Partial hyperbolicity, heterodimensional cycles, and transitivity

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In these notes we study the interplay between the notions of heterodimensional cycles, weak forms of hyperbolicity, and transitivity.

We begin by studying in Section 1 the *topological explosion* of non-normally hyperbolic planar horsehsoes. The explosion of these horseshoes explain how invariant manifolds (stable or unstable) associated to a hyperbolic bundle of dimension k may topologically behave as a manifold of dimension $k+1$. Using these ideas we study in Section 2 the unfolding of heterodimensional cycles and explain how (in a very simple situations) the unfolding of these cycles may lead to non-hyperbolic dynamics (the phenomenon of intermingled homoclinic classes of different indices).

The constructions in Sections 1 and 2 lead to the notion of blender introduced and discussed in Section 3. We discuss the role of blender in the context of robustly transitive dynamics.

In Section 4, we study robust transitive dynamics. We first present some examples of non-hyperbolic robustly trnasitive diffeomorphisms. Then we formulate the main result about theses systems claiming that they always are weakly transitive. We deduce some dynamical consequences from this. In this constructions, we review the argument of Mañé that asserts that in absence of domination one can create either sinks or sources after perturbations.

1 Topological explosions of non-normally hyperbolic horseshoes

In this section, we see how a planar horseshoe in $\mathbb{R}^3$ (i.e., a horseshoe contained in a plane) can be perturbed to obtain a *thick* horseshoe. Suppose, for instance, that the stable direction of the horseshoe has dimension one, then there is a small three dimensional cube around the saddle of the horseshoe such that the stable manifold of the saddle of the horseshoe separates this cube into two parts: roughly, it is not possible to join the top and the bottom of the cube by a cylinder without intersecting this stable manifold. In other words, the one dimensional stable manifold behaves, in some topological sense, as a two dimensional one. This enables to suppose that both invariant manifolds of the horseshoe are two-dimensional.

The argument for proving the assertions above involves a reduction to one-dimensional dynamics and is the key of the blenders-like constructions in Section 2 and 3 for obtaining robust transitivity.

1.1 Non-normally hyperbolic horseshoes

We begin by recalling a few basic facts concerning hyperbolicity and homoclinic classes. For other forms of weak hyperbolicity (partial hyperbolicity, dominated splittings and their properties, see the appendix in Section 5). The construction in this section is exported from [6].

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Given a $\mathcal{C}^r$-diffeomorphism $f\colon M \to M$, $1 \le r \le \infty$, we say that a compact f-invariant set $\Lambda_f \subset M$ is *hyperbolic* if there is a splitting $E^s_{\Lambda_f} \oplus E^u_{\Lambda_f}$ of the tangent bundle $T_{\Lambda_f}M$ and there are constants $C > 0$, $\lambda < 1$, such that

$$||D_x f^n(v)|| \le C\lambda^n ||v|| \quad \text{and} \quad ||D_x f^{-n}(w)|| \le C\lambda^n ||w||$$

for all $v \in E^s_x$, $w \in E^u_x$, $x \in \Lambda_f$ and $n \ge 1$. We call Λ_f a *basic set* if it is *transitive* (i.e. there is a dense orbit of f in Λ_f) and *isolated* (i.e. $\Lambda_f = \cap_{i\in\mathbb{Z}} f^i(U)$ for some neighbourhood U of Λ_f) and contains a dense subset of periodic points. An important feature of basic sets is their persistence under perturbation of the dynamics: there is a neighbourhood $\mathcal{V}^r$ of f in $\mathrm{Diff}^r(M)$ such that for all $g \in \mathcal{V}^r$

$$\Lambda_g = \bigcap_{i\in\mathbb{Z}} g^i(U) \quad \text{(the \textit{continuation} of } \Lambda_f)$$

is a basic set of g and, moreover, $g|\Lambda_g$ is *conjugate* to $f|\Lambda_f$: there is a homeomorphism $h\colon \Lambda_f \to \Lambda_g$ such that $(g|\Lambda_g) \circ h = h \circ (f|\Lambda_f)$.

The homoclinic class of a hyperbolic periodic orbit p of a diffeomorphism f, denoted by $H(p, f)$, is the closure of the transverse intersections of its stable and unstable manifolds. This set coincides with the closure of the periodic orbits q *homoclinically related* to p, i.e., the stable manifold of p transversely meets the unstable one of q and the unstable manifold of p transversely meets the stable one of q. We observe, that a homoclinic class always is an f-invariant transitive set.

Since our construction is local, it is no restriction to consider $M = \mathbb{R}^{n+1}$, $n \ge 2$, and we do so from now on. We begin by taking a $\mathcal{C}^r$-diffeomorphism F of $\mathbb{R}^n$ with a basic set Σ (a *horseshoe*) such that $F|\Sigma$ is conjugate to the full shift on two symbols. Next, we let $1 < \lambda < 2$ and consider the diffeomorphism

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}, \quad (X, x) \mapsto (F(X), \lambda x).$$

Note that f also has a horseshoe Let P be some fixed point of F in Σ. Then $(P, 0)$ is a hyperbolic fixed point of f and

$$W^s((P,0), f) = W^s(P,F) \times \{0\} \text{ and } W^u((P,0), f) = W^u(P,F) \times \mathbb{R}.$$

For simplicity, we assume that every expanding eigenvalue of $DF(P)$ is larger than 2 and then the strong unstable manifold of $(P, 0)$ is

$$W^{uu}((P,0), f) = W^u(P,F) \times \{0\}.$$

Since $W^s(P,F)$ and $W^u(P,F)$ meet transversely at some $Q \in \mathbb{R}^n$, the manifolds $W^s((P,0),F)$ and $W^{uu}((P,0),F)$ have a quasi-transverse intersection at $(Q,0)$. Now we consider arcs of $\mathcal{C}^r$-diffeomorphisms $\{f_t\}_{t\in[-1,1]}$, with $f_0 = f$, unfolding generically this intersection and we prove that the continuation Λ_t of $\Lambda_0 = \Lambda_f$ satisfies the following:

Geometrical property: *There is a cube $C = [-1,1]^s \times [-1,1]^u \times [-1,1]$ and a constant $\kappa > 0$ such that the stable manifold of P transversely intersects any stip of the form $x_0^s \times [-1,1] \times [r_1, r_2]$ for all $r_1, r_2 \in (0, \kappa)$.* This implies, that $W^s(P)$ transversely meets any *vertical* cylinder contained in C and joining the top and the bottom of C. See Lemma 1.1

As we already said, we start with a diffeomorphism F of $\mathbb{R}^n$, $n \geq 2$, exhibiting a horseshoe Σ

$$\Sigma = \bigcap_{i \in \mathbb{Z}} F^i(R), \qquad \text{with } R = [-1,1]^n \text{ say,}$$

and $F^{-1}(R) \cap R$ consisting of two connected components $\hat{D}_1$, $\hat{D}_2$. We take F to be affine on each of these components: there are s, $u \geq 1$, with $s+u=n$, and linear maps $S_i \colon \mathbb{R}^s \to \mathbb{R}^s$, $U_i \colon \mathbb{R}^u \to \mathbb{R}^u$, such that

$$DF|\hat{D}_i = \begin{pmatrix} S_i & 0 \\ 0 & U_i \end{pmatrix} \text{ and } ||S_i||, ||U_i^{-1}|| < 1 \text{ for } i = 1,\, 2.$$

In particular, $\hat{D}_i = [-1,1]^s \times D_i$, $D_i \subset [-1,1]^u$, for $i = 1$, 2. In what follows (x^s, x^u) are the usual coordinates in $R = [-1,1]^s \times [-1,1]^u$ and we suppose the fixed point P of F in $\hat{D}_1$ to be located at $(0^s, 0^u)$.

Define the smooth arc $\{f_t\}_{t \in [-1,1]}$ of diffeomorphisms of $\mathbb{R}^{n+1}$ by

$$f_t(x^s, x^u, x) = \begin{cases} (F(x^s, x^u), \lambda x) \text{ if } x^u \in D_1, \\ (F(x^s, x^u), \lambda x - t) \text{ if } x^u \in D_2. \end{cases}$$

We let $1 < \lambda < 2$ and $||U_1^{-1}|| < 1/2$ so that the fixed point $\mathcal{O} = (0^s, 0^u, 0)$ of f_t has

$$W^{uu}_{\text{loc}}(\mathcal{O}, f_t) = \{0^s\} \times [-1,1]^u \times \{0\} \text{ and } W^{s}_{\text{loc}}(\mathcal{O}, f_t) = [-1,1]^s \times \{0^u\} \times \{0\}.$$

On the other hand, given any $x^s \in [-1,1]^s$ there are x^s_1, $x^s_2 \in [-1,1]^s$ such that

$$\begin{aligned} f_t(\{x^s\} &\times [-1,1]^u \times \{x\}) \supset \\ &\supset (\{x^s_1\} \times [-1,1]^u \times \{\lambda x\}) \cup (\{x^s_2\} \times [-1,1]^u \times \{\lambda x - t\}) \end{aligned}$$

for every $x \in \mathbb{R}$. Hence,

$$f_t(W^{uu}_{\text{loc}}(\mathcal{O}, f_t)) \supset \{0^s_2\} \times [-1,1]^u \times \{-t\}$$

(note that $0^s_1 = 0^s$) and so the arc $\{f_t\}_{t \in [-1,1]}$ unfolds generically the quasi-transverse intersection of $W^s(\mathcal{O}, f_0)$ and $W^{uu}(\mathcal{O}, f_0)$ at $(0^s_2, 0^u, 0)$.

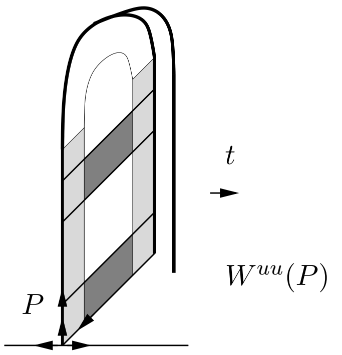


Figure 1: Non-normally hyperbolic horseshoe and its unfolding

Denote by Λ_t the continuation for f_t, small t, of the basic set $\Lambda_0 = \Sigma \times \{0\}$ of f_0. Observe that Λ_t coincides with the closure of the set $H(\mathcal{O}, f_t) = W^s(\mathcal{O}, f_t) \cap W^u(\mathcal{O}, f_t) \cap R$ of all (transverse) homoclinic points of $\mathcal{O}$ in R.

Lemma 1.1. *Let $t > 0$ (resp. $t < 0$) be close to zero and $J \subset [0,t]$ (resp. $J \subset [t,0]$) be an open interval. Then there are $x^s \in [-1,1]^s$ and $j \geq 0$ such that*

$$f_t^j(\{0^s\} \times [-1,1]^u \times J) \supset \{x^s\} \times [-1,1]^u \times \{\lambda^{-1} t\}.$$

As a consequence, $(\{0^s\} \times [-1,1]^u \times J) \cap H(\mathcal{O}, f_t) \neq \emptyset$.

This Lemma gives,

$$\pi\left(\overline{H(\mathcal{O}, f_t)}\right) \supset [0,t], \quad \text{where } \pi : \ [-1,1]^s \times [-1,1]^u \times \mathbb{R} \to \mathbb{R}, \ \pi(x^s, x^u, x) = x.$$

Proof of the lemma: We suppose $t > 0$, the case $t < 0$ being completely analogous. Define s by

$$\lambda\, s - t = s, \qquad s = \frac{t}{\lambda - 1} > t.$$

Consider the affine functions

$$\begin{aligned} \pi_{1,t} : \ & [0, \lambda^{-1} s] \to [0, s], \ \pi_{1,t}(y) = \lambda y, \\ \pi_{2,t} : \ & [\lambda^{-1} t, t] \to [0, s], \ \pi_{2,t}(y) = \lambda y - t. \end{aligned}$$

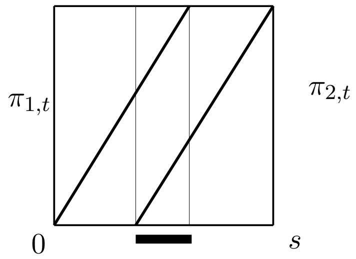


Figure 2: One-dimensional associated dynamics

Note that $\pi_{2,t}$ is well defined since $1 < \lambda < 2$ and that there is an overlapping interval $[\lambda^{-1} t, \lambda^{-1} s]$ where the two maps are defined. By (1)

$$\begin{aligned} f_t(\{x^s\} \times [-1,1]^u \times \{x\}) &\supset \{x_1^s\} \times [-1,1]^u \times \{\pi_{1,t}(x)\} \quad \text{if } x \in [0, \lambda^{-1} t], \\ f_t(\{x^s\} \times [-1,1]^u \times \{x\}) &\supset \{x_2^s\} \times [-1,1]^u \times \{\pi_{2,t}(x)\} \quad \text{if } x \in [\lambda^{-1} t, t]. \end{aligned}$$

The dynamical interpretation of this maps is the following. Take any vertical disk $V = \{x^s\} \times [-1,1]^u \times \{\kappa\}$ in the cube $C = [-1,1]^s \times [-1,1]^u \times [0,s]$. Then the image $f_t(V) \cap C$ has one or two connected components (if the point κ is in the overlapping region one has two componentes, otherwise one has just one component) which are vertical disks and whose central coordinate is given by the maps $\pi_{i,t}$.

Write $I_0 = J$ and $z_0 = 0^s$. If I_0 contains $\lambda^{-1} t$ then there is nothing to prove. Hence, we may assume that either $I_0 \subset (0, \lambda^{-1} t)$ or $I_0 \subset (\lambda^{-1} t, t)$. We let $i = 1$ in the first case and $i = 2$ in the second one and we write $I_1 = \pi_{i,t}(I_0)$ and $z_1 = 0_i^s$. Then, by (2),

$$f_t(\{0^s\} \times [-1,1]^u \times I_0) \supset \{z_1\} \times [-1,1]^u \times I_1$$

As above, if I_1 contains $\lambda^{-1}t$ then we are done. Otherwise we apply the previous procedure inductively: for each $j \geq 1$, if $\lambda^{-1}t \notin I_{j-1}$ then we construct an open interval $I_j \subset [0,t]$ and a point $z_j \in [-1,1]^s$ so that

$$I_j = \pi_{1,t}(I_{j-1}) \text{ if } I_{j-1} \subset [0,\lambda^{-1}t] \quad \text{and} \quad I_j = \pi_{2,t}(I_{j-1}) \text{ if } I_{j-1} \subset [\lambda^{-1}t, t] \quad \text{and}$$
$$f_t(\{z_{j-1}\} \times [-1,1]^u \times I_{j-1}) \supset (\{z_j\} \times [-1,1]^u \times I_j).$$

Since $\text{length}\,(I_j) = \lambda \cdot \text{length}\,(I_{j-1})$ and $\lambda > 1$ there must be a first j such that $\lambda^{-1}t \in I_j$. This ends the proof of the first part of the Lemma.

