A C^1−generic dichotomy for diffeomorphisms: Weak forms of hyperbolicity or infinitely many sinks or sources

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A Ricardo Mañé (1948-1995), por todo su trabajo

Abstract

We show that, for every compact n-dimensional manifold, $n \geq 1$, there is a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms for which the homoclinic class of any periodic saddle of f verifies one of the following two possibilities: Either it is contained in the closure of an infinite set of sinks or sources (Newhouse phenomenon), or it presents some weak form of hyperbolicity called *dominated splitting* (this is a generalization of a bidimensional result of Mañé [Ma$_3$]). In particular, we show that any C^1−robustly transitive diffeomorphism admits a dominated splitting.

Résumé

Généralisant un résultat de Mañé sur les surfaces [Ma$_3$], nous montrons que, en dimension quelconque, il existe un sous-ensemble résiduel de $\mathrm{Diff}^1(M)$ de difféomorphismes pour lesquels la classe homocline de toute selle périodique hyperbolique possède deux comportements possibles : ou bien elle est incluse dans l'adhérence d'une infinité de puits ou de sources (phénomène de Newhouse), ou bien elle présente une forme affaiblie d'hyperbolicité appelée une *décomposition dominée*. En particulier nous montrons que tout difféomorphisme C^1−robustement transitif possède une décomposition dominée.

Introduction

Context

The Anosov-Smale theory of uniformly hyperbolic systems has played a double role in the development of the qualitative theory of dynamical systems. On one hand, this theory shows that chaotic and random behavior can appear in a stable way for deterministic systems depending on a very small number of parameters. On the other hand, the chaotic

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systems admit in this theory a quasi-complete description from the ergodic point of view, moreover the hyperbolic attractors satisfy very simple statistical properties (see [Si], [Ru], and [BoRu]): For Lebesgue almost every point in the topological basin of the attractor, the time average of any continuous function along its orbit converges to the spatial average of the function by a probability measure whose support is the attractor.

However, since the end of the 60s, one knows that this hyperbolic theory does not cover a dense set of dynamics: There are examples of open sets of non-hyperbolic diffeomorphisms. More precisely,

- For every compact surface S there exist non-empty open sets of $\mathrm{Diff}^2(S)$ of diffeomorphisms whose non-wandering set is not hyperbolic (see [N_1]).
- Given any compact manifold M, with $\dim(M) \geq 3$, there are non-empty open subsets of $\mathrm{Diff}^1(M)$ of diffeomorphisms whose non-wandering set is not hyperbolic (see, for instance, [AS] and [So] for the first examples).

In the 2−dimensional case, at least in the C^1-topology, the heart of this phenomenon is closely related with the appareance of homoclinic tangencies: For every compact surface S the set of C^1-diffeomorphisms with homoclinic tangencies is C^1-dense in the complement in $\mathrm{Diff}^1(S)$ of the closure of the Axiom A diffeomorphisms (that is a recent result in [PuSa]).

Even if in this work we are concerned with the C^1-topology, let us recall that Newhouse showed (see [N_1]) that generic unfolding of homoclinic tangencies create C^2-locally residual subsets of $\mathrm{Diff}^2(S)$ of diffeomorphisms having an infinite set of sinks or sources. Along this paper, by C^r-*Newhouse phenomenon* we mean the coexistence of infinitely many sinks or sources in a C^r-locally residual subset of $\mathrm{Diff}^r(M)$.

The main motivation of this article comes from the following result of Mañé (see [Ma_3] (1982)), which gives, for C^1-generic diffeomorphisms of surfaces, a dichotomy between hyperbolic dynamics and Newhouse phenomenon:

Theorem (Mañé) *Let S be a closed surface. Then there is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1(S)$, $\mathcal{R} = \mathcal{R}_1 \coprod \mathcal{R}_2$, such that every $f \in \mathcal{R}_1$ verifies the Axiom A and every $f \in \mathcal{R}_2$ has infinitely many sinks or sources.*

Recall that a diffeomorphism of a manifold M is *transitive* if it has a dense orbit in the whole manifold. Such a diffeomorphism is called C^r-*robustly transitive* if it belongs to the C^r-interior of the set of transitive diffeomorphisms. Since transitive diffeomorphisms have neither sinks nor sources, a direct consequence from Mañé's result is the following:

Every C^1-robustly transitive diffeomorphism on a compact surface admits a hyperbolic structure on the whole manifold, i.e. it is an Anosov diffeomorphism.

Let us observe that Mañé's result has no direct generalization to higher dimensions: For every $n \geq 3$ there are compact $n-$dimensional manifolds supporting C^1-robustly transitive non-hyperbolic diffeomorphisms (in particular, without sources and sinks). All the examples of such diffeomorphisms, successively given by [Sh] (1972) on the torus T^4, by [Ma_2] (1978) on T^3, by [BD_1] (1996) in many other manifolds (those supporting a transitive Anosov flow or of the form $M \times N$, where M is a manifold with an Anosov diffeomorphism and N any compact manifold), by [B] (1996) and [BoVi] (1998) examples in T^3 without any hyperbolic expanding direction and examples in T^4 without any hyperbolic direction, present some weak form of hyperbolicity, the newer the examples the weaker the form of hyperbolicity, but always exhibiting some remaining weak form of hyperbolicity. Let us be more precise.

Recall first that an invariant compact set K of a diffeomorphism f on a manifold M is hyperbolic if the tangent bundle $TM|_K$ of M over K admits a f_*−invariant continuous splitting $TM|_K = E^s \oplus E^u$, such that f_* uniformly contracts the vectors in E^s and uniformly expands the vectors in E^u. This means that there is $n \in \mathbb{N}$ such that $\|f_*^n(x)|_{E^s(x)}\| < 1/2$ and $\|f_*^{-n}(x)|_{E^u(x)}\| < 1/2$ for any $x \in K$ (where $||\cdot||$ denotes the norm).

The examples of C^1−robustly transitive diffeomorphisms f in [Sh] and [Ma$_2$] let invariant a splitting $TM = E^s \oplus E^c \oplus E^u$, where f_* contracts uniformly the vectors in E^s and expands uniformly the vectors in E^u. Moreover, this splitting is dominated (roughly speaking, the contraction (resp. expansion) in E^s (resp. E^u) is stronger than the contraction (resp. expansion) in E^c, for details see Definition 0.1 below), and the central bundle E^c is one dimensional. The examples in [BD$_1$] admit also such a nonhyperbolic splitting, but the central bundle may have any dimension. The diffeomorphisms in [B] have no stable bundle E^s and admit a splitting $E^c \oplus E^u$, where the restriction of f_* to E^c is not uniformly contracting, but it uniformly contracts the area. Finally, [BoVi] gives examples of robustly transitive diffeomorphisms on T^4 without any uniformly stable or unstable bundles: They let invariant some dominated splitting $E^{cs} \oplus E^{cu}$, where the derivative of f contracts uniformly the area in E^{cs} and expands uniformly the area in E^{cu}.

Roughly speaking, in this paper we see that, if a transitive set does not admit a dominated splitting, then one can create as many sinks or sources as one wants in any neighbourhood of this set. In particular, C^1−robustly transitive diffeomorphisms always admit some dominated splitting.

Before stating our results more precisely, let us mention two previous results on 3−manifolds which are the roots of this work:

- [DPU] shows that there is an open and dense subset of C^1−robustly transitive 3−dimensional diffeomorphisms f admitting a dominated splitting $E^1 \oplus E^2$ such that at least one of the two bundles is uniformly hyperbolic (either stable or unstable). In that case, by terminology, f is *partially hyperbolic.* Moreover, [DPU] also gives a semi-local version of this result defining C^1−robustly transitive sets. Given a C^1−diffeomorphism f, a compact set K is C^1−*robustly transitive* if it is the maximal f−invariant set in some neighbourhood U of it and if, for every g C^1−close to f, the maximal invariant set $K_g = \bigcap_{\mathbb{Z}} g^n(U)$ is also transitive.

- [BD$_2$] gives examples of diffeomorphisms f on 3−manifolds having two saddles P and Q with a pair of contracting and expanding complex (non-real) eigenvalues, respectively, which belong in a robust way to the same transitive set Λ_f. Clearly, this simultaneous presence of complex contracting and expanding eigenvalues prevents the transitive set Λ_f from admitting a dominated splitting! Then [BD$_2$] shows that, for a C^1−residual subset of such diffeomorphisms, the transitive set Λ_f is contained in the closure of the (infinite) set of sources or sinks.

The results of these two papers seem to go in opposite directions, but here we show that they describe two sides of the same phenomenon. In fact,we give here a framework which allows us to unify these results and generalize them in any dimension: In the absence of weak hyperbolicity (more precisely, of a dominated splitting) one can create an arbitrarily large number of sinks or sources.

In the non-hyperbolic context, the classical notion of basic pieces (of the Smale's spectral decomposition theorem) is not defined and an important problem is to understand what could be a good substitute of it. The elementary pieces of non-trivial transitive dynamics are the homoclinic classes of hyperbolic periodic points, which are exactly the basic sets in Smale theory. Actually, [BD_2] shows that, C^1-generically, two periodic points belong to the same transitive set if and only if their two homoclinic classes are equal [1]. The hyperbolic-like properties of these homoclinic classes are the main subject of this paper.

Finally, we also see that some of our arguments can be adapted almost straightforwardly in the volume preserving setting. Let us now state our results in a precise way.

Statement of the results

Our first theorem asserts that given any hyperbolic saddle P its homoclinic class either admits an invariant dominated splitting or can be approximated (by $\mathcal{C}^1-$perturbations) by arbitrarily many sources or sinks.

Definition 0.1 *Let f be a diffeomorphism defined on a compact manifold M, K an $f-$invariant subset of M, and $TM|_K = E \oplus F$ an f_*-invariant splitting of TM over K, where the fibers E_x of E have constant dimension. We say that $E \oplus F$ is a* dominated splitting *(of f over K) if there exists $n \in \mathbb{N}$ such that*

$$\|f_*^n(x)|_E\| \, \|f_*^{-n}(f^n(x))|_F\| < 1/2.$$

We write $E \prec F$, or $E \prec_n F$ if we want to emphasize the role of n, and we speak of $n-$dominated splitting.

Let us make two comments on this definition. First, the invariant set K is not supposed to be compact and the splitting is not supposed to be continuous. However, if K admits a dominated splitting, it is always continuous and can be extended uniquely to the closure $\bar{K}$ of K (these are classical results, for details see Lemma 1.4 and Corollary 1.5). Moreover, the existence of a dominated splitting is equivalent to the existence of some continuous strictly-invariant cone field over $\bar{K}$; this cone field can be extended to some neighbourhood U of $\bar{K}$ and persists by C^1-perturbations. Thus there is an open neighbourhood of f of diffeomorphisms for which the maximal invariant set in U admits a dominated splitting. In that sense, the existence of a dominated splitting is a C^1-robust property.

Given a hyperbolic saddle P of a diffeomorphism f we denote by $H(P, f)$ the *homoclinic class* of P, i.e. the closure of the transverse intersections of the invariant manifolds of P.

[1] Recently, some substantial progress has be done for understanding the elementary pieces of dynamics of nonhyperbolic diffeomorphisms. In [Ar_1] and [CMP] it is showed that, for C^1-generic diffeomorphims or flows, any homoclinic class is a maximal transitive set. Moreover, any pair of homoclinic classes are either equal or disjoint. On the other hand, [BD_3] constructs examples of C^1-locally generic $3-$dimensional diffeomorphisms with maximal transitive sets without periodic orbits.

This set is transitive and the set Σ of hyperbolic periodic points $Q \in H(P, f)$ of f, whose stable and unstable manifolds intersect transversally the invariant manifolds of P, is dense in $H(P, f)$.

Theorem 1 *Let P be a hyperbolic saddle of a diffeomorphism f defined on a compact manifold M. Then*

- *either the homoclinic class $H(P, f)$ of P admits a dominated splitting,*
- *or given any neighbourhood U of $H(P, f)$ and any $k \in \mathbb{N}$ there is g arbitrarily C^1-close to f having k sources or sinks arbitrarily close to P, whose orbits are included in U.*

If P is a hyperbolic periodic point of f then, for every g C^1-close to f, there is a hyperbolic periodic point P_g of g close to P (this point is given by the implicit function theorem), called the *continuation* of P for g. From Theorem 1 we get the following two corollaries.

Corollary 0.2 *Under the hypotheses of Theorem 1, one of the following two possibilities holds,*

- *either there are a C^1-neighbourhood $\mathcal{U}$ of f and a dense open subset $\mathcal{V} \subset \mathcal{U}$ such that $H(P_g, g)$ has a dominated splitting for any $g \in \mathcal{V}$,*
- *or there exist diffeomorphisms g arbitrarily C^1-close to f such that $H(P_g, g)$ is contained in the closure of infinitely many sinks or sources.*

Corollary 0.3 *There exists a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every $f \in \mathcal{R}$ and any hyperbolic periodic saddle P of f, the homoclinic class $H(P, f)$ satisfies one of the following possibilities:*

- *either $H(P, f)$ has a dominated splitting,*
- *or $H(P, f)$ is included in the closure of the infinite set of sinks and sources of f.*

Problem *Is there a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that the homoclinic class of any hyperbolic periodic point P is the maximal transitive set containing P (i.e. every transitive set containing P is included in $H(P, f)$)? Moreover, when is $H(P, f)$ locally maximal?* [2]

[2] Observe that the first part of the problem was positively answered in [Ar_1] and [CMP], (recall footnote 1). Using these results, [Ab] shows that there is a C^1-residual set of diffeomorphisms such that the number of homoclinic classes is well defined and locally constant. Moreover, if this number is finite, the homoclinic classes are locally maximal sets and there is a filtration whose levels correspond to homoclinic classes.

Actually, we prove a quantitative version of Theorem 1 relating the strength of the domination with the size of the perturbations of f that we consider to get sinks or sources (see Proposition 2.6). This quantitative version is one of the keys for the next two results.

Note first that the creation of sinks or sources is not compatible with the C^1-robust transitivity of a diffeomorphism. We apply Hayashi's connecting lemma (see [Ha] and Section 2) to get, by small perturbations, a dense homoclinic class in the ambient manifold. Then using the quantitative version of Theorem 1 we show:

Theorem 2 *Every C^1-robustly transitive set (or diffeomorphism) admits a dominated splitting.*

Mañé's theorem for surface diffeomorphisms mentioned before gives a C^1-generic dichotomy between hyperbolicity and C^1-Newhouse phenomenon. It is now natural to ask what happens, in any dimension, *far from* C^1-Newhouse phenomenon.

Theorem 3 *Let f be a diffeomorphism such that the cardinal of the set of sinks and sources is bounded in a C^1-neighbourhood of f. Then there exist $l \in \mathbb{N}$ and a C^1-neighbourhood $\mathcal{V}$ of f such that, for any $g \in \mathcal{V}$ and every periodic orbit P of g whose homoclinic class $H(P,g)$ is not trivial, $H(P,g)$ admits an $l-$dominated splitting.*

A long term objective is to get a spectral decomposition theorem in the nonuniformly hyperbolic case for diffeomorphisms far from Newhouse phenomenon. Having this goal in mind, we can reformulate Theorem 3 as follows:

Under the hypotheses of Theorem 3, for every g sufficiently C^1-close to f there are compact invariant sets $\Lambda_i(g)$, $i \in \{1, \dots, \dim(M)-1\}$, such that:

- *Every $\Lambda_i(g)$ admits an $l-$dominated splitting $E_i(g) \prec_l F_i(g)$ with $\dim(E_i(g)) = i$,*
- *every non-trivial homoclinic class $H(Q,g)$ is contained in some $\Lambda_i(g)$.*

Unfortunately, this result has two disadvantages. First, the $\Lambda_i(g)$ are supposed to be neither transitive nor disjoint. Moreover, the nonwandering set $\Omega(g)$ is not *a priori* contained in the union of the $\Lambda_i(g)$ (but every homoclinic class of a periodic orbit in $(\Omega(g) \setminus \bigcup_i \Lambda_i(g))$ is trivial). So we are still far away from a completely satisfactory spectral decomposition theorem[3]. In view of these comments the following problem arises in a natural way.

Problem *Let $\mathcal{U} \subset \mathrm{Diff}^1(M)$ be an open set of diffeomorphisms for which the number of sinks and sources is uniformly bounded. Is there some subset $\mathcal{U}_0 \subset \mathcal{U}$, either dense and open or residual in $\mathcal{U}$, of diffeomorphisms g such that $\Omega(g)$ is the union of finitely many disjoint compact sets $\Lambda_i(g)$ having a dominated splitting?*

Let us observe that Newhouse phenomenon is not incompatible with the existence of a dominated splitting if we do not have any additional information on the action of f_* on the subbundles of the splitting. Actually, using Mañé's ergodic closing lemma (see [Ma_3])

[3]Fortunately, the results in the footnote 2 gave a spectral decomposition for generic diffeomorphisms with finitely many homoclinic classes.

we will get some control on the action of the derivative f_* on the volume induced on the subbundles. For that we need to introduce dominated splittings having more than two bundles. An invariant splitting $TM|_K = E_1 \oplus \cdots \oplus E_k$ is *dominated* if $\bigoplus_1^j E_i \prec \bigoplus_{j+1}^k E_i$ for every j. In this case we use the notation $E_1 \prec E_2 \prec \cdots \prec E_k$.

By Proposition 4.11, there is a unique dominated splitting, called *finest dominated splitting,* such that any dominated splitting is obtained by regrouping its subbundles by packages corresponding to intervals.

Theorem 4 *Let $\Lambda_f(U)$ be a C^1-robustly transitive set and $E_1 \oplus \cdots \oplus E_k$, $E_1 \prec \cdots \prec E_k$, be its finest dominated splitting. Then there exists $n \in \mathbb{N}$ such that $(f_*)^n$ contracts uniformly the volume in E_1 and expands uniformly the volume in E_k.*

This result synthesizes previous results in lower dimensions of [Ma$_3$] and [DPU] on robustly transitive diffeomorphisms (or sets) and it shows that, in the list of robustly transitive diffeomorphisms above, each example corresponds to the worst pathological case in the corresponding dimension. Observing that if E_1 or E_k has dimension one, then it is uniformly hyperbolic (contracting or expanding) then, for robustly transitive diffeomorphisms, we have:

- In dimension 2 the dominated splitting has necessarily two one dimensional bundles, so the diffeomorphism is hyperbolic and then Anosov (Mañé's result above).

- In dimension 3 at least one of the bundles has dimension 1 and so it is hyperbolic and the diffeomorphism is partially hyperbolic (see [DPU]). In this dimension, the finest decomposition can contain *a priori* two or three bundles and in the list above there are examples of both of these possibilities.

- In higher dimensions the extremal subbundles may have dimension strictly bigger than one and so they are not necessarily hyperbolic: This is exactly what happens in the examples in [BoVi].

Theorem 4 motivates us to introduce the notions of *volume hyperbolicity* and *volume partial hyperbolicity,* as the existence of dominated splittings, say $E \prec G$ and $E \prec F \prec G$, respectively, such that the volume is uniformly contracted on the bundle E and expanded on G. We think that this notion could be the best substitute for the hyperbolicity in a nonuniformly hyperbolic context.

The volume partial hyperbolicity is clearly incompatible with the existence of sources or sinks. However, in the proof of Theorem 4 , at least for the moment, we need the robust transitivity to obtain the partial volume hyperbolicity. Bearing in mind this comment and our previous results, let us pose some questions:

Problems

1. *In Theorem 1, is it possible to replace the notion of dominated splitting by the notion of volume partial hyperbolicity?* [4]

[4] In this direction, using the techniques in this paper, [Ab] shows the volume hyperbolicity of the homoclinic classes of generic diffeomorphisms having finitely many homoclinic classes.

2. *Is the notion of volume hyperbolicity (or volume partial hyperbolicity) sufficient to assure the generic existence of finitely many Sinai-Ruelle-Bowen (SRB) measures whose basins cover a total Lebesgue measure set? More precisely:*

   Let f be a C^1-robustly transitive diffeomorphism of class C^2 on a compact manifold M. Does it exist g close to f having finitely many SRB measures such that the union of their basins has total Lebesgue measure in M?

For ergodic properties of partially hyperbolic systems (mainly existence of SRB measures) we refer the reader to [BP], [BoVi], and [ABV].

Let us observe that in the measure preserving setting (also volume preserving) the notion of stably ergodicity (at least in the case of C^2-diffeomorphisms) seems to play the same role that the robust transitivity in the topological setting. See the results in [GPS] and [PgSh] which, in rough terms, show that weak forms of hyperbolicity may be necessary for stably ergodicity and go a long way in guaranteeing it. Actually, in [PgSh] it is conjectured that stably ergodic diffeomorphisms are open and dense among the partially hyperbolic C^2-volume preserving diffeomorphisms. See [BPSW] for a survey on stably ergodicity and [DW] for recent progress on the previous conjecture. Our next objective is precisely the study of C^1-volume preserving diffeomorphisms.

Although this paper is not devoted to conservative diffeomorphisms some of our results have a quite straightforward generalization into the conservative context. This means that the manifold is endowed with a smooth volume form ω; then we can speak of *conservative* (i.e. volume preserving) diffeomorphisms. We denote by $\mathrm{Diff}^1_\omega(M)$ the set of C^1-conservative diffeomorphisms.

A first challenge is to get a suitable version of the generic spectral decomposition theorem by Mañé (dichotomy between hyperbolicity and Newhouse phenomenon) for conservative diffeomorphisms. Obviously, since conservative diffeomorphisms have neither sinks nor sources, one needs to replace sinks and sources by *elliptic points* (i.e. periodic points whose derivatives have some eigenvalue of modulus one). Very little is known in this context. First, there is a unpublished result by Mañé (see [Ma_1]) which says that C^1-generically, area preserving diffeomorphisms of compact surfaces are either Anosov or have Lyapunov exponents equal to zero for almost every orbit (see also [Ma_4] for an outline of a possible proof)[5]. Mañé also announced a version of his theorem in higher dimensions for symplectic diffeomorphisms, see [Ma_1] [6]. Unfortunately, as far as we know, there are no available complete proofs of these results. See also the results by Newhouse in [N_2] where he states a dichotomy between hyperbolicity (Anosov diffeomorphisms) and existence of elliptic periodic points.

