Collision, explosion and collapse of homoclinic classes

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Abstract

Homoclinic classes of generic C^1-diffeomorphisms are maximal transitive sets and pairwise disjoint. We here present a model explaining how two different homoclinic classes may intersect, failing to be disjoint. For that we construct a one-parameter family of diffeomorphisms $(g_s)_{s\in[-1,1]}$ with hyperbolic points P and Q having nontrivial homoclinic classes, such that, for $s<0$, the classes of P and Q are disjoint, for $s=0$, the classes collide and their intersection is a saddle-node, and, for $s>0$, after an explosion, the two classes are equal. Our constructions involve bifurcations through heterodimensional and saddle-node cycles.

Introduction

In this paper we study the collision of non-trivial homoclinic classes *via* saddle-node bifurcations and the dynamics before and after this collision. The main motivation of this paper comes from recent results about *maximal transitive sets*: for generic[1] C^1-diffeomorphisms, the homoclinic classes are either disjoint or equal ([Ar] and [CMP]). Our objective is to understand how two homoclinic classes may be non-disjoint and different as well as the dynamical consequences of this pathology.

Let us start by recalling some definitions. Given a diffeomorphism f, an f-invariant set Λ is *transitive* if there is an $x\in\Lambda$ whose forward orbit is dense in Λ, i.e., $\Lambda=\overline{\cup_{i\in\mathbb{N}} f^i(x)}$. A transitive set is *maximal* if it is a maximal element of the family of all transitive sets partially ordered by inclusion. Observe that any transitive set is contained in a maximal one. A transitive set Λ is *saturated* if it contains every transitive set Σ such that $\Lambda\cap\Sigma\neq\emptyset$. Clearly, every saturated transitive set is also maximal. The *homoclinic class* of a saddle P of f, denoted by $H(P,f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Every homoclinic class is a transitive set, not necessarily maximal nor saturated.

The problem of characterizing and describing (for a large class of systems) maximal and saturated transitive sets is a key problem in dynamics. In fact, these saturated transitive sets are the natural candidates for playing the role of the elementary pieces of dynamics (similar to the role of the basic sets in the Smale hyperbolic theory, [Sm]). Recently, [Ab] states that for generic C^1-diffeomorphisms f having finitely many different homoclinic classes the non-wandering set of f, $\Omega(f)$, is the disjoint union of such classes. Moreover, these classes verify a weak form of hyperbolicity (existence of a dominated splitting, see [BDP]) and are the maximal invariant sets of a fixed filtration (see Section 6.3) independent of the generic diffeomorphism in a neighborhood of f.

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[1]by a generic diffeomorphism we mean a diffeomorphism in a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$.

Consider a closed manifold M and denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the usual uniform topology. In [Ar], it is proved that homoclinic classes of generic diffeomorphisms are *maximal transitive sets.* [CMP] generalizes this result by proving that homoclinic classes of generic diffeomorphisms are saturated transitive sets. Thus homoclinic classes of generic diffeomorphisms are either equal or disjoint. We observe that there are locally generic diffeomorphisms having saturated transitive sets without periodic orbits (so which are not homoclinic classes), see [BD$_2$]. The goal of this paper is to give examples of homoclinic classes which are not saturated transitive sets, presenting an explanation of how this pathology arises. In fact, we exhibit homoclinic classes which are not contained in any saturated transitive set. For simplicity, we consider diffeomorphisms defined on three manifolds, but our constructions can be carried out to higher dimensions after straightforward modifications.

We construct a diffeomorphism f with saddles P and Q with *Morse index* (dimension of the unstable bundle) one and two such that their homoclinic classes are nonhyperbolic, nontrivial, and maximal transitive, and whose intersection is just a saddle-node. So these classes are not saturated transitive sets. In fact, as mentioned above, we will prove that they are not contained in any saturated transitive set.

Theorem A. *Let M be a 3-dimensional closed manifold. There exist an open set $W \subset M$ and a family of diffeomorphisms $(g_s)_{s\in[-1,1]}$, $g_s\colon M \to M$, such that, for every s, the diffeomorphism g_s has hyperbolic fixed points P and Q of Morse indices 1 and 2 such that the maximal invariant set of g_s in W, denoted by Λ_s, verifies the following:*

- *For every small $s < 0$, the set $\Lambda_s \cap \Omega(g_s)$ is the disjoint union of the homoclinic classes $H(P, g_s)$ and $H(Q, g_s)$, where $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and locally maximal.*
- *For $s = 0$, $\Lambda_0 = \Lambda_0 \cap \Omega(g_0) = H(P, g_0) \cup H(Q, g_0)$, where $H(P, g_0)$ and $H(Q, g_0)$ are locally maximal and $H(P, g_0) \cap H(Q, g_0) = \{S\}$, where S is a saddle-node fixed point.*
- *For every small $s > 0$, $\Lambda_s = \Lambda_0 \cap \Omega(g_0) = H(P, g_s) = H(Q, g_s)$.*

This result means that the homoclinic classes of P and Q *collide* at $s = 0$ and thereafter *explode* (the point P that does not belong to $H(Q, g_0)$ is in $H(Q, g_s)$ for every small positive s, and the same holds for the point Q and $H(P, g_s)$). Finally, the homoclinic classes also *collapse*: $H(Q, g_s) = H(P, g_s)$ for positive s.