As for the second one, it is now a direct consequence. Observe that

$$f_t(\{x^s\} \times D_2 \times \{\lambda^{-1}t\}) \cap W^s_{\text{loc}}(\mathcal{O}, f_t) \neq \emptyset$$

and so $\{x^s\} \times [-1,1]^u \times \{\lambda^{-1}t\}$ intersects $W^s(\mathcal{O}, f_t)$ at some point Q_t. Then, taking $j \geq 0$ as above,

$$f_t^{-j}(Q_t) \in (\{0^s\} \times [-1,1]^u \times J) \subset W^u(\mathcal{O}, f_t),$$

which also means that $f_t^{-j}(Q_t) \in H(\mathcal{O}, f_t)$. This completes our argument. □

2 Heterodimensional cycles and robust transitivity

In this section we study heterodimensional cycles and see how the unfolding of these cycles may lead to the phenomenon of robust transitive dynamics. The ideas in this section are, in some sense, close to the ones in the previous section. The idea is to reduce the dynamics to the analysis of a system of (one-dimensional) iterated functions. Using these functions we see how an invariant manifold of dimension one may topologically behave as a manifold of dimension two.

We call *heterodimensional cycle* the geometric configuration corresponding to two periodic points P and Q of Morse unstable index (dimension of the unstable bundle) $p < q$, respectively, such that the stable manifold of each point meets the unstable manifold of the other. In most cases we consider $q = p + 1$. Such cycles were first considered by Newhouse, Palis in [26], and were systematically studied in the series of papers [9, 10, 12, 13, 14, 15, 16]. This elementary phenomenon, which occurs for diffeomorphisms in any dimension larger than 2, is one of the main mechanisms for non-hyperbolicity. For instance, the homoclinic classes of P and Q may coincide, for open sets in parameter space, in which case they can not be hyperbolic.

At the heart of heterodimensional cycles and the kinds of dynamics they yield, are certain transitive hyperbolic sets that we call *blenders*. This was introduced in [2], motivated by [10]. Topologically blenders are Cantor sets, but their distinctive geometric feature is that their embedding in the manifold is such that their stable set behaves as if it were of dimension higher than its actual topological dimension. More precisely, if p is the unstable index of the blender Γ, then the closure of the $(n-p)-$dimensional stable manifold of a periodic point $P \in \Gamma$ (in other words, the stable set of Γ) may intersect an unstable manifold of dimension $p-1$ in a robust fashion. This property as a key ingredient in a semi-local argument for building robustly transitive dynamics.

In Section 3 we present an affine model of a blender, and then explore various generalizations of this model. In Section 3.2 we show that the unfolding of heterodimensional cycles yields blenders, leading to periodic orbits of different indices being heteroclinically related in a robust way.

2.1 Heterodimensional cycles

Consider a closed manifold M and a diffeomorphism f defined on M and having a pair of hyperbolic saddles P and Q with different indices, that is, different dimensions of their unstable subspaces. Assume $W^s(P)$ and $W^u(Q)$ have non-empty intersection, and the same for $W^u(P)$ and $W^s(Q)$. We say that f has a *heterodimensional cycle* associated to P and Q. Of course, heterodimensional cycles can only exist in dimension bigger or equal to 3.

2.1.1 Explosion of homoclinic classes

The main result in this section depicts heterodimensional cycles as a powerful mechanism for explosion of homoclinic classes of periodic points and creation of robust non-hyperbolic transitive sets. The assumptions are not minimal, we shall see later that this kind of behavior is typical of a large class of cycles.

For simplicity, we suppose that the periodic points P and Q in the cycle are actually fixed points. In addition, we always assume

- (codimension 1) the saddles P and Q have indices p and $q = p + 1$, respectively,
- (quasi-transversality) the manifolds $W^s(P)$ and $W^u(Q)$ intersect transversely, and the intersection between $W^u(P)$ and $W^s(Q)$ is quasi-transverse: $T_xW^s(Q) \cap T_xW^u(P) = \{0\}$ for every intersection point x.

The next theorem asserts that the homoclinic classes of P and Q often explode, and become intermingled (non-empty intersection) when the cycle is unfolded.

A parametrized family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms *unfolds generically* a heterodimensional cycle of $f = f_0$ if there are open disks $K_t^u \subset W^u(P_t)$ and $K_t^s \subset W^s(Q_t)$, depending continuously on t, such that $K_0^u \cap K_0^s$ contains a point of quasi-transverse intersection, and the distance between K_t^s and K_t^u increases with positive velocity when t increases. Here P_t and Q_t denote the continuations for f_t of the periodic points P and Q.

Theorem 1 ([10]). *There is a non-empty open set of parametrized C^∞ families of diffeomorphisms $(f_t)_{t\in[-1,1]}$ unfolding generically a heterodimensional cycle of $f = f_0$, such that for all small positive t*

1. *the transverse intersection between $W^s(P_t)$ and $W^u(Q_t)$ is contained in the homoclinic class of Q_t;*
2. *and the homoclinic class of P_t is contained in the homoclinic class of Q_t.*

Let d be the dimension of the ambient manifold M. The key fact behind Theorem 1 is that, for every small positive t, the $(d-p)$-dimensional manifold $W^s(P_t)$ is contained in the closure of $W^s(Q_t)$. This makes that the stable manifold of Q_t, which has dimension $(d-p-1)$, behaves in fact as a manifold of dimension $d-p$, that is one unit bigger. This type of phenomenon has been synthesized in the notion of blender, which we discuss in Section 3.

The proof of the theorem, which we are going to sketch next, relies on a reduction of the dynamics to a family of iterated functions systems on the interval. Besides the standing hypotheses of codimension 1 and quasi-transversality above, the main conditions defining the open set in the theorem are *connectedness* of the transverse intersection γ, *non-criticality* or partial hyperbolicity of the dynamics on γ, and *small distortion* of the transition from the neighborhood of Q to that of P. This will be explained in the sequel, but let us point out right away that these conditions may hold simultaneously for a parametrized family f_t and for its inverse f_t^{-1}: one gets open sets of families for which the homoclinic classes of P_t and Q_t coincide after the bifurcation.

2.1.2 A model heterodimensional cycle

To explain the ideas of the proof of Theorem 1 while avoiding technical difficulties, we consider here a simple situation of a cycle in $\mathbb{R}^3$, which displays the main features of the general case we are dealing with. In this section, we construct a model one-parameter family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms unfolding a heterodimensional cycle. The study of the semi-local dynamics of f_t will be reduced to the analysis of a one-parameter family of endomorphisms with infinitely many discontinuities which describe the dynamics of f_t in the *central direction,* see Section 2.1.3.

Consider a diffeomorphism f with a heterodimensional cycle having the following dynamical configuration. In local coordinates in $\mathbb{R}^3$, the cycle is associated to saddle fixed points $Q = (0,0,0)$ and $P = (0,1,0)$ of indices 2 and 1, respectively, verifying the following conditions:

Partially hyperbolic (semi-local) dynamics of the cycle:

- In the cube $[-1,1] \times [-1,2] \times [-1,1]$ the diffeomorphism has the form
$$f(x,y,z) = (\lambda_s x, F(y), \lambda_u z),$$
where $F\colon [-1,2] \to (-1,2)$ is an increasing map with exactly two fixed points, a source at 0 and a sink at 1, and $0 < \lambda_s < d_m < 1 < d_M < \lambda_u$, where $0 < d_m < F'(x) < d_M$ for all $x \in [0,1]$.

- There is $\delta > 0$ such that F is linear in $[-\delta,\delta]$ and affine in $[1-\delta, 1+\delta]$. We denote by $\beta > 1$ and $0 < \lambda < 1$, the derivative of F at 0 and 1, respectively.

Observe that $[-1,1] \times \{(0,0)\} \subset W^s(Q)$, $\{0\} \times [0,1) \times [-1,1] \subset W^u(Q)$, $\{(0,0)\} \times [-1,1] \subset W^u(P)$, and $[-1,1] \times (0,1] \times \{0\} \subset W^s(Q)$. Thus the curve $\gamma = \{0\} \times (0,1) \times \{0\}$ (called *connexion*) is a normally hyperbolic curve contained in $W^u(Q) \pitchfork W^s(P)$.

Existence and unfolding of the cycle:

- **The cycle:** There exist $k_0 \in \mathbb{N}$ and a small neighborhood U of $(0,1,-1/2) \in W^u(P)$ such that the restriction of f^{k_0} to U is a translation,
$$f^{k_0}(x,y,z) = (x - 1/2, y - 1, z + 1/2).$$
Thus $f^{k_0}(0,1,-1/2) = (-1/2,0,0) \in W^u(Q)$ and $W^s(Q)$ and $W^u(P)$ meet throughout the orbit of the heteroclinic point $(-1/2,0,0)$. By construction, $(-1/2,0,0)$ is a quasi-transverse heteroclinic point.

- **The unfolding of the cycle:** Consider the family $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ of diffeomorphisms coinciding with f in $[-1,1] \times [-1,2] \times [-1,1]$ and such that the restriction of $f_t^{k_0}$ to U is of the form
$$f_t^{k_0}(x,y,z) = (x - 1/2, y - 1 + t, z + 1/2) = f^{k_0}(x,y,z) + (0,t,0).$$
So, for $t > 0$, $\{(-1/2,t)\} \times [-1,1] \subset W^u(P, f_t)$ and $x_t = (-1/2,t,0)$ is a transverse homoclinic point of P (for f_t). Similarly, $y_t = (-1/2,0,0)$ is a transverse homoclinic point of Q.

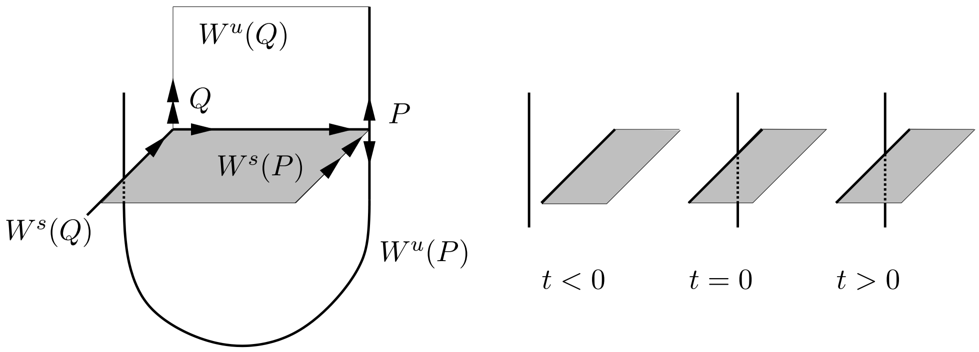


Figure 3: The model cycle and its unfolding

Consider a small *neighborhood of the heterodimensional cycle associated with* f_0, that is, an open set W containing the connexion $\gamma = \{0\} \times [0,1] \times \{0\}$ and the f_0-orbit of the heteroclinic point $(-1/2, 0, 0)$. For small t, let Λ_t be the *maximal* f_t*-invariant set in* W, $\Lambda_t = \cap_{n\in\mathbb{Z}} f_t^n(W)$. Consider also the forward and backward f_t-invariant sets in W, $\Lambda_t^+ = \cap_{n\geq 0} f_t^{-n}(W)$ and $\Lambda_t^- = \cap_{n\geq 0} f_t^n(W)$.

Fix a small positive ρ and consider the fundamental domains of F given by $D^+ = [\beta^{-1}\rho, \rho]$ and $D^- = [1-\rho, \lambda(1-\rho)]$ contained in the neighborhoods of 0 and 1 where F is affine. We can choose ρ such that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$ (for notational simplicity let us put $N = 1$). The map F^N is the *transition from* 0 *to* 1 [1]. Assume that that $F'(1) = \lambda$ and $F'(0) = \beta$ verify:

(T1) $F'(x) \geq \dfrac{1}{2}\,\dfrac{1-\lambda}{1-\beta^{-1}}$, for all $x \in D^+$,

(T2) $(1-\lambda) < \beta^{-1}$, and

(T3) $\dfrac{(1-\lambda)\,\lambda}{2\,(1-\beta^{-1})\,\beta} = \ell > 1$.

To get condition (T1) it is enough to consider F with small distortion. For conditions (T2) and (T3) it is enough to take β close enough to 1^+. Our first result is:

Theorem 2.1. *For every* $t > 0$ *sufficiently small,* $H(P, f_t) \subset H(Q, f_t)$*.*

In fact, it is possible also prove the inclusion $\Lambda_t \subset H(Q, f_t)$.

[1] This transition plays a key role for determining the dynamics after unfolding the cycle and it is determined by the *Mather invariant* of F: in a neighborhood of 0, the map F is the time-one of the vector field $X(y) = (\log\beta)\, y\, \frac{\partial}{\partial y}$, whose flow is $y \mapsto \beta^t\, y$. Similarly, in a neighborhood of 1, F is the time-one of $Y(y) = (\log\lambda)\,(y-1)\,\frac{\partial}{\partial y}$. Consider, for y close to 0^+, an large n such that $F^n(x)$ is close to 1^- and write $DF^n(y)(X(y)) = \mu(y)\, Y(F^n(y))$. Using the local F-invariance of X and Y (near 0 and 1), one has that $\mu(x)$ does not depend on n and that $\mu(x) = \mu(F(x))$. The function μ is the Mather invariant of F which describes its distortion. For instance, if μ is identically 1, then F is exactly the exponential of a vector field.

2.1.3 Expanding one-dimensional dynamics associated to the cycle

For each small $t > 0$, consider the *scaled fundamental domains* $D_t^{\pm}$ defined as follows: let $D_t^- = [1-t, \lambda(1-t)]$ and define k_t as the smaller $k \in \mathbb{N}$ with $F^{-k}(D_t^-) \subset [0,t]$. We define

$$D_t^+ = [a_t, b_t] = [\beta^{-1}(b_t), b_t] = F^{-k_t}(D_t^-), \quad \text{where} \quad \beta^{-2} t < a_t < b_t \le t.$$

We next define an expanding map R_t, $R_t\colon D_t^+ \to D_t^+$, with discontinuities describing the central dynamics of the return map of f_t defined on $[-1,1] \times D_t^+ \times [-1,1]$. First, for each small $t > 0$ define the *transition map T_t from D_t^+ to D_t^-* by

$$T_t\colon D_t^+ \to D_t^-, \quad x \mapsto T_t(x) = F^{k_t}(x).$$

Lemma 2.2. *The map T_t verifies $T_t'(x) > \ell > 1$ for all $x \in D_t^+$, where ℓ is as in condition* **(T3)**.