Related to the announced results of Mañé, there is the following question in [He]:

Conjecture (Herman) *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative diffeomorphism of a compact manifold M. Assume that there is a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_\omega(M)$ such that for any $g \in \mathcal{U}$ and every periodic orbit x of g the matrix $g^n_*(x)$ (where n is the period of x) has at least one eigenvalue of modulus different from one. Then f admits a dominated splitting.*

[5]Recently, [Bc] gives a complete proof of this result. For a generalization to higher dimensions see [BcVi_1].

[6]See [Ar_2] for progress on this subject in dimension four.

The following results give partial answers to this question:

Theorem 5 *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative diffeomorphisms of an $N-$dimensional manifold M. Then there is $l \in \mathbb{N}$ such that,*

- *either there is a conservative $\varepsilon - C^1-$perturbation g of f having a periodic point x of period $n \in \mathbb{N}$ such that $g^n_*(x) = Id$,*
- *or for any conservative diffeomorphism g $\varepsilon-C^1-$close to f and every periodic saddle x of g the homoclinic class $H(x,g)$ admits an $l-$dominated splitting.*

Theorem 6 *Let f be a conservative diffeomorphism defined on a compact $N-$dimensional manifold. Then there are two possibilities:*

- *Either given any $k \in \mathbb{N}$ there is a conservative diffeomorphism g arbitrarily C^1-close to f having k periodic orbits whose derivatives are the identity.*
- *Or the manifold M is the union of finitely many (less than $N-1$) invariant compact (*a priori *nondisjoint) sets having a dominated splitting.*

Observe that if in Theorem 6 above the diffeomorphism f is transitive and the second possibility of the dichotomy occurs, then one of the invariant compact sets has to be the whole manifold (one of them contains a dense orbit). This means that, in the transitive case, Theorem 6 gives a complete positive answer to Herman's conjecture:

Corollary 0.4 *Let $f \in \mathrm{Diff}^1_\omega(M)$ be a conservative transitive diffeomorphism of a manifold M. Assume that there is a neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^1_\omega(M)$ such that for any $g \in \mathcal{U}$ and every periodic orbit x of g the matrix $g^n_*(x)$ (where n is the period of x) has at least one eigenvalue of modulus different from one. Then f admits a dominated splitting.*

Let us observe that if f is transitive and there is some periodic point x of f such that $f^n_*(x) = Id$, n is the period of x, then given any $\varepsilon > 0$ there is a C^1-perturbation $g \in \mathrm{Diff}^1_\omega(M)$ of f such that its totally elliptic points (derivative equal to the identity) are $\varepsilon-$dense in M.

A conservative diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$ is *robustly transitive in* $\mathrm{Diff}^1_\omega(M)$ if there is $\varepsilon > 0$ such that every $\varepsilon-$ perturbation $g \in \mathrm{Diff}^1_\omega(M)$ of f is transitive. Observe that *a priori* the robust transitivity in $\mathrm{Diff}^1_\omega(M)$ does not imply the robust transitivity in $\mathrm{Diff}^1(M)$.

Conjecture *Let f be a robustly transitive diffeomorphism in $\mathrm{Diff}^1_\omega(M)$. Then f admits a nontrivial dominated splitting defined on the whole M.*

In view of Corollary 0.4, to prove this conjecture one needs to to show that a robustly transitive diffeomorphism f can not have periodic points x whose derivative $f^n_*(x)$ is the identity.

Finally, we observe that the control of the volume in the subbundles is almost straightforward for conservative systems:

Proposition 0.5 *Let f be a conservative diffeomorphism and $E \oplus F$, $E \prec F$, be a dominated splitting of TM. Then f_* contracts uniformly the volume in E and expands uniformly the volume in F.*

Main ideas of the proofs

Mañé's paper [Ma$_3$] combines two main ingredients: Systems of matrices and the Ergodic Closing Lemma. He first considers the linear maps induced by the derivative of a diffeomorphism f over the orbits of its periodic points, thus obtaining a *system of matrices.* He shows that (in his context) a system of matrices admits a dominated splitting if it is not possible to perturb it to get a matrix with some eigenvalue of modulus one. By a lemma of Franks', see [F] and Section 1, each perturbation of the system of matrices over a finite number of periodic orbits corresponds to a C^1-perturbation of f and vice versa. Hence the existence of a dominated splitting also holds for C^1-diffeomorphisms. Finally, to get the uniform expansion and contraction on the subbundles of the splitting he uses his Ergodic Closing Lemma (see [Ma$_3$]).

Our proof uses these two tools introduced by Mañé. Using Franks' Lemma we translate the problem of the existence of a dominated splitting for diffeomorphisms into the same problem for abstract linear systems. However, the systems of matrices in [Ma$_3$] do not contain some relevant dynamical informations about f that we need. Actually, the solution of this difficulty is probably the subtlest point of our arguments, so let us be somewhat more precise:

On one hand, in the context in [Ma$_3$] all the periodic points have the same *index* (dimension of the stable bundle), thus the system of matrices has a natural splitting (the one corresponding to the stable and unstable bundles of f_*). Then if this splitting is not dominated one gets a perturbation of it having one eigenvalue of modulus 1. On the other hand, in our case there are points having different indices. Moreover, points having eigenvalues of modulus 1 are not forbidden. So we need some extra arguments to conclude our proof. In fact, we need to control *all* the eigenvalues to create sources or sinks.

The additional argument, that comes from the dynamics, is a property of our linear systems called *transitions.* Given two periodic points P and Q in the same *homoclinic class* (i.e. their invariant manifolds intersect transversally) there are periodic orbits passing first arbitrarily close to P, and thereafter arbitrarily close to Q, and so on. These orbits can be chosen upon arbitrary sequence of times (the orbit spends k_1 iterates close to P then, after a bounded number of iterates, it becomes close to Q and remains k_2 iterates close to Q, and so on). So we define a structure we call transitions which translates this dynamical behavior into the world of the abstract linear systems. This property allows us consider the product of matrices of the system corresponding to different orbits as a matrix of the system. In fact, the transitions endow the linear system with a "semigroup-like" structure. Clearly, this is not the case for general linear systems.

Finally, after introducing the linear systems with transitions, the proof of the existence of a dominated splitting involves only arguments of linear algebra. Precisely, this algebraic approach has allowed us to improve previous results by stating them in higher dimensions and by eliminating the robust transitivity hypothesis.

The problem of the existence of points with different indices already appeared in [DPU], where it has been solved by considering only robustly transitive sets, thus any perturbation of the dynamics remains transitive. This additional hypothesis in [DPU] allows to jump from the dynamical world to the abstract linear world, here do some perturbation, and then jump back to the dynamical world do a new perturbation, and so on. In our context we have no control of the variation of a homoclinic class after dynamical perturbations. So it is crucial for Theorem 1 that all the perturbations we do "live in the world of abstract linear systems" and do not modify the underlying dynamics (that is here possible because Mañé's linear systems have been enriched with the transitions).

In our proof, assuming that there is no dominated splitting, we perform a series of perturbations of the linear system, as a final result of such perturbations we get a linear system having a homothety. It is only then that we realize this linear system as a diffeomorphism using Franks' Lemma, and the point corresponding to the homothety becomes a sink or a source of the diffeomorphism.

Finally, for the control of the volume in the extremal subbundles (Theorem 4) we use the Ergodic Closing Lemma, which gives a dynamical perturbation having a periodic point reflecting the lack of volume expansion or contraction of the bundles. Unfortunately, without any additional hypothesis, this point has *a priori* nothing to do with the initial homoclinic class. This explains why Theorem 4 only holds for robustly transitive systems.

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Contents

1 Linear systems with transitions

Let f be a diffeomorphism. By Franks' Lemma below (see for instance [F]), to any perturbation A of the derivative f_* along the orbits of finitely many periodic points corresponds a diffeomorphism g, C^1-close to f, such that $g_* = A$ along these orbits. This lemma allows us to consider perturbations of the derivative f_* keeping unchanged the dynamics of f, in order to get a suitable derivative along some periodic orbits. The aim of this section is to define in details the framework (*periodic linear systems*) which gives a precise meaning of this kind of perturbations, and to translate into this language the dynamical properties that we will need (specially the notion of *transitions*, see Definition 1.6). Finally, we prove that the homoclinic classes define periodic linear system with transitions (Lemma 1.9) and we state an easy (but typical) consequence of the existence of transitions (Lemma 1.10).

Before beginning this section let us state precisely the Franks' Lemma:

Lemma (Franks) *Suppose that E is an $f-$invariant finite set and B is an $\varepsilon-$perturbation of f_* along E, then there is a diffeomorphism g $\varepsilon-\mathcal{C}^1-$close to f, coinciding with f out of an arbitrarily small neighbourhood of E, equal to f in E, and such that g_* coincides with B in E.*

Let us point out that Franks' Lemma is the key which allows to translate results on linear systems to the dynamical context and it will be ofen used in this paper.

1.1 Linear systems: Topology and linear changes of coordinates

Let Σ be a topological space and f a homeomorphism defined on Σ. Consider a locally trivial vector bundle (of finite dimension) $\mathcal{E}$ over Σ. Denote by $\mathcal{E}_x$ the fiber of $\mathcal{E}$ at $x \in \Sigma$. We assume that the dimension of the fibers $\mathcal{E}_x$, $\dim(\mathcal{E}_x)$, does not depend on $x \in \Sigma$. In what follows, we call this number *dimension of the bundle* $\mathcal{E}$, denoted by $\dim(\mathcal{E})$.

An *euclidian metric* $|\cdot|$ *on the bundle* $\mathcal{E}$ is a collection of euclidian metrics on the fibers $\mathcal{E}_x$, $x \in \Sigma$, *a priori* not depending continuously on x.

We denote by $\mathcal{GL}(\Sigma, f, \mathcal{E})$ the set of maps $A\colon \mathcal{E} \to \mathcal{E}$ such that for every $x \in \Sigma$ the induced map $A(x,\cdot)$ is a linear isomorphism from $\mathcal{E}_x \to \mathcal{E}_{f(x)}$, thus $A(x,\cdot)$ belongs to $\mathcal{L}(\mathcal{E}_x, \mathcal{E}_{f(x)})$ and is invertible. For each $x \in \Sigma$ the euclidian metrics on $\mathcal{E}_x$ and $\mathcal{E}_{f(x)}$ induce a norm (always denoted by $|\cdot|$) on $\mathcal{L}(\mathcal{E}_x, \mathcal{E}_{f(x)})$:

$$|B(x,\cdot)| = \sup\{|B(x,v)|, v \in \mathcal{E}_x, |v| = 1\}.$$

Let now $A \in \mathcal{GL}(\Sigma, f, \mathcal{E})$ and define $|A| = \sup_{x\in\Sigma} |A(x,\cdot)|$. Observe that, for any $A \in \mathcal{GL}(\Sigma, f, \mathcal{E})$ its inverse A^{-1} belongs to $\mathcal{GL}(\Sigma, f^{-1}, \mathcal{E})$. So we can define $|A^{-1}|$ in the same way. Finally, the *norm* of a $A \in \mathcal{GL}(\Sigma, f, \mathcal{E})$ is $\|A\| = \sup\{|A|, |A^{-1}|\}$.

Definition 1.1 *A* linear system[7] *is a* $4-$*uple* $(\Sigma, f, \mathcal{E}, A)$ *where* Σ *is a topological space,* f *is a homeomorphism of* Σ*,* $\mathcal{E}$ *is an Euclidean bundle over* Σ*,* A *belongs to* $\mathcal{GL}(\Sigma, f, \mathcal{E})$*, and* $\|A\| < \infty$.

In what follows, for the sake of simplicity, we sometimes denote by A a linear system $(\Sigma, f, \mathcal{E}, A)$ if there is no ambiguity on Σ, f, and $\mathcal{E}$.

Example 1 *Let* f *be a diffeomorphism defined on a riemannian manifold* M *and* $\Sigma \subset M$ *an* $f-$*invariant subset. Consider the restriction to* Σ *of the tangent bundle,* $\mathcal{E} = TM|_\Sigma$*. The riemannian metric on* M *induces an Euclidean structure on* $\mathcal{E}$*. Then* $(\Sigma, f|_\Sigma, \mathcal{E}, f_*|_\mathcal{E})$ *is the* natural linear system induced by f over Σ.

We denote by $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ the space of linear systems over $(\Sigma, f, \mathcal{E})$ such that $||A|| < \infty$ endowed with the distance defined by

$$d(A,B) = \sup\{|A-B|, |A^{-1} - B^{-1}|\}, \quad A,\, B \in \mathcal{GL}^\infty(\Sigma, f, \mathcal{E}).$$

We can now define an $\varepsilon-$*perturbation* of A as a linear system $\tilde{A}$, defined over $(\Sigma, f, \mathcal{E})$, such that $d(A, \tilde{A}) < \varepsilon$.

Very elementary arguments of linear algebra show that any perturbation of a linear system can be obtained by composing it with linear maps close to identity. More precisely, let $A \in \mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ and consider some linear system $E \in \mathcal{GL}^\infty(\Sigma, Id_\Sigma, \mathcal{E})$. Then $E \circ A$ and $A \circ E$ (defined in the obvious way) belong to $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$. Moreover, if E is close to the identity linear system $(\Sigma, Id_\Sigma, \mathcal{E}, Id_\mathcal{E})$, then $E \circ A$ and $A \circ E$ are also close to A.

Consider now some change of the Euclidean metrics on the fibers. Assume that the matrices of the changes of coordinates (from an orthonormal basis of the initial metric to an orthonormal basis of the new metric) and their inverses are uniformly bounded on Σ.

[7]After writing this paper, we realized that this notion corresponds to the classical concept of *linear cocycle* over the homeomorphism f.

Then every linear system in the initial metric induces a new system (for the new metric). Moreover, this change of metrics keeps invariant the topology of the set of linear systems. Let us be a little bit more precise.

Let $\mathcal{E}$ denote an Euclidean bundle on a topological space Σ endowed with the Euclidean metric $|\cdot|$. Denote by $\mathcal{E}_1$ the same bundle, but now endowed with a different Euclidean metric $|\cdot|_1$. Denote by $P: \mathcal{E} \to \mathcal{E}_1$ the identity map considered as a morphism of bundles. Using the metrics $|\cdot|$ and $|\cdot|_1$ we can define the norms $|P|$ and $|P^{-1}|$. Write $\|P\| = \sup\{|P|, |P^{-1}|\}$. If $\|P\| < \infty$, the canonical bijection $Id: \mathcal{GL}(\Sigma, f, \mathcal{E}) \to \mathcal{GL}(\Sigma, f, \mathcal{E}_1)$ induces an homeomorphism from $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ (with the distance d) to $\mathcal{GL}^\infty(\Sigma, f, \mathcal{E}_1)$ (with the corresponding distance d_1). These two simple facts are put together in the following lemma.

Lemma 1.2

1. *Given $K > 0$ and $\varepsilon > 0$ there is $\delta > 0$ such that for any linear system $(\Sigma, f, \mathcal{E}, A)$, $A \in \mathcal{GL}^\infty(\Sigma, f, \mathcal{E})$ and $\|A\| < K$, and every $\delta-$perturbation of the identity $(\Sigma, id_\Sigma, \mathcal{E}, E)$, one has that $E \circ A$ and $A \circ E$ are $\varepsilon-$perturbations of A.*

2. *For every $K > 0$, $K_0 > 0$, and $\varepsilon > 0$ there are $K_1 > 0$, $\delta > 0$ satisfying the following property:*

   Consider a pair of Euclidean bundles $\mathcal{E}$ and $\mathcal{E}_1$ over Σ endowed with the metrics $|\cdot|$ and $|\cdot|_1$, and the isomorphism of bundles $P: \mathcal{E} \to \mathcal{E}_1$ induced by the identity on Σ (i.e. given $x \in \Sigma$ the map $P(x, \cdot)$ is a linear isomorphism from $\mathcal{E}_x$ to $(\mathcal{E}_1)_x$). Assume that $\|P\| < K_0$. Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system such that $\|A\|$ is bounded by K.

   Let $B = P \circ A \circ P^{-1}$, then $(\Sigma, f, \mathcal{E}_1, B)$ is a linear system bounded by K_1. Moreover, any $\delta-$perturbation of B is conjugate by P to some $\varepsilon-$perturbation of A.

Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and $n \in \mathbb{N}$. The *$n-th$ iterate of A*, denoted by $A^{(n)}$, is the linear system over $(\Sigma, f^n, \mathcal{E})$ defined by $A^{(n)}(x) = A(f^{n-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x)$.

Consider an $f-$invariant subset Σ' of Σ and the restriction of the linear bundle $\mathcal{E}$ to Σ', then A induces canonically a linear system over $(\Sigma', f|_{\Sigma'}, \mathcal{E}|_{\Sigma'})$ called *linear subsystem induced by A over Σ'.*

1.2 Special linear systems

Along this work, the linear systems we consider will be often endowed with some additional structures: In some cases they are *continuous,* and most of them are *periodic.* We also consider *systems of matrices.* Finally, the most important additional structure we will introduce is the notion of *transitions.* Let us now present the three first quite natural structures. Due to its specific and subtle nature we postpone to the next paragraph the notion of transition. This key definition will deserve special attention and care.

In the sequel, Σ is a topological space, f a homeomorphism of Σ, and $\mathcal{E}$ a locally trivial vector bundle over Σ endowed with an Euclidean metric $|\cdot|$ on the fibers. A linear system $(\Sigma, f, \mathcal{E}, A)$ is *continuous* if the Euclidean structure on the fibers varies continuously and the function $A: \mathcal{E} \to \mathcal{E}$ is continuous.

The linear system $(\Sigma, f, \mathcal{E}, A)$ is *periodic* if all the orbits of f are periodic. In this case we let $M_A(x): \mathcal{E}_x \to \mathcal{E}_x$ the product of the $A(f^i(x))$ along the orbit of x. More precisely, let $p(x)$ be the period of $x \in \Sigma$, then

$$M_A(x) = A(f^{p(x)-1}(x)) \circ \cdots \circ A(x) = A^{(p(x))}(x).$$

Finally, $(\Sigma, f, \mathcal{E}, A)$ is a *system of matrices* if the Euclidean bundle $\mathcal{E}$ is the trivial bundle $\Sigma \times \mathbb{R}^N$, where $\mathbb{R}^N$ is endowed with the canonical Euclidean metric. In this case every linear map $A(x)$ is canonically identified with an element of $GL(N, \mathbb{R})$.

Let $(\Sigma, f, \mathcal{E}, A)$ be an (*a priori* non-continuous) linear system. It will be sometimes useful to fix an orthonormal basis on each fiber $\mathcal{E}_x$ (this basis does not depend, in general, continuously on the point $x \in \Sigma$). These basis give an (a priori non-continuous) trivialization of the Euclidean bundle $\mathcal{E}$. So in these new coordinates A can be considered as a system of matrices. Two systems of matrices *define the same linear system* if at each point there exists an orthonormal change of coordinates conjugating the two systems.

1.3 Dominated splittings

The definition of dominated splitting for an invariant set of a diffeomorphism (see Definition 0.1) can be directly generalized for linear systems as follows. Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system, an *invariant subbundle* is a collection of linear subspaces $F(x) \subset \mathcal{E}_x$ whose dimensions do not depend on x and such that $A(F(x)) = F(f(x))$. An $A-$*invariant splitting* $F \oplus G$ is given by two invariant subbundles such that $\mathcal{E}_x = F(x) \oplus G(x)$ at each $x \in \Sigma$.

Definition 1.3 *Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and $\mathcal{E} = F \oplus G$ an $A-$invariant splitting. We say that $F \oplus G$ is a* dominated splitting *if there exists $n \in \mathbb{N}$ such that*

$$\|A^{(n)}(x)|_F\| \, \|A^{(-n)}(f^n(x))|_G\| < 1/2$$

for every $x \in \Sigma$. We write $F \prec G$.

*If we want to emphasize the role of n then we say that $F \oplus G$ is an $n-$*dominated splitting *and write $F \prec_n G$.*

Finally, the dimension of the dominated splitting *is the dimension of the subbundle F.*

Suppose now that $(\Sigma, f, \mathcal{E}, A)$ is a **continuous** linear system, then any dominated splitting can be obtained by considering subsystems induced by A over dense subsets $\Sigma' \subset \Sigma$. More precisely,

Lemma 1.4 *Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous linear system such that there is a dense $f-$invariant subset $\Sigma_1 \subset \Sigma$ whose corresponding linear subsystem admits an $l-$dominated splitting. Then $(\Sigma, f, \mathcal{E}, A)$ admits an $l-$dominated splitting.*

More generally, suppose that there is a sequence of (non necessarily continuous) systems $(\Sigma, f, \mathcal{E}, A_k)$ converging to $(\Sigma, f, \mathcal{E}, A)$ such that for every k there is a dense invariant subset $\Sigma_k \subset \Sigma$ where A_k admits an $l-$dominated splitting. Then A admits an $l-$dominated splitting in the whole Σ.

Finally, any dominated splitting of a continuous linear system is continuous.

Proof: Given $x \in \Sigma$ consider a sequence (x_k), $x_k \in \Sigma_k$, converging to x. For a fixed k we have an $l-$dominated splitting $E_k \oplus F_k$. Taking a subsequence we can assume that the dimensions of these spaces are independent of k and that the sequences $E_k(x_k)$ and $F_k(x_k)$ converge to some subspaces $E(x)$ and $F(x)$.

By definition of $l-$dominance, given any k, $u_k \in E_k(x_k)$, and $v_k \in F_k(x_k)$, we have

$$2\,\frac{\|A_k^l(u_k)\|}{\|u_k\|} \leq \frac{\|A_k^l(v_k)\|}{\|v_k\|}.$$

By the continuity of A and the convergences of $A_k \to A$, $x_k \to x$, $E_k(x_k) \to E(x)$, and $F_k(x_k) \to F(x)$, we get

$$2\,\frac{\|A^l(u)\|}{\|u\|} \leq \frac{\|A^l(v)\|}{\|v\|}$$

for every $u \in E(x)$ and $v \in F(x)$. So these two spaces are transverse.