In the previous theorem the open set W is a level of a *filtration,* (see Section 6.3). Theorem A now implies the following:

Theorem B. *Under the hypotheses of Theorem A, the homoclinic classes $H(P, g_0)$ and $H(Q, g_0)$ are not saturated and they are not contained in any saturated transitive set.*

Our construction involves *saddle-node bifurcations* and *heterodimensional cycles.* We introduce a codimension-two bifurcation, the *saddle-node heterodimensional cycles,* and study the *lateral homoclinic classes of a saddle-node.* Let us explain all that in details.

Consider a diffeomorphism f having two hyperbolic fixed points P and Q with Morse indices 1 and 2, respectively. Then, f has a *heterodimensional cycle associated to P and Q* if the 2-dimensional stable manifold of P and unstable manifold of Q, denoted by $W^s(P, f)$ and $W^u(Q, f)$, have a non-empty transverse intersection, and the 1-dimensional unstable manifold of P, $W^u(P, f)$, and stable one of Q, $W^s(Q,f)$, have a quasi-transverse intersection throughout the orbit of a point x_0, i.e., $T_{x_0}W^s(Q,f)+T_{x_0}W^u(P,f)=T_{x_0}W^s(Q,f)\oplus T_{x_0}W^u(P,f)$, thus $\dim(T_{x_0}W^s(Q,f)+T_{x_0}W^u(P,f))=2$. A heterodimensional cycle is depicted in Figure 1 in Section 1. Bifurcations through heterodimensional cycles have been systematically studied in the series of papers [D_1, D_2, DR_1, DR_2, DU, DR_4, DR_5].

A *saddle-node* S of a diffeomorphism f is a periodic point (we here assume to be fixed) such that the derivative of f at S has 1 as its only eigenvalue in the unitary circle. We consider saddle-nodes of *saddle-type* (i.e., the derivative of f at S simultaneously has eigenvalues inside and outside the unitary circle). Thus the tangent bundle of M at S has a Df-invariant splitting $E^{ss}\oplus E^c\oplus E^{uu}$, where E^{ss} (resp. E^{uu}) is the bundle spanned by the eigenvectors associated to the contracting (resp. expanding) eigenvalues, and E^c is the eigenspace associated to the eigenvalue 1 (in our context, all these spaces have dimension 1). By the theory of invariant manifolds, see [HPS], there exist the *strong stable* and *unstable* manifolds of the saddle-node, defined as the unique f-invariant manifolds tangent at S to E^{ss} and to E^{uu} and denoted by $W^{ss}(S,f)$ and $W^{uu}(S,f)$, respectively.

Motivated by the fact that (generic) saddle-nodes (of saddle type) simultaneously behave as points of index two and one (the stable and unstable manifolds of the saddle-node have both dimension 2), we introduce *saddle-node heterodimensional cycles.* A diffeomorphism f has a saddle-node heterodimensional cycle associated to a saddle-node S and the saddle P of Morse index one if the (two-dimensional) unstable manifold of S and stable manifold of P have nonempty transverse intersection and the (one-dimensional) invariant manifolds $W^{ss}(S,f)$ and $W^u(P,f)$ have a quasi-transverse intersection along the orbit of some point. A saddle-node heterodimensional cycle is depicted in Figure 4 in Section 4. One similarly defines saddle-node heterodimensional cycles associated to a (saddle-type) saddle-node S and a saddle Q of Morse index two.

Roughly speaking, in our construction we consider a diffeomorphism f simultaneously having two saddle-node heterodimensional cycles. We consider a two parameter family $(f_{t,s})_{t,s\in[-1,1]}$ of diffeomorphisms such that $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, one associated to a saddle-node S and a saddle P of Morse index one and other one associated to a saddle Q of index one and the saddle-node S. The parameter t describes the unfolding of the cycles (relative motion between compact parts of $W^u(P,f_{t,0})$ and $W^{ss}(S,f_{t,0})$ and of $W^s(Q,f_{t,0})$ and $W^{uu}(S,f_{t,0})$). The parameter s describes the unfolding of the saddle-node: for positive s there are two saddles S_s^+ and S_s^- of indices 2 and 1, colliding at $s=0$ to the saddle-node S and disappearing for negative s. We see that, fixed any small $\bar t>0$, for $s>0$ (before the collapse of the saddles), $H(P,f_{\bar t,s})=H(S_s^+,f_{\bar t,s})$ and $H(Q,f_{\bar t,s})=H(S_s^-,f_{\bar t,s})$ for all small positive s. Moreover, $H(P,f_{\bar t,s})\cap H(Q,f_{\bar t,s})=\emptyset$. At the saddle-node bifurcation we have $H(P,f_{\bar t,0})\cap H(Q,f_{\bar t,0})=\{S\}$. Finally, for $s<0$, after the disappearing of the saddles, $H(P,f_{\bar t,s})=H(Q,f_{\bar t,s})$. See the results in Section 6. Theorem A follows by considering the arc $g_s=f_{\bar t,-s}$. To deduce Theorem B from Theorem A, we consider a filtration having the open set W as a level and analyze the orbits of recurrent points of Λ_s.

In forthcoming papers, we will illustrate how this type of bifurcation naturally appear as secondary bifurcations in the unfolding of heterodimensional cycles and give a model for the collision, explosion, and collapse of (nontrivial) hyperbolic homoclinic classes, see [DR_6].

