0 obtained by collapsing the sink S_μ with some accessible saddle P_μ of Λ_μ. The limit set of f_0 in D is the homoclinic class of P_0, denoted by Λ_0, which is conjugate to Λ_μ ($\mu < 0$).

An arc of diffeomorphisms $(f_\mu)_{\mu \in I}$ has an *isolated saddle-node horseshoe bifurcation at* μ_0 if f_{μ_0} has a saddle-node horseshoe Λ_{μ_0} and the limit set of f_μ is hyperbolic for all $\mu \neq \mu_0$. Observe that an isolated saddle-node horseshoe cycle is necessarily noncritical. See [CK] for an example of such a bifurcation.

We prove that the unfolding of a one critical saddle-node horseshoe cycle can accomplish secondary isolated saddle-node horseshoe cycles where the dynamics is completely controlled:

Theorem A. *Given a two dimensional disk D there is an open subset of $\mathcal{O}(D)$ of arcs $(f_\mu)_{\mu \in [-1,1]}$ unfolding a critical one saddle-node cycle at $\mu = 0$ such that there is a sequence of intervals $H_k \subset (0,1)$ accumulating on 0 and a sequence of parameters $\beta_k \in H_k$ such that*

- *every arc $(f_\mu)_{\mu \in H_k}$ has an isolated saddle-node horseshoe bifurcation at β_k, and*
- *for every $\mu \in H_k$ the topological entropy of f_μ in D, $h(f_\mu|_D)$, is constant (say equal to h_k) and h_k is an strictly increasing sequence converging to $\log(2) = h(f_0|_D)$.*

This theorem gives an explanation to the intuitive idea that a one saddle-node horseshoe bifurcation is a destroying dynamics bifurcation.

Let us observe that the bifurcations at the parameters β_k correspond to isolated bifurcations via non-critical saddle-node cycles, similar to those considered in [CK]. In particular, for each k, the diffeomorphisms $f_\mu \in H_k$ do not exhibit homoclinic tangencies.

Concerning the problem of the variation of the entropy in saddle-node horseshoe cycles, let us observe that, at least in the context of C^1-diffeomorphisms, every diffeomorphism corresponding to a point of entropy variation is approximated by diffeomorphisms with homoclinic tangencies, see [PS]. This result suggests that in the case of non-critical saddle-node horseshoe cycles the entropy is constant throughout the bifurcation. In fact, this is what happens in the case considered in [CK], where the horseshoes before and after the bifurcation are conjugate. In our case, the variation of the entropy is due to the creation of secondary homoclinic tangencies at parameters μ which do not belong to the intervals H_k.

To prove Theorem A we construct a Markov partition for the limit sets of f_μ in D, denoted by Λ_μ, $\mu \in H_k$, (see Section 9) and calculate explicitly its transition matrix (see Section 9.2). Using this matrix we compute the topological entropy of $f_\mu|_D$ in Proposition 9.2.

A key step in our constructions is the one dimensional model for isolated saddle-node horseshoe cycles (analysis of the so-called *ghost dynamics*, see Section 3.3). More precisely, as we have mentioned, there is an interval I in $W^u(P_0) \cap W^s(P_0)$ tangent to the strong stable foliation. The naive idea of our construction is to modify the dynamics of f_0 in I (this modification does not alter the limit set of f_0) in such a way the induced ghost dynamics has an isolated one-dimensional non-critical saddle-node horseshoe, see the example in Section 2. We will see that this bifurcation pattern will be repeated infinitely many times (the same pattern up the period) at every parameter β_k, $\beta_k \to 0^+$, leading to the unfolding of (two-dimensional) isolated saddle-node horseshoes cycles.

We also exhibit an open class of arcs unfolding critical two saddle-node horseshoes increasing the entropy after the bifurcation. More precisely, consider a disk D and the subset $\mathcal{T}(D)$ of saddle-node arcs of $\mathcal{C}^\infty$-diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ such that every f_μ maps the disk D into its interior and

- for $\mu < 0$, the limit set of f_μ in D is the disjoint union of two sinks S^1_μ and S^2_μ, a hyperbolic saddle set Λ_μ conjugate to a complete shift of two symbols, and a saddle Q_μ.
- For $\mu = 0$, there is a *saddle-node point* P_0 obtained by collapsing the sink S^1_μ with some accessible saddle P_μ of Λ_μ. Denote by Λ_0 the homoclinic class of the saddle-node P_0. The limit set of f_0 in the disk D is the disjoint union of Λ_0, a saddle Q_0 and a sink S^2_0. Moreover, at $\mu = 0$ there is a critical two saddle-node horseshoe cycle involving the hyperbolic saddle Q_0 and Λ_0.

Proposition B. *There is an open subset of $\mathcal{T}(D)$ of arcs $(f_\mu)_{\mu\in[-1,1]}$ such that $h(f_\mu|_D) > \log(2) = h(f_0|_D)$ for all $\mu > 0$ small enough.*

Using the construction in the proof of this proposition, in the next result we prove that the dynamics in the sequel of a critical one saddle-node horseshoe cycle is not necessarily destroyed monotonously:

Theorem C. *Given a two dimensional disk D there is an open subset of $\mathcal{O}(D)$ of arcs $(f_\mu)_{\mu\in[-1,1]}$ unfolding a critical one saddle-node horseshoe cycle at $\mu = 0$ such that there is a sequence of intervals $H_k \subset (0,1)$ accumulating on 0 and a sequence of parameters $\beta_k \to 0$, $\beta_k \in H_k$, such that*

- *every arc $(f_\mu)_{\mu\in H_k}$ has a critical two saddle-node horseshoe cycle at β_k, and*
- *let $H_k \setminus \{\beta_k\} = H^-_k \cup H^+_k$ ($\mu^- < \mu^+$ for all $\mu^\pm \in H^\pm_k$), then $h(f^-_\mu|_D) < h(f^+_\mu|_D)$ for all $\mu^\pm \in H^\pm_k$. Moreover, $h(f_\mu|_D)$ is constant for $\mu \in H^-_k$ (say equal to h_k) and h_k is an strictly increasing sequence converging to $\log(2) = h(f_0|_D)$.*

The proof of this theorem is similar to the one of Theorem A, but here we choose the dynamics of f_0 in I in such a way that the induced ghost dynamics exhibits a one dimensional critical two saddle-node horseshoe cycle as the one in Proposition B. In this way, for the parameters β_k the arc unfolds two-dimensional saddle-node cycles exactly as in Proposition B.

In view of the results above let us pose the following problem:

Problem: *Consider a critical one saddle-node horseshoe Λ which is the limit set in a disk D and an arc of diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ unfolding generically the saddle node of Λ at $\mu = 0$.*

Suppose that $\mu = 0$ is a first bifurcation (i.e., f_μ is hyperbolic for every $\mu < 0$). Is $h(f_\mu|_D) < h(f_0|_D)$ for all small $\mu > 0$?

Observe that to solve this problem it is necessary to get estimates for the topological entropy of Hénon-like attractors, for some estimates see [NP].

Let us observe that it is possible to prove similar results to Theorems A and C in the case of critical one saddle-node cycles (to deal with this situation is much more simple than the case of saddle-node horseshoes considered here). In this case, the analogous of Theorem A is the existence of a sequence of intervals H_n, $H_n \to 0$, containing a saddle-node parameter β_n, where the entropy is a constant h_n, $h_n \to 0$ as $n \to \infty$, and such that $(f_\mu)_{\mu \in H_n}$ unfolds a noncritical saddle-node horseshoe at every β_n. The analogous of Theorem C is that the topological entropy does not depend monotonously on the parameter μ.

This paper is organized as follows. We begin by introducing in Section 2 a one-dimensional model for the unfolding of saddle-node horseshoes motivated by the DA-maps in [W]. In Section 3 we introduce the main tools for the local analysis of saddle-nodes (*transition maps* and *reparametrizations*) and the one-dimensional *ghost returns* associated to saddle-node cycles. In Section 4 we study these ghost returns in the case of saddle-node horseshoe cycles. The dynamics of the ghost returns have two parts: a critical part (studied in Sections 5 and 7) and a noncritical part (analyzed in Section 6). The two dimensional ghost returns are introduced in Section 8. In Section 9, using the ghost returns, we construct Markov partitions for a selected family of parameter intervals. These partitions enable us to calculate the topological entropy and to prove Theorem A. In Section 10 we introduce a class of critical two saddle-node horseshoes for which the topological entropy increases after the bifurcation (Proposition B). Using this family we prove Theorem C.

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2 DA-type bifurcations for one dimensional horseshoes

A great part of the constructions in this paper relies on the reduction to the problem to a one-dimensional heuristic model. In order to do our constructions more clear, in this section we give a simple one-dimensional model which is the key to get the hyperbolicity of the limit set in the general two-dimensional setting. This model is motivated by the construction of the Derived from Anosov diffeomorphisms in [W].

Let $(\phi_\mu)_{\mu \in [-1,1]}$ be a one parameter family of interval maps, $\phi_\mu \colon [0,1] \to [0,1]$, inducing a family of differentiable endomorphisms defined on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ satisfying the following conditions:

a) The map ϕ_μ is smooth for every $\mu \in [-1,1]$, ϕ_0 has a saddle-node $P = P_0$, a sink $S = S_0$ and a source $Q = Q_0$, with $0 < S < Q < P < 1$.

b) For every $\mu \in (-\varepsilon, \varepsilon)$, the map ϕ_μ has exactly two critical points c_1 and c_2 (for simplicity we omit their dependence on μ) whose images are in the basin of attraction of the sink S_μ (the hyperbolic continuation of S) and $0 < S_\mu < c_1 < Q_\mu < P_0 < c_2 < 1$.

c) As a map from the interval $[0,1]$ into itself, ϕ_μ has three discontinuities, say d_1, d_2, d_3 (depending continuously on μ, again such a dependence is omitted), such that $c_1 < P < d_1 < c_2 < d_2 < d_3 < 1$, ϕ_μ is monotone (decreasing) in (d_2, d_3) and surjective in $[d_2, d_3]$.

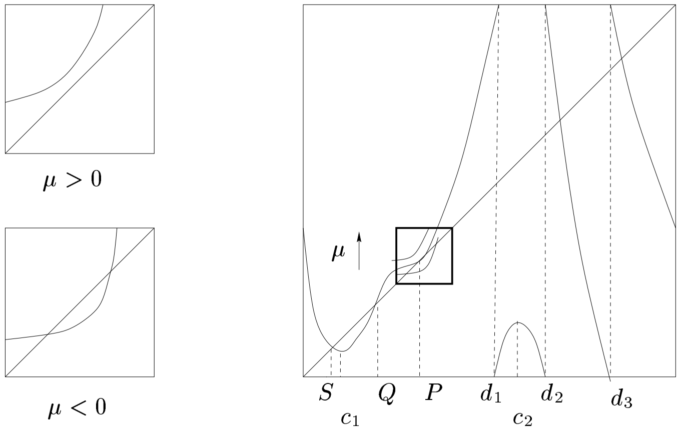

Figure 3: DA-like bifurcation

d) There are small neighborhoods N_1 and N_2 of the criticalities c_1 and c_2, and N_3 and N_4 of the sink S and of the saddle-node P such that, for every x outside $N = \cup_{i=1}^4 N_i$ and every $\mu \in (-\varepsilon, \varepsilon)$, one has $|\phi'_\mu(x)| > 3$, and $N_1 \cup N_2$ is mapped by ϕ_μ into the basin of attraction of $S_{\{\mu\}}$.

e) The arc (ϕ_μ) unfolds generically the saddle-node P at $\mu = 0$ (see Section 3.1 for the precise definition), having no fixed points inside N_4 for all $\mu > 0$ and two fixed points, a sink S'_μ and a source Q'_μ, $S'_\mu < Q'_\mu$, inside N_4, for $\mu < 0$.

f) For positive values of μ close to zero, there exists a positive integer $N(\mu)$ such that the number $n(\mu, x)$ of iterations by ϕ_μ that a point $x \in N_4$ needs to leave N_4 (i.e., $\phi^j_\mu(x) \in N_4$ for all $0 \leq j \leq n(\mu, x) - 1$, and $\phi_\mu^{n(\mu,x)}(x) \notin N_4$) is bounded from above by $N(\mu)$. Moreover, for every $x \in N_4$, $|D\phi_\mu^{\{}n(\mu, x)\}(x)| > 1/2$.

Condition (c) means that ϕ_0 has a one dimensional saddle-node horseshoe containing P. More precisely, an endomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ has a *one-dimensional saddle-node horseshoe* Λ if the set Λ is locally maximal for f, it contains a saddle-node periodic point, the restriction of f to Λ is conjugate to a complete one-sided shift of $k \geq 2$ symbols, and every periodic point of f in Λ is hyperbolic with the exception of a saddle-node. Observe that this saddle-node horseshoe is *noncritical* (the orbits of the critical points do not meet the stable manifold of the saddle-node).

Condition (f) implies that for every small $\mu < 0$ the endomorphism ϕ_μ has a (hyperbolic) one-dimensional horseshoe containing Q_μ.

The existence of non trivial open sets of families verifying (a)–(f) follows from the local form of the saddle node and the construction of the transition maps (see Section 3.2).

Proposition 2.1. *For every family $(\phi_\mu)_{\mu \in (-\varepsilon, \varepsilon)}$ satisfying conditions (a)–(f) above, the limit set of $\{\phi\}_\mu$ is the disjoint union of:*

- *the sink S_μ and a hyperbolic horseshoe Λ_μ containing the source Q_μ, for $\mu > 0$;*

- *the sink S_0, the source Q_0, and a saddle-node horseshoe Λ_0 containing the saddle-node P_0, for $\mu = 0$;*
- *the two sinks S_μ and S'_μ, the source Q_μ and a hyperbolic horseshoe Λ_μ containing the source Q'_μ, for $\mu < 0$.*

Moreover, the horseshoes Λ_μ are conjugate to a complete one-sided shift of three symbols for all small $|\mu|$.

Proof: Since the proposition follows from quite standard arguments we just sketch its proof. Observe that we can assume (without loss of generality) that $\phi_0(0) > P$. Define the points $\bar{P}$, $\bar{Q}'_\mu$ ($\mu < 0$) and $\bar{Q}_\mu$ ($\mu > 0$) as the only points in $(0, c_1)$ with $\phi_0(\bar{P}) = P$, $\phi_\mu(\bar{Q}) = Q_\mu$, and $\phi_\mu(\bar{Q}'_\mu) = Q'_\mu$ (here we are assuming that $|\mu|$ is small enough). Observe that

- for $\mu = 0$, every point in $I_0 = (\bar{P}, P)$ different from Q and $\bar{Q}$ is either in the stable manifold of S or P, ($\mu = 0$),
- for $\mu < 0$, every point in $I_\mu = (\bar{Q}'_\mu, Q'_\mu)$ different from Q_μ and $\bar{Q}_\mu$ is either in the stable manifold of S_μ or S'_μ, and
- for $\mu > 0$, every point in $I_\mu = (\bar{Q}_\mu, Q_\mu)$ is in the stable manifold of S_μ.

For every small μ, define the set Λ_μ as the maximal invariant set of ϕ_μ in $(\mathbb{S}^1 \setminus I_\mu)$. By construction, every point which is not in Λ_μ is either in the stable manifold of the sinks S_μ and S'_μ and of the saddle node P, or it is in the preorbits of Q_μ and Q'_μ (according with the choice of the parameter).

The hyperbolicity of Λ_μ for $\mu > 0$ follows from the fact that there is $M(\mu) > 0$ such that, for all $x \in \Lambda_\mu$, there exists a number $m(\mu, x) \leq M(\mu)$ such that $|D\phi_\mu^{m(\mu,x)}(x)| > 3/2$, where $m(\mu, x) = n(\mu, x) + 1$ if $x \in N_4$ and $m(\mu, x) = 1$ if otherwise, recall (d) and (f) above. The hyperbolicity for $\mu < 0$ follows similarly, and the same holds for the almost hyperbolicity of Λ_0 (see the end of the next paragraph).

To see that the horseshoes Λ_μ are conjugate to a one-sided complete shift of three symbols it is enough to consider the Markov partition defined by the three intervals of $(\mathbb{S}^1 \setminus I_\mu \cup \phi_\mu^{-1}(I_\mu))$. Observe that ϕ_0 is expanding (except at the point P) on each interval of the partition and that for $\mu < 0$ the map ϕ_μ is expanding on each interval of the partition. This also gives the hyperbolicity (or almost hyperbolicity) of Λ_μ, $\mu \geq 0$.

Finally, by construction, the image by ϕ_μ of each interval of the partition is the convex hull (considered as an interval of $[0, 1]$) of the intervals of the partition.

The construction of the Markov partitions implies that the horseshoes Λ_μ are topologically conjugate to the one-sided shift of three symbols.

□

Remark 2.2. *Under natural conditions, the construction above holds in the case of saddle-node arcs (ϕ_μ) having two critical points and an arbitrarily big number of discontinuities d_i such that $\phi_0([d_i, d_{i+1}]) = [0, 1]$ and the restriction of ϕ_0 to $[d_i, d_{i+1}]$ is expanding for every interval $[d_i, d_{i+1}]$ without criticalities. In fact, in this paper we will consider the case where the number of discontinuities and of expanding surjective intervals goes to infinity.*

3 Local dynamics of saddle-nodes: transition maps and ghost dynamics

In this section we define saddle-node bifurcation for diffeomorphisms, state its known properties, and introduce the tools for the analysis of the local dynamics at the saddle-node (adapted vector fields and transitions).

3.1 Saddle-node arcs

The arc $(f_\mu)_{\mu\in[0,1]}$ has a saddle-node bifurcation at $(P_b, b) \in M \times [0,1]$ if there is $k \geq 1$ such that $f_b^k(P_b) = P_b$ and $Df_b^k(P_b)$ has an eigenvalue one and the other eigenvalue has modulus different than one. From now on we take $k = 1$ and $b = 0$. The saddle-node is *attracting* if all the eigenvalues of $Df_0(P_0)$ are contained in the unit circle. Here we consider attracting saddle-nodes. Then there is a Df_0-invariant splitting of the tangent space at P_0, $T_{P_0}M = E^s \oplus E^c$, where $\dim(E^c) = 1$, $Df_0(P_0)|E^c$ is the identity, and $Df_0(P_0)|E^s$ is a contraction.

By the invariant manifold theory, see [HPS], there are one-dimensional center manifolds W_0^c tangent to E^c at P_0, which are locally invariant (i.e., $f_0(W_0^c) \cap W_0^c$ contains a neighbourhood of P_0 in W_0^c). We consider non-degenerate saddle-nodes, meaning that (for a suitable choice of the central manifold) $(f_0|W_0^c)''(P_0) \neq 0$.

By the normal hyperbolicity of W_0^c, the continuation W_μ^c of W_0^c is defined for all small $|\mu|$. In appropriate coordinates, the restriction of f_μ to W_μ^c is of the form

$$f_\mu(x) = x + \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3), \quad \alpha \neq 0.$$

We suppose that $\alpha > 0$. Thus $\{x < 0\} \subset W^s(P_0)$, $\{x > 0\} \subset W^u(P_0)$, f_μ has no fixed points close to P_0 for $\mu > 0$, and, for every small $\mu < 0$, the diffeomorphism f_μ has a sink S_μ and a saddle P_μ, $S_\mu < P_\mu$ (in the natural ordering of W_μ^c), collapsing at $\mu = 0$ to the saddle-node P_0.

By [HPS] there is a strong stable foliation $\mathcal{F}^{ss}$ of $W_{loc}^s(P_0)$, defined as the unique codimension one f_0-invariant foliation of $W_{loc}^s(P_0)$ such that $T_{P_0}(F^{ss}(P_0))$ is E^s, where $F^{ss}(x)$ is the leaf of $\mathcal{F}^{ss}$ containing x.

The leaf of the strong stable foliation containing P_0 is its *strong stable manifold,* denoted by $W^{ss}(P_0)$. Associated to this foliation there is the projection π^s along its leaves from $W_{loc}^s(P_0)$ into an interval of the form (say) $[A, P_0] \subset W_0^c(P_0)$.

3.2 Transition maps

The main tool for the local analysis of the dynamics of a saddle-node are the *transition maps* introduced in [NPT]. For that one considers *adapted saddle-node vector fields,* i.e., a *saddle-node arc of vector fields* $X(\mu, x)$ of the form

$$X(\mu, x) = \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3), \quad \alpha > 0,$$

and such that $f_\mu(x) = X_1(\mu, x)$ for all $x \in W_\mu^c$, where $X_s(\mu, \cdot)$ is the time-s map of the vector field $X(\mu, \cdot)$, see [DRoV, Section 2.1].

In each component of $(W_0^c \setminus \{P_0\})$ consider fundamental domains of f_0, say $D^- = [A, f_0(A)] \subset W^s(P_0)$ and $D^+ = [B, f_0(B)] \subset W^u(P_0)$, where $A < 0 < B$ and $X(0, x) > 0$ for all $x \in$

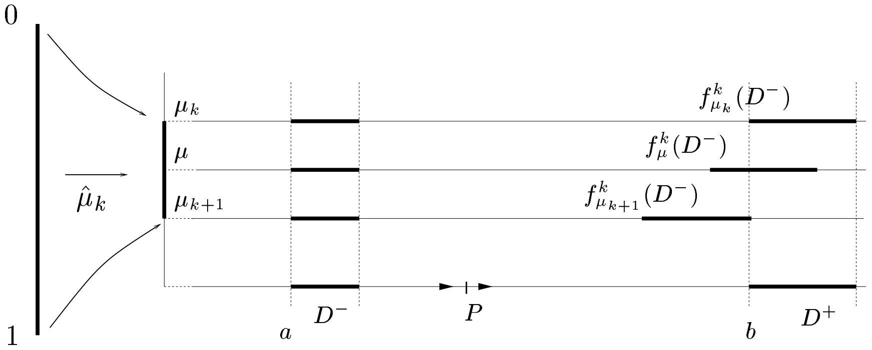

Figure 4: Dynamical normalizations of parameter space

$([f_0^{-2}(A), f_0^2(B)] \setminus \{0\})$. For every big $k \in \mathbb{N}$ define the parameter $\mu_k > 0$ by the relation

$$X_k(\mu_k, A) = B.$$

Then the parameters μ_k are uniquely defined (for big k) and form a decreasing sequence converging to 0^+ as $k \to \infty^+$. Moreover, by [DRoV, Proposition 2.1], for k big enough the maps σ_k defined by

$$\sigma_k \colon [\mu_{k+1}, \mu_k] \to [0,1], \quad X_{k+\sigma_k(\mu)}(\mu, A) = B. \tag{1}$$

are C^∞ decreasing diffeomorphisms. Denote by $\hat{\mu}_k \colon [0,1] \to [\mu_{k+1}, \mu_k]$ the inverse of σ_k.

These reparametrizations verify the following distortion property:

Proposition 3.1. ([DRoV, Proposition 2.2]). *Given any $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and every Borel set A of $[0,1]$ we have*

$$(1-\varepsilon)|A| \leq \frac{|\hat{\mu}_k(A)|}{|[\mu_{k+1}, \mu_k]|} \leq (1+\varepsilon)|A|,$$

where $|A|$ denotes the Lebesgue measure of the set A.

We define the sequence of transitions $(T_k)_k$ by

$$T_k \colon [0,1] \times D^- \to \mathbb{R}, \quad T_k(\sigma, x) = f^k_{\hat{\mu}_k(\sigma)}(x).$$

These transitions verify the following properties:

Proposition 3.2. ([DRoV, Theorems 2.3 and 2.6]).

- *The sequence $(T_k)_k$ converges in the C^∞-topology to the* transition map

  $$T_\infty \colon [0,1] \times D^- \to \mathbb{R}, \quad T_\infty(\sigma, x) = X_{t(x)-\sigma}(0, B),$$

  where $t(x)$ is given by the relation $X_{t(x)}(0, A) = x$.

- *Consider a small neighbourhood* V *of* 0 *in* $\mathbb{R}$ *and (in local coordinates) the set* $D^- \times V \subset W^s_{loc}(P_0)$. *Define the maps*

$$\tilde{T}_k, \tilde{T}_\infty \colon [0,1] \times D^- \times V \to \mathbb{R} \times V, \qquad \begin{cases} \tilde{T}_k(\sigma, x, y) = f^k_{\hat{\mu}_k(\sigma)}(x, y), \\ \tilde{T}_\infty(\sigma, x, y) = (T_\infty(\sigma, x), 0). \end{cases}$$

Then $(\tilde{T}_k)_k$ *converges to* $\tilde{T}_\infty$ *in the* C^∞*-topology.*

3.3 Ghost dynamics

In [DRoV] (following [NPT]) it is studied (among other things) global aspects of the dynamics of one saddle-node cycles based in the one dimensional *ghost return map* R_∞ defined as a composition of T_∞, an iterate of the bifurcating diffeomorphism, and the projection (on the central manifold of the saddle-node) along the leaves of the strong stable foliation. In that case, the return map is defined in the whole fundamental domain of the center stable manifold of the saddle-node. In the setting of attracting critical one saddle-node horseshoe cycles, it is not possible to define such a return in a complete fundamental domain, so one needs to adapt this construction.

The corresponding step in our setting are the ghost return maps defined from an open (maximal) subinterval J of the fundamental domain D^+ into the whole D^+ as follows.

Observe first that $W^u(P_0)$ is not contained in $W^s(P_0)$. We consider the open subinterval J of D^+ whose extremes are points of transverse intersection between $W^u(P_0)$ and $W^{ss}(P_0)$ and whose interior is contained in $W^s(P_0)$. We can suppose that $f_0^2(J) \subset W^s_{loc}(P_0)$. The one-parameter family $R_\infty(\sigma, \cdot)$, $\sigma \in [0,1]$, of *ghost return maps* is defined by

$$R_\infty(\sigma, \cdot) \colon J \to f_0^{-1}(D^+) \cup D^+, \quad x \mapsto R_\infty(\sigma, x) = T_\infty(\sigma, f_0^{s(x)}(\pi^s f_0^2(x))), \tag{2}$$

where π^s is the projection along the leaves of the strong stable manifold of $W^s(P_0)$ from $W^s_{loc}(P_0)$ into the central stable manifold of P_0 and $s(x)$ is the iterate of $\pi^s(f_0(x))$ in $(A, f_0(A)] \subset D^-$, where $s(x) \to \infty$ as $x \to \partial J$.

Identifying the points of $f_0^{-1}(D^+)$ and D^+ in the same orbit, we get a map from J into the whole D^+. In Figure 5 it is illustrated the case of a ghost return map (in a saddle-node horseshoe) with infinitely many branches.

The relevant property of the ghost return maps is that, for positive parameters μ close to 0^+, an important part of the dynamics of f_μ (up to a reparametrization and up to the period) is close to the one of this family of endomorphisms (this is a consequence of Proposition 3.2).

4 Critical Saddle-node horseshoes

4.1 The saddle-node horseshoe arc

The beginning of our construction is identical to the one in [DRoV, Section 4.2], so we just recall the main steps of this construction, we refer to [DRoV, pages 56-57] for the details.

Consider a rectangle R and affine two-dimensional horseshoe $f \colon R \to R$, $R = R_1 \cup \cdots \cup R_5$, where $R_1, \ldots, R_5$ are compact sub-rectangles of R with pairwise disjoint interiors and such that

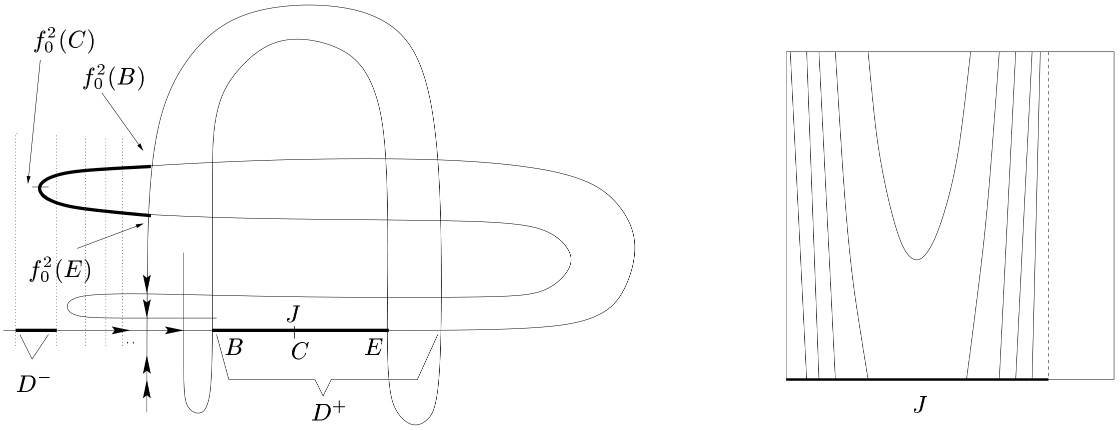

Figure 5: Global dynamics and ghost return maps.

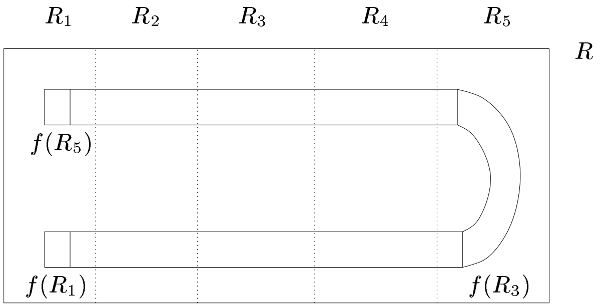

Figure 6: The map f

$f(R_1 \cup R_5) \subset \mathrm{int}(R_1)$, $f(R_3) \subset \mathrm{int}(R_5)$ and

$$Df|_{R_1} = \begin{pmatrix} \phi & 0 \\ 0 & \lambda \end{pmatrix}, \quad Df|_{R_2} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad Df|_{R_4 \cup R_5} = \begin{pmatrix} -\kappa_2 & 0 \\ 0 & -\lambda \end{pmatrix},$$

where $\kappa_1, \kappa_2 > 1$, $0 < \lambda < 1$, and ϕ is some smooth real map (see Figure 6).

We assume that the fixed point P of f in R_2 is $(s, 0)$, $s > 0$, and that its (local) unstable manifold is contained in $\mathbb{R} \times \{0\}$. Moreover, f has a unique fixed point in R_1, which is a sink $S = (s, 0)$, $s < 0$ such that $R_1 \subset W^s(S)$ (thus $R_5 \subset W^s(S)$). The maximal invariant set of f in R consists of a sink S and a horseshoe Λ (containing P). We now modify the first coordinate of f in order to collapse the saddle P and the sink S *via* a saddle-node bifurcation P_0. In this way we get an arc of diffeomorphisms $(f_\mu)_{\mu \in [-1,1]}$ unfolding the saddle-node at $\mu = 0$ such that for negative μ the maximal invariant set of f_μ in R is the union of a sink S_μ (the continuation of S) and a hyperbolic horseshoe Λ_μ containing P_μ and conjugate to a complete shift of two symbols (Λ_μ and P_μ are the continuation of the horseshoe Λ and the saddle P). For $\mu = 0$ we get a saddle-node horseshoe Λ_0 which is the maximal invariant set of f_0 in R.

We perform the construction above in such a way vertical segments in $R_1 \cup R_2 \cup R_4 \cup R_5$ are mapped by f_μ into vertical segments.

Let us assume that $R = [-2,2] \times [-1,1]$, the saddle-node is $(0,0)$, the vertical direction corresponds to the stable direction, $\Lambda_0 \subset [0,1] \times [0,1]$, $\{0\} \times [0,1] \subset W^c_0(+)$ and $\{0\} \times [-1,0] \subset W^c_0(-)$.

4.2 Ghost dynamics and return maps

Let $W^c_0(+)$ and $W^c_0(-)$ be the two connected components of $(W^c_0 \setminus \{P_0\})$, where $W^c_0(-) \subset W^s(P_0)$ corresponds to the points of W^c_0 with negative central coordinate. Consider a pair of points B and E in $W^c_0(P_0) \cap W^{ss}(P_0)$ satisfying the following:

- $B \in R_2$, $E \in R_4 \cap (B, f_0(B))$ ($[B, f_0(B)]$ is the segment joining B and $f_0(B)$ in W^c_0), and (B, E) is contained in the interior of $W^s(P_0)$,
- $f^2_0(B)$, $f^2_0(E) \in W^{ss}_{loc}(P_0)$, and $f^2_0([B,E])$ is a parabolic curve contained in $W^s_{loc}(P_0)$ transverse to the strong stable foliation of $W^s_{loc}(P_0)$ except at a point $f^2_0(C)$, $C \in (B,E)$, corresponding to a quadratic tangency. See Figure 5.

We let $\pi^s(f^2_0(C)) = A \in W^c_0(-)$ and consider the fundamental domains $D^- = [A, f_0(A)]$ of $W^c_0(-)$ and $D^+ = [B, f_0(B)]$ of $W^c_0(+)$. As in Section 3.2, associated to D^- and D^+ we define the parameters μ_k, the reparametrizations $\hat{\mu}_k$ and σ_k, and the transitions T_k and T_∞. Since $\pi^s f^2_0((B,E)) = [A, P_0)$ and $f^2_0((B,E))$ is a parabolic curve, given any $x \in (B,E)$ there is a first $s(x)$, $s(x) \leq 0$, such that

$$f_0^{s(x)}(\pi^s(f^2_0(x))) \in D^-,$$

where $s(x) \to -\infty$ as $x \to \{B, E\}$. The *ghost return map* now is defined by

$$R_\infty \colon [0,1] \times (B,E) \to f_0^{-1}(D^+) \cup D^+, \quad R_\infty(\sigma, x) = T_\infty(\sigma, f_0^{s(x)}(\pi^s f^2_0(x))), \tag{3}$$

where T_∞ is defined as in Proposition 3.2. As above, identifying points in $f_0^{-1}(D^+) \cup D^+$ in the same orbit we can think of this return map as a map from (B,E) into D^+. See Figure 5 for the induced one dimensional map.

Observe that $\pi^s(f^2_0((B,E))) = \cup_{i=0}^\infty f^i_0(D^-)$. Hence, for each $\sigma \in [0,1]$ and $i \geq 0$ there are exactly two points $d^-_i(\sigma)$ and $d^+_i(\sigma)$ in (B,E) , $d^-_i(\sigma) < d^+_i(\sigma)$, such that

$$t(\pi^s(f^2_0(d^\pm_i(\sigma)))) = i + \sigma, \quad \text{i.e.,} \quad X_{i+\sigma}(0, A) = \pi^s(f^2_0(d^\pm_i(\sigma))).$$

For $i = 0$, we have that $d^+_0(\sigma)$ (resp. $d^-_0(\sigma)$) is decreasing (resp. increasing) function with $d^+_0(0) = d^-_0(0) = C = \pi^s(f^2_0(C))$. Observe that $d^+_i(\sigma)$ (resp. $d^-_i(\sigma)$) is an increasing sequence (resp. decreasing) converging to E (resp. B).

For each $\sigma \in [0,1]$ let $\Delta_0(\sigma) = [d^-_0(\sigma), d^+_0(\sigma)]$ and, for $i > 0$, let $\Delta^+_i(\sigma) = [d^+_{i-1}(\sigma), d^+_i(\sigma)]$ and $\Delta^-_i(\sigma) = [d^-_i(\sigma), d^-_{i-1}(\sigma)]$. The following properties hold for $R_\infty(\sigma, \cdot)$:

(S) Surjective branches: For each $\sigma \in [0,1]$ and $i \geq 1$, the map $R_\infty(\sigma, \cdot) \colon \Delta^\pm_i(\sigma) \to D^+$ is surjective and injective in the interior of $\Delta^\pm_i(\sigma)$. Moreover, $R_\infty(\sigma, \cdot)$ is increasing in $\Delta^+_i(\sigma)$ and decreasing in $\Delta^-_i(\sigma)$.

(C) Criticalities and unimodal branch: For every $\sigma \neq 0$, the restriction of $R_\infty(\sigma, \cdot)$ to $\Delta_0(\sigma)$ is unimodal and C is its critical point, which is quadratic and generic.

We have that the maps $R_\infty(\sigma, \cdot)$ are very expansive close to E and B:

Lemma 4.1. *The derivative of $R_\infty(\sigma,\cdot)$ verifies $|D_x(R_\infty(\sigma,\cdot))| \to \infty$ as $x \to E$ or $x \to B$ for every $\sigma \in [0,1]$.*

Proof: Since the size of the fundamental domains $[x, f(x)]$ of the restriction of f to the central stable manifold of P goes to zero as x goes to P, we have that $|\Delta_i^\pm(\sigma)| \to 0$ as $i \to \infty$. By the mean value theorem, for each $\Delta_i^\pm(\sigma)$, $i \geq 1$ and $\sigma \in [0,1]$, there is $x_i^\pm(\sigma) \in \Delta_i^\pm(\sigma)$, with

$$|D_{x_i^\pm}(R_\infty(\sigma,\cdot))| = \frac{|D^+|}{|\Delta_i^\pm(\sigma)|}.$$

Since $|\Delta_i^\pm(\sigma)| \to 0$ as $i \to \infty$ this implies that $|D_{x_i^\pm(\sigma)}(R_\infty(\sigma,\cdot))| \to \infty$ as $i \to \infty$.

The bounded distortion of the saddle-node now implies that there is $K > 0$ such that

$$\frac{|D_x(R_\infty(\sigma,\cdot))|}{|D_y(R_\infty(\sigma,\cdot))|} > K \quad \text{for all } \sigma \in [0,1],\ x, y \in \Delta_i^\pm(\sigma), \text{ and } i \geq 2,$$

see Claim 5.5 for details. This ends the proof of the lemma. □

A subinterval J of (B, E) *is mapped into k fundamental domains of* W_0^c if there is $x \in W_0^c(-)$ such that $[x, f_0^k(x)] \subset \pi^2(f_0^2(J))$. Observe that given any $\kappa \in \mathbb{N}$ there is $\varepsilon_\kappa > 0$, $\varepsilon_\kappa \to 0$ as $\kappa \to \infty$, such that the interval $K(\varepsilon_\kappa) = [B + \varepsilon_\kappa, E - \varepsilon_\kappa] \subset (B, E)$ is mapped by f_0 into κ fundamental domains of W_0^c. Fix now a big κ and consider ε_κ and $K(\varepsilon_\kappa)$ as above. Take small $\varepsilon > 0$ such that $[C - \varepsilon, C + \varepsilon] \subset K(\varepsilon_\kappa)$ if κ is big enough (and then ε_κ is small).

By construction, for a fixed small ε_κ, given any $\sigma \in [0,1]$ and $y \in D^+$, the set of points $x \in (K(\varepsilon_\kappa) \setminus (C - \varepsilon, C + \varepsilon))$ with $R_\infty(\sigma, x) = y$ has (at least) $(2\kappa - 3)$ elements. Thus, fixed a big κ, we can modify f_0 in $f_0(K(\varepsilon_\kappa))$ to have that $|D_x(R_\infty(\sigma, x))| > 4$ for all $x \in K(\varepsilon_\kappa) \setminus (C - \varepsilon, C + \varepsilon)$ and $\sigma \in [0,1]$. Observing that $f_0(K(\varepsilon_\kappa))$ is contained in the stable manifold of the saddle-node, we have that the previous perturbation can be done without modifying the limit set of f_0 in R. The previous comments and Lemma 4.1 are summarized in the following condition:

(E) Uniform expansion: $|D_x(R_\infty(\sigma, x))| > 4$ for all $x \in (B, E) \setminus (C - \varepsilon, C + \varepsilon)$ and $\sigma \in I$.

4.3 Return maps. Approximation by the ghost dynamics

4.3.1 One dimensional return maps

For every small $|\mu|$ there are defined the continuations E_μ, B_μ and C_μ of E, B and C as follows: E_μ and B_μ are the points of W_μ^c close to B and E with $f_\mu^2(E_\mu), f_\mu^2(B_\mu) \in F^{ss}(0)$ and $f_\mu([B_\mu, E_\mu])$ is tangent to the strong stable foliation at $f_\mu^2(C_\mu)$. For simplicity, let us omit the dependence on μ of the points B_μ, E_μ and C_μ.

For small $\mu > 0$ define the *return map* R_μ as follows,

$$R_\mu \colon [B, E] \to D^+, \quad R_\mu(x) = f_\mu^{s_\mu(x)}(\pi^s(f_\mu^2(x))),$$

where $s_\mu(x)$ is the first forward iterate of $\pi^s(f_\mu^2(x))$ in D^+.