Finally, it remains to check that these two spaces are uniquely defined and give an invariant splitting. Observe first that $A(E(x))$ and $A(F(x))$ are the limits of the (same) subsequences before of $E_k(f(x_k))$ and $F_k(f(x_k))$. Then for any $m \in \mathbb{Z}$ we get

$$2^m\,\frac{\|A^{ml}(u)\|}{\|u\|} \leq \frac{\|A^{ml}(v)\|}{\|v\|}$$

for every $u \in E(x)$ and $v \in F(x)$. Now a standard dynamical argument asserts that the spaces $E(x)$ and $F(x)$ verifying this inequality are uniquely determined by their dimensions.

To complete the proof, observe that the unicity of the dominated splitting above gives the continuity. $\square$

Corollary 1.5 *Let f be a diffeomorphism defined on a compact manifold M and Λ an $f-$invariant set. Assume that there are $l \in \mathbb{N}$, $i \in 1, \ldots, \dim(M) - 1$, and a sequence of diffeomorphisms f_n converging to f in the C^1-topology such that*

- *every f_n has a periodic orbit x_n such that $H(x_n, f_n)$ admits an $l-$dominated splitting of dimension i,*
- *the set Λ is included in the topological upper limit set of the $H(x_n, f_n)$, i.e.*

$$\limsup_{n\to\infty}(H(x_n, f_n)) = \bigcap_{n=1}^{\infty} \mathrm{closure}(\bigcup_{i>n} H(x_n, f_n)).$$

Then Λ admits an $l-$dominated splitting of dimension i.

Proof: Consider the topological set

$$I = \{0\} \cup \{\frac{1}{n}, n \in \mathbb{N} \setminus \{0\}\}.$$

In $M \times I$ we consider the union

$$(\Lambda \times \{0\}) \cup \bigcup_{1}^{+\infty} (H(x_n, f_n) \times \{\frac{1}{n}\}).$$

The differentials of f and f_n define in a natural way a linear system on this set, which is continuous because the f_n converge to f in the C^1-topology. Moreover, $\bigcup_1^{+\infty}(H(x_n, f_n) \times \{\frac{1}{n}\}$ is a dense subset (because Λ is contained in the topological upper limit set of the $H(x_n, f_n)$) and the system over $\bigcup_1^{+\infty}(H(x_n, f_n) \times \{\frac{1}{n}\}$ admits an $l-$dominated splitting. To finish the proof it is now enough to apply Lemma 1.4. $\square$

1.4 Periodic linear systems with transitions

To saddles P and Q of the same index which are linked by transverse intersections of their invariant manifolds (i.e. they are *homoclinically related*) belong to the same transitive hyperbolic set. So they are accumulated by other periodic orbits whose orbits spend an arbitrarily long time close to P, thereafter close to Q, and so on. In fact, the existence of Markov partitions shows that fixed any finite sequence of times there is a periodic orbit expending alternately the times of the sequence close to P and Q, respectively. Moreover, the transition time (between a neighbourhood of P and a neighbourhood of Q) can be chosen bounded. This property will allow us to scatter in the whole homoclinic class of P some properties of the periodic points Q of this class.

The aim of this section is to translate this property into the language of linear systems, introducing the concept of *linear system with transitions.* Then we shall deduce some direct consequences of the existence of such transitions. Let us go into the details of our constructions. We begin by giving some definitions.

Given a set $\mathcal{A}$, a *word with letters* in $\mathcal{A}$ is a finite sequence of elements of $\mathcal{A}$, its *length* is the number of letters composing it. The set of words admits a natural a semi-group structure: The product of the word $[a] = (a_1, \dots, a_n)$ by $[b] = (b_1, \dots, b_k)$ is $[a][b] = (a_1, \dots, a_n, b_1, \dots, b_k)$. We say that a word $[a]$ *is not a power* if $[a] \neq [b]^k$ for every word $[b]$ and $k > 1$.

In this section $(\Sigma, f, \mathcal{E}, A)$ is a periodic linear system of dimension N: Recall that every $x \in \Sigma$ is periodic for f, $p(x)$ denotes its period, and $M_A(x)$ denotes the product $A^{(p(x))}(x)$ of A along the orbit of x.

If (Σ, f, A) is a periodic system of matrices (in $GL(N, \mathbb{R})$), then for any $x \in \Sigma$ we write $[M]_A(x) = (A(f^{p(x)-1}(x)), \dots, A(x))$; which is a word with letters in $GL(N, \mathbb{R})$. Hence the matrix $M_A(x)$ is the product of the letters of the word $[M]_A(x)$.

Definition 1.6 *Given $\varepsilon > 0$, a periodic linear system $(\Sigma, f, \mathcal{E}, A)$ admits $\varepsilon-$transitions if for every finite family of points $x_1, \dots, x_n = x_1 \in \Sigma$ there is an orthonormal system of coordinates of the linear bundle $\mathcal{E}$ (so that $(\Sigma, f, \mathcal{E}, A)$ can be now considered as a system of matrices (Σ, f, A)), and for any $(i, j) \in \{1, \dots, n\}^2$ there exist $k(i, j) \in \mathbb{N}$ and a finite word $[t^{i,j}] = (t_1^{i,j}, \dots, t_{k(i,j)}^{i,j})$ of matrices in $GL(N, \mathbb{R})$, satisfying the following properties:*

1. *For every $m \in \mathbb{N}$, $\iota = (i_1, \dots, i_m) \in \{1, \dots, n\}^m$, and $a = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m$ consider the word*

$$[W(\iota, a)] = [t^{i_1,i_m}][M_A(x_{i_m})]^{\alpha_m}[t^{i_m,i_{m-1}}][M_A(x_{i_{m-1}})]^{\alpha_{m-1}} \cdots [t^{i_2,i_1}][M_A(x_{i_1})]^{\alpha_1},$$

   where the word $w(\iota, a) = ((x_{i_1}, \alpha_1), \dots, (x_{i_m}, \alpha_m))$ with letters in $M \times \mathbb{N}$ **is not a power**. *Then there is $x(\iota, a) \in \Sigma$ such that*

- *The length of $[W(\iota,a)]$ is the period $p(x(\iota,a))$ of $x(\iota,a)$.*
- *The word $[M]_A(x(\iota,a))$ is $\varepsilon-$close to $[W(\iota,a)]$ and there is an $\varepsilon-$perturbation $\tilde{A}$ of A such that the word $[M]_{\tilde{A}}(x(\iota,a))$ is $[W(\iota,a)]$.*

2. *One can choose $x(\iota,a)$ such that the distance between the orbit of $x(\iota,a)$ and any point x_{i_k} is bounded by some function of α_k which tends to zero as α_k goes to infinity.*

Given ι and a as above, the word $[t^{i,j}]$ is an *$\varepsilon-$transition from x_j to x_i*. We call *$\varepsilon-$transition matrices* the matrices $T_{i,j}$ which are the product of the letters composing $[t^{i,j}]$.

Remark 1.7 *Consider points $x_1,\dots,x_{n-1},x_n = x_1 \in \Sigma$ and $\varepsilon-$transitions $[t^{i,j}]$ from x_j to x_i. Then we have that*

1. *for every positive $\alpha \geq 0$ and $\beta \geq 0$ the word $([M]_A(x_i))^\alpha\,[t^{i,j}]\,([M]_A(x_j))^\beta$ is also an $\varepsilon-$transition from x_j to x_i,*
2. *for any i, j, and k the word $[t^{i,j}][t^{j,k}]$ is an $\varepsilon-$transition from x_k to x_i,*
3. *As a consequence of the two items above, the words $W(\iota,\alpha)$ in Definition 1.6 are $\varepsilon-$transitions from x_{i_1} to itself. Moreover, the set of such $\varepsilon-$transitions forms a semigroup.*

Definition 1.8 *We say that a periodic linear system* admits transitions *if for any $\varepsilon > 0$ it admits $\varepsilon-$transitions.*

The following lemma justifies the introduction of the notion of transition for studying homoclinic classes:

Lemma 1.9 *Let P be a hyperbolic saddle of index k (dimension of its stable manifold). The derivative f_* induces a continuous periodic linear system with transitions on the set Σ of hyperbolic saddles in $H(P,f)$ of index k and homoclinically related to P.*

Proof: Fix any $\varepsilon > 0$ and a finite family $x_1,\dots,x_n$ in Σ. As the x_i are homoclinically related to P, there is a compact transitive hyperbolic subset K of $H(P,f)$ containing all the x_i. So this set K can be covered by a Markov partition with arbitrarily small rectangles. We can now choose orthonormal systems of coordinates in $T_x(M)$, $x \in K$, such that the orthonormal basis depend continuously on x when the points are in the same rectangle.

Let (K,f,A) be the system of matrices defined on K by writing f_* in this system of coordinates. Now, using the continuity of f_*, and by subdividing if necessary the rectangles of the Markov partition, we can assume that, for any x and y in the same rectangle, one has

$$\|A(x) - A(y)\| < \varepsilon \quad \text{and} \|A^{-1}(x) - A^{-1}(y)\| < \varepsilon.$$

The transitions from x_i to x_j are now obtained considering the derivative of f along any orbit in K going from the rectangle containing x_i to the rectangle containing x_j. □

The next lemma shows how a property at one point of a system with transitions can be scatter to a dense subset:

Lemma 1.10 (Scattering Property) *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Fix $\varepsilon > \varepsilon_0 > 0$ and assume that there exist an ε_0-perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is either a dilatation (i.e. all its eigenvalues have modulus bigger than 1) or a contraction (i.e. all its eigenvalues have modulus less than 1).*

Then there are a dense $f-$invariant subset $\tilde{\Sigma}$ of Σ and an $\varepsilon-$perturbation $\hat{A}$ of A such that for any $y \in \tilde{\Sigma}$ the linear map $M_{\hat{A}}(y)$ is either a dilatation or a contraction (according to the choice before).

Proof: Write $\varepsilon_1 = \varepsilon - \varepsilon_0$, take some point z in Σ, and consider two ε_1-transitions $T_{x,z}$ (from z to x) and $T_{z,x}$ (from x to z). For a fixed $\delta > 0$, by definition of transitions, there is $n(z,\delta)$ such that for any $n > 0$ there are $y_n \in \Sigma$, with $d(y_n, z) < \delta$, and an ε_1-deformation A' of A along the orbit of y_n such that

$$M_{A'}(y_n) = T_{z,x} \circ M_A(x)^n \circ T_{x,z} \circ M(z)^{n(z,\delta)}.$$

Define $\hat{M}_n$ by

$$\hat{M}_n = T_{z,x} \circ M_{\tilde{A}}(x)^n \circ T_{x,z} \circ M(z)^{n(z,\delta)}.$$

We can now choose n big enough so that $\hat{M}_n$ is either a dilatation or a contraction (according to $M_{\tilde{A}}(x)$). Thus by an ε_1-perturbation $\hat{A}$ of A' along the orbit of y_n we can get $M_{\hat{A}}(y) = \hat{M}_n$.

Since we are not requiring the continuity of $\hat{A}$, we can build it as above, that is, orbit by orbit considering points in a dense subset. This ends the proof of the lemma. □

2 Quantitative results: Proofs of the theorems.

In this section we state, in terms of linear systems (Proposition 2.1) and in terms of diffeomorphisms (Proposition 2.6), quantitative results on the existence of dominated splittings (giving the strength of the dominance).

Proposition 2.1 gives a dichotomy between the existence of a dominated splitting for a linear system and the existence of perturbations of the system with homotheties. This proposition is divided into two main steps: Proposition 2.4, asserting that the lack of dominance allows to create complex eigenvalues, and Proposition 2.5, which says that sufficiently many complex eigenvalues allow us to get homotheties. These propositions will be proven in the next two sections.

In this section we deduce from Proposition 2.1 most of the results announced in the introduction.

2.1 Reduction of the study of the dynamics to a problem on linear systems

Proposition 2.1 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $l > 0$ such that any continuous periodic $N-$dimensional linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K (i.e. $\|A\| < K$) and having transitions satisfies the following,*

- *either A admits an $l-$dominated splitting,*

- *or there are an $\varepsilon-$perturbation $\tilde{A}$ of A and a point $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is an homothety.*

The proof of Proposition 2.1 is divided in two main steps: In the first one, we show that, if $(\Sigma, f, \mathcal{E}, A)$ is a linear system with transitions such that no dense-subsystem of it admits an $l-$dominated splitting, then we can perturb A to get *a lot of complex eigenvalues.* In the second step, we see that, if we can obtain sufficiently many complex eigenvalues, then we can perturb the system to get a homothety (which will be either a contraction or a dilatation). Let us state precisely these two steps. We begin with some definitions.

Definition 2.2 *Let $M \in GL(N, \mathbb{R})$ be a linear isomorphism of $\mathbb{R}^N$ such that M has some complex eigenvalue λ, i.e. $\lambda \in \mathbb{C} \setminus \mathbb{R}$. We say that λ has* rank $(i, i+1)$ *if there is a $M-$invariant splitting of $\mathbb{R}^N$, $F \oplus G \oplus H$, such that:*

- *Every eigenvalue σ of $M|_F$ (resp. $M|_H$) has modulus $|\sigma| < |\lambda|$ (resp. $|\sigma| > |\lambda|$),*
- $\dim(F) = i - 1$ *and* $\dim(H) = N - i - 1$,
- *the plane G is the eigenspace of λ.*

Definition 2.3 *A periodic linear system* $(\Sigma, f, \mathcal{E}, A)$ has a complex eigenvalue of rank $(i, i+1)$ *if there is $x \in \Sigma$ such that the matrix $M_A(x)$ has a complex (non-real) eigenvalue of rank $(i, i+1)$.*

Proposition 2.1 is a direct consequence of Propositions 2.4 and 2.5 below:

Proposition 2.4 *For every $\varepsilon > 0$, $N \in \mathbb{N}$, and $K > 0$ there is $l \in \mathbb{N}$ satisfying the following property:*

Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic $N-$dimensional linear system with transitions such that its norm $\|A\|$ is bounded by K. Assume that there exists $i \in \{1, \ldots, N-1\}$ such that every $\varepsilon-$perturbation $\tilde{A}$ of A has no complex eigenvalues of rank $(i, i+1)$.

Then $(\Sigma, f, \mathcal{E}, A)$ admits an $l-$dominated splitting $F \oplus G$, $F \prec_l G$, with $\dim(F) = i$.

Proposition 2.5 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Given $\varepsilon > \varepsilon_0 > 0$ assume that, for any $i \in \{1, \ldots, N-1\}$, there is an ε_0-perturbation of A having a complex eigenvalue of rank $(i, i+1)$. Then there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ is a homothety with ratio of modulus different from* 1.

The key of the proof of Proposition 2.4 is a $2-$dimensional argument of Mañé that we present in Section 3. The proof in higher dimensions consists of an inductive argument which allows us to reduce the dimension of the linear space by considering some quotients (roughly speaking, considering projections). Using this inductive procedure we finally arrive at a two-dimensional space. The lemmas in Section 4.1 allow us to make these successive reductions of dimension. The proofs of Propositions 2.4 and 2.5 are in Section 5.

Now using Proposition 2.1 we proof most of the results announced in the introduction.

2.2 Proofs of the Theorems

Let us first explain why Proposition 2.1 implies Theorem 1. Actually, this proposition implies the following quantitative version of Theorem 1, which is our main (but a little bit technical) result:

Proposition 2.6 *For every $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $l(\varepsilon, K, N) \in \mathbb{N}$ such that for any diffeomorphism f defined on a riemannian $N-$dimensional manifold M such that the derivatives f_* and f_*^{-1} are bounded by K, and any saddle P of f with a non-trivial homoclinic class $H(P, f)$, the following holds:*

- *Either the homoclinic class $H(P, f)$ admits an $l(\varepsilon, K, N)-$dominated splitting,*
- *or for every neighbourhood U of $H(P, f)$ and $k \in \mathbb{N}$ there is g $\varepsilon-C^1-$close to f having k sources or sinks whose orbits are contained in U.*

2.2.1 Proofs of Theorem 1 and Proposition 2.6

As Proposition 2.6 implies directly Theorem 1, it remains to see that Proposition 2.6 follows from Proposition 2.1.

For that consider a diffeomorphism f, such that $\|f_*\|$ and $\|f_*^{-1}\|$ are bounded by K, and a periodic saddle P of f with a nontrivial homoclinic class. Let

$$\Sigma = H(P, f), \quad \mathcal{E} = TM|_\Sigma, \quad \text{and} \quad A = (f_*)|_\Sigma.$$

Then $(\Sigma, f, \mathcal{E}, A)$ is a continuous linear system. Denote by $\Sigma' \subset \Sigma$ the set of saddles homoclinically related to P (in particular, having the same index as P). Observe that Σ' is a dense $f-$invariant subset of Σ. Moreover, by Lemma 1.9, the subsystem induced by A over Σ' admits transitions.

If A admits an $l-$dominated splitting over Σ' then, by Lemma 1.4, such a splitting can be extended to an $l-$dominated splitting on the whole $\Sigma = H(P, f)$, and we are done.

Now take the constant $l > 0$ given by Proposition 2.1 corresponding to K, $N = \dim(M)$, and $\varepsilon/2$. If A does not admit an $l-$dominated splitting over Σ', then Proposition 2.1 says that there is an $\varepsilon/2-$perturbation $\tilde{A}$ of A and a point $x \in \Sigma'$ such that $M_{\tilde{A}}(x)$ is a homothety. We can suppose that (up to an arbitrarily small perturbation) this homothety is either a dilatation or a contraction. Assume, for instance, the first possibility.

As the system admits transitions, by Lemma 1.10, there is a dense subset of Σ' of points y admitting $\varepsilon-$deformations $\hat{A}$ along their orbits such that the corresponding linear map $M_{\hat{A}}(y)$ is a dilatation. Choose now an arbitrarily large (but finite) number of such points y, and denote by E this set of periodic orbits.

The proofs of Proposition 2.6 (thus of Theorem 1) follows now immediately from the Franks' Lemma. □

We now prove the corollaries of Theorem 1 in the Introduction.

2.2.2 Proof of Corollary 0.2

Let P be a hyperbolic saddle of a diffeomorphism f and $\mathcal{U}$ a neighbourhood of f where P admits a continuation P_g for every $g \in \mathcal{U}$.

Denote by $\mathcal{DS}$ the set of diffeomorphisms $g \in \mathcal{U}$ for which $H(P_g, g)$ admits a dominated splitting. If the closure of the interior of $\mathcal{DS}$, $cl(int(\mathcal{DS}))$, is a neighbourhood of f then the first possibility in the corollary holds and we are done. Otherwise, for any $\varepsilon > 0$ there is g_1, $\varepsilon/2-$close to f in the C^1-topology, in the complement of $cl(int(\mathcal{DS}))$. Thus g_1 has an open neighbourhood $\mathcal{U}_1$ such that for any $g \in \mathcal{U}_1$ there is h arbitrarily close to g such that $H(P_h, h)$ does not admit any dominated splitting.

Given a set $E \subset M$ and $\delta > 0$, let $V(E, \delta)$ be the set of points of M at distance strictly less than δ from E. We now construct inductively sequences $\varepsilon_i > 0$ and $g_i \in \mathcal{U}_1$ satisfying the following properties:

1. $H(P_{g_i}, g_i)$ has no dominated splitting,
2. g_{i+1} is $\varepsilon_i/2-$close to g_i in the C^1-topology,
3. there is a finite set $\mathcal{S}_{i+1}$ of periodic sinks or sources of g_{i+1} such that $H(P_{g_i}, g_i) \subset V(\mathcal{S}_{i+1}, \varepsilon_i/2)$,
4. $\varepsilon_{i+1} < \varepsilon_i/2$,
5. for all g $\varepsilon_{i+1}-$close to g_{i+1} the set of sinks or sources $\mathcal{S}_{i+1}$ has a continuation $\mathcal{S}_{i+1}(g)$ such that $H(P_{g_i}, g_i) \subset V(\mathcal{S}_{i+1}(g), \varepsilon_i)$.

Let us first end the proof of the corollary using the sequences (ε_i) and (g_i) above. The sequence (g_i) is a Cauchy sequence in $\mathrm{Diff}^1(M)$, so it converges to some C^1-diffeomorphism h. Moreover, from $\varepsilon_{i+1} < \varepsilon_i/2$ and the $\varepsilon_i/2-$proximity of g_{i+1} to g_i, we get that h is ε_i-close to g_i for all i. Therefore, by item (5), the set of sources or sinks $\mathcal{S}_{i+1}(h)$ is well defined and $H(P_{g_i}, g_i) \subset V(\mathcal{S}_{i+1}(h), \varepsilon_i)$ for every i.

Consider now the set $\mathcal{S}(h) = \bigcup_1^\infty(\mathcal{S}_i(h))$ consisting of sinks or sources. By construction, the closure of $\mathcal{S}(h)$ contains the topological upper limit set of the $H(P_{g_i}, g_i)$, that is

$$\text{closure}\,(\mathcal{S}(h)) \supset \limsup_{i\to\infty} H(P_{g_i}, g_i) = \bigcap_{i=1}^{\infty} \text{closure}\,(\bigcup_{j>i} H(P_{g_j}, g_j)).$$

Finally, by definition of homoclinic class and since the transverse intersections vary continuously, this upper limit set contains $H(P_h, h)$, so $H(P_h, h)$ is contained in the closure of the set of sinks or sources of h. Thus h is the diffeomorphism in the statement of the corollary.