Let us say a few words about our constructions. As mentioned, our setting necessarily corresponds to a non-generic situation, so we focus our attention on an example (we have not done any effort for generality). We begin by presenting (in Section 1) a model for the unfolding of a heterodimensional cycle. This model (motivated by [D_1] and [BD_1]) allows us to give a rather transparent explanation of the dynamics in the unfolding of a heterodimensional cycle by reducing it to the study of the dynamics of an iterated system of functions defined on an interval, this is done in Section 2. Recall that the dynamics of a (linear) Smale horseshoe is given by two affine expanding maps of the interval (say $I = [0,1]$) whose domains of definition are two disjoint closed subintervals of I (say $[0,1/3]$ and $[1/3,1]$). The interval $(1/3,2/3)$ is the main gap of the horseshoe and corresponds to points in the basin of attraction of a sink. The affine model for heterodimensional cycles is a system of iterated functions with infinitely many maps F_i defined on subintervals I_i of I which are non-disjoint (the interior of the intervals I_i are pairwise disjoint, but I_i and I_{i+1} have a common extreme). Thus in this model there are no gaps and there are no escaping points.

In Section 3, we prove that, after unfolding the cycle, the dynamics of the model family is non-hyperbolic: the point of index 1 in the cycle belongs to the homoclinic class of the point of index 2 in the cycle. Here, using the one-dimensional reduction, we give a shorter and clearer proof of the results in [D$_1$]. Since our constructions rely heavily on this proof and there is not any written version of this approach, we have decided to include a short description of it.

In Section 4, we introduce the *lateral homoclinic classes of a saddle-node* S of a diffeomorphism f as above, $H^+(S,f)$ and $H^-(S,f)$, respectively defined as the closure of the transverse intersections $W^u(S,f) \pitchfork W^{ss}(S,f)$ and $W^s(S,f) \pitchfork W^{uu}(S,f)$. These lateral homoclinic classes essentially behave as the usual ones. We see that for arcs f_t unfolding at $t = 0$ the saddle-node heterodimensional cycle (associated to a saddle P of index one and the saddle-node S) one has $H(P,f_t) \subset H^+(S,f_t)$ for all small positive t. Moreover, under mild conditions, one also gets $H^+(S,f_t) = H(P,f_t)$ for all small $t > 0$. The inclusion $H(P,f_t) \subset H^+(S,f_t)$ follows adapting (in a rather straightforward way) the results for the model family in Section 3. For the inclusion $H^+(S,f_t) \subset H(P,f_t)$ we need new ingredients that we borrow from [D$_2$].

Using the results in Sections 3 and 4, we get a complete description of the homoclinic classes $H(P,f_{\bar{t},s})$ and $H(Q,f_{\bar{t},s})$ before the collapse of the saddles S_s^+ and S_s^- to the saddle-node. Finally, to study $H(P,f_{\bar{t},s})$ and $H(Q,f_{\bar{t},s})$ after the collision, we introduce new systems of iterated functions and analyze their dynamics.

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1 Heterodimensional cycles: a model family

In this section, we construct a model one-parameter family $(f_t)_{t\in[-1,1]}$ of diffeomorphisms unfolding a heterodimensional cycle. The study of the semi-local dynamics of f_t will be reduced to the analysis of a one-parameter family of endomorphisms with infinitely many discontinuities which describe the dynamics of f_t in the *central direction,* see Section 2.

Consider a diffeomorphism f with a heterodimensional cycle having the following dynamical configuration. In local coordinates in $\mathbb{R}^3$, the cycle is associated to saddle fixed points $Q = (0,0,0)$ and $P = (0,1,0)$ of indices 2 and 1, respectively, verifying the following conditions:

Partially hyperbolic (semi-local) dynamics of the cycle:

- In the cube $[-1,1] \times [-1,2] \times [-1,1]$ the diffeomorphism has the form

$$f(x,y,z) = (\lambda_s x, F(y), \lambda_u z),$$

where $F\colon [-1,2] \to (-1,2)$ is an increasing map with exactly two fixed points, a source at 0 and a sink at 1, and $0 < \lambda_s < d_m < 1 < d_M < \lambda_u$, where $0 < d_m < F'(x) < d_M$ for all $x \in [0,1]$.

- There is $\delta > 0$ such that F is linear in $[-\delta, \delta]$ and affine in $[1-\delta, 1+\delta]$. We denote by $\beta > 1$ and $0 < \lambda < 1$, the derivative of F at 0 and 1, respectively.

Observe that $[-1,1]\times\{(0,0)\} \subset W^s(Q)$, $\{0\}\times[0,1)\times[-1,1] \subset W^u(Q)$, $\{(0,0)\}\times[-1,1] \subset W^u(P)$, and $[-1,1] \times (0,1] \times \{0\} \subset W^s(Q)$. Thus the curve $\gamma = \{0\} \times (0,1) \times \{0\}$ (called *connexion*) is a normally hyperbolic curve contained in $W^u(Q) \pitchfork W^s(P)$.