Observe that, with the notation in Section 4.2, for $\mu > 0$,the map R_μ has only finitely many surjective branches. More precisely, for each $\mu > 0$, the function $s_\mu(x)$ is upper bounded by $N(\mu) = s_\mu(C)$, where $N(\mu) \to \infty$ as $\mu \to 0^+$.

Associated to the maps R_μ consider the sequence of one-parameter families of returns defined by

$$R_k \colon [0,1] \times [B,E] \to D^+, \quad R_k(\sigma, x) = R_{\hat{\mu}_k(\sigma)}(x), \tag{4}$$

where $\hat{\mu}_k \colon [0,1] \to [\mu_{k+1}, \mu_k]$ are the reparametrizations in Section 3.2.

Observe that the numbers $s(x)$ in the definition R_∞ (see equation (3)) and $s_\mu(x)$ above are uniformly bounded from below and above (independently of small $\mu \geq 0$) in each compact subset of (B,E) (bounds depend only on the choice of the compact subset). This fact and Proposition 3.2 imply the following:

Lemma 4.2. *Let K be a compact subset of (B,E). Then the restriction of R_k to $[0,1]\times K$ converges in the C^∞-topology to the restriction of R_∞ to $[0,1] \times K$.*

Since $R_\mu(\cdot) = R_k(\sigma, \cdot)$ if $\mu = \sigma_k(\sigma)$, Lemma 4.2 and the uniform expansivity property (E) immediately give the following:

Lemma 4.3. *Let $\varepsilon > 0$ as in property (E). Then there is $k_0 \in \mathbb{N}$ such that $|D_x(R_\nu(\cdot))| > 3$ for all $x \notin (C-\varepsilon, C+\varepsilon))$ and $\nu \in \sigma_k([0,1])$, $k \geq k_0$.*

In what follows, the compact set $K(\varepsilon_\kappa)$ containing $(C-\varepsilon, C+\varepsilon)$ will remain fixed. In the next section we will analyze the dynamics of $R_\infty(\sigma, \cdot)$ in the critical set $(C-\varepsilon, C+\varepsilon)$ for certain sets of parameters. In Section 6. we will study the dynamics in $[B,E] \setminus K(\varepsilon_\kappa)$ (this dynamics will be strongly expansive). Observe that to study the dynamics in $D^+ \setminus K(\varepsilon_\kappa)$ it is not possible to use (directly) the results in [DRoV] (Propositions 3.2).

5 The dynamics of the critical region

Lemma 5.1. *There are a strictly increasing sequence of natural numbers (r_k) and a decreasing sequence of parameters $(\nu_k)_k$, $\nu_k \in (\mu_{r_k+1}, \mu_{r_k}]$, $\nu_k \to 0^+$ as $k \to \infty$, such that*

$$R_{\nu_k}(B) = R_{\nu_k}(E) = B. \tag{5}$$

Moreover, the parameters ν_k are uniquely defined if k is sufficiently big.

Proof: By definition of E, B and of the returns R_ν, to prove the lemma it is enough to see that there is sequence of parameters $(\nu_k)_k$ such that

$$f^k_{\nu_k}(0) = f^k_{\nu_k}(\pi^s(f^2_{\nu_k}(B))) = f^k_{\nu_k}(\pi^s(f^2_{\nu_k}(E))) = f^k_{\nu_k}(B). \tag{6}$$

By construction, the parameters μ_k in Section 3.2 verify $f^k_{\mu_k}(A) = B$. Take a big $k > 0$, by continuity of the map $\mu \mapsto f_\mu^{-k-1}(B)$ and since $f_{\mu_{k+1}}^{-k-1}(B) = A < 0 < f_0^{-k-1}(B)$, there is a first $r_k \in \mathbb{N}$ such that

$$f_{\mu_{r_k}}^{-k-1}(B) \leq 0 < f_{\mu_{r_k}}^{-k}(B), \qquad B \leq f_{\mu_{r_k}}^{k+1}(0) < f_{\mu_{r_k}}(B).$$

Consider now the continuous map

$$h_k\colon [\mu_{r_k+1}, \mu_{r_k}] \to \mathbb{R}, \quad h_k(\mu) = f_\mu^{k+1}(0).$$

Observe that, due to the local form of the saddle-node, the map $h_k(\cdot)$ is strictly increasing (if k is big enough). Thus, by definition of r_k,

$$h_k(\mu_{r_k+1}) < B \leq h_k(\mu_{r_k}).$$

Thus, there is $\nu_k \in (\mu_{r_k+1}, \mu_{r_k}]$ such that $h_k(\nu_k) = f_{\nu_k}^{k+1}(0) = B$. Since $\pi^s(f_{\nu_k}^2(B)) = \pi^s(f_{\nu_k}^2(E)) = 0$, this ends the proof of (6), and hence, of the first part of the lemma.

Since (for big k) the maps h_k are increasing the parameter ν_k is unique. By construction, the sequence (ν_k) is decreasing and $\nu_k \to 0^+$ as $k \to \infty$. The proof of the lemma is now complete. □

We associate to each parameter ν_k in Lemma 5.1 the number m_k defined by

$$f_{\nu_k}^{-m_k}(0) \in (A, f_0(A)].$$

This choice and the definition of μ_{r_k} and r_k imply that

$$r_k = m_k + k. \tag{7}$$

Given $\nu_k \in (\mu_{r_k+1}, \mu_{r_k}]$, let $\sigma_k^* = \sigma_{r_k}(\nu_k) \in [0, 1)$. The sequence (σ_k^*) has a convergent subsequence, converging to some $\sigma^* \in [0, 1]$. For notational simplicity, let us assume that the initial sequence (σ_k^*) is convergent.

We now modify the dynamics of f_0 in the interior of the rectangle R_5, without modifying the forward orbits of B and E, to make C a fixed point of $R_\infty(\sigma^*, \cdot)$. This perturbation preserves the limit set of f_0 in R.

Define $\sigma_1 \in (0, 1)$ by $X_{\sigma_1}(0, B) = C$. If $\sigma^* = \sigma_1$, then the critical point C is a fixed point of $R_\infty(\sigma_1, \cdot)$, and there is nothing to do. Otherwise, since $f_0(C) \in R_5$, after modifying the dynamics of f_0 in R_5, we can assume that

$$\begin{aligned} \pi^s(f_0^2(C)) &= X_\tau(0, A), \quad \text{where } \tau = \sigma_1 + \sigma^*, \text{ if } \sigma_1 + \sigma^* < 1, \text{ or} \\ \pi^s(f_0^2(C)) &= X_\tau(0, A), \quad \text{where } \tau = \sigma_1 + \sigma^* - 1, \text{ if } \sigma_1 + \sigma^* > 1. \end{aligned}$$

In both cases, $\tau \in (0, 1)$ and $\pi^s(f_0^2(C)) \in [A, f_0(A)) \subset D^-$. In the first one, we have

$$T_\infty(\sigma^*, \pi^s(f_0^2(C))) = T_\infty(\sigma^*, X_\tau(0, A)) = X_{\tau-\sigma^*}(0, B) = X_{\sigma_1}(0, B) = C.$$

In the second case, we have

$$T_\infty(\sigma^*, \pi^s(f_0^2(C))) = T_\infty(\sigma^*, X_\tau(0, A)) = X_{\tau-\sigma^*}(0, B) = X_{\sigma_1-1}(0, B) = X_1(0, C) \simeq C,$$

where, in the last identity, we identify points of $D^+ \cup f_0^{-1}(D^+)$ in the same orbit. The definition of R_∞ in Section 3.3 implies that $R_\infty(\sigma^*, C) = C$.

Observe that the perturbation above can be done in such a way that the resulting ghost returns $R_\infty(\sigma, \cdot)$ verify conditions (S), (C), and (E) in Section 4.2.

Since C is a quadratic critical point of $R_\infty(\sigma^*,\cdot)$, we can assume that the constant $\varepsilon > 0$ in (E) is such that

$$R_\infty(\sigma^*,[C-\varepsilon, C+\varepsilon]) \subset (C-\varepsilon/6, C+\varepsilon/6).$$

By the continuous dependence of $R_\infty(\sigma,\cdot)$ on σ, there is $\delta > 0$ such that

$$R_\infty(\sigma,[C-\varepsilon, C+\varepsilon]) \subset (C-\varepsilon/5, C+\varepsilon/5), \quad \text{for all } \sigma \in I^* = (\sigma^*-\delta, \sigma^*+\delta). \tag{8}$$

Since $\sigma_k^* \to \sigma^*$, there is $\xi > 0$ with $[\sigma_k^*-\xi, \sigma_k^*+\xi] \subset (\sigma^*-\delta, \sigma^*+\delta) \subset (0,1)$, for every k big enough. Let

$$J_k = \hat{\mu}_{r_k+1}([\sigma_k^*-\xi, \sigma_k^*+\xi]) \subset I_k^* = \hat{\mu}_{r_k+1}([\sigma_0-\delta, \sigma_0+\delta]) \subset [\mu_{r_k+1}, \mu_{r_k}].$$

By (8), the definition of R_ν, and Proposition 3.2, we get

$$R_\nu([C-\varepsilon, C+\varepsilon]) \subset (C-\varepsilon/4, C+\varepsilon/4), \quad \text{for all } \nu \in J_k \text{ and } k \text{ big enough.} \tag{9}$$

Proposition 5.2. (Choice of parameters) *There are ε and $\xi > 0$ and sequences of natural numbers r_k, of parameters $\nu_k = \hat{\mu}_{r_k}(\sigma_k^*)$, and of intervals $J_k = \hat{\mu}_{r_k}(\sigma_k^*-\xi, \sigma_k^*+\xi)$ such that for every $\nu \in J_k$ it holds*

- *$R_\nu(C-\varepsilon, C+\varepsilon) \subset (C-\varepsilon/3, C+\varepsilon/3)$,*
- *$|D_x R_\nu| > 3$ for all $x \in ([B,E] \setminus (C-\varepsilon, C+\varepsilon))$.*

The first part of the proposition is stated in (9). The second part follows from Lemma 4.3 and the lemma below:

Lemma 5.3. *Given any $M > 2$ there are $\kappa = \kappa(M) > 0$ and $k_0 = k_0(M) \in \mathbb{N}$ such that*

$$|R'_\nu(x)| > M,$$

for every $x \in [B, B+\kappa] \cup [E-\kappa, E]$, ν in J_k, and $k \geq k_0$.

Remark 5.4. *Lemma 5.3 implies that B is a hyperbolic fixed point of R_{ν_k}, so it has a continuation $\tilde{B}_\nu$, which is fixed and hyperbolic for R_ν, for every ν in a small vicinity of ν_k. Similarly, in this parameter neighbourhood, it is defined the hyperbolic continuation $\tilde{E}_\nu$ of E, with $R_\nu(\tilde{E}_\nu) = \tilde{B}_\nu$.*

Proof: The main ingredient of the proof of the lemma is the following bounded distortion property. Consider $\nu > 0$, $x \in [f_\nu^2(A), f_\nu(B)] \subset W_\nu^c$, and $y \in [f_\nu^{-1}(x), x]$. Let $x^* = f_\nu^{-\ell(x)}(x)$, $\ell(x) > 0$, be the first element of the backward f_ν-orbit of x in D^- and write $y^*(x) = f_\nu^{-\ell(x)}(y)$.

Claim 5.5. (Bounded distortion property) *There is $K_0 > 0$ such that for every small $\nu > 0$ and every pair of points x and y with $x \in [f_\nu^2(A), f_\nu(B)]$ and $f_\nu^{-1}(x) \leq y < x$, the following holds:*

$$|[y^*(x), x^*]| \geq K_0\, |[y, x]|.$$

Proof of the claim: Write

$$I^* = [y^*(x), x^*] \subset \Delta^* = [f_\nu^{-1}(x^*), x^*],\ I = f_\nu^{\ell(x)}(I^*) = [y, x] \subset \Delta = f_\nu^{\ell(x)}(\Delta^*) = [f_\nu^{-1}(x), x].$$

Then, there are points $z \in I^*$, $w \in \Delta^*$ such that

$$\frac{|I|}{|\Delta|} = \frac{|D_z f_\nu^{\ell(x)}|}{|D_w f_\nu^{\ell(x)}|}\, \frac{|I^*|}{|\Delta^*|} \leq \exp(K_1 \sum_{i=0}^{\ell(x)-1} |f_\nu^i(z) - f_\nu^i(w)|)\, \frac{|I^*|}{|\Delta^*|},$$

where K_1 is an uniform upper bound of the derivative of $\log |(D_z f_\nu)|$ in the segment $[f_\nu^{-1}(A), f_\nu(B)]$ (for all small $|\nu|$). Observe that the fundamental domain Δ^* contains the interval J bounded by z and w. Thus,

$$\sum_{i=0}^{\ell(x)-1} |f_\nu^i(J)| \leq |[f_\nu^{-1}(A), f_\nu(B)]| \leq K_2,$$

where K_2 is independent of ν. Finally,

$$\frac{|I|}{|\Delta|} \leq \exp(K_1 \sum_{i=0}^{\ell(x)-1} |f_\nu^i(J)|)\, \frac{|I^*|}{|\Delta^*|} \leq \exp(K_1\, K_2)\, \frac{|I^*|}{|\Delta^*|} \leq K_3\, \frac{|I^*|}{|\Delta^*|}.$$

The claim follows observing that $|\Delta|$ is upper bounded and $|\Delta^*|$ is lower bounded away from 0. $\square$

To end the proof of the lemma, fix small $\nu > 0$ and take big k and small $\kappa > 0$ such that $C \notin [B, B+\kappa] \cup [E-\kappa, E] = \Xi_\kappa$. Write $\Gamma_\kappa = \pi^s(f_\nu^2(\Xi_\kappa))$. Then there are constants $K_1, \dots, K_5 > 0$, independent of small $\nu > 0$, satisfying the following:

- **e1)** $K_1 < |D_x(\pi^s(f_\nu^2))| < K_2$ for every $x \in \Xi_\kappa$ (just observe that $C \notin \Xi_\kappa$),
- **e2)** $\pi^s(f_\nu^2(\Xi_\kappa)) \subset [-K_3\, \kappa, K_3\, \kappa] = I(\kappa)$ (just recall that $\pi^s(f_{\nu_k}^2(B)) = \pi^s(f_{\nu_k}^2(E)) = 0$),
- **e3)** $|f_\nu(z) - z| < K_4\, (\kappa^2 + \nu)$ for all $z \in I(\kappa)$ and small $\nu > 0$ (just use the local coordinates of the saddle-node), and
- **e4)** Given any $z \in I(\kappa)$, define $z^* = f_\nu^{-\ell(z)}(z) \in D^-$ as the first element of the backward orbit of z in D^-, then $K_5^{-1}\, |D^-| < |[z^*, f_\nu(z^*)]| < K_5\, |D^-|$.

Consider now $z \in I(\kappa)$. Comparing the sizes of the fundamental domains $[z^*, f_\nu(z^*)]$ and $[z, f_\nu(z)]$ (see (e3) and (e4)) and using Claim 5.5, we get $K_6 > 0$ (independent of small $\nu > 0$) such that

$$|D_z(f_\nu^{-\ell(z)})| \geq \frac{K_6}{\kappa^2 + \nu}, \quad \text{for every } z \in I(\kappa). \tag{10}$$

On the other hand, $f_\nu^{r_k}(D^-) \cap D^+ \neq \emptyset$ for every $\nu \in J_k$. Again, by using Claim 5.5, we get $K_7 > 0$ (independent of small $\nu > 0$) such that

$$K_7 < |D_{z^*} f_\nu^{r_k}| \quad \text{for every } z \in I(\kappa). \tag{11}$$

Given $x \in [B, E]$, let $y(x) = \pi^s(f_\nu^2(x))$, and observe that $R_\nu(x) = f_\nu^{r_k - \ell(y(x))}(y(x)^*)$. Thus, from (e1), (10) and (11), there is $K_8 > 0$ (independent of small $\nu > 0$) such that

$$|D_x R_\nu| \geq \frac{K_8}{\kappa^2 + \nu}, \quad \text{for every } x \in \Xi_\kappa \text{ and } \nu \in J_k.$$

The lemma now follows by taking κ and ν_k small enough. □

6 Dynamics in the noncritical region

In this section, we prove that, for every parameter $\nu \in J_k$, we have an appropriate control of the dynamics of the returns to the fundamental domains $D^+(\nu) = [\tilde{B}_\nu, f_\nu(\tilde{B}_\nu)]$, $\tilde{B}_\nu \to B_{\nu_k}$ as $\nu \to \nu_k$, of the central manifold, where $\tilde{B}_\nu$ is the continuation of the hyperbolic periodic point B_{ν_k} of R_{ν_k}, (see Remark 5.4). Similarly, the point $\tilde{E}_\nu$ is the point close to E_{ν_k} with $R_\nu(\tilde{E}_\nu) = R_\nu(\tilde{B}_\nu) = \tilde{B}_\nu$. We claim that the continuations of these points are defined for every parameter in the interval J_k (provided that ξ is small). This follows from the next lemma (see Remark 6.2).

Lemma 6.1. *Fixed $\varepsilon > 0$ the parameter $\nu(\varepsilon)$ close to ν_k defined by $R_{\nu(\varepsilon)}(B_{\nu_k}) = B_{\nu_k} + \varepsilon$ is of the form $\nu(\varepsilon) = \nu_k + \ell\,\varepsilon\,\nu_k^{3/2}$, for some constant ℓ uniformly bounded from below and above (independently of k). Moreover, the parameter $\nu(\varepsilon)$ is uniquely defined.*

Proof: For each $\nu \in J_k$ define the map

$$\Psi(\nu) = \int_0^{B_{\nu_k}} \frac{dx}{X(\nu, x)}.$$

The parameters $\nu(\varepsilon)$ and ν_k verify the following relations:

$$\int_0^{B_{\nu_k}+\varepsilon} \frac{dx}{X(\nu(\varepsilon), x)} = \Psi(\nu(\varepsilon)) + \int_{B_{\nu_k}}^{B_{\nu_k}+\varepsilon} \frac{dx}{X(\nu(\varepsilon), x)} = k, \qquad \int_0^{B_{\nu_k}} \frac{dx}{X(\nu_k, x)} = \Psi(\nu_k) = k.$$

Thus,

$$|\Psi(\nu(\varepsilon)) - \Psi(\nu_k)| = \int_{B_{\nu_k}}^{B_{\nu_k}+\varepsilon} \frac{dx}{X(\nu(\varepsilon), x)} \in [k_1\,\varepsilon, k_2\,\varepsilon], \tag{12}$$

for some constants k_1 and $k_2 > 0$ independent of small ν (observe that $|X(\nu, \cdot)|$ is uniformly bounded away from zero in the interval $[f_0^{-2}(B), f_0^2(B)]$ for every small $|\nu|$).

By [DRoV, Lemma 2.4], $\Psi'(\nu)$ is of order of $\nu^{-3/2}$ (small ν). Therefore, $|\Psi(\nu(\varepsilon)) - \Psi(\nu_k)|$ is of order of $\nu_k^{-3/2}|\nu(\varepsilon) - \nu_k|$. The first part of the lemma now follows from (12). Observe that the estimates on the derivative of Ψ also gives the uniqueness of $\nu(\varepsilon) \in J_k$. □

Remark 6.2. *The proof of Lemma 6.1 implies that the parameters* μ_k *are of order of* $\mu_{k+1} + \mu_{k+1}^{3/2}$. *Thus there is a subinterval* L_k *of* $J_k \subset [\mu_{r_k+1}, \mu_{r_k}]$ *where the continuations* $\tilde{B}_\nu$ *and* $\tilde{E}_\nu$ *of* B_{ν_k} *and* E_{ν_k} *are defined. and constants* k_1 *and* $k_2 > 0$ *(independent of* k*) such that*

$$k_1 \le \frac{|L_k|}{\mu_{r_k} - \mu_{r_k+1}} \le k_2.$$

In particular, by Proposition 3.1, there is $\tau > 0$ *such that* $\sigma_{r_k}([\sigma_k^* - \tau, \sigma_k^* + \tau]) \subset L_k$. *By shrinking* τ *(if necessary), we can assume that* $\tau = \xi$, *where* ξ *is as in Proposition 5.2 (i.e., we can take* $L_k = J_k$*).*

Given a parameter $\nu \in J_k$ and $i \in \{0, \dots, m_k\}$, $r_k = k + m_k$, let $\mathcal{D}_\nu^i$ be the set of points in $\Delta(\nu) = [\tilde{B}_\nu, \tilde{E}_\nu]$ such that

$$\mathcal{D}_\nu^i = \{x \in \Delta(\nu) \quad \text{such that} \quad f_\nu^{k+m_k-i}(\pi^s f_\nu^2(x)) = f_\nu(\tilde{B}_\nu)\}, \text{ i.e. } R_\nu(x) = \tilde{B}_\nu\}.$$

Since $f_\nu^2([\tilde{B}_\nu, \tilde{E}_\nu])$ is a parabolic curve transverse to the strong stable foliation except at the point $f_\nu^2(C)$, by the definition of m_k (see (7)), we have that

- **(r$_1$)** $\mathcal{D}_\nu^i = \emptyset$ for all $i > m_k$,
- **(r$_2$)** $\mathcal{D}_\nu^i = \{d_i^-(\nu), d_i^+(\nu)\}$ where $d_i^-(\nu) < d_i^+(\nu)$, for every $i \in \{0, 1, \dots, m_k\}$,
- **(r$_3$)** $[d_i^-(\nu), d_i^+(\nu)] \subset (d_{i+1}^-(\nu), d_{i+1}^+(\nu))$, $d_{m_k}^-(\nu) = \tilde{B}_\nu$ and $d_{m_k}^+(\nu) = \tilde{E}_\nu$,
- **(r$_4$)** for $i > 0$, let $\Delta_i^- = [d_i^-, d_{i-1}^-]$ and $\Delta_i^+ = [d_{i-1}^+, d_i^+]$, then $R_\nu(\Delta_i^\pm(\nu)) = [\tilde{B}_\nu, f_\nu(\tilde{B}_\nu)] = D^+(\nu)$ for all $i \in \{1, 2, \dots, m_k\}$.

Remark 6.3. *The points* $d_i^\pm(\nu)$ *are the continuations, for* R_ν, *of the discontinuities of* R_∞. *They are defined for every big* k *and every* $\nu \in J_k$ *(see Lemma 4.2).*

Next proposition summarizes the main properties of the return maps R_ν:

Proposition 6.4. *For every* k *sufficiently big and* $\nu \in J_{_k}$ *consider the subinterval* $\Delta(\nu) = [\tilde{B}_\nu, \tilde{E}_\nu]$ *of* $D^+(\nu)$ *and the return map* R_ν *defined by*

$$R_\nu \colon \Delta(\nu) \to D^+(\nu), \quad x \mapsto f_\nu^{s_\nu(x)}(\pi^s f_\nu^2(x)),$$

where $s_\nu(x)$ *is the first forward iterate of* $\pi^s f_\nu^2(x)$ *in* $D^+(\nu)$. *The* return map R_ν *satisfies the following properties for every* $\nu \in J_k$*:*

- **a) invariance of the critical region:** $R_\nu([C - \varepsilon, C + \varepsilon]) \subset (C - \varepsilon, C + \varepsilon)$,
- **b) surjective expansive branches:** *There is a finite set of points, the* discontinuities of R_ν, $\mathcal{D}_\nu = \{d_i^-(\nu), d_i^+(\nu)\}_{i=0}^{m_k}$, *with*

$$d_{m_k}^-(\nu) < \cdots < d_1^-(\nu) < d_0^-(\nu) < C < d_0^+(\nu) < d_1^-(\nu) < \cdots < d_{m_k}^+(\nu)$$

such that

b$_1$) $[C-\varepsilon, C+\varepsilon] \subset [d_0^-(\nu), d_0^+(\nu)] = \Delta_0(\nu)$, *$\Delta_0(\nu)$ is the* critical branch,

b$_2$) *let* $\Delta_i^+(\nu) = [d_{i-1}^+(\nu), d_i^+(\nu)]$ *and* $\Delta_i^-(\nu) = [d_i^-(\nu), d_{i-1}^-(\nu)]$, *then* $R_\nu(\Delta_i^\pm(\nu)) = D^+(\nu)$, *and* R_ν *is increasing in* $\Delta_i^+(\nu)$ *and decreasing in* $\Delta_i^-(\nu)$,

b$_3$) $|D_x R_\nu| > 3$ *for all* $x \in (\Delta(\nu) \setminus (C-\varepsilon, C+\varepsilon)$.

Remark 6.5. *In this proposition, we take the map* R_ν *being bi-valuated in the set* $\mathcal{D}_\nu$, *considering its extension with* $R_\nu|_{\Delta_i^\pm}(d_i^\pm(\nu)) = \{\tilde{B}_\nu, f_\nu(\tilde{B}_\nu)\}$. *This is the same that to consider* $D^+(\nu)$ *as a circle identifying its two extremes.*

Proof: Bearing in mind properties (r$_1$)-(r$_4$), we have that items (a), (b$_1$) and (b_3) of Proposition 6.4 were proved Proposition 5.2. To get (b$_2$), observe that, since $C \notin \Delta_i^\pm(\nu)$, $i \neq 0$, the restrictions of R_ν to $\Delta_i^\pm(\nu)$ are monotone maps. The fact that these maps are increasing in $\Delta_i^+(\nu)$ and decreasing in $\Delta_i^-(\nu)$ follows observing that $\pi^s f_\nu$ preserves the orientation in $(C, \tilde{E}_\nu]$ and reverses the orientation in $[\tilde{B}_\nu, C)$. $\square$

In the next section we describe the dynamics of R_ν, $\nu \in J_k$, in the critical region $(C-\varepsilon, C+\varepsilon)$.

7 The dynamics of the return maps R_ν in the critical region

We now perturb the initial saddle-node diffeomorphism f_0 in a small neighbourhood of $f_0(C)$ (without modifying the limit set of f_0) in such a way that the restriction of $R_\infty(\sigma^*, \cdot)$ to the neighbourhood $(C-\varepsilon, C+\varepsilon)$ of C, (σ^* and ε as in equation (8) and Proposition 5.2), satisfies the following conditions:

(c$_1$) the point C is the (unique) critical point of $R_\infty(\sigma, \cdot)$ for all $\sigma \in [0,1]$ and $R_\infty(\sigma^*, C) < C$,

(c$_2$) there are $\delta \in (0, \varepsilon)$ and $\tau > 0$ such that

$$R_\infty(\sigma^*, [C-\delta, C+\delta]) = [R_\infty(\sigma^*, C), C+\tau], \quad R_\infty(\sigma^*, C \pm \delta) = C + \tau,$$

(c$_3$) $|R'_\infty(\sigma^*, x)| > 2$ for all $x \in ((C-\varepsilon, C+\varepsilon) \setminus (C-\delta, C+\delta))$, (this is possible due to condition (b$_3$) in Proposition 6.4),

(c$_4$) the restriction of $R_\infty(\sigma^*, \cdot)$ to $[C-\delta, C+\delta]$ is a unimodal map having (exactly) three fixed points: a sink S, a repellor Q, and a saddle-node P, where $S < C < Q < P$, such that

- C is in the basin of attraction of the sink S,
- let Q' and P' be the points in $(C-\delta, C)$ with $R_\infty(\sigma^*, Q') = Q$ and $R_\infty(\sigma^*, P') = P$, then $(P', Q') \cup (Q, P] \subset W^s(P)$ and $(Q', Q) \subset W^s(S)$.

For σ close to σ^*, we can define the continuations $\delta^\pm(\sigma)$ of $\pm\delta$ by $R_\infty(\sigma, C \pm \delta^\pm(\sigma)) = C + \tau$. Observe that, by definition, the maps $R_\infty(\sigma, \cdot)$ are *($\sigma - \sigma^*$)-translations* of $R_\infty(\sigma^*, \cdot)$. Thus for parameters σ close to σ^* one has the following:

(c_5) For $\sigma > \sigma^*$, the restriction of $R_\infty(\sigma, \cdot)$ to $[C - \delta^-(\sigma), C + \delta^+(\sigma)]$ has (exactly) to fixed points, a sink S_σ (the continuation of S), whose basin of attraction contains the critical point C, and a repellor Q_σ (the continuation of Q). Moreover, if Q'_σ is the point in $(C - \delta^-(\sigma), C)$ with $R_\infty(\sigma, Q'_\sigma) = Q_\sigma$, then $(Q'_\sigma, Q_\sigma) \subset W^s(S_\sigma)$.

(c_6) For $\sigma < \sigma^*$, the saddle-node P of $R_\infty(\sigma^*, \cdot)$ splits into a sink P^ℓ_σ and a source P^r_σ, $P^\ell_\sigma < P^r_\sigma$. The restriction of $R_\infty(\sigma, \cdot)$ to $[C - \delta^-(\sigma), C + \delta^+(\sigma)]$ has (exactly) four fixed points, two sinks S_σ and P^ℓ_σ, and two repellors Q_σ and P^r_σ, where $S_\sigma < C < Q_\sigma < P^\ell_\sigma < P^r_\sigma$. As before, define $(P^{\ell,r}_\sigma)'$ by $R_\infty(\sigma, (P^{\ell,r}_\sigma)') = P^{\ell,r}_\sigma$. Then $((P^r_\sigma)', Q'_\sigma) \cup (Q_\sigma, P^r_\sigma) \subset W^s(P^\ell_\sigma)$ and $(Q'_\sigma, Q_\sigma) \subset W^s(S_\sigma)$.

(c_7) The dynamics in a neighbourhood of the saddle-node is expanding: There is an interval N containing the saddle node P in its interior, $Q \notin N$, such that

- $R'_\infty(\sigma^*, x) > 3/2$, for all $x \in ([Q, C + \delta] \setminus N)$,
- For values of $\sigma > \sigma^*$ close to σ^*, there exists a positive integer $N(\sigma)$ such that the number $n(\sigma, x)$ of iterations by $R_\infty(\sigma, \cdot)$ that a point $x \in N$ needs to leave N (i.e., $R^j_\infty(\sigma, x) \in N$ for all $0 \leq j \leq n(\sigma, x) - 1$, and $R^{n(\sigma,x)}_\infty(\sigma, x) \notin N$) is bounded from above by $N(\sigma)$. Moreover, for very $x \in N$, $|(R^{n(\mu,x)}_\infty(\sigma, x))'| > 1/2$.

Observe that this condition is similar to hypothesis (f) in Section 2.

We now fix small $\epsilon > 0$ with $(\sigma^* - \epsilon, \sigma^* + \epsilon) \subset (\sigma^* - \xi, \sigma^* + \xi)$ and let $H_k = \sigma_k(\sigma^* - \epsilon, \sigma^* + \epsilon)$.

Since the set $[C - \delta, C + \delta]$ is a compact subset of (B, E) we can apply Lemma 4.2 to the sequence of return maps $R_k(\sigma, \cdot) = R_{\hat{\mu}_k(\sigma)}(\cdot)$, for parameters $\sigma \in (\sigma^* - \epsilon, \sigma^* + \epsilon)$. Since $R_k(\sigma, \cdot)$ converges to $R_\infty(\sigma, \cdot)$ and the saddle-node is a codimension one bifurcation we have the following lemma:

Lemma 7.1. *For every k sufficiently big, there is $\beta_k \in H_k$, $\beta_k = \sigma_k(\eta_k)$, $\eta_k \to \sigma^*$ as $k \to \infty$, such that R_{β_k} has a saddle-node at β_k whose local dynamics satisfies conditions (c_5) and (c_6) above.*

8 Two dimensional return maps

In this section, we introduce the two dimensional return maps. For small $\mu > 0$, consider the points $\tilde{B}_\mu$ and $\tilde{E}_\mu$ introduced in Remark 5.4 and the critical point C_μ in Section 4.3.1 (from now on, we omit the dependence on μ) and consider the sub-rectangles $[\tilde{B}, \tilde{E}]^s$ and $(D^+)^s$ of R defined as follows. Define $F^v(\tilde{B})$, $F^v(\tilde{E})$, and $F^v(f_\mu(\tilde{B}))$ as the connected components of $(F^{ss}(\tilde{B}_\mu) \cap R)$ containing the points $\tilde{B}$, $\tilde{E}$ and $f_\mu(\tilde{B})$. The rectangles $[\tilde{B}, \tilde{E}]^s$ and $(D^+)^s$ are the sub-rectangles of R whose vertical sides are $F^v(\tilde{B})$ and $F^v(\tilde{E})$, and $F^v(\tilde{B})$ and $F^v(f_\mu(\tilde{B}))$, respectively.

The *return map* $\tilde{R}_\mu$ is defined by

$$\tilde{R}_\mu \colon [\tilde{B}, \tilde{E}]^s \to (D^+)^s, \quad \tilde{R}_\mu(x, y) = f_\mu^{s_\mu(x,y)}(f_\mu^2(x, y)),$$

where $s_\mu(x, y)$ is the first forward iterate of (x, y) in $(D^+)^s$.

As in the case of the one-dimensional returns, consider the sequence of one-parameter families of two dimensional returns defined by

$$\tilde{R}_k\colon [0,1]\times[\tilde{B},\tilde{E}]^s \to (D^+)^s, \quad \tilde{R}_k(\sigma,x)=\tilde{R}_{\hat{\mu}_k(\sigma)}(x). \tag{13}$$

Also define

$$\tilde{R}_\infty\colon [0,1]\times[\tilde{B},\tilde{E}]^s \to (D^+), \quad \tilde{R}_\infty(\sigma,x)=T_\infty(\pi_s(f_\mu^{s_\mu(x,y)}(f_\mu^2(x,y)))). \tag{14}$$

Using Proposition 3.2, we have the following:

Lemma 8.1. *Let K be a compact subset contained in the interior of $[\tilde{B},\tilde{E}]^s$. Then the restriction of $\tilde{R}_k$ to $[0,1]\times K$ converges in the C^∞-topology to the restriction of $\tilde{R}_\infty$ to $[0,1]\times K$.*

Using Lemma 4.3, we get:

Lemma 8.2. *There are κ and $k_0\in\mathbb{N}$ such that $|D_z(\tilde{R}_\nu((v_1,0))|\geq 3|v_1|$, for all $z\notin[C-\varepsilon,C+\varepsilon]^s$, every vector $(v_1,0)$, and $\nu\in\sigma_k([0,1])$, $k\geq k_0$, where the set $[C-\varepsilon,C+\varepsilon]^s$ is defined in the natural way.*

As in Lemma 7.1, we get a sequence of parameters β_k corresponding to saddle-node bifurcations of $\tilde{R}(\mu,\cdot)$ (for simplicity we use the same notation). Recall that $H_k=\sigma_k(\sigma^*-\epsilon,\sigma^*+\epsilon)$ and denote by H_k^+ and H_k^- the two connected components of $(H_k\setminus\{\beta_k\})$, where $\mu^-<\mu^+$ for all $\mu^\pm\in H_k^\pm$. By Lemma 8.1, we have:

Remark 8.3. *Let $\mu=\sigma_k(\sigma)\in H_k$. Denote by Q_μ the hyperbolic saddle of $\tilde{R}_\mu=\tilde{R}_k(\sigma,\cdot)$ corresponding to the continuation of the repeller Q_σ of $\tilde{R}_\infty(\sigma,\cdot)$. Observe that Q_μ is a saddle of period r_k+2 of the diffeomorphism f_μ.*

- *For $\mu\in H_k^+$, let Q'_μ be the continuation for $\tilde{R}_\mu=\tilde{R}_k(\sigma,\cdot)$ of the preperiodic point Q'_σ of $\tilde{R}_\infty(\sigma,\cdot)$. Such a continuation is the first intersection (starting from Q_μ) of the unstable manifold of Q_μ (for f_μ) and the stable manifold of Q_μ (for f_μ) given by the connected component of $W^s(Q_\mu)\cap R$ containing Q_μ. Thus, Q'_μ is a transverse homoclinic point of Q_μ (for f_μ).*
- *For the saddle-node parameter $\mu=\beta_k$, let P_{β_k} be the saddle-node of $\tilde{R}_{\beta_k}=\tilde{R}_\infty(\eta_k,\cdot)$. Observe that P_{β_k} is a periodic saddle-node (of period r_k+2) of f_{β_k}. Define P'_{β_k} as the continuation of P'_{σ^*} obtained as the first intersection (starting from P_{β_k}) of the strong unstable manifold of P_{β_k} (for f_μ) and strong stable manifold of P_{β_k} (for f_μ) given by the connected component of $W^{ss}(P_{\beta_k})\cap R$ containing P_{β_k}.*
- *For $\mu\in H_k^-$, we define as above the continuations $P_\mu^{r,\ell}$ and $(P_\mu^r)'$ of the hyperbolic periodic points $P_\sigma^{r,\ell}$ and of the preperiodic points $(P_\sigma^{r,\ell})'$ of $\tilde{R}_\infty(\sigma,\cdot)$, recall ($c_6$) in Section 7. The continuation $(P_\mu^r)'$ is a homoclinic point of P_μ.*

By shrinking the size of the parameter interval J containing σ^*, we can assume that there are intervals $\tilde{I}\subset D^+$ and $\hat{I}\subset D^-$ such that, for all $\sigma\in J$, one has that

- the interval $\tilde{I}$ contains C and S_σ and it is contained in $W^s(S_\sigma)$,

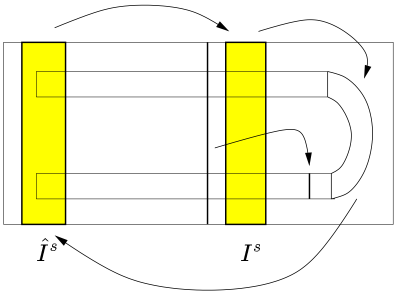

Figure 7: The critical region

- $R_\infty(\sigma, \tilde{I}) \subset \tilde{I}$ and C and S_σ belong to the interior of $T_\infty(\sigma, \hat{I})$.

Consider the sub-rectangles of R defined by

$$\tilde{I}^s = \tilde{I} \times [-1, 1] \quad \text{and} \quad \hat{I}^s = \hat{I} \times [-1, 1],$$

and the vertical and horizontal foliations $\mathcal{F}^h$ and $\mathcal{F}^v$ of R whose leaves are defined by

$$F^h_{y_0} = \{(x, y_0) \in R\} \quad \text{and} \quad F^v_{x_0} = \{(x_0, y) \in R\}.$$

Observe that, by construction, $f^2_\mu(C)$ belongs to the interior of $\hat{I}^s$ for all $|\mu|$ small enough. Thus, if the initial saddle-node horseshoe is sufficiently contracting in the vertical direction, we can assume that $(f_\mu(F^h_{y_0}) \setminus \hat{I}^s)$ is transverse to $\mathcal{F}^v$ for every y_0, and that the angle between the image of $\mathcal{F}^h$ by f_μ and $\mathcal{F}^v$ outside $\hat{I}^s$ is bigger than (for instance) $\pi/6$, see Figure 7.

Since, for every $\sigma \in J$, the rectangle $\tilde{I}^s$ is contained in the stable manifold of S_σ for $\tilde{R}_\infty(\sigma, \cdot)$, Lemma 8.1 implies that:

Remark 8.4. *Consider any $\mu \in H_k$, big k. Then, $(\hat{I}^s \cup \tilde{I}^s) \subset W^s(S_\mu)$.*

In view of this remark, for parameters $\mu \in H_k$, we can focus our attention on the maximal invariant set of f_μ in $(R \setminus \tilde{I}^s)$. Observe that in this region we have a dynamics *without criticalities.* Let G_0 be the unique hyperbolic fixed point of the initial saddle-node horseshoe Λ_0. Denote by G_μ the continuation of G_0, for small μ, and by Σ_μ the homoclinic class of G_μ. Observe that $\Sigma_0 = \Lambda_0$.

Proposition 8.5. *The limit set in the rectangle R of f_μ is hyperbolic, for every $\mu \in (H_k \setminus \beta_k)$, k big enough. More precisely,*

- *for every $\mu \in H^-_k$, the limit set of f_μ in R is the disjoint union of Σ_μ, and the orbits of the sinks S_μ and P^ℓ_μ and of the saddle Q_μ,*

- *for every $\mu \in H^+_k$, the limit set of f_μ in R is the disjoint union of Σ_μ and the orbit of the sink S_μ.*

For $\mu = \beta_k$, the limit set of f_{β_k} in R is the union of Σ_{β_k} and the orbits of the sink S_{β_k} and of the saddle Q_{β_k}. Moreover, Σ_{β_k} is a saddle-node horseshoe containing the saddle-node P_{β_k}.