To end the proof of the corollary it remains to build the sequences (ε_i) and (g_i) above. We proceed inductively, assume that ε_j and g_j are defined for every $j \le i$. Consider some finite set $\Sigma_i \subset H(P_{g_i}, g_i)$ of saddles such that $H(P_{g_i}, g_i) \subset V(\Sigma_i, \varepsilon_i)$. By item (1), applying Theorem 1 finitely many times, we can create a sink or a source close to each point of Σ_i, obtaining a diffeomorphism g_{i+1} which is $\varepsilon_i/2-$close to g_i and has a set of sinks or sources $\mathcal{S}_{i+1}$ containing $H(P_i, g_i)$ in its $\varepsilon_i/2-$neighbourhood $V(\mathcal{S}_{i+1}, \varepsilon_i/2)$. Thus g_{i+1} satisfies items (2) and (3). Having in mind the definition of $\mathcal{U}_1$, we can suppose (after a new perturbation if necessary) that $H(P_{g_{i+1}}, g_{i+1})$ has no dominated splitting, i.e. g_{i+1} satisfies item (1). Then, using the continuous variation of the finite set $\mathcal{S}_{i+1}(g)$ in a small neighbourhood of g_{i+1}, we can choose $\varepsilon_{i+1} < \varepsilon_i/2$ (item (4)) verifying item (5) above. This ends the proof of the corollary. □

2.2.3 Proof of Corollary 0.3

Recall first that there are dense open subsets $\mathcal{O}_n$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f for which the set $\mathcal{P}(f,n)$ of periodic points of period less than n is finite and hyperbolic. Remark also that the cardinal of $\mathcal{P}(f,n)$ is locally constant in $\mathcal{O}_n$ and that the set $\mathcal{P}(f,n)$ depends continuously on f.

Denote by $\Sigma(n,f) \subset \mathcal{P}(f,n)$ the set of saddles with non-trivial homoclinic class. Then there is a dense open subset $\tilde{\mathcal{O}}_n \subset \mathcal{O}_n$ of diffeomorphisms f such that $\Sigma(n,f)$ has locally constant cardinal and depends continuously on f.

Claim *There is a residual subset $\mathcal{R}_n \subset \tilde{\mathcal{O}}_n$ of diffeomorphisms f such that for any $P \in \Sigma(n,f)$ either $H(P,f)$ admits a dominated splitting or P belongs to the closure of the set of sinks or sources.*

Proof of the claim: Consider any open subset $\mathcal{O} \subset \tilde{\mathcal{O}}_n$ where the periodic points in $\Sigma(n,f)$ are continuous functions of f. So let us denote by Σ_n the (finite) set of these functions: Given $P \in \Sigma_n$ and $f \in \mathcal{O}$ we denote by P_f the corresponding periodic point of f.

For a fixed $P \in \Sigma_n$ let $\mathcal{DS}(P)$ be the set of $f \in \mathcal{O}$ such that $H(P_f, f)$ admits a dominated splitting. Let $\mathcal{U}(P)$ be the complement in $\mathcal{O}$ of the closure of the interior of $\mathcal{DS}(P)$. Let $\mathcal{U}(P,i)$ be the set of $f \in \mathcal{U}(P)$ for which there is a sink or a source Q_f (of any period) with $d(P_f, Q_f) < 1/i$: This set is open and, by Theorem 1, dense in $\mathcal{U}(P)$. Therefore the intersection

$$\mathcal{R}_n(P) = \bigcap_{i=1}^{\infty} (\mathcal{U}(P,i) \cup \mathrm{int}(\mathcal{DS}(P)))$$

is a residual subset of $\mathcal{O}$. Write

$$\mathcal{R}_n(\mathcal{O}) = \bigcap_{P \in \Sigma_n} \mathcal{R}_n(P),$$

by construction, the set $\mathcal{R}_n(\mathcal{O})$ is a residual subset of $\mathcal{O}$ consisting of diffeomorphisms f satisfying the conclusion in the claim. Thus to end the proof of the claim it suffices to consider the set $\mathcal{R}_n$ obtained as the union (over all the open sets $\mathcal{O} \subset \tilde{\mathcal{O}}_n$) of the $\mathcal{R}_n(\mathcal{O})$.
□

We are now ready to end the proof of the corollary. Consider $\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{R}_n$. By the claim, the set $\mathcal{R}$ is a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that for any saddle P of f there are two possibilities, either $H(P,f)$ has a dominated splitting, or P is in the closure of the set $\mathcal{S}(f)$ of sinks or sources of f (remark that if the homoclinic class of P is trivial then it admits a dominated splitting because P is hyperbolic).

In the first case we are done. In the second one, we need to check that the whole homoclinic class of P is contained in the closure of the sinks or sources. So assume that $H(P, f)$ does not admit any dominated splitting. Observe that for every saddle Q homoclinically related to P one has $H(Q, f) = H(P, f)$, thus $H(Q, f)$ has no dominated splitting. As $f \in \mathcal{R}$, we have just seen that Q is in the closure of $\mathcal{S}(f)$. Since the set of saddles homoclinically related with P is dense in $H(P, f)$ we have that $H(P, f)$ itself is contained in the closure of $\mathcal{S}(f)$. So $\mathcal{R}$ is the residual set announced in the Corollary 0.3 and the proof is complete. $\square$

2.2.4 Proof of Theorem 2

Let us first prove this theorem in the case of robustly transitive diffeomorphisms. There are two reasons for that. First, the proof of this case is simpler than the proof in the case of transitive sets (i.e. the general case). Second, proceeding in this way can emphasize the additional difficulties and subtleties of the proof for transitive sets.

Proof of Theorem 2 for robustly transitive diffeomorphisms. Consider a C^1-robustly transitive diffeomorphism f and an open neighbourhood $\mathcal{U}$ of f such that any $g \in \mathcal{U}$ is transitive. Reducing the size of $\mathcal{U}$ if necessary, we can assume that there are $K > 0$ and $\varepsilon > 0$ such that every $\varepsilon-$perturbation h of any $g \in \mathcal{U}$ is transitive and the differentials h_* and h_*^{-1} are bounded by K.

Recall that by the Pugh's Closing Lemma [P] there is a residual subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms whose non-wandering set is the closure of the hyperbolic periodic points. So there is a residual subset $\mathcal{R}_0$ of $\mathcal{U}$ of diffeomorphisms g having a dense set of hyperbolic saddles (note that due to the transitivity the diffeomorphisms in $\mathcal{U}$ have neither sinks nor sources).

Moreover, [BD$_2$, Theorem B] says that there is a residual set $\mathcal{R}_1$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that two periodic points of f belong to the same transitive set if and only if their homoclinic classes are equal. Thus for any $g \in \mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1$ and every periodic point P_g of g the homoclinic class $H(P_g, g)$ is the whole manifold M.

By the robust transitivity of the g, given by the choice of ε, it is not possible to create a sink or a source by an $\varepsilon-$perturbation of any $g \in \mathcal{R}$. So Proposition 2.6 gives l such that every $g \in \mathcal{R}$ admits an $l-$dominated splitting on $M = H(P_g, g)$. Finally, choosing a sequence $g_n \in \mathcal{R}$ converging to f, Corollary 1.5 ensures that f admits an $l-$dominated splitting, ending the proof of the theorem for robustly transitive diffeomorphisms. $\square$

Proof of Theorem 2, general case. Let $\Lambda_f = \bigcap_{-\infty}^{+\infty} f^i(\bar{U}) \subset U$ be a C^1-robustly transitive set in some open set U. Assume that Λ_f is not reduced to a single hyperbolic orbit (in that case we have nothing to do).

The proof follows essentially along the arguments in the robust transitive case before, but we need to pay special attention to the following fact: In the transitive case ($U = M$) all the orbits we consider are automatically in $\Lambda_f = M$ (that is a tautology), but *a priori* this does not happen when $U \neq M$. Let us go into the details of the proof of this case.

Let $\mathcal{U}$ be a C^1-open neighbourhood of f such that, for every $g \in \mathcal{U}$, the maximal invariant set $\Lambda_g = \bigcap_{-\infty}^{+\infty} g^i(\bar{U})$ is transitive. As above, using the Pugh's Closing Lemma and [BD$_2$, Theorem B], we get a residual subset $\mathcal{R}_0$ of $\mathcal{U}$ of diffeomorphisms g such that the hyperbolic periodic points of Λ_g are dense in Λ_g and have the same homoclinic classes. Thus, for every $g \in \mathcal{R}_0$, the set Λ_g is included in the homoclinic class $H(P, g)$ of some periodic point P. However, that is the special difficulty of this case, we do not know *a priori* if $H(P,g)$ is contained in U (in the robust transitive case that is obvious: $U = M$!)

To solve this problem denote by $H(P,g,U)$ the points of the closure of the transverse intersections of the invariant manifolds of P whose orbits remain in U. We call this set the *homoclinic class of P in U*. So $H(P,g,U)$ is a transitive compact subset of Λ_g and, since $\Lambda_f \subset U$, it is far from the boundary of U.

Lemma 2.7 *There is a residual set $\mathcal{R}_2$ of $\mathcal{U}$ such that for any $g \in \mathcal{R}_2$ there is a periodic point P such that $H(P,g,U) = \Lambda_g$.*

Proof: The proof is identical to [BD$_2$, Theorem B] using the following version of Hayashi's Connecting Lemma. So we do not go into the details.

Theorem (Hayashi's Connecting Lemma) *Let M be a compact manifold, U an open set of M, V an open set relatively compact in U, and f a diffeomorphism defined on M.*

Assume that there are periodic saddles P and Q whose orbits are contained in U, a sequence of points x_i converging to some point $x \in W^u_{loc}(P,f)$, and a sequence of positive integers n_i such that $f^{n_i}(x_i)$ converges to some $y \in W^s_{loc}(Q,f)$. Suppose also that for any i and every $m \in \{0,\dots,n_i\}$ one has $f^m(x_i) \in V$.

Then there is g arbitrarily C^1-close to f such that $x \in W^u(P,g) \cap W^s(Q,g)$. Moreover, the whole orbit of x is contained in U and $g^n(x) = y$ for some $n > 0$.

□

To prove Theorem 2 we apply Proposition 2.6, so we need to see that the sets $\Lambda_g = H(P_g,g,U)$ given by Lemma 2.7 are not all reduced to the single periodic orbit P_g.

Lemma 2.8 *There is g arbitrarily C^1-close to f having at least two different hyperbolic periodic orbits contained in U.*

Using Lemma 2.7, this implies that one can choose g such that $\Lambda_g = H(P_g,g,U)$ is not reduced to the periodic orbit P_g.

Proof: We are assuming that Λ_f is not reduced to a single periodic orbit (this is just the trivial case). Hence, if f has at least two periodic orbits the result is immediate: After perturbation we can make these orbits hyperbolic ones.

So it remains to consider the case where f has not periodic orbits. We know that there are diffeomorphisms g close to f having periodic orbits. We argue by contradiction, suppose that every g (with periodic points) close to f has only one periodic orbit Q. Then considering an isotopy from g to f we get a bifurcation of this periodic orbit. After a perturbation, we get a saddle-node, a flip, or a Hopf bifurcation. In these three cases, a new perturbation gives two periodic orbits: A saddle-node and a flip split into two hyperbolic periodic points, and a Hopf point into a periodic point and an invariant circle (in this case to get a new periodic point it is enough modify the rotation number of the restriction of the map to the invariant circle). □

By definition of robust transitivity, there are no perturbations of f having sinks or sources whose orbits are contained in U. Take a sequence $g_n \to f$ of diffeomorphisms such that $\Lambda_{g_n} = H(P_{g_n}, g_n, U)$ is nontrivial. Proposition 2.6 implies that there is $l \in \mathbb{N}$ such that

Λ_{g_n} admits an $l-$dominated splitting for any n large enough. Corollary 1.5 now implies that these dominated splittings induce an $l-$dominated splitting on $\limsup_{n\to\infty}(\Lambda_{g_n})$, and so on Λ_f. Now the proof of the theorem is complete. $\square$

2.2.5 Proof of Theorem 3

This theorem is a direct consequence of Proposition 2.6. We argue as follows, let $m \in \mathbb{N}$, $K > 0$, and $\varepsilon > 0$ such that any diffeomorphism g which is $\varepsilon-$close to f has less than m sinks and sources, and g_* and $(g_*)^{-1}$ are both uniformly bounded by K. Let l_0 be the constant $l(K, \varepsilon/2, \dim(M))$ given by Proposition 2.6. Then, for every g $\varepsilon/2-$close to f and any saddle P of g having a non-trivial homoclinic class $H(P,g)$, we have that $H(P,g)$ has an l_0-dominated splitting. $\square$

To prove Theorem 4 we need new arguments of a very different nature: The notion of *finest dominated splitting* and the *ergodic closing lemma* of Mañé, so let us postpone its proof until Section 6 of this paper.

We also postpone until the end of the paper (Section 7) the results about conservative diffeomorphisms.

3 Two-dimensional linear systems

In this section we give a version (Proposition 3.1) of Proposition 2.1 for two dimensional systems (without requiring transitions) following an argument essentially due to Mañé in [Ma$_3$]. In the next sections, using an argument of reduction of the dimension of the system (quotients and restrictions of linear systems, see Section 4.1) we deduce from this two dimensional result the general version of it (see Section 5).

Proposition 3.1 *Given any $K > 0$ and $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that for every two-dimensional linear system $(\Sigma, f, \mathcal{E}, A)$, with norm $\|A\|$ bounded by K and such that the matrices $M_A(x)$ preserve the orientation, one has that*

- *either A admits an $l-$dominated splitting,*
- *or there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that $M_{\tilde{A}}(x)$ has a complex (non-real) eigenvalue.*

The difference between Proposition 3.1 and Proposition 2.1 (in the case of $2-$dimensional systems) is that here we get a complex eigenvalue instead of a homothety. In fact, if the system admits transitions then one can use this complex eigenvalue to get homotheties (this will be done later in any dimension, see Proposition 2.5).

We begin the proof of Proposition 3.1 by a very elementary lemma whose proof we omit:

Lemma 3.2 *For every $\alpha > 0$ and every matrix $M \in GL_+(2,\mathbb{R})$ having two different eigenspaces E_1 and E_2 whose angle is less than α there is $s \in [-1,1]$ such that $R_{s\,\alpha} \circ M$ has a complex (non-real) eigenvalue (here $R_{t\,\alpha}$ denotes the rotation of angle $t\,\alpha$).*

In what follows, for notational simplicity, let us write $I_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}$.

Lemma 3.3 *For every $\alpha > 0$ and $\mu > 0$ there is $c > 1$ verifying the following property: Consider the matrices*

$$B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \; and \, C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \quad such\ that \ \frac{|b_1|}{|b_2|} > c \ \ and \ \ \frac{|b_1\, c_1|}{|b_2\, c_2|} < 1.$$

Then the angle between the eigenvectors of the matrix $D = B \circ I_\mu \circ C$ is less than α.

Proof: Observe first that $(1,0)$ is an eigenvector of the matrix D. The heuristic idea of the proof is very simple: Consider the vector $(1,\beta)$, for some *small* $\beta \le 2/(c\,\mu)$ fixed. As $|b_1/b_2|$ and $|c_2/c_1|$ are large (i.e. greater than c) the vectors $B^{-1}(1,\beta)$ and $C(1,\beta)$ are almost vertical (angle with the vertical less than μ). The role of the matrix I_μ now is to send the direction of $C(1,\beta)$ into the direction of $B^{-1}(1,\beta)$, thus $(1,\beta)$ is an eigenvector of D.

The precise calculations are not more complicated: Let $(1,\beta)$, $\beta \ne 0$, be some eigenvector of D not parallel to $(1,0)$ (i.e. associated to the eigenvalue $b_2\,c_2$). Then β satisfies

$$|\beta| = \frac{|b_2\,c_2 - b_1\,c_1|}{|\mu\, b_1\, c_2|} = \frac{|b_2/b_1 - c_1/c_2|}{\mu} < \frac{2}{c\,\mu}.$$

This completes the proof of the lemma. $\square$

Consider a periodic system of matrices (Σ, f, A) in $GL_+(2,\mathbb{R})$ such that all the matrices of the system are diagonal. Thus the canonical splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is invariant. Given $x \in \Sigma$ denote by $\sigma(x)$ and $\lambda(x)$ the eigenvalues of $M_A(x)$ associated to the vertical direction $(\{0\} \times \mathbb{R})$ and the horizontal direction $(\{0\} \times \mathbb{R})$, respectively. Up to a trivial change of coordinates, one can assume that for any $x \in \Sigma$, the eigenvalue $\sigma(x)$ of $M_A(x)$ is bigger in modulus than the eigenvalue $\lambda(x)$.

Lemma 3.4 *For any $\varepsilon > 0$, $\alpha > 0$, and $K > 0$ there is $l \in \mathbb{N}$ with the following property:*

Consider a periodic system (Σ, f, A) of diagonal matrices in $GL_+(2,\mathbb{R})$ as above, bounded by K such that $|\sigma(x)| \ge |\lambda(x)|$ for every $x \in \Sigma$.

Suppose that the splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is not $l-$dominated. Then there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that the angle between the eigenspaces of $M_{\tilde{A}}(x)$ is less than α.

Proof: Let us write

$$A(x) = \begin{pmatrix} a(x) & 0 \\ 0 & b(x) \end{pmatrix}.$$

Observe that if the splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$ is $l-$dominated then for every $x \in \Sigma$ one has

$$2\,|\prod_0^{l-1} a(f^i(x))| < |\prod_0^{l-1} b(f^i(x))|.$$

Recall that, by Lemma 1.2, there is $\mu > 0$ (depending on ε and K) such that multiplying matrices $A(x)$ by diagonal matrices $\mu-$close to identity one gets $\varepsilon/3-$perturbations of A.

Suppose first that there is x in Σ such that

$$|\sigma(x)| \leq (1+\mu)^{2\,p(x)}\,|\lambda(x)|, \qquad \text{where } p(x) \text{ is the period of } x.$$

Then multiplying the matrices $A(f^i(x))$ by some matrix of the form

$$\begin{pmatrix} 1+\nu & 0 \\ 0 & 1/(1+\nu) \end{pmatrix}, \quad \text{for some } \nu \in [0,\mu],$$

we get an $\varepsilon/3-$perturbation A' of A such that $M_{A'}(x)$ is an homothety. So given any pair of (different) directions of $\mathbb{R}^2$, there is an arbitrarily small perturbation $\tilde{A}$ of A' such that $M_{\tilde{A}}(x)$ has two eigenvectors parallel to such directions. This ends the proof of the lemma in this first case.

So we can now assume that

$$|\sigma(x)| > (1+\mu)^{2p(x)}\,|\lambda(x)| \quad \text{ for every } x \in \Sigma.$$

Consider the constant c, given by Lemma 3.3, associated to α and μ, and l such that

$$(1+\mu)^l > 2\,c.$$

We show that, for any system of matrices (Σ, A, f) bounded by K, l is the constant announced in the statement of the lemma.

Recall the observation in the beginning of the proof of the lemma, since the canonical splitting $\mathbb{R}^2 = \mathbb{R}\oplus\mathbb{R}$ is not $l-$dominated, then there is $x \in \Sigma$ such that

$$2\,|\prod_0^{l-1} a(f^i(x))| \geq |\prod_0^{l-1} b(f^i(x))|.$$

Assume first that $l < p(x)$. Given $y \in \Sigma$ let

$$\tilde{a}(y) = (1+\mu)\,a(y) \text{ if } y = f^i(x), i \in \{0,\cdots,l-1\}, \quad \tilde{a}(y) = a(y) \text{ if } i \in \{l,\ldots,p(x)-1\}.$$

Consider now the $\varepsilon/3-$perturbation $\tilde{A}$ of A given by

$$\tilde{A}(y) = \begin{pmatrix} \tilde{a}(y) & 0 \\ 0 & b(y) \end{pmatrix}.$$

Let $B = \tilde{A}^{(l)}(x)$ (the product of the matrices of the system $\tilde{A}$ along the orbit of x from time 0 to time $l-1$) and $C = \tilde{A}^{(p(x)-l)}(f^l(x))$. Then, we get

$$M_{\tilde{A}}(x) = C\circ B \quad \text{and} \quad M_{\tilde{A}}(f^l(x)) = B\circ C.$$

Observe that B and C verify the hypotheses of Lemma 3.3, so the angle between the eigenvectors of the matrix $D = B\circ I_\mu \circ C$ is less than α.

Denote by $\bar{A}$ the $\varepsilon/3-$perturbation of $\tilde{A}$ obtained modifying only the matrix $\tilde{A}(f^{-1}(x))$ $= A(f^{-1}(x))$ by replacing this matrix by $\bar{A}(f^{-1}(x)) = I_\mu \circ A(f^{-1}(x))$. Then the angle between the eigenvectors of $M_{\bar{A}}(f^l(x)) = D$ is less than α.

To finish the proof of lemma it remains the case $l \geq p(x)$. Remark that l can not be a multiple of $p(x)$, hence $l = k\,p(x) + l_0$, for some $k \geq 1$ and $1 \leq l_0 < p(x)$. By hypothesis,

$$(1+\mu)^{2\,p(x)}\,|\prod_{0}^{p(x)-1} a(f^i(x))| < |\prod_{0}^{p(x)-1} b(f^i(x))|.$$

Since we are assuming that the splitting is not $l-$dominated, we also have

$$2\,|\prod_{0}^{l-1} a(f^i(x))| \geq |\prod_{0}^{l-1} b(f^i(x))|.$$

So we get that

$$|\prod_{0}^{l_0-1} a(f^i(x))| > \frac{(1+\mu)^{2\,k\,p(x)}}{2}\,|\prod_{0}^{l_0-1} b(f^i(x))|.$$

Finally, note that $2\,k\,p(x) > l$ and recall the choice of l (i.e. $(1+\mu)^l > 2\,c$), thus

$$\frac{(1+\mu)^{2\,k\,p(x)}}{2} > \frac{(1+\mu)^l}{2} > c.$$

Thus, as in the previous case, we can apply Lemma 3.3 to the matrices $B = A^{(l_0)}(x)$ and $C = A^{(k\,p(x)-l_0)}(f^{l_0}(x))$. This completes the proof of the lemma. $\square$

3.1 Proof of Proposition 3.1

The proof of Proposition 3.1 follows almost directly from Lemmas 1.2, 3.2, and 3.4. Let us go into the details.