Existence and unfolding of the cycle:

- **The cycle:** There exist $k_0 \in \mathbb{N}$ and a small neighborhood U of $(0,1,-1/2) \in W^u(P)$ such that the restriction of f^{k_0} to U is a translation,
$$f^{k_0}(x,y,z) = (x-1/2, y-1, z+1/2).$$
Thus $f^{k_0}(0,1,-1/2) = (-1/2,0,0) \in W^u(Q)$ and $W^s(Q)$ and $W^u(P)$ meet throughout the orbit of the heteroclinic point $(-1/2,0,0)$. By construction, $(-1/2,0,0)$ is a quasi-transverse heteroclinic point.

- **The unfolding of the cycle:** Consider the family $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ of diffeomorphisms coinciding with f in $[-1,1]\times[-1,2]\times[-1,1]$ and such that the restriction of $f_t^{k_0}$ to U is of the form
$$f_t^{k_0}(x,y,z) = (x-1/2, y-1+t, z+1/2) = f^{k_0}(x,y,z) + (0,t,0).$$
So, for $t > 0$, $\{(-1/2,t)\}\times[-1,1] \subset W^u(P, f_t)$ and $x_t = (-1/2,t,0)$ is a transverse homoclinic point of P (for f_t). Similarly, $y_t = (-1/2,0,0)$ is a transverse homoclinic point of Q.

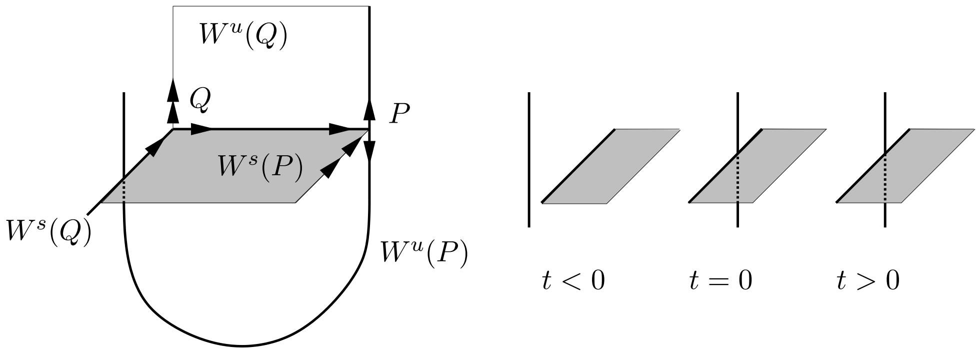

Figure 1: The model cycle and its unfolding

Consider a small *neighborhood of the heterodimensional cycle associated with* f_0, that is, an open set W containing the connexion $\gamma = \{0\}\times[0,1]\times\{0\}$ and the f_0-orbit of the heteroclinic point $(-1/2,0,0)$. For small t, let Λ_t be the *maximal* f_t*-invariant set in* W, $\Lambda_t = \cap_{n\in\mathbb{Z}} f_t^n(W)$. Consider also the forward and backward f_t-invariant sets in W, $\Lambda_t^+ = \cap_{n\geq 0} f_t^{-n}(W)$ and $\Lambda_t^- = \cap_{n\geq 0} f_t^n(W)$.

Fix a small positive ρ and consider the fundamental domains of F given by $D^+ = [\beta^{-1}\rho, \rho]$ and $D^- = [1-\rho, \lambda(1-\rho)]$ contained in the neighborhoods of 0 and 1 where F is affine. We can choose ρ such that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$ (for notational simplicity let us put $N = 1$). The map F^N is the *transition from* 0 *to* 1 [2]. Assume that that $F'(1) = \lambda$ and $F'(0) = \beta$ verify:

(T1) $F'(x) \geq \dfrac{1}{2}\,\dfrac{1-\lambda}{1-\beta^{-1}}$, for all $x \in D^+$,

(T2) $(1-\lambda) < \beta^{-1}$, and

(T3) $\dfrac{(1-\lambda)\,\lambda}{2\,(1-\beta^{-1})\,\beta} = \ell > 1.$

To get condition (T1) it is enough to consider F with small distortion. For conditions (T2) and (T3) it is enough to take β close enough to 1^+. In fact, later we will consider the case where Q is a saddle-node (saddle-node heterodimensional cycles) and $\beta = 1$, see Section 4. Our first result is:

Theorem 1.1. *For every $t > 0$ sufficiently small, $H(P, f_t) \subset H(Q, f_t)$ and $\Lambda_t \subset H(Q, f_t)$.*

This theorem was stated in [D$_1$]. Here we give a more conceptual prove of it, which enables us to introduce some technical tools to be used systematically later on. First, in Section 2 we will introduce the system of iterated functions associated to the cycle (this approach is motivated by [DR$_5$]). In Section 3, we deduce the theorem from the results in Section 2.