Proof: Given $x \in D_t^+$ let $k_t = n_t(x) + 1 + m_t(x)$, where $F^{n_t(x)}(x) \in D^+$ and $F^i(x) \notin D^+$ for all $1 \le i < n_t(x)$. We claim that

$$\frac{1}{\beta\, t} \le \beta^{n_t(x)} \le \frac{\beta^2}{t} \quad \text{and} \quad \lambda\, t \le \lambda^{m_t(x)} \le \frac{t}{\lambda}. \tag{1}$$

For first inequalities just note $x \in D_t^+ \subset (\beta^{-2}\, t, t]$ and $\beta^{n_t(x)}\, x \in [\beta^{-1}, 1]$. The other ones follow analogously. As $T_t(x) = \lambda^{m_t(x)}\, F(\beta^{n_t(x)}\, x)$, hypotheses (T1) and (T3) and the estimates in (1) give

$$|T_t'(x)| = \lambda^{m_t(x)}\, |F'(x)|\, \beta^{n_t(x)} \ge (\lambda\, t) \left(\frac{1}{2}\, \frac{1-\lambda}{1-\beta^{-1}}\right) \left(\frac{1}{\beta\, t}\right) = \frac{1}{2}\, \frac{\lambda\,(1-\lambda)}{\beta\,(1-\beta^{-1})} = \ell > 1,$$

as we claimed. □

Since $T_t(x) \in D_t^- = [1-t, 1-\lambda t]$ for all $x \in D_t^+$, we can define the map G_t by

$$G_t\colon D_t^+ \to [0,t], \quad x \mapsto G_t(x) = T_t(x) + (t-1).$$

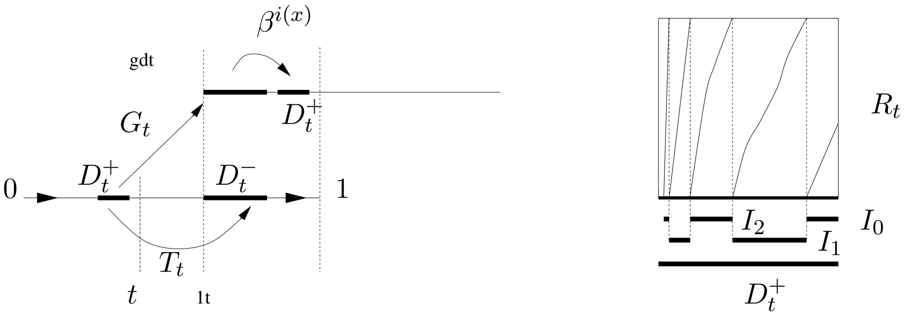


Figure 4: The expanding maps T_t, G_t and R_t

Remark 2.3. *The map G_t is monotone increasing and $G_t(D_t^+) = [0, t(1-\lambda)]$.*

Claim 2.4. *Let $(a_t, b_t] = \tilde{D}_t^+ \subset D_t^+$. Given $x \in \tilde{D}_t^+$ let $i(x) \in \mathbb{Z}$ be the minimum i with $\beta^i(G_t(x)) \in D_t^+$. Then there is $i_0 > 0$ (maximum with such property) such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: Recall first that $b_t \in (\beta^{-1} t, t]$. On the other hand, from $(1-\lambda) < \beta^{-1}$ (condition (T2)) and Remark 2.3, $G_t(\tilde{D}_t^+) = (0, t\,(1-\lambda)] \subset (0, \beta^{-1} t)$. Thus the right extreme of $G_t(\tilde{D}_t^+)$ is less than the left extreme of D_t^+, hence $i(x) > 0$ for all $x \in \tilde{D}_t^+$, ending the proof of the claim. $\square$

Finally, the return map R_t is defined by

$$R_t \colon \tilde{D}_t^+ \to D_t^+, \quad R_t(x) = \beta^{i(x)}(G_t(x)) = \beta^{i(x)}\,(T_t(x) + (t-1)).$$

Next we study the dynamics of R_t: the map R_t is uniformly expanding and has (infinitely many) discontinuities where the lateral derivatives are well defined. These discontinuities will play a key role in our constructions. The definition of $i_0 \in \mathbb{N}$ in Claim 2.4 implies that $\beta^{-i_0}(a_t) \in G_t(D_t^+)$. For each $i \geq i_0$ define $d_i \in \tilde{D}_t^+$ by $G_t(d_i) = \beta^{-i}(a_t)$. By construction, the sequence $(d_i)_{i \geq i_0}$ corresponds to the discontinuities of R_t and verifies the following:

- $d_{i+1} < d_i$ and $d_i \to a_t$,
- Let $[d_{i+1}, d_i] = I_i$, $i > i_0$, and $I_{i_0} = [d_{i_0}, b_t]$. The map R_t is continuous and strictly increasing in the interior of each interval I_i. We continuously extend R_t to the whole I_i, obtaining a bi-valuated return map R_t with $R_t(d_i) = \{a_t, F(a_t) = b_t\}$ for all $i > i_0$. In particular, the restriction of R_t to any I_i, $i > i_0$, is onto. We let $R_t(b_t) = c_t \leq b_t$.

The main properties of R_t are summarized in the next lemma.

Lemma 2.5. *The restriction of R_t to each interval $I_i, i > i_0$, is onto and $R_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = d_i$ this means that the lateral derivatives of R_t at x are greater than ℓ). Moreover, $0 \in G_t(R_t(d_i))$ for all $i \geq i_0$.*

The expansiveness of R_t follows from Lemma 2.2 and Claim 2.4. Condition $0 \in G_t(R_t(d_i))$ follows from $a_t \in R_t(d_i)$ and $0 \in G_t(a_t)$.

Lemma 2.6. *Consider small $t > 0$ and an open subinterval J of $\tilde{D}_t^+$. Then there is $k \in \mathbb{N} \cup \{0\}$ such that $R_t^k(J)$ contains a discontinuity of R_t. In particular, there is $x \in J$ such that $G_t(R_t^k(x)) = 0$.*

Proof: If the interval J contains a discontinuity we are done. Otherwise, let $i > 0$ be such that the intervals J, $R_t(J), \dots, R_t^i(J)$ do not contain discontinuities. Thus, for each $k \in \{0, \dots, i\}$, there is $i_k \geq i_0$ such that $R_t^k(J) \subset I_{i_k}$. Lemma 2.5 implies that $|R_t^k(J)| \geq \ell^k |J|$, $\ell > 1$, for all $k \in \{0, \dots, i\}$. Since the size of the intervals I_i is upper bounded, this inequality implies that there is a first $m \in \mathbb{N}$ such that $R_t^m(J)$ is not contained in any I_i, thus it intersects the set of discontinuities of R_t. $\square$

The maximal invariant set: Proof of Theorem 2.1 Next proposition is the main technical result of this section. Heuristically, it means that the one-dimensional stable manifold of Q topologically behaves as a two-dimensional manifold.

Proposition 2.7. *For every small $t > 0$ and every two-disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^s(Q, f_t) \pitchfork \chi \neq \emptyset$. In particular, $W^s(P, f_t)$ is contained in the closure of $W^s(Q, f_t)$.*

Proof of the inclusion $H(P,f_t) \subset H(Q,f_t)$ in Theorem 2.1. By the definition of $H(P,f_t)$, it suffices to see that any $x \in W^s(P,f_t) \pitchfork W^u(P,f_t)$ is accumulated by homoclinic points of Q. By the configuration of the cycle, $W^u(P,f_t) \subset \text{closure}\,(W^u(Q,f_t))$, thus, given any $x \in H(P,f_t)$ and any $n > 0$, there is a disk Δ_n, contained in $W^u(Q,f_t)$ and in the ball of radius $1/n$ centered at x, whose interior meets transversely $W^s(P,f_t)$. By Proposition 2.7, $\Delta_n \pitchfork W^s(Q,f_t) \neq \emptyset$. Thus there is $y_n \in \Delta_n \cap H(Q,f_t)$. By construction, $y_n \to x$, proving the inclusion $H(P,f_t) \subset H(Q,f_t)$.

Proof of Proposition 2.7 We now go into the details of the proof of Proposition 2.7. We first introduce some definitions.

- A set $\Delta \subset [-1,1] \times [-1,2] \times [-1,1]$ is a *vertical strip* if $\Delta = \{x_1\} \times [l_1,l_2] \times [r_1,r_2]$, where $l_1 < l_2$ and $r_1 < 0 < r_2$. The segment $\{x_1\} \times [l_1,l_2] \times \{0\}$ is the *basis* of Δ. The *width* and the *height* of Δ are $w(\Delta) = (l_2 - l_1)$ and $h(\Delta) = (r_2 - r_1)$. The strip Δ is *complete* if $r_2 = 1$ and $r_1 = -1$, *well located* if $[l_1,l_2]$ is contained in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.
- A subset $J \subset [-1,1] \times [-1,2] \times [-1,1]$ is a *vertical segment* if $J = \{x_1\} \times \{l_1\} \times [r_1,r_2]$, where $r_1 < 0 < r_2$. The point $(x,\ell_1,0)$ is the basis of J. The height of J is $h(J) = (r_2 - r_1)$. As above, the segment J is *complete* if $r_1 = -1$ and $r_2 = 1$, *well located* if l_1 is in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.
- A vertical segment J (resp. strip Δ) is *at the right of Q* if $l_1 \in (0,1]$.

Given an interval $\alpha \subset [-1,2]$, let $\Delta(\{x\} \times \alpha \times \{0\}) = \{x\} \times \alpha \times [-1,1]$ be the *unique* complete vertical strip with basis $\{x\} \times \alpha \times \{0\}$. Similarly, $J(x,y,0)$ is the unique complete vertical segment with basis $(x,y,0)$. The next algorithm associates to perfect segments and strips their successors:

Algorithm 2.8. *Let $\Delta = \Delta(\{x\} \times \alpha \times \{0\})$ be a perfect strip and define $\mathcal{G}_t(\Delta)$ as the perfect strip such that:*

- *the basis of $\mathcal{G}_t(\Delta)$ is of the form $(\{x'\} \times G_t(\alpha) \times \{0\})$, where $x' = \lambda_s^{k_t}\, x - 1/2$,*
- *$\mathcal{G}_t(\Delta)$ is contained in $f_t^{k_t}(\Delta)$ (where $F^{k_t}(D_t^+) = D_t^-$).*

Suppose now that α does not contain discontinuities, i.e. $\alpha \subset (d_{i+1}, d_i)$ for some i. Define $\mathcal{R}_t(\Delta)$ as the perfect strip such that:

- *the basis of $\mathcal{R}_t(\Delta)$ is of the form $(\{\hat{x}\} \times R_t(\alpha) \times \{0\})$, where $\hat{x} = \lambda_s^i(\lambda_s^{k_t}\, x - 1/2)$,*
- *$\mathcal{R}_t(\Delta)$ is contained in $f_t^{k_t+i}(\Delta)$.*

Similarly, to a perfect segment $J = J(x,y,0)$ we associate perfect segments $\mathcal{G}_t(J)$ and $\mathcal{R}_t(J)$ (provided $y \neq d_i$ for all i).

The strips $\mathcal{G}_t(\Delta)$ and $\mathcal{R}_t(\Delta)$ in Algorithm 2.8 are obtained as follows. Given a set A and a point $x \in A$, let $C(x,A)$ be the connected component of A containing x. Take a small neighborhood V of $f_0^{k_0}(\{0,1\} \times [-1,1]) \subset W^u(P,f_0)$, k_0 as in the definition of the cycle in Section 2, then

$$\begin{aligned} \mathcal{G}_t(\Delta) &= C(f_t^{k_t}(x,y,0), f_t^{k_t}(\Delta) \cap V) \cap [-1,1]^3), \\ \mathcal{R}_t(\Delta) &= C(f_t^{i}(x',y',0), f_t^{k_t+i}(\Delta) \cap V) \cap [-1,1]^3), \end{aligned}$$

where $(x, y, 0)$ is any point in the basis of Δ, $x' = (\lambda_s^{k_t} x - 1/2)$, and $y' \in G_t(\alpha)$. The construction for the successors of segments is analogous.

Lemma 2.9. *The manifold $W^u(P, f_t)$ contains a perfect segment for all small $t > 0$.*

Proof: Consider the transverse homoclinic point $x_t = (-1/2, t, 0)$ of P. Recall that $t \geq b_t$ and $\beta^{-1} t \in D_t^+ = [a_t, b_t]$. Let us assume that $t > b_t$, and thus $\beta^{-1} t \in (a_t, b_t)$ (the case $t = b_t$ follows similarly, so it will be omitted). Consider the complete vertical segment $\mathcal{R}_t(J)$, where $J = J(-(\lambda_s^{-1}/2), \beta^{-1} t, 0) \subset W^u(P, f_t)$. If $R_t(\beta^{-1} t)$ belongs to the interior of D_t^+, then $\mathcal{R}_t(J) \subset W^u(P, f_t)$ is the announced segment. Otherwise, $R_t(\beta^{-1} t) = b_t$ and there is a homoclinic point of P of the form $(x', b_t, 0)$. Using the λ-lemma and the product structure of the cycle, one gets homoclinic points $(x_n, y_n, 0)$ of P and complete segments $J_n = J(x_n, y_n, 0) \subset W^u(P, f_t)$ such that $x_n \to x'$, $y_n \to b_t$, and y_n is increasing. Thus y_n is in the interior of D_t^+ for every big n and $J_n \subset W^u(P, f_t)$ is perfect. $\square$

For clearness we first prove Proposition 2.7 in the following special case:

Proposition 2.10. *Let $\chi \subset [-1, 1] \times [-1, 2] \times [-1, 1] = \{x\} \times A$, where A is a disk of $\mathbb{R}^2$ whose interior contains a point of the form $(y, 0)$ with $y \in (0, 2)$. Then χ intersects transversely $W^s(Q, f_t)$.*

We claim that is enough to prove the Proposition 2.10 for perfect strips:

Lemma 2.11. *Let Δ be a perfect strip. Then there is $k \in \mathbb{N}$ such that $f_t^k(\Delta) \pitchfork W^s(Q, f_t) \neq \emptyset$.*

Proof: Suppose that $\Delta = \Delta(\{x_0\} \times \alpha \times \{0\})$, α in the interior of D_t^+. By Lemma 2.6, there exist y_0 in the interior of α and $k \in \mathbb{N}$ such that $G_t(R_t^k(y_0)) = 0$. Thus the vertical strip $\mathcal{G}_t(\mathcal{R}_t^k(\Delta))$ (contained in the forward orbit of Δ) intersects transversely $[-1, 1] \times \{(0, 0)\} \subset W^s(Q, f_t)$. $\square$

Proof of Proposition 2.10: By Lemma 2.9, $W^u(P, f_t)$ contains a perfect vertical segment J. Since, by definition, χ meets transversely $W^s(P, f_t)$, the λ-lemma implies that forward orbit of χ contains a sequence of complete strips χ_n accumulating to J. Thus χ_n contains a perfect strip for all n large. Lemma 2.11 implies that χ_n (and thus χ) transversely meets $W^s(Q, f_t)$. $\square$

Proof of Proposition 2.7: We can assume that χ is transverse to $W^s_{loc}(P, f_t)$ and contained in $[-1, 1] \times [-1, 2] \times [-1, 1]$. If χ contains a subset of the form $\{x\} \times A$, where A is an open subset of $\mathbb{R}^2$ containing a point $(0, y)$ with $y \in (0, 2)$, Proposition 2.10 implies the result. For the general case, consider a point $(x_0, y_0, 0)$, $y_0 \in (0, 2)$, in the interior of $\chi \pitchfork W^s_{loc}(P, f_t)$ and for every big n the vertical strip

$$\Sigma_n = \{x_0\} \times [y_0 - 1/n, y_0 + 1/n] \times [-1/n, 1/n].$$

The strips Σ_n verify the hypotheses of Proposition 2.10, hence there is $(x_0, y_n, z_n) \in \Sigma_n \pitchfork W^s(Q, f_t)$ such that $H_n = [-1, 1] \times \{(y_n, z_n)\} \subset W^s(Q, f_t)$. Since $(x_0, y_n, z_n) \to (x_0, y_0, 0)$, it is immediate that H_n meets transversely χ for all large n, ending the proof of the proposition. $\square$

2.2 Unfolding heterodimensional cycles

The dynamics associated to the unfolding of general heterodimensional cycles is very rich, and still far from being completely understood. However, a good number of results are available, especially for the (codimension-1) quasi-transversal case. Here we review some of those results.