For a fixed $\varepsilon > 0$ and $K > 0$, consider a system of matrices (Σ, f, A) in $GL_+(2,\mathbb{R})$, bounded by K, such that it is not possible to create a complex eigenvalue by an $\varepsilon-$perturbation of A. We prove that such a system is $l-$dominated. This clearly implies the proposition.

By Lemma 1.2, there is $\alpha = \alpha(K,\varepsilon) > 0$ such that the composition of the system with a rotation of angle less than α gives an $\varepsilon/2-$perturbation of the system. So Lemma 3.2 ensures that for any $x \in \Sigma$ the angle between the eigenspaces of $M_A(x)$ is bigger than α. This means that there are $K_0 = K_0(\alpha)$ and a family of matrices $P(x)$, $x \in \Sigma$, with $P(x)$ and $P^{-1}(x)$ bounded by K_0, such that the eigenspaces of $P(x) \circ M_A(x) \circ P(x)^{-1}$ are orthogonal.

Let $B = P \circ A \circ P^{-1}$ be the system defined by $B(x) = P(f(x)) \circ A(x) \circ P(x)^{-1}$. By Lemma 1.2, there are $K_1 = K_1(K, K_0)$ and $\delta = \delta(K, K_0, \varepsilon)$ such that B and B^{-1} are bounded by K_1 and any $\delta-$perturbation of B is obtained by conjugating by P some $\varepsilon-$perturbation of A.

By construction, all the eigenspaces of the matrices $M_B(x)$ are orthogonal. Thus, by an orthonormal change of coordinates, we can assume that B let invariant the canonical splitting $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$, and that the eigenvalue of $M_B(x)$ corresponding to the vertical direction is bigger (in modulus) than the eigenvalue associated to the horizontal direction. Finally, there is no possible to create a complex eigenvalue by a $\delta-$perturbation of B

Fix $K_2 = K_2(K_1)$ such that any $\delta-$perturbation of B is bounded by K_2. Consider now $\alpha_2 = \alpha_2(K_2)$ such that the composition by a rotation of angle at most α_2 of a linear system bounded by K_2 gives a $\delta/3-$perturbation. Then (following Lemma 3.2) given any $\delta/3-$perturbation $\tilde{B}$ of B the angles between the eigenspaces of $M_{\tilde{B}}(x), x \in \Sigma$, are bigger than α_2. So, from Lemma 3.4, we get $l_0(\delta, K_1, \alpha_2)$ such that B is l_0-dominated.

Using that $A = P^{-1} \circ B \circ P$, where P is bounded by K_0, we get $l(l_0, K_0)$ such that A is $l-$dominated.

Now, to conclude the proof it is enough to remark that all the constants introduced in the proof are functions of K and ε. □

4 Invariant subbundles: Reduction of the dimension and the finest dominated splitting

4.1 Quotient of linear systems and restriction to subbundles

The proof of Proposition 2.4 uses successively Proposition 3.1 on 2−dimensional subbundles. Our construction also involves an argument (considering quotient spaces) that allows us to reduce the dimension of the ambient space. We pay special attention to the invariant subbundles of a linear system, and we will often need to compare the action of the system on them. This motivates the introduction of the notions of *restriction* and *quotient* of a linear system.

Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system and F an invariant subbundle of $\mathcal{E}$ (with constant dimension). We denote by A_F the *restriction of* A *to* F and by A/F the *quotient of* A *along* F endowed with the metric of the orthogonal complement $F^\perp$ of F, i.e. given a class $[v]$ we let

$$|[v]| = |v_F^\perp|, \quad \text{where } v = v_F^\perp + v_F,\ v_F^\perp \in F^\perp, \text{ and } v_F \in F.$$

Write A in blocks of the form
$$\begin{pmatrix} A_F & B \\ 0 & C \end{pmatrix},$$
since $C = A/F = (P_{F^\perp} \circ A)/F$, where $P_{F^\perp}$ is the orthogonal projection on $F^\perp$, we have the following lemma:

Lemma 4.1 *Given any $\varepsilon > 0$ one has that*

- *every $\varepsilon-$perturbation of A_F is the restriction of an $\varepsilon-$perturbation of A keeping invariant the other eigenvalues (but not necessarily the eigenspaces). Actually, A/F is not modified.*
- *Any $\varepsilon-$perturbation of A/F is the quotient of an $\varepsilon-$perturbation of A keeping invariant A_F.*

The definition of domination of a linear system, recall Definition 1.3, has a direct generalization for pairs of invariant subbundles. Suppose that E and F are two invariant subbundles of a linear system $(\Sigma, f, \mathcal{E}, A)$, we say that E *is $l-$dominated by* F if for every $x \in \Sigma$ one has

$$\|A_E^{(l)}(x)\| \, \|A_F^{(-l)}(f^l(x))\| < 1/2.$$

In this case we write $E \prec F$ or $E \prec_l F$. It is easy to see that this dominance implies that for any $x \in \Sigma$ the intersection $E(x) \cap F(x)$ is reduced to the zero-vector. So this definition is equivalent to say that $E \oplus F$ is an $l-$dominated splitting of the system $A_{E\oplus F}$.

In what follows we will often use the following very easy lemma whose proof we omit:

Lemma 4.2 *For any $K > 0$ and $l > 0$ there are $K_0 > 0$, $l_0 > 0$, and $K_1 > 0$ satisfying the following property.*

Consider a linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K with an invariant splitting $E \prec_l F$. Then there is a linear change of coordinates P, bounded by K_0, such that the bundles $P(E)$ and $P(F)$ are orthogonal and invariant by the system defined by $B = P \circ A \circ P^{-1}$. Moreover, $P(E) \prec_{l_0} P(F)$ (for B) and B is bounded by K_1.

One of the main difficulties in the proof of Proposition 2.4 comes from the fact that the dominance has not a good behavior if one consider sums of dominated subbundles. The following remark illustrates this difficulty:

Remark 4.3 *There exist linear maps and invariant bundles $\mathcal{E} = E \oplus F \oplus G$ such that $E \prec F$ and $E \prec G$, but the splitting $E \oplus (F \oplus G)$ is not dominated.*

This difficulty comes from the relationship between dominance and *angles* (the angles between two bundles of a dominated splitting can not be *very* small), and the following easy geometric observation: One can have simultaneously *big* angles between E and F, $\angle(E,F)$, and E and G, $\angle(E,G)$, but an arbitrarily small angle $\angle(E, F \oplus G)$. See the next example, and Lemma 4.2 and Remark 4.5 below.

Example 2 *Consider the quotient Σ of $\mathbb{Z} \times \mathbb{N}$ by the relation $(n,m) = (n+3m, m)$ and the map $f\colon \Sigma \to \Sigma$ given by $(n,m) \mapsto (n+1, m)$. Denote by $\mathcal{E}$ the trivial bundle of fiber $\mathbb{R}^3$. Define the linear map A (acting on $\mathbb{R}^3$) by*

$$A(n,m) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{if } -m \le n < 0, \, , \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{if } 0 \le n < m, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{if } m \le n < 2m. \end{cases}$$

Then we have that $(\Sigma, f, \mathcal{E}, A)$ is a periodic linear system.

Observe that the directions $e_1\mathbb{R}$, $(e_2 + e_3)\mathbb{R}$, and $e_3\mathbb{R}$ define a splitting of $\mathcal{E}(0,m)$ by eigenspaces of the matrix $M_A(0,m)$. We now define E, F, and G at the point (n,m) as the images of these spaces by A^n.

By construction, $\mathcal{E}_0 = E \oplus F \oplus G$ is an invariant splitting and A induces an isometry on E and a dilatation of ratio bigger than 2 on G and F. So we get $E \prec_1 F$ and $E \prec_1 G$. However, this system does not admit any dominated splitting: There are vectors $e_2 \in F \oplus G$ which are contracted during an arbitrarily large time.

The following lemma explains how we solve the difficulty above.

Lemma 4.4 *Consider any $K > 0$ and $l \in \mathbb{N}$. Then there exists L with the following property: Given any linear system $(A, f, \mathcal{E}, \Sigma)$ such that $\|A\|$ is bounded by K with an invariant splitting $E \oplus F \oplus G$, one has*

1. *$(E \prec_l F$ and $E/F \prec_l G/F) \Rightarrow E \prec_L (F \oplus G)$,*
2. *$(F \prec_l G$ and $E/F \prec_l G/F) \Rightarrow (E \oplus F) \prec_l G)$.*

Proof: Observe that Lemma 4.2 allows us to assume that (up to a bounded change of coordinates) E is orthogonal to F and E/F is orthogonal to G/F (in the quotient space). This means that E is ortoghonal to $P_{F^\perp}(G)$, thus E is also orthogonal to G.

Hence at each point $x \in \Sigma$ we can choose an orthonormal basis $(e_i(x))$ of $\mathcal{E}_x$ such that $e_i(x) \in E(x)$ for $i \leq \dim(E(x))$, and $e_i(x) \in F(x)$ for $\dim(E(x)) < i \leq \dim(E(x)) + \dim(F(x))$. Moreover, $G(x)$ is contained in the subspace spanned by the $e_i(x)$ for $i > \dim(E(x))$. So this subspace is $F(x) \oplus G(x)$ and it is invariant by A. Using the basis $(e_i(x))$ we can write the matrices of the system A as follows,

$$A(x) = \begin{pmatrix} A_E(x) & 0 & 0 \\ 0 & A_F(x) & B(x) \\ 0 & 0 & C(x) \end{pmatrix},$$

where C is $A/(E \oplus F)$ and B is bounded by a constant K_1 depending only on K and l. Write

$$D = A_{F \oplus G} = \begin{pmatrix} A_F & B \\ 0 & C \end{pmatrix}.$$

We can now consider D as a linear system over $F \oplus G$, which allows us to iterate D. We prove that there is L (depending on K and l) such that

$$\|A_E^L(x)\| \, \|D^{-L}(x)\| < 1/2,$$

that is exactly the first assertion in the lemma ($E \prec_L F \oplus G$). To prove this claim observe that

$$D^{-l} = \begin{pmatrix} A_F^{-l} & H \\ 0 & C^{-l} \end{pmatrix}, \quad \text{where} \quad H = -\sum_{j=1}^{l} A_F^{-j} \circ B \circ C^{j-l-1}.$$

Therefore all the matrices in the definition of D^{-l} are bounded by some constant K_2 depending on K and K_1, and so depending on K and l. An elementary calculation shows that for $i \geq 1$

$$D^{-il} = \begin{pmatrix} A_F^{-il} & \sum_0^{i-1} A_F^{-jl} \circ H \circ C^{(j+1-i)l} \\ 0 & C^{-il} \end{pmatrix}.$$

By hypotheses,

$$\|A_E^l\|\,\|A_F^{-l}\| < 1/2 \quad \text{and} \quad \|A_E^l\|\,\|C^{-l}\| < 1/2,$$

the last inequality follows immediately from $E/F \prec_l G/F$. Thus

$$\sup\{\|A_F^{-i\,l}\|,\, \|C^{-i\,l}\|\} < \frac{1}{2^i\,\|A_E^{i\,l}\|}.$$

An elementary estimate shows that the norm of a matrix of the form $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ is bounded by $\sup\{\|X\|, \|Z\|\} + \|Y\|$. So we get that

$$\|D^{-i\,l}\| \leq \frac{1}{2^i\,\|A_E^{i\,l}\|}\,(1 + 2\,i\,K_1)$$

It is now immediate that

$$\|A_E^{i\,l}\|\,\|D^{-i\,l}\| < \frac{1}{2^i}(1 + (2\,i\,K_1)).$$

Taking i big enough, this product is clearly less than $1/4$, ending the proof of our claim. This completes the first part of the lemma, the second one follows analogously, so we omit it. $\square$

Remark 4.5

- *The relations $\prec$ and $\prec_l$ are (strict and partial) order relations on the set of $A-$invariant subbundles of a linear system $(\Sigma, f, \mathcal{E}, A)$: The transitivity and the strict anti-symmetry of these relations are clear.*

- *Let E and G be two subbundles of $\mathcal{E}$, the angle $\angle(E,G)$ between E and G is the infimum of the angles (u,v), where $u \in E_x, v \in G_x, x \in \Sigma$. By Lemma 4.2, if $E \prec_l G$ then the angle $\angle(E,G)$ between E and G is greater than some constant α depending on K and l.*

- *Let E and F be two subbundles such that $E \prec F$ for $A_{E\oplus F}$. Let G be an $A-$invariant subbundle such that $G \cap E = G \cap F = 0$ and that the angles $\angle(E,G)$ and $\angle(F,G)$ are bigger than some $\alpha > 0$. Then E/G is dominated by F/G. Moreover, the constant of domination depends only on the bound K of A, the constant of domination of the splitting $E \prec F$, and the angle α. The easy idea of the proof is the following: As the angle $\angle(E,G)$ is greater than α, then the metrics of E/G (i.e. the metric in $G^\perp$) and of E are obtained by a bounded change of coordinates. The same is true for F. It is now enough to go back to the definition of domination.*

As a corollary we now get,

Lemma 4.6 *Let $K > 0$ and $l \in \mathbb{N}$. There is L such that for any linear system $(\Sigma, f, \mathcal{E}, A)$ bounded by K and any invariant subbundles E, F, and G, such that $E \prec_l F$ and $F \prec_l G$, one has $E \prec_L F \oplus G$ and $E \oplus F \prec_L G$.*

Proof: As the $E \prec_l F$ and $F \prec_l G$ there is a constant α (depending only on K and l) such that the angles $\angle(E,F)$ and $\angle(F,G)$ are greater than α. By transitivity of the relation $\prec_l$ one has $E \prec_l G$. So, by the third part of the remark, we get l_1 (depending on K, l, and α) such that $E/F \prec_{l_1} G/F$. Now the corollary follows directly from Lemma 4.4 $\square$

4.2 The finest dominated splitting

Lemma 4.4 allows us to define the notions of *undecomponible* and *finest dominated splittings.* Let us recall that an invariant splitting $\mathcal{E} = E_1 \oplus \cdots \oplus E_k$ is dominated if for any $1 \le j \le k-1$ one has $\bigoplus_1^j E_i \prec \bigoplus_{j+1}^k E_i$.

Corollary 4.7 *A splitting $\mathcal{E} = \bigoplus_{i=1}^k E_i$ is dominated if and only if $E_i \prec E_{i+1}$ for every $i \in \{1, \dots, k-1\}$.*

Proof: Observe that if $\mathcal{E} = \bigoplus_{i=1}^k E_i$ is dominated then one has $E_i \prec E_{i+1}$ straightforwardly from the definition. To prove the converse we argue inductively (on the number k of subbundles of the splitting). For $k = 3$ the corollary is exactly Lemma 4.6. Assume now that the lemma is true for $k-1$.

We have to prove that if $E_i \prec E_j$ for every $i < j$ then the splitting is dominated, that is $\bigoplus_1^i E_j \prec \bigoplus_{i+1}^k E_j$ for all $0 < i < k$. Assume first that $i > 1$. Applying the induction hypothesis to the restrictions of A to $\bigoplus_2^k E_j$ and $\bigoplus_1^i E_j$, we get $\bigoplus_2^i E_j \prec \bigoplus_{i+1}^k E_j$ and $E_1 \prec \bigoplus_2^i E_j$. Now the dominance $\bigoplus_1^i E_j \prec \bigoplus_{i+1}^k E_j$ follows applying Lemma 4.6 to the subbundles E_1, $\bigoplus_2^i E_j$, and $\bigoplus_{i+1}^k E_j$.

Finally, if $i = 1$ we apply the induction hypothesis to $\bigoplus_1^{k-1} E_j$ and $\bigoplus_2^k E_j$. This gives $E_1 \prec \bigoplus_2^{k-1} E_j$ and $\bigoplus_2^{k-1} E_j \prec E_k$. Now the conclusion follows as above applying Lemma 4.6 to these three subbundles. $\square$

Corollary 4.7 above allows us to use the notation $E_1 \prec E_2 \prec \cdots \prec E_k$ to denote a dominated splitting $\bigoplus_1^k E_j$.

Lemma 4.8 *Let $E_1 \prec \cdots \prec E_k \prec F \prec G_1 \cdots \prec G_m$ be a dominated splitting such that A_F admits a dominated splitting $F_1 \prec F_2$. Then $E_1 \oplus \cdots \oplus E_k \oplus F_1 \oplus F_2 \oplus G_1 \oplus \cdots \oplus G_m$ is a dominated splitting.*

Proof: It suffices to verify that $E_k \prec F_1$ and $F_2 \prec G_1$. $\square$

Definition 4.9 *Let $\bigoplus_1^k E_i$ and $\bigoplus_1^m F_j$ be two dominated splittings of $\mathcal{E}$, $\mathcal{E} = \bigoplus_1^k E_i$ and $\bigoplus_1^m F_j$. We say that the splitting $\bigoplus_1^m F_j$* is finer *than $\bigoplus_i^k E_i$ if every E_i is the direct sum of some of the F_j, i.e. $E_i = \bigoplus_{j_1}^{j_2} F_j$. In this case we write $\bigoplus_1^m F_j \sqsubseteq \bigoplus_1^k E_i$.*

Remark 4.10 *Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system of dimension N.*

- *The relation $\sqsubseteq$ is an order relation on the set of dominated splittings of a linear system $(\Sigma, f, \mathcal{E}, A)$.*

- *Given any finite sequence n_i with $\sum_i n_i = N$, there is at most one dominated splitting $E_1 \prec \cdots \prec E_k$ such that $\dim E_i = n_i$ for every i (actually this easy assertion is a direct consequence of the next lemma). As a consequence, the set of dominated splittings of $(\Sigma, f, \mathcal{E}, A)$ is finite. So there are splittings which are minimal for the relation $\sqsubseteq$, these splittings are called* undecomponible. *Moreover, given any splitting we can subdivide it to get a minimal one, i.e. every splitting is bigger, $\sqsupseteq$, than at least one undecomponible splitting.*

- *Following Lemma 4.8, a dominated splitting $\bigoplus_1^k E_i$ is undecomponible if and only if none of the subsystems A_{E_i} admits dominated decomposition.*

Proposition 4.11 *For every linear system $(\Sigma, f, \mathcal{E}, A)$ there is a unique undecomponible dominated splitting, called* the finest dominated splitting.

Proof: The first step to prove this proposition is the following lemma.

Lemma 4.12 *Let $E_1 \prec \cdots \prec E_k$ and $F_1 \prec \cdots \prec F_m$ be two dominated splittings of $\mathcal{E}$. Then either $E_1 \subset F_1$ and $\bigoplus_2^m F_i \subset \bigoplus_2^k E_i$, or $F_1 \subset E_1$ and $\bigoplus_2^k E_i \subset \bigoplus_2^m F_i$. As a consequence, if $E_1 \neq F_1$ then either E_1 or F_1 admits a dominated splitting.*

Proof: First we prove that for any $x \in \Sigma$ one of the linear spaces $E_1(x)$ and $F_1(x)$ is contained in the other one. We argue by contradiction, suppose, contrary to our hypotheses, that there are

$$x \in \Sigma, \quad u \in (E_1(x) \setminus F_1(x)), \quad \text{and} \quad v \in (F_1(x) \setminus E_1(x)).$$

As $\bigoplus_1^k E_i$ is dominated, and $u \in E_1$ and $v \notin E_1$, for every i big enough we get

$$\frac{\|A^{(i)}(x)(u)\|}{\|u\|} \leq \frac{\|A^{(i)}(x)(v)\|}{2\,\|v\|}.$$

Similarly, the dominance of $\bigoplus_1^m F_i$, $u \notin F_1(x)$, and $v \in F_1(x)$ imply (for big i)

$$\frac{\|A^{(i)}(x)(v)\|}{\|v\|} \leq \frac{\|A^{(i)}(x)(u)\|}{2\|u\|}.$$

These two inequalities give a contradiction. So we have proven that for any $x \in \Sigma$ one has $E_1(x) \subset F_1(x)$ or vice versa. As the dimensions of these spaces do not depend on x, we have that either $E_1 \subset F_1$ or $F_1 \subset E_1$.

The same arguments give the other inclusion, that is either $\bigoplus_2^m F_i \subset \bigoplus_2^k E_i$, or $\bigoplus_2^k E_i \subset \bigoplus_2^m F_i$.

Finally, assume now that $E_1 \neq F_1$ and, for instance, $E_1 \subset F_1$. Then F_1 is transverse to $\bigoplus_2^k E_i$, so we get a splitting $F_1 = E_1 \oplus (F_1 \cap \bigoplus_2^k E_i)$. This splitting is clearly invariant and dominated. This finishes the proof. $\square$

We are now ready to prove Proposition 4.11:

Let $\bigoplus_1^k E_i$ and $\bigoplus_1^m F_j$ be two undecomponible dominated splittings. Using Lemma 4.12 above we get that $E_1 = F_1$ and that $\bigoplus_2^k E_i = \bigoplus_2^m F_j$. Consider now the restriction $B = A_{\bigoplus_2^m F_j}$. Now $E_2 \prec \cdots \prec E_k$ and $F_2 \prec \cdots \prec F_m$ are two undecomponible dominated splittings of B, so $E_2 = F_2$. We conclude the proof repeating k times this argument. $\square$

4.3 Transitions and invariant spaces

Lemma 4.13 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with $\varepsilon-$transitions and $\varepsilon_0 > \varepsilon$. Consider an invariant subset $\Sigma_0 \subset \Sigma$ such that there is a dominated splitting $E_1 \prec \cdots \prec E_k$ defined over Σ_0. Then for every finite subset $\Lambda = \{x_1, \ldots, x_m\}$ of Σ_0 there are ε_0-transitions such that the corresponding linear maps (transitions) T_{x_j,x_i} map $E_l(x_i)$ on $E_l(x_j)$ for any $l \in \{1, \ldots, k\}$ and x_j, $x_i \in \Lambda$.*

Before proving this lemma let us observe that as a direct consequence of it we get the following corollary.