2 Expanding one-dimensional dynamics associated to the cycle

For each small $t > 0$, consider the *scaled fundamental domains* $D_t^\pm$ defined as follows: let $D_t^- = [1-t, \lambda(1-t)]$ and define k_t as the smaller $k \in \mathbb{N}$ with $F^{-k}(D_t^-) \subset [0, t]$. We define

$$D_t^+ = [a_t, b_t] = [\beta^{-1}(b_t), b_t] = F^{-k_t}(D_t^-), \quad \text{where} \quad \beta^{-2}t < a_t < b_t \leq t.$$

We next define an expanding map R_t, $R_t\colon D_t^+ \to D_t^+$, with discontinuities describing the central dynamics of the return map of f_t defined on $[-1,1] \times D_t^+ \times [-1,1]$. First, for each small $t > 0$ define the *transition map T_t from D_t^+ to D_t^-* by

$$T_t\colon D_t^+ \to D_t^-, \quad x \mapsto T_t(x) = F^{k_t}(x).$$

Lemma 2.1. *The map T_t verifies $T_t'(x) > \ell > 1$ for all $x \in D_t^+$, where ℓ is as in condition* **(T3)***.*

[2]This transition plays a key role for determining the dynamics after unfolding the cycle and it is determined by the *Mather invariant* of F, [Ma]: in a neighborhood of 0, the map F is the time-one of the vector field $X(y) = (\log \beta)\, y\, \frac{\partial}{\partial y}$, whose flow is $y \mapsto \beta^t\, y$. Similarly, in a neighborhood of 1, F is the time-one of $Y(y) = (\log \lambda)\,(y-1)\,\frac{\partial}{\partial y}$. Consider, for y close to 0^+, an large n such that $F^n(x)$ is close to 1^- and write $DF^n(y)(X(y)) = \mu(y)\, Y(F^n(y))$. Using the local F-invariance of X and Y (near 0 and 1), one has that $\mu(x)$ does not depend on n and that $\mu(x) = \mu(F(x))$. The function μ is the Mather invariant of F which describes its distortion. For instance, if μ is identically 1, then F is exactly the exponential of a vector field.

Proof: Given $x \in D_t^+$ let $k_t = n_t(x) + 1 + m_t(x)$, where $F^{n_t(x)}(x) \in D^+$ and $F^i(x) \notin D^+$ for all $1 \le i < n_t(x)$. We claim that

$$\frac{1}{\beta t} \le \beta^{n_t(x)} \le \frac{\beta^2}{t} \quad \text{and} \quad \lambda t \le \lambda^{m_t(x)} \le \frac{t}{\lambda}. \tag{1}$$

For first inequalities just note $x \in D_t^+ \subset (\beta^{-2} t, t]$ and $\beta^{n_t(x)} x \in [\beta^{-1}, 1]$. The other ones follow analogously. As $T_t(x) = \lambda^{m_t(x)} F(\beta^{n_t(x)} x)$, hypotheses (T1) and (T3) and the estimates in (1) give

$$|T_t'(x)| = \lambda^{m_t(x)} |F'(x)| \beta^{n_t(x)} \ge (\lambda t) \left(\frac{1}{2} \frac{1-\lambda}{1-\beta^{-1}}\right) \left(\frac{1}{\beta t}\right) = \frac{1}{2} \frac{\lambda (1-\lambda)}{\beta (1-\beta^{-1})} = \ell > 1,$$

as we claimed. $\square$

Since $T_t(x) \in D_t^- = [1-t, 1-\lambda t]$ for all $x \in D_t^+$, we can define the map G_t by

$$G_t \colon D_t^+ \to [0, t], \quad x \mapsto G_t(x) = T_t(x) + (t-1).$$

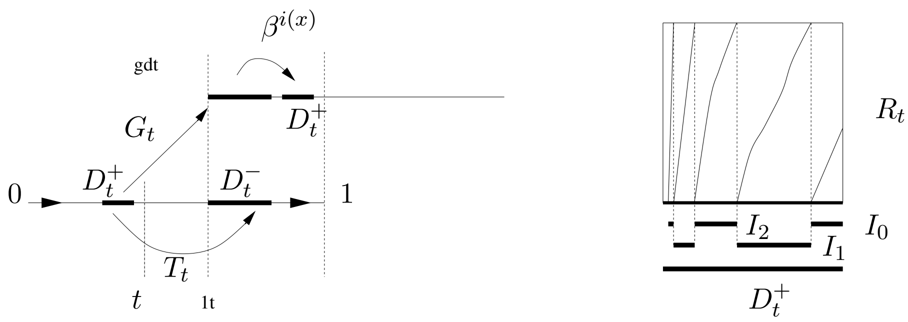

Figure 2: The expanding maps T_t, G_t and R_t

Remark 2.2. *The map G_t is monotone increasing and $G_t(D_t^+) = [0, t(1-\lambda)]$.*

Claim 2.3. *Let $(a_t, b_t] = \tilde{D}_t^+ \subset D_t^+$. Given $x \in \tilde{D}_t^+$ let $i(x) \in \mathbb{Z}$ be the minimum i with $\beta^i(G_t(x)) \in D_t^+$. Then there is $i_0 > 0$ (maximum with such property) such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: Recall first that $b_t \in (\beta^{-1} t, t]$. On the other hand, from $(1-\lambda) < \beta^{-1}$ (condition (T2)) and Remark 2.2, $G_t(\tilde{D}_t^+) = (0, t(1-\lambda)] \subset (0, \beta^{-1} t)$. Thus the right extreme of $G_t(\tilde{D}_t^+)$ is less than the left extreme of D_t^+, hence $i(x) > 0$ for all $x \in \tilde{D}_t^+$, ending the proof of the claim. $\square$

Finally, the return map R_t is defined by

$$R_t \colon \tilde{D}_t^+ \to D_t^+, \quad R_t(x) = \beta^{i(x)}(G_t(x)) = \beta^{i(x)} (T_t(x) + (t-1)).$$