Finally, the restrictions of the diffeomorphisms f_μ to the sets Σ_μ are topologically conjugate for all $\mu \in H_k$.

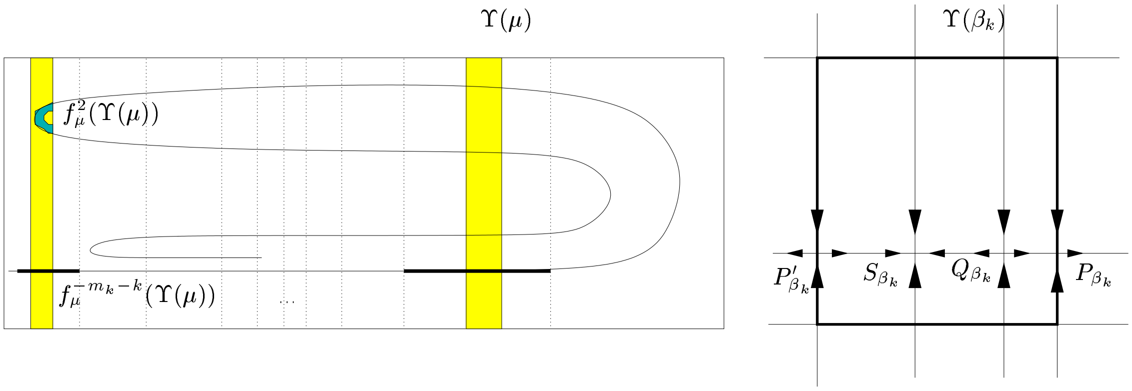

Figure 8: The rectangles $\Upsilon(\mu)$

The first part of the proof of the proposition follows using the arguments in [DRoV, Section 4.2], where a standard cone-field construction is used to prove the hyperbolicity of invariant sets in the unfolding of saddle-node horseshoes. Since these are folklore constructions in bifurcation theory, we just sketch the main ideas of the proof of the proposition. To prove the topological part of the proposition, we will construct Markov partitions for the sets Σ_μ. In rough terms, we have *essentially* the same partition for all the parameters μ in H_k, and the partition for parameters in H_{k+1} is obtained by adding a new rectangle to the partition for $\mu \in H_k$. These partitions will allow us to compute *exactly* the topological entropy of f_μ in R ($\mu \in H_k$). We begin by constructing the Markov partitions. This is done in the next section.

9 Markov Partitions for the sets Σ_μ

9.1 Rectangles of the partition

Fix a big k and, for each $\mu \in H_k$, consider the sub-rectangle $\Upsilon(\mu)$ of R defined as follows and depicted in Figure 8. Given a set A and a point $x \in A$, denote by $C(x, A)$ the connected component of A containing x.

- For $\mu \in H_k^+$, the rectangle $\Upsilon(\mu)$ is the closure of $C(S_\mu, W^s(S_\mu) \cap R)$.
- For $\mu = \beta_k$, the rectangle $\Upsilon(\beta_k)$ is the closure of

$$C(S_{\beta_k}, W^s(S_{\beta_k}) \cap R) \cup C(P_{\beta_k}, W^s(P_{\beta_k}) \cap R) \cup C(P'_{\beta_k}, W^s(P_{\beta_k}) \cap R).$$

- For $\mu \in H_k^-$, the rectangle $\Upsilon(\mu)$ is the closure of the union of

$$C(S_\mu, W^s(S_\mu) \cap R) \cup C(P_\mu^\ell, W^s(P_\mu^\ell) \cap R) \cup C((P_\mu^\ell)', W^s(P_\mu^\ell) \cap R).$$

The *left and right vertical sides* $\partial_\ell \Upsilon(\mu)$ and $\partial_r \Upsilon(\mu)$ of $\Upsilon(\mu)$ are defined as follows,

- $\partial_\ell \Upsilon(\mu) = C(Q'_\mu, W^s(Q_\mu) \cap R)$ and $\partial_r \Upsilon(\mu) = C(Q_\mu, W^s(Q_\mu) \cap R)$, for $\mu \in H_k^+$,

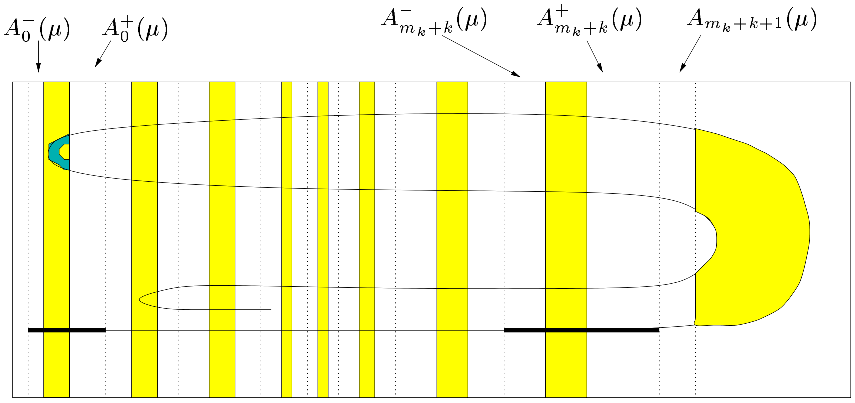

Figure 9: The recatangles $A_j^{\pm}(\mu)$

- $\partial_\ell \Upsilon(\beta_k) = C(P'_{\beta_k}, W^{ss}(P_{\beta_k}) \cap R)$ and $\partial_r \Upsilon(\mu) = C(P_{\beta_k}, W^{ss}(P_{\beta_k}) \cap R)$, for $\mu = \beta_k$,
- $\partial_\ell \Upsilon(\mu) = C((P^r_\mu)', W^s(P^r_\mu) \cap R)$ and $\partial_r \Upsilon(\mu) = C(P^r_\mu, W^s(P^r_\mu) \cap R)$, for $\mu \in H_k^-$.

Note that, for any $\mu \in H_k$, the positive limit set of any point visiting the interior of $\Upsilon(\mu)$ is the orbit of one of the following periodic points: S_μ if $\mu \in H_k^+$, S_{β_k} or Q_{β_k} for $\mu = \beta_k$, and S_μ or Q_μ or P_μ^ℓ if $\mu \in H_k^-$. Thus, the nontrivial part of the dynamics of f_μ is given by points whose orbit avoid $\Upsilon(\mu)$. In particular,

$$\Sigma_\mu \subset \bigcap_{i \in \mathbb{Z}} f_\mu^i(R \setminus \Upsilon(\mu)). \tag{15}$$

In fact, we will get the equality in the previous inclusion. Consider the closed sub-rectangle $R(\mu)$ of R whose left and right vertical sides are the segments $C(\tilde{B}_\mu, W^s(\tilde{B}_\mu) \cap R)$ and $C(f_\mu(\tilde{B}_\mu), W^s(\tilde{B}_\mu) \cap R)$, respectively. Let

$$\Upsilon^-(\mu) = C(\tilde{B}_\mu, R(\mu) \setminus \Upsilon(\mu)) \quad \text{and} \quad \Upsilon^+(\mu) = C(f_\mu(\tilde{B}_\mu), R(\mu) \setminus \Upsilon(\mu)).$$

We are now ready to construct the Markov partition of Σ_μ. For $j \in \{0, 1, \ldots, m_k + k\}$, we let

$$\begin{aligned} A_j^-(\mu) &= C(f_\mu^{-(m_k+k)+j}(\tilde{B}_\mu), f_\mu^{-(m_k+k)+j}(\Upsilon^-(\mu)) \cap R), \text{ and} \\ A_j^+(\mu) &= C(f_\mu^{-(m_k+k)+j}(f_\mu(\tilde{B}_\mu)), f_\mu^{-(m_k+k)+j}(\Upsilon^+(\mu)) \cap R). \end{aligned}$$

To complete the Markov partition of Σ_μ, we need to consider the rectangle $A_{m_k+k+1}(\mu)$ defined as follows: the left vertical side of $A_{m_k+k+1}(\mu)$ is $C(f_\mu(\tilde{B}_\mu), W^s(f_\mu(\tilde{B}_\mu)) \cap R)$. Before defining the right side of $A_{m_k+k+1}(\mu)$, let $W^s_\mu = W^s(Y_\mu)$, ($Y_\mu \in \{Q_\mu, P_{\beta_k}, P^r_\mu\}$) and observe that the images by f_μ of the vertical sides of $\Upsilon(\mu)$ are contained in the same connected component of $W^s_\mu \cap R$. Finally, the other vertical side of $A_{m_k+k+1}(\mu)$ is $C(f_\mu(Y_\mu), W^s_\mu \cap R)$. The family of rectangles $A_j^{\pm}(\mu)$ is depicted in Figure 9.

Lemma 9.1. *The family of rectangles*

$$\mathbb{M}_k(\mu) = \{A_0^+(\mu), A_1^\pm(\mu), \dots, A^\pm_{m_k+k}(\mu), A_{m_k+k+1}(\mu)\}$$

is a Markov partition for Σ_μ.

Proof: By (15), the family of rectangles $\mathbb{M}'_k(\mu) = \{A_0^\pm(\mu), A_1^\pm(\mu), \dots, A^\pm_{m_k+k}(\mu), A_{m_k+k+1}(\mu)\}$ cover Σ_μ (by now, we also consider the set $A_1^-(\mu)$, we will discard this set later). Thus it is enough to check that:

- The two vertical sides of each rectangle of $\mathbb{M}'_k(\mu)$ are strong stable leaves (in fact, stable manifolds of hyperbolic periodic points).
- The image of each vertical side of any rectangle of $\mathbb{M}'_k(\mu)$ is contained in the interior of the vertical side of a rectangle of $\mathbb{M}'_k(\mu)$, and the image of any horizontal side of any rectangle of $\mathbb{M}'_k(\mu)$ intersects the left and the right vertical sides of any rectangle of $\mathbb{M}'_k(\mu)$ whose interior it intersects, and this image is transverse to the vertical lines.

By construction, these two assertions are obvious for the sets $A_i^\pm(\mu)$, $i = 0, \dots, m_k + k$. To prove these assertions for $A_{m_k+k+1}(\mu)$, observe first that its left vertical side is mapped into $F^s(0)$, which is, by construction, the left vertical side of $A^-_{m_k}(\mu)$. On the other hand, the image by f_μ of right vertical side of $A_{m_k+k+1}(\mu)$ is contained in the left vertical side of $A_0^+(\mu)$. To see that just recall that, by construction, the right and the left vertical sides of $A_{m_k+k+1}(\mu)$ and the left vertical side of $A_1^+(\mu)$ are contained in the stable manifold of the saddle Y_μ of period $m_k + k + 2$.

Finally, the rectangle $A_0^-(\mu)$ can be discarded in order to get a Markov partition of Σ_μ: the set $f_\mu(A_0^-(\mu))$ is contained in $A_1^-(\mu)$ and that there are no points of Σ_μ whose orbits meet $A_0^-(\mu)$. The proof of the lemma is now complete. □

9.2 Transition matrices of the Markov partitions

Take any $\mu \in H_k$ and let $\ell_k = 2(m_k + k + 1)$ be the number of rectangles of the Markov partition $\mathbb{M}_k(\mu)$. The transition matrix $T_k = (a_{i,j})$ associated to $\mathbb{M}_k(\mu)$ is a $\ell_k \times \ell_k$ matrix $(a_{i,j})$ such that $a_{i,j} = 1$ if the image of the j-th rectangle of $\mathbb{M}_k(\mu)$ intersects the interior of the i-th rectangle (in this case, the transition from the j-th to the i-th rectangle is allowed), and $a_{i,j} = 0$ otherwise.

Observe that, by construction and the arguments in the proof of Lemma 9.1, the only transitions allowed are the following;

- for every $0 \le j \le m_k + k - 1$, from $A_j^+(\mu)$ to $A_{j+1}^+(\mu)$ and from $A_j^-(\mu)$ to $A_{j+1}^-(\mu)$,
- from $A^-_{m_k+k}(\mu)$ to $A_{m_k+k+1}(\mu)$,
- $A^+_{m_k+k}(\mu)$ to the rectangles $A^\pm_{m_k+i}(\mu)$, for every $0 \le i \le k$, and to A_{m_k+k+1},
- from $A_{m_k+k+1}(\mu)$ to $A_0^+(\mu)$ and $A_i^\pm(\mu)$, for every $1 \le i \le m_k$.

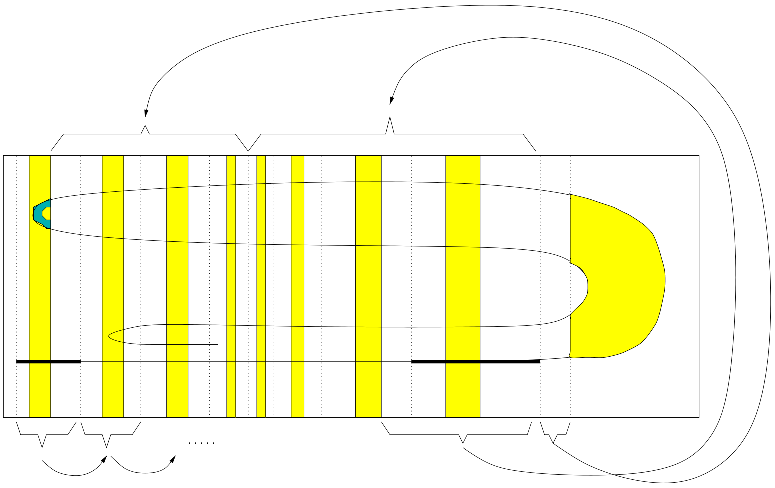

Figure 10: Transitions in $\mathbb{M}_k(\mu)$

So, the matrix $T_k = (a_{i,j})$ has the form:

$$T_k = (a_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{bmatrix} A_0^+(\mu) \\ A_1^-(\mu) \\ A_1^+(\mu) \\ A_2^-(\mu) \\ A_2^+(\mu) \\ \vdots \\ A_{m_k}^+(\mu) \\ A_{m_k+1}^-(\mu) \\ \vdots \\ A_{m_k+k}^+(\mu) \\ A_{m_k+k+1}(\mu) \end{bmatrix} \tag{16}$$

Note that this matrix does not depend on $\mu \in H_k$. Also note, for future uses, that there are $(2m_k - 1)$ non-zero elements in the ℓ_k-th column, and $(2k+1)$ in the $(\ell_k - 1)$-th one.

9.3 Hyperbolicity of the set Σ_μ

As we have mentioned, the proof of the hyperbolicity of Σ_μ (the homoclinic class of G_μ), $\mu \neq \beta_k$, follows by using quite standard cone field arguments (see also [DRoV, Section 4.2]). So, we just sketch this proof.

Let us first consider parameters $\mu \in H_k^+$. Observe that in the vertical direction we have an uniform contraction. Using this contraction and a cone field argument, we get the stable direction.

To get the unstable direction, we will see that there are $\sigma > 1$ and $N_\mu \geq 1$ such that, for every $z \in \Lambda_\mu$ and every vector $v = (1,0)$, there is $m \in \{1, \dots N_\mu\}$ such that $Df_\mu^m(1,0) = (h,v)$, where $|h| \geq \sigma$. To prove this claim, the only difficulty to bypass is the presence of criticalities (which are far from Σ_μ, by construction, recall Remark 8.4). In fact, the only points where we may have delicate estimates to get expansion are those whose orbits meet $A_{m_k+k+1}(\mu)$. Observe that such orbits also meet either $A^+_{m_k+k}(\mu)$ or $A^-_{m_k+k}(\mu)$. Moreover, every $x \in A^-_{m_k+k}(\mu)$ returns to $A^\pm_{m_k+k}(\mu)$ after i iterates, $i \in [k, m_k + k + 2]$. Thinking of the map R_μ as the horizontal component of f^i_μ, Proposition 8.2 gives the announced expansion. Finally, for a point in $A^+_{m_k+k}(\mu)$ its first iterate is expanding. This ends the sketch of the proof of the hyperbolicity of Σ_μ, for $\mu \in H^+_k$.

For parameters in H^-_k the proof follows similarly, but we need to think on the iterates of the orbits close to the saddle-node as a unique iterate, which is expanding, recall the construction for the one dimensional model family in Section 2. This standard construction is also done in [CK]. We omit the details.

Finally, the fact that Λ_{β_k} has only a nonhyperbolic periodic point (the saddle-node P_{β_k}) follows similarly.

Observe that our constructions also imply that the stable and unstable manifolds of periodic points points of Σ_μ intersects the horizontal and the vertical sides, respectively, of any rectangle of the partition $\mathbb{M}_k(\mu)$. So, every pair of periodic points of Σ_μ are homoclinically related. Finally, using the arguments above, one can prove that $\Sigma_\mu = \cap_{i\in\mathbb{Z}} f^i_\mu(R \setminus \Upsilon(\mu))$ (recall equation (15)). This equality implies that:

- for $\mu \in H^-_k$, the limit set of f_μ in R is the disjoint union of the hyperbolic sets Σ_μ, S_μ, Q_μ and P^ℓ_μ,
- for $\mu \in H^+_k$, the limit set of f_μ in R is the disjoint union of the hyperbolic sets Σ_μ and S_μ,
- for $\mu = \beta_k$, the limit set of f_μ in R is the disjoint union of the saddle-node horseshoe Σ_{β_k} and S_{β_k}.

9.4 Topological Entropy

In this section, we compute the topological entropy of the restriction of f_μ to Σ_μ (which is equal to the topological entropy of the restriction of f_μ to R).

Proposition 9.2. *The topological entropy $h(f_\mu)$ of the restriction of f_μ to Σ_μ is equal to a constant h_k, for every $\mu \in H_k$ (big k). Moreover, the sequence (h_k) is increasing and verifies*

$$\log\left(\frac{2\,\ell_k}{\ell_k+2}\right) < h_k < \log(2),$$

where $\ell_k = 2(m_k + k)$ is the number of rectangles of the Markov partition $\mathbb{M}_k(\mu)$.

Proof: Let σ_ℓ be the eigenvalue of the transition matrix T_k in equation (16) with biggest modulus. Recall that $h(f_\mu)$ is $\log(|\sigma_\ell|)$, see for instance [M, Corollary 7.7]. We will show that

$$\frac{2\,\ell_k}{\ell_k+2} < |\sigma_\ell| < 2.$$

Since the set Σ_μ contains horseshoes (just observe that the continuation of the hyperbolic fixed point G_0 of Λ_0 has transverse homoclinic points), $h(f_\mu)$ is strictly positive. Hence, the matrix T_k has an eigenvalue of modulus bigger than one. So, to find σ_ℓ, we can focus on those eigenvalues of T_k having modulus bigger than one.

Lemma 9.3. *Let λ be an eigenvalue of T_k such that $|\lambda| > 1$. Then every eigenvector $X = (x_1, x_2, \ldots, x_\ell)$ associated to λ satisfies $x_{2i} = x_{2i-1}$ for every $i = 1, \ldots, \ell/2$.*

Proof: Observe that, for every X and $\lambda \neq 0$ with $(a_{i,j})X^t = \lambda X^t$, the equations below hold:

$$\begin{array}{rcl} -\lambda\, x_1 + x_\ell & = & 0 \\ -\lambda x_2 + x_\ell & = & 0 \\ x_1 - \lambda\, x_3 + x_\ell & = & 0 \\ x_2 - \lambda\, x_4 + x_\ell & = & 0 \\ \vdots & \vdots & \vdots \\ x_{2m_k-2} - \lambda\, x_{2m_k} + x_\ell & = & 0 \\ x_{2m_k-1} - \lambda\, x_{2m_k+1} + x_{\ell-1} & = & 0 \\ \vdots & \vdots & \vdots \\ x_{2m_k+2k-2} - \lambda\, x_{2m_k+2k} + x_{\ell-1} & = & 0. \end{array} \tag{17}$$

In particular, we have the following:

$$\begin{array}{rcl} x_2 - x_1 & = & 0 \\ x_4 - x_3 & = & 0 \\ \vdots & \vdots & \vdots \\ x_{2m_k} - x_{2m_k-1} & = & 0 \\ x_{2m_k+1} - x_{2m_k} & = & \dfrac{x_{\ell-1} - x_\ell}{\lambda} \\ x_{2m_k+2} - x_{2m_k+1} & = & \dfrac{x_{\ell-1} - x_\ell}{\lambda^2} \\ \vdots & \vdots & \vdots \\ x_\ell - x_{\ell-1} & = & \dfrac{x_{\ell-1} - x_\ell}{\lambda^k}. \end{array} \tag{18}$$

The last equation implies that if $(x_{\ell-1} - x_\ell) \neq 0$, then $|\lambda| = 1$, which is a contradiction. So, we can assume that $x_\ell = x_{\ell-1} = 1$ (the option $x_\ell = x_{\ell-1} = 0$ leads to the zero vector, which is not an eigenvector). The lemma follows arguing recursively. $\square$

Returning to the original system in (17), putting $x_\ell = x_{\ell-1} = 1$, and recalling that $\ell = 2m_k + 2$, we find:

$$\begin{array}{rcl} x_2 & = & \lambda^{-1} \\ x_4 & = & \lambda^{-1} + \lambda^{-2} \\ \vdots & \vdots & \vdots \\ x_{\ell-2} & = & \sum_{i=1}^{\ell/2-1} \lambda^{-i} \\ x_{\ell-2} + 1 & = & \lambda. \end{array} \tag{19}$$

By the last two equations above, it follows that

$$-\lambda^{-(\ell/2)} - \lambda + 2 = 0 \quad \Longrightarrow \quad \lambda^{(\ell/2)+1} - 2\,\lambda^{\ell/2} + 1 = 0.$$

Consider now the polynomial

$$\Pi_\ell(\lambda) = \lambda^{(\ell/2)+1} - 2\,\lambda^{\ell/2} + 1.$$

Observe that $\lambda = 1$ is a root of Π_ℓ and that

$$\Pi'_\ell(\lambda) = \lambda^{(\ell/2)-1}((\ell/2)+1)\,\lambda - \ell)$$

has only one non-zero root $\lambda_0 = (2\,\ell)/(\ell+2) > 0$. Since $\Pi_\ell(x) \to +\infty$ as $x \to +\infty$, it is immediate to check that λ_0 is the only minimum of Π_ℓ. More precisely, if ℓ is large enough, then λ_0 is close to 2^- and has negative image by Π_ℓ (observe that $\Pi_\ell(1) = 0$, $\lambda_0 > 1$ and that Π_ℓ is decreasing in the interval $(1, \lambda_0)$). These properties, together with the expression of Π_ℓ, imply that Π_ℓ has a unique root λ_1 bigger that λ_0. Since $\Pi_\ell(2) = 1$, we have

$$\lambda_1 \in (\lambda_0, 2) \subset (\frac{2\,\ell}{\ell+2}, 2).$$

Observe that $\Pi_\ell(0) = 1$ and that 0 and λ_0 are the only critical points of Π_ℓ. Thus, if the degree of Π_ℓ is even, there are no negative roots for Π_ℓ and λ_1 is the biggest eigenvalue, in modulus, of Π_ℓ. On the other hand, if the degree of Π_ℓ is odd, the polynomial Π_ℓ has (exactly) one negative root, say λ_2. Since $\Pi_\ell(-1) = -2$ and $\Pi_\ell(0) = 1$ we have that $\lambda_2 \in (-1, 0)$. Thus, $|\lambda_2| < \lambda_1$ and λ_1 is biggest eigenvalue in modulus of Π_ℓ.

By construction, the root λ_1 of Π_ℓ corresponds to an (the biggest in modulus) eigenvalue of T_k. Thus, $\lambda_1 = \sigma_\ell$, ending the proof of the first part of the proposition.

To end the proof of the proposition, we need to see that the sequence (h_k) is increasing. Consider $\ell = \ell_k$ and ℓ_{k+1} and observe that $\ell_{k+1} = \ell + 2n$ for some $n \geq 1$. Then, since $\sigma_\ell > 1$,

$$\Pi_{\ell+2n}(\sigma_\ell) = (\sigma_\ell)^{(\ell/2)+n+1} - 2\,(\sigma_\ell)^{(\ell/2)+n} + 1 < (\sigma_\ell)^{(\ell/2)+n+1} - 2\,(\sigma_\ell)^{(\ell/2)+n} + (\sigma_\ell)^n < (\sigma_\ell)^n\,\Pi_\ell(\sigma_\ell) = 0.$$

Thus, since $\lim_{x\to\infty} \Pi_{\ell+2n}(x) = \infty$ and $\Pi_{\ell+2n}$ is increasing in $(\sigma_{\ell+2n}, \infty)$, $\sigma_{\ell+2n} > \sigma_\ell$, ending the proof of the proposition. □

10 Increasing the entropy after unfolding saddle-node horseshoes

In this section, we consider critical two saddle-node horseshoe cycles and give an informal explanation (we just sketch the constructions omitting the technical details) of how the topological entropy may increase after the unfolding of this class of cycles (Proposition B). Using this construction, we show how the entropy may be non-monotone in the unfolding of one critical saddle-node horseshoes (Theorem C).

10.1 Entropy and the unfolding of critical two saddle-node horseshoe cycles

Consider a rectangle R as in Section 4.1 and a diffeomorphism f such that $f(R)$ is mapped in the interior of R and the limit set of f in R is the disjoint union of:

- a fixed sink S, having two real positive multipliers of different modulus,

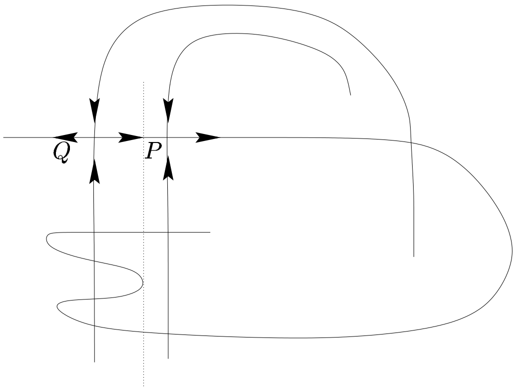

Figure 11: Critical two saddle-node horseshoe

- a saddle-node horseshoe Λ conjugate to a complete shift of two symbols which is the homoclinic class of the saddle node P of Λ,
- a fixed saddle Q having a trivial homoclinic class such that one branch of $W^u(Q)$ is contained in $W^s(P)$ and the other branch contained in $W^s(P)$,
- the unstable manifold of the saddle-node P verifies the following properties:
  - $W^u(P)$ intersects transversely $W^s(Q)$ throughout the orbit of four points,
  - $W^u(P)$ intersects stable manifold of the saddle-node and is tangent to the strong stable foliation of $W^s(S)$ along the orbit of two points,
  - $W^u(P)$ is tangent to the strong stable foliation of $W^s(P)$ along the orbit of a point.

The geometry of this class of two cycles is depicted in Figure 11.

The cycle above can be constructed in such a way the following holds:

Proposition 10.1. *Consider an arc of diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ unfolding generically the two saddle-node cycle above at $\mu = 0$ in such a way f_μ is hyperbolic for negative μ.*

Then, for every $\mu > 0$ small enough, there is m_μ, $m_\mu \to \infty$ as $\mu \to 0^+$, such that

$$h(f_\mu|_R) \geq \frac{\log(2^{m_\mu}+2)}{m_\mu}.$$

In particular, $h(f_\mu|_R) > \log(2) = h(f_0|_R)$.

Observe that this result is just a reformulation of Proposition B.

Proof: To prove the proposition, given a small $\mu > 0$, we consider disjoint rectangles R_1, R_2, R_3, and R_4 as in Figure 12, where the dynamics in R_1 and R_2 contains a hyperbolic horseshoe Λ_μ conjugate to a complete shift of two symbols. The sets R_3 and R_4 are small rectangles having one vertical side contained in the stable manifold of the saddle Q, the horizontal sides are small segments contained in the unstable manifold of Q. By the λ-lemma, there exists m such that f^m has horseshoes in R_3 and in R_4, see Figure 12.

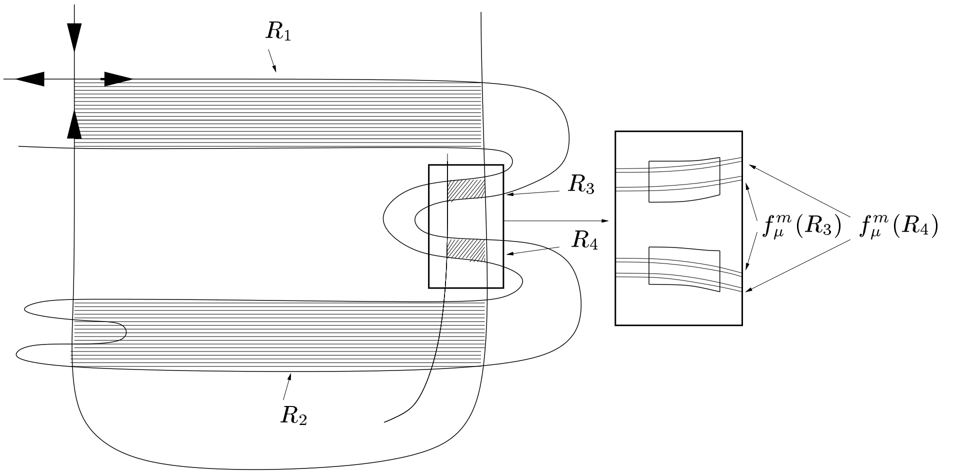

Figure 12: The rectangles R_i and their images

Moreover, one can take the rectangles above in such a way the maximal invariant set of f_μ in $\hat{R} = R_1 \cup R_2 \cup R_3 \cup R_4$ is hyperbolic. The construction above imply that, for every k, the diffeomorphism $(f_\mu^m)^k$ has at least $(2^m + 2)^k$ different hyperbolic fixed points in $\hat{R}$. Thus

$$h(f_\mu^m) \geq \log(2^m + 2).$$

The proposition follows from the fact that $m\, h(f_\mu) = h(f_\mu^m)$. □

10.2 Non-monotonicity of the topological entropy for critical one saddle-node horseshoes

In this section, we sketch the proof of Theorem C. Let us first recall that, in the construction for critical one saddle-node horseshoes cycles, we consider the interval $(B, E) \subset W^u(P)$, P the saddle-node, and E and B two points of transverse intersections between $W^u(P)$ and $W^{ss}(P)$ with $(B, E) \subset W^s(P)$. With these ingredients we induce a ghost return dynamics such that the unidimensional map $R_\infty(\sigma, \cdot)$ unfolds a (one-dimensional) noncritical two saddle-node horseshoe cycle at some parameter σ^*. Following this idea, consider a subinterval I of (B, E) and modify the dynamics of f in $f^2(I)$ in such a way the new ghost return $R_\infty(\sigma, \cdot)$ unfolds a (one-dimensional) two critical saddle-node horseshoe cycle at σ^*. In Figure 13 is depicted this type of one dimensional cycle.

Using this one dimensional cycle, we get that for the parameters β_k (defined exactly as in Lemma 7.1) the arc f_μ unfolds (two-dimensional) critical two saddle-node horseshoe cycles similar to the one in the previous section.

Define H_k^- and H_k^+ as in Remark 8.3. The calculations for the topological entropy for parameters in H_k^- are exactly the same as in Section 9.4, and they lead to the same result: $h(f_{\nu_k}|_R) < h(f_{\nu_{k+1}}|_R) < \log(2)$ for all $\nu_k \in H_k^-$ and $\nu_{k+1} \in H_{k+1}^-$.

On the other hand, the calculations in the previous section (Proposition 10.1) imply that $h(f_\mu|_R) > h(f_\nu|_R)$ for all $\mu \in H_k^+$ and $\nu \in H_k^-$ (big k). This construction ends the sketch of the proof of Theorem C.

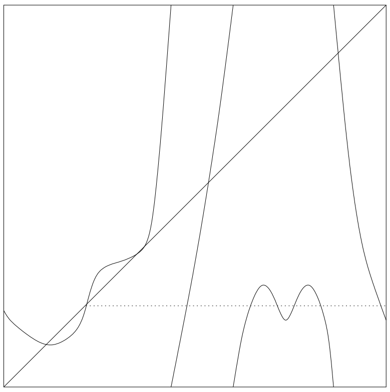

Figure 13: Two critical saddle-node horseshoe cycle for $R_\infty(\sigma^*, \cdot)$

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Tame and wild C^1-dynamics

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(I) MOTIVATIONS

(I) Spectral decomposition of the limit set $L(f)$ for C^1-diffeomorphisms

M^2 a compact and closed surface.

Diff$^1(M^2)$ the space of C^1-diffeomorphisms (usual topology).

Theorem (Mañé, 82):

For C^1-generic surface diffeomorphisms (diffeos in a residual subset):

A) hyperbolic dynamics:

$L(f) =$ Axiom A $\oplus$ no-cycles,

B) Newhouse coexistence phenomenon:

f has infinitely many sinks or sources.

Hyperbolic case: There is a spectral decomposition of $L(f)$:

$$L(f) = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_n$$

Λ_i **basic set** (elementary piece of dynamics)

Properties of basic sets Λ:

- Transitive (i.e. dense orbit)

  indecomposability

- Hyperbolicity: $T_\Lambda M = E^s \oplus E^u$.

- Local maximal (isolation),

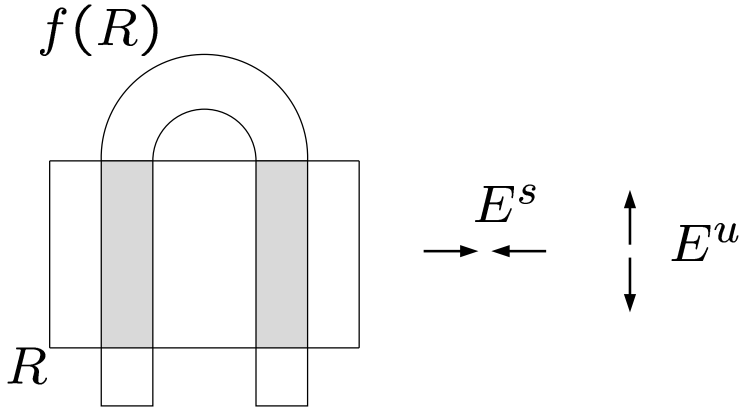
  $\Lambda = \bigcap_{n\in\mathbb{Z}}^n (U)$, U nbhood of Λ.

- Homoclinic class,

  $\Lambda = \overline{W^s(P) \pitchfork W^u(P)} = H_P(f)$.

(nontrivial) **Models** of basic sets:

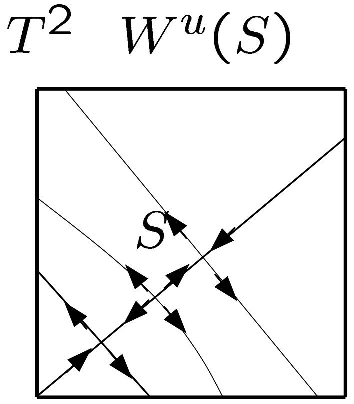
Smale's horseshoes,

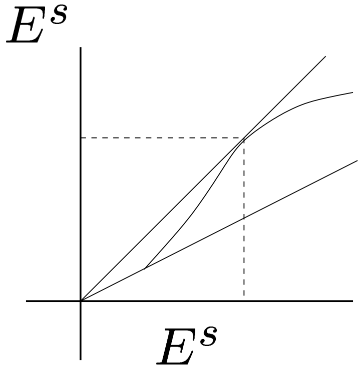


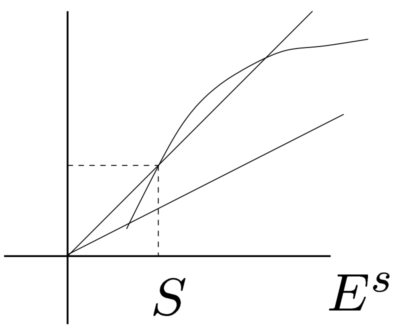
Anosov diffeomorphisms,

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Derived from Anosov attractors,

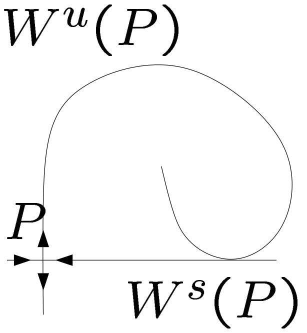






Remarks and goals:

- Is it possible to eliminate (B) in the Thm.?. The only known mechanism for persistent tangencies for surface diffeos is the unfolding of C^2 homoclinic tangencies (Distortion properties).



- Reformulate the theorem in higher dimensions (false) (see nonhyperbolic robustly transitive diffeos)

- Give examples leading to persistent coexistence of infinitely many sinks for C^1 diffeos (higher dimensions)

Remarks and restictions: Why C^1-top.?

Good perturbation results are avaliable: Short list with hints.

- **Pugh closing lemma:** (closing orbits of recurrent points) this allows to estructurate the dynamics over the pdic. pts.

- **Hayashi's connecting lemmas:** creation of cycles

- **Mañé's ergodic closing lemma:** useful to prove hyperbolic-like properties for invariant splittings.

- **Frank's lemma:** dynamical realization of the derivative (it relates co-cycles and dynamics).

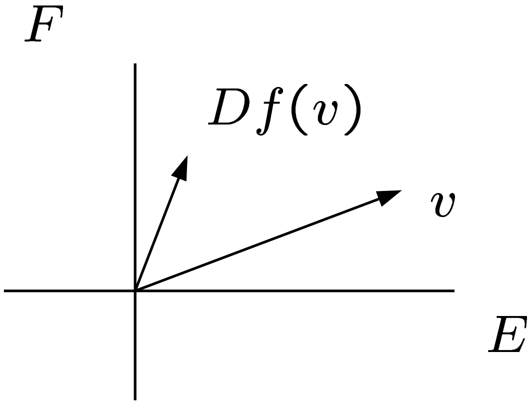
Forms of hyperbolicity

Hyperbolicity.

Dominated splitting:

$T_\Lambda M = E \oplus F$, Df-invariant $\oplus$ const. dim.

$$\exists \ell \in \mathbb{N} \quad \text{s.t } \|Df^\ell|_{E(x)}\| \cdot \|Df^{-\ell}|_{F(f^\ell(x))}\| < \frac{1}{2}$$



Partial hyperbolicity:

(dominated) $\oplus$ (E or F hyperbolic)
Strong part. hyp splt, $E \oplus F \oplus G$:
E and G hyperbolic
$(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ dominated

Robustly transitive diffeos

Transitive diffeos: f has a dense orbit in the whole manifold'

Models:

- irrational rotations (non-persistent)
- Anosov diffeos (persistent)

Robustly transitive C^1-diffeos: any C^1-pertb. of f is transitive.

There are non-hyperbolic robustly transitive diffeos in dimension ≥ 3

$\Longrightarrow$

Mañé's Thm is false in higher dimensions!: transitive diffeos have not neither sinks nor sources.

Examps of non-hyp robst trans diffeos

- Skew products in $T^4 = T^2 \times T^2$ (Shub)

  (Anosov × DAnosov),

  strong part. hyper., $TM = E^s \oplus E^c \oplus E^u$,

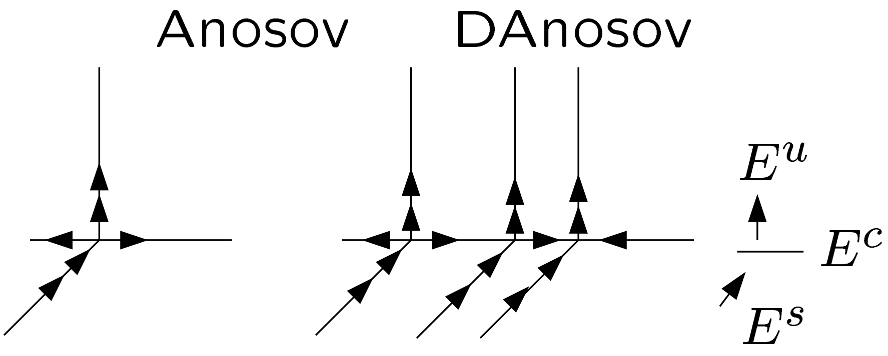
  1-dim central direction,

- DAnosov in T^3 (Mañé):

  a transitive set contains saddles of indices 2 and 1,

  strong part. hyperb., $TM = E^s \oplus E^c \oplus E^u$,

  1-dim central direction.



saddle-node bif.