Corollary 4.14 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transition and $E \oplus F$ a dominated splitting. Then the induced systems A_E and A_F admit transitions.*

Proof: By hypothesis, there are $\varepsilon-$transitions $[t^0_{x_j,x_i}]$ from x_i to x_j for every pair of points in Λ. By Remark 1.7, for any n_1, $n_2 \geq 0$ the word

$$[M]_A(x_j)^{n_2}\, [t^0_{x_j,x_i}]\, [M]_A(x_i)^{n_1}$$

is also a transition . Moreover, any $(\varepsilon_0 - \varepsilon)-$perturbation of an $\varepsilon-$transition is also a ε_0-transition.

Taking an arbitrarily small perturbation of $[t^0_{x_j,x_i}]$ we can assume that its corresponding matrix $\tilde{T}^0_{j,i}$ maps $E_k(x_i)$ transversally to $\bigoplus_1^{k-1} E_m(x_j)$. Thus, by the dominance $\bigoplus_1^{k-1} E_m \prec E_k$, taking n_2 large enough, we have that

$$\left((M_A(x_j))^{n_2} \circ \tilde{T}^0_{j,i}\right)(E_k(x_i)) \quad \text{is arbitrarily close to} \quad E_k(x_j).$$

So there is a small perturbation $M_{\tilde{A}}(x_j)$ of $M_A(x_j)$ such that

$$\left(M_{\tilde{A}}(x_j) \circ (M_A(x_j))^{n_2} \circ \tilde{T}^0_{j,i}\right)(E_k(x_i)) = E_k(x_j).$$

Denote by $T^1_{j,i}$ the linear map corresponding to this new transition. Now the consider pre-image of $E_k(x_j)$ by $T^1_{j,i}$ and observe that

$$\left(M_A(x_i)^{-n_1} \circ (T^1_{j,i})^{-1}\right)(E_k(x_j)) = E_k(x_i).$$

As above, since $\bigoplus_1^{k-1} E_m \prec E_k$, for n_1 sufficiently large we get that

$$\left(M_A(x_i)^{-n_1} \circ (T^1_{j,i})^{-1}\right)\left(\bigoplus_1^{k-1} E_m(x_j)\right) \quad \text{is arbitrarily close to} \quad \bigoplus_1^{k-1} E_m(x_i).$$

So, arguing as before, the announced transition $T_{j,i}$ is obtained composing (at the right) $T^1_{j,i} \circ M_A(x_i)^{n_1}$ with a small perturbation of $M_A(x_i)$ mapping $E_k(x_i)$ into itself and $\bigoplus_1^{k-1} E_m(x_i)$ into $\left(M_A(x_i)^{-n_1} \circ (T^1_{j,i})^{-1}\right)(\bigoplus_1^{k-1} E_m(x_j))$.

Next we consider the restriction of A to $\bigoplus_1^{k-1} E_m$ to get a new perturbation of the transition mapping $E_k(x_i)$, $E_{k-1}(x_i)$, and $\bigoplus_1^{k-2} E_m(x_i)$ into the corresponding spaces for x_j. The inductive pattern of our construction is now clear. $\square$

4.4 Diagonalizable systems.

Definition 4.15 *A periodic linear system $(\Sigma, f, \mathcal{E}, B)$ of dimension N is* diagonalizable *if for every $x \in \Sigma$ the matrix $M_B(x)$ has only real positive eigenvalues with multiplicity 1.*

Denote by $\lambda_1(x) < \cdots < \lambda_N(x)$ the eigenvalues of $M_B(x)$ and by $E_i(x)$ the one dimensional eigenspace corresponding to $\lambda_i(x)$. Then $\mathcal{E} = \bigoplus_1^N E_i$ is an invariant splitting of B.

The next result says that every periodic system with transitions can be approximated by a diagonalizable subsystem (defined on a dense subset). More precisely,

Lemma 4.16 *Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Then for any $\varepsilon > 0$ there is a diagonalizable $\varepsilon-$perturbation $\tilde{A}$ of A defined on a dense invariant subset $\tilde{\Sigma}$ of Σ.*

Proof: Observe first that any matrix $M \in GL(n, \mathbb{R})$ can be perturbed (by an arbitrarily small perturbation) to get $\tilde{M}$ having only eigenvalues (complex or real) with multiplicity one and such that any pair of eigenvalues with the same modulus are complex and conjugate, having rational argument. So there is k such that $\tilde{M}^k$ has only real positive eigenvalues, but some of them may have multiplicity 2 (those coming from a complex eigenvalue of $\tilde{M}$). As above, by a small perturbation of $\tilde{M}^k$, we get a matrix M_1 having only real positive eigenvalues with multiplicity 1.

Take $x \in \Sigma$ and write $M = M_A(x)$. By the comments above, there is an $\varepsilon/10-$perturbation $\tilde{M}$ of M having only multiplicity one eigenvalues. Denote by M_1 the perturbation of an appropriate power of $\tilde{M}$ as the one built above (i.e having only real eigenvalues with multiplicity one). Let T be an $\varepsilon/10-$transition matrix from x into itself. So for any positive n_1 and n_2 there are $y \in \Sigma$ and a $3\varepsilon/10-$perturbation $\tilde{A}$ of A along the orbit of y such that

$$M_{\tilde{A}}(y) = M_1^{n_2} \circ T \circ M_1^{n_1}.$$

So, taking n_1 and n_2 sufficiently big, we can now repeat the proof of Lemma 4.13 to get a $4\varepsilon/10-$transition T_1 preserving all the eigenspaces of M_1. These spaces are $1-$dimensional, thus they are the eigenspaces of T_1 and their corresponding eigenvalues are real.

Recall (see Remark 1.7) that if T_1 is a transition from x into itself then T_1^2 is also a transition. Therefore, replacing if necessary T_1 by T_1^2, we can assume that T_1 has only real positive eigenvalues with multiplicity one. Observe that T_1 and M_1 have the same one dimensional eigenspaces, thus T_1 commutes with M_1. So, for k big enough, $M_1^k \circ T_1$ has only positive real eigenvalues with multiplicity 1. Moreover, there are some point z (whose orbit passes arbitrarily close to x) and an $\varepsilon/2-$perturbation $\hat{A}$ of A along the orbit of z such that

$$M_{\hat{A}}(z) = M_1^k \circ T_1.$$

This proves that the set $\tilde{\Sigma}$ of points $z \in \Sigma$ admitting an $\varepsilon-$perturbation along their orbits such that $M_{\hat{A}}(z)$ has only positive real eigenvalues with multiplicity 1 is dense in Σ, ending the proof of the lemma. □

In what follows, to prepare the proof of Theorem 4, we will give some volume controlling versions of our lemmas, as we do in the next remark:

Remark 4.17 *Let $(\Sigma, f, \mathcal{E}, A)$ be a linear system with transitions such that there is some point $x_0 \in \Sigma$ such that the modulus of the jacobian of the matrix $M_A(x_0)$, denoted by $J(M_A(x_0))$, is greater than one. Then we can choose the diagonalizable $\varepsilon-$perturbation $\tilde{A}$ of A given by Lemma 4.16 such that the modulus of the jacobian $J(M_{\tilde{A}}(p))$ is bigger than one for every p in the dense subset $\tilde{\Sigma}$ of Σ.*

Proof: First, by using the transitions and the existence of the point x_0 with $J(M_A(x_0)) > 1$, we get a dense subset $\hat{\Sigma}$ of Σ such that $J(M_A(x)) > 1$ for all $x \in \hat{\Sigma}$. Now we can repeat the proof of Lemma 4.16 above: Taking the exponents n_1 and n_2 large enough we get that (in the proof of the lemma) the moduli of the jacobian of the corresponding matrices $M_{\tilde{A}}(y)$ are greater than one. □

5 Dominated splittings, complex eigenvalues of rank $(i, i+1)$, and homotheties

In this section we prove Propositions 2.4 and 2.5. To prove Proposition 2.4, we use Lemma 4.4 and consider successive quotients of linear systems in order to get complex eigenvalues (obtained by using the arguments of the 2−dimensional case, see Section 3). Then the transitions and the existence of complex eigenvalues of any rank allow us to "mix all the eigenvalues" of some matrix, obtaining the homothety announced in Proposition 2.5.

5.1 Getting complex eigenvalues of any rank

Lemma 5.1 *Given $K > 0$ and $\varepsilon > 0$ there is $l \in \mathbb{N}$ such that for any diagonalizable linear periodic system $(\Sigma, f, \mathcal{E}, B)$ of dimension N and bounded by K, and any $1 \leq i \leq N-1$ one has:*

- *Either there is an $\varepsilon-$perturbation of B having a complex eigenvalue of rank $(i, i+1)$,*
- *or*
$$E_j/(E_{j+1} \oplus \cdots \oplus E_k) \prec_l E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k)$$
for every $j \leq i \leq k$.

Proof: Fix $\varepsilon > 0$ and let l be the dominance constant given by Proposition 3.1.

If $E_j/(E_{j+1} \oplus \cdots \oplus E_k)$ is $l-$dominated by $E_{k+1}/(E_{j+1} \oplus \cdots \oplus E_k)$ we are done. Otherwise, by Proposition 3.1, we can perturb the quotient to get eigenvalues $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} = \alpha$. Moreover, by Lemma 4.1, this perturbation of the quotient gives a perturbation $\tilde{B}$ of B having a pair of eigenvalues $\tilde{\lambda}_j = \tilde{\lambda}_{k+1} = \alpha$ and preserving the eigenvalues of the restriction of B to $E_{j+1} \oplus \cdots \oplus E_k$ (we denote such eigenvalues by λ_i).

Consider a small isotopy B_t producing this perturbation (i.e. $B_0 = B$ and $B_1 = \tilde{B}$) and denote by $\lambda_{j,t}$ and $\lambda_{k+1,t}$ the continuations of the eigenvalues λ_j and λ_{k+1} at time t, so

$$\lambda_{j,0} = \lambda_j, \quad \lambda_{j,1} = \tilde{\lambda}_j, \quad \lambda_{k+1,0} = \lambda_{k+1}, \quad \text{and} \quad \lambda_{k+1,1} = \tilde{\lambda}_{k+1}.$$

Moreover, we can assume that for every $0 \leq t < 1$ one has $\lambda_{j,t} < \lambda_{k+1,t}$ (otherwise we stop the isotopy at the first t such that $\lambda_{j,t} = \lambda_{k+1,t}$). Then, by continuity, following this isotopy there are three possibilities:

- at some stage t of the isotopy, $\lambda_{j,t} = \lambda_{i+1} \leq \lambda_{k+1,t}$,
- at some stage t of the isotopy, $\lambda_{j,t} \leq \lambda_i = \lambda_{k+1,t}$,
- for $t = 1$, $\lambda_i < \lambda_{j,1} = \alpha = \lambda_{k+1,1} < \lambda_{i+1}$.

In each of these cases a small perturbation gives a complex eigenvalue of rank $(i, i+1)$, ending the proof of the lemma. $\square$

To deduce Proposition 2.4 from Lemma 5.1 above we will apply successively Lemma 4.4. Even if the arguments in the proof of the proposition are rather simple and the main difficulty of it is of combinatorial type, for clearness and to organize this combinatorics, we divide the proof into two steps:

Lemma 5.2 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $L_0 \in \mathbb{N}$ such that for every diagonalizable periodic linear system $(\Sigma, f, \mathcal{E}, B)$, of dimension N and bounded by K, and for any $1 \leq i \leq N-1$ one has:*

- *Either B admits an $\varepsilon-$perturbation having a complex eigenvalue of rank $(i, i+1)$*
- *or*

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_{L_0} \bigoplus_{i+1}^{N} E_k/_{\bigoplus_{j+1}^{i} E_m}$$

for every $j \leq i$.

Before proving the lemma observe that if $j = i$ the second item of the lemma means that $E_i \prec_{L_0} \bigoplus_{i+1}^{N} E_k$.

Proof: Assume that every $\varepsilon-$perturbation of B has no complex eigenvalues of rank $(i, i+1)$. From Lemma 5.1, taking $k = i$ and $k = i+1$ we get l such that for every $j < i$ one has

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_l E_{i+1}/_{\bigoplus_{j+1}^{i} E_m} \quad \text{and} \quad E_j/_{\bigoplus_{j+1}^{i+1} E_m} \prec_l E_{i+2}/_{\bigoplus_{j+1}^{i+1} E_m}.$$

We now apply Lemma 4.4 taking

$$E = E_j/_{\bigoplus_{j+1}^{i} E_m}, \quad F = E_{i+1}/_{\bigoplus_{j+1}^{i} E_m}, \quad \text{and} \quad G = E_{i+2}/_{\bigoplus_{j+1}^{i} E_m}.$$

Observe first that

$$E/F = E_j/_{\bigoplus_{j+1}^{i+1} E_m} \quad \text{and} \quad G/F = E_{i+2}/_{\bigoplus_{j+1}^{i+1} E_m}.$$

By the comments above, $E \prec_l F$ and $E/F \prec_l G/F$. Thus by Lemma 4.4 there is l_1 such that

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_{l_1} (E_{i+1} \oplus E_{i+2})/_{\bigoplus_{j+1}^{i} E_m}.$$

Moreover, this holds for every $j \leq i$. Taking $k = i+2$ in Lemma 5.1, one gets

$$E_j/_{\bigoplus_{j+1}^{i+2} E_m} \prec_l E_{i+3}/_{\bigoplus_{j+1}^{i+2} E_m}.$$

As before, applying Lemma 4.4 to

$$E = E_j/_{\bigoplus_{j+1}^{i} E_m}, \quad F = (E_{i+1} \oplus E_{i+2})/_{\bigoplus_{j+1}^{i} E_m}, \quad \text{and} \quad G = E_{i+3}/_{\bigoplus_{j+1}^{i} E_m},$$

one gets l_2 such that

$$E_j/_{\bigoplus_{j+1}^{i} E_m} \prec_{l_2} (E_{i+1} \oplus E_{i+2} \oplus E_{i+3})/_{\bigoplus_{j+1}^{i} E_m}.$$

The inductive procedure to prove the lemma is now clear. $\square$

Lemma 5.3 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $L \in \mathbb{N}$ such that for every diagonalizable periodic periodic system $(\Sigma, f, \mathcal{E}, B)$, of dimension N and bounded by K, and any $1 \le i \le N-1$ one has that:*

- *Either B admits an $\varepsilon-$perturbation having a complex eigenvalue of rank $(i, i+1)$,*
- *or*
$$\bigoplus_{1}^{i} E_j \prec_L \bigoplus_{i+1}^{N} E_j.$$

Proof: Assume that it is not possible to perturb B to get a complex eigenvalue of rank $(i, i+1)$. Once more, the proof is an inductive argument using alternately Lemmas 5.2 and 4.4 above.

By Lemma 5.2, applied to $j = i$ and to $j = i-1$, one knows

$$E_i \prec_{L_0} \bigoplus_{i+1}^{N} E_k \quad \text{and} \quad E_{i-1}/E_i \prec_{L_0} \bigoplus_{i+1}^{N} E_k/E_i.$$

So Lemma 4.4 gives L_1 such that $(E_{i-1} \oplus E_i) \prec_{L_1} \bigoplus_{i+1}^{N} E_k$. By Lemma 5.2, taking $j = i-2$, one gets

$$E_{i-2}/(E_{i-1} \oplus E_i) \prec_{L_0} \bigoplus_{i+1}^{N} E_k/(E_{i-1} \oplus E_i).$$

Thus applying Lemma 4.4 to E_{i-2}, $E_{i-1} \oplus E_i$, and $\bigoplus_{i+1}^{N} E_k$ one obtains

$$(E_{i-2} \oplus E_{i-1} \oplus E_i) \prec_{L_1} \bigoplus_{i+1}^{N} E_k.$$

The lemma now follows by a very simple induction. $\square$

5.2 End of the proof of Proposition 2.4

Recall that Proposition 2.4 says that if there is no perturbation of the system with a complex eigenvalue of rank $(i, i+1)$ then the system has an $l-$dominated splitting of dimension i.

Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic linear system, of dimension N and bounded by K, having transitions. Assume that there are $\varepsilon > 0$ and $i \in \{1, \dots, N-1\}$ such that every $\varepsilon-$perturbation of A has no complex eigenvalues of rank $(i, i+1)$.

Choose a sequence ε_n such that $\varepsilon/2 > \varepsilon_n \to 0$. As the system $(\Sigma, f, \mathcal{E}, A)$ has transitions, using Lemma 4.16 we get dense subsets Σ_n of Σ and diagonalizable ε_n-perturbations B_n of A defined on Σ_n. Note that B_n is a diagonalizable periodic linear system of dimension N bounded by $K_1 = K + \varepsilon/2$. Thus, by hypotheses, it is impossible to create complex eigenvalues of rank $(i, i+1)$ by $(\varepsilon/2)-$perturbations of it.

By Lemma 5.3, there is $L = L(K_1, \varepsilon/2, N)$ such that every system B_n above admits an $L-$dominated splitting $E_n \oplus F_n$, with $E_n \prec_L F_n$ and $\dim(E_n) = i$. Finally, as the system A is continuous, the sets Σ_n are dense in Σ, and $\|B_n - A\| \to 0$, Lemma 1.4 ensures that A admits an $L-$dominated splitting $E \prec_L F$ with $\dim(E) = i$. The proof of the proposition is now complete. $\square$

5.3 Proof of Proposition 2.5

The naive idea of the proof of Proposition 2.5 is to use the transitions to multiply matrices corresponding to different points of Σ having complex eigenvalues of different rank. In this way one distributes homogeneously the action of the eigenvalues in the whole fibers, obtaining homotheties. The main step of the proof is the following lemma:

Lemma 5.4 *Let $(\Sigma, f, \mathcal{E}, A)$ be a continuous periodic linear system of dimension N with transitions. Fix $0 < \varepsilon_0$ and assume that there are $i \in \{1, \dots, N-1\}$ and an ε_0-perturbation of A having a complex eigenvalue of rank $(i, i+1)$.*

Then for every $0 < \varepsilon_1 < \varepsilon_0$ there is a point $p \in \Sigma$ such that for every $1 \leq i < N$ there is an ε_1-transition $[t^i]$ from p to itself with the following properties:

- *There is an ε_1-perturbation $[M]_{\tilde{A}}(p)$ of the word $[M]_A(p)$ such that the corresponding matrix $M_{\tilde{A}}(p)$ has only real positive eigenvalues with multiplicity 1. Denote by $\tilde{\lambda}_1 < \cdots < \tilde{\lambda}_N$ such eigenvalues and by $E_i(p)$ their respective (1−dimensional) eigenspaces.*
- *There is an $(\varepsilon_0 + \varepsilon_1)-$perturbation $[\tilde{t}^i]$ of the transition $[t^i]$ such that the corresponding matrix $\tilde{T}^i$ satisfies*
  - *$\tilde{T}^i(E_j(p)) = E_j(p)$ if $j \notin \{i, i+1\}$,*
  - *$\tilde{T}^i(E_i(p)) = E_{i+1}(p)$ and $\tilde{T}^i(E_{i+1}(p) = E_i(p)$.*

In fact we have also a stronger (volume controlling) version of this lemma:

Remark 5.5 *Under the hypotheses of Lemma 5.4, suppose in addition that there is a point $x \in \Sigma$ such that the modulus of the jacobian $J(M_A(x))$ is bigger than one. Then we can choose the point p and the perturbation $\tilde{A}$ in the lemma such that $J(M_{\tilde{A}}(p)) > 1$.*

Before proving Lemma 5.4 and Remark 5.5 let us deduce Proposition 2.5 from them.

5.3.1 End of the proof of Proposition 2.5

The hypothesis of the proposition (existence of ε_0-perturbations with complex eigenvalues of any rank) imply that we can apply Lemma 5.4 to the system $(\Sigma, f, \mathcal{E}, A)$ for all $i \in \{1, \dots, N-1\}$. To prove the proposition we show that for any $\varepsilon > \varepsilon_0$ there are an $\varepsilon-$perturbation $\hat{A}$ of A and a point $x \in \Sigma$ for which $M_{\hat{A}}(x)$ is an homothety.

Chosen $0 < \varepsilon_1 < (\varepsilon - \varepsilon_0)/10$, let p and $[t^i]$ be the point and the ε_1-transition from p to itself given by Lemma 5.4.

Observe that the action of the perturbed transition $\tilde{T}^i$ (which are $(\varepsilon_0+\varepsilon_1)-$perturbations of T_i) on the finite set $\{E_i(p)\}_{1\leq j\leq N}$ of eigenspaces of $M_{\tilde{A}}(p)$ is the transposition $(i, i+1)$ which interchanges $E_i(p)$ and $E_{i+1}(p)$, keeping invariant the others $E_j(p)$. Recall that a *transposition* is an order 2 permutation, thus it is equal to its inverse. Moreover, the transpositions $(i, i+1)$, $i \in \{1, \dots, N-1\}$, generate the group of (all) permutations of the finite set $\{E_j(p)\}_{1\leq j\leq N}$.

Given $0 \leq k < N$ denote by σ_k the cyclic permutation defined by $\sigma_k(E_j(p)) = E_{j+k}(p)$, where the sum $i + j$ is considered in the cyclic group $\mathbb{Z}/N\mathbb{Z}$.

As a direct consequence of the previous comments, we have that for every $0 < k < N$ there exists an element $[\tilde{S}_k]$ in the semi-group generated by the transitions $[\tilde{t}^i]$ such that its action on the finite set $\{E_i(p)\}_{1\leq j\leq N}$ is the permutation σ_k, i.e. if $\tilde{S}_k$ is the matrix corresponding to the word $[\tilde{S}_k]$ then one has $\tilde{S}_k(E_i(p)) = \sigma_k(E_i(p))$.

Let $[S_k]$ be the word of matrices corresponding to the perturbation $[\tilde{S}_k]$ in the semi-group generated by the initials $[t^j]$. As the $[t^j]$ are ε_1-transitions from p to itself, any word in the semi-group generated by the $[t^j]$ is also an ε_1-transition (recall Remark 1.7). In particular, the $[S_k]$ are ε_1-transitions from p to itself. (For completeness let us write $[S_0] = [S_N]$ the empty word whose corresponding matrix is by convention the identity).