Next we study the dynamics of R_t: the map R_t is uniformly expanding and has (infinitely many) discontinuities where the lateral derivatives are well defined. These discontinuities will play a key role in our constructions. The definition of $i_0 \in \mathbb{N}$ in Claim 2.3 implies that $\beta^{-i_0}(a_t) \in G_t(D_t^+)$. For each $i \ge i_0$ define $d_i \in \tilde{D}_t^+$ by $G_t(d_i) = \beta^{-i}(a_t)$. By construction, the sequence $(d_i)_{i \ge i_0}$ corresponds to the discontinuities of R_t and verifies the following:

- $d_{i+1} < d_i$ and $d_i \to a_t$,
- Let $[d_{i+1}, d_i] = I_i$, $i > i_0$, and $I_{i_0} = [d_{i_0}, b_t]$. The map R_t is continuous and strictly increasing in the interior of each interval I_i. We continuously extend R_t to the whole I_i, obtaining a bi-valuated return map R_t with $R_t(d_i) = \{a_t, F(a_t) = b_t\}$ for all $i > i_0$. In particular, the restriction of R_t to any I_i, $i > i_0$, is onto. We let $R_t(b_t) = c_t \leq b_t$.

The main properties of R_t are summarized in the next lemma.

Lemma 2.4. *The restriction of R_t to each interval $I_i, i > i_0$, is onto and $R'_t(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = d_i$ this means that the lateral derivatives of R_t at x are greater than ℓ). Moreover, $0 \in G_t(R_t(d_i))$ for all $i \geq i_0$.*

The expansiveness of R_t follows from Lemma 2.1 and Claim 2.3. Condition $0 \in G_t(R_t(d_i))$ follows from $a_t \in R_t(d_i)$ and $0 \in G_t(a_t)$.

Lemma 2.5. *Consider small $t > 0$ and an open subinterval J of $\tilde{D}_t^+$. Then there is $k \in \mathbb{N} \cup \{0\}$ such that $R_t^k(J)$ contains a discontinuity of R_t. In particular, there is $x \in J$ such that $G_t(R_t^k(x)) = 0$.*

Proof: If the interval J contains a discontinuity we are done. Otherwise, let $i > 0$ be such that the intervals J, $R_t(J), \ldots, R_t^i(J)$ do not contain discontinuities. Thus, for each $k \in \{0, \ldots, i\}$, there is $i_k \geq i_0$ such that $R_t^k(J) \subset I_{i_k}$. Lemma 2.4 implies that $|R_t^k(J)| \geq \ell^k |J|$, $\ell > 1$, for all $k \in \{0, \ldots, i\}$. Since the size of the intervals I_i is upper bounded, this inequality implies that there is a first $m \in \mathbb{N}$ such that $R_t^m(J)$ is not contained in any I_i, thus it intersects the set of discontinuities of R_t. $\square$

3 The maximal invariant set: Proof of Theorem 1.1

Next proposition is the main technical result of this section. Heuristically, it means that the one-dimensional stable manifold of Q topologically behaves as a two-dimensional manifold.

Proposition 3.1. *For every small $t > 0$ and every two-disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^s(Q, f_t) \pitchfork \chi \neq \emptyset$. In particular, $W^s(P, f_t)$ is contained in the closure of $W^s(Q, f_t)$.*

Proof of the inclusion $H(P, f_t) \subset H(Q, f_t)$ in Theorem 1.1. By the definition of $H(P, f_t)$, it suffices to see that any $x \in W^s(P, f_t) \pitchfork W^u(P, f_t)$ is accumulated by homoclinic points of Q. By the configuration of the cycle, $W^u(P, f_t) \subset \text{closure}\,(W^u(Q, f_t))$, thus, given any $x \in H(P, f_t)$ and any $n > 0$, there is a disk Δ_n, contained in $W^u(Q, f_t)$ and in the ball of radius $1/n$ centered at x, whose interior meets transversely $W^s(P, f_t)$. By Proposition 3.1, $\Delta_n \pitchfork W^s(Q, f_t) \neq \emptyset$. Thus there is $y_n \in \Delta_n \cap H(Q, f_t)$. By construction, $y_n \to x$, proving the inclusion $H(P, f_t) \subset H(Q, f_t)$.

3.1 Proof of Proposition 3.1

We now go into the details of the proof of Proposition 3.1. We first introduce some definitions.

- A set $\Delta \subset [-1, 1] \times [-1, 2] \times [-1, 1]$ is a *vertical strip* if $\Delta = \{x_1\} \times [l_1, l_2] \times [r_1, r_2]$, where $l_1 < l_2$ and $r_1 < 0 < r_2$. The segment $\{x_1\} \times [l_1, l_2] \times \{0\}$ is the *basis* of Δ. The *width* and the *height* of Δ are $w(\Delta) = (l_2 - l_1)$ and $h(\Delta) = (r_2 - r_1)$. The strip Δ is *complete* if $r_2 = 1$ and $r_1 = -1$, *well located* if $[l_1, l_2]$ is contained in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.