(saddle-node bif. does not creates sinks/sources)

- Skew products in any dimension (B-D)

  (pertb. of Anosov × identity)

  strong part. hyper., $TM = E^s \oplus E^c \oplus E^u$,

  k-dim central direction,

- pertb. of time-1 of Anosov v. f. (B-D)

  (isotopic to the identity)

- DAnosov in T^3 (Bonatti)

  partial hyperbolicity, $TM = E^s \oplus E^c$,

  2-dim central direction,

  only one hyperbolic direction.

- DAnosov in T^4 (Bonatti-Viana)

  no hyperbolic directions (no partially hyperbolic),

  $TM = E \oplus F$, dominated splitting.

Remark: In a reformulation of Mañé's Thm.: one needs to consider forms of hyperbolicity weaker than partial hyperbolcity.

(II) Persistence of sinks and sources for C^1-diffeomorphisms

Theorem (Bonatti-D, 99):

Diff$^1(M^3)$.

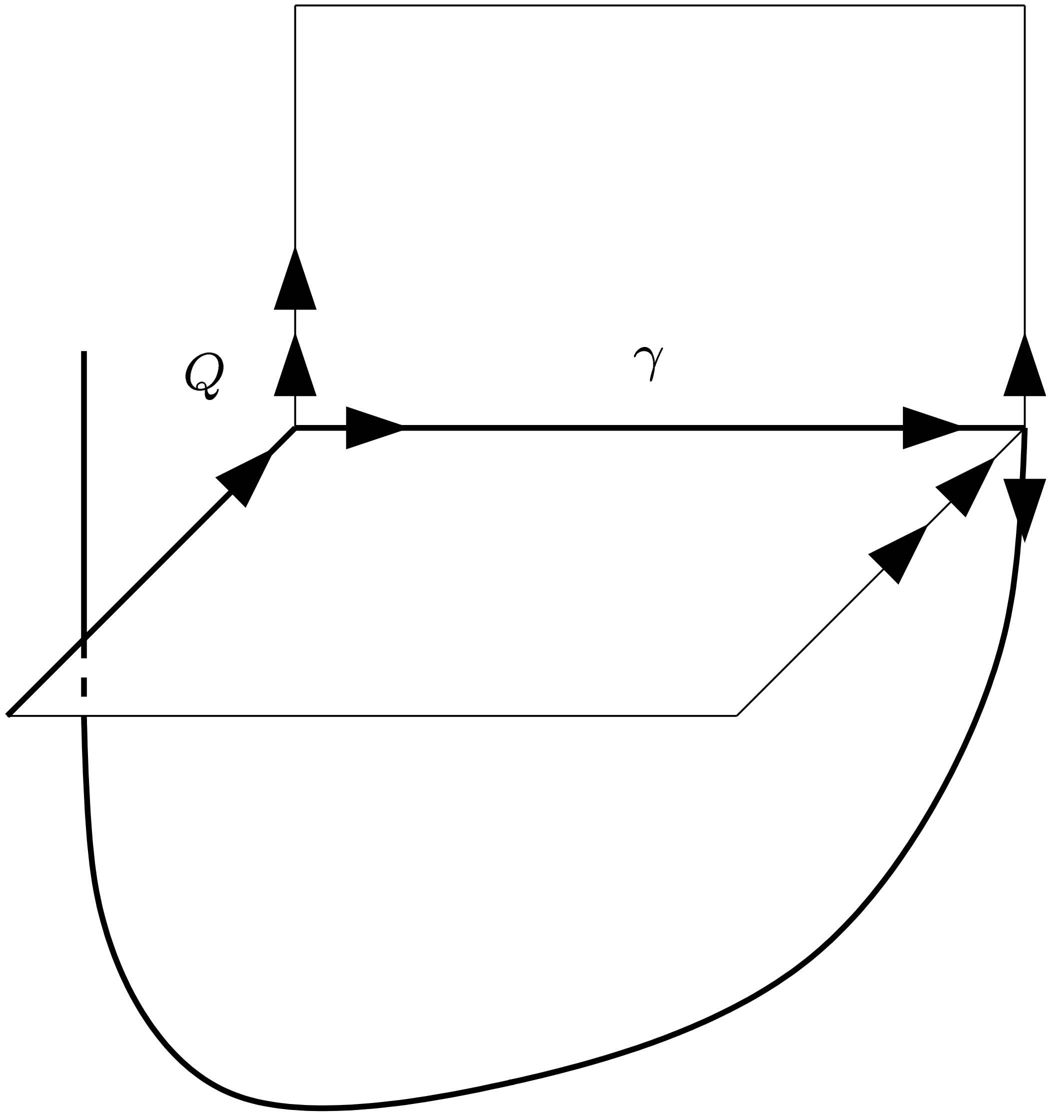
There are U open and $R \subset U$ residual in U s.t. every f in R has simultaneously infinitely many sinks and sources.

Ingredients:

- **heterodimensional cycles**,
- **non-dominated homoclinic classes.**

Bifurc. via heterodim. cycles (dim 3)

Take saddles P and Q of indices 1 and 2
$W^s(P) \pitchfork W^u(Q) \neq \emptyset$,
$W^u(P)$ and $W^s(Q)$ have a quasi-transverse intersection.



Theorem (D-Rocha, 00):

Diff$^1(M^n)$, $n \geq 3$.

f has a heterodim. cycle as in the figure

There is U open in Diff$^1(M^n)$ s.t.

- the closure the U contains f,
- for every $g \in U$, $H_P(g) = H_Q(g)$

  (non-hyperb. homoclinic classes).

The proof only involves orbits contained in a neighborhood V of the cycle (semi-local argument).

The dynamics after the bifurcation *essentially* given by the dynamics in the curve γ.

For bifurcations from hyperbolic dynamics (f.i. Morse-Smale)

$$\Lambda_g = H_P(g) = H_Q(g) = \cap_{i\in\mathbb{Z}} g^i(V)$$

V is a nbhood of the cycle.

Key for New Ph in surfaces: persistence of tangencies.

Key for persist. of cycles in heterodim bifs:
$W^u(Q)$ (2-dim), $W^u(Q) \subset \overline{W^u(P)}$.
naive implication: $W^u(P)$ is 2-dim
(i.e. P has indices 2 and 1).

Non-dominated homoclinic classes

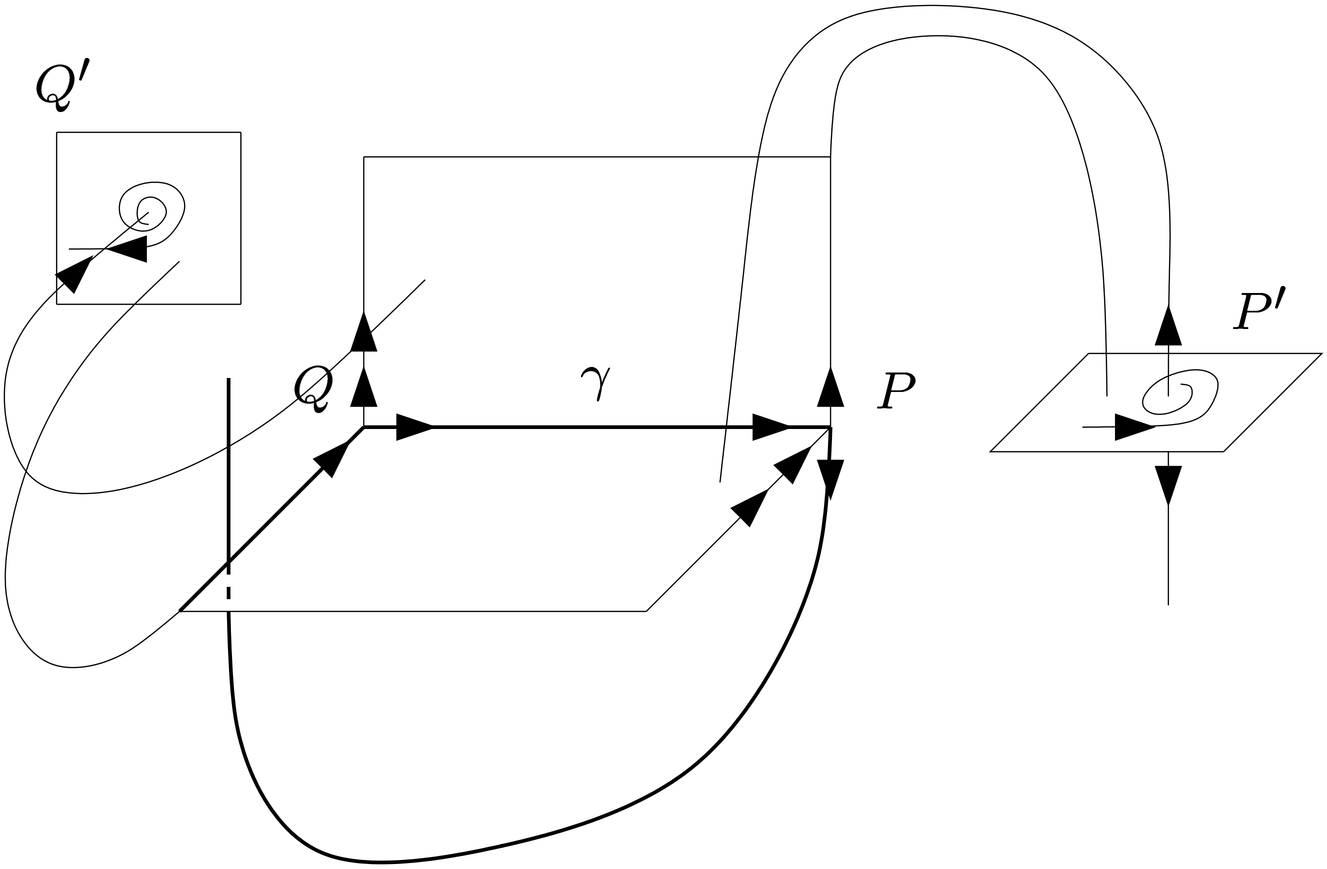
Complete the figure as follows:

Suppose that there are saddles P' and Q'

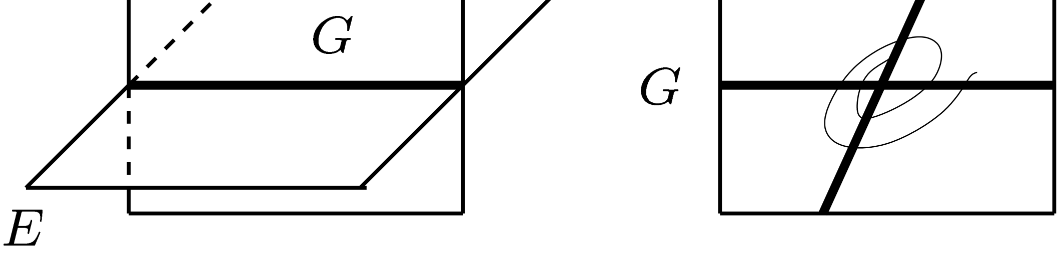
- **(i)** P' of index 1 homoclinic. related to P,

  thus $H_P(f) = H_{P'}(f)$
- **(ii)** P' has a non-rela contrac. eigenv.,
- **(iii)** Q' of index 2 homoclinic. relat. to Q,

  thus $H_Q(f) = H_{Q'}(f)$,
- **(iv)** Q' has a non-real expand. eigenv.,



(i) $\oplus$ (iii) $\Longrightarrow$ $\Lambda_g = H_{P'}(g) = H_P(g) = H_Q(g) = H_{Q'}(g)$ for all $g \in U$.



(ii) $\oplus$ (iv) $\Longrightarrow$ the transitive set Λ_g does not admit any dominated splitting $T_{\Lambda_g}M = E \oplus F$.

$\dim(E) = 2 \Longrightarrow F(Q') \subset E^u(Q')$, $E^u(Q')$ has an invariant direction, contradicting complex eigenvalues.

$\dim(F) = 2 \Longrightarrow E(P') \subset E^s(Q')$...

Proposition:

(P, P', Q, Q', and U as above) $\oplus$

jacobian(P') $> 1 >$ jacobian(Q')

There is $R \subset U$, R residual in U s.t.

$$\begin{aligned}&\Lambda_g = H_{P'}(g) = \cdots = H_{Q'}(g) \subset \\ &\subset \{\text{sinks and sources of } g\}\end{aligned}$$

(Λ_g ∞) $\Longrightarrow$ ∞-many sinks and sources)

Motivated by the previous result **Generalization:**

Theorem (Bonatti-D.-Pujals, 99):

Generically in Diff$^1(M^k)$ there is a **dichotomy:**

(d) $H_P(f)$ has a dominated splitting,

- dim $M = 2$ hyp.,
- dim $M = 3$ part. hyp.

(∞) $H_P(f)$ is contained in the closure of an infinite set of sinks and sources.

Problem: To study the dynamics of homoclinic classes withou domination (pathological dynamics associated to such a configuration)

(III) Notion of elementary piece of dynamics (non-hyperbolic setting). Chasing a sepectral decomposition

Theorem ? (Naive reform. of Mañés Th):

For C^1-generic diffeomorphisms

A) $L(f) = \Lambda_1 \cup \cdots \cup \Lambda_k$, $\Lambda_i \cap \Lambda_j = \emptyset$,

Λ_i elementary piece of dynamics with some weak hyperbolicity.

B) pathological dynamics

Newh. coexist. phenom. and worse,

(related to homoclinic classes without domination).

Chasing candidates for of elementary piece of dynamics (those must satisfy most of the properties of the basic sets)

Models:

- non-hyperbolic transitive sets in the unfolding of heterod. cycles.
- Nonhyperbolic transitive sets.

Properties:

- Transitivity: **Maximal transitive sets**
- Homoclinic classes,
- Local maximality: **Robust transitive sets**
- Weak hyperbolicity: **Hyperbolicity, partial hyperbolicity, dominated splittings**.

Thm (Arnaud, 99, Car-Mor-Pac, 01)

Generically in $\mathrm{Diff}^1(M)$ every homoclinic class

is a maximal **saturated** transitive set.

Λ **is saturated:** Σ transitive and $\Sigma \cap \Lambda \neq \emptyset$ $\Longrightarrow \Sigma \subset \Lambda$.

Corollary Generically in $\mathrm{Diff}^1(M)$:,

- either $H_P(f) \cap H_Q(f) = \emptyset$,
- or $H_P(f) = H_Q(f)$

Good news: For generic diffeomorphisms, the homoclinic classes induce a partition on (a priori) a part of the limit set.

Natural and naive (related) questions (with negative answers):

Generically, is $L(f)$ the union of (infinitely many) homoclinic classes?

Are the saturated sets of generic diffeomorphisms homoclinic classes?

Theorem (Negative answers) (B.-D., 01)

There are U open in Diff$^1(M)$

R residual in U

s. t. every f in R has a

(nontrivial) maxi. satur. trans. set Λ_f,

Λ_f is a Cantor (∞) set,

the dynamics of f over Λ_f is **minimal**

Minimal: Every orbit of Λ_f is dense in Λ_f. There are not periodic points.

The set Λ_f is obtained via a (infinite) renormalization process.

The set Λ_f is an adding machine like set.

(IV) The retrieved spectral decomposition (non-hyperbolic setting)

R residual subset of $\mathrm{Diff}^1(M)$ where

$$\boxed{H_P(f)\cap H_Q(f)=\emptyset \text{ or } H_P(f)=H_Q(f).}$$

Define:

$\#R\to\mathbb{N}\cup\infty,\quad f\to\#(\{$ hmclnc classes$\}$

The map # is locally constant in R.

(T) Tame diffeos: $\#(f)\in\mathbb{N}$,

(W) Wild diffeos: $\#(f)=\infty$ (for instance, Newh. Phen.)

Theorem (Retrieved spectral decomposition) (Abdenur, 01)

For tame diffeos f:

$$L(f) = H_{P_1}(f) \cup \cdots \cup H_{P_m}(f),$$

every $H_{P_i}(f)$ has a dominated splitting.

There is a **filtration** $\{M_i\}_i$

($M_i \subset M_{i+1}$ and $f(M_i) \subset \text{int}(M_i)$)

s.t. for every $g \in T$ close to f,

$$H_{P_i}(g) = \bigcap_{k\in\mathbb{Z}} g^k(M_{i+1} \setminus M_i).$$

Corollary: The naive reformulation of the Mañé's Thm is correct.

(II) WILD DYNAMICS. HMCLNC CLSS WITHOUT DOMINATION

dim $M = 3$, higher dimensions similar, but much more complicated

Wild homoclinic classes (def. motivated by the constructions leading to Newh. Ph. in the C^1-top.)

The class $H_P(f)$ is **wild** if it has hyp. prdc. pts. A and B of indices 1 and 2 s.t.

- A has a contracting non-real eigenvalue,
- B has an expanding non-real eigenvalue,
- jacobian $(A) > 1 >$ jacobian (B).

For every g close to f, $A, B \in H_P(g)$.

(Special class of non-dominated hmclnc class)

This condition is much more stronger than to be wild (f.i. infinitely many independent saddles in a normally hyperbolic manifold).

B set of diffeos with a wild homocl. class (Rmk: B is open and $B \cap R \subset W$).

Theorem (Strong forms of Newhouse phenomenon) (B.-D., 01)

There is a residual subset S of B s.t. every $f \in S$ simultaneously has ∞-many:

- sinks and sources,
- saddles with trivial hmclnc clss.,
- wild hmclnc clss.,
- partially hyp. attracators and repellors,
- minimal saturated transitive sets.

(I) Step (BDP arguments)

$H_P(f)$ does not have a domin. split., then it is possible to create a sink/source.

Homothety lemma:

$H_P(f)$ does not have a dominated splitting,

For all $\varepsilon > 0$ there are:

a prdc. pt. $x \in H_P(f)$ with a ε-dense orbit in $H_P(f)$

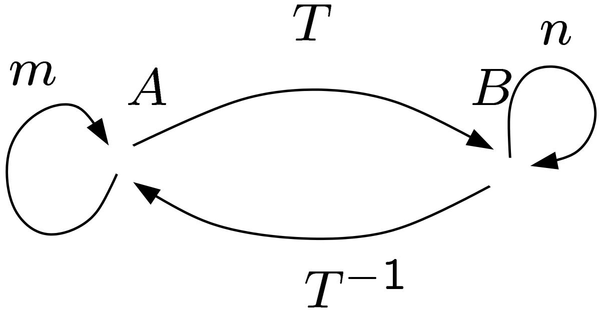
g C^1-close to f and a nbhood V of x s. t.

$$g^m/V = \text{homothety}, \quad m = \text{the period of } x.$$

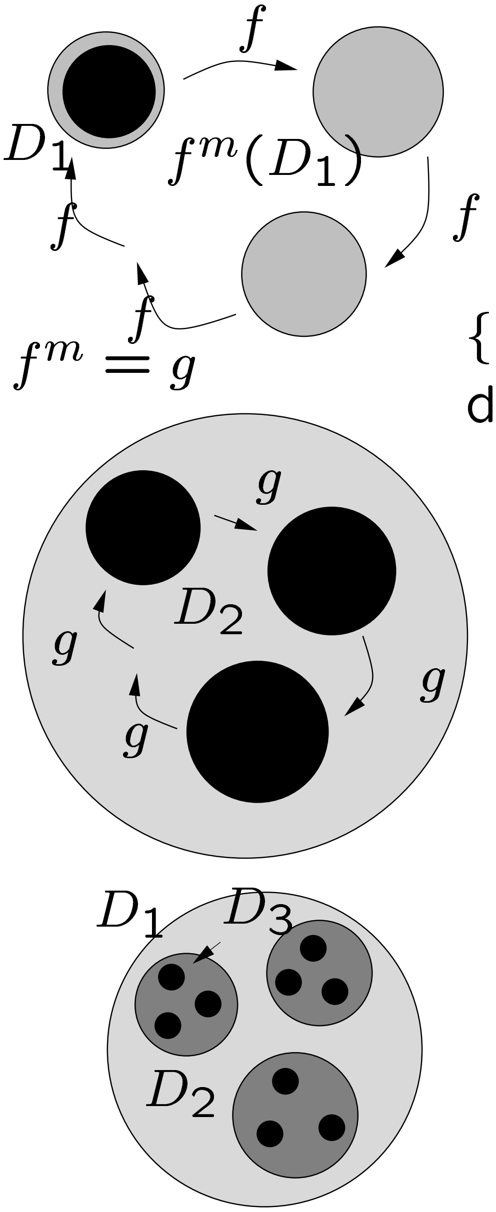
The ratio of the homothety depends on the jacobians of the prdc. pts. in $H_P(f)$.

Using pts. A and B with jacobians greater and less than 1 one gets the identity.

Construction involves **transitions** (main technical tool in BDP)



(II) Inductive and renormalization-like proc



$\{D_i\}$ nested fam.
diam$(D_i) \to 0$

$\Lambda = \bigcap_{i=1}^{\infty} \Delta_i$

Λ is a Cantor set
Every orbit is dense in Λ

Routes to wild dynamics

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ABSTRACT

We present dynamical scenarios leading to wild dynamics. We first discuss the C^2 persistent coexistence of infinitely many sinks associated to homoclinic tangencies of surface diffeomorphisms (Newhouse coexistence phenomenon). We next study heterodimensional cycles in higher dimensions and apply these results to get the C^1 coexistence phenomenon. Finally, we introduce the notion of dominated splitting and explain how the lack of domination of the non-wandering set may lead to stronger forms of the C^1 coexistence phenomenon.

2000 Mathematics Subject Classification: 37G25, 37G30, 37G35, 37D30
Key words: dominated splitting, heterodimensional cycle, homoclinic class, homoclinic tangency, minimal set, (Newhouse) coexistence phenomenon, transitive set

1. Introduction

Throughout this paper M denotes a finite dimensional closed Riemannian manifold and $\mathrm{Diff}^k(M)$, $k \geq 1$, the space of C^k diffeomorphisms of M endowed with the usual

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[*] Enrique Outerelo was my M. Sc. advisor in 1984. Before that he was my instructor in the courses of General and Differential Topology at the Universidad Complutense of Madrid. The course of General Topology was a true eye-opener for me. Back in 1984, good times, Juan Manuel López de Sá and Roberto Moriyón organized an enthusiastic Dynamical Systems Seminar in the Universidad Complutense held every Saturday morning (it seems to me that surviving these long sessions was some sort of a miracle, maybe being a lot younger helped). Enrique attended these seminars and I asked him whether he would be my thesis advisor. Following a crazy (but good) idea of López de Sá, we studied a paper on wild dynamics, the so-called Newhouse phenomenon, the subject of the present article. Although Enrique was not familiar with this subject he agreed to supervise my thesis. The M. Sc. thesis was the first step in my research on wild dynamics, my main and favorite topic of research since then. It is a great pleasure for me to thank warmly Enrique for his advice and support during those years.

[†] I thank my co-authors C. Bonatti, E. R. Pujals, J. Rocha, R. Ures and M. Viana for their long-time collaboration since a long time (most of the results presented here are its consequence). I also thank S. Volchan for a careful reading of the first version of this paper and his useful comments.

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topology. By a C^k *generic diffeomorphism* we mean a diffeomorphism in a residual subset of $\mathrm{Diff}^k(M)$ (i.e., a set containing a countable intersection of open and dense subsets).

Recently, Abdenur stated in [1] a dichotomy for generic C^1 diffeomorphisms: those diffeomorphisms having finitely many *elementary pieces of dynamics* in the ambient manifold, the *tame* diffeomorphisms, and those with infinitely many elementary pieces (including an uncountable number), the *wild* ones. Our goal is to describe dynamical scenarios for the occurrence of wild dynamics. We will see how the wild dynamics is related to *non-dominated dynamics,* the *persistence of cycles,* and *homoclinic tangencies* (we give precise definitions later on). We will not discuss the notion of *elementary piece of dynamics* (for a broad discussion on this topic see [7, Chapter 10] and [2]). Naively speaking and avoiding technicalities, one aims to split the part of the ambient manifold supporting the non-trivial dynamics (recurrences) into invariant, independent, indecomposable, and maximal pieces of dynamics (which will be the elementary pieces). Archetypical examples of such pieces are the *basic sets* in the Smale hyperbolic theory.

We present in Section 3 the first (chronologically) example of wild dynamics: the persistent coexistence of infinitely many sinks for surface C^2 diffeomorphisms, associated to homoclinic tangencies. This phenomenon (the so-called Newhouse coexistence phenomenon) is typically C^2 and relies on the notion of *thickness* (a kind of fractal dimension) of a hyperbolic set. In Section 4.1, we study *heterodimensional cycles* and how persistently non-hyperbolic sets arise from these cycles. In Section 4.2, we apply these results to get the coexistence phenomenon in the C^1 topology in higher dimensions. Finally, in Section 5 we introduce the notion of dominated splitting and present dynamical configurations (non-dominated homoclinic classes) leading to specially interesting cases of wild dynamics: diffeomorphisms displaying persistently and simultaneously infinitely many non-trivial homoclinic classes, minimal sets, non-hyperbolic attractors, and sinks (for instance).

2. Definitions and background

Let us recall some notions of dynamics. Given a diffeomorphism $f\colon M \to M$, an f-invariant closed set Λ (i.e., $f(\Lambda) = \Lambda$) is *hyperbolic* if the tangent bundle of M over Λ has a (continuous) Df-invariant splitting $E^s \oplus E^u$ such that there are constants $0 < \lambda < 1$ and $c > 0$ with

$$|D_x f^n(v^s)| \le c\,\lambda^n\,|v^s| \quad \text{and} \quad |D_x f^{-n}(v^u)| \le c\,\lambda^n\,|v^u|,$$

for all $n \in \mathbb{N}$, every point $x \in \Lambda$, and any pair of vectors $v^s \in E^s_x$ and $v^u \in E^u_x$, where E^i_x is the fiber of the bundle E^i at x $(i = s, u)$ and $|\cdot|$ is the Riemannian metric. The bundles E^s and E^u are the *stable* and *unstable* directions of Λ. Any hyperbolic set Λ of a diffeomorphism f verifies the following two properties: (1) its continuation is defined (i.e., diffeomorphisms g close to f have a hyperbolic set Λ_g close to Λ), and (2) the angles between the stable and unstable bundles over Λ are (uniformly) bounded away from zero. Prototypes of hyperbolic sets, besides the simplest case of orbits of hyperbolic periodic points, are the Smale horseshoe (see Figure 1 below) and the whole two dimensional torus for the Anosov maps, a model example being the map induced on $\mathbb{T}^2$ by the linear map of $\mathbb{R}^2$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The *non-wandering set* of a diffeomorphism f, denoted by $\Omega(f)$, is the set of points $x \in M$ such that, for every neighbourhood U of x, there is $n = n(U) > 0$ with $f^n(U) \cap U \neq \emptyset$. The non-wandering set $\Omega(f)$ is f-invariant and closed. Moreover, it contains the sets of periodic and limit points. In fact, C^1 generically, $\Omega(f)$ is the closure of the periodic points of f, [31]. A diffeomorphism f verifies the *Axiom A* if its non-wandering set is hyperbolic and equal to the closure of its periodic points. In the Axiom A case, the non-wandering set $\Omega(f)$ is the union of finitely many pairwise disjoint *homoclinic classes,* which are the *basic sets* of the Smale spectral decomposition of $\Omega(f)$, see [35]. Axiom A diffeomorphisms and basic sets are archetypes of tame diffeomorphisms and elementary pieces of dynamics, respectively.

Recall that the *homoclinic class* of a (hyperbolic) saddle point p, $H(p, f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of p. For instance, the horseshoe and the whole two torus are the homoclinic class of any saddle of them. Homoclinic classes are always *transitive sets* (i.e., f-invariant sets being the closure of the orbit of some point). The stable manifold of x, $W^s(x, f)$, is the set of points w such that $\operatorname{dist}(f^n(x), f^n(w)) \to 0$ as $n \to \infty$ (here, d is the distance induced by the Riemannian metric). The unstable manifold of x, $W^u(x, f)$, is defined by $W^u(x, f) = W^s(x, f^{-1})$.

In dimension greater than or equal to three there are manifolds M and open sets $\mathcal{N} = \mathcal{N}(M)$ of $\operatorname{Diff}^1(M)$ consisting of *transitive diffeomorphisms* f (the whole manifold is the closure of the orbit of some point, in particular, $\Omega(f) = M$) having saddles of different indices (the *index* of a saddle is the dimension of its unstable bundle), see [34, 19, 3, 8]. Since, for every $f \in \mathcal{N}$ it holds $\Omega(f) = M$, it follows that $\Omega(f)$ is not hyperbolic (just note that in a transitive hyperbolic set the dimension of the unstable bundle is constant). In particular, every $f \in \mathcal{N}$ does not verify the Axiom A. Therefore, Axiom A diffeomorphisms fail to be dense in $\operatorname{Diff}^1(M)$ in these cases. In fact, this holds in $\operatorname{Diff}^k(M)$, for any M of dimension $n \geq 3$ and any $k \geq 1$. In Section 3, we outline the Newhouse construction showing that Axiom A diffeomorphisms are not dense in the space of C^2 diffeomorphisms. The density of Axiom A maps in the C^1 topology (for surface diffeomorphisms) remains an open problem.

An important difference between dynamics in dimension two and in higher dimensions arises from the following simple fact. First, note that in dimension two any non-trivial transitive set (i.e., the set is not a periodic orbit) cannot contain either sinks or sources. Hence, for surface diffeomorphisms, all the (hyperbolic) periodic points of a transitive set are saddles, in particular, every periodic point has index one. In higher dimensions, for instance in dimension three, the saddles may have either index two or one, so, a priori, a transitive set may contain saddles with different indices. In fact, in the construction of non-hyperbolic transitive diffeomorphisms in [19], one starts with a hyperbolic transitive diffeomorphism (Anosov) whose saddles have index two, thereafter performing a saddle-node bifurcation, then saddles of index one are created without breaking the transitivity, obtaining transitive diffeomorphisms with saddles of indices one and two. The previous argument gives the following heuristic principle: in a persistently transitive set of a surface diffeomorphism the saddles cannot bifurcate. In Section 4.1, we describe another semi-local mechanism (heterodimensional cycles) for obtaining transitive sets containing saddles with different indices (in dimension three or higher).

3. Two dimensional C^2 wild dynamics: persistence of tangencies and infinitely many sinks

The constructions above suggest that is simpler to construct open sets of diffeomorphisms which do not verify the Axiom A seems to be simpler in higher dimensions than in surfaces. We begin by presenting the Newhouse construction of an open set $\mathcal{T}$ of C^2 diffeomorphisms which do not verify the Axiom A, [23]. In this construction (which is local and can be done in any surface) the C^2 regularity is essential. The set $\mathcal{T}$ consists of diffeomorphisms having tangencies: for every $g \in \mathcal{T}$, there are points x and y in some hyperbolic subset of $\Omega(g)$ (a horseshoe) such that the stable manifold of x and the unstable one of y are tangent throughout the orbit of a point $z \in \Omega(g)$. By definition of invariant manifolds, $W^s(x,g) = W^s(z,g)$ and $W^u(y,g) = W^u(z,g)$. Since in the hyperbolic case $T_w W^j(w,g) = E^j_w$, $j = s,u$, it follows that any $g \in \mathcal{T}$ does not verify the Axiom A: at the tangency point $z \in \Omega(g)$ it is not possible to exhibit a hyperbolic splitting (the stable and unstable directions should coincide).

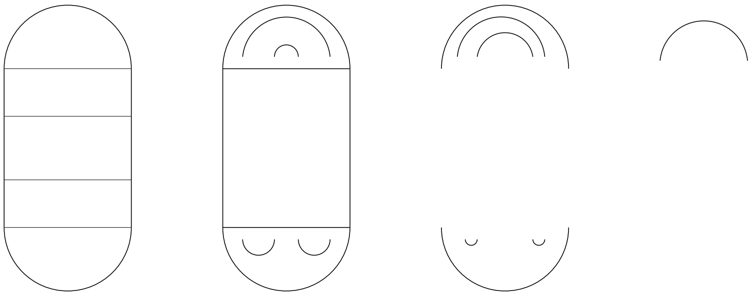

Figure 1: A linear horseshoe and its local perturbation

We outline the construction in the simplest situation. Consider a diffeomorphism f having a linear Smale horseshoe Λ as depicted in Figure 1, where (local) stable manifolds are horizontal segments and (local) unstable manifolds are vertical ones. We next modify the dynamics of f far from the horseshoe to create a point z of tangency between the stable and unstable manifolds of a saddle p of the horseshoe. In this construction, corresponding to the creation of a *one-cycle* via a *homoclinic tangency,* (we will discuss these notions later on), the point z is wandering, i.e. $z \notin \Omega(f)$.

Locally at $z = (z_1, z_2)$ the stable manifold of p is a horizontal segment γ_s and the unstable one a parabola γ_u. The next step is to unfold the tangency, considering an arc of diffeomorphisms f_t, $t \geq 0$ and $f_0 = f$, preserving the horseshoe Λ and the local stable manifold of p containing z, and translating the parabolic segment γ_u in the vertical direction by the vector $(0, t)$. In the unfolding new homoclinic points of the saddle p are created (this corresponds to a Ω-explosion). We now explain how the persistence of tangencies arises, for that a new hypothesis on the horseshoe is necessary.

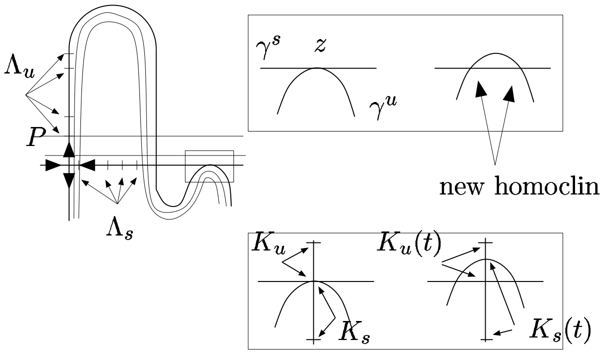

Figure 2: The homoclinic tangency

Suppose that the linear horseshoe Λ is constructed in such a way that $p = (0, 0)$, $\{0\} \times [0, 1] \subset W^u(p, f)$, $[0, 1] \times \{0\} \subset W^s(p, f)$ (these segments are the local invariant manifolds of p, $W^s_{loc}(p, f)$ and $W^u_{loc}(p, f)$), and $z = (z_1, 0)$. Moreover, the horseshoe Λ is the product of the Cantor sets $\Lambda_s = \Lambda \cap W^s_{loc}(p, f)$ and $\Lambda_u = \Lambda \cap W^u_{loc}(p, f)$. Consider the curve $\Upsilon = \{z_1\} \times [-\epsilon, +\epsilon]$ containing z and the (Cantor) sets $K_s = W^u(\Lambda_s, f) \cap \Upsilon$ and $K_u = W^s(\Lambda_u, f) \cap \Upsilon$. Then $K_u \subset \Upsilon^+ = \{z_1\} \times [0, +\epsilon]$ and $K_s \subset \Upsilon^- = \{z_1\} \times [-\epsilon, 0]$. The same construction for parameters $t > 0$ gives

$$\begin{aligned} (z_1, 0) &\in K_u(t) = W^s(\Lambda_u, f_t) \cap \Upsilon \subset \{z_1\} \times [0, +\epsilon]; \\ (z_1, t) &\in K_s(t) = W^u(\Lambda_u, f_t) \cap \Upsilon \subset \{z_1\} \times [-\epsilon, t]. \end{aligned}$$

Moreover, the sets $K_s(t)$ and $K_u(t)$ are *linked,* meaning that their convex hulls have non-empty intersection ($\{z_1\} \times [0, t] \subset K_s(t) \cap K_u(t)$). Take the set $B = \{t \geq$

$0\colon K_u(t) \cap K_s(t) \neq \emptyset\}$. This set corresponds to parameters t such that the stable and the unstable manifolds of the horseshoe Λ of f_t have a tangency. The key point is the following: there is $t_0 > 0$ such that $[0, t_0] \subset B$ (i.e., $[0, t_0]$ is an interval of persistent tangencies associated to Λ). Moreover, for $t > 0$, the tangency points belong to the non-wandering set of f_t. To perform this construction, that gives an interval of tangencies, we use the notion of *thickness* of a Cantor set of $\mathbb{R}$.

The thickness of a Cantor set of $\mathbb{R}$ is a fractal dimension, defined in the same spirit as the more usual notions of Hausdorff dimension and limit capacity. It corresponds to the relation between the lengths of the intervals we remove in the inductive construction of the set and the lengths of the two remaining adjacent connected components. Consider a Cantor set $K \subset \mathbb{R}$ whose convex hull is (to say) the interval $[0,1]$ and a presentation of it, that is an enumeration of the connected components $\{G_i\}_{i\in\mathbb{N}}$ of $[0,1] \setminus K$ (the *gaps* of K). Let $F_0 = [0,1]$ and, for $i \geq 1$, $F_i = F_{i-i} \setminus G_i$, where each F_i is the union of $(i+1)$ pairwise disjoint closed intervals. Let B_i^r and B_i^ℓ the components of F_i intersecting the boundary of G_i, called *bridges* of G_i. The *thickness of the presentation* $\{G_i\}$ of K is

$$\tau(K, \{G_i\}) = \inf_{i \geq 1} \left\{ \min \frac{|B_i^r|}{|G_i|}, \frac{|B_i^\ell|}{|G_i|} \right\}.$$

The thickness of the Cantor set K, $\tau(K)$, is the supremum of $\tau(K, \{G_i\})$ taken over all the presentations $\{G_i\}$ of K. In our context, the importance of thickness is given by the next lemma.

Lemma 3.1 (Gap Lemma, [25, 28]) *Let K_1 and K_2 be Cantor sets with $\tau(K_1) \cdot \tau(K_2) > 1$ and such that their convex hulls are non disjoint. Then either $K_1 \cap K_2 \neq \emptyset$ or K_1 is contained in a gap K_2 or vice-versa.*

In the constructions in [23, 24] one starts considering a *thick horseshoe* Λ, which means with the notation above that $\tau(\Lambda_s) \cdot \tau(\Lambda_u) > 1$. Using bounded distortion arguments and that the holonomies along the stable and the unstable manifolds of Λ are C^1 (this is a deep assertion typical of two-dimensional dynamics), one gets $\tau(K_i(t)) = \tau(\Lambda_i)$, $i = s, u$, hence $\tau(K_s(t)) \cdot \tau(K_u(t)) > 1$. Since, by construction, the Cantor sets $K_s(t)$ and $K_u(t)$ are linked (in particular, no gap of $K_u(t)$ contains $K_s(t)$ and vice-versa) Lemma 3.1 implies that, for all $t \geq 0$, $K_s(t) \cap K_u(t) \neq \emptyset$, obtaining a parameter interval of persistence of tangencies, thus ending the proof in the model.

The proof in the general case follows along similar ideas, involving the fact that the thickness of a hyperbolic set Λ_f of a C^2 surface diffeomorphism f (defined as the sum of the the thickness of the sets $\Lambda_f \cap W^s_{loc}(p, f)$ and $\Lambda_f \cap W^u_{loc}(p, f)$, where p is any saddle of Λ_f) continuously depend on the dynamics (for g close to f consider the continuation Λ_g of Λ_f). Hence, for perturbations of the linear model the condition on the product of the thickness persists, and we can repeat the construction.

The previous constructions can be summarized as follows. An open set $\mathcal{U}$ of $\text{Diff}^k(M)$, $k \geq 1$, has *persistence of homoclinic tangencies* if there is a continuous map Λ defined on a dense subset $\mathcal{D}$ of $\mathcal{U}$ associating to each $f \in \mathcal{D}$ a transitive hyperbolic set Λ_f containing points x and y such that $W^s(x,f)$ and $W^u(y,f)$ have a tangency. A hyperbolic set Λ_f is called *thick* if there is a saddle $p \in \Lambda_f$ such that $\tau(\Lambda_f \cap W^s_{loc}(p,f)) \cdot \tau(\Lambda_f \cap W^u_{loc}(p,f)) > 1$ (this definition does not depend on the choice of p).