By definition of transitions, for any $n \in \mathbb{N}$ there is a point $x_n \in \Sigma$ such that the word $[M]_A(x_n)$ is ε_1-close to the word $[W_n]$ defined by

$$\begin{aligned}[W_n] = \;& [S_1]\,[M]^n_A(p)\,[S_{N-1}]\,[S_1]\,[S_{N-1}]\,[S_2]\,[M]^n_A(p)\,[S_{N-2}]\,[S_2]\,[S_{N-2}]\cdots \\ & \cdots[S_{N-2}]\,[M]^n_A(p)\,[S_2]\,[S_{N-2}]\,[S_2]\,[S_{N-1}]\,[M]^n_A(p)\,[S_1]\,[S_{N-1}]\,[S_1]\,[M]^n_A(p).\end{aligned}$$

Let us state some properties justifying the introduction of this word:

i) Given any i, the matrix $\tilde{S}_{N-i} \circ \tilde{S}_i$ acts trivially on the set of spaces $\{E_j(p)\}$. Let us denote by $\mu_{i,j}$ the eigenvalue of $\tilde{S}_{N-i} \circ \tilde{S}_i$ corresponding to the eigenspace $E_j(p)$.

ii) Recall that $\tilde{S}_i$ maps $E_j(p)$ into $E_{i+j}(p)$ and that $M_{\tilde{A}}(p)$ is diagonal in the basis corresponding to the directions $E_k(p)$, denote by $\tilde{\lambda}_k$ the eigenvalue of $M_{\tilde{A}}(p)$ corresponding to such a direction. Hence every $E_j(p)$ is an eigenspace of $\tilde{S}_{N-i} \circ (M_{\tilde{A}}(p))^n \circ \tilde{S}_i$ whose corresponding eigenvalue is $\mu_{i,j}\,\tilde{\lambda}^n_{j+i}$.

iii) By the two items before, for every j the space $E_j(p)$ is an eigenspace of the matrix

$$\tilde{W}_{i,n} = \tilde{S}_{N-i} \circ (M_{\tilde{A}}(p))^n \circ \tilde{S}_i \circ \tilde{S}_{N-i} \circ \tilde{S}_i$$

whose corresponding eigenvalue is $\mu^2_{i,j}\,\tilde{\lambda}^n_{j+i}$. (Recall that $\tilde{W}_{0,n} = M_{\tilde{A}}(p)^n$ by convention).

Proposition 2.5 is now an immediate consequence of the following claim:

Claim *For every $n > n_0$ sufficiently large there is an $\varepsilon-$perturbation $\hat{A}$ of A along the orbit of x_n such that $M_{\hat{A}}(x_n)$ is a homothety of ratio $\tilde{\Lambda}^n$, where*

$$\tilde{\Lambda} = \prod_1^N \tilde{\lambda}_i = J(M_{\tilde{A}}(p)).$$

Proof: Denote by $[\tilde{W}_n]$ the word obtained from $[W_n]$ by putting a $\tilde{}$ above any letter S and A; this word is an $(\varepsilon_0 + \varepsilon_1)-$perturbation of the word $[W_n]$, so it is an $(\varepsilon_0 + 2\,\varepsilon_1)-$perturbation of the word $[M]_A(x_n)$. Moreover, using the notation in the item (iii) above, the corresponding matrix $\tilde{W}_n$ is the product

$$\tilde{W}_n = \tilde{W}_{N-1,n} \circ \cdots \circ \tilde{W}_{1,n} \circ \tilde{W}_{0,n}.$$

So, by item (iii) above, for every j, the one-dimensional space $E_j(p)$ is an eigenspace of $\tilde{W}_n$ and its corresponding eigenvalue $\tilde{\lambda}_{j,n}$ is

$$\tilde{\lambda}_{j,n} = \prod_{i=0}^{N-1} \tilde{\lambda}_{j+i}^n \prod_{i=0}^{N-1} \mu_{i,j}^2 = \prod_{i=1}^{N} \tilde{\lambda}_i^n \prod_{i=0}^{N-1} \mu_{i,j}^2.$$

Writing

$$C_j = \prod_{i=0}^{N-1} \mu_{i,j}^2 > 0 \quad \text{and} \quad \tilde{\Lambda} = \prod_1^N \tilde{\lambda}_i,$$

we get that, for any $n \in \mathbb{N}$, the eigenvalue $\tilde{\lambda}_{j,n}$ is

$$\tilde{\lambda}_{j,n} = C_j\, \tilde{\Lambda}^n.$$

This means that the matrix $\tilde{W}_n$ is the product of a homothety $(\tilde{\Lambda}^n \cdot Id)$ with a matrix B which does not depend on n and let invariant every one-dimensional space $E_i(p)$. So the matrix B commutes with every $\tilde{W}_{i,n}$. Finally, by construction, all the eigenvalues of B are positive.

Denote by

$$C_{n,j} = (C_j)^{-\frac{1}{n}}.$$

Clearly, when n becomes very large the $C_{n,j}$ are arbitrarily close to 1. Consider the matrix B_n having the $E_j(p)$ as eigenspaces and the $C_{n,j}$ as the corresponding eigenvalues. Denote by $[M]_{\hat{A}}(p)$ the word obtained from $[M]_{\tilde{A}}(p)$ by replacing its first letter $\tilde{A}(p)$ (at the right) by $\tilde{A}(p) \circ B_n$. For n large enough this new word is an ε_1-perturbation of $[M]_{\tilde{A}}(p)$, so by item (i) of Lemma 5.4 it is also a $2\,\varepsilon_1-$perturbation of $[M]_A(p)$. Now, the matrix corresponding to $[M]_{\hat{A}}(p)$ is $M_{\tilde{A}}(p) \circ B_n$. As B_n commutes with $M_{\tilde{A}}(p)$, and recalling the definitions of B_n and $C_{n,j}$, we get that

$$(M_{\tilde{A}}(p) \circ B_n)^n = M_{\tilde{A}}^n(p) \circ B^{-1}.$$

As a conclusion, the word $[\hat{W}_n]$ obtained by changing the initial subword $[M]_{\tilde{A}}^n(p)$ of $\tilde{W}_n$ by $[M]_{\hat{A}}(p)$ is $(\varepsilon_0 + 2\,\varepsilon_1) < \varepsilon$ close to the word $[M]_A(x_n)$, and its corresponding matrix $\hat{W}_n = \tilde{W}_n \circ B^{-1} = \tilde{\Lambda}^n \cdot Id$ is a homothety. This ends the proof of the claim. □

After reading carefully the proof before, one has the following remark which will play a key role to control the volume in the next section.

Remark 5.6 *Under the hypotheses of Proposition 2.5, suppose in addition that there is $x_0 \in \Sigma$ such that the modulus of the jacobian $J(M_A(x_0))$ is bigger than one. Then we can choose the perturbation $\tilde{A}$ of A and the point $x \in \Sigma$ in the proposition such that $M_{\tilde{A}}(x)$ is a homothety with ratio of modulus bigger than one.*

Proof: It is enough to repeat the proof of Proposition 2.5 bearing in mind the volume controlling Remark 5.5: So the point p and the first perturbation $\tilde{A}$ can be chosen such that $1 < J(M_{\tilde{A}}(p)) = \tilde{\Lambda}$. To conclude the proof, it is now enough to recall that (with the notation of the proof of Proposition 2.5) $J(M_{\hat{A}}(x_n)) = \tilde{\Lambda}^n$ (see the claim in this proof). $\square$

To end the proof of Proposition 2.5 it remains to prove Lemma 5.4. This is done in the next section.

5.3.2 Proof of Lemma 5.4 (and Remark 5.5)

The proof of Lemma 5.4 follows from the ideas of the proof of Lemma 4.13, based on the following fact: Given a vector v, a pair of matrices T and M, and the eigenvector w associated to the largest (in modulus) eigenvalue of M, it is very easy to map v into w by an arbitrarily small perturbation of $M^n \circ T$, if n is large enough. So, using this fact, given a dominated splitting, a simple inductive argument allows us to get transitions preserving it. The difficulty here is that we want to get a transition interchanging two spaces of a dominated splitting (in our case the eigenspaces of a diagonal matrix). For that we will use the complex eigenvalues, which enable us to map an arbitrary vector into the eigenvector corresponding to the weaker eigenvalue. Let us explain all that in details.

Fix any $0 < \varepsilon_1 < \varepsilon_0$. We now build some $\varepsilon_1/10-$perturbation $\tilde{A}$ of A, modifying the initial system along a finite number of orbits. In the next paragraphs we describe this perturbation along each orbit.

First, by Lemma 4.16, there are a point $p \in \Sigma$ and a perturbation $\tilde{A}$ of A along the orbit of p such that the corresponding matrix $M_{\tilde{A}}(p)$ is diagonalizable and has only positive real eigenvalues with multiplicity 1. Denote by $\lambda_1 < \cdots < \lambda_N$ such eigenvalues and by $E_j(p)$ the corresponding eigenspaces. Moreover, by Remark 4.17, if the linear system A satisfies the hypothesis of Remark 5.5, i.e. existence of some point with jacobian greater than one, one can choose the point p and the perturbation $\tilde{A}$ such that $J(M_{\tilde{A}}(p)) = \prod_{i=1}^{N} \lambda_i$ is strictly bigger than one.

By hypotheses, there is a point $p_i \in \Sigma$ (whose orbit is disjoint from the one of p) and an ε_0-perturbation $\tilde{A}$ (along the orbit of p_i) such that the matrix $M_{\tilde{A}}(p_i)$ has a complex eigenvalue of rank $(i, i+1)$. Now we fix $\varepsilon_1/10-$transitions $[t_{i,0}]$ and $[t_{0,i}]$ from p to p_i and from p_i to p.

To simplify the notations, we write $[M] = [M]_A(p)$, $[M_i] = [M]_A(p_i)$, and M and M_i for the corresponding matrices. We also write $[\tilde{M}]$, $[\tilde{M}_i]$, $\tilde{M}$, and $\tilde{M}_i$ for the corresponding perturbations of the words and matrices.

Observe that, once more by Remark 1.7, for every i and positive n_1, n_2, and n_3 the word

$$[t_i(n_1, n_2, n_3)] = ([M])^{n_3}[t_{0,i}]([M_i])^{n_2}[t_{i,0}][M]^{n_1}$$

is also an $\varepsilon_1/10-$transition from p to itself of the system A.

The proof of Lemma 5.4 (and of Remark 5.5) now follows immediately from the next result:

Lemma 5.7 *There are $n_1 \geq 0$, $n_2 \geq 0$, and $n_3 \geq 0$ such that the word $[t_i(n_1, n_2, n_3)]$ defined above admits an $(\varepsilon_0+\varepsilon_1)-$perturbation such that the corresponding matrix $\tilde{T}_i(n_1, n_2, n_3)$ let invariant every E_j, $j \notin \{i, i+1\}$, and interchanges E_i and E_{i+1}.*

Proof: For a fixed $i \in \{1, \dots, N\}$ consider the ε_0-perturbation $\tilde{A}$ of A along the orbit of p_i constructed above and denote by $E(p_i) \oplus F(p_i) \oplus G(p_i)$ its invariant splitting (over the orbit of p_i) where $F(p_i)$ is the $2-$dimensional eigenspace corresponding to the complex (conjugate) eigenvalues of $\tilde{M}_i$ and $E(p_i) \prec F(p_i) \prec G(p_i)$. Since $M_{\tilde{A}}(p)$ is diagonalizable we have the splitting

$$E(p) = \bigoplus_{1}^{i-1} E_j(p), \quad F(p) = E_i(p) \oplus E_{i+1}(p), \quad \text{and} \quad G(p) = \bigoplus_{i+1}^{N} E_j(p),$$

where $E_j(p)$ is the eigenspace associated to λ_j.

As in Lemma 4.13, replacing, if necessary, the transitions $[t_{0,i}]$ and $[t_{i,0}]$ by words of the form $[M]^n[t_{0,i}][M_i]^n$ and $[M_i]^n[t_{i,0}][M]^n$ for some big n, we can assume that $[t_{i,0}]$ admits a $(\varepsilon_0+\varepsilon_1/10)-$perturbation $[\tilde{t}_{i,0}]$ such that the corresponding matrix $\tilde{T}_{i,0}$ maps the splitting $E(p) \prec F(p) \prec G(p)$ into $E(p_i) \prec F(p_i) \prec G(p_i)$. Conversely, we can also suppose that the matrix $\tilde{T}_{0,i}$ of the $(\varepsilon_0+\varepsilon_1/10)-$perturbation $[\tilde{t}_{0,i}]$ of $[t_{i,0}]$ maps $E(p_i) \prec F(p_i) \prec G(p_i)$ into $E(p) \prec F(p) \prec G(p)$.

Our next objective is to get two different one-dimensional subspaces of $F(p_i)$ and a perturbation of $\tilde{M}_i$ interchanging such subspaces. For that write

$$E_i(p_i) = \tilde{T}_{0,i}^{-1}(E_i(p)) \quad \text{and} \quad E_{i+1}(p_i) = \tilde{T}_{i,0}(E_{i+1}(p)).$$

As $E_i(p)$ is a subspace of $F(p)$, $E_i(p_i)$ is a (non-invariant!) subspace of $F(p_i)$. The same argument shows that $E_{i+1}(p_i)$ is also a non-invariant subspace of $F(p_i)$.

Recall that $\tilde{M}_i$ has a pair of complex (non-real) eigenvalues whose eigenspace is $F(p_i)$. So it is an exercise to get $m > 0$ and an $\varepsilon_1/10-$perturbation $\hat{M}_i$ of $\tilde{M}_i$, preserving the splitting $E(p_i) \oplus F(p_i) \oplus G(p_i)$, such that $\hat{M}_i^m(E_{i+1}(p_i)) = E_i(p_i)$.

Consider now the linear map

$$B_0 = \tilde{T}_{0,i} \circ \hat{M}_i^m \circ \tilde{T}_{i,0} \colon \mathcal{E}_p \to \mathcal{E}_p,$$

recall that $\mathcal{E}_p$ denotes the fiber of p. By construction

i) B_0 preserves the splitting $E(p) \oplus F(p) \oplus G(p)$,

ii) $B_0(E_{i+1}(p)) = E_i(p)$.

iii) $B_0(E_i(p))$ is a straightline (in the plane $F(p)$) different from $E_i(p)$ and so transverse to $E_i(p)$

Observe that these three properties also hold for $\tilde{M}^k \circ B_0$ for every $k > 0$.

Recalling that $E_i(p) \oplus E_{i+1}(p)$ is a dominated splitting over the orbit of p (for the perturbated system whose matrix is $\tilde{M}$), using the transversality in item (iii), we have that if $k > 0$ is large enough then $\tilde{M}^k(E_i(p))$ is arbitrarily close to $E_{i+1}(p)$. So we can choose $k > 0$ and an $\varepsilon_1/10-$perturbation $\hat{M}$ of $\tilde{M}$ such that $B_1 = \hat{M} \circ \tilde{M}^k \circ B_0$ satisfies the following two properties:

- B_1 preserves the splitting $E(p) \oplus F(p) \oplus G(p)$,
- $B_1(E_i(p)) = E_{i+1}(p)$ and $B_1(E_{i+1}(p)) = B_1(E_i(p)$.

Note that these properties of B_1 are also verified by every map of the form $\tilde{M}^{k_1} \circ B_1 \circ \tilde{M}^{k_2}$ (k_1 and $k_2 > 0$). Applying the arguments of the proof of Lemma 4.13 to the restrictions of B_1 to $E(p)$ and $G(p)$ we get k_1 and $k_2 > 0$, and $\varepsilon_1/10-$perturbations $[N_1]$ of the word $[\tilde{M}]^{k_1}$, and $[N_2]$ of $[\tilde{M}]^{k_2}$, coinciding with $[\tilde{M}]^{k_1}$ and $[\tilde{M}]^{k_2}$ on $F(p)$, such that

$$N_1 \circ B_1 \circ N_2(E_j(p)) = E_j(p) \quad \text{for every } j \notin \{i, i+1\}.$$

To finish the proof of the lemma it suffices to observe that, by construction, the matrix $N_1 \circ B_1 \circ N_2$ corresponds to a word which is a $(\varepsilon_0 + 3\,\varepsilon_1/10)-$perturbation $(\varepsilon_0 + 3\varepsilon_1/10 < \varepsilon_0 + \varepsilon_1)$ of

$$[\tilde{M}]^{k_1}\,[\tilde{M}]\cdots[\tilde{M}]\,[t_{0,i}]\,[\tilde{M}_i]^k\,[t_{i,0}]\,[\tilde{M}]\cdots[\tilde{M}]\,[\tilde{M}]^{k_2}.$$

Thus this word is an ε_1-perturbation of $[t_i(n_1, n_2, n_3)]$ for some n_1, n_2, and n_3. Now the proof of the lemma is complete. □

6 Finest dominated splitting and control of the jacobian in the extremal bundles: Proof of Theorem 4

6.1 Control of jacobian over periodic points

Let $(\Sigma, f, \mathcal{E}, A)$ be a periodic linear system with transitions. Suppose that $F_1 \oplus F_2 \oplus \cdots \oplus F_{k-1} \oplus F_k$, $F_1 \prec F_2 \prec \cdots \prec F_{k-1} \prec F_k$, is the finest dominated splitting of this system. We call F_1 and F_k *extremal bundles of the dominated splitting.* Denote by A_i the restriction of A to the subbundle F_i. The goal of this section is to prove some estimates on the determinant of A_i. Let us begin with the following result:

Lemma 6.1 *For any $K > 0, N \in \mathbb{N}, L \in \mathbb{N}$, and $\varepsilon > 0$ there is $l > 0$ with the following property:*

Consider a periodic linear system $(\Sigma, f, \mathcal{E}, A)$, of dimension N and bounded by K, having an $L-$dominated splitting $E \oplus F$ such that

- *the subbundle E does not admit any nontrivial $l-$dominated splitting, and*
- *there is a point $p \in \Sigma$ such that $\det(M_{A_E}(p)) > 1$.*

Then there are an $\varepsilon-$perturbation $\tilde{A}$ of A and $x \in \Sigma$ such that all the eigenvalues of $M_{\tilde{A}}(x)$ have modulus strictly bigger than 1.

Clearly, there is a version of this lemma where the bundle F has no $l-$dominated splitting and $\det(M_{A_F}(p)) < 1$, then the moduli of the eigenvalues of the perturbation $\tilde{A}$ are strictly smaller than 1.

We will apply twice this lemma to the finest dominated splitting of a system, first taking $E = F_1$ and $F = F_2 \oplus \cdots \oplus F_k$, and second taking $E = F_1 \oplus \cdots \oplus F_{k-1}$ and $F = F_k$. Let us now prove the lemma.

Proof: Note first that there is $\delta_L > 0$ such that, for every system C δ_L-close to A, we can define the continuation $E_C \oplus F_C$ of the splitting $E \oplus F$, which is also dominated (recall the comments after Definition 0.1).

Let $\tau = \inf\{\varepsilon, \delta_L\}$. Consider the $l_0 > 0$ given by Proposition 2.4 associated to $\tau/2$, and fix $l = 2\,l_0$.

Now take a system $(\Sigma, f, \mathcal{E}, A)$ satisfying the hypotheses of the lemma. ¿From Corollary 4.14, the system A_E induced by A on the subbundle E admits transitions and, by hypothesis, does not admit any $l-$dominated splitting. In particular, by Proposition 2.4, for every $0 \leq i < \dim(E)$, there is a $\tau/2-$perturbation B_i of A_E having a complex eigenvalue of rank $(i, i+1)$, i.e. there is $x_i \in \Sigma$ such that $M_{B_i}(x_i)$ has a complex eigenvalue of rank $(i, i+1)$.

Moreover, by hypothesis, there is a point $p \in \Sigma$ for which the modulus of the jacobian $J(M_{A_E})(p)$ is strictly bigger than one.

The previous comments mean that taking $\varepsilon_0 = \tau/2$, the system (Σ, f, E, A_E) satisfies all the hypotheses of Proposition 2.5 and Remark 5.6. So there is a $\tau-$perturbation $\tilde{A}_E$ of A_E and a point $x \in \Sigma$ such that $M_{\tilde{A}_E}(x)$ is a homothety of ratio bigger than one.

Finally, by Lemma 4.1, $\tilde{A}_E$ is the restriction to E of some $\tau-$perturbation $\tilde{A}$ of A which coincides with A over F. Since the splitting $E \oplus F$ is dominated for $\tilde{A}$, we get that all the eigenvalues of $M_{\tilde{A}}(x)$ associated to F have necessarily modulus bigger than one. This ends the proof of the lemma. $\Box$

Theorem 4 follows from the next proposition that is a consequence of the Ergodic Closing Lemma in [Ma3] and whose proof (very similar to Mañé's argument in [Ma3] to get hyperbolicity) we postpone until the next subsection:

Proposition 6.2 *Let f be a diffeomorphism and $\Lambda_f(U)$ an $f-$invariant compact set which is maximal invariant in some neighbourhood U of it. Suppose that $E \oplus F$, $E \prec F$, is a dominated splitting of $T_{\Lambda_f(U)}M$ for f_*.*

By shrinking U, if necessary, there is a C^1-neighbourhood $\mathcal{U}$ of f such that for every $g \in \mathcal{U}$ the maximal invariant set $\Lambda_g(U)$ of $g \in \mathcal{U}$ has a dominated splitting $E_g \oplus F_g$ which is the continuation of $E \oplus F$. Then one has that

- *either there are an arbitrarily small C^1-perturbation g of f and a hyperbolic periodic point $p \in \Lambda_g(U)$ such that g_* expands the volume on $E_g(p)$,*
- *or $f_*|_E$ contracts uniformly the volume.*

Let us now end the proof of Theorem 4.

Proof of Theorem 4: Let $\Lambda_f(U)$ be a (non-trivial) robustly transitive set. Denote by N the dimension of the ambient manifold and let K be a strict upper bound of the norms of f_* and f_*^{-1}. Take $\varepsilon > 0$ such that for every g $\varepsilon - C^1-$close to f the set $\Lambda_g(U)$ is transitive and the norms of g_* and g_*^{-1} are bounded by K.