- A subset $J \subset [-1,1]\times[-1,2]\times[-1,1]$ is a *vertical segment* if $J = \{x_1\}\times\{l_1\}\times[r_1,r_2]$, where $r_1 < 0 < r_2$. The point $(x,\ell_1,0)$ is the basis of J. The height of J is $h(J) = (r_2 - r_1)$. As above, the segment J is *complete* if $r_1 = -1$ and $r_2 = 1$, *well located* if l_1 is in the interior of D_t^+, and *perfect* if it simultaneously is complete and well located.

- A vertical segment J (resp. strip Δ) is *at the right of* Q if $l_1 \in (0,1]$.

Given an interval $\alpha \subset [-1,2]$, let $\Delta(\{x\}\times\alpha\times\{0\}) = \{x\}\times\alpha\times[-1,1]$ be the *unique* complete vertical strip with basis $\{x\}\times\alpha\times\{0\}$. Similarly, $J(x,y,0)$ is the unique complete vertical segment with basis $(x,y,0)$. The next algorithm associates to perfect segments and strips their successors:

Algorithm 3.2. *Let $\Delta = \Delta(\{x\}\times\alpha\times\{0\})$ be a perfect strip and define $\mathcal{G}_t(\Delta)$ as the perfect strip such that:*

- *the basis of $\mathcal{G}_t(\Delta)$ is of the form $(\{x'\}\times G_t(\alpha)\times\{0\})$, where $x' = \lambda_s^{k_t}\, x - 1/2$,*
- *$\mathcal{G}_t(\Delta)$ is contained in $f_t^{k_t}(\Delta)$ (where $F^{k_t}(D_t^+) = D_t^-$).*

Suppose now that α does not contain discontinuities, i.e. $\alpha \subset (d_{i+1}, d_i)$ for some i. Define $\mathcal{R}_t(\Delta)$ as the perfect strip such that:

- *the basis of $\mathcal{R}_t(\Delta)$ is of the form $(\{\hat{x}\}\times R_t(\alpha)\times\{0\})$, where $\hat{x} = \lambda_s^i(\lambda_s^{k_t}\, x - 1/2)$,*
- *$\mathcal{R}_t(\Delta)$ is contained in $f_t^{k_t+i}(\Delta)$.*

Similarly, to a perfect segment $J = J(x,y,0)$ we associate perfect segments $\mathcal{G}_t(J)$ and $\mathcal{R}_t(J)$ (provided $y \neq d_i$ for all i).

The strips $\mathcal{G}_t(\Delta)$ and $\mathcal{R}_t(\Delta)$ in Algorithm 3.2 are obtained as follows. Given a set A and a point $x \in A$, let $C(x,A)$ be the connected component of A containing x. Take a small neighborhood V of $f_0^{k_0}(\{0,1\}\times[-1,1]) \subset W^u(P,f_0)$, k_0 as in the definition of the cycle in Section 1, then

$$\begin{aligned} \mathcal{G}_t(\Delta) &= C(f_t^{k_t}(x,y,0), f_t^{k_t}(\Delta)\cap V)\cap[-1,1]^3), \\ \mathcal{R}_t(\Delta) &= C(f_t^{i}(x',y',0), f_t^{k_t+i}(\Delta)\cap V)\cap[-1,1]^3), \end{aligned}$$

where $(x,y,0)$ is any point in the basis of Δ, $x' = (\lambda_s^{k_t}\, x - 1/2)$, and $y' \in G_t(\alpha)$. The construction for the successors of segments is analogous.

Lemma 3.3. *The manifold $W^u(P,f_t)$ contains a perfect segment for all small $t > 0$.*

Proof: Consider the transverse homoclinic point $x_t = (-1/2, t, 0)$ of P. Recall that $t \geq b_t$ and $\beta^{-1}\, t \in D_t^+ = [a_t, b_t]$. Let us assume that $t > b_t$, and thus $\beta^{-1} t \in (a_t, b_t)$ (the case $t = b_t$ follows similarly, so it will be omitted). Consider the complete vertical segment $\mathcal{R}_t(J)$, where $J = J(-(\lambda_s^{-1}/2), \beta^{-1}t, 0) \subset W^u(P,f_t)$. If $R_t(\beta^{-1}t)$ belongs to the interior of D_t^+, then $\mathcal{R}_t(J) \subset W^u(P,f_t)$ is the announced segment. Otherwise, $R_t(\beta^{-1}t) = b_t$ and there is a homoclinic point of P of the form $(x', b_t, 0)$. Using the λ-lemma and the product structure of the cycle, one gets homoclinic points $(x_n, y_n, 0)$ of P and complete segments $J_n = J(x_n, y_n, 0) \subset W^u(P,f_t)$ such that $x_n \to x'$, $y_n \to b_t$, and y_n is increasing. Thus y_n is in the interior of D_t^+ for every big n and $J_n \subset W^u(P,f_t)$ is perfect. $\square$

For clearness we first prove Proposition 3.1 in the following special case:

Proposition 3.4. *Let $\chi \subset [-1,1] \times [-1,2] \times [-1,1] = \{x\} \times A$, where A is a disk of $\mathbb{R}^2$ whose interior contains a point of the form $(y,0)$ with $y \in (0,2)$. Then χ intersects transversely $W^s(Q,f_t)$.*

We claim that is enough to prove the Proposition 3.4 for perfect strips:

Lemma 3.5. *Let Δ be a perfect strip. Then there is $k \in \mathbb{N}$ such that $f_t^k(\Delta) \pitchfork W^s(Q,f_t) \neq \emptyset$.*