Theorem 3.1 (Newhouse, [23]) *Let M be a closed surface and $f \in \mathrm{Diff}^2(M)$ a diffeomorphism having a thick hyperbolic set Λ_f with a homoclinic tangency associated to a saddle $p \in \Lambda_f$. Then there is an open set $\mathcal{U} \subset \mathrm{Diff}^2(M)$ with persistence of homoclinic tangencies whose closure contains f.*

Theorem 3.1 is the key step in the proof of the following result:

Theorem 3.2 (Newhouse, [24]) *Given any surface M there are an open set $\mathcal{U}$ of $\mathrm{Diff}^2(M)$ and a residual subset $\mathcal{S}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks.*

The proof of this theorem has two steps. For the first one, recall that a saddle p of a surface diffeomorphism f is *dissipative* if the absolute value of the product of the eigenvalues of D_pf^n (n the period of p) is less than one. Let f be a diffeomorphism with a (quadratic) homoclinic tangency associated to a dissipative saddle p. Then there is a diffeomorphimsm g arbitrarily close to f having a sink (a periodic attracting point). By a *quadratic homoclinic tangency* associated to p we mean that the stable and unstable manifolds of p have a non-transverse intersection at some point, where the invariant manifold have a quadratic contact. The second step (the most difficult one) is to prove that the unfolding of a homoclinic tangency leads to persistence of tangencies.

Consider an open set $\mathcal{U}$ of persistence of tangencies and let $\mathcal{D}$ be a dense subset of $\mathcal{U}$ of diffeomorphisms with homoclinic tangencies (we assume that the tangencies are associated to dissipative saddles). To get a residual subset $\mathcal{S}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks, note first that to have a sink is an open property. Thus the set $\mathcal{S}_k$ of diffeomorphisms with k (different) sinks is open. To prove Theorem 3.2 we see that each $\mathcal{S}_k$ is dense in $\mathcal{U}$ and let $\mathcal{S} = \cap \mathcal{S}_k$ (by construction, $\mathcal{S}$ is residual in $\mathcal{U}$). The density follows inductively, it is enough to see that by the first step every diffeomorphism in the set $\mathcal{D}$ can be approximated by a diffeomorphism with a sink. The density of $\mathcal{D}$ in $\mathcal{U}$ gives the density of $\mathcal{S}_1$. The inductive pattern is now clear: to construct $\mathcal{S}_2$ one produces new homoclinic tangencies (this is possible by the density of $\mathcal{D}$), and by unfolding the tangency one gets a new sink preserving the previous one.

The constructions in this section rely on the notion of thickness and are genuinely C^2. A result of Ures [36], claims that hyperbolic sets of C^1 generic diffeomorphisms have zero thickness, thus the Newhouse construction of persistence of tangencies cannot be carried out in the C^1 topology. Thence to construct (if possible) open sets of C^1 surface diffeomorphisms not satisfying the Axiom A new ingredients are necessary.

In fact, the density of C^1 surface diffeomorphisms verifying the Axiom A is an open question (such a density is false in higher dimensions). This is a central problem in two dimensional dynamics. In this context, recently Pujals and Sambarino proved that, for C^1 surface diffeomorphisms, the union of those with homoclinic tangencies and the ones verifying the Axiom A form a dense subset of $\mathrm{Diff}^1(M)$, see [32].

The Newhouse coexistence result was generalized to higher dimensions by Palis and Viana, [29], and Romero, [33]. The result in [29] state the coexistence of infinitely many sinks for homoclinic tangencies associated to *sectionally dissipative* saddles p (i.e., the product of any pair of eigenvalues of $D_p f^n$ (n the period of p) has modulus less than one, in particular the unstable manifold of p is one-dimensional). The paper [33] proves persistence of tangencies without any restriction on the eigenvalues of the saddle, exhibiting normally hyperbolic surfaces where the restricted dynamics have homoclinic tangencies (i.e., the problem is reduced to a two-dimensional one).

Finally, an important phenomenon associated to homoclinic tangencies is the existence of Hénon-like strange attractors with *some persistence* (contrary to the case of sinks, the continuation of a non-hyperbolic attractor is not well defined). See the paper by Mora and Viana [22] for precise statements. Combining the (weak) persistence of Hénon-like attractors and the persistence of tangencies at homoclinic bifurcations, Colli obtained (a version of) the coexistence phenomenon for Hénon-like attractors at homoclinic tangencies, see [9] for details and precise statements.

For a complete proof of the main results in this section and a broad discussion of homoclinic tangencies, besides the original papers by Newhouse, we refer the reader to the book by Palis and Takens [28, Chapters 3 and 6].

4. C^1 coexistence phenomenon in three manifolds

4.1. Heterodimensional cycles. Intermingled homoclinic classes

Let f be an Axiom A diffeomorphism and $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ (different) basic sets of the spectral decomposition of the non-wandering set of f. The diffeomorphism f has a *k-cycle* associated to $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ if the unstable manifold of Λ_i (the union of the unstable manifolds of the points of Λ_i) meets the stable one of Λ_{i+1} for all $i = 1, \dots, k$, where $i_{k+1} = 1$. The cycle is *equidimensional* if all the basic sets in the cycle have the same index (the dimension of the unstable bundle, which coincides with the index of any saddle of the set). Otherwise the cycle is *heteredimensional.* Heterodimensional cycles can only occur in dimension greater than or equal to three. The horseshoe in Section 3 with a tangency is an example of equidimensional one-cycle.

We now discuss the dynamics at heterodimensional cycles in the simplest situation. Consider a diffeomorphism f on a three manifold with a cycle associated to fixed points p and q of indices one and two such that the eigenvalues of $D_p f$ and $D_q f$ verify $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and $0 < \beta_s < 1 < \beta_c < \beta_u$, respectively. Moreover, the transverse intersection between the two dimensional manifolds $W^s(p, f)$ and $W^u(q, f)$ contains a curve γ (the *connection*) with endpoints p and q. The curve γ is simultaneously transverse to the strong stable foliation of $W^s(p,f)$ and to the strong unstable foliation of $W^u(q,f)$ (the unique f-invariant one-dimensional foliations of $W^s(p,f)$ and $W^u(q,f)$ whose leaves through p and q are tangent to the eigenspaces of λ_s and β_u). Finally, the intersection between the one-dimensional manifolds $W^u(p,f)$ and $W^s(q,f)$ is quasi-transverse and occurs throughout the orbit of a *heteroclinic point* x: $W^u(p,f) \cap W^s(p,f) = \{f^k(x)\}_{k\in\mathbb{Z}}$ and $T_xW^u(p,f) + T_xW^s(q,f)$ is a direct sum. See Figure 3. The cycle in Figure 3 is an example of a *connected* and *non-critical* one.

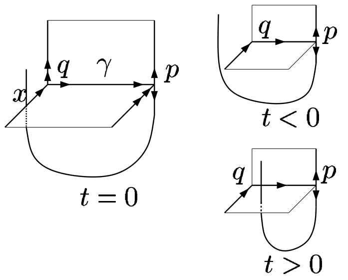

Figure 3: A heterodimensional cycle and its unfolding

In the context of equidimensional cycles (homoclinic tangencies) in surfaces, the dynamics after unfolding the cycle mainly depends on the fractal dimensions (thickness, Hausdorff dimension, and limit capacity) of the hyperbolic sets in the cycle (see the series of papers [23, 26, 27, 30]). Moreover, the occurrence of wild dynamics is related to the persistence of cycles (homoclinic tangencies in this setting). In the case of heterodimensional cycles, as the one described above, the situation is rather different and the dynamics after the unfolding of the cycle is essentially determined by the restriction of the bifurcating diffeomorphism f to the connection curve γ. This claim is suggested by the series of papers [10, 13, 15], where the dynamics in the sequel of the unfolding is reduced to the analysis of a system of iterated functions generated by f and a translation. For an expository explanation of the dynamics at heterodimensional cycles see [7, Chapter 6] and [16].

When the distortion of the restriction of f to the connection γ is small, after unfolding the cycle the homoclinic classes of p and q are persistently intermingled and non-hyperbolic transitive sets containing simultaneously p and q are created throughout the bifurcation. The definition of an arc $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ unfolding a heterodimensional cycle at $t = 0$, $f_0 = f$, is similar to the one of a homoclinic tangency: there are compact parts $K^u(t)$ of $W^u(p,f_t)$ and $K^s(t)$ of $W^s(q,f_t)$ depending continuously with t and intersecting at the heteroclinic point x for $t = 0$, moving with positive velocity one with respect the other one (for instance and in local coordinates, $x = (0,0,0)$, $K^s(t) = [-1,1] \times \{(0,0)\}$ and $K^u(t) = \{(0,t)\} \times [-1,1]$), see Figure 3.

Theorem 4.1 ([10, 11]) *There is an open set $\mathcal{O}$ of arcs of diffeomorphisms $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ unfolding a heterodimensional cycle at $t=0$, associated to saddles p and q of indices r and $(r+1)$, such that the homoclinic classes of p and q coincide for all positive t, i.e., $H(p,f_t)=H(q,f_t)$, $t\in(0,\varepsilon]$.*

The cycles in Theorem 4.1 are similar to those in Figure 3 (non-critical and connected). The open set $\mathcal{O}$ includes arcs of diffeomorphisms being *Morse-Smale* at the bifurcation (i.e., the non-wandering set of f_0 is finite and hyperbolic, thus the basic sets containing p and q are both trivial). It is interesting to compare Theorem 4.1 with the results about homoclinic tangencies, where to get persistent non-hyperbolic dynamics after the unfolding of the cycle (i.e., f_t is non-hyperbolic for every $t>0$) one needs to consider tangencies associated to thick hyperbolic sets.

An important point in Theorem 4.1 is that the only assumption on the dynamics of the bifurcating diffeomorphism f_0 involved in its proof (besides the type of geometry of the curve $\gamma\subset W^s(p,f_0)\cap W^u(q,f_0)$) is that the restriction of f_0 to the connection γ has small distortion. These conditions are compatible with other hypotheses on the global dynamics of f_0, for instance, the homoclinic classes of p and q may contain other saddles (we will use this fact latter).

The main step of the proof of Theorem 4.1 (in three manifolds) is the following property of the one-dimensional unstable manifold of p: *for every $t>0$, the closure of $W^u(p,f_t)$ contains the two-dimensional unstable manifold $W^u(q,f_t)$ of q.* Heuristically, this means that the stable and unstable manifolds of p have both (topological) dimension two, thus the saddle p behaves simultaneously as a point of indices one and two. Similarly, the two dimensional manifold $W^s(p,f_t)$ is contained in the closure of the one dimensional manifold $W^s(q,f_t)$, so the saddle q also *has* indices two and one. This is the main step for proving that the homoclinic classes of p and q coincide.

Roughly, the property $W^u(q,f_t)\subset \text{closure}(W^u(p,f_t))$ persistently (for all $t>0$) plays a role analogous to the thick horseshoe in the proof of the persistence of tangencies. The key point here (and the main difference) is that the property $W^u(q,f_t)\subset \text{closure}(W^u(p,f_t))$ persistently only requires C^1 regularity, while the construction of depends heavily on the C^2 differentiability of the dynamics.

The construction in [10] of homoclinic classes containing saddles having different indices persistently was generalized and systematized in [3] in order to construct open sets of non-hyperbolic transitive diffeomorphisms. In [3] is introduced the notion of *blender,* a topological *plug* depending only on semi-local properties, guaranteeing that the closure of the unstable manifold of a saddle of index r contains (C^1 persistently) the unstable manifold of a point of index $(r+1)$. For an expository construction of blenders see [7, Chapter 6.2]. See also a much more elementary discussion in [12].

Finally, recall that the constructions in Section 3 of wild diffeomorphisms had two steps. First, the unfolding of a tangency associated to a thick horseshoe leads to persistence of tangencies, and using such a persistence of tangencies one gets coexistence of infinitely many sinks. Second, by unfolding any homoclinic tangency thick horseshoes with tangencies are created. Combining these two steps one has that persistence of tangencies and coexistence of infinitely many sinks are inherent to the unfolding of a tangency. There is a somewhat parallel situation in the heterodimensional setting. First, we can think of property $W^u(q, f_t) \subset \text{closure}(W^u(p, f_t))$ (where the indices of p and q are 1 and 2, respectively) of the diffeomorphisms in Theorem 4.1 as playing the role of the thick horseshoes. Second, for cycles *far from homoclinic tangencies* (i.e., the bifurcating diffeomorphism cannot be perturbed to get a tangency associated to the saddles in the cycle), after unfolding the cycle (and arbitrarily close to the bifurcating diffeomorphism) one gets new diffeomorphisms with $W^u(q, f_t) \subset \text{closure}(W^u(p, f_t))$. Thus persistence of intermingled homoclinic classes of different indices is a phenomenon inherent to the unfolding of heterodimensional cycles. See [14] for details and precise statements.

4.2. C^1 coexistence phenomenon on three manifolds

We next discuss the C^1 coexistence phenomenon in dimension 3 (the arguments can be adapted to higher dimensions after minor changes). The main result is:

Theorem 4.2 ([4]) *Let M be a manifold of dimension three. There are an open set $\mathcal{N}$ of $\text{Diff}^1(M)$ and a residual subset $\mathcal{S}$ of $\mathcal{N}$ such that every $f \in \mathcal{S}$ has infinitely many sinks.*

The key point is to get C^1 persistence of tangencies (associated to sectionally dissipative saddles). Observe that, in dimension three, the unfolding of a tangency does not necessarily lead to the creation of sinks: consider a diffeomorphism F defined as the product of a surface diffeomorphism $f: S \to S$ with a tangency associated to a dissipative saddle and a strong (linear) expansion. Thus the tangency of F is not sectionally dissipative. The unfolding of this tangency leads to diffeomorphisms with saddles of index one: the saddles have a two dimensional stable bundle corresponding to a sink of the restriction to S and a one-dimensional unstable direction given by the transverse expansion.

The first step in the proof of Theorem 4.2 is to get C^1 persistence of (heterodimensional) cycles. Theorem 4.1 gives an open set $\mathcal{H}$ of $\text{Diff}^1(M)$ of diffeomorphisms f having saddles p and q of indices one and two such that their homoclinic classes are equal. On the other hand, the Hayashi's connecting lemma implies the following:

Lemma 4.1 (Hayashi, [17]) *Let Σ be a transitive set of a diffeomorphisms f containing a pair of saddles a and b. Then there is g arbitrarily C^1 close to f such that $W^s(a, g) \cap W^u(b, g) \neq \emptyset$.*

We apply Lemma 4.1 to the diffeomorphisms in $\mathcal{H}$. Let $f \in \mathcal{H}$ and consider the transitive set $\Sigma_f = H(p, f) = H(q, f)$ and any pair of saddles $p', q' \in \Sigma_f$ of indices one and two. Then there is g arbitrarily C^1 close to f with a heterodimensional cycle associated to p' and q'. To prove this claim we apply Lemma 4.1 twice. First, taking $a = p'$ and $b = q'$, we get g close to f such that $W^s(p', g)$ meets $W^u(q', g)$. As both invariant manifolds have dimension two, we can assume that the intersection is transverse and thus persistent: $W^s(p',h) \cap W^u(q',h) \neq \emptyset$, for all h close to g. Applying now Lemma 4.1 to the g, taking $a = q'$ and $b = p'$, we get h close to g (thus to f) with $W^u(p',h) \cap W^s(q',h) \neq \emptyset$. As, by construction, $W^s(p',h) \cap W^u(q',h) \neq \emptyset$, the diffeomorphism h has a heterodimensional cycle. In other words, the diffeomorphisms of $\mathcal{H}$ having heterodimensional cycles form a dense subset of it.

To get persistence of tangencies in $\mathcal{H}$ we need an extra hypotheses on the dynamics. As we mentioned above, the semi-local hypotheses in Theorem 4.1 are compatible with other features of the dynamics. Suppose that the saddle q is *homoclinically related* to a saddle r having a pair of non-real expanding eigenvalues. This means that $W^s(q,f)$ and $W^u(r,f)$ meet transversely, and similarly for $W^u(q,f)$ and $W^s(r,f)$. The homoclinic relation implies that the homoclinic classes of q and r are equal. Thus $H(p,f) = H(q,f) = H(r,f)$. We now apply Lemma 4.1 to p and r, obtaining a heterodimensional cycle associated to p and r. The facts that r has a expanding non-real eigenvalue and that $W^s(p,f)$ meets transversely $W^u(q,f)$ imply that the stable manifold of p spirals around the stable manifold of r. This property together with the heterodimensional cycle condition allows us to get (after a perturbation) a tangency associated to p. In other words, if there is a saddle with a pair of non-real expanding eigenvalues homoclinically related to q, then $\mathcal{H}$ is an open set of C^1 persistence of tangencies. Taking the saddle p to be sectionally dissipative we can repeat (essentially) the construction in Section 3, obtaining Theorem 4.2.

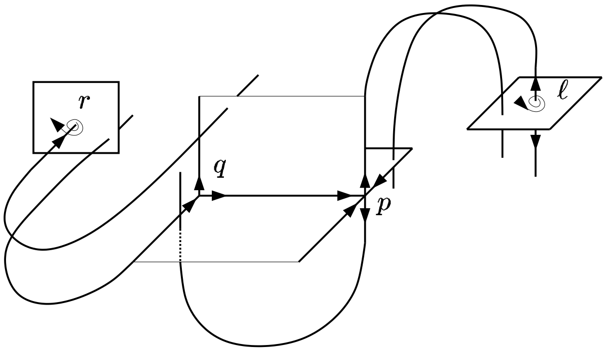

Figure 4: Non-dominated homoclinic classes

One can perform the construction above assuming also the existence of a saddle ℓ homoclinically related to p with a pair of non-real contracting eigenvalues (see Figure 4). This leads to persistence of homoclinic tangencies associated to q. Thus, when q is sectionally expansive (i.e., sectionally dissipative for f^{-1}) we also get a residual subset of $\mathcal{H}$ of diffeomorphisms having infinitely many sources. Hence, under this assumption, intersecting the two residual sets we have obtained, we get a new residual subset of $\mathcal{H}$ of diffeomorphisms simultaneously having infinitely many sinks and sources. In fact, the last construction is a particular case of a more general result we are going to discuss in the next section: the dynamics at non-dominated homoclinic classes.

5. Non-dominated dynamics: a sample of wild dynamics

In the previous paragraph, we exhibited an open set of $\mathrm{Diff}^1(M)$ consisting of diffeomorphisms f having homoclinic classes with saddles r and ℓ of different indices and having non-real eigenvalues. This type of homoclinic class is the prototype of *non-dominated dynamics* and leads to the C^1 Newhouse coexistence phenomenon and other stronger forms of wild dynamics.

We first introduce the definition of *dominated splitting,* an extension of the notion of hyperbolicity, introduced by Liao [18] and Mañé [20] that played a fundamental role in the proof of the stability conjecture, [21]. Consider a diffeomorphism $f\colon M \to M$ and an f-invariant closed set Λ of it. A Df-invariant splitting $T_\Lambda M = E \oplus F$ of the tangent bundle of M over Λ is dominated if there is $\ell \in \mathbb{N}$ such that

$$\frac{|D_x f^\ell(v)|}{|D_x f^\ell(w)|} \le \frac{1}{2},$$

for every point $x \in \Lambda$ and any pair of unitary vectors $v \in E$ and $w \in F$. Moreover, the dimensions of the fibers E_x and F_x are independent of the point x of Λ.

In this definition, the derivative of f in the bundle E (similarly in F) may exhibit contractions and expansions (according to the point). Assume, for simplicity, that $\ell = 1$, the key point in the definition is that any expansion of Df in E is weaker than the corresponding expansion in F (similarly, contractions in F are weaker than the ones in E). The definition also implies that positive Df-iterations of vectors v, $v \notin E$, converge to the F direction.

The next result means that non-dominated homoclinic classes are a natural habitat for wild dynamics:

Theorem 5.1 (C^1 generic dichotomy for homoclinic classes, [6]) *Let M be a closed manifold. There is a residual subset $\mathcal{R}$ of* $\mathrm{Diff}^1(M)$ *of diffeomorphisms f such that, for every saddle p of f, the homoclinic class $H(p,f)$ of p satisfies the following;*

- *either $H(p,f)$ has a dominated splitting (weak hyperbolicity),*
- *or $H(p,f)$ is contained in the closure of an infinite set of sinks or sources of f (coexistence phenomenon).*

In this theorem and in the weak hyperbolic case, if M is a surface then the homoclinic class is hyperbolic (this was proved by Mañé in [20]). When the dimension of the manifold is three, the splitting is *partially hyperbolic* (i.e., its one-dimensional bundle is either uniformly contracting or expanding). Finally, it is an open question whether for C^1 surface diffeomorphisms the coexistence phenomenon can be eliminated.

Theorem 5.1 gives a new path leading to the coexistence phenomenon. Take a three dimensional diffeomorphism f with (fixed) saddles $\bar{p}$ and $\bar{q}$ of such that:

- the index of $\bar{p}$ is one and $D_{\bar{p}}f$ has a pair of non-real contracting eigenvalues,
- the index of $\bar{q}$ is two and $D_{\bar{p}}f$ has a pair of non-real expanding eigenvalues,
- there is a C^1-neighborhood $\mathcal{V}$ of f such that $H(\bar{p}, g) = H(\bar{q}, g)$ for all $g \in \mathcal{V}$.

Then there is a residual subset of $\mathcal{V}$ of diffeomorphisms with infinitely many sinks or sources (in fact, the closure of such points contains the infinite set $H(\bar{p}, g) = H(\bar{q}, g)$). Note that here we do not make any assumption on the sectionally dissipativiness or expansiviness of the saddles $\bar{p}$ and $\bar{q}$ as in Section 4.2. Let us explain this claim.

By construction, the transitive set $\Sigma_g = H(\bar{p}, g) = H(\bar{q}, g)$ does not admit any dominated splitting for all $g \in \mathcal{V}$. Otherwise, assume by contradiction that $E \oplus F$ is a dominated splitting of Σ_g such that E is one dimensional. Consider the stable bundle $E^s_{\bar{p}}$ of $\bar{p}$ (the eigenspace associated to the contracting eigenvalue). Then, necessarily (this follows from the domination of the splitting) $E \subset E^s_{\bar{p}}$, thus $D_{\bar{p}}g$ leaves invariant a one dimensional direction of $E^s_{\bar{p}}$, which is incompatible with the fact that $D_{\bar{p}}g$ has non-real contracting eigenvalues. The contradiction follows similarly when F is one dimensional by considering $\bar{q}$ and its unstable direction. Theorem 5.1 now implies that there is a residual subset of $\mathcal{V}$ consisting of diffeomorphisms with infinitely many sinks or sources. In fact, if the set Σ_f contains a saddle a (the index of a now is irrelevant) such that the Jacobian of $D_a f^n$ (n the period of a) is less (resp. bigger) than one, then one gets infinitely many sinks (resp. sources). Again, we may have both situations simultaneously.

The sets $H(\bar{p}, f) = H(\bar{q}, f)$ in the previous construction are examples of non-dominated homoclinic classes. We end this section explaining how the dynamics at a non-dominated class may be extremely rich when an extra hypothesis is assumed:

- for every $f \in \mathcal{V}$, the set $\Sigma_f = H(\bar{p}, f) = H(\bar{q}, f)$ contains a pair of saddles with Jacobians greater and less than one (where such saddles are homoclinically related to $\bar{p}$ or $\bar{q}$).

A class verifying this hypotheses is named *wild homoclinic class.* For this type of classes there is the following coexistence phenomenon.

Theorem 5.2 ([5]) *Let $\mathcal{W}$ be an open set of (three dimensional) C^1 diffeomorphisms f having a wild homoclinic class $H(p, f)$. Then there is a residual subset $\mathcal{R}$ of $\mathcal{W}$ such that, for every $g \in \mathcal{R}$, the set $H(p, g)$ is simultaneously contained in the closure of infinitely many pairwise disjoint:*

- *saturated transitive sets with minimal dynamics,*

- *non-trivial uniformly hyperbolic attractors and repellers,*
- *non-trivial partially hyperbolic attractors and repellers,*
- *wild homoclinic classes.*
- *infinitely many sinks and sources.*

Let us recall that an (infinite) f-invariant closed set Λ is *minimal* if every orbit of it is dense in the whole Λ (in particular, Λ does not contain periodic points). We say that a transitive set Υ is *saturated* if it contains any transitive set intersecting it (thus these sets are maximal transitive).

The main idea in the proof of Theorem 5.2 is to observe that there are diffeomorphisms g close to the initial f whose dynamics in small balls are the identity (the precise definition of this property corresponds to the notion of *universal dynamics* in [5]). The proof also involves a renormalization-like inductive argument: (roughly) using the universal dynamics one recovers, in small scale and up to the period, the initial wild dynamics.

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Heterodimensional

Cycles

umalca 2004, cancún, méxico

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(I) <u>HETERODIMENSIONAL CYCLES</u>

$f\colon M \to M$, $\dim M \geq 3$.

P hyperbolic saddle of f, **<u>index of P</u>**,
ind(P) = dimension of the unstable bundle.

P, Q hyperbolic periodic saddles of f
dimension variability: $\boxed{\text{ind}(P) < \text{ind}(Q)}$

<u>heterodimensional cycle</u> related to
P and Q:

$$\underbrace{W^s(P) \cap W^u(Q)}_{\textbf{(a)}} \neq \emptyset \neq \underbrace{W^u(P) \cap W^s(Q)}_{\textbf{(b)}}$$

Typical (generic) situation:

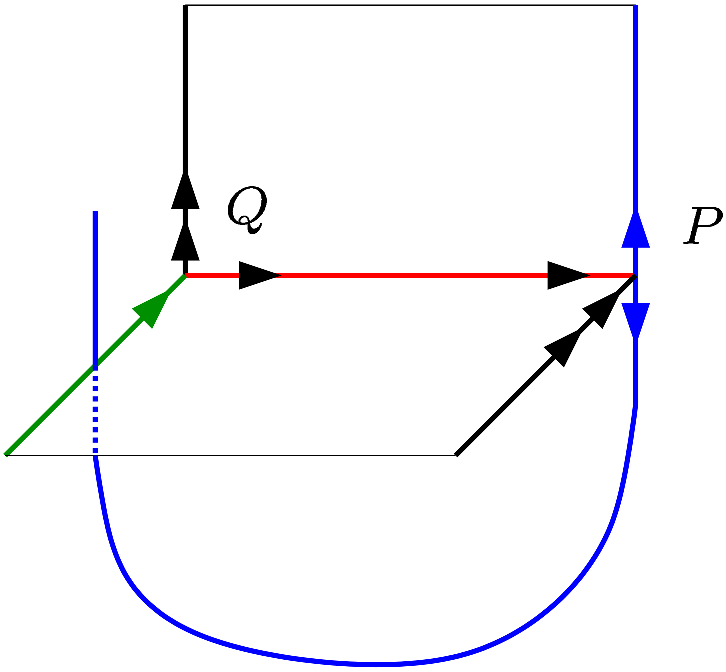
(a) transverse intersection;

(b) quasi-transverse:
for all $x \in W^s(Q) \cap W^u(P)$ heteroclinic point:

$$T_xW^u(P) + T_xW^s(Q) = T_xW^u(P) \oplus T_xW^s(Q)$$

Codimension one-case: $\boxed{\text{ind}(P)+1 = \text{ind}(Q)}$

(for simplicity: $\dim(M)=3$, $\text{ind}(Q)=2$).



homoclinic class of P:

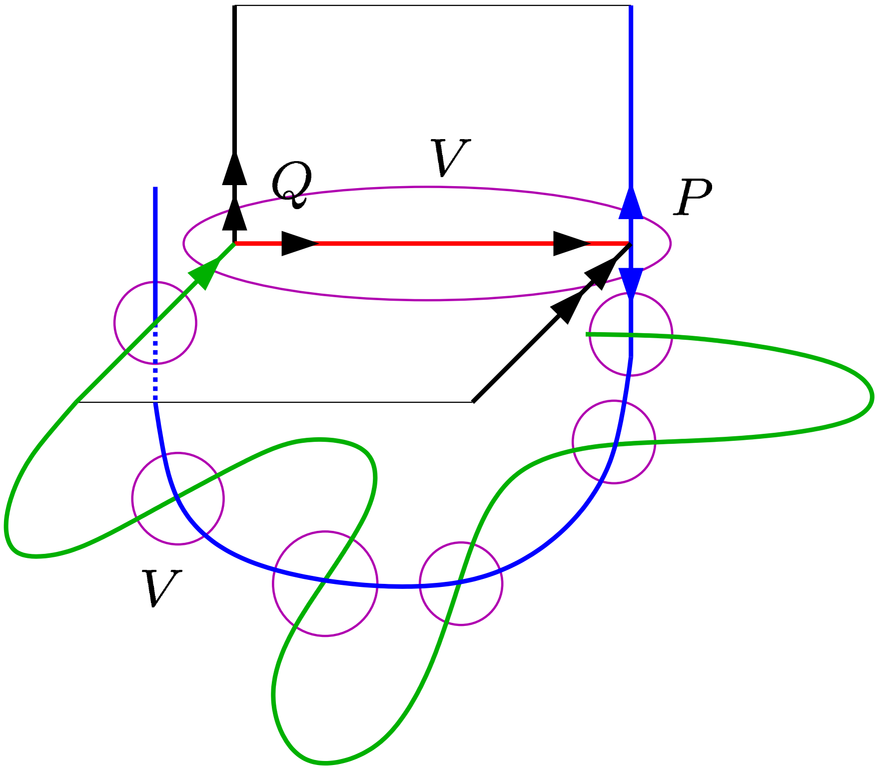
$$\boxed{H(P(f) = \overline{W^s(P) \pitchfork W^u(P)}}$$

- transitive (dense orbit)
- periodic points of the same index as P are dense (homoclinically related)
- (in general) it may contain points of different indices (non-hyperbolicity)

Goal: Dynamics of nearby diffeomorphisms in a **neighborhood of the cycle** after the unfolding (explosion of the homoclinic classes of P and Q)

neighborhood of the cycle:
(small) open set V containing all elements of the cycle, orbits of the:

- saddles P and Q,
- heteroclinic points of P and Q:
  $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$



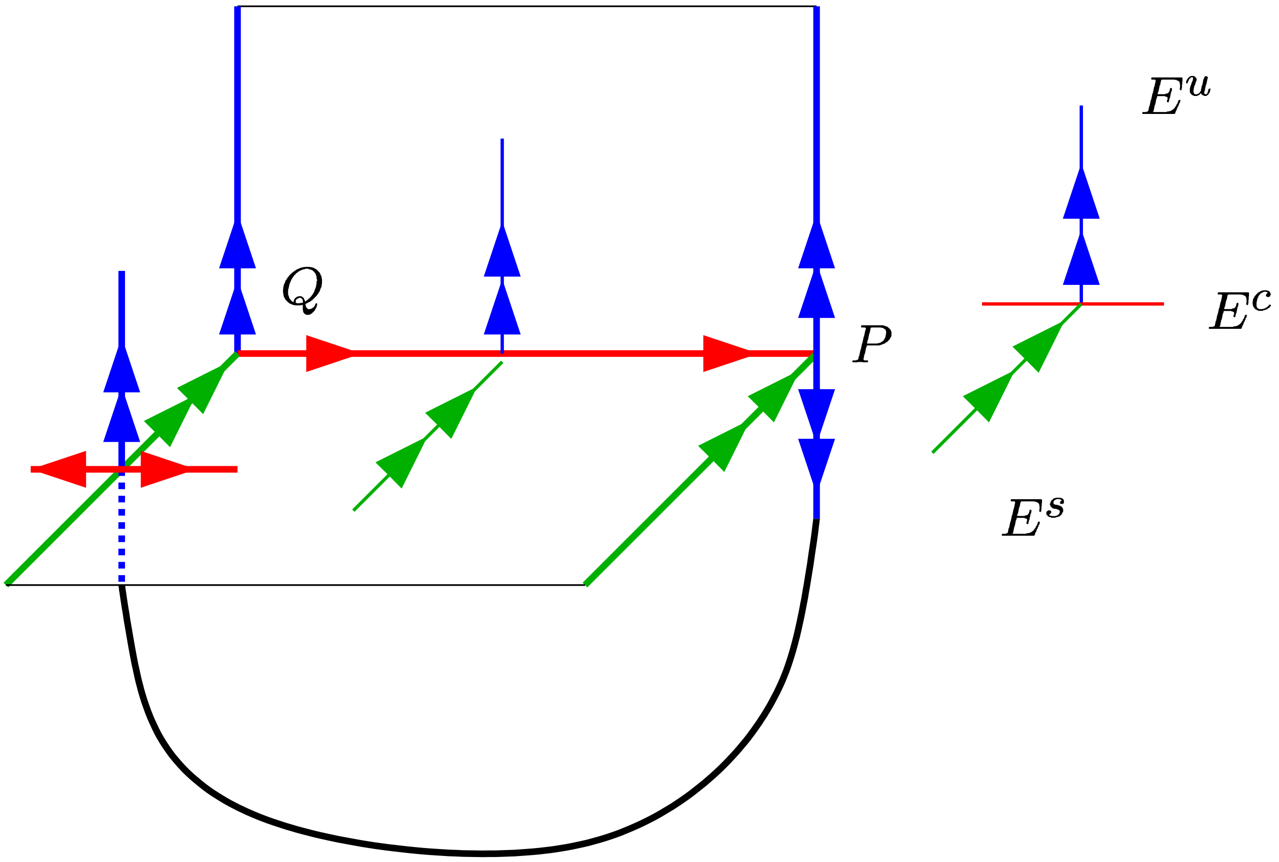
Partially hyperbolic case: g close to f,

$$\Lambda_g = \bigcap_{\mathbb{Z}} g^i(V)$$

Λ_g maximal inv. set of g in V,

$$T(\Lambda_g) = E^s \oplus E^c \oplus E^u$$

E^s, E^u unif. hyper. (stable and unstable),
E^c one-dimensional.



Strategy (Palis-Takens):

- Consider $(f_t)_{t=[-1,1]}$:

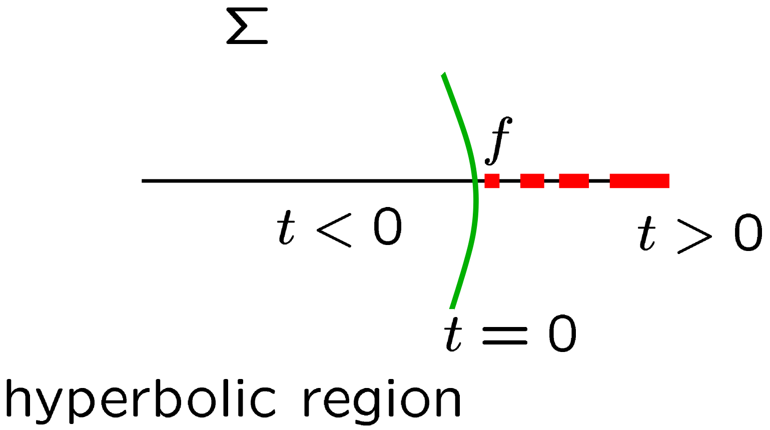
– starting in a hyperbolic region (f.i., Morse-Smale relative to neigh. of the cycle)

– crossing transversely its boundary Σ for $t = 0$ at $f_0 = f$

(here: the component of the boundary containing f has codimension one)

- Study the dynamics for $t > 0$ small.

- Interesting dynamics: those displayed by diffeomorphism f_t for a positive measure set of parameters $t > 0$.



(II) DYNAMICS ARISING FROM HETERODIMENSIONAL CYCLES

$\mathcal{P}$ a dynamical property:

$$\mathbb{P} = \{t > 0\colon f_t \text{ satisfies } \mathcal{P}\}$$

Prevalent: $\liminf_{t\to 0+} \frac{|\mathbb{P}\cap[0,t]|}{t} > 0.$

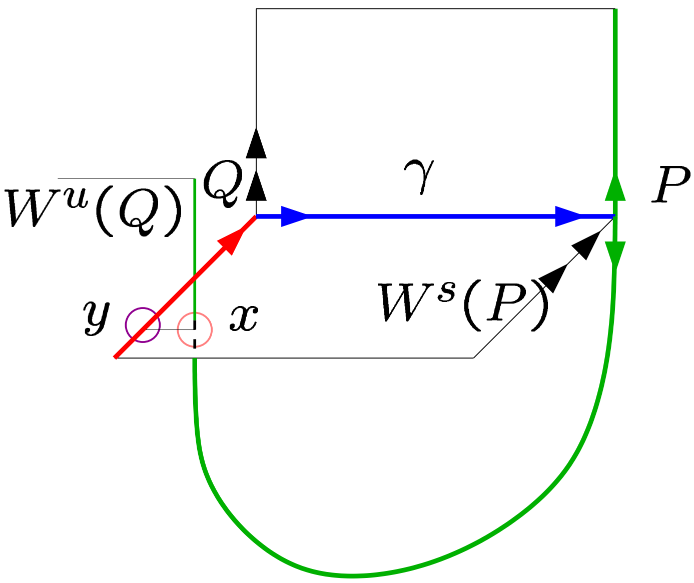
Absolutely Prev.: $\liminf_{t\to 0+} \frac{|\mathbb{P}\cap[0,t]|}{t} = 1.$

Persistent: $|\mathbb{P} \cap [0,t]| > 0$ for all $t > 0$.

(Some) typical features after unfolding a partially hyperbolic heterodimensional cycle:

(1) intermingled homoclinic classes:

$$H(P, f_t) \cap H(Q, f_t) \neq \emptyset$$



Typically: $H(P,f_t)=H(Q,f_t)$, $\gamma\subset H(P,f_t)$

(Carballo et al.) C^1-**Generically:**

$$H(P,f_t)\cap H(Q,f_t)\neq\emptyset \implies H(P,f_t)=H(Q,f_t)$$

(2) hyperbolicity: $\Lambda_{f_t}=H(P,f_t)\cup H(Q,f_t)$ the classes are hyperbolic (indices 1 and 2);

(3) collisions of homoclinic classes (secondary saddle-node biffurcations);

(4) persistence of heterodimensional cycles (thick one-dimensional Cantor sets).

Thm A (D-Rocha):

For every $(f_t)_{t\in[-1,1]}$ unfolding a partially hyperbolic heterodimensional cycle there is a sequence of parameter intervals B_n, $B_n \to 0^+$, such that:

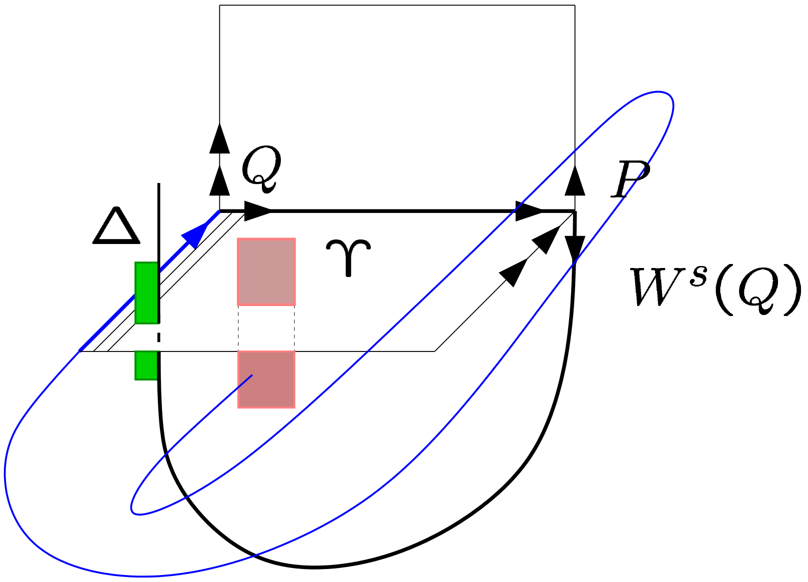
A) $\forall t \in B_n$: $H(P,f_t) = H(Q,f_t)$, and Λ_{f_t} is transitive (containing P and Q).