The example of heterodimensional cycle in the previous section is *connected*, meaning that the transverse intersection $W^s(P) \cap W^u(Q)$ contains a smooth invariant curve γ joining P to Q, and *non-critical*, meaning that the closed invariant set $\gamma \cup \{P, Q\}$ admits a Df-invariant dominated splitting of the form $E^u \oplus T\gamma \oplus E^s$. Finally, the restriction to the connection γ has *small distortion*. Due to non-criticality, no homoclinic tangencies related to either P or Q may occur in the unfolding of those cycles. As we are going to see, other types of cycles may exhibit very different dynamical behavior. For instance, [13] shows that the unfolding of *critical* cycles may lead to the phenomenon of persistence of tangencies. We observe that this construction does not involve fractal dimensions. On the other hand, in the same paper it is shown that the unfolding of *non-connected* cycles may lead to Morse-Smale dynamics after the bifurcation: essentially, one considers cycles such that the connected components of γ are small, compared to the length of fundamental domains of f near P and Q.

Finally, [14] exhibits connected non-critical cycles with *large distortion*, and for which there are sets of parameters t with positive Lebesgue density at $t = 0$ (the density can be made arbitrarily close to 1) such that the limit set of f_t in the neighborhood of the cycle is hyperbolic and coincides with the disjoint union of the homoclinic classes of P_t and Q_t.

3 Blenders

While the arguments in Section 2.1.2 require a fair amount of differentiability, the conclusion of Theorem 1 does extend to the C^1 topology, as we shall see in Section 3.2. One way to prove this is using the notion of blender, which we discuss here.

Blenders were introduced in [3], and have been used to obtain persistence of cycles and transitivity. In this regard, they play a role similar to that of thick Cantor sets in the phenomenon of persistence of tangencies in [24, 25]. With the advantage that, unlike thick Cantor sets, blenders are robust already in the C^1 category.

Let us point out that what makes blenders so useful is not so much the dynamics they carry, but rather their geometry. Indeed, in simple terms, *a blender is a hyperbolic basic set such that convenient projections of its stable, or unstable, set have larger topological dimension than the stable, or unstable, set itself.* One such model appeared already in Section 1. This was a hyperbolic Cantor $H \subset \mathbb{R}^3$, with 1-dimensional stable manifolds and such that the projection of its stable set $W^s(H)$ onto the plane generated by the stable and the weak unstable directions contained a whole open set. In fact, the topologically 1-dimensional set $W^s(H)$ intersects any curve roughly parallel to the strong unstable direction. See also the construction in Section 2.1.2 Blenders are occur naturally in the unfolding of non-critical heterodimensional cycles. For instance, in the model considered in Section 2.1.2, a blender is obtained by considering those points having an expanding return to the reference fundamental domain D^t_Q. These points form a saddle hyperbolic set with 1-dimensional stable direction. The main step of the proof of Theorem 1 consisted in checking that the (one-dimensional) local stable manifold of the saddle Q transversely intersects any disk transverse to the 2-dimensional manifold $W^s(P)$. Since the points in the blender are homoclinically related to Q, it is not hard to deduce the same assertion for the stable manifold of any point of the blender.

3.1 A simplified model

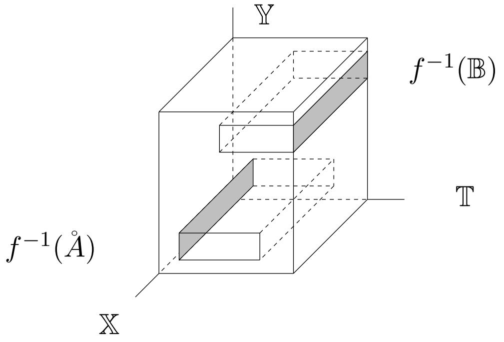
We start by describing a simple example displaying the main features of blenders. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism of $\mathbb{R}^2$ such that the rectangle $R = [0,1]^2$ is a Smale horseshoe for f:

- $R \cap f(R)$ is the union of the vertical sub-rectangles $A = I_a \times [0,1]$ and $B = I_b \times [0,1]$, where I_a and I_b are closed intervals contained in $(0,1)$.
- $R \cap f^{-1}(R)$ is the union of the horizontal sub-rectangles $f^{-1}(A) = [0,1] \times J_a$ and $f^{-1}(B) = [0,1] \times J_b$, where J_a and J_b are closed intervals contained in $(0,1)$.
- The restrictions of f to $f^{-1}(A)$ and to $f^{-1}(B)$ are affine maps with linear part $\begin{pmatrix} \pm\frac{1}{3} & 0 \\ 0 & \pm 3 \end{pmatrix}$. In particular, such restrictions preserve the horizontal and vertical directions.

Denote by $q = (x_0, y_0)$ the unique fixed point of f in A. Consider now a diffeomorphism $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that in $[-1,1] \times R$ is of the form

$$F(t,x,y) = \begin{cases} (\frac{5}{4}\,t, f(x,y)) & \text{if } (x,y) \in f^{-1}(A) \text{ and } t \in [-1,1], \\ (\frac{5}{4}\,t - \frac{1}{2}, f(x,y)) & \text{if } (x,y) \in f^{-1}(B) \text{ and } t \in [-1,1]. \end{cases}$$

The dynamics of F in the cube $\Gamma = [-1,1] \times R$ is hyperbolic, with $\mathbb{T} \oplus \mathbb{Y}$ as unstable direction ($\mathbb{T}$ is weak unstable and $\mathbb{Y}$ is strong unstable), and $\mathbb{X}$ as stable direction. The *blender* is the maximal invariant set inside Γ. See Figure 5.



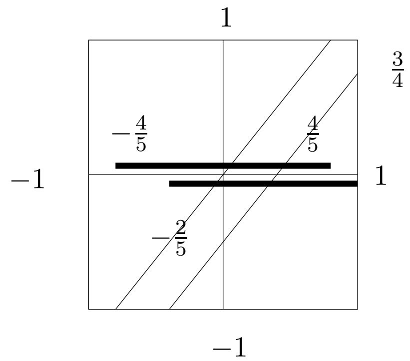


Figure 5: Blender with one-dimensional reduction

The distinctive geometric feature of blenders is stated in the next lemma. For the statement we need some terminology. Note that $\Gamma \cap F(\Gamma)$ contains the cubes $\mathring{A} = [-1,1] \times A$ and $\mathbb{B} = [-1, \frac{3}{4}] \times B$ and that $Q = (0, x_0, y_0) = (0, q) \in \mathring{A}$ is a hyperbolic fixed saddle of F of index 2. Let $W^s_{loc}(Q) = \{0\} \times [0,1] \times \{y_0\}$ be the connected component of $W^s(Q) \cap \Gamma$ that contains Q.

- A *vertical segment to the right of* $W^s_{loc}(Q)$ is a segment $\sigma_{t,x} = \{t\} \times \{x\} \times [0,1]$, where $x \in [0,1]$ and $0 < t \leq 1$.
- A *vertical strip to the right of* $W^s_{loc}(Q)$ is a rectangle $\Delta = [t_1, t_2] \times \{x\} \times [0,1]$, where $x \in [0,1]$ and $0 < t_1 < t_2 \leq 1$. The number $w(\Delta) = (t_2 - t_1)$ is the *width of* Δ.

Lemma 3.1. *(distinctive property) Every vertical strip Δ to the right of $W^s_{loc}(Q)$ intersects $W^s(Q)$.*

Proof: We show that if Δ is a vertical strip to the right of $W^s_{loc}(Q)$ then either $F(\Delta)$ intersects $W^s_{loc}(Q)$, or else $F(\Delta) \cap \Gamma$ contains a vertical strip Δ_1 to the right of $W^s_{loc}(Q)$ with $w(\Delta_1) = (5/4)\, w(\Delta)$.

To deduce the lemma from this claim, start with Δ as in the statement and consider its forward iterates. For as long as these iterates do not intersect the local stable manifold of Q, they contain vertical strips whose widths grow exponentially fast. Since the width is bounded above by 1, some iterate $F^n(\Delta)$ must intersect $W^s_{loc}(Q)$. This implies that Δ intersects the (global) stable manifold of Q. Hence, we are left to prove the claim in the previous paragraph.

Consider any vertical strip $\Delta = [t_1, t_2] \times \{x\} \times [-1, 1]$ to the right of $W^s_{loc}(Q)$. Write $\Delta_A = F(\Delta) \cap \mathring{A}$ and $\Delta_B = F(\Delta) \cap \mathbb{B}$.

- If $t_2 \le \frac{4}{5}$ then $[\frac{5}{4}\, t_1, \frac{5}{4}\, t_2] \subset (0, 1]$. It follows that Δ_A is a vertical strip to the right of $W^s_{loc}(Q)$, with width $w(\Delta_A) = \frac{5}{4}\, w(\Delta)$. Thus the second possibility in the claim holds.
- If $t_2 \in (\frac{4}{5}, 1]$ then $\frac{5}{4}\, t_2 - \frac{1}{2} \in (\frac{1}{2}, \frac{3}{4}]$. By the definition of F, there is x' such that

$$\Delta_B = F(\Delta) \cap \mathbb{B} = \{[\frac{5}{4}\, t_1 - \frac{1}{2}, \frac{5}{4}\, t_2 - \frac{1}{2}]\} \times \{x'\} \times [0, 1].$$

  There are now two subcases:

- If $\frac{5}{4}\, t_1 - \frac{1}{2} > 0$ then Δ_B is a vertical strip to the right of $W^s_{loc}(Q)$, with width $w(\Delta_B) = (5/4)\, w(\Delta)$. So, the second possibility in the claim holds in this case too.
- Otherwise $\frac{5}{4}\, t_1 - \frac{1}{2} \le 0$ and $\frac{5}{4}\, t_2 - 1/2 \ge 0$. Then Δ_B meets $W^s_{loc}(Q) = \{0\} \times [0, 1] \times \{y_0\}$, which corresponds to the first possibility in the claim.

The proof is complete. □

As we did in the previous section, we may consider the quotient of the dynamics by the sum of the strong stable and strong unstable directions. This yields an iterated functions system $\mathcal{G}(\phi_1, \phi_2)$ with two generators, as follows:

- for $t \in [-\frac{4}{5}, \frac{4}{5}]$ consider $\phi_1(t) = \frac{5}{4}\, t$, corresponding to the restriction of F to the domain $F^{-1}(\mathring{A}) = [-\frac{4}{5}, \frac{4}{5}] \times f^{-1}(A)$,
- for $t \in [-\frac{2}{5}, 1]$ consider $\phi_2(t) = \frac{5}{4}\, t - \frac{1}{2}$, corresponding to the restriction of F to the domain $F^{-1}(\mathbb{B}) = [-\frac{2}{5}, 1] \times f^{-1}(B)$.

Notice that both maps ϕ_1 and ϕ_2 are expanding, and their domains of definition overlap on the interval $[-\frac{2}{5}, \frac{4}{5}]$. This reflects the fact that given any vertical segment $\sigma = \{t\} \times \{x\} \times [0, 1]$ with t in the overlapping interval, there are x_1 and x_2 such that the vertical segments $\{\phi_1(t)\} \times \{x_1\} \times [0, 1]$ and $\{\phi_2(t)\} \times \{x_2\} \times [0, 1]$ are the two connected components of $F(\sigma) \cap \Gamma$.

Lemma 3.1 may now be reformulated for the iterated functions system, as follows. The proof is similar to the previous case.

Lemma 3.2. *Given any open interval $\alpha \subset [0, 1]$ there is $\phi \in \mathcal{G}(\phi_1, \phi_2)$ such that $0 \in \phi(\alpha)$.*

To illustrate the dynamical usefulness of blenders, we close this section with the following direct consequence of Lemma 3.1:

Lemma 3.3. *Suppose that there is a hyperbolic periodic point P of F of index 1 whose one-dimensional unstable manifold crosses Γ along a vertical segment to the right of $W^s_{loc}(Q)$. Then $W^s(P) \subset \overline{W^s(Q)}$.*

So the one-dimensional stable manifold of Q looks like a 2-dimensional manifold, as its closure contains the 2-dimensional $W^s(P)$.

3.1.1 Relaxing the construction

So far we discussed a very simple affine model of blender. Of course, for most applications we need more flexible versions of the previous constructions, robust under C^1 small perturbations. Just as an example, in the context of Lemma 3.3 we would like to have the conclusion when the segment is just approximately vertical. We are going to explain how this flexibilization is carried out, starting with perturbations of the affine model, and to indicate other ways in which the construction may be generalized. Motivated by this, we discuss possible definitions of the notion of blender.

In the affine model before, $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{Y}$ is a dominated splitting on Γ, with $\mathbb{X}$ as strong stable, $\mathbb{Y}$ as strong unstable, and $\mathbb{T}$ as central (weak unstable) direction. Hence, there are cone fields $\mathcal{C}^s$, $\mathcal{C}^{uu}$, $\mathcal{C}^u$ around the directions $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{T} \oplus \mathbb{Y}$, respectively, strictly invariant under the derivative DF. These cone fields allow us to define *almost vertical segments (to the right of $W^s_{loc}(Q)$)*, as curve segments tangent to $\mathcal{C}^{uu}$ and crossing Γ (on the right hand side of the local stable manifold), as well as *almost vertical strips (to the right of $W^s_{loc}(Q)$)*, as 2-dimensional disks through Γ that are tangent to $\mathcal{C}^u$ and foliated by almost vertical segments (to the right of $W^s_{loc}(Q)$). The notion of width extends to these almost vertical strips: the length of the shortest curve transverse to $\mathcal{C}^{uu}$ joining the two boundary components inside the strip.

The families of almost vertical segments and strips are invariant under forward iteration, in the same sense as before for truly vertical segments and strips. Moreover, the widths of almost vertical strips not intersecting $W^s_{loc}(Q)$ increase exponentially under iteration, again in the same sense as before. This means that the proof of Lemma 3.1 carries through to this context, only with extra technical difficulties: *every almost vertical strip at the right of $W^s_{loc}(Q)$ intersects $W^s(Q)$.*

Now, the cone fields remain invariant for any diffeomorphism in a C^1 neighborhood of F. Moreover, the width estimate is also robust under any small C^1 perturbation, because it depends only on the fact that the derivative expands all vectors in the cone $\mathcal{C}^u$. This means that the conclusion of the previous paragraph is true for any map near F: *the distinctive blender property is a robust property!*

Blenders from a constructive viewpoint. One way the construction of blenders may be relaxed, with respect to the affine model in Section 3.1, is allowing for more complicated combinatorics: any number of dynamical components, with variable return times to the ambient box.

Consider a C^1 diffeomorphism F and a box $\Gamma = D^k \times [-1,1] \times D^l$, where $D^i = [-1,1]^i$, in the ambient space, endowed with cone fields $\mathcal{C}^s$ and $\mathcal{C}^{uu} \subset \mathcal{C}^u$ around the directions of D^k, D^l, and $[-1,1] \times D^l$, respectively. Assume that there are sub-boxes $(\Gamma_i)_{i=1}^k$ of Γ, and return times $(n_i)_{i=1}^k$ such that every return map $F^{n_i}\colon \Gamma_i \to \Gamma$ preserves both cone fields and expands uniformly all vectors in $\mathcal{C}^u$. Define as above almost vertical (l-dimensional) disks, tangent to $\mathcal{C}^{uu}$, and almost vertical ($l+1$-dimensional) strips, tangent to $\mathcal{C}^u$ and foliated by almost vertical disks crossing Γ.