The proof is by contradiction. Let $F_1 \oplus F_2 \oplus \cdots \oplus F_k$ be the finest dominated splitting of f_*. Write $E = F_1$ and $F = F_2 \oplus \cdots \oplus F_k$, and fix L such that the splitting $E \oplus F$ is $2L-$dominated. Now let $l > 0$ be the constant associated to K, N, $2L$, and $\varepsilon/2$ in Lemma 6.1.

If $f_*|E$ does not contract uniformly the volume in E then, by Proposition 6.2, there are g arbitrarily close to f and a hyperbolic periodic point $p \in \Lambda_g(U)$ such that g_* expands the volume in $E_g(p)$. Since, by hypothesis, $\Lambda_g(U)$ is C^1-robustly transitive, using Lemma 2.7 we can assume (after a perturbation) that the relative homoclinic class $H(p,g,U)$ of p is the whole $\Lambda_g(U)$.

Consider the dense subset $\Sigma \subset \Lambda_g(U)$ consisting of all the hyperbolic periodic points of $\Lambda_g(U)$ homoclinically related to p. Then g induces the periodic linear system $(\Sigma, g, T_\Sigma M, g_*)$, of dimension N and bounded by K, with transitions, see Lemma 1.9.

Moreover, if g is close enough to f, $E(g) \oplus F(g)$ is an $L-$dominated splitting of this system and the bundle $E(g)$ does not admit any $l-$dominated splitting (this last assertion follows from Lemma 1.4). So applying Lemma 6.1, we get an $\varepsilon/2-$perturbation B of $(\Sigma, g, T_\Sigma M, g_*)$ and a periodic point $q \in \Sigma$ such that all the eigenvalues of $M_B(q)$ are bigger than one (in modulus). Using Franks' Lemma we get that the (non-trivial) maximal invariant set in U of some $\varepsilon-$perturbation g of f contains a repeller (precisely the point q). This contradicts the choice of ε (robust transitivity of $\Lambda_g(U)$).

To end the proof of the theorem it remains to get the uniform expansion of the volume in F_k, this follows as above by replacing f by f^{-1}. $\square$

6.2 Mañé's Ergodic Closing Lemma: Proof of Proposition 6.2

Definition 6.3 *Let f be a diffeomorphism defined on a compact manifold M endowed with a Riemannian metric d. A point x is* well closable *(for f) if for every $\varepsilon > 0$ there are g $\varepsilon - C^1-$close to f and a periodic point y of g such that $d(f^i(x), g^i(y)) < \varepsilon$ for every $0 \le i < k(y)$, where $k(y)$ is the period of y. We denote by $\mathcal{W}(f)$ the set of well closable points of f.*

We have the following result (see [Ma$_3$]),

Theorem (Ergodic Closing Lemma) *Let f be a diffeomorphism and μ an $f-$invariant probability. Then $\mu-$almost every point is well closable, i.e. $\mu(\mathcal{W}(f)) = 1$.*

Suppose now that $\Lambda_f(U)$ is a locally maximal set in a neighborhood U of it and that $E \oplus F$, $E \prec F$, is a dominated splitting of f_* over $T_{\Lambda_f(U)}M$. Recall that the bundle E is continuous (see Lemma 1.4), thus $(\Lambda_f(U), f, E, f_*|_E)$ is a continuous linear system.

Since E is endowed with a continuous Euclidian metric, we can define the modulus of the determinant of $f_*|_E$, the *jacobian of f on E,* denoted by $|J(f,E)|\colon \Lambda_f(U) \to \mathbb{R}$, which is a continuous (and so integrable) positive function. Thus $\log(|J(f,E)|)\colon \Lambda_f(U) \to \mathbb{R}$ is well defined and continuous. Moreover, by shrinking U, if necessary, we have that for every g close enough to f there is defined a dominated continuation $E_g \oplus F_g$ of the splitting $E \oplus F$. Thus we can define the function $\log(|J(g,E_g)|)$ depending continuously on g (observe that the subbundles E_g and F_g depend continuously on g).

The first step to prove Proposition 6.2 is the following lemma.

Lemma 6.4 *Assume there is an $f-$invariant probability measure μ supported on $\Lambda_f(U)$ such that*

$$\int \log(|J(f,E)|)\, d\mu \ge 0.$$

Then there are g arbitrarily C^1-close to f and a periodic orbit $y \in \Lambda_g(U)$ of g where g_* expands the volume of E_g, i.e.

$$|J(g^k, E_g)(y)| > 1, \quad \textit{where } k \textit{ is the period of } y.$$

Proof: Observe first that we can assume that μ is ergodic: Otherwise it is enough to consider the decomposition of μ into ergodic invariant measures, then for at least one of them the integral of $\log(|J(f,E)|)$ is also positive.

By the Ergodic Closing Lemma, there is a $\mu-$generic point x which is well closable. If x is periodic we have nothing to do. In the other case, there are sequences of diffeomorphisms g_n converging to f in the C^1-topology, of periodic points y_n (of period k_n) of g_n, and of numbers $\varepsilon_n \to 0$, such that

$$d(f^i(x), g_n^i(y_n)) < \varepsilon_n, \quad \text{for every } 0 \leq i < k_n - 1.$$

In particular, if ε_n is small enough, this ensures that the point y_n belongs to $\Lambda_{g_n}(U)$. Moreover,

$$\frac{1}{k_n} \sum_0^{k_n-1} \log(|J(g_n, E_{g_n}|)(g_n^i(y_n)) \to \int \log(|J(f,E)|)\, d\mu \geq 0.$$

So given any $\delta > 1$, taking n large enough, we have that the linear system defined over the orbit of y_n obtained by multiplying $(g_n)_*|E_{g_n}$ by the scalar δ expands the volume on $E_{g_n}(y_n)$. The conclusion in the lemma now follows immediately using the Franks' Lemma (see the very beginning of Section 1). $\square$

Using Lemma 6.4 we have that Proposition 6.2 (thus Theorem 4) is a direct consequence of:

Lemma 6.5 *Let $E \oplus F$, $E \prec F$, be a dominated splitting of f over $\Lambda_f(U)$. Assume that for any $N \in \mathbb{N}$ the jacobian $|J(f^N, E)|$ is not bounded from above by one uniformly. Then there is an $f-$invariant measure μ such that*

$$\int \log(|J(f,E)|)\, d\mu \geq 0.$$

Proof: By hypothesis, given any $N > 0$ there exists some point $x_N \in \Lambda_f(U)$ such that $|J(f^N, E)(x)| \geq 1$. Write

$$\mu_N = \frac{1}{N} \sum_0^{N-1} \delta(f^i(x_N)),$$

where $\delta(z)$ is the Dirac measure at the point z. As the space of probabilities is compact for the weak topology, there is a subsequence N_i such that μ_{N_i} converges weakly to some probability measure μ.

A classical elementary argument proves that μ is $f-$invariant: $f_*(\mu) - \mu$ is the weak limit of $\frac{1}{N_i}(\delta(f^{N_i}(x_{N_i})) - \delta(x_{N_i}))$, which converges to 0. Finally, observing that

$$\int \log(|J(f,E)|)\, d\mu_N = \frac{1}{N} \sum_0^{N-1} \log(|J(f,E)(f^i(x_N))|) = \frac{1}{N} \log(|J(f^N,E)(x_N)|) \geq 0,$$

one deduces immediately that $\int \log(|J(f,E)|\,d\mu \geq 0$. $\square$

7 The conservative case

In this section we translate some of our constructions into the conservative context (volume preserving C^1-diffeomorphisms).

In what follows the compact manifold M is endowed with a smooth volume form ω, and we denote by $\mathrm{Diff}^1_\omega(M)$ the set of C^1-diffeomorphisms preserving this volume form ω endowed with the usual C^1-topology. Following the traditional terminology, a *conservative diffeomorphism* is an element of $\mathrm{Diff}^1_\omega(M)$. Here we only consider manifolds of dimension N strictly bigger than one.

Observe that, given a linear system $(\Sigma, f, \mathcal{E}, A)$, using the Euclidian metric on the bundle $\mathcal{E}$, we can define the modulus of the determinant of the linear maps $A(x)$, denoted by $J_A(x) = |\det A(x)|$.

Definition 7.1 *A periodic linear system $(\Sigma, f, \mathcal{E}, A)$ is* conservative *if $J_A(x) = 1$ for all $x \in \Sigma$.*

Remark 7.2 *Given a linear system $(\Sigma, f, \mathcal{E}, A)$ of dimension N we define its* conservative part *A^c by*

$$A^c(x) = J_A(x)^{\frac{-1}{N}} \cdot A(x).$$

*Clearly, if A is conservative then $A = A^c$. The map $A \mapsto A^c$ is continuous. Therefore for any $K > 0$ and $\varepsilon > 0$ there is $\varepsilon_1 > 0$ such that if A is a conservative system bounded by K and B is a (*a priori *nonconservative) ε_1-perturbation of it, then B^c is $\varepsilon-$close to A.*

Finally, if the initial system A is continuous, periodic, and with transitions, then the same holds for its conservative part A^c.

We begin with a straightforward corollary of Proposition 2.1:

Proposition 7.3 *For any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $l > 0$ such that for every continuous periodic conservative linear system $(\Sigma, f, \mathcal{E}, A)$, of dimension N and bounded by K, with transitions one has that,*

- *either A admits an $l-$dominated splitting,*
- *or there is a conservative $\varepsilon-$perturbation $\tilde{A}$ of A such that $M_{\tilde{A}}(x)$ is the identity for some $x \in \Sigma$.*

Proof: According to Proposition 2.1, there is l with the following property:

Let A be a continuous periodic conservative linear system, of dimension N and bounded by K, and admitting transitions. If A does not admit any $l-$dominated splitting then there is an ε_1-perturbation B of A such that the matrix $M_B(x)$ is a homothety for some $x \in \Sigma$. Then, by Remark 7.2, B^c is a conservative $\varepsilon-$perturbation of A and $M_{B^c}(x)$ is a homothety with determinant 1 (because the system is conservative), thus it is the identity. This completes the proof of the proposition. $\square$

To proceed our proofs in the conservative setting we need suitable versions of our perturbations lemmas (Franks' Lemma and Hayashi's Connecting Lemma) for volume preserving diffeomorphisms.

Proposition 7.4 (Conservative Franks' Lemma) *Consider a conservative diffeomorphism f and a finite $f-$invariant set E. Assume that B is a conservative $\varepsilon-$perturbation of f_* along E. Then for every neighbourhood V of E there is a conservative diffeomorphism g arbitrarily C^1-close to f coinciding with f on E and out of V, and such that g_* is equal to B on E.*

As we did not find any precise reference for this probably well known result, so let us give here the sketch of its proof:

Proof: The proof is based on the following elementary fact of linear algebra:

Lemma 7.5 *For any $N > 1$ and $\varepsilon > 0$ there is a neighbourhood $\mathcal{G}$ of the identity in $SL(N, \mathbb{R})$ such that any matrix $A \in \mathcal{G}$ can be written as a product $B_1 \circ B_2 \circ \cdots \circ B_{4N-4}$, where the $B_i = P_i \circ R_i \circ P_i^{-1}$, P_i and R_i are $\varepsilon-$close to the identity in $SL(N, \mathbb{R})$, and R_i is a rotation.*

Proof: We just give the main steps of the proof. All the matrices we consider will be in $SL(N, \mathbb{R})$. We have the following properties:

- Every matrix A close to identity can be written in the form $L \circ L^{-1} \circ A$, where L is diagonal with real eigenvalues of multiplicity one and close to 1, $L^{-1} \circ A$ is diagonalizable and has eigenvalues close to 1, and the matrix of change of coordinates is close to identity.

- Any diagonal matrix in $SL(N, \mathbb{R})$ can be written as the product of $N - 1$ diagonal matrices whose eigenvalues are all equal to one, except two of them (inverse one to the other).

- Any diagonal matrix D close to the identity, $D \in SL(2, \mathbb{R})$, can be written as $R \circ R^{-1} \circ D$, where R is a rotation, $R^{-1} \circ D$ is conjugate to a rotation by some matrix P, and the matrices R, $R^{-1} \circ D$, and P are close to identity.

The lemma follows immediately from these three properties. □

Observe that Franks' Lemma consists of local perturbations around finitely many points. Taking local charts at these points we can consider that the volume is the Lebesgue measure (Leb), see for instance [Mo]. So the conservative Franks' Lemma follows from the next lemma:

Lemma 7.6 *For every $N \in \mathbb{N}$ and $\varepsilon > 0$ there is a neighbourhood $\mathcal{G}$ of the identity in $SL(N, \mathbb{R})$ such that for every $A \in \mathcal{G}$ there is $h \in \mathrm{Diff}^1_{Leb}(\mathbb{R}^N)$ satisfying the following properties:*

- *h coincides with the identity outside the unit ball at the origin,*

- *$h(0) = 0$ and $h_*(0) = A$,*

- $\|h_* - Id\| < \varepsilon$.

To prove this lemma it is enough to see that its proof is very easy if A is a rotation or conjugate to a rotation. Since Lemma 7.5 allows us to write A as the product of $4N-4$ of such maps, all them close to identity, the general case follows from the simple first case. $\square$

Exactly as Proposition 2.6 follows from Franks' Lemma and Proposition 2.1, we deduce from Proposition 7.4 (conservative Franks' lemma) and Propositions 7.3 the following conservative version of Proposition 2.6:

Proposition 7.7 *Given any $K > 0$, $N > 0$, and $\varepsilon > 0$ there is $l(\varepsilon, K) \in \mathbb{N}$ such that for every conservative diffeomorphism f on a Riemannian $N-$dimensional manifold M, with derivatives f_* and f_*^{-1} bounded by K, and any saddle p of f having a non-trivial homoclinic class $H(p, f)$, one has that:*

- *Either the homoclinic class $H(p, f)$ admits an $l(\varepsilon, K)-$dominated splitting,*
- *or for every neighbourhood U of $H(p, f)$ and $k \in \mathbb{N}$ there are a conservative diffeomorphism g $\varepsilon-C^1-$close to f and k periodic points x_i of g arbitrarily close to p, whose orbits are contained in U, such that the derivatives $g_*^{n_i}(x_i)$ are the identity (n_i is the period of x).*

Observe that this proposition implies Theorem 5 (in fact, this proposition is a quantitative version of this theorem).

7.1 Proof of Theorem 6

The first step to prove the theorem is the following lemma,

Lemma 7.8 *There is a residual subset $\mathcal{R} \subset \mathrm{Diff}^1_\omega(M)$ of diffeomorphisms f such that the non-trivial homoclinic classes of hyperbolic periodic points of f are dense in M.*

Proof: Let us begin by recalling that, for conservative diffeomorphisms, the recurrent points are dense in M. Moreover, using the C^1-closing lemma in the conservative case (see [Rb]), one has that the conservative diffeomorphisms whose periodic orbits are hyperbolic and dense in the ambient manifold form a dense subset of $\mathrm{Diff}^1_\omega(M)$. Moreover, by the continuity of the hyperbolic periodic orbits, this dense subset is in fact residual.

Suppose that p is a hyperbolic periodic point of a diffeomorphism g such that its periodic points are hyperbolic and dense in the manifold. Then, given fundamental domains D^s and D^u of $W^s(p, g)$ and $W^u(p, g)$, there is a sequence of periodic points q_i converging to some $z \in D^u$ such that $g^{k_i}(q_i) \to y \in D^s$ for some sequence $k_i \to +\infty$. This implies that the invariant manifolds of any periodic point p of g satisfy the hypotheses of the Conservative Connecting Lemma of Xia [X]:

Theorem (Conservative Connecting Lemma) *Let M be a compact manifold, g a conservative diffeomorphism, and p a hyperbolic periodic point of $g \in \mathrm{Diff}^1_\omega(M)$.*

Suppose that there are sequences of points (x_i), $x_i \to z \in W^u(p)$, and integers (n_i) such that $n_i \to \infty$ and $g^{n_i}(x_i) \to y \in W^s(p)$.

Then there is $h \in \mathrm{Diff}^1_\omega(M)$ arbitrarily C^1-close to g such that $W^u(p_h, h)$ intersects transversely $W^s(p_h, h)$ at z and $h^k(z) = y$ for some $k > 0$.

Now a classical argument (see, for instance, the proof of [BD$_2$, Proposition 1.1]) gives that there is a dense open subset $\mathcal{R}_n$ of $\mathrm{Diff}^1_\omega(M)$ of diffeomorphism whose non-trivial homoclinic classes are $\frac{1}{n}-$dense. To end the proof of the lemma it is enough to take $\mathcal{R} = \bigcap_{n>0} \mathcal{R}_n$. □

Now to end the proof of Theorem 6 consider a diffeomorphism $f \in \mathrm{Diff}^1_\omega(M)$ and $\varepsilon > 0$ such that there is no $\varepsilon-$perturbation of f having periodic points whose derivative is the identity. Thus, by Proposition 7.7, there is l such that every non-trivial homoclinic class of any conservative $\varepsilon/2-$perturbation g of f admits an $l-$dominated splitting. For such a g we define $\Lambda_i(g)$, $i = 1, \ldots, N-1$, as the closure of union of the non-trivial homoclinic classes with an $l-$dominated splitting $E \prec F$ of dimension i (i.e. $\dim(E) = i$). By Lemma 1.4, this invariant compact set $\Lambda_i(g)$ has an $l-$dominated splitting.

Using Lemma 7.8, we can take a sequence of diffeomorphisms $g_n \in \mathcal{R}$ as above converging to f. Then, for each n, the union of the $\Lambda_i(g_n)$ is the whole manifold. We define $K_i(f)$ as the topological upper limit set of the $\Lambda_i(g_n)$, i.e.

$$K_i(f) = \limsup_{n\to\infty} \Lambda_i(g_n) = \bigcap_k \mathrm{closure}(\bigcup_{n\geq k} \Lambda_i(g_n)).$$

Again by Lemma 1.4, the set $K_i(f)$ admits an $l-$dominated splitting. Finally, by construction, the manifold M is the union of the $K_i(f)$.

The arguments above shows that, if f does not admit any $\varepsilon-$pertubation with a periodic orbit whose derivative is the identity, then M is the union of finitely many invariant compact sets with $l-$dominated splittings. Otherwise, there is a conservative perturbation g of f and a homoclinic class of a periodic point of g whose induced periodic linear system can be perturbed to get one point such that its linear map is the identity. Using the transitions we can get an arbitrarily large number of such points. Now the result follows from the conservative Franks' Lemma (Proposition 7.4). □

7.2 Volume properties of dominated splittings of conservative systems

We end this paper by giving some volume properties of dominated splitting of conservative linear systems.

Proposition 0.5 is a direct consequence of the following lemma:

Lemma 7.9 *Let $(\Sigma, f, \mathcal{E}, A)$ be a conservative linear system with an $l-$dominated splitting $E \oplus F$, $E \prec_l F$. Then*

$$|\det(A^l_E)(x))| \leq \frac{1}{\sqrt{2}} \quad and \quad |\det(A^l_F)(x))| \geq \sqrt{2},$$

for every $x \in \Sigma$, recall that $A^l(x) = A(f^{l-1}(x)) \circ \cdots \circ A(x)$.

Proof: Since the system is conservative,

$$|\det(A_E^l)(x)\det(A_F^l)(x)| = 1.$$

In particular, the second inequality in the lemma follows from the first one.

We argue by contradiction. If the conclusion in the lemma does not hold then there is $x \in \Sigma$ such that $|\det((A_E^l)(x))| > \frac{1}{\sqrt{2}}$. Thus the modulus of the determinant of the matrix $(\sqrt{2})^{\frac{1}{\dim(E)}} \cdot A_E^l(x)$ is bigger than 1. So this matrix expands at least one vector. Thus there is some unit vector $u \in E(x)$ such that

$$\|A_E^l(x)(u)\| > 2^{-\frac{1}{2\dim(E)}}.$$

As the system is $l-$dominated, given any unit vector $v \in F(x)$ we have

$$\|A_F^l(x)(v)\| > 2^{1-\frac{1}{2\dim(E)}} \geq \sqrt{2}.$$

In particular,

$$\left(|\det A_F^l(x)| > \sqrt{2}^{\dim(F)} \geq \sqrt{2}\right) \Longrightarrow \left(|\det(A_E^l)(x)\ \det(A_F^l)(x)| > 1\right),$$

contradicting that the system is conservative.

□

The next lemma implies immediately Proposition 0.5:

Lemma 7.10 *Let f be a conservative diffeomorphism with a dominated splitting $E \oplus F$, $E \prec F$. Then there is ℓ such that*

$$|\det(f_*^\ell(x)|_E)| < \frac{1}{2} \quad \textit{and} \quad |\det(f_*^{-\ell}(x))|_F| > \frac{1}{2} \quad \textit{for every } x \in M.$$

Proof: Observe first that, using the conservative version of the Closing Lemma, after a perturbation, we can assume that for every k the periodic points of f of period bigger than k are dense in M.

The proof now is by contradiction. Suppose that the thesis is false, then for every $t > 0$ there is $g_t \in \mathrm{Diff}^1_\omega(M)$ close to f and a periodic point P_t such that

$$g_t^n(P_t) = P_t \quad \text{and} \quad \det((g_t^n)_*(P_t)|_E) > (1-t)^n.$$

It is not hard to see that we can assume that the periods of the points P_t go to infinite as $t \to 0$ (otherwise, after perturbation, one gets a linear system A and periodic point x such that $M_A(x)$ has an eigenvalue of modulus bigger or than 1 in E and an eigenvalue of modulus less than 1 in F, contradicting the dominance of the splitting).

Taking $t = 1/n$, arguing as in Lemma 7.10, we perturb each $(g_{1/n})_*$ along the orbit of $P_{1/n}$ to get a linear system B close to f_* such that $E \oplus F$ is a dominated splitting of B which does not satisfies Lemma 7.9. This gives a contradiction. □

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