B) (prevalent non-hyperb., first bifurc.).

$$\liminf_{t\to 0} \frac{|B\cap[-t,t]|}{2\,t} > 0; \qquad B = \cup B_n.$$

Main ingredient: **Blender-like construction**

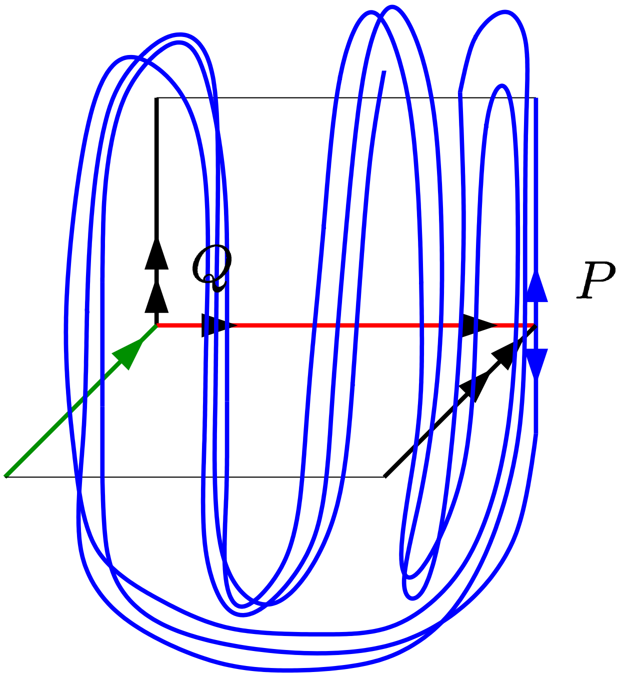
$W^u(Q)$ intersects *every* 2-disk Υ crossing transversely $W^s(P)$.

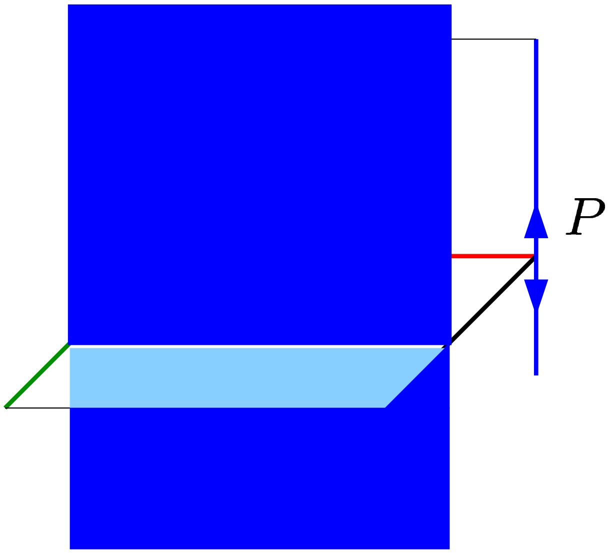


Consequence:
$H(P, f_t) \subset H(Q, f_t)$
just take disks $\Delta \subset W^u(Q, f_t)$ whose boundary contain a homoclinic point of P).

Heuristically:

the one-dimensional manifold $W^u(Q)$ *is* (topologically) two-dimensional manifold. Similarly for $W^s(P)$.
two-fold behaviour: P and Q *are* simultaneously points of indices one and two.





Applications of blenders (& het. cycl):

- Construction of **robustly transitive** C^1 diffeomorphisms (Bonatti, D):

$\mathcal{T}(M)$: $\{f$: s.t. $\exists$ C^1-nbhood $\mathcal{U}$ of f s.t. any $g \in \mathcal{U}$ is transitive (dense orbit in M)$\}$

- Properties of robustly transitive sets and diffeos (B,D,Pujals,R):

(open and dense) the indices of the periodic points form an interval (in $\mathbb{N}$)

- Additional hypotheses: **(3-bundle) partial hyperbol.** $TM = E^s \oplus E^c \oplus E^u$, with E^c 1-dim, E^s, E^u unif. hyperb.

$$\boxed{\mathcal{S}(M)\colon \{f \in \mathcal{T} \oplus \text{ part. hyp.}\}}$$

$\Longrightarrow$ strong stable $\mathcal{F}^s$ and unstable $\mathcal{F}^u$ foliations (tangent to E^s and E^u) are defined

- Properties of the strong invariant foliations (B,D,Ures):

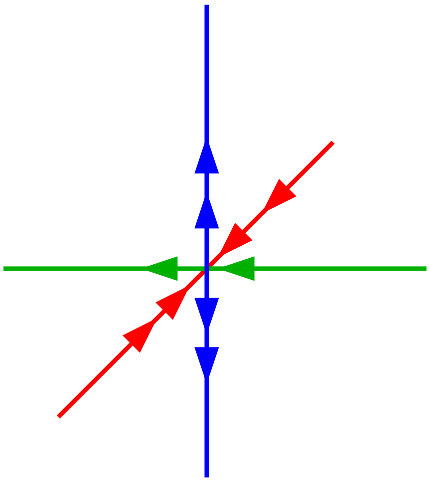
Open and dense in $\mathcal{S}(M)$: Either the strong stable $\mathcal{F}^s$ or the unstable $\mathcal{F}^u$ foliation is minimal (every leaf is dense).

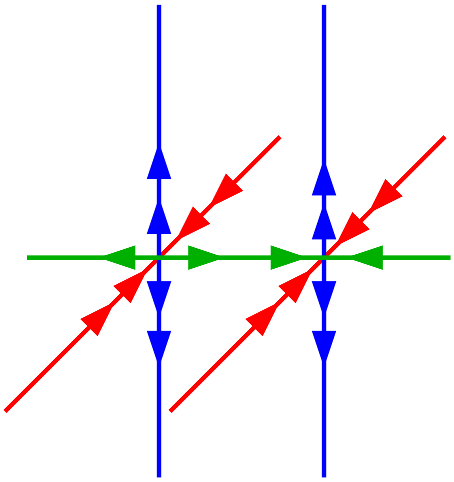
- **Natural habitat** for heterodim. cycles:

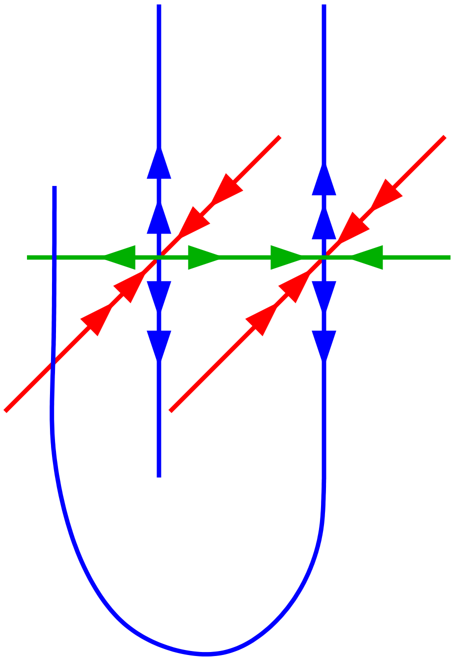
(1) creation of saddle-nodes
(transitivity $\oplus$ non-hyperbolicity $\oplus$ cycles (connecting lemma))

(2) unfolding of the saddle-node bifurcation: saddles with different indices

(3) creation of a (heterodimensional) cycle
(connecting lemma $\oplus$ transitivity))







Given an arc $(f_t)_{t\in[-1,1]}$ unfolding a cycle at $t = 0$:

$$\boxed{H(s) = \{t \in [0, s] : f_s \text{ hyperbolic } \}}$$

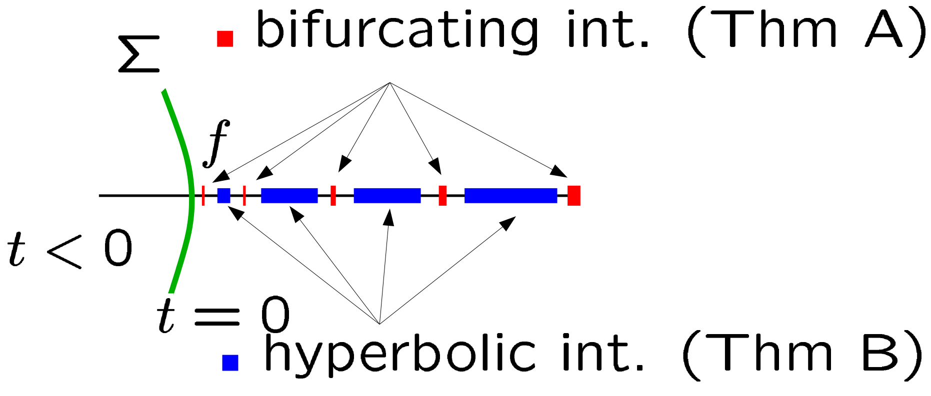
hyperbolic: Axiom A $\oplus$ no-cycles

Thm B (D-Rocha):

For every $\varepsilon > 0$ there is an open set of $\mathcal{A}_\epsilon$ of arcs $(f_t)_{t\in[-1,1]}$ unfolding a partially hyperbolic heterodimensional cycle such that:

$$\liminf_{t\to 0} \frac{|H(t) \cap [0, t]|}{t} > 1 - \varepsilon$$

(by the previous theorem this limit is < 1)
hyperbolicity is prevalent but not absolute

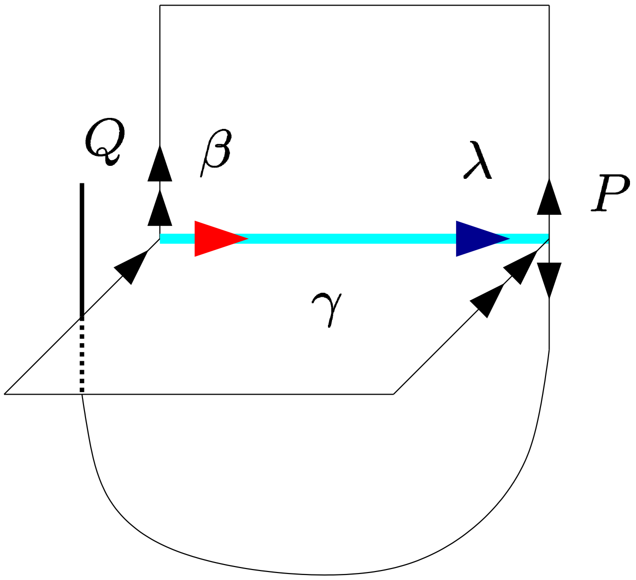


(III) LIMIT DYNAMICS

Dynamics after the bifurcation *depends* on the **central** dynamics of the bifurc. diffeo.

Ingredients:

(a) central eigenvalues of P and Q (corresponding to the central direction E^c)
(b) transition from Q to P throughout the curve $\gamma = W^s(P) \pitchfork W^u(Q)$ (restriction f/γ)



Mather invariant:

f/γ close to 0:

time-one of $\boxed{X(y) = (\log\beta)\, y \frac{\partial}{\partial y}}$

flow: $\boxed{y \mapsto \beta^t\, y}$

f/γ close to 8:

time-one of $\boxed{Y(y) = (\log\lambda)\, (y - 8) \frac{\partial}{\partial y}}$

flow: $\boxed{y \mapsto \beta^t\, y}$

y near 0^+, big n with $f^n(y)$ near 8^-:

$$\boxed{Df^n(y)(X(y)) = \mu(y)\, Y(f^n(y))}$$

local f-invariance (X, Y near $0, 8$),

- independent of n,
- $\mu(y) = \mu(f(y))$.

μ measures the distortion.

$\mu(y)$: Mather invariant of f/γ

Heuristic principle:

- small distortion: intermingled homoclinic classes, for $t > 0$: $H(P, f_t) = H(Q, f_t)$. (D).
- For Thm B one needs a big distortion (as $\varepsilon \to 0$, distortion $\to \infty$).
- Thm. A: dynamics in *intermediate* cases.

Strategy:

- projection to the central direct. (curve γ),
- fix a (reference) fundam. domain D^+ of γ, consider returns to D^+.

Definition of the limit dynamics:

Simplifying assumptions:

- $\gamma = [0, 8]$,
- f linear in $[0, 1]$, $x \mapsto \beta x$,
- f affine in $[7, 8]$, $x \mapsto \lambda\,(x - 8) + 8$,
- Fix: $D^+ = [\beta^{-1}, 1]$, $D^- = [7, 8 - \lambda]$,
- $f^N(D^+) = D^-$. $F = f^N$ **transition**.

To determine the dynamics one studies the **returns** of points to D^+ by iterations of f_t.

Quotient dynamics: After identifying the stable and unstable directions

(Quotient) Family of returns to D^+:

a one-parameter family of system of iterated functions (∞-many maps) generated by:

- two affine maps (dynamics at Q and P),

$$\boxed{x \mapsto \beta\, x} \qquad \boxed{x \mapsto \lambda\,(x-8)+8}$$

- a transition $\boxed{\text{the map } F}$,
- a translation (induced by the parameter t after reparametrization).

- **Step 1:** Transition
- **Step 2:** Affine iterates

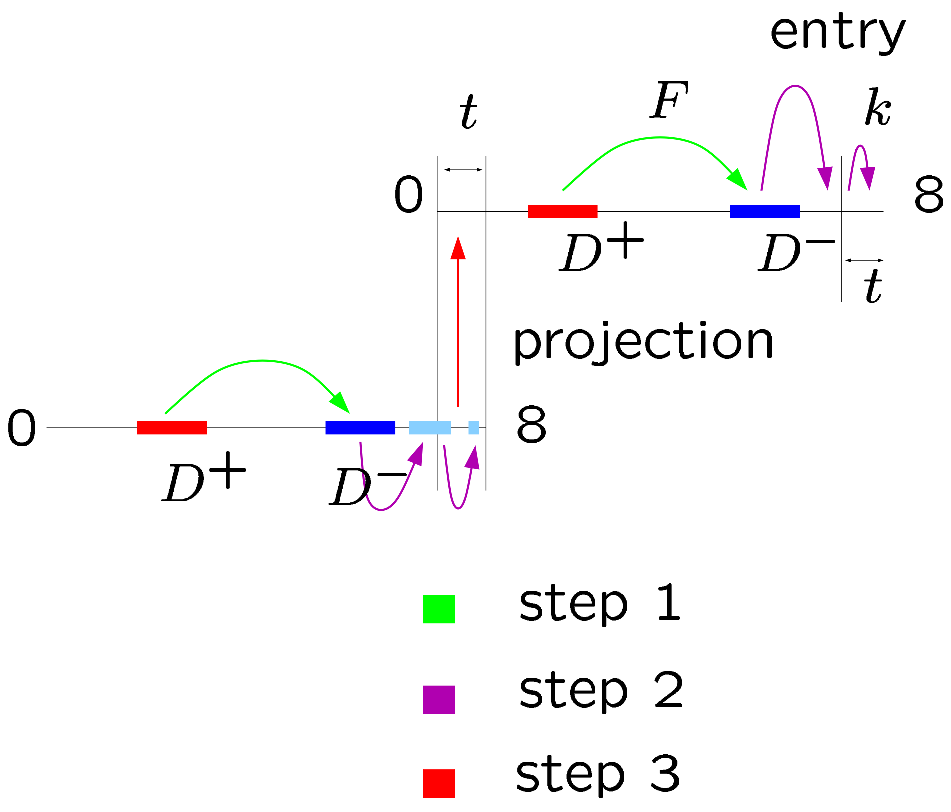
Part 1: approx. to $P=8$ at distance t (entry iterates $\lambda^{\text{entry}} \simeq t$),
Part 2: further iterates ($k \geq 0$).

Steps 1–2: $\boxed{\lambda^{\text{entry}+k}(F(x)-8)+8}$

- **Step 3:** effect of the parameter t (acting as a translation) and projection

Steps 1–3: $$\boxed{\boxed{(\lambda^{\text{entry}+k}(F(x)-8)+8)} + t - 8}$$

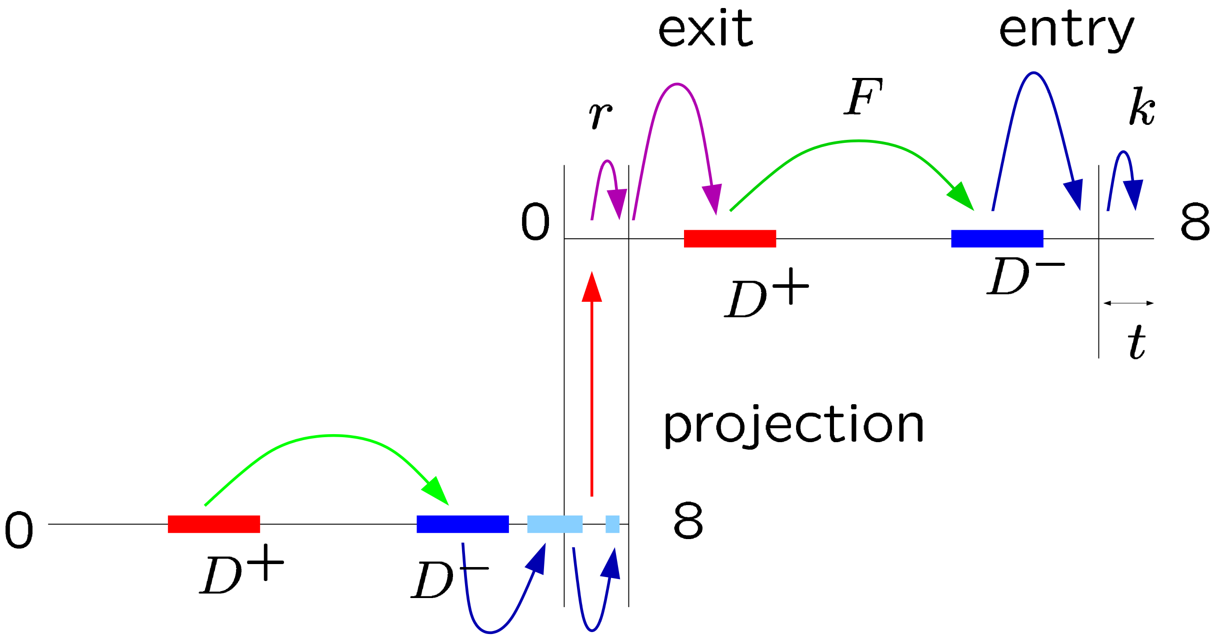
Steps 1–3: $$\boxed{\boxed{\lambda^{\text{entry}+k}(F(x)-8)} + t}$$



- **Step 4:** Linear iterates

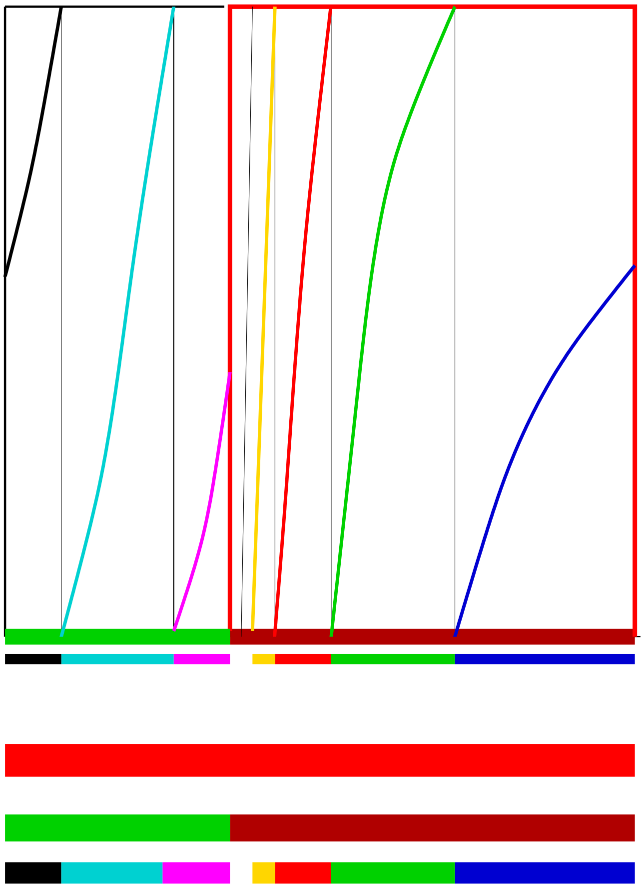
Part 1: Separation at distance t from $Q = 0$

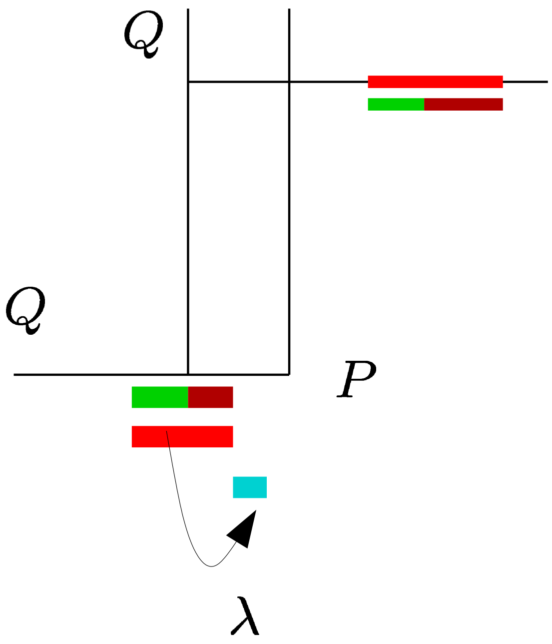
Part 2: Return to D^+, entry iterates $\beta^{\text{entry}} \simeq 1/t$.

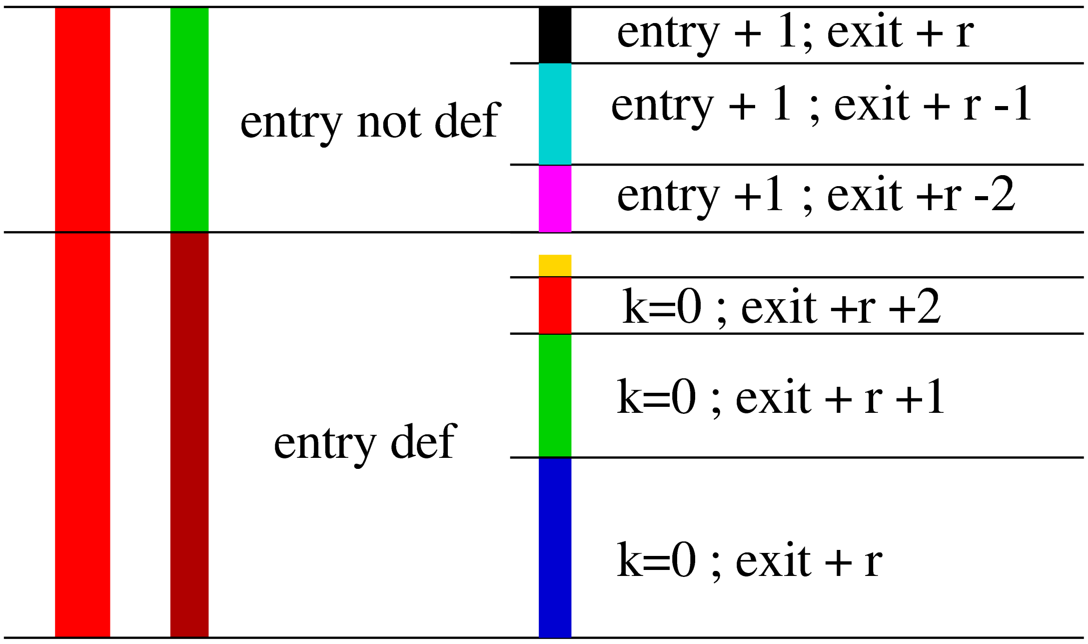


Steps 1–4:

$$\boxed{\beta^{\text{exit}+r}\left(\ \boxed{\lambda^{\text{entry}+k}(F(x)-8)+t}\ \right)}$$




Q
Q
P
λ


entry not def
entry + 1; exit + r
entry + 1 ; exit + r -1
entry +1 ; exit +r -2
entry def
k=0 ; exit +r +2
k=0 ; exit + r +1
k=0 ; exit + r

Special simplified case:

$$\lambda^{\text{entry}} = t, \qquad \beta^{\text{exit}} = 1/t$$

Returns for the parameter t (of type k, r):

$$\boxed{\beta^r(\lambda^k(F(x) - 8)) + 1}$$

Re-scaling effect: $s = t - \mu$, small $\mu > 0$.

One-param. family iterated functions $R_\mu^{k,r}$

$$\boxed{R_\mu^{k,r}(x) = \beta^r(\ \lambda^k(F(x) - 8)) + 1 + \beta^r \mu}$$

Remark: k is free and $r = r(n, x)$.
Typically: $k \to \infty$, $r(n, x) = 0$.
$k = 0$, there are x with $r(n, x)$ arbitrarily big.

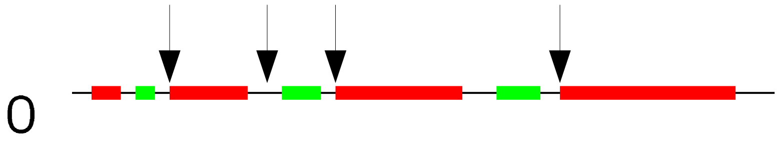
To get $H(P, f_t) = H(Q, f_t)$:

$$\boxed{R_\mu^{0,r},\ R_\mu^{1,r} \text{ uniformly expanding}}$$

Thm B: F has a big distortion.

(IV) COLLISIONS OF HOMOCLINIC CLASSES

Next step: to understand the transition from hyperbolic to bifurcating intervals
(Theorems A and B)



🟥 hyperbolicity

🟩 non-hyperbolicity

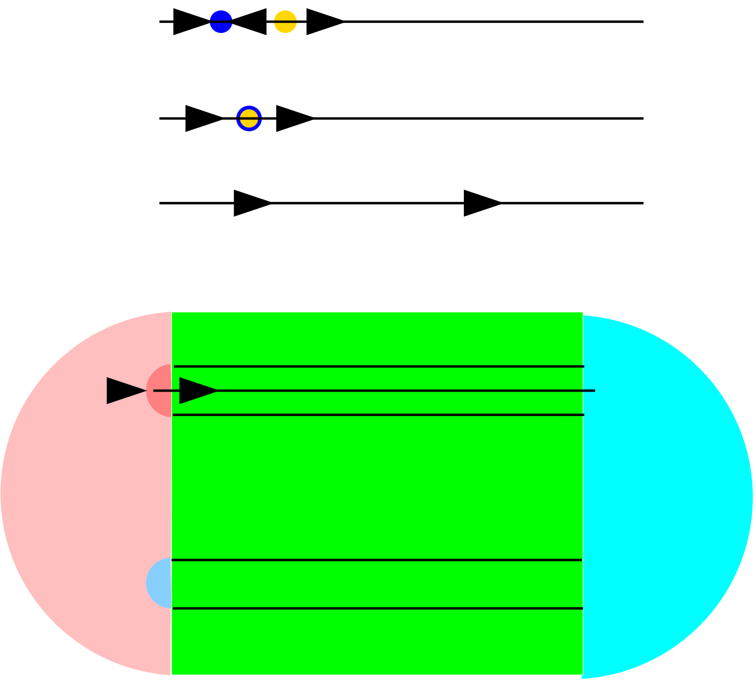
▼ secondary bifurcations

Secondary bifurcations:

- new heterodimensional cycles;
- loss of hyperbolicity of some saddle (saddle-node bifurcations)
- (others ?)

Interesting: the saddle is involved in the dynamics

- saddle-node cycles and horseshoes;



- derived from Anosov;

- collisions of homoclinic classes

dimension two: derived from Anosov, saddle-node horeseshoes, always a class is trivial

Goal collisions of hyperb. homoclinic classes (at heterodimensional cycles)

Model (product structure) (D. Rocha):

the arc $(f_t)_{t\in[-1,1]}$ unfolds a (partially hyperbolic) heterodimensional cycle at $t = 0$ such that there are sequences $(t_n), (t_n^+), (t_n^-)$, $t_n, t_n^{\pm} \to 0^+$, with

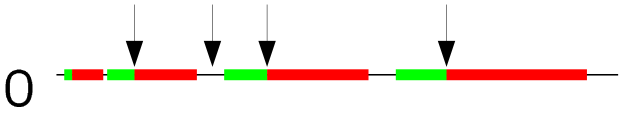
- f_t is hyperbolic if $t \in (t_n, t_n^+)$, (in particular $H(P, f_t)$ and $H(Q, f_t)$ are hyperbolic (thus disjoint)),

- for $t = t_n$ there is a **collision of the homoclinic classes** of P and Q:
$H(P, f_{t_n}) \cap H(Q, f_{t_n}) \neq \emptyset$ (the orbit of a saddle-node)

- there is **persistence of heterodimensional cycles** in (t_n^-, t_n^+)

Persistence of (heterodimensional) cycles (in the open interval I): there is a dense subset of I such that every f_t has a heterodimensional cycle.

- the intervals (t_n^-, t_n) and (t_n, t_n^+) are both prevalent are the bifurcation.



🟥 hyperbolicity

🟩 persistence of cycles

▼ collision bif.

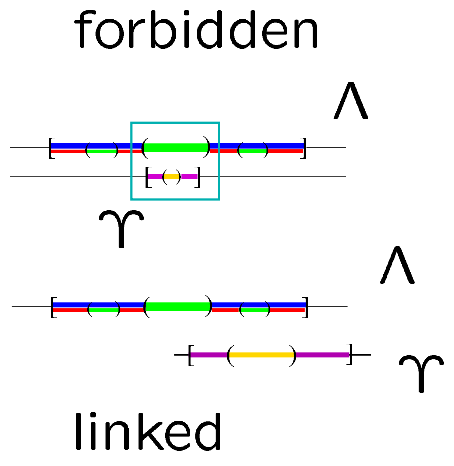
Ingredients:

- construct F such that $R_0^{0,0}$ unfolds a saddle-node;

- creation of arbitrarily thick hyperbolic sets Υ (of index 2) immediately before the unfolding of the saddle-node (the unfolding is for decreasing t):
Υ is dynamically defined, associated to the system of iterated functions
(non-critical dynamics)

- the homoclinic class of P contains hyperbolic sets Λ with thicknes uniformly bounded from zero.

- **Gap Lemma**:
$\tau(\Lambda)\cdot\tau(\Upsilon) > 1$ and each set is not contained in a gap of the other one, then $\Lambda \cap \Upsilon \neq \emptyset$.



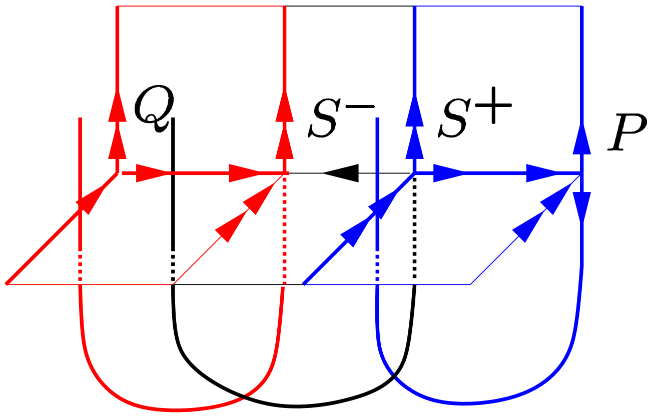
- the product dynamics gives a cycle associated to Υ and Λ.

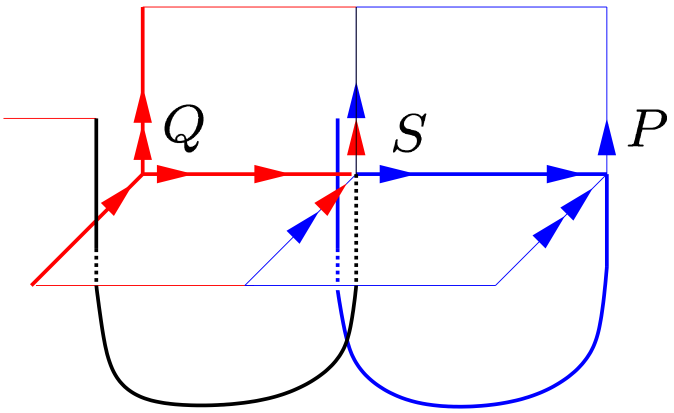
Remark: (D. Santoro). There is a model family for the collision of partially hyperbolic (non-hyp.) homoclinic classes $H(P)$ and $H(Q)$ leading to the collapse of the classes:

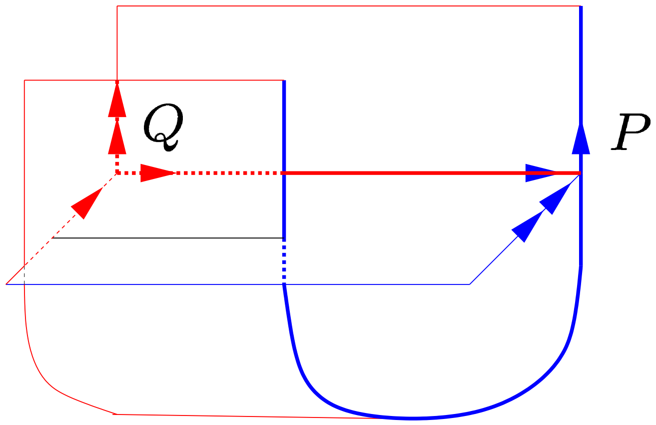
$$\boxed{H(P) = H(Q) \text{ after the collision}}$$

Methods for the collapse $\neq$.

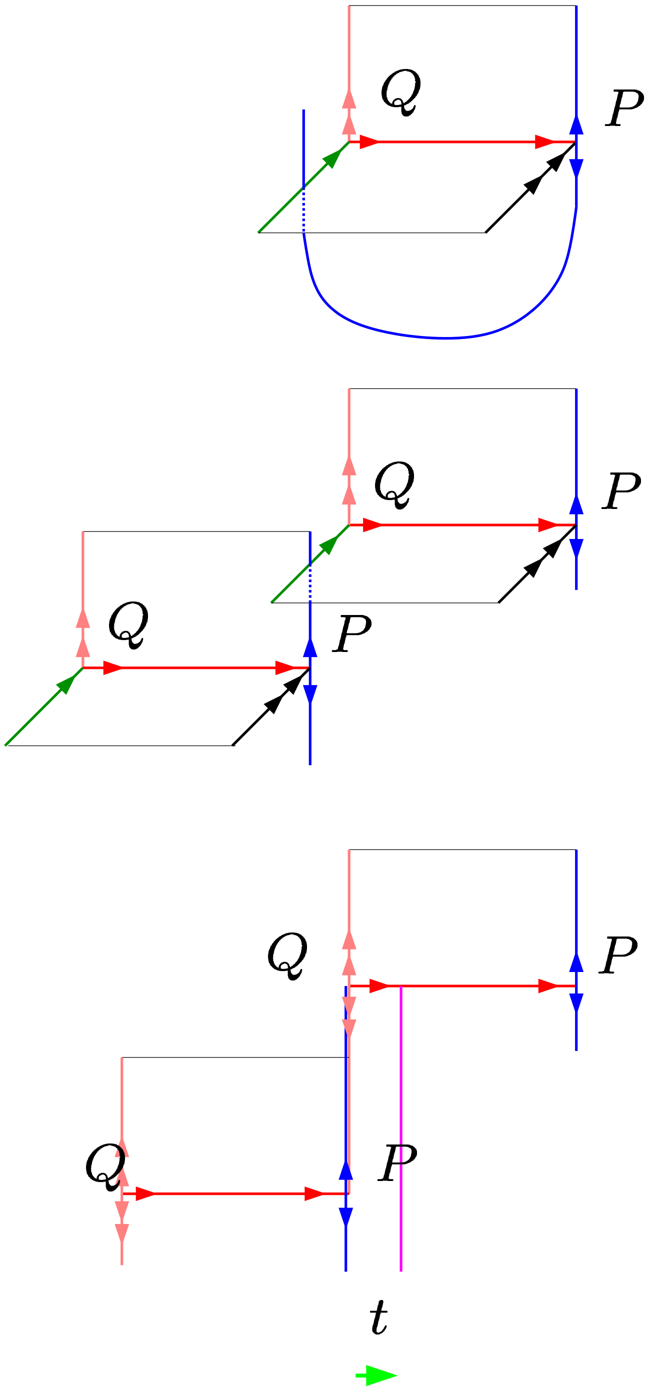
collision of non-hyperbolic homoclinic classes



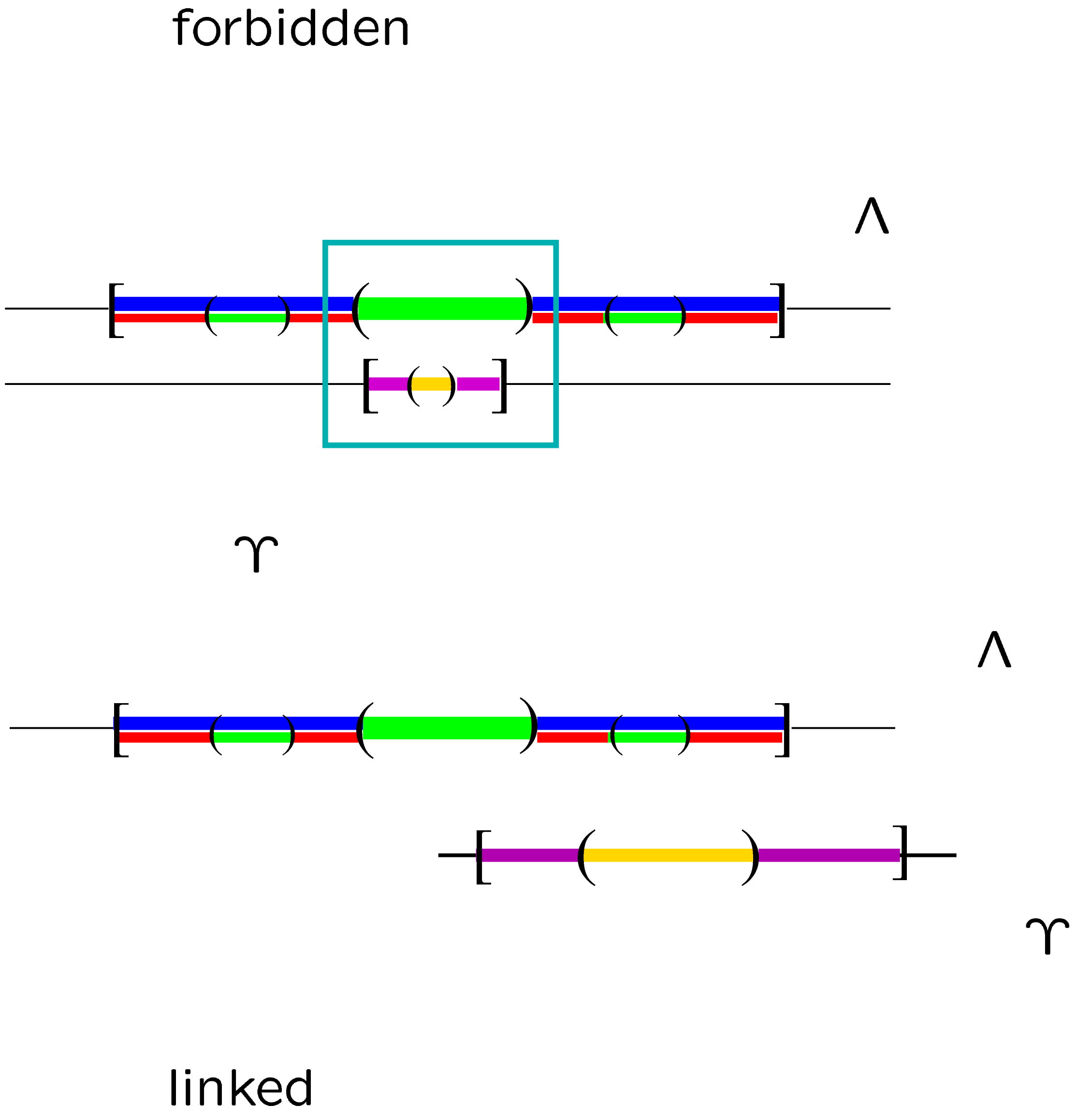




model cycle (one-dim reduction)



parameter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On maximal transitive sets of generic diffeomorphisms

Christian Bonatti & Lorenzo J. Díaz *

November 7, 2002

Abstract

We construct locally generic C^1-diffeomorphisms of 3-manifolds with maximal transitive Cantor sets without periodic points. The locally generic diffeomorphisms constructed also exhibit strongly pathological features generalizing the Newhouse phenomenon (coexistence of infinitely many sinks or sources). Two of these features are: coexistence of infinitely many nontrivial (hyperbolic and nonhyperbolic) attractors and repellors, and coexistence of infinitely many nontrivial (nonhyperbolic) homoclinic classes.