Define the width $w(\mathcal{S})$ of an almost vertical strip $\mathcal{S}$, again as before. We turn the type of property in Lemma 3.1 into our first definition of a blender:

Definition 3.4. (Γ, F) *is a* blender associated to a periodic saddle Q of index $l+1$ *if for every almost vertical strip $\mathcal{S}$ crossing Γ there is $i \in \{1, \dots, k\}$ such that*

- *either $F^{n_i}(\mathcal{S} \cap \Gamma_i)$ contains an almost vertical strip $\mathcal{S}'$ intersecting $W^s_{loc}(Q)$*
- *or $F^{n_i}(\mathcal{S} \cap \Gamma_i)$ contains an almost vertical strip $\mathcal{S}'$ with $w(\mathcal{S}') \geq \lambda\, w(\mathcal{S})$, where $\lambda > 1$ is independent of the strip $\mathcal{S}$.*

The periodic point Q in the definition does not need to belong to the cube Γ: our arguments just require that $W^s(Q)$ contains a disk Δ crossing Γ and tangent to the cone field $\mathcal{C}^s$. The reader may check that the unfolding of the heterodimensional cycle in Section 2.1.2 yields a blender inside a box close to the fixed point Q but not necessarily containing it.

In the particular case where the maps F^{n_i} preserve the D^k and D^l directions, as was the case for the affine blender, one may consider the quotient dynamics of F by the sum $D^k \oplus D^l$. In this way we obtain, for each i, a one-dimensional expanding map $\phi_i : I_i \to [0,1]$ induced by F^{n_i} on a compact $I_i \subset [0,1]$. (Γ, F) being a blender corresponds to the maps ϕ_i having the covering property: $\cup_i \mathrm{inter} I_i = (0,1)$. A point $x \in [0,1]$ is *periodic* for the iterated functions system $\mathcal{G}(\phi_1, \dots, \phi_k)$ if $\phi(x) = x$ for some $\phi \in \mathcal{G}$. Extending Lemma 3.2, we have

Proposition 3.5. *Given any system $\mathcal{G}(\phi_1, \dots, \phi_k)$ of iterated expanding functions with the covering property, there is a finite set $\mathcal{F} = \{x_1, \dots, x_s\}$, $1 \leq s < k$, of periodic points such that, for any open interval $\alpha \subset [0,1]$, there is $\phi \in \mathcal{G}$ with $\phi(\alpha) \cap \mathcal{F} \neq \emptyset$.*

Blenders from an operational viewpoint. We now adopt a different point of view: we will think of them as tools for proving transitivity. Thus we propose another definition which takes into account their properties, rather than their construction.

Definition 3.6. *Let $F : M \to M$ be a diffeomorphism and Q be a hyperbolic saddle of index $l+1$. We say that F* has a blender associated to *Q if there is a C^1 neighborhood $\mathcal{U}$ of F, and a C^1 open set $\mathcal{D}$ of embeddings of an l-dimensional disk D^l into M, such for every $G \in \mathcal{U}$, every disk $D \in \mathcal{D}$ intersects the closure of $W^s(Q_G)$, where Q_G is the continuation of the periodic point Q for G.*

The set $\mathcal{D}$ in the definition is the *superposition region* of the blender. For instance, in the simplified model, $l = 1$ and the superposition region is the family of all almost vertical segments (that is, tangent to the vertical cone field and crossing the box Γ) to the right of $W^s_{loc}(Q)$. Bearing in mind the distinctive property of blenders, one says that the blender is *activated* by a saddle P of index l if the unstable manifold of P contains a disk of the superposition region.

Lemma 3.7. *Let F be a diffeomorphism having a blender associated to a saddle Q of index $l+1$. Suppose that the blender is activated by a saddle P of index l. Then, for every diffeomorphism G in a C^1 neighborhood of F, the closure of $W^s(Q_G)$ contains $W^s(P_G)$, where P_G and Q_G are the continuations of P and Q for G.*

The point with this result is that it means that the $(d-l-1)$-dimensional manifold $W^s(Q_G)$ of Q_G looks like it contains a manifold with higher dimension, namely $W^s(P_G, G)$.

Proof: Let $\mathcal{D}$ the superposition region of the blender. As compact parts of invariant manifolds depend continuously on the diffeomorphism, $W^u(P_G)$ contains a disk $D_G \in \mathcal{D}$, for all G close to F. Let U be any open set intersecting $W^s(P_G)$. By the λ-lemma, for every n big enough, $G^n(U)$ contains a C^1-open set of disks arbitrarily close to D_G. Then these disks are in the superposition region. By definition of blender, it follows that $W^s(Q_G)$ intersects $G^n(U)$. As the stable manifold is invariant, it also intersects U. $\square$

3.2 Partially hyperbolic cycles

We are going to see that the formation of intermingled homoclinic classes is a frequent phenomenon in the unfolding of cycles, even in the C^1 topology. Consider a diffeomorphism f with a cycle associated to saddles P and Q of indices p and $p+1$, respectively. We say that the cycle is *C^k-far from homoclinic tangencies* if every diffeomorphism g which is C^k close to f does not have homoclinic tangencies associated either of the continuations P_g and Q_g of P and Q.

Theorem 2 (D., Rocha [15]). *Let f be a C^1 diffeomorphism with a heterodimensional cycle associated to saddles P and Q of indices p and $q = p+1$. Suppose that the cycle is C^1-far from homoclinic tangencies. Then there is an open set $\mathcal{U} \subset \mathrm{Diff}^1(M)$ whose closure contains f and such that the homoclinic classes of P_g and Q_g coincide for all $g \in \mathcal{U}$.*

The conclusion of this theorem also holds for non-critical connected cycles, without any assumption on the distortion. In that case we even have the following version for one-parameter families generically unfolding the cycle: *Given $t > 0$, let $B(t)$ be the interior of the set of parameters $s \in [-t, t]$ such that the homoclinic classes of P_s and Q_s coincide. Then $B(t)$ has positive Lebesgue measure for every $t > 0$.* Moreover, if the cycle corresponds to a first bifurcation, the relative density of $B(t)$ at the bifurcation is positive.

Theorem 2 is a consequence of the following proposition, which uses blenders with several generators.

Proposition 3.8. *Let f be a diffeomorphism as in Theorem 2. Then there is an open set $\mathcal{V} \subset$ $\mathrm{Diff}^1(M)$ whose closure contains f such that for every g in $\mathcal{V}$ there are blenders $\Gamma(g)$ and $\Gamma(g^{-1})$, defined for g and g^{-1}, respectively, such that:*

- *$\Gamma(g)$ is associated to a hyperbolic periodic point R_g homoclinically related to Q_g and is activated by P_g.*
- *$\Gamma(g^{-1})$ is associated to a hyperbolic periodic point S_g homoclinically related to P_g and is activated by Q_g.*

Observe that $\Gamma(g^{-1})$ is defined for g^{-1}: to be activated by Q_g means that the stable manifold of Q_g contains a (p-dimensional) disk of the superposition region. Since R_g and Q_g (respectively, S_g and P_g) are homoclinically related, Lemma 3.7 and Proposition 3.8 imply that, for every g in the open set $\mathcal{V}$,

$$W^s(P_g) \subset \overline{W^s(R_g)} = \overline{W^s(Q_g)} \quad \text{and} \quad W^u(Q_g) \subset \overline{W^u(S_g)} = \overline{W^u(P_g)}.$$

This is the main step towards showing that the homoclinic classes of P_g and Q_g coincide.

4 Robust transitivity

A diffeomorphisms $f : M \to M$ is *robustly transitive* if every diffeomorphism in a C^1 neighborhood has dense orbits. More generally, an invariant compact set Λ is robustly transitive if there is an open neighborhood U of Λ and a neighborhood $\mathcal{V}$ of the diffeomorphism f in $\mathrm{Diff}^1(M)$ such that for every $g \in \mathcal{V}$ the maximal invariant set Λ_g of g in $\bar{U}$, $\Lambda_g = \cap_{i\in\mathbb{Z}} g^i(U)$, is a transitive set contained in U and $\Lambda_f = \Lambda$. These sets were introduced in [11] as a generalization of the basic sets of the Smale theory. The description of these systems is an important challenge: on the one hand, being robust, they can not be ignored in any global picture of dynamical systems; on the other hand, their dynamics is often not stable, indeed many different bifurcations may occur inside the set, without affecting transitivity. However, such bifurcations can never produce sinks nor sources. In the case of surface diffeomorphisms this implies that the periodic points inside a robustly transitive set are persistently hyperbolic, in a uniform fashion:

Theorem 3 (Mañé [23]). *Every robustly transitive set of a surface C^1 diffeomorphism is a hyperbolic set.*

Section 4.1 presents several constructions of robustly transitive diffeomorphisms. A point to be remarked about all these examples is that, while they need not be uniformly hyperbolic, they all have some weak form of hyperbolicity: at least, a dominated splitting of the tangent bundle. Indeed, the main conclusion in this chapter is that *existence of a dominated splitting is a necessary condition for robust transitivity.* (See Theorem 4). The basic idea behind this statement is the observation that in the absence of a dominated splitting one can perturb the derivative at some periodic orbit, to turn that orbit into an attractor or a repeller, thus, breaking transitivity. A consequence of this main result is that homoclinic classes of C^1-generic diffeomorphisms either admit a dominated splitting or are contained in the closure of (infinitely many) periodic attractors or repellers.

In fact, in some situations (especially in low dimensions) one has the stronger property of partial hyperbolicity, and that is better understood. The main reason is that may use strong stable and strong unstable foliations as very useful tools for understanding topological and ergodic properties of the system.

4.1 Examples of robust transitivity

The best known examples of robustly transitive diffeomorphisms are the Anosov diffeomorphisms, especially the linear ones, that is the diffeomorphisms induced on the torus $\mathbb{T}^n$ by a hyperbolic map in $GL(n, \mathbb{Z})$. By focus here on non-hyperbolic examples. The first examples of these diffeomorphisms were constructed in the seventies by Shub [31] and Mañé [22], respectively, on $\mathbb{T}^4$ and on $\mathbb{T}^3$. Their examples are partially hyperbolic with a dominated splitting of the tangent bundle $TM = E^s \oplus E^c \oplus E^u$ into 3 non-trivial subbundles, the central subbundle E^c being one-dimensional and non-hyperbolic. Much later, [2] and [7] gave new classes of robustly transitive diffeomorphisms, on much more general manifolds and also satisfying weaker hyperbolicity conditions.

Let us list the three classes of known examples:

(A) Perturbations of the time-1 of any transitive Anosov flow.

These examples are isotopic to identity, and they exist in many manifolds, other than the tori, already in dimension 3. It is not known which 3-manifolds do support such diffeomorphisms.

(B) Skew-products: $\mathbb{T}^n \times N \to \mathbb{T}^n \times N$, $(x, y) \mapsto (A(x), B(x, y))$ and their perturbations, where A is a transitive Anosov diffeomorphism, N is any compact manifold, and the behavior in the y direction is dominated by that of A along its stable and unstable directions.

In these examples the non-hyperbolic central direction may be chosen tangent to the fibers $\{x\} \times N$ and so $\dim E^c = \dim N$, providing examples of robustly transitive diffeomorphism having a central direction of any dimension.

(C) Robustly transitive diffeomorphisms derived from Anosov, that is, obtained by deforming Anosov torus diffeomorphisms by isotopy. They may be constructed

- (in dim ≥ 3) without any stable subbundle, that is, $TM = E^u \oplus E^{cs}$, where E^{cs} is indecomposable and non-hyperbolic. Analogously, without any unstable subbundle.
- (in dim ≥ 4) without any hyperbolic (stable or unstable) subbundle, that is, $TM = E^{cu} \oplus E^{cs}$ is a dominated splitting where E^{cs} and E^{cu} are indecomposable and non-hyperbolic.

We see in Theorem 4 that these two last constructions correspond to the weakest possible hyperbolicity for robustly transitive systems.

4.1.1 An example of Shub

Consider Anosov diffeomorphisms A and B defined on the torus $\mathbb{T}^2$ such that the product map $(A, B)\colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ is partially hyperbolic with central bundle E^c tangent to the fibers $\{x\} \times \mathbb{T}^2$ (for that it is enough to choose $A = B^n$ for any $n > 1$). Assume that A has fixed points p and q and B has a fixed point r.

Consider a map $f\colon \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$ of the form $f(x, y) = (A(x), F_x(y))$, partially hyperbolic with E^c tangent to the fibers $\{x\} \times \mathbb{T}^2$, and also that $F_p = B$.

Proposition 4.1. *The diffeomorphism f is robustly transitive.*

Proof: As the central foliation $\mathcal{F}^c$ (tangent to the subbundle E^c) is differentiable, Theorem 14 implies that the pair $(\mathcal{F}^c, f)$ is structurally stable. It follows that the stable manifold and the unstable manifold of the continuation of the fiber $\{p\} \times \mathbb{T}^2$ are both dense in $\mathbb{T}^4$ for every C^1 perturbation of f. One deduces that the invariant manifolds of the fixed point (p, r) are both robustly dense in $\mathbb{T}^4$. This implies that f is robustly transitive: the stable manifold of the fixed point intersect any open set U of $\mathbb{T}^4$, thus the iterates $f^n(U)$, $n > 0$ accumulate arbitrarily large disks in the unstable manifold, which, on their turn, intersect any open set V. That is, $f^n(U) \cap V \neq \emptyset$. $\square$

To ensure that these examples are non-hyperbolic, it is enough to take $F_q\colon \mathbb{T}^2 \to \mathbb{T}^2$ having periodic points with different Morse indices (a saddle and a sink): then f also has points with different Morse indices, and transitivity implies that it can not be hyperbolic.

4.1.2 An example of Mañé

Consider a linear Anosov map $A\colon \mathbb{T}^3 \to \mathbb{T}^3$ having three real eigenvalues, $0 < \lambda_1 < 1 < \lambda_2 < \lambda_3$ and $3 < \lambda_3$. Let $\mathcal{F}^c$ be the linear foliation corresponding to the eigenvalue λ_2. Fix small domains $C' \subset C$

with C' much smaller than C. Consider a partially hyperbolic perturbation f of A preserving the foliation $\mathcal{F}^c$, having it as its central foliation, and coinciding with A outside C'.

Assume that the expansion of f in the strong-unstable direction remains bigger than 3. If C is small enough, there is a constant $L > 0$ such that, for every unstable segment σ of length $(\sigma) \geq L$, the set $(f(\sigma) \setminus C)$ contains an unstable segment of length bigger than L. On the other hand, given any unstable segment σ the lengths of its forward iterates $f^n(\sigma)$ are eventually bigger than L. This shows that every unstable segment contains some point x_0 whose forward orbit intersects C finitely many times only.

We claim that, given any small central segment α through such a point x_0, the lengths of its forward iterates increase exponentially. This is obvious while the iterates are shorter than the distance from C' to the complement of C, just because $f^n(\alpha)$ remains outside C' (discard finitely many iterates if necessary), where the central direction is actually expanding. When the segments are long just observe that the fraction of $f^n(\alpha)$ inside C' is very small: expansion of the big sub-segments outside C' suffices for expansion of the total length.

Given an open set $U \subset M$ take an unstable segment $\sigma \subset U$, a point $x_0 \in \sigma$ visiting C finitely many times only, and a central segment $\alpha \subset U$ through x_0. By Theorem 14 each leaf of $\mathcal{F}^c$ is dense in $\mathbb{T}^3$, in a robust fashion. So, the fact that the length of $f^n(\alpha)$ goes to infinity implies that the orbit of α is dense, and so the same is true for the orbit of U. This gives (robust) transitivity of f.