We prove that these phenomena are associated to the existence of a homoclinic class $H(P, f)$ with two specific properties:

- in a C^1-robust way, the homoclinic class $H(P, f)$ does not admit any dominated splitting,
- there is a periodic point P' homoclinically related to P such that the Jacobians of P' and P are greater than and less than one, respectively.

1 Introduction

One of the main problems in dynamical systems is to describe the limit set for a large class of diffeomorphisms. The ideal objective is to split the limit set into finitely many pieces of dynamics which are mutually independent, maximal, and dynamically indecomposable. The next step should be to give a description (as complete as possible) of each piece.

In the case of hyperbolic diffeomorphisms, the Smale theory gives a complete topological description of the limit set: the limit set of a hyperbolic diffeomorphism is the union of finitely many pairwise disjoint hyperbolic *transitive sets* (existence of a dense orbit), these sets are the *basic pieces* of the Smale theory. These basic sets are locally maximal and correspond to *homoclinic classes* of periodic points, see [Sm]. These notions correspond to the ideas of dynamical independence, maximality, and indecomposability.

The Smale diffeomorphisms fail to be dense in the space of C^1-diffeomorphisms if the dimension of the ambient manifold is strictly greater than two, see for instance the first examples in [AS] in dimension greater than or equal to 4, adapted for 3-manifolds in [Si], and its density remains an open problem for surface C^1-diffeomorphisms. Thus, in the nonhyperbolic setting, it is natural to try to decompose the limit set of a large class of diffeomorphisms f as the union of (preferably) finitely many elementary pieces of dynamics playing the role of the basic pieces of the Smale theory.

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For the role of *elementary pieces of dynamics* of a diffeomorphism f defined on a compact manifold M there are three natural candidates:

- the *robustly transitive sets* introduced in [DPU]: an f-invariant set Λ is robustly transitive if there is a neighborhood U of it such that the set $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(\overline{U})$ is transitive and contained in U for every g C^1-close to f and $\Lambda_f = \Lambda$.

  A robustly transitive set Λ is good for the role of elementary piece of dynamics if the neighborhood U in the definition can be chosen being a filtrating neighborhood, that is, $U = V \setminus \overline{W}$, where V and W are open open sets such that $f(\overline{V}) \subset V$ and $f(\overline{W}) \subset W$. Otherwise, the robustly transitive set may be included in greater transitive set.

- Given any periodic saddle P of f, the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P is a transitive set, *the homoclinic class* of P, denoted by $H(P, f)$.

- The *maximal transitive sets,* that is, sets which are maximal in the family of all transitive compact invariant sets of f ordered by inclusion. As the closure of the union of an increasing (for $\subset$) family of transitive sets is transitive, Zorn's Lemma ensures that any transitive set is contained in a maximal one.

These three notions are closely related but, in general, they are not equivalent. We adopt here the *generic* or *locally generic* point of view: a property $\mathcal{P}$ is locally generic if there exist a non-empty open set $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ and a residual subset $\mathcal{R}$ of $\mathcal{U}$ on which $\mathcal{P}$ is satisfied.

- By definition, every robustly transitive set associated to a filtrating neighbourhood is maximal transitive. Moreover, as a consequence of the Connecting Lemma in [H], robustly transitive sets of generic diffeomorphisms are *relative homoclinic classes,* where the relative homoclinic class of a saddle P in a neighbourhood U is the closure of the transverse homoclinic points of P whose orbits are contained in U. However, in general, homoclinic classes (relative or not) fail to be robustly transitive sets (see, for instance, [BD$_2$]).

- For generic C^1-diffeomorphisms, every homoclinic class $H(P, f)$ is a maximal transitive set (see [Ar]) verifying a stronger property: every transitive set intersecting $H(P, f)$ is contained in it (see [CMP][1]).

Here we prove that generic diffeomorphisms may have maximal transitive sets which are neither homoclinic classes nor robustly transitive sets (this result will be restated in a more precise and stronger formulation in the next section, see *Theorem A*):

Theorem *Given any compact* 3*-manifold* M *there is a locally residual subset* $\mathcal{F}(M)$ *of* $\mathrm{Diff}^1(M)$ *of diffeomorphisms* f *having maximal transitive Cantor sets* Λ_f *without periodic points. In particular, the maximal transitive sets* Λ_f *are not homoclinic classes.*

---

[1] This stronger notion corresponds to the so-called *maximal transitive sets* in [CMP]. The name *saturated transitive set* we use here seems to be more coherent with the usual terminology, see Section 1.1.

The sets Λ_f in the theorem are *infinitely renormalizable* according to the terminology in [BGLT] or *adding machines,* see for instance [BS].

Let us observe that, following [CMP], for generic diffeomorphisms, every maximal transitive set which is not a homoclinic class does not contain periodic points. So we define *aperiodic maximal transitive sets* to be the maximal transitive sets without periodic points. However, as a classical consequence of the Closing Lemma, see [Pu], every aperiodic maximal transitive set of a generic diffeomorphism is contained in the closure of the set of periodic points.

The construction of the aperiodic maximal transitive sets in the theorem involves the coexistence of infinitely many (different) homoclinic classes. Actually, due to a recent result in [Ab], the coexistence of infinitely many different (maybe trivial) homoclinic classes is a necessary condition for the (locally generic) existence of aperiodic maximal transitive sets. More precisely, [Ab] states that, in a residual set of $\mathrm{Diff}^1(M)$, the cardinality of the set of homoclinic classes is locally constant. Moreover, for those generic diffeomorphisms with finitely many homoclinic classes, there is a spectral decomposition theorem analogous to the one in the hyperbolic case (see [Sm]): the non-wandering set is the union of finitely many pairwise disjoint homoclinic classes exhibiting a weak form of hyperbolicity (existence of a dominated splitting, see Definition 3.8). Furthermore, as in the hyperbolic case, the homoclinic classes are the maximal invariant sets in neighborhoods corresponding to levels of a filtration. In particular, the maximal transitive sets of generic diffeomorphisms with finitely many homoclinic classes are homoclinic classes. This result in [Ab] generalizes the generic dichotomy (hyperbolicity versus infinitely many sinks or sources) obtained in [Ma] for generic C^1-diffeomorphisms of surfaces.

Theorem A (locally generic existence of aperiodic maximal transitive sets) is consequence of a generalization of the so-called *Newhouse Phenomenon* (coexistence of infinitely many sinks or sources for locally generic diffeomorphisms). In the case of C^2-diffeomorphisms, this phenomenon is associated to the unfolding of homoclinic tangencies, see [N_1], [N_2] and [PT] for surface diffeomorphisms and [PV] and [Tj] in higher dimensions. In the C^1-setting the existence of such a phenomenon for surface diffeomorphisms remains an open question.

In higher dimensions, the C^1-Newhouse phenomenon (as well the locally generic existence of aperiodic maximal transitive sets) is related to the existence of *wild homoclinic classes,* see [BD_2]. The homoclinic class $H(P, f)$ of a periodic point P is *wild* if (in a robust way) it does not admit any dominated . A sufficient condition (in dimension 3) is that it contains (in a stable way) a pair of hyperbolic periodic points of *indices* (dimension of the stable bundle) two and one whose derivatives have non-real eigenvalues and Jacobians greater than one and less than one, respectively. Here we will consider an additional hypotheses: we consider an open set $\mathcal{W}_0$ consisting of diffeomorphisms f having a saddle P such that the homoclinic class $H(P, f)$ contains a periodic point P' homoclinically related to P, such that the Jacobian of f (at the period) at the points P and P' is greater than 1 and less than one < 1, respectively, see Definition 1.1.

In the next section, in Theorem B we will give a much more precise statement of the following result:

Theorem *Generic diffeomorphisms $f \in \mathcal{W}_0$ with a wild homoclinic class satisfy the following universal property: for every open set $\mathcal{O}$ of diffeomorphisms of the disk D^3, there are infinitely many disjoint periodic disks on which the first return map of f is smoothly conjugate to some element of $\mathcal{O}$.*

As a consequence we get:

Corollary *Locally generic diffeomorphisms $f \in \mathcal{W}_0$ having a wild homoclinic class display infinitely many times (in periodic disks with pairwise disjoint orbits) any robust property of C^1-diffeomorphisms of the disk D^3. In particular, these diffeomorphisms exhibit simultaneously infinitely many:*

- *pairwise disjoint non-trivial homoclinic classes,*
- *infinitely many non-trivial hyperbolic and non-hyperbolic attractors, and*
- *infinitely many non-trivial hyperbolic and non-hyperbolic repellors.*

For a more complete list of pathological forms of the Newhouse Phenomenon see Corollary C.

In the proof of the previous theorem (see also Theorem B), the key technical result is that *given any wild homoclinic class $H(Q, f)$ of a diffeomorphism f there are diffeomorphisms g arbitrarily C^1-close to f with a periodic point P of arbitrarily large period n and arbitrarily close to $H(Q, f)$ such that the derivative $Dg^n(P)$ is the identity,* (see Theorem 3.3). This claim follows by using strongly the arguments in [BDP]. This assertion allows us to prove that given any diffeomorphism f with a wild homoclinic class $H(Q, f)$ and any open set $\mathcal{O}$ of diffeomorphisms of the disk D^3 into itself preserving the orientation, there are disks D arbitrarily close to $H(Q, f)$ and a C^1-perturbation g of f such that the disks D are periodic for g (i.e., $D, g(D), \dots, g^{m-1}(D)$ are pairwise disjoint and $g^m(D) \subset D$, for some $m \geq 1$) and the restriction of g^m to D is differentially conjugate to some element of $\mathcal{O}$.

Let us now explain how Theorem A follows from Theorem B. Consider any generic diffeomorphism f with a wild homoclinic class. First, by definition, the property of having a wild homoclinic class is an open one. So, Theorem B implies that there are small periodic disks where the first return map has a new wild homoclinic class. In other words, by rescaling a generic diffeomorphism with a wild homoclinic class, one obtains a new generic diffeomorphism with a wild homoclinic class. Arguing inductively, one gets an infinite decreasing sequence of small periodic disks D_k on which the first return maps of f have wild homoclinic classes. Finally, the aperiodical maximal transitive sets of the theorem are obtained as the intersection of the orbits of the disks D_k.

1.1 Definitions and precise statement of results

Before stating precisely our results let us give some definitions.

Let M be a compact closed manifold and $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms of M endowed with the usual C^1-uniform topology. Consider $f \in \mathrm{Diff}^1(M)$, we say that an f-invariant set Λ is *transitive* if it is the closure of the forward orbit of some point x of Λ. A transitive set Λ is *maximal* if it is a maximal element of the family of all compact transitive sets of f ordered by inclusion. A maximal transitive set is *aperiodic* if it does not contain any periodic orbit. A transitive set is *saturated* if every transitive set Σ intersecting Λ is contained in Λ (this means that the transitive set is saturated by the equivalence relation generated by the relation of belonging to the same transitive set). Observe that, since the union of two transitive sets may fail to be transitive, it is not guaranteed that every transitive set is contained in a saturated one (recall that it is contained in a maximal one), see for instance the examples in [DS] of diffeomorphisms with a pair of different homoclinic classes with non-empty intersection whose union is not contained in any transitive set.

An f-invariant set Λ of a diffeomorphism f is *minimal* if the orbit of any point of Λ is dense in itself (or equivalently, Λ has no proper f-invariant sets). In particular, an infinite minimal set Λ does not contain periodic orbits. An f-invariant set Λ is *uniquely ergodic* if it supports only one f-invariant probability measure (which is necessarily ergodic). Observe that there are minimal sets where the restriction of f is not uniquely ergodic (see [Fu]).

The *homoclinic class* of a hyperbolic periodic point P of f, denoted by $H(P,f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Every homoclinic class $H(P,f)$ is an f-invariant transitive set and the hyperbolic periodic points of the same *index* as P are dense in $H(P,f)$ (the *index* of a hyperbolic periodic point Q of f is the dimension of the stable bundle of f at Q, i.e., the number of eigenvalues of modulus less than one of $Df^{p(Q)}(Q)$ counted with multiplicity, where $p(Q)$ is the period of Q).

Two hyperbolic periodic points P_1 and P_2 of f are *homoclinically related* if $W^s(P_1) \pitchfork W^u(P_2) \neq \emptyset$ and $W^u(P_1) \pitchfork W^s(P_2) \neq \emptyset$ (where $W^s(P_1) \pitchfork W^u(P_2)$ denotes the transverse intersection between $W^s(P_1)$ and $W^u(P_2)$). In particular, the indices of P_1 and P_2 are equal and their homoclinic classes coincide. In fact, the homoclinic class of a hyperbolic periodic point P is the closure of the periodic points homoclinically related to P.

Finally, a compact set Λ is *Lyapunov stable* for f if for every neighborhood U of Λ there is a neighborhood V of Λ such that $f^i(V) \subset U$ for all $i \geq 0$. Observe that every Lyapunov stable transitive set Λ is saturated: any transitive set intersecting Λ is contained in any neighborhood of Λ. This also holds if Λ is Lyapunov stable for f^{-1}.

A set $\mathcal{F}$ of $\mathrm{Diff}^1(M)$ is *residual* in an open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ if there are open subsets $(\mathcal{V}_n)_{\in \mathbb{N}}$ of $\mathrm{Diff}^1(M)$ such that every $\mathcal{V}_n$ is dense in $\mathcal{U}$ and $\cap_n \mathcal{V}_n \subset \mathcal{F}$. We say that a set $\mathcal{F}$ is *locally residual* if there is a non-empty open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$ in which $\mathcal{F}$ is residual.

Theorem A *Given any closed 3-manifold M there is a locally residual subset $\mathcal{F}(M)$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f having aperiodic maximal transitive Cantor sets Λ_f. Moreover, the set Λ_f is minimal and uniquely ergodic for every $f \in \mathcal{F}(M)$. In particular, the maximal transitive sets Λ_f are not homoclinic classes. Finally, the sets Λ_f are simultaneously Lyapunov stable for f and f^{-1}, hence the sets Λ_f are saturated.*

The constructions in the proof of this theorem also implies that every diffeomorphism $f \in \mathcal{F}(M)$ simultaneously has uncountable aperiodic maximal transitive sets. This remark was pointed out to us by F. Beguin.

Observe that given any hyperbolic periodic point P_f of f there is a C^1-neighborhood $\mathcal{V}$ of f such that the continuation P_g of P_f is defined for every g in $\mathcal{V}$.

Definition 1.1 *Consider a diffeomorphism f defined on a closed three manifold M and a hyperbolic periodic point P_f of f.*

1. *The homoclinic class of P_f is* wild *if there is a neighborhood $\mathcal{V}_0$ of f such that, for any $g \in \mathcal{V}_0$, the homoclinic class $H(P_g, g)$ does not admit any dominated splitting (see Definition 3.8 for the notion of dominated splitting).*

2. *The homoclinic class $H(P_f, f)$ belongs to $W_0(f)$ if there are two hyperbolic periodic points Q_f and P'_f and a neighborhood $\mathcal{V}$ of f where the continuations P_g, P'_g and Q_g are defined for all $g \in \mathcal{V}$ such that*

(a) *Q_g belongs to $H(P_g, g)$,*
(b) *the point P_g has index two and a contracting non-real multiplier,*
(c) *the point Q_g has index one and an expanding non-real multiplier,*
(d) *the point P'_g has index two and is homoclinically related to P_g,*
(e) $\mathrm{Jac}_{g^{\mathrm{per}(P_g)}}(P_g) > 1$ *and* $\mathrm{Jac}_{g^{\mathrm{per}(P'_g)}}(P'_g) < 1$, *where* $\mathrm{Jac}_g(R)$ *denotes the Jacobian of g at the point R.*

We define homoclinic classes of points of index 1 in $W_0(f)$ analogously.

Notice that items (a)–(c) of the definition of homoclinic classes in $W_0(f)$ imply that the homoclinic class $H(P_f, f)$ is wild.

Denote by $\mathcal{W}(M)$ the set of diffeomorphisms of $\mathrm{Diff}^1(M)$ having a wild homoclinic class, and by $\mathcal{W}_0(M)$ the set of diffeomorphisms f having an homoclinic class $H(P_f, f) \in W_0(f)$. By the previous comment, $\mathcal{W}_0(M) \subset \mathcal{W}(M)$.

By definition, the sets $\mathcal{W}(M)$ and $\mathcal{W}_0(M)$ are open in $\mathrm{Diff}^1(M)$. The fact that the set $\mathcal{W}_0(M)$ is non-empty can be proved exactly as in [BD$_2$, Section 3.2], where it is proved that the set $\mathcal{W}(M)$ is non-empty, see the appendix (Section 6) in this paper for details.

The arguments in [BD$_2$] show that C^1-generic diffeomorphisms of $\mathcal{W}_0(M)$ exhibit simultaneously infinitely many sinks and infinitely many sources. In fact, in [BDP] it is proved that, for generic diffeomorphisms, any homoclinic class either has a dominated splitting or is contained in the closure of the (infinite) set of sinks and sources. Thus generic diffeomorphisms of $\mathcal{W}(M)$ (i.e., with a wild homoclinic class) have infinitely many sinks or sources.

Remark 1.2 *The proof of Theorem A shows that diffeomorphisms with aperiodic maximal transitive sets are generic in the set $\mathcal{W}_0(M)$.*

We prove that generic diffeomorphisms of $\mathcal{W}_0(M)$ are the universal models for 3-dimensional dynamics, in the sense that they satisfy the following universal property.

Let D^n be the compact ball of radius 1 in $\mathbb{R}^n$ and denote by $\mathrm{Diff}^+_{\mathrm{int}}(D^n)$ the space of orientation preserving C^1-diffeomorphisms $\phi\colon D^n \to \mathrm{int}(D^n)$ endowed with the usual C^1-topology.

Definition 1.3 (Universal Dynamics) *A diffeomorphism f has universal dynamics at an f-invariant set Σ if for every open subset $\mathcal{O}$ of $\mathrm{Diff}^+_{\mathrm{int}}(D^n)$ and every point $x \in \Sigma$ there are sequences of pairwise disjoint disks $(D_k)_k$ and of natural numbers $(n_k)_k$ and a constant $K > 0$ such that:*

(U1) *the disks $D_k, f(D_k), \dots, f^{n_k-1}(D_k)$ are pairwise disjoint,*

(U2) *the disks D_k satisfy the following:*

- $\mathrm{diam}(D_k) \to 0$ *and* $D_k \to x$ *as* $k \to \infty$,
- $(\sup\{\mathrm{diam}(f^i(D_{2k}))\}_{i=0}^{n_{2k}-1}) \to 0$ *as* $k \to \infty$, *and*
- $(\sup\{\mathrm{diam}(f^{-i}(D_{2k+1}))\}_{i=0}^{n_{2k+1}-1}) \to 0$ *as* $k \to \infty$,

(U3) $f^{n_{2k}}(D_{2k}) \subset \mathrm{int}(D_{2k})$ *and the restriction $f^{n_{2k}}|_{D_{2k}}$ of $f^{n_{2k}}$ to D_{2k} is differentially conjugate to some $g_{2k} \in \mathcal{O}$,*

(U4) *$f^{-n_{2k+1}}(D_{2k+1}) \subset \mathrm{int}(D_{2k+1})$ and the restriction $f^{-n_{2k+1}}|_{D_{2k+1}}$ is differentially conjugate to some $g_{2k+1} \in \mathcal{O}$.*

The following result is the key of all the constructions in this paper.

Theorem B *There is a residual subset $\mathcal{U}(M)$ of $\mathcal{W}_0(M)$ of diffeomorphisms f with universal dynamics at any wild homoclinic class in $W_0(f)$.*

Theorem A will follow from Theorem B. We shall see that for every f in the residual subset $\mathcal{U}(M)$ of $\mathcal{W}_0(M)$ there are natural numbers n_k and disks D_k as in the definition of Universal Dynamics where the induced dynamics of f (i.e., the restriction of f^{n_k} or f^{-n_k} to D_k, according to the case) belongs to $\mathcal{W}_0(D_k)$ (the set of wild diffeomorphisms g of D_k having a homoclinic class in $W_0(g)$). This fact allows us to apply Theorem B to this small disk D_k. Arguing inductively and repeating the previous construction infinitely many times, we will get maximal transitive sets $\Phi(f)$ which are *infinitely renormalizable:* there are sequences of nested disks $(\Delta_k)_k$ and of natural numbers (n_k) satisfying conditions (U1)–(U4) above such that

$$\Phi(f) = \bigcap_{2k} \hat{\Delta}_{2k}, \quad \text{where} \quad \hat{\Delta}_{2k} = \bigcup_{i=0}^{n_{2k}-1} f^i(\Delta_{2k}).$$

By construction, the Cantor set $\Phi(f)$ is minimal, maximal transitive, and Lyapunov stable for f and f^{-1}, see Proposition 4.2 and Section 4 for the details of this construction

A property $\mathcal{P}$ of a diffeomorphism f is a *robust property* if every diffeomorphism in some C^1-neighborhood of f verifies $\mathcal{P}$.

Similarly, a property $\mathcal{P}$ is *generic* in an open set $\mathcal{V}$ of diffeomorphisms if there is a residual subset $\mathcal{R}$ of $\mathcal{V}$ consisting of diffeomorphisms satisfying $\mathcal{P}$. Finally, a property $\mathcal{P}$ is *locally generic* if it is generic in some non-empty open set of $\mathrm{Diff}^1(M)$. As a direct consequence of Theorem B we now have the following heuristic principle.

Heuristic principle: *Given any robust or locally generic property $\mathcal{P}$ of $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$ there is a residual subset $\mathcal{R}_{\mathcal{P}}(M)$ of $\mathcal{W}_0(M)$ of diffeomorphisms satisfying $\mathcal{P}$. Moreover, this property is displayed by every $f \in \mathcal{R}_{\mathcal{P}}(M)$ in infinitely many disjoint periodic disks.*

In the next corollary we summarize some of these dynamical robust properties we consider more relevant:

Corollary C *There is a residual subset $\mathcal{R}(M)$ of the set $\mathcal{W}_0(M)$ contained in the set of wild diffeomorphisms consisting of diffeomorphisms f having simultaneously infinitely many:*

a) *Sinks and sources.*

b) Independent saddles, *(i.e., saddles whose homoclinic classes consist only of a periodic orbit).*

c) *Non-trivial uniformly hyperbolic transitive attractors and repellors.*

d) *Non-trivial partially hyperbolic robustly transitive attractors and repellors.*

e) *Homoclinic classes containing persistently saddles of different indices.*

f) *Wild homoclinic classes $\Lambda_i \in W_0(f)$.*

g) *Aperiodic maximal transitive Cantor sets $\mathcal{F}_i$ where f is minimal. Moreover, the sets $\mathcal{F}_i$ are simultaneously Lyapunov stable for f and f^{-1}, hence every $\mathcal{F}_i$ is saturated.*

Moreover, the wild homoclinic classes Λ_i and the aperiodic maximal transitive sets $\mathcal{F}_i$ can be chosen such that

$$\overline{\bigcup \Lambda_i} = \overline{\bigcup \mathcal{F}_i}.$$

Let us observe that we do not know if last the equality in the corollary holds for all the aperiodic maximal transitive sets.

Our constructions can be carried out in higher dimensions. First, a trivial way to get aperiodic maximal transitive sets in higher dimensions consists of multiplying the dynamics on D^3 by a hyperbolic transverse dynamics in such a way the disk turns out to be normally hyperbolic. On the other hand, a more interesting approach is to get universal dynamics in higher dimensions by defining wild homoclinic classes in $W_0(f)$ similarly: these classes contain in a robust way periodic saddles of any possible index $(2, \dots, \dim(M) - 1)$, possess complex (non-real) eigenvalues of any rank (see [BDP] for the definition of rank), and, for any $k \in [2, \dim M]$, has homoclinically related points of index k having Jacobians greater than and less than one.

Recall that a *topological attractor* of a diffeomorphism f is a compact set Γ such that there is a compact neighbourhood Δ of it such that $f(\Delta)$ is contained in the interior of Δ and $\cap_{i \geq 0} f^i(\Delta) = \Gamma$. By construction, the aperiodic maximal transitive sets in Corollary C are a countable intersection of (non-transitive) attractors. Thus, in the terminology introduced in [Hu], these sets are *quasi-attractors.* Recall that the aperiodic maximal transitive Cantor sets here are Lyapunov stable for f and f^{-1}, so they have trivial stable and unstable basins. This fact gives counter-examples[2] (in the C^1-setting) to the following question posed in [Hu]:

Are generic chain recurrent quasi-attractors attractors?

In the case of *tame diffeomorphisms,* i.e., diffeomorphisms having finitely many pairwise disjoint homoclinic classes and whose number is locally generically constant, in [CM] it is proved that the union of the basins of the (topological) attractors is open and dense in the ambient manifold. This result, the question in [Hu], and our constructions (in the world of the wild diffeomorphisms) suggest the following problems concerning the size of the basins of attraction of the aperiodic maximal transitive sets with minimal dynamics in this paper.

Problems:

- *Consider a diffeomorphism f in the residual subset $\mathcal{R}(M)$ of $\mathcal{W}_0(M)$ in Corollary C. Is there an aperiodic maximal transitive set $\mathcal{F}$ of f whose basin of attraction has non-empty interior? Or a weaker version of this question; does the basin of attraction of $\mathcal{F}$ contain a residual subset in some non-empty open subset of M?*
- In the opposite direction of the previous questions, there is the following natural problem.

  For generic diffeomorphisms f of $\mathrm{Diff}^1(M)$*, is the union of the basins of the (topological) transitive attractors of f dense in the whole manifold?*

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[2] This consequence of the results in this paper has been pointed out to us by M.-C. Arnaud and by a referee

For generic diffeomorphisms f, the set of periodic points of f is dense in the nonwandering set of f, in particular any topological attractor of f contains periodic points. Hence, a positive answer to this problem will imply a negative answer to the first one.

This paper is organized as follows. In the next section we sketch the proof of Theorem B. In Section 3 we give the complete proof of Theorem B. This section has two preparatory parts, the first one concerning perturbations of the identity (see Section 3.1) and the second one about dynamical perturbations of the derivative (see Section 3.2). In Section 4 we deduce Theorem A from Theorem B. Corollary C is proved in Section 5. Finally, in the Appendix (Section 6) we outline the construction of the wild homoclinic classes in $W_0(f)$.

2 Sketch of the proof of Theorem B

Consider an open subset $\mathcal{W}_0$ of $\mathcal{W}_0(M)$ of diffeomorphisms f having a wild homoclinic class $H(P_f, f) \in W_0(f)$, where P_f is a saddle of index 2 depending continuously on $f \in \mathcal{W}_0$. Observe that $\mathcal{W}_0(M)$ can be written as union of the sets of the form of $\mathcal{W}_0$, so that it is enough to prove Theorem B for the set $\mathcal{W}_0$.

The proof of Theorem B consists of the following steps:

Step 1: We begin by proving that given any open subset $\mathcal{O}$ of $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$ and any open set V of M, there are an arbitrarily small C^1-perturbation h of the identity map in V, a small disk $D \subset V$, and $k > 0$ such that the $(k-1)$ first iterates of D by h are pairwise disjoint, $h^k(D) \subset \mathrm{int}(D)$, and $h^k|_D$ is differentially conjugate to some $g \in \mathcal{O}$. This is done in Proposition 3.1.

Given $f \in \mathcal{W}_0$ and $\varepsilon > 0$ consider the set $\Sigma(\varepsilon, f)$ of hyperbolic periodic points R of f such that

- R is homoclinically related to P_f, and
- $\dfrac{\log\,(\mathrm{Jac}\, f^{\mathrm{per}(R)}(R))}{\mathrm{per}(R)} \in (-\varepsilon, \varepsilon)$.

Step 2: The next step is to see that, for every $\varepsilon > 0$ and every $f \in \mathcal{W}_0$, the set $\Sigma(\varepsilon, f)$ is dense in $H(P_f, f)$, see Lemma 3.6.

Step 3: Using the point Q_f of index one with a non-real expanding eigenvalue in the definition of wild homoclinic class in $W_0(f)$, we have that $H(P_f, f)$ does not admit any dominated splitting for all $f \in \mathcal{W}_0$. Step 2 now implies that, for every $f \in \mathcal{W}_0$ and every $\varepsilon > 0$, the set $\Sigma(\varepsilon, f)$ does not admit any dominated splitting (see Definition 3.8 and Lemma 3.9).

Step 4: Using the fact that $\Sigma(\varepsilon, f)$ does not admit any dominated splitting, Proposition 2.1 in [BDP] implies that, for every $\delta > 0$, there are a point $P_1 \in \Sigma(\varepsilon, f)$ and a δ-perturbation of the derivative of f throughout the orbit of P_1 (without modifying neither the f-orbit of P_1 nor the Jacobian of f at P_1) such that the linear map $A\colon T_{P_1}M \mapsto T_{P_1}M$ corresponding to the product of the perturbed derivatives along the orbit of P_1 is a homothety. By definition of $\Sigma(\varepsilon, f)$, the linear map A verifies

$$\frac{\log(\det(A))}{\mathrm{per}(P_1)} \in [-\varepsilon, \varepsilon].$$

So, after a new ε-perturbation, we can assume that $\det(A) = 1$, i.e., the linear map A is the identity.

Step 5: By a lemma of Franks (see Lemma 3.5), we can perform the previous perturbation of the derivative of f dynamically, obtaining a diffeomorphism g arbitrarily C^1-close to f with a periodic point P_1 whose derivative is the identity. After a new small perturbation, if necessary, we can assume that $g^{\mathrm{per}(P_1)}$ is the identity map in a small neighborhood V of P_1, see Theorem 3.3.

Step 6: The end of the proof of Theorem B involves an inductive argument (see Section 3.3). In rough terms, we apply Step 1 to $g^{\mathrm{per}(P_1)}$ and check that the disk D in Step 1 can be taken arbitrarily close to any point $x \in H(P_f, f)$. This will be done using the density of $\Sigma(\varepsilon, f)$ in $H(P_f, f)$ and the constructions in [BDP]. Finally, we get in the disk D a point with a wild homoclinic class and repeat the procedure above (this gives the inductive pattern).

3 Proof of Theorem B

3.1 Perturbations of the identity map

Proposition 3.1 *Let $D \subset M$ be a compact ball and ϕ any diffeomorphism in $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$. Then every neighborhood of the identity map of D contains a diffeomorphism f such that*

- *f coincides with the identity in a neighborhood of the boundary of D,*
- *there are a disk $D_0 \subset D$ and a natural number n_0 such that $D_0, f(D_0), \dots, f^{n_0-1}(D_0)$ are pairwise disjoint sets, $f^{n_0}(D_0) \subset \mathrm{int}(D_0)$, and the restriction of f^{n_0} to D_0 is differentially conjugate to some $\phi \in \mathcal{O}$.*

This proposition follows from the following classical result:

Theorem 3.2 *Let $\mathrm{Diff}^1_0(D^3)$ be the group of C^1-diffeomorphisms of the disk D^3 coinciding with the identity in a neighborhood of the boundary ∂D^3 of D^3. Then every diffeomorphism $g \in \mathrm{Diff}^1_0(D^3)$ is isotopic to the identity by a path $g_t \in \mathrm{Diff}^1_0(D^3)$.*

As we did not find a precise reference for this result[3], we will briefly discuss it and show how Theorem 3.2 follows from:

Theorem (J. Cerf, [Ce]) *Any orientation preserving diffeomorphism of the sphere S^3 is isotopic to the identity.*

Sketch of proof of Theorem 3.2: Choose some point $O \in S^3$ and denote by $\mathrm{Diff}^1_0(S^3)$ the set of C^1-diffeomorphisms of S^3 which coincide with the identity in a neighborhood of O. Notice that the blowing down of the boundary ∂D^3 induces a homeomorphism between $\mathrm{Diff}^1_0(D^3)$ and $\mathrm{Diff}^1_0(S^3)$. Consider a diffeomorphism $f \in \mathrm{Diff}^1_0(S^3)$. By the theorem above, it is isotopic to the identity by a path of diffeomorphism k_t in $\mathrm{Diff}^1(S^3)$, $k_0 = f$ and $k_1 = id$. Then $k_t(O)$ is a closed loop in S^3,

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[3] If we did not require the diffeomorphisms to be the identity close to the boundary, it is the result of Milnor:

Theorem ([Mi, Lemma 2, Chapter 6]) *Every orientation preserving diffeomorphism $g: D^3 \to D^3$ is isotopic to the identity.*

homotopic to the trivial loop. Using this fact, it is not hard to verify that there is another isotopy h_t, $h_0 = f$ and $h_1 = id$, such that $h_t(O) = O$ for every t. Then the derivative $Dh_t(O)$ defines a loop in $GL(3, \mathbb{R})$ whose fundamental group is $\mathbb{Z}/2\mathbb{Z}$, generated by "running once around an axis". So, by composing if necessary the path h_t with a loop from identity to identity, one can assume that the loop $Dh_t(O)$ is trivial in $GL(3, \mathbb{R})$. This allows us to build another isotopy g_t, $g_0 = f$ and $g_1 = id$, such that $g_t(O) = O$ and $Dg_t(O) = id_O \in GL(T_O(S^3))$. Considering now a small neighborhood U_ε of O, one gets that g_t is uniformly C^1-close to the identity on U_ε. This allows to crush the diffeomorphisms g_t to get the identity on a small neighborhood of O, getting an isotopy $f_t \in \mathrm{Diff}_0^1(S^3)$, $f_0 = f$ and $f_1 = id$. The result now follows using the homeomorphism between $\mathrm{Diff}_0^1(D^3)$ and $\mathrm{Diff}_0^1(S^3)$ above. □

Proof of the proposition: Observe first that, up to a choice of a diffeomorphism $\psi: D \to D^3$, one can assume that D is equal to D^3.

Fix ϕ in $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$ and choose a diffeomorphism g of D^3 coinciding with the identity in a neighborhood of the boundary of D^3 and such that there is a disk $\hat{D}$ in the interior of D^3 such that the restriction of g to $\hat{D}$ is smoothly conjugate to ϕ.

The previous theorem implies that g can be written as the composition $g_k \circ \cdots \circ g_1 \circ g_0$ of finitely many diffeomorphisms g_i arbitrarily close to the identity.

We first perturb the identity map of D^3 to get a diffeomorphism h equal to identity in a neighborhood of the boundary ∂D^3, and a periodic round disk $D_0 \subset D^3$ of arbitrarily large period $n_0 \geq k$ such that

- $D_0, h(D_0), \ldots, h^{n_0-1}(D_0)$ are pairwise disjoint,
- $h: h^i(D_0) \to h^{i+1}(D_0)$ is an isometry for every $i \in \{0, \ldots, n_0 - 1\}$,
- $h^{n_0}: D_0 \to D_0$ is equal to the identity.

We now write $g = g_{n_0-1} \circ \cdots \circ g_0$, where, for $0 \leq i \leq k$, the g_i are as above and, for $i > k$, we take g_i equal to the identity. Consider an affine bijective map $H: D_0 \to D^3$, and define

$$\tilde{g}_i = H^{-1} \circ g_i \circ H: D_0 \to D_0.$$

Replace now the restriction of h to each disk $h^i(D_0)$ by the map $h^{i+1} \circ \tilde{g}_{i-1} \circ h^{-i}$ and denote by f the resulting diffeomorphism. Since the restrictions of h to the iterates $h^i(D_0)$ are isometries and the maps g_i are C^1-close to identity, the diffeomorphism f is C^1-close to h. Observe that, by construction, the restriction of f^{n_0} to D_0 is

$$f^{n_0} = h^{n_0} \circ \tilde{g}_{n_0-1} \circ \cdots \circ \tilde{g}_0 = H^{-1} \circ g_{n_0-1} \circ \cdots \circ g_0 \circ H = H^{-1} \circ g \circ H,$$

where the second identity follows recalling that the restriction of h^{n_0} to D_0 is the identity. Since g is differentially conjugate to ϕ, it is clear that the disk D_0, the natural number n_0, and the map f verify all properties in the proposition, ending the proof of the proposition. □

3.2 Perturbations of the derivative at a periodic point

Theorem 3.3 *Consider the open subset $\mathcal{W}_0$ of $\mathcal{W}_0(M)$ of diffeomorphisms f having a wild homoclinic class $H(P_f, f) \in W_0(f)$, where the point P_f has index 2 and depends continuously on $f \in \mathcal{W}_0$. Then for every diffeomorphism f in $\mathcal{W}_0$ and every $\varepsilon > 0$ there are:*

- *a hyperbolic periodic point x homoclinically related to P_f (thus $x \in H(P_f, f)$) whose orbit is ε-dense in $H(P_f, f)$,*
- *a small neighborhood V of x and an ε-perturbation g of f along the orbit of x, such that the sets $V, g(V), \ldots, g^{p(x)-1}(V)$ are pairwise disjoint and the restriction of $g^{p(x)}$ to V is the identity map ($p(x)$ is the period of x). Moreover, the orbits of x by f and g coincide.*

Given a periodic point x of f we let

$$J(x, f) = \frac{\log(\mathrm{Det}(Df^{p(x)}(x)))}{p(x)}.$$

For given $\varepsilon > 0$ and $f \in \mathcal{W}_0$, denote by $\Sigma(\varepsilon, f)$ the set of periodic points x homoclinically related to P_f whose orbit is ε-dense in $H(P_f, f)$ and such that $J(x, f) \in (-\varepsilon, \varepsilon)$.

To prove Theorem 3.3 we analyze the linear cocycle defined by the differential of f over the wild homoclinic class $H(P_f, f)$, its restriction to $\Sigma(\varepsilon, f)$, and the perturbations of this cocycle. A lemma of Franks (Lemma 3.5) will allow us to realize such perturbations of the cocycle as C^1-perturbations of f. The main step to prove Theorem 3.3 is now the following proposition:

Proposition 3.4 *Let $\mathcal{W}_0 \subset \mathcal{W}_0(M)$ as in Theorem 3.3. Then for every $f \in \mathcal{W}_0$ and every $\varepsilon > 0$ there are a hyperbolic periodic point x in $\Sigma(\varepsilon, f)$ and an ε-perturbation*

$$A(f^i(x)) \colon T_{f^i(x)}M \to T_{f^{i+1}(x)}M$$

of the derivative of Df along the orbit of x such that the product

$$M_A(x) = A(f^{p(x)-1}) \circ \cdots \circ A(x)$$

is the identity map.

Before proving the proposition, let us prove Theorem 3.3.

Proof of the theorem: We begin by stating the announced lemma of Franks:

Lemma 3.5 ([Fr], [Ma]) *Consider a C^1-diffeomorphism φ and a φ-invariant finite set Σ. Let L be an ε-perturbation of $D\varphi$ along Σ (i.e., the linear maps $L(x)$ and $D\varphi(x)$ are ε-close for all $x \in \Sigma$). Then for every neighborhood U of Σ there is a diffeomorphism ϕ, C^1-ε-close to φ, such that:*

- *$\varphi(x) = \phi(x)$ if $x \in \Sigma$ or if $x \notin U$,*
- *$D\phi(x) = L(x)$ for all $x \in \Sigma$.*

Applying Lemma 3.5 to a diffeomorphism f in $\mathcal{W}_0$, the set $\Sigma = \{x, f(x), \ldots, f^{p(x)-1}(x)\}$ (x as in Proposition 3.4), and the linear map $L = A$ (A is the perturbation of the derivative in Proposition 3.4), we get a perturbation g of f preserving the orbit of x with $Dg^{p(x)}(x) = Id$.

The first part of the theorem follows recalling that, by definition of $\Sigma(\varepsilon, f)$, the orbit of x is ε-dense in $H(P_f, f)$.

For the second part of the theorem, observe that, after a new perturbation of g throughout the orbit of x, we get a neighborhood V of x such that V, $g(V)$ and $g^{p(x)-1}(V)$ are pairwise disjoint and the restriction of $g^{p(x)}$ to V is the identity. Observe that, by construction, the orbits of x by f and g are the same.

The proof of Theorem 3.3 is now complete. □

The proof of Proposition 3.4 is a small variation of the main technical result in [BDP]: *Given any non-trivial homoclinic class H that does not admit any dominated splitting (see Definition 3.8) there are a periodic point $x \in H$ and a perturbation of the derivative of f along the orbit of x such that the product of the perturbed derivatives along this orbit is a homothety,* (see [BDP, Proposition 2.1]). Here we use the fact that the points in $\Sigma(\varepsilon, f)$ have Jacobians close to 1 to get the homothety being the identity. The main difficulty here is to check that the set $\Sigma(\varepsilon, f)$ verifies the hypotheses in [BDP].