To asure that this construction yields non-hyperbolic maps, Mañé considered a fixed point p of A and performed a saddle-node bifurcation along a central leaf $\mathcal{F}^c$ inside a small neighborhood C' of p. In this way, f has periodic points with different Morse indices 2 and 1.

4.1.3 A local criterium for robust transitivity

In this section we explain the method used in [2] and [15] for proving robust transitivity. Let us start with the following simple criterium, which we used in Shub's example.

Lemma 4.2. *(Basic transitivity criterium) Suppose f has a hyperbolic periodic point p whose invariant manifolds are both (robustly) dense in M. Then f is (robustly) transitive.*

In what follows, we restrict ourselves to 3-dimensional manifolds. See Section 3 for details and notation about blenders. In the next statement note that $W^s(p)$ and $W^u(q)$ are two-dimensional.

Proposition 4.3. *(Blender criterium) Assume that*

1. *f is partially hyperbolic on the whole manifold and there is a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ into three non-trivial subbundles such that E^u is uniformly expanding and E^s is uniformly contracting.*

2. *There are a constant $L > 0$ and hyperbolic periodic points p and q with Morse indices 1 and 2, respectively, such that the stable manifold $W^s(p)$ of p intersects every unstable segment of length bigger than L, and the unstable manifold $W^u(q)$ of q intersects every stable segment of length bigger than L.*

3. *The point q is homoclinically related to a blender Γ such that $E^c \mid \Gamma$ is uniformly expanding, and the one-dimensional unstable manifold $W^u(p)$ of p intersects every superposition region of Γ (we say that p activates Γ).*

Then f is robustly transitive.

Proof: From assumption 2 it follows that $W^s(p)$ intersects all unstable segments, since the length of forward iterates of unstable segments goes to infinity, and the stable manifold is invariant. This implies that the stable manifold of p is dense in M.

By compactness of the family of unstable segments of length $2L$, and the fact that the unstable foliation and the stable manifold of p depend continuously on the dynamics, one easily gets that $W^s(p, g)$ intersects every unstable segment of length $2L$ for all g close to f. As before, this implies the density of $W^s(p, g)$. In other words, the stable manifold of p is robustly dense in M. The same arguments give robust density of the unstable manifold of q.

Using the principle of superposition of the blenders from Section 3, we deduce that the closure of the stable manifold of q robustly contains the stable manifold of p, thus it is robustly dense in M. At this point we know that both invariant manifolds of q are robustly dense in M, and so the basic criterium in Lemma 4.2 implies that f is robustly transitive. □

The blender criterium is used in [2] to prove that both

- the time-1 map of a transitive Anosov flow and
- the product of an Anosov map by the identity on a compact manifold N

are approximated by (non-hyperbolic) robustly transitive diffeomorphisms. For the first situation one perturbs the dynamics along a closed orbit of the flow to get a pair of periodic orbits with different Morse index. In the second situation, one uses a chain of blenders to find a periodic point of index u (the dimension of the unstable manifold of the Anosov map) such that the closure of its unstable manifold contains the unstable manifold of a point of index $\dim N + u$.

4.2 Robust transitivity without hyperbolic directions

All robustly transitive diffeomorphisms described above have hyperbolic directions, that is, either uniformly contracting or uniformly expanding. Here we describe the construction in [7] of robustly transitive diffeomorphisms without any hyperbolic direction. This construction follows closely the one in Section 4.1.2, so we just emphasize the main differences.

Consider a linear Anosov map A of the torus $\mathbb{T}^4$ having 4 real eigenvalues, $0 < \lambda_1 < \lambda_2 < 1 < \lambda_3 < \lambda_4$. Fix small domains $C_1' \subset C_1$ and $C_2' \subset C_2$, with C_1 and C_2 disjoint. Fix an A-invariant cone field $\mathcal{C}^{cu}$ and an A^{-1}-invariant cone field $\mathcal{C}^{cs}$, around the planes corresponding to the expanding eigenvalues λ_3, λ_4 and the contracting eigenvalues λ_1, λ_2, respectively. Now consider diffeomorphisms f coinciding with A outside $C_1 \cup C_2$ and satisfying

1. Df preserves $\mathcal{C}^{cu}$ and expands area uniformly in this cone field.
2. Df^{-1} preserves $\mathcal{C}^{cs}$ and expands area uniformly in this cone field.
3. Df expands uniformly the norm of vectors in $\mathcal{C}^{cu}$ outside C_1'.
4. Df^{-1} expands uniformly the norm of vectors in $\mathcal{C}^{cs}$ outside C_2'.

By choosing sufficiently thin cone fields, we verify that there exists a constant $L > 0$ such that every center unstable disk D_1 (tangent to $\mathcal{C}^{cu}$) of radius bigger than L intersects any center stable disk D_2 (tangent to $\mathcal{C}^{cs}$) of radius larger than L.

Arguing as in Mañé's example, but using uniform expansion of area instead of uniform expansion of norms, we show that every center unstable disk D^1 contains a point whose forward orbit meets C_1 only finitely many times. This allows us to show that $f^n(D_1)$ contains a center unstable disk of radius bigger than L, for every large $n > 0$. The same argument shows that the large negative iterates of any center stable disk D_2 contains a center stable disk of radius bigger that L.

Finally, given any pair of open sets U and V, we may choose a center unstable disk $D_1 \subset U$ and a center stable disk $D_2 \subset V$. Thus, forward iterates of U and backward iterates of V intersect eventually: $f^n(U) \cap f^{-m}(V)$ for all large n and m. This implies that f is transitive.

In order to ensure that f does not preserve any hyperbolic subbundle, we choose the diffeomorphism such that C_1 contains a fixed point p_1 of f of index 2 with non-real contracting eigenvalues together with a fixed point q_1 of index 3. This prevents the existence of uniformly contracting subbundles: the presence of p_1 implies that a stable subbundle would have to be of dimension at least 2, while q_1 implies that the dimension is at most 1. We also choose f so that C_2 contains a fixed point p_2 of index 2 with non-real expanding eigenvalues together with a fixed point q_2 of index 1. Arguing as before, this prevents the existence of any uniformly expanding subbundle.

4.3 Consequences of robust transitivity

The main result in this section is the following generalization of Theorem 3 to arbitrary dimension:

Theorem 4 (Bonatti, D., Pujals, Ures [4, 11]). *Every robustly transitive set of a C^1 diffeomorphism is volume hyperbolic.*

An invariant set Λ is called *volume hyperbolic* if there exists a splitting $T_\Lambda M = E^1 \oplus \cdots \oplus E^k$ into Df-invariant subbundles such that

1. the splitting is *dominated*: there exists $m \geq 1$ such that
$$\|Df^n \mid E^i(x)\| \, \|(Df^n \mid E^j(x))^{-1}\| \leq 1/2$$
for every $n \geq m$, $i > j$, and $x \in \Lambda$.

2. the extreme subbundles E^1 and E^k are uniformly volume expanding and volume contracting: for all $n \geq m$ and $x \in \Lambda$,
$$|\det(Df^{-n} \mid E^1(x))| \leq 1/2 \quad \text{and} \quad |\det(Df^n \mid E^k(x))| \leq 1/2$$

In dimension 2, the only possibility is for the splitting to be into two 1-dimensional subbundles $(2 = 1 + 1)$, and then volume hyperbolicity is the same as hyperbolicity. Therefore, Theorem 4 contains Theorem 3. Moreover, in dimension 3 there are two possibilities, $3 = 2 + 1$ and $3 = 1 + 1 + 1$. In either case, at least one of the subbundles is 1-dimensional and hyperbolic: either uniformly expanding or uniformly contracting. Thus, in this case robust transitivity implies partial hyperbolicity [11]: at least one of the invariant subbundles is hyperbolic. These results are best possible, by the examples in Section 4.1.

As mentioned before, the way to prove Theorem 4 is by showing that the lack of a dominated splitting leads to the creation of a sink or source, after a convenient perturbation. If the lack of dominated splitting is persistent, this yields infinitely many coexisting sinks or sources! We may apply this idea to a transitive set naturally associated to every hyperbolic saddle point p of a diffeomorphism f, its *homoclinic class*. This is defined as the closure of the transverse intersections between its invariant manifolds and denoted by $H(p, f)$.

Theorem 5 (Bonatti, D., Pujals [4])). *There exists a residual subset $\mathcal{R}$ of* $\mathrm{Diff}^1(M)$ *such that for every diffeomorphism $f \in \mathcal{R}$ and periodic point p of f,*

- *$H(p, f)$ is contained in the closure of infinitely many sources or sinks,*
- *or else $H(p, f)$ admits a dominated splitting.*

In the rest of this section we explain the main ideas in the proofs of the theorems above. Roughly speaking, Theorem 4 follows from an idea in linear algebra: in the absence of domination, the eigenspaces of a linear map obtained by multiplication of several linear maps are very unstable: by small perturbation of each of the factors, one can mix the eigenvalues in order to get a homothetic transformation, which then leads to a new sink or a new source.

4.3.1 Lack of domination and creation of sinks or sources

We begin the exposition with the, simpler, 2-dimensional case. The argument we are going to present, due to Mañé [23], shows the following: *if a set $\mathcal{P}$ of hyperbolic periodic saddles of a diffeomorphism of a compact surface does not admit a dominated splitting then it is possible to turn one of such periodic points to be a new sink or source, by a C^1 small perturbation.*

We begin with an elementary property of matrices in $GL(2, \mathbb{R})$. Denote by R_α the rotation of angle α in $\mathbb{R}^2$.

Lemma 4.4. *Consider an orientation preserving matrix $A \in GL(2, \mathbb{R})$ having two different real eigenvalues and whose eigenspaces form an angle less than $\varepsilon > 0$. Then there is $t \in [-\varepsilon, \varepsilon]$ such that $R_t \circ A$ has a pair of conjugate complex eigenvalues.*

A way to prove this is to consider the action h_t of $R_t \circ A$ on the projective space $\mathbb{RP}^1$, identified with the circle S^1. By hypothesis, the map h_0 has two hyperbolic fixed points, at distance less than ε. Moreover, outside the small interval between these fixed points, the sign of $h_0(x) - x$ is constant, say positive. So, $h_0(x) - x > -\varepsilon$ for every x in S^1. Consequently, $h_\varepsilon(x) - x > 0$ for all x in S^1. This implies that the rotation number of the circle map h_t is not constant in $[0, \varepsilon]$. Choosing t in this interval such that h_t has non-zero rotation number, we get that the eigenvalues of $R_t \circ A$ cannot be real, as claimed.

Now consider the natural splitting $T_pM = E^u(p) \oplus E^s(p)$ into eigenspaces, defined over $\mathcal{P}$.

Lemma 4.5. *Suppose that the angle* $\mathrm{angle}(E^s(p), E^u(p))$ *between $E^s(p)$ and $E^u(p)$ is not bounded away from 0 for $p \in \mathcal{P}$. Then there are g arbitrarily C^1-close to f and $p \in \mathcal{P}$ such that p is either a sink or a source of g.*

We need the following result that allows to perform dynamically perturbations of the linear co-cycle defined by the derivative of a diffeomorphism f. It may also be stated uniformly for any g in a neighborhood of f.

Theorem 6 (Franks [17]). *Let f be a C^1-diffeomorphism defined on a closed manifold M and consider any $\delta > 0$. Then there is $\varepsilon > 0$ such that, given any finite set Σ, any neighborhood U of Σ, and any linear maps*

$$A_x \colon T_xM \to T_{f(x)}M\,, \quad x \in \Sigma$$

such that every A_x is ε-close to the derivative of f at x, there is a diffeomorphism g which is δ-close to f in the C^1 topology, coinciding with f on $M \setminus U$ and on Σ, and such that the derivative of g at any point $x \in \Sigma$ is A_x.

By the previous theorem, any small perturbation of the derivative of a diffeomorphism f along a periodic orbit p can be realized dynamically, that is, as the derivative of a C^1-nearby diffeomorphism g for which p is still a periodic point. Then, it is enough to see that there are arbitrarily small perturbations of the derivative of f along the orbit of some $p \in \mathcal{P}$ such that the resulting matrix has non-real eigenvalues of absolute value different from 1.

This is deduced from Lemma 4.4 as follows. Choose a point $p \in \mathcal{P}$ such that $\mathrm{angle}(E^s(p), E^u(p))$ is less than some small fixed ε and let n be the period of p. Then compose the derivative $Df(f^{n-1}(p))$ with a rotation $R_t \colon T_p(M) \to T_p(M)$ for an appropriate t, so that the resulting linear map

$$R_t \cdot Df(f^{n-1}(p)) \cdot Df(f^{n-2}(p)) \cdots Df(p)$$

has complex conjugate eigenvalues. Up to a final perturbation, the absolute value may be taken different from 1.

Lemma 4.5 completes the argument when the angle between E^s and E^u is not bounded away from 0. So, we are left to consider the case in which this angle is uniformly bounded from below on $\mathcal{P}$. In this case, up to a uniformly bounded change of the Riemannian metric, there are unitary vectors $e^s(p) \in E^s(p)$ and $e^u(p) \in E^u(p)$ forming an orthonormal basis of T_pM. In this basis, the derivative $Df(p)$ is given by a diagonal matrix. The set $\mathcal{P}$ is called *hyperbolic in the period* if there is $0 < \lambda < 1$ such that

$$\|Df^{n_p}(p)e^s(p)\| \leq \lambda^{n_p} \quad \text{and} \quad \|Df^{-n_p}(p)e^u(p)\| \leq \lambda^{n_p}$$

for every $p \in \mathcal{P}$, where n_p is the period of p.

Lemma 4.6. *Suppose $\mathcal{P}$ is not hyperbolic in the period. Then there exists a small C^1-perturbation g of f and a periodic point $p \in \mathcal{P}$ such that p is either a sink or a source of g.*

This follows by multiplying each matrix $Df(p)$ by a homothetic transformation of ratio $(1+\varepsilon)$ or $(1-\varepsilon)$, depending on which direction hyperbolicity fails, and then applying Theorem 6.

Finally, consider the case where $\mathcal{P}$ is hyperbolic in the period. As we are assuming that $\mathcal{P}$ does not admit a dominated splitting, there are arbitrarily large n and $p \in \mathcal{P}$ such that $n < n_p$ and

$$\frac{\|Df^n(p)e^u(p)\|}{\|Df^n(p)e^s(p)\|} < 2.$$

Since $\mathcal{P}$ is hyperbolic in the period, $n_p - n$ must go to infinite as n goes to infinity.

Lemma 4.7. *Assume the splitting $E^u \oplus E^s$ over $\mathcal{P}$ is not dominated. Then there is an arbitrarily small perturbation of the derivative of f along the orbit of some point $p \in \mathcal{P}$ such that the angle between the eigenspaces of the corresponding matrix is arbitrarily small.*

This reduces the problem to the situation in Lemma 4.5, thus ending the proof of the statement made at the beginning of this section. So let us now explain the proof of this lemma.

Consider n and p as in the inequality above. Since n is large and the matrices of the derivatives are diagonal, by multiplying each $Df(f^i(p))$, $0 \leq i < n$ by

$$C = \begin{pmatrix} \dfrac{1}{1+\varepsilon} & 0 \\ 0 & 1+\varepsilon \end{pmatrix}$$

we can make the quotient $\|Df^n(p)(e^u(p))\|/\|Df^n(p)(e^s(p))\|$ arbitrarily small, Denote $A(f^i(p)) = C \cdot Df(f^i(p))$. Then consider

$$B = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix}$$

and replace $A(p)$ by $A'(p) = A(p) \cdot B$. At this stage the direction $E^u(p)$ is no longer invariant, but our perturbations do keep $E^s(p)$ invariant. The linear map $A(f^{n-1}(p)) \circ \cdots \circ A(f(p)) \circ A'(p)$ sends $E^s(p)$ to $E^s(f^n(p))$, and $E^u(p)$ to some straight line $\tilde{E}(f^n(p))$ forming a small angle with $E^s(f^n(p))$, provided n is large enough. Through a final perturbation at times $> n$ we are going to recover the invariance of E^u, while keeping E^s invariant. This means they will again be the eigenspaces and so, since at time n their angle is small, the proof will be complete.

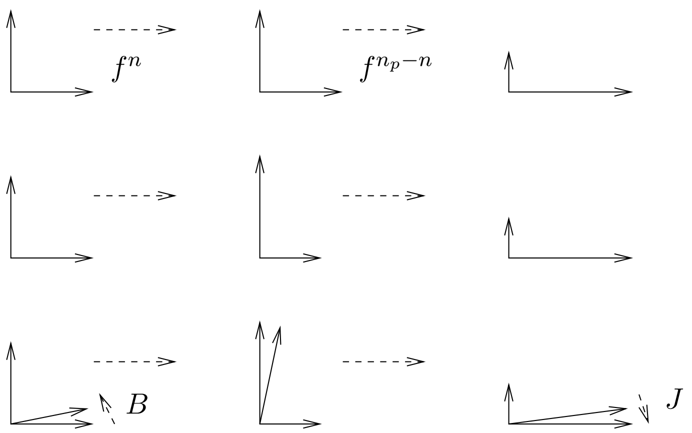


Figure 6: Getting small angles between eigenspaces

So, let us explain this last step, with the aid of Figure 6. Since the periodic orbit is hyperbolic at the period, the image of $e^u(p) + \varepsilon e^s(p)$ under $A^{n_p}(p)$ makes a small angle with $E^u(p)$. In other words, $A(f^{n_p-1}(p)) \circ \cdots A(f(p)) \circ A'(p)$ maps $E^s(p)$ to itself and $E^u(p)$ to a subspace $\tilde{E}^u(p)$ forming a small angle with $E^u(p)$. Then there exists a matrix J close to the identity mapping $E^s(p)$ into $E^s(p)$ and $\tilde{E}^u(p)$ into $E^u(p)$. Thus, $E^s(p)$ and $E^u(p)$ are eigenspaces of

$$J \circ A(f^{n-1}(p))) \circ A(f^{n-2}(p))) \circ \cdot \circ A(f(p)) \circ A'(p).$$

This is the final perturbation mentioned above, and so the proof of the lemma is complete.

4.3.2 Dominated splittings *vs.* homothetic transformations

The previous ideas extend to arbitrary dimension. One gets that, given a set $\mathcal{P}$ of hyperbolic periodic saddles all with the same Morse index (the same dimension of the unstable direction), if the natural splitting induced by the stable and unstable directions is not dominated then it is possible to change the index of some saddle by a small C^1 perturbation.

However, in dimension larger than 2 such bifurcations do not break transitivity, and they do occur in robustly transitive sets. Towards proving Theorem 4, we are going to show that *in the absence of any dominated splitting it is possible to perturb the derivative of f along a periodic orbit* *in order to turn into a homothetic transformation and, after an additional small perturbation, to create a sink or a source for a diffeomorphism close to f.* An important tool is Franks' lemma (Theorem 6), according to which C^1 small perturbations of the derivative can always be realized dynamically.

The main novelty with respect to the previous arguments is the introduction of *transitions* between different periodic orbits, which permit a kind of concatenation of such orbits and corresponding multiplication of eigenvalues. In a few words, while in the 2-dimensional situation presented above we deal with a linear cocycle defined over period points and their perturbations only, the arguments that follow use strongly the dynamics on a homoclinic class and the continuity of the derivative: the transitions correspond to heteroclinic orbits.

We begin with another argument of linear algebra which is crucial in the proof. Consider a basis $\beta = \{v_1, \ldots, v_n\}$ of $\mathbb{R}^n$ and let L_i be the subspace spanned by v_i. Consider a linear map $A \in GL(\mathbb{R}, n)$ which is diagonal relative to the basis β, and $(n-1)$ linear maps $A_1, A_2, \ldots, A_{n-1} \in GL(\mathbb{R}, n)$ such that each A_j

- leaves invariant the subspaces $L_1, \ldots, L_{j-1}, L_{j+2}, \ldots, L_n$ and
- permutes L_j and L_{j+1}, that is, $A_j(L_j) = L_{j+1}$ and $A_j(L_{j+1}) = L_j$.

For arbitrarily small neighborhoods $\mathcal{A}$ of A and $\mathcal{A}_i$ of each A_i, consider all possible products of matrices in $\mathcal{A} \cup (\cup_i \mathcal{A}_i)$. Then some of these products are homothetic transformations. The idea is that, as the matrices A_i generate all permutations of the eigenspaces of A, it is possible to mix the eigenvalues of A to obtaining a matrix having essentially the same rate of expansion in all directions.

In the model we have in mind, the matrices A and A_i correspond to the derivatives $Df^k(p)$, $k = \mathrm{per}(p)$ at periodic points p of a diffeomorphism f. Of course, to multiply derivatives corresponding to different points does not make dynamical sense. However, properties of homoclinic classes $H(p, f)$ allow us to give a sense to this procedure.

First of all, there is a dense subset K in $H(p, f)$ consisting of periodic points homoclinically related to p. As a dominated splitting extends to the closure of its domain (see Appendix 5), absence of dominated splitting on $H(p, f)$ implies the same for K. Thus, it is enough to consider this set K of periodic orbits.

Next, given two fixed or periodic hyperbolic saddles x and y in K, and given any family $\{n_1, m_1, \ldots, n_k, m_k\}$ of arbitrarily big natural numbers, there are a natural number r (independent of n_i and m_i) and a saddle $z \in H(p, f)$ of period $t = kr + \sum_i (n_i + m_i)$ whose orbit spends alternately n_i consecutive iterates close to x and m_i consecutive iterates close to y. This means that the derivative of $Df^t(z)$ is, roughly, of the form

$$Df^t(z) = (T_2 \circ Df^{m_k}(q) \circ T_1 \circ Df^{n_k}(p)) \circ \cdots \circ (T_2 \circ Df^{m_1}(q) \circ T_1 \circ Df^{n_1}(p))$$

where T_1 and T_2 are called the *transitions* from x to y and from y to x, respectively. These transitions correspond to a bounded number of iterations, independent of n_i and m_i, so that their contribution to the product becomes negligible as n_i and m_i go to infinite.

Thus, it makes sense to multiply derivatives corresponding to different periodic points if they are homoclinically related. Besides the resulting linear maps correspond, roughly, to the derivative at another periodic saddle in the homoclinic class. Similar arguments show that in any homoclinic class there is a dense subset of periodic points whose derivative at the period can be made diagonal, with positive eigenvalues of multiplicity 1, by a small perturbation of the derivative along the orbit.

The construction above shows that a robust transitive set always admits a dominated splitting. Then the *finest* such splitting is well-defined. To get volume hyperbolicity it remains to prove uniform volume expansion and volume contraction along the two extreme subbundles. This is an application of the next results. We first state the Pugh closing lemma:

Theorem 7 (Pugh [29, 30]). *Let f be a C^1 diffeomorphism defined on a compact manifold M and $x \in M$ a non-wandering point of f. There exists $g \in \mathrm{Diff}^1(M)$ arbitrarily close to f in the C^1 topology for which x is a periodic point.*

Let $\Sigma(f)$ be the set of points $x \in M$ such that given any C^1 neighborhood $\mathcal{U}$ of f and any $\varepsilon > 0$ there exist $g \in \mathcal{U}$, coinciding with f outside the ε-neighborhood of the f-orbit of x, and there exists a g-periodic point $y \in M$ such that

$$d(f^i(x), g^i(y)) \le \varepsilon \quad \text{for all } 0 \le i \le \mathrm{per}(y).$$

Theorem 8 (Mañé [23]). *For any C^1 diffeomorphism f of a compact manifold, the set $\Sigma(f)$ has full probability for f.*

A major application of this statement is to express in terms of periodic orbits dynamical behaviors that are detected by invariant measures. The following argument, which is part of Mañé's proof of the 2-dimensional version of Theorem 4 in [23], is a good illustration:

Let f be a surface diffeomorphism with a dominated splitting $E \oplus F$ and suppose the sub-bundle F is *not* uniformly expanding. Then there are arbitrarily long orbit segments over which expansion along F is arbitrarily weak. Any weak accumulation point of Dirac averages along such orbit segments is an invariant measure for which the sub-bundle F is not expanding. Pugh's closing lemma permits to close a typical orbit of such a measure. However, not knowing whether the periodic orbit remains close to the original one, one loses control on the dilation along F. On the contrary, the ergodic closing lemma ensures that the derivative of the perturbed map g is, at best, weakly expanding along the continuation F_g of F for g over the orbit of the periodic point y. By another small perturbation (this is Franks' lemma, that we shall discuss in a while) we can now turn any slight expansion into a contraction, which means that y becomes a sink.

We are not ready to consider the finest dominated splitting above. Since E^1 does not admit any dominated sub-splitting, the previous arguments ensure that by a small perturbation we can produce a homothetic transformation along E^1. If E^1 is not uniformly volume expanding, by using the ergodic closing lemma we get a homothetic transformation along E^1 at a periodic point whose volume expansion rate is less than $1 + \delta$. So, maybe after a new perturbation, we get a homothetic transformation with contraction rate strictly smaller than 1. By domination, this implies that all the subbundles E_i are contracting. This means that the periodic point is really a source, which is a contradiction.

4.3.3 On the dynamics of robustly transitive sets

Here our goal is to deduce dynamical properties of robustly transitive sets, such as the kind of bifurcations they may exhibit, or the types of saddles they may contain, from the analysis of the finest dominated splitting. The arguments in the previous section imply that *if it is not possible* *to change, by perturbation, the index of a periodic point in a robustly transitive set Λ, then Λ is hyperbolic.* In the same spirit, the following result explains what happens when it is not possible to increase the indices of the periodic saddle points of Λ: it relates the maximal and minimal indices with the hyperbolicity of the subbundles in the finest dominated splitting.

Theorem 9 (Bonatti, D., Pujals, Rocha [5]). *Consider $k \geq 1$ and an open subset U of a closed manifold. Let $\mathcal{U}$ be an open subset of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that $\Lambda_f(U)$ is robustly transitive, has a dominated splitting $E_f \oplus F_f$ with $\dim E_f = k$, and does not contain saddles of Morse index less than k. Then E_f is uniformly expanding for every diffeomorphism f in $\mathcal{U}$.*

Theorem 10 (Bonatti, D., Pujals, Rocha [5]). *There is an open and dense subset $\mathcal{I}(U)$ of $\mathcal{R}(U)$ such that for all $g \in \mathcal{I}(U)$ the set of indices of the hyperbolic saddles of $\Lambda_g(U)$ is an interval in $\mathbb{N}$.*

The main ingredient in the proof is a notion of transitions associated to a heterodimensional cycle. These are similar to the transitions associated to homoclinically related periodic points in Section 4.3.2, but there are also some subtle differences that we now explain. We first recall the connecting lemma:

Theorem 11 (Hayashi [18]). *Given two periodic points p and q, if there are orbits visiting the neighborhoods of some point in the unstable manifold of p and some point in the stable manifold of q then, by an arbitrarily C^1-small perturbation, one can make the two invariant manifolds intersect each other.*

The connecting lemma was a decisive ingredient in the recent progress towards understanding C^1 generic dynamical systems. On the other hand, many of these applications require variations of the original statement. In particular, the following question posed itself quite naturally: Assume the unstable manifold of p and the stable manifold of q accumulate on some point x. *Is there a C^1 perturbation of the system for which the two invariant manifolds intersect each other ?* This has been answered recently, assuming x is not periodic itself, independently by Arnaud [1], Hayashi [19], and Wen, Xia [32]. Their statements are slightly different, here we quote [1]:

Theorem 12. *Let f be a diffeomorphism on a compact manifold, $x \in M$ be a non-periodic point, and $\mathcal{U}$ be a C^1 neighborhood of f. Then there exist $N \geq 1$ such that for any neighborhood V of x there exists another neighborhood $W \subset V$ such that given any points y and z outside $\cup_{0 \leq n < N} f^n(V)$ and such that W contains some forward iterate of y and some backward iterate of z, there exists $g \in \mathcal{U}$ coinciding with f outside $\cup_{0 \leq n < N} f^n(V)$ and for which z is in the forward orbit of y, passing through V.*

Given two periodic points x and y of Λ with different indices, using robust transitivity and the connecting lemma, Theorem 12, we may create a heterodimensional cycle involving x and y. Then there is a perturbation of f having a periodic orbit which remains arbitrarily large periods close to x and close to y, with bounded transition times from one saddle point to the other. The derivative on this periodic orbit is a small perturbation of the product of arbitrarily large powers of the derivatives at x and y by a pair of bounded matrices, the *transitions*. One difference with respect to Section 4.3.2 is that here the new periodic orbits passes close to x and y only once, it closes the period already at the first return. A second difference is that in the homoclinic case the transitions are for the initial diffeomorphism, whereas here they are defined for perturbations of it.

Another important topic is the occurrence of homoclinic tangencies inside robustly transitive sets. A heuristic principle (see [5] for precise statements) is that *if the finest dominated splitting of* $\Lambda_f(U)$ *involves a non-hyperbolic subbundle of dimension bigger than* 1 *then there is* g *close to* f *such that* $\Lambda_g(U)$ *contains a homoclinic tangency.* Moreover, it is possible to determine the index of the hyperbolic saddle associated to the tangency. On the other hand, if all non-hyperbolic subbundles of the finest dominated splitting are 1-dimensional then it is not possible to create homoclinic tangencies inside the robustly transitive set by small perturbations of the system. In this case the prototypical bifurcations are heterodimensional cycles and saddle-node and period-doubling bifurcations.

Let us observe that homoclinic tangencies are indeed compatible with robust transitivity because, in higher dimensions, they need not lead to the creation of sinks or sources, unless the saddle point involved in the tangency is sectionally dissipative, see [27].

5 Appendix: Weak forms of hyperbolicity

There are several notions extending the concept of uniform hyperbolicity in the following spirit: one still requires the existence of an invariant splitting of the tangent bundle, but one allows one or more of the subbundles involved to be such that the derivative exhibits mixed contracting/neutral/expanding behavior along them. Typically, these non-hyperbolic subbundles concentrate the dynamical difficulties: often one can project the dynamics along the remaining hyperbolic directions thus reducing the study to the central part.

Brin, Pesin [8] and Hirsch, Pugh, Shub [20] considered *partially hyperbolic systems*, that is, admitting an invariant splitting of the tangent bundle as the sum $E^s \oplus E^c \oplus E^u$ of three subbundles, where E^s is uniformly contracting, E^u is uniformly expanding, and the central direction E^c is "in between" the other two (one of the subbundles may be trivial). Examples include the time-1 map of an Anosov flow and the product of an Anosov diffeomorphism by the identity. In the context of the stability conjecture, Liao [21], Mañé [23], Pliss [28] were led to the more general notion of *dominated splitting* into two subbundles: one of them is definitely more contracted (or less expanded) than the other, after a uniform number of iterates.