To prove Proposition 3.4 we need to state some properties of the sets $\Sigma(\varepsilon, f)$. This will be done in the next two lemmas.

Lemma 3.6 *Consider any $\varepsilon > 0$ and $f \in \mathcal{W}_0$. Then $\Sigma(\varepsilon, f)$ is a dense subset of $H(P_f, f)$.*

Proof: Recall that, by definition of wild homoclinic class in $W_0(f)$, we have that $J(P_f, f) > 1$ and that $H(P_f, f)$ contains a point P'_f homoclinically related to P_f with $J(P'_f, f) < 1$. This implies that for every $\delta > 0$ there is a hyperbolic basic set[4] $H_\delta \subset H(P_f, f)$ containing P_f and P'_f which is δ-dense in $H(P_f, f)$. We choose a Markov partition of H_δ by rectangles of small size such that the variation of $\log(\det(Df(x))$ in each rectangle is less than $\varepsilon/2$.

Now, given any periodic point $x \in H_\delta$, $J(x, f)$ is determined (up to an error of $\varepsilon/2$) by its itinerary. For each n_1 and n_2 (big enough) there is a periodic point x of period $n_1 + n_2 + k$ spending n_1 consecutive iterates in the same rectangles that the iterates of P_f, n_2 consecutive iterates in the rectangle containing the iterates of P'_f, and k iterates passing through all rectangles of the partition, where k is bounded independently of n_1 and n_2. Choosing appropriated n_1 and n_2, we get a point $x \in H_\delta$ such that $J(x, f)$ is ε-close to 0 and whose orbit intersects all the rectangles of the partition. As we can take the size of the rectangles of the partition and δ arbitrarily small, we get that the orbit of x is ε-dense in $H(P_f, f)$. Thus the periodic point x belongs to $\Sigma(\varepsilon, f)$. The same proof gives the density of $\Sigma(\varepsilon, f)$ in $H(P_f, f)$. This ends the proof of the lemma. □

We have the following lemma which follows straightforwardly from the proof of Lemma 3.6:

Lemma 3.7 *Consider any $\varepsilon > 0$ and $f \in \mathcal{W}_0$. Then given any finite subset Ξ of $\Sigma(\varepsilon, f)$ there is a basic transitive set Λ containing Ξ whose periodic points belong to $\Sigma(\varepsilon, f)$.*

Proof: Observe that all the points in the set Ξ are homoclinically related and, for every $x \in \Xi$, $|J(x, f)| < (\varepsilon - \delta)$ for some $\delta > 0$. Thus there is a basic set $\Lambda' \subset H(P_f, f)$ containing Ξ. Consider now a Markov partition of Λ' by sufficiently small rectangles (as in Lemma 3.6, this allows us to control the Jacobians up to an error less than $\delta/2$). Finally, the set Λ corresponds to the points in Λ' whose itineraries are determined by a sub-shift of finite type associated to the rectangles of the partition. As in Lemma 3.6, the choice of the sub-shift is done to have a suitable control of the Jacobians. The proof of the lemma is now complete. □

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[4] A *basic set* is a transitive hyperbolic compact set with *local product structure,* or, equivalently, which is the maximal invariant set in some neighborhood of it.

Let us recall the definition of *dominated splitting.*

Definition 3.8 *A dominated splitting of an f-invariant set Λ is a Df-invariant splitting $E \oplus F$ defined over $T_\Lambda M$ such that the fibers of E and F have constant dimension and there is $k \geq 1$ such that for every $x \in \Lambda$ one has*

$$||Df^k|_{E(x)}|| \cdot ||Df^{-k}|_{F(f^k(x))}|| < \frac{1}{2},$$

that is, the derivative Df expands the vectors in F uniformly more than the vectors in E. Then we say that F dominates *E and write $E \prec F$.*

Lemma 3.9 *Consider any diffeomorphism $f \in \mathcal{W}_0$ and $\varepsilon > 0$. Then the set $\Sigma(\varepsilon, f)$ does not admit any dominated splitting.*

Proof: By Lemma 3.6, the set $\Sigma(\varepsilon, f)$ is dense in $H(P, f)$. Observing that a dominated splitting defined in a set Λ always admits a dominated extension to the closure of Λ (see, for instance, [BDP, Lemma 1.4]), it is enough to see that $H(P_f, f)$ does not admit any dominated splitting for every $f \in \mathcal{W}_0$.

We argue by contradiction, let $f \in \mathcal{W}_0$ and suppose that there is a dominated splitting

$$T_{H(P_f,f)}M = E \oplus F, \quad E \prec F.$$

The fact that P_f has a non-real contracting eigenvalue implies that $\dim(E) = 2$. Similarly, the fact that $Q_f \in H(P_f, f)$ has a non-real expanding eigenvalue implies that $\dim(F) = 2$. Thus $\dim(E) + \dim(F) = 4 > 3 = \dim(M)$, which is a contradiction. □

We can now finish the proof of Proposition 3.4:

Proof of the proposition: Let $f \in \mathcal{W}_0$. Consider the continuous linear cocycle $\mathcal{D}$ induced by Df over $\Sigma(\varepsilon, f)$. By Lemma 3.9, this cocycle does not admit any dominated splitting. As all orbits in $\Sigma(\varepsilon, f)$ are periodic, we have a *continuous linear periodic system,* according to the terminology in [BDP, Section 1.1].

Lemma 3.7 and the proof of [BDP, Lemma 1.9] imply that the cocycle $\mathcal{D}$ admits *transitions,* see [BDP, Definitions 1.6 and 1.8][5]. This implies that the cocycle $\mathcal{D}$ over $\Sigma(\varepsilon, f)$ verifies the hypotheses of [BDP, Proposition 2.1]. So, given any $\delta > 0$, there are a point $x \in \Sigma(\varepsilon, f)$ and a δ-perturbation $\mathcal{B}$ of $\mathcal{D}$ along the orbit of x such that $M_{\mathcal{B}}(x)$ is a homothety (recall that $M_{\mathcal{B}}(x) = B(f^{p(x)-1}) \circ \cdots \circ B(x))$. Moreover, if δ is small enough,

$$J(x, \mathcal{B}) = \frac{\log(\det(M_B(x)))}{p(x)} \in [-2\varepsilon, 2\varepsilon].$$

By multiplying B along the orbit of x by a number λ close to 1 (in fact, $\lambda = e^{-|J(x,\mathcal{B})|/3}$, where $3 = \dim(M)$) we get a new perturbation $\mathcal{A}$ of $\mathcal{D}$ along the orbit of x such that $M_A(x) = Id$. This ends the proof of the proposition. □

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[5] The precise definition of transitions is somewhat technical and we do not use explicitly it here. This notion is a formalization of the following intuitive idea: consider a finite set X of periodic orbits of a hyperbolic basic set Λ. Then, there are periodic points of Λ whose orbits visit a small neighborhood of some orbit of X, remain there an arbitrarily large time and go (in a uniformly bounded time) to the neighborhood of another orbit of X, remain there an arbitrarily large time, and so on, until the period. The derivatives at these orbits are essentially products of the derivatives at the orbits of X with some correcting terms, which are bounded matrices called *transitions* and correspond to the itineraries out of a small neighborhood of X. Here Lemma 3.7 asserts that any finite set of $\Sigma(\varepsilon, f)$ is contained in a basic set whose periodic orbits are in $\Sigma(\varepsilon, f)$, providing the existence of transitions.

3.3 End of the proof of Theorem B

Given an open subset $\mathcal{O}$ of $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$ and $\varepsilon > 0$ let $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ be the subset of $\mathcal{W}_0$ ($\mathcal{W}_0$ is the open subset of $\mathcal{W}_0(M)$ in Theorem 3.3) of diffeomorphisms f such that there are a neighborhood $\mathcal{U}_f$ of f in $\mathcal{W}_0$, a disk D, and a natural number $n > 0$, such that every $g \in \mathcal{U}_f$ satisfies the following properties:

($\mathcal{P}$1) the iterates $g^i(D)$, $i = 0, \dots, n-1$, are pairwise disjoint,

($\mathcal{P}$2) $g^n(D)$ is contained in the interior of D,

($\mathcal{P}$3) for every point $x \in H(P_g, g)$ there is $i \in \{0, \dots, n-1\}$ such that $g^i(D)$ is contained in the ball $B(x, \varepsilon)$,

($\mathcal{P}$3') the iterates $g^i(D)$, $i = 0, \dots, n-1$, are contained in the ε-neighborhood of $H(P_g, g)$,

($\mathcal{P}$4) the diameters of the family of disks $\{g^i(D)\}_{i=-n}^{n-1}$ are upper bounded by $2\,\varepsilon$,

($\mathcal{P}$5) the restriction of g^n to D is differentially conjugate to some $\varphi \in \mathcal{O}$.

We denote by $\mathcal{P}^-(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ the set of diffeomorphisms f in $\mathcal{W}_0$ whose inverse f^{-1} belongs to $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$.

Remark 3.10

1. *The sets $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ and $\mathcal{P}^-(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ are both open (even if the property $\mathcal{P}3$ is not open): by definition, any $f \in \mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ admits a neighbourhood $\mathcal{U}_f$ of diffeomorphisms verifying ($\mathcal{P}$1–$\mathcal{P}$5), thus $\mathcal{U}_f \subset \mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$.*

   A priori, these sets may be empty. In the next lemma we will see that they are dense in $\mathcal{W}_0$.

2. *For every $f \in \mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$, there is a neighborhood $\mathcal{V}$ of f where the natural number n is constant, and the disk D and the conjugacy between the restriction of g^n to D and some element of $\mathcal{O} \subset \mathrm{Diff}^+_{\mathrm{int}}(D^3)$ can be chosen depending continuously on g. We call this conjugacy the* identification *between D and D^3.*

Lemma 3.11 *The sets $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ and $\mathcal{P}^-(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ are open and dense in $\mathcal{W}_0$ for every $\varepsilon > 0$ and every open set $\mathcal{O} \subset \mathrm{Diff}^+_{\mathrm{int}}(D^3)$.*

Proof: We prove the lemma for $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$, the proof for $\mathcal{P}^-(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ is the same. By Remark 3.10, it is enough to prove the density of $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ in $\mathcal{W}_0$.

Observe that the map $g \mapsto H(P_g, g)$ is lower semi-continuous. So there is a residual subset $\mathcal{S}_1 \subset \mathcal{W}_0$ where this map is continuous. Thus to prove the lemma it is enough to see that given any $f \in \mathcal{S}_1$ there is an arbitrarily small perturbation of f in $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$.

Given $f \in \mathcal{S}_1$, consider $\varepsilon_0 > 0$ such that, for every diffeomorphism g ε_0-close to f, the Hausdorff distance between the homoclinic classes $H(P_f, f)$ and $H(P_g, g)$ is less than $\varepsilon/10$. By Theorem 3.3, given any $0 < \varepsilon_1 < \inf\{\varepsilon_0/10, \varepsilon/10\}$, there are

- a hyperbolic periodic point $y \in H(P_f, f)$ homoclinically related to P_f whose orbit is ε_1-dense in $H(P_f, f)$,

- an ε_1-perturbation f_1 of f along the orbit of y and a neighborhood V of y such that $V, f_1(V), \dots, f_1^{p(y)-1}(V)$ are pairwise disjoint, the restriction of $f_1^{p(y)}$ to V is the identity, and the orbits of y by f and f_1 coincide.

Applying Proposition 3.1 to a small compact ball D contained in V and the open set $\mathcal{O}$, we get a ε_1-perturbation F of the identity in the disk D that can be extended to the identity outside the disk D, a disk $D_0 \subset D$ (with diameter less than ε_1) and k, such that the map $f_2 = F \circ f_1$ and the natural number $m = p(y) \cdot k$ satisfy the following properties:

(d1) the disks $\{f_2^i(D_0)\}_{i=0}^{m-1}$ are pairwise disjoint,

(d2) $f_2^m(D_0) \subset \operatorname{int}(D_0)$ and the restriction of f_2^m to D_0 is differentially conjugate to an element of $\mathcal{O}$ (recall that $f_1^{p(y)}$ is the identity in D, so $f_2^{p(y)}$ coincides with F in D),

(d3) the disk $f_2^i(D_0)$ is contained in the interior of the ball $B(y_i, \varepsilon_1)$ of radius ε_1 centered at y_i, where $y_i = f_1^i(y) = f^i(y)$ and $i \geq 0$,

(d4) the diameters of the family of disks $\{f_2^i(D_0)\}_{i=-m}^{m-1}$ are strictly upper bounded by $2\operatorname{diam}(D_0) = 2\varepsilon_1$.

Observe that conditions (d1), (d2), (d3) and (d4) are open ones.

The previous assertion implies that there is a non-empty open set $\mathcal{O}^1$ such that every $f_2 \in \mathcal{O}^1$ is an ε_1-perturbation of f_1 and verifies (d1)–(d4) above, where in these conditions the disk D_0, the natural number m, and the points y_i do not depend on the diffeomorphisms, that is, the disks $\{f_2^i(D_0)\}_{i=0}^{m-1}$ are pairwise disjoint, $f_2^m(D_0) \subset \operatorname{int}(D_0)$, the restriction of f_2^m to D_0 is differentially conjugate to an element of $\mathcal{O}$, and $f_2^i(D_0)$ is contained in the interior of $B(y_i, \varepsilon_1)$, $i \in \{0, 1, \dots, m-1\}$.

Let us state a remark about the disk D_0 we will use later.

Remark 3.12 *The orbit of the disk D_0 is disjoint from $H(P_{f_2}, f_2)$.*

This remark follows by observing that the point P_{f_2} does not belong to the segment of orbit $V, f_2(V), \dots, f_2^{p(y)-1}(V)$ and that the disk D_0 is strictly forward f_2-invariant and contained in $V \cup f_2(V) \cup \dots \cup f_2^{p(y)-1}(V_2)$.

To conclude the proof of the lemma it is enough to check that every $f_2 \in \mathcal{O}^1$ is an ε_0-perturbation of f and belongs to $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$. The first assertion follows immediately observing that f_2 is a $(3\varepsilon_1)$-perturbation of f and that $\varepsilon_1 < \varepsilon_0/10$.

For the second one, observe that, by construction, f_2 verifies ($\mathcal{P}$1), ($\mathcal{P}$2), ($\mathcal{P}$4) and ($\mathcal{P}$5) in the definition of $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$. So it remains to see that f_2 also verifies ($\mathcal{P}$3) and ($\mathcal{P}$3').

Let us show that it verifies ($\mathcal{P}$3) that is, given any point $x_2 \in H(P_{f_2}, f_2)$ there is $i \in \{0, 1, \dots, m-1\}$ such that $f_2^i(D_0) \subset B(x_2, \varepsilon)$.

To see why this is so observe that, by the choice of ε_0, the Hausdorff distance between $H(P_{f_2}, f_2)$ and $H(P_f, f)$ is less than $\varepsilon/10$. Thus, given any $x_2 \in H(P_{f_2}, f_2)$ there is $x \in H(P_f, f)$ with $d(x_2, x) < \varepsilon/10$. Hence, by the choice of y (the f-orbit of y is ε_1-dense in $H(P_f, f)$ and is equal to the f_1-orbit), there is some i such that $f^i(y) = f_1^i(y) = y_i \in B(x_2, \varepsilon_1 + \varepsilon/10)$. Therefore,

$$B(y_i, \varepsilon_1) \subset B(x_2, 2\varepsilon_1 + \varepsilon/10) \subset B(x_2, \varepsilon).$$

Finally, by condition (d3) above, there is i such that

$$f_2^i(D_0) \subset B(y_i, \varepsilon_1) \subset B(x_2, \varepsilon).$$

In the same way, the property ($\mathcal{P}$3') follows from the lower semi continuity of the homoclinic class $H(P_g, g)$ and from property (d3).

Since the previous argument holds for every sufficiently small $\varepsilon_0 > 0$, the proof of the density of $\mathcal{P}(\mathcal{O}, \varepsilon, \mathcal{W}_0)$ in $\mathcal{W}_0$ is complete. This also ends the proof of the lemma. □

Consider now a countable basis $\mathcal{O}_n$ of the topology of $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$ and a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to 0$ as $n \to +\infty$. Define the set

$$\mathcal{U}_0 = \bigcap_{i,j \in \mathbb{N}} (\mathcal{P}(\mathcal{O}_i, \varepsilon_j, \mathcal{W}_0) \cap \mathcal{P}^-(\mathcal{O}_i, \varepsilon_j, \mathcal{W}_0)).$$

By Lemma 3.11, the set $\mathcal{U}_0$ is a countable intersection of dense open subsets of $\mathcal{W}_0$, thus it is a residual subset of $\mathcal{W}_0$.

The first step of the proof of Theorem B is to see that it holds for one fixed wild homoclinic class $H(P_f, f) \in \mathcal{W}_0(f)$. This is a consequence of the next lemma:

Lemma 3.13 *Every $f \in \mathcal{U}_0$ has universal dynamics at $H(P_f, f)$ (recall Definition 1.3).*

Proof: Given $f \in \mathcal{U}_0$ denote by $D^+_{i,j}$ and $m^+_{i,j}$ (resp. $D^-_{i,j}$ and $m^-_{i,j}$) the disk and the natural number (period of the disk) associated to f by the condition $f \in \mathcal{P}(\mathcal{O}_i, \varepsilon_j, \mathcal{W}_0)$ (resp. $f \in \mathcal{P}^-(\mathcal{O}_i, \varepsilon_j, \mathcal{W}_0)$), and so verifying the conditions ($\mathcal{P}$1)–($\mathcal{P}$5).

Claim 1 *There are two sequences $j(i)$ and $k(i)$ converging to $+\infty$as $i \to +\infty$ and satisfying the following property:*

- *given any integer $i \in \mathbb{N}$ denote by δ_{2i} and δ_{2i+1} the disks $D^+_{i,j(i)}$ and $D^-_{i,k(i)}$, respectively, and consider the unions $\Delta_{2i} = \bigcup_{n=0}^{+\infty} f^n(\delta_{2i})$ and $\Delta_{2i+1} = \bigcup_{n=0}^{+\infty} f^{-n}(\delta_{2i+1})$. Then the sets Δ_ℓ, $\ell \in \mathbb{N}$, are pairwise disjoint.*

Before proving the claim, let us conclude the proof of Lemma 3.13. Consider any open set $\mathcal{O}$ of $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$ and any point $x \in H(P_f, f)$. As the set $\mathcal{O}_i$ are a basis of the topology of $\mathrm{Diff}^+_{\mathrm{int}}(D^3)$, there is an infinite sequence $i_n \to +\infty$ such that $\mathcal{O}_{i_n} \subset \mathcal{O}$. Property ($\mathcal{P}$3) implies that, for any integer i, the disk δ_{2i} has a positive iterate D_{2i} contained in the ball $B(x, \varepsilon_{j(i)})$ and the disk δ_{2i+1} has a negative iterate D_{2i+1} contained in the ball $B(x, \varepsilon_{j(i)})$. Now one easily verifies that the sequence D_k, $k \in \mathbb{N}$, verifies all the conditions (U$_1$)–(U$_4$) in the definition of the universal dynamics, concluding the proof of the lemma.

We now prove the claim. By Remark 3.12, the homoclinic class $H(P_f, f)$ and the forward f-orbit of $D^+_{i,j}$ are compact disjoint sets, thus there is $\mu^+_{i,j} > 0$ such that the distance between $H(P_f, f)$ and $\bigcup_{k \geq 0} f^k(D^+_{i,j})$ is greater than $2\,\mu^+_{i,j}$. We argue analogously with $H(P_f, f)$ and the backward f-orbit of $D^-_{i,j}$, obtaining a positive lower bound $2\,\mu^-_{i,j}$ for the distance between $H(P_f, f)$ and $\bigcup_{k \leq 0} f^k(D^-_{i,j})$.

We build the sequences $j(i)$ and $k(i)$ inductively. Assume that $j(0), k(0), \dots, j(i), k(i)$ are defined, we define $j(i+1)$ and $k(i+1)$ as follows: we choose $j(i+1)$ such that

$$\varepsilon_{j(i+1)} < \inf\{\mu^+_{\ell, j(\ell)}, \mu^-_{\ell, k(\ell)}, \ell = 0, \dots, i\}.$$

Property ($\mathcal{P}3$') asserts that the union of the positive iterates of the disk $D^+_{i,j(i+1)}$ is contained in the $\varepsilon_{j(i+1)}$ neighborhood of $H(P_f,f)$ so that $\Delta_{2(i+1)}$ is disjoint from the Δ_ℓ, $\ell \leq 2i+1$. In the same way, we choose $k(i+1)$ such that

$$\varepsilon_{k(i+1)} < \inf\{\mu^+_{i+1,j(i+1)}, \inf\{\mu^+_{\ell,j(\ell)}, \mu^-_{\ell,k(\ell)}, \ell = 0,\dots,i\}\}.$$

This concludes the proof of the claim. □

To end the proof of Theorem B we need to get the universal property for all the wild homoclinic classes in $\mathcal{W}_0(f)$ (f in a residual subset of $\mathcal{W}_0(M)$). For that consider a countable basis $\mathcal{G}_n$ of the set of wild diffeomorphisms of $\mathcal{W}_0(M)$ and, for each n, consider all the pairs $(\mathcal{G}_n, P\colon \mathcal{G}_n \to M)$ where P is a continuous map associating to each f a hyperbolic periodic point $P(f)$ of f with a wild homoclinic class. Observe that this set of pairs is countable. Thus we can index them in the form $\mathcal{Y}_i = (\mathcal{G}_{n(i)}, P_i)$.

By Lemma 3.13, there is a residual subset $\mathcal{Z}_i$ of $\mathcal{G}_{n(i)}$ such that every $f \in \mathcal{Z}_i$ has universal dynamics at $H(P_i(f), f)$.

Consider now $\mathcal{V}_i = \mathcal{Z}_i \cup (\mathcal{W}_0(M) \setminus \mathcal{G}_{n(i)})$. This set is residual in $\mathcal{W}_0(M)$. Then $\mathcal{V} = \bigcap_{i\in\mathbb{N}} \mathcal{V}_i$ is residual in $\mathcal{W}_0(M)$ and any $f \in \mathcal{V}$ has universal dynamics at each wild homoclinic class of $\mathcal{W}_0(f)$. This completes the proof of Theorem B. □

4 Aperiodic maximal transitive Cantor sets

In this section we prove Theorem A.

4.1 Notations

By definition, every $f \in \mathcal{W}_0(M)$ has a saddle P_f such that $H(P_f,f)$ is a wild homoclinic class in $\mathcal{W}_0(f)$. Moreover, for every $f \in \mathcal{W}_0(M)$ there is a neighborhood where the points P_g depend continuously on g. Using the metrizability of $\mathrm{Diff}^1(M)$, one gets a dense open subset $\mathcal{W}_1(M) \subset \mathcal{W}_0(M)$ and a continuous function $f \mapsto P_0(f)$ defined on $\mathcal{W}_1(M)$ such that $H(P_0(f),f) \in \mathcal{W}_0(f)$ for all $f \in \mathcal{W}_1(M)$.

We denote by $\mathcal{W}_0(D^3) \subset \mathrm{Diff}^+_{\mathrm{int}}(D^3)$ the subset of diffeomorphisms f having a wild homoclinic class in $\mathcal{W}_0(f)$ and, as above, consider a dense open subset $\mathcal{W}_1(D^3)$ of $\mathcal{W}_0(D^3)$ of diffeomorphisms with a point $P(f)$ depending continuously on $f \in \mathcal{W}_1(D^3)$ and having a wild homoclinic class in $\mathcal{W}_0(f)$.

Given $\varepsilon > 0$, consider now the sets $\mathcal{P}(M,\varepsilon) = \mathcal{P}(\mathcal{W}_1(D^3), \varepsilon, \mathcal{W}_1(M))$ defined as in Section 3.3 and associated to the open sets $\mathcal{O} = \mathcal{W}_1(D^3)$ and $\mathcal{W}_0 = \mathcal{W}_1(M)$. By Lemma 3.11, the set $\mathcal{P}(M,\varepsilon)$ is open and dense in $\mathcal{W}_1(M)$ for every $\varepsilon > 0$.

We observed in Remark 3.10 that the disks $D_1(f)$ and the natural numbers $k_1(f)$ verifying ($\mathcal{P}1$)–($\mathcal{P}5$), as well as the identification of $D_1(f)$ with D^3 conjugating $f^{k_1(f)}$ to some $g_f \in \mathcal{W}_1(D^3)$, can be chosen varying locally continuously. Moreover, as the restriction of $f^{k_1(f)}$ to $D_1(f)$ is conjugate to some $g_f \in \mathcal{W}_1(D^3)$ (up to the identification $D_1(f) = D^3$), f has a periodic point $P_1(f)$ corresponding to $P(g_f)$, depending continuously on f and having a wild homoclinic class in $\mathcal{W}_0(f)$. So there is a dense open subset $\mathcal{Q}_1(M,\varepsilon) \subset \mathcal{P}(M,\varepsilon)$ and a continuous function Ψ_1 defined on $\mathcal{Q}_1(M,\varepsilon)$ by

$$f \mapsto \Psi_1(f) = (D_1(f), k_1(f), P_1(f)),$$

where

- $D_1(f)$ and $k_1 = k_1(f) > 1$ are the disk and the natural number given by the definition of $\mathcal{P}(M,\varepsilon)$, satisfying $(\mathcal{P}1)$–$(\mathcal{P}5)$, and such that the restriction of f^{k_1} to the disk $D_1(f)$ is differentially conjugate to some $g \in \mathcal{W}(D^3)$, recall condition $(\mathcal{P}5)$. Observe that, by $(\mathcal{P}4)$, the diameters of the disks $f^i(D_1(f))$, $i \in \{-k_1(f), \dots, k_1(f)\}$, are upper bounded by $2\,\varepsilon$.
- $P_1(f) \in D_1(f)$ is a point with a wild homoclinic class in $W_0(f)$.

We define analogously the set $\mathcal{Q}_1^+(D^3,\varepsilon) \subset \mathcal{W}_0(D^3)$ and the maps $D_1(f), k_1(f)$ and $P_1(f)$. We define $\mathcal{Q}_1^-(D^3,\varepsilon)$ as the set of diffeomorphisms f such that f^{-1} belongs to $\mathcal{Q}_1^+(D^3,\varepsilon)$.

4.2 The inductive process

For starting our construction we fix a sequence $\varepsilon_i > 0$, $i \in \mathbb{N}^*$, converging to 0 as $i \to \infty$.

Let $\mathcal{Q}_1 = \mathcal{Q}_1(M,\varepsilon_0)$. Notice that for any $f \in \mathcal{Q}_1$ there is $\varepsilon_1(f)$ such that for any disk Δ' of diameter less than $2\,\varepsilon_1(f)$ in D^3 one has the the following property: *denote by $\Delta \subset D_1(f)$ the disk corresponding to Δ' via the identification, then the iterates $f^i(\Delta)$, $i \in \{-k_1(f), \dots, k_1(f)\}$, have diameters bounded by ε_1.*

As the identification of $D_1(f)$ with D^3 depends continuously on f, the number $\varepsilon_1(f)$ can be chosen locally independent of f. So there is a dense open subset $\mathcal{R}_1$ of $\mathcal{Q}_1$ where $\varepsilon_1(f)$ is locally constant.

Define the set $\mathcal{Q}_2$ as the subset of $\mathcal{R}_1$ of diffeomorphisms f such that the restriction of $f^{k_1(f)}$ to $D_1(f)$ is identified to some element $g_f \in \mathcal{Q}_1^-(D^3,\varepsilon_1(f))$. Remark that, by Lemma 3.11, the set $\mathcal{Q}_2$ is open and dense in $\mathcal{R}_1$. Consider $f \in \mathcal{Q}_2$ and let

- $D_2(f)$ be the disk identified with $D_1(g_f)$,
- $k_2(f) = k_1(f) \cdot k_1(g_f) \geq 2 \cdot k_1(f)$,
- $P_2(f) \in D_2(f)$ be the point identified with $P_1(g_f)$ (with a wild homoclinic class in $W_0(g_f)$).

Lemma 4.1 *The diameter of $f^i(D_2(f))$ is upper bounded by ε_1 for every $i \in \{-k_2(f), \dots, k_2(f)\}$ and every $f \in \mathcal{Q}_2$.*

Proof: Write $i = j + l \cdot k_1(f)$, with $j \in \{-k_1(f), \dots, k_1(f)\}$ and $l \in \{-k_2(g_f), \dots, k_2(g_f)\}$. The disk $g^l(D_1(g_f))$ has diameter less than $2\,\varepsilon_1(f)$ (recall$(\mathcal{P}3)$), and the corresponding disk (via the identification) in $D_1(f)$ is $f^{l\cdot k_1(g_f)}(D_2(f))$. The choice of $\varepsilon_1(f)$ now implies that $f^j(f^{l\cdot k_1(g_f)}(D_2(f)))$ has diameter less than ε_1, finishing the proof of the lemma. □

We now construct inductively nested sequences $\mathcal{Q}_n$ of subsets of $\mathcal{W}_0(M)$ (with continuous functions $D_n(f), k_n(f)$ and $P_n(f)$) and $\mathcal{R}_n \subset \mathcal{Q}_n$ (open and dense in $\mathcal{Q}_n$ and with a locally constant function $\varepsilon_n(f)$) in the following way:

- $\mathcal{Q}_{n+1}$ is the subset of $\mathcal{R}_n$ of diffeomorphisms f such that the restriction of $f^{k_n(f)}$ to $D_n(f)$ is identified (conjugate) with some $g_f \in \mathcal{Q}_1^{\pm}(D^3,\varepsilon_n(f))$, where ± is the sign of $(-1)^n$.

- Thereafter we construct $\mathcal{R}_{n+1} \subset \mathcal{Q}_{n+1}$, open and dense in $\mathcal{Q}_{n+1}$, with a locally constant function ε_{n+1} such that for every disk Δ' of diameter less than $\varepsilon_{n+1}(f)$ in D^3 the corresponding disk $\Delta \subset D_{n+1}(f)$ via the identification is such that the iterates $f^i(\Delta)$, $i \in \{-k_{n+1}(f), \dots, k_{n+1}(f)\}$, have diameters bounded by ε_{n+1}.

By construction, the sets and the functions above satisfy the following properties:

1. $\mathcal{Q}_n$ is open and dense in $\mathcal{Q}_{n-1}$, so it is also open and dense in $\mathcal{W}_0(M)$,
2. $k_{n+1}(f) = m_n \cdot k_n(f)$ for some $m_n \geq 2$,
3. the diameters of $f^{-k_n(f)}(D_n(f)), \dots, f^{k_n(f)}(D_n(f))$ are upper bounded by ε_{n-1}, in particular, these diameters uniformly go to 0 as $n \to \infty$,
4. if n is even then the disks $D_n(f), f(D_n(f)), \dots, f^{k_n(f)-1}(D_n(f))$ are pairwise disjoint and $f^{k_n(f)}(D_n(f)) \subset \mathrm{int}(D_n(f))$,
5. if n is odd then the disks $D_n(f), f^{-1}(D_n(f)), \dots, f^{-k_n(f)+1}(D_n(f))$ are pairwise disjoint and $f^{-k_n(f)}(D_n(f)) \subset \mathrm{int}(D_n(f))$,
6. for every even (resp. odd) number n the positive (resp. negative) f-orbit of $D_n(f)$ is contained in the negative (resp. positive) orbit of $D_{n-1}(f)$.

Consider now the residual subset $\mathcal{R} = \bigcap_{n\geq 1} \mathcal{Q}_n$ of $\mathcal{W}_0(M)$. For each $f \in \mathcal{R}$ define the set

$$\Phi(f) = \bigcap_{n\in\mathbb{N}} \bigcup_{0}^{k_{2n}(f)} f^i(D_{2n}(f)).$$

Theorem A follows from the next proposition:

Proposition 4.2 *Consider f in the residual subset $\mathcal{R}$ of $\mathcal{W}_0(M)$. Then the following holds:*

- *$\Phi(f)$ is an f-invariant Cantor set.*
- *The forward orbit of any point $x \in \Phi(f)$ is dense in $\Phi(f)$, that is, f is minimal on $\Phi(f)$ (in particular, $\Phi(f)$ is transitive and aperiodic).*
- *$\Phi(f)$ is a maximal transitive set and it is saturated.*
- *$\Phi(f)$ is Lyapunov stable for f and f^{-1}.*
- *$\Phi(f)$ is uniquely ergodic.*

Proof: The majoration by ε_{n-1} of the diameters of the disks $f^i(D_n(f))$, $i = -k_n, \dots, k_n(f)$, implies that $\Phi(f)$ is totally disconnected. The fact that $D_n(f)$ contains $m_n > 1$ disks of the orbit of D_{n+1} implies that the set $\Phi(f)$ has no isolated points. Finally, it is compact as intersection of compact sets. Therefore the set $\Phi(f)$ is a Cantor set.

Observe that, for every n, the set $\bigcup_0^{k_{2n}(f)-1} f^i(D_{2n}(f))$ (resp. $\bigcup_0^{k_{2n+1}(f)-1} f^{-i}(D_{2n+1}(f))$) is a forward (resp. backward) invariant (ε_{2n-1})-neighborhood (resp. (ε_{2n})-neighborhood) of $\Phi(f)$, thus the set $\Phi(f)$ is f-invariant and Lyapunov stable for f and f^{-1}.

Observe that, by construction, the set $\Phi(f)$ does not contain periodic points: each orbit of $\Phi(f)$ meets any disk $D_n(f)$, whose $k_n(f)-1$ first iterates are disjoint, thus its period is at least $k_n(f)$ for all n. As $k_n(f) \to +\infty$, there is no periodic points in $\Phi(f)$. Moreover, by construction, each orbit of $\Phi(f)$ meets each component of $\bigcup_0^{k_{2n}(f)-1} f^i(D_{2n}(f))$, implying that each orbit of $\Phi(f)$ is (ε_{2n-1})-dense for every n. Since $\varepsilon_n \to 0$, this implies the density of any forward orbit (the proof for backward orbits is analogous). So the restriction of f to $\Phi(f)$ is minimal and aperiodic.

The Lyapunov stability of $\Phi(f)$ implies that $\Phi(f)$ is maximal and saturated.

By construction, the time average of any orbit of $\Phi(f)$ in a disk $f^i(D_n(f))$ is equal to $\frac{1}{k_n(f)}$. One deduces from this fact that given any continuous function $\theta\colon \Phi(f) \to \mathbb{R}$, its Birkhoff sums along two orbits converge to the same number. Thus $\Phi(f)$ supports a unique invariant measure, which is necessarily uniquely ergodic. The proof of the proposition is now complete. □

5 Proof of Corollary C

To prove Corollary C, observe that the property of having sinks, sources, non-trivial hyperbolic or partially hyperbolic robustly transitive attractors or repellors, homoclinic classes containing persistently points of different indices, and wild homoclinic classes in $W_0(f)$ are robust properties, and that to exhibit aperiodic maximal transitive set is a locally generic property, recall Theorem A. For the independent saddles, we can guarantee their robust existence using filtrations.

The unique difficulty to prove the corollary is to verify that these phenomena may appear in D^3: that is clear for sinks, sources and independent saddles. In [BD$_2$] it is explained how to get a wild homoclinic class in $W_0(f)$, see also the appendix (Section 6). For hyperbolic attractors of dimension 1 one can consider, for example, the product of a Plykin attractor in D^2 by a transverse contraction, or the Smale's solenoid (as far as we know, it is not possible to have 2-dimensional hyperbolic attractors in D^3). For partially hyperbolic attractors, we consider the map defined on the solid torus $D^2 \times S^1$ obtained as the product of a Plykin attractor in D^2 by the identity map in S^1. In [BD$_1$] it is proved that this map can be perturbed to get a robustly transitive partially hyperbolic (nonhyperbolic) attractor.

By Remark 1.2, the existence of wild homoclinic class in D^3 implies the local genericity of diffeomorphisms of D^3 having aperiodic maximal transitive sets.

The heuristic principle now implies that these phenomena appear generically infinitely many times in $\mathcal{W}_0(M)$.

Finally, observe that all the aperiodic maximal transitive sets in our construction are contained in the closure of the union of infinitely many wild homoclinic classes, and that the universal dynamics at the wild homoclinic classes allow us to get (generically) the other inclusion.

6 Appendix

In this section we sketch the construction of wild homoclinic classes in $W_0(f)$ whose existence was claimed in Section 1.1. This construction is a minor modification of the one in [BD$_2$, Section3.2] involving blenders. Let us sketch this construction in the simplest situation.

In rough terms, a blender of a diffeomorphism f is a cube Γ endowed with f-invariant cone fields C^s, C^u and C^{uu}, $C^{uu} \subset C^u$, such that for every point $x \in \Gamma$ with $f(x) \in \Gamma$ the derivative of f uniformly contracts the vectors in C^s and uniformly expands the vectors in C^u. We assume that the cube Γ contains a hyperbolic fixed saddle Q of index 1 whose local stable manifold transversely meets the two sides of Γ *parallel* to the unstable cone field. Using the cone field C^{uu}, one defines vertical segments through Γ as curves joining the *top* and the *bottom* faces of the cube and whose tangent vectors are in C^u. Using the manifold $W^s_{loc}(Q)$, one speaks of vertical segments at the right and at the left of $W^s_{loc}(Q)$. Using this terminology, we say that (Γ, f) is a blender associated to Q if the closure of the stable manifold of Q intersects every vertical segment through Γ at the (say) right of Q.

Suppose now that (Γ, f) is a blender associated to a saddle Q of index 1 as above and that there is a hyperbolic fixed saddle P of index 2 whose one-dimensional unstable manifold contains a vertical segment through Γ at the right of $W^s_{loc}(Q)$. Then we say that P *activates* the blender (Γ, f). In such a case, as a consequence of the λ-lemma, the whole stable manifold of P is contained in the closure of $W^s(Q)$. Moreover, if $W^s(P)$ and $W^u(Q)$ have a nonempty transverse intersection, the whole homoclinic class of P is contained in Q.

We observe that to have a blender is a C^1-open property, this mainly follows since the existence of the cone fields as above is C^1-persistent and compact parts of invariant manifolds depend continuously on the diffeomorphism.

A simple way to get blenders is to consider a pair of saddles P and Q as above related by a heterodimensional cycle which is far from tangencies. Recall that a diffeomorphism f has heterodimensional cycle associated to P and Q if $W^u(P)$ and $W^s(Q)$ meet quasi-transversely along an orbit and $W^s(P)$ and $W^u(Q)$ have a non-empty transverse intersection. This cycle is far from tangencies if every diffeomorphism C^1-close to f has no homoclinic tangencies associated to P or Q. For this result see [DR]. In fact, in [DR] it is proved that, if f has a cycle far from tangencies associated to P and Q there is an open set $\mathcal{B}$ of diffeomorphisms, $f \in \overline{\mathcal{B}}$, such that for every $g \in \mathcal{B}$ there are blenders (Γ, f) associated to Q and activated by P, and (Γ, f^{-1}) associated to P and activated by Q. Using the first blender one has $H(P_g, g) \subset H(Q_g, g)$, and the second one gives $H(Q_g, g) \subset H(P_g, g)$ for all $g \in \mathcal{B}$.

The previous construction is semi-local and it only involves the separatrices of $W^s(Q)$ and of $W^u(P)$ involved in the cycle. Thus it can be carried out satisfying also the following.

- The saddle P is homoclinically related to a saddle P' having a pair of contracting non-real eigenvalues and such that the Jacobian of f at P' is bigger than one (for that it is enough to take P' having a big expansion in the unstable direction).
- The saddle Q is homoclinically related to a saddle Q' having a pair of expanding non-real eigenvalues and such that the Jacobian of f at Q' is less than one.

Since these conditions are C^1-open, after shrinking $\mathcal{B}$ if necessary, we have that, for every $g \in \mathcal{B}$,

$$H(P'_g, g) = H(P_g, g) = H(Q_g, g) = H(Q'_g, g).$$

Thus the homoclinic class $H(P_g, g) = H(Q_g, g)$ is wild and belongs to $W_0(g)$ in the sense of Definition 1.1. This ends our sketch of the construction of wild homoclinic classes.

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