5.1 Dominated splittings

5.1.1 Definition and properties

Definition 5.1. *Let* $f : M \to M$ *be a diffeomorphism on a closed manifold* M *and* K *be any* f*-invariant set. A splitting* $T_xM = E_1(x) \oplus \cdots \oplus E_k(x)$*,* $x \in K$ *of the tangent bundle over* K *is* dominated *if it is invariant under the derivative* Df *and there exists* $\ell \in \mathbb{N}$ *such that for every* $i < j$*, every* $x \in K$*, and every pair of unit vectors* $u \in E_i(x)$ *and* $v \in E_j(x)$*, one has*

$$\frac{\|Df^\ell(x)u\|}{\|Df^\ell(x)v\|} < \frac{1}{2} \tag{2}$$

and the dimension of $E_i(x)$ *is independent of* $x \in K$ *for every* $i \in \{1, \ldots, k\}$.

We write $E_i \prec E_j$. In some contexts we need to specify the strength of the domination, measured by the number ℓ of iterates needed to get the inequality in the definition. Then we write $E_i \prec_\ell E_j$ and say that the splitting is ℓ*-dominated.*

The previous definition may be formulated, equivalently, as follows: there exist $C > 0$ and $\lambda < 1$ such that

$$\frac{\|Df^n(x)u\|}{\|Df^n(x)v\|} < c\lambda^n \quad \text{for all } n \geq 1. \tag{3}$$

When using this notion, λ and c may be taken as measures of the strength of the domination.

Let us list some useful elementary properties of dominated splittings. An important ingredient in the proofs of the properties *extension to a neighborhood* and *persistence* is a characterization of existence of a dominated splitting in terms of invariant cone-fields.

Uniqueness: The dominated splitting is unique if one fixes the dimensions of the subbundles: Let d be the dimension of the ambient manifold and let

$$d = n_1 + n_2 + \cdots + n_k\,.$$

Then there is at most one dominated splitting $T_K M = E_1 \oplus E_2 \oplus \cdots \oplus E_k$ over K with dimension of E_i equal to n_i for all i. It is clear that without the dimension requirement, there may be more than one dominated splitting. For instance, if $E^s \oplus E^c \oplus E^u$ is dominated then so are $E \oplus E^u$ and $E^s \oplus F$, where $E = E^s \oplus E^c$ and $F = E^c \oplus E^u$.

Transversality: The angles between any two subbundles of a dominated splitting are uniformly bounded from zero. In fact, the angle between the spaces $\oplus_{i\in I} E_i$ and $\oplus_{j\in J} E_j$ is uniformly bounded from zero, for any disjoint sets $I, J \subset \{1, \ldots, k\}$.

Continuity: Every dominated splitting is continuous, meaning that the subbundles $E_i(x)$ depend continuously on the point x.

Extension to the closure: Every ℓ-dominated splitting over a set K may be extended to an ℓ-dominated splitting over the closure of K. Even more, if g_n is a sequence of diffeomorphisms converging to f and K_n is a g_n-invariant set with an ℓ-dominated splitting, with dimensions independent of n, then the f-invariant set

$$K = \limsup_n K_n = \bigcap_N \overline{\bigcup_{n\geq N} K_n}$$

admits an ℓ-dominated splitting.

Extension to a neighborhood: Let K be an f-invariant set with a dominated splitting. Then such a splitting can be extended in a dominated way to the maximal invariant set of f in a neighborhood of K.

Persistence: Every dominated splitting persists under C^1-perturbations. More precisely, consider an f-invariant set K with an ℓ-dominated splitting. Then, for every $\varepsilon > 0$ there is a neighborhood U of K and a C^1-neighborhood $\mathcal{U}$ of f such that, for every $g \in \mathcal{U}$, the maximal g-invariant set in the closure of U admits an $(\ell - \varepsilon)$-dominated splitting ($(\ell - \varepsilon)$-dominated splitting meaning that $\|Df^\ell(x)u\|/\|Df^\ell(x)v\| < \frac{1}{2} + \varepsilon$), having the same dimensions of the initial dominated splitting over K.

Clustering: Suppose that $E_1 \oplus \cdots \oplus E_k$ is a dominated splitting and let $1 \leq i < j \leq k$, with either $i > 1$ or $j < k$. Then

$$E_1 \oplus \cdots \oplus E_{i-1} \oplus F \oplus E_{j+1} \oplus \cdots \oplus E_k\,,$$

where $F = E_i \oplus \cdots \oplus E_j$, is a dominated splitting.

Sub-splittings: Let $E_1 \oplus \cdots \oplus E_k$ be a dominated splitting and suppose that $E_j = F_1 \oplus \cdots \oplus F_m$ is a dominated splitting. Then

$$E_1 \oplus \cdots \oplus E_{j-1} \oplus F_1 \oplus \cdots \oplus F_m \oplus E_{j+1} \oplus \cdots \oplus E_k$$

also is a dominated splitting.

These properties motivate the following notion introduced in [4]:

Proposition 5.2. *Let K be an f-invariant set admitting a dominated splitting. Then there is a unique* finest dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_k$ *on K such that no E_i admits a dominated splitting. Any other invariant splitting is obtained by considering a partition of $\{1, \ldots, k\}$ into sub-intervals, and clustering the corresponding subbundles.*

The following notions are relevant in the study of robustly transitive dynamics:

Definition 5.3. *Let f be a diffeomorphism and K be an f-invariant set having a dominated splitting $T_K M = E_1 \oplus \cdots \oplus E_k$. We say that the splitting and the set K are*

- partially hyperbolic *if there is $n \geq 1$ such that the restriction of Df^n either contracts uniformly E_1 or expands uniformly E_k.*
- volume hyperbolic *if there is $n \geq 1$ such that the Jacobian of $Df^n \mid E_1$ is strictly less than 1 (uniform volume contraction) and the Jacobian of $Df^n \mid E_k$ is strictly greater than 1 (uniform volume expansion).*

We say that the diffeomorphism f is partially/volume hyperbolic if the ambient space M is a partially/volume hyperbolic set for f.

Remark 5.4. *An invariant set is volume hyperbolic if and only if its finest dominated splitting is volume hyperbolic. If a volume hyperbolic splitting has at least one of the extremal bundles E_1 or E_k of dimension 1 then it is partially hyperbolic.*

Just to give some flavor of the dynamical consequences of these properties, let us quote a result on bifurcations of invariant sets with a dominated splitting, from [5]: *Let f be a diffeomorphism having a dominated splitting into one-dimensional subbundles, defined on the whole ambient manifold. Then no diffeomorphism close to f exhibits homoclinic tangencies.*

5.1.2 Proofs of the elementary properties:

Consider a diffeomorphism $f : M \to M$ on a closed manifold and let K be an f-invariant set.

Transversality: We consider the case of two subbundles. Assume that the tangent bundle over K contains two subbundles E and F, not necessarily continuous, and some $\ell \geq 1$ such that $E \prec_\ell F$ on K. We first show that the angle between E and F is uniformly bounded from zero. For doing that it is enough to prove that the difference $\|u - v\|$, $u \in E(x)$, $v \in F(x)$, $\|u\| = \|v\| = 1$, is larger than some constant independent of $x \in K$, u and v. Assume otherwise. Then there are sequences

$u_n \in E(x_n)$ $v_n \in F(x_n)$ of unit vector with $u_n - v_n$ going to 0. As the derivative of f is bounded, one may find a sequence $m_n \to +\infty$ such that

$$\frac{1}{2} < \frac{\|Df^{m_n}(u_n)\|}{\|Df^{m_n}(v_n)\|} < 2,$$

which contradicts the ℓ-domination. This proves the first part of the transversality property.

Clustering: Assume now that K contains 3 subbundles E, F, and G, and there exits $\ell \geq 1$ such that $E \prec_\ell F$ and $F \prec_\ell G$. As the angle between E and F is uniformly bounded from zero, by the previous argument, there is $\alpha > 0$ such that any unit vector $u \in E(x) \oplus F(x)$, $x \in K$ may be written as $u = u_E + u_F$ with $\|u_E\| + \|u_F\| < \alpha$. Fix $k \geq 1$ such that $2^k > 2\alpha$. Then, for any unit vector $v \in G(x)$,

$$\frac{\|Df^{k\ell}(u)\|}{\|Df^{k\ell}(v)\|} < \frac{1}{2^k}\alpha < \frac{1}{2}.$$

So we proved $(E \oplus F) \prec_{k\ell} G$. The same proof give that $E \prec (F \oplus G)$. Now a simple induction argument gives the complete clustering property.

Uniqueness: Assume that $E \prec F$ and $G \prec H$ are two dominated splittings over K such that $\dim E \leq \dim G$. We argue, by contradiction, that $E \subset G$. Actually, this is a stronger result than what we claimed, useful for proving uniqueness of the finest dominated splitting. A simpler proof of the uniqueness statement we claimed is contained in the proof of the continuity and extension properties below.

So, assume $E(x) \not\subset G(x)$ for some $x \in K$ and consider some unit vector $u \in E(x) \setminus G(x)$. Write $u = u_G + u_H$ with $u_H \neq 0$. Then the positive iterates of u grow at the same rate as those of u_H. Write also $u_H = v_E + v_F$. If v_F did not vanish then the positive iterates of v_F would grow at the same rate as those of u_H, that is, at the same rate as the iterates of u, which would contradict $E \prec F$. Therefore, $v_F = 0$, which means that $u_H \in E(x) \cap H(x)$. Since $\dim E \leq \dim G$ and we are assuming that they do not coincide, there is some unit vector $w \in G(x) \setminus E(x)$. Write $w = w_E + w_F$ with $w_F \neq 0$. Then the positive iterates of $w \in G$ grow at the same rate as those of w_F, and so they grow exponentially faster than the iterates of u_H (because $u_H \in E(x)$). Since u_H is also in $H(x)$, this contradicts the domination $G \prec H$, and completes our argument. This gives the uniqueness of a dominated splitting into two subbundles, once dimensions are fixed.

Continuity and extension to the closure: Let $E \oplus F$ be an ℓ-dominated splitting over K. Let $x_n \in K$ be a sequence converging to some point x. Up to choosing subsequences one may assume that the spaces $E(x_n)$ and $F(x_n)$ converge to subspaces $E(x)$ and $F(x)$. A simple continuity argument show that, for any $k \in \mathbb{N}$, for any unit vectors $u \in E_x$ $v \in F(x)$ one has

$$\frac{\|Df^{k\ell}(u)\|}{\|Df^{k\ell}(v)\|} < \frac{1}{2^\ell} \quad \text{and} \quad \frac{\|Df^{-k\ell}(v)\|}{\|Df^{-k\ell}(u)\|} < \frac{1}{2^\ell}. \tag{4}$$

This directly proves that $E(x) \oplus F(x) = T_xM$. The first part of (4) characterizes $E(x)$ uniquely (once its dimension is fixed): the iterates of any unit vector u in $E(x)$ grow slower than those of any unit vector $w \notin E(x)$. In the same way, the second part of (4) characterizes $F(x)$ uniquely.

This now implies that, for any other sequence y_n converging to x, any accumulation point of $E(y_n)$ coincides with $E(x)$ and, in the same way, any accumulation point of $F(y_n)$ coincides with $F(x)$. As a consequence, one gets that the splitting on K extends by continuity to the closure of K. At the same time, this argument proves that the subbundles are continuous on K. The assumption that the splitting is invariant on K implies that invariance on the closure as well. Finally, (4) shows that this invariant extension is still dominated.

The argument extends easily to any dominated splitting with any number of subbundles, using clustering: the previous argument shows that each splitting $(E_1 \oplus \cdots \oplus E_i) \oplus (E_{i+1} \oplus \cdots \oplus E_k)$, and by intersecting the subbundles in all these extensions one obtains the extension of the original splitting to the closure of K.

Extension to a neighborhood and persistence Consider a ℓ-dominated splitting $E \oplus F$ on K. Extend it to a continuous splitting $TM = E \oplus F$ in a neighborhood U of K, not necessarily invariant. Then consider the cone fields $\mathcal{C}_\alpha$ on U consisting of all the vectors such that the norm of the component in the F-direction is larger than α times the norm of the component in the E-direction. The domination property implies that, for any $x \in K$, $Df^\ell(\mathcal{C}_1(x)) \subset \mathcal{C}_2(f^\ell(x))$. As a consequence, there is a neighborhood $V \subset U$ of K and a C^1-neighborhood $\mathcal{U}$ of f such that for any $g \in \mathcal{U}$ and any $x \in V$ one has $Dg^\ell(\mathcal{C}_1(x)) \subset \mathcal{C}_{2-\varepsilon}(g^\ell(x))$. This implies that the maximal invariant set of g in V has a dominated splitting with the same dimensions of the initial one, and with almost the same strength.

5.1.3 Invariant foliations

Let f be a C^r diffeomorphism on a closed manifold M admitting a dominated splitting $TM = E^u \oplus E^c \oplus E^s$ where Df is uniformly expanding along E^u and uniformly contracting along E^s (f is partially hyperbolic).

Theorem 13 (Brin, Pesin [8], Hirsch, Pugh, Shub [20]). *Assuming* $\dim E^u > 0$*, there exists a unique family* $\mathcal{F}^u$ *of injectively* C^r *immersed submanifolds* $\{\mathcal{F}^u(x) : x \in M\}$ *such that* $x \in \mathcal{F}^u(x)$ *and* $\mathcal{F}^u(x)$ *is tangent to* E^u_x *at every* $x \in M$*. This family is invariant, in the sense that* $f(\mathcal{F}^u(x)) = \mathcal{F}^u(f(x))$ *for all* $x \in K$*, and the leaves* $\mathcal{F}^u(x)$ *are uniformly contracted by some negative iterate of* f*. Moreover,* $\mathcal{F}^u$ *is a continuous lamination of* M*, meaning that every point has a continuous local chart which trivializes the leaves and is uniformly* C^r *along each leaf.*

$\mathcal{F}^u$ is the *strong unstable foliation* and its leaves $\mathcal{F}^u(x)$ are called strong unstable manifolds. Local strong unstable manifolds are neighborhoods of the point inside the corresponding leaf. By reversing time one gets strong stable foliations and (local) strong stable manifolds, assuming $\dim E^s > 0$. More generally, strong stable and strong unstable foliations exist for any invariant subset $K \subset M$ admitting a dominated splitting as above. The family $\mathcal{F}^u$ is indexed by $x \in K$. These strong invariant foliations are continuous laminations, in the same sense as before.

The following general structural stability result assumes that the initial central foliation is *plaque expansive*: given ε there exists δ such that if x_n and y_n are δ-pseudo orbits with all jumps along the central leaves, and they remain within distance ε for all times, then x_n and y_n are in the same local central leaf for all n.

Theorem 14 (Hirsch, Pugh, Shub [20]). *If f admits a plaque expansive central foliation $\mathcal{F}^c$ then $(f, \mathcal{F}^c)$ is structurally stable. The hypothesis of plaque expansiveness holds, in particular, if $\mathcal{F}^c$ is of class C^1.*

The ergodic properties of orbits propagate along stable and unstable sets, since forward time averages are constant on stable sets and backward time averages are constant on unstable sets. In order to use this idea, often called the Hopf argument, one needs the holonomies of foliations to preserve zero measure sets. This *absolute continuity* property does hold for the strong stable and strong unstable foliations of partially hyperbolic C^2 diffeomorphisms [8].

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