Proof: Suppose that $\Delta = \Delta(\{x_0\} \times \alpha \times \{0\})$, α in the interior of D_t^+. By Lemma 2.5, there exist y_0 in the interior of α and $k \in \mathbb{N}$ such that $G_t(R_t^k(y_0)) = 0$. Thus the vertical strip $\mathcal{G}_t(\mathcal{R}_t^k(\Delta))$ (contained in the forward orbit of Δ) intersects transversely $[-1,1] \times \{(0,0)\} \subset W^s(Q,f_t)$. □

Proof of Proposition 3.4: By Lemma 3.3, $W^u(P,f_t)$ contains a perfect vertical segment J. Since, by definition, χ meets transversely $W^s(P,f_t)$, the λ-lemma implies that forward orbit of χ contains a sequence of complete strips χ_n accumulating to J. Thus χ_n contains a perfect strip for all n large. Lemma 3.5 implies that χ_n (and thus χ) transversely meets $W^s(Q,f_t)$. □

Proof of Proposition 3.1: We can assume that χ is transverse to $W^s_{loc}(P,f_t)$ and contained in $[-1,1] \times [-1,2] \times [-1,1]$. If χ contains a subset of the form $\{x\} \times A$, where A is an open subset of $\mathbb{R}^2$ containing a point $(0,y)$ with $y \in (0,2)$, Proposition 3.4 implies the result. For the general case, consider a point $(x_0,y_0,0)$, $y_0 \in (0,2)$, in the interior of $\chi \pitchfork W^s_{loc}(P,f_t)$ and for every big n the vertical strip

$$\Sigma_n = \{x_0\} \times [y_0 - 1/n, y_0 + 1/n] \times [-1/n, 1/n].$$

The strips Σ_n verify the hypotheses of Proposition 3.4, hence there is $(x_0,y_n,z_n) \in \Sigma_n \pitchfork W^s(Q,f_t)$ such that $H_n = [-1,1] \times \{(y_n,z_n)\} \subset W^s(Q,f_t)$. Since $(x_0,y_n,z_n) \to (x_0,y_0,0)$, it is immediate that H_n meets transversely χ for all large n, ending the proof of the proposition. □

3.2 The maximal f_t-invariant set

To prove the second part of Theorem 1.1 (the inclusion $\Lambda_t \subset H(Q,f_t)$), let V_0 be the connected component of the neighborhood of the cycle W containing the heteroclinic point $(-1/2,0,0)$. There are two types of points of Λ_t: (a) those points whose orbit does not meet V_0 (i.e., the set $\gamma = \{0\} \times [0,1] \times \{0\} \subset W^s(P,f_t) \cap W^u(Q,f_t)$) and (b) those having an iterate in V_0.

We claim that every point of type (a) belongs to $H(Q,f_t)$: given any $(0,x,0)$, $x \in (0,1)$, consider the disk $\Delta_n = \{0\} \times [x-1/n, x+1/n] \times [-1/n,+1/n] \subset W^u(Q,f_t)$ satisfying the hypothesis of Proposition 3.1. Hence $\Delta_n \pitchfork W(Q,f_t) \neq \emptyset$ and thus $\Delta_n \cap H(Q,f_t) \neq \emptyset$. Since this holds for all $n \in \mathbb{N}$, $(0,x,0) \in H(Q,f_t)$.

For points $w \in \Lambda_t$ of type (b), after replacing w by some iterate of it, we can assume that $w \in V_0$. Consider the sequence $(n_i(w))_{i \in \mathcal{I}_t(w)}$ associated to w, where $\mathcal{I}_t(w) \subset \mathbb{Z}$ is an interval in $\mathbb{Z}$, inductively defined as follows: let $n_0(w) = 0$ and, assuming defined $n_j(w)$, $j \geq 0$, we define $n_{j+1}(w)$ as the first integer $k > (n_j(w)+1)$ such that $f_t^k(w) \in V_0$. If the forward orbit of w does not return to V_0 for every $k > (n_j(w)+1)$ then j is the right extreme of $\mathcal{I}_t(w)$. We argue analogously for negative j: assuming defined $n_j(x)$, $j \leq 0$, $n_{j-1}(w)$ is the first negative integer $k < (n_j(w)-1)$ with $f_t^k(w) \in V_0$. If the backward orbit of w does not return to V_0 for every $k < (n_j(w)-1)$, then j is the left extreme of $\mathcal{I}_t(w)$.

Consider the subset $\mathcal{I}_t^+(\infty)$ (resp. $\mathcal{I}_t^-(\infty)$) of $\Lambda_t \cap V_0$ of points w such that $\mathcal{I}_t(w)$ is not upper (resp. lower) bounded. Let $\mathcal{I}_t^+(b)$ be the subset of $\Lambda_t \cap V_0$ of points w such that $\mathcal{I}_t(w)$ is upper bounded. The set $\mathcal{I}_t^-(b)$ is defined similarly. Let $\mathcal{I}_t^\pm(\infty) = \mathcal{I}_t^+(\infty) \cap \mathcal{I}_t^-(\infty)$ and $\mathcal{I}_t^\pm(b) = \mathcal{I}_t^+(b) \cap \mathcal{I}_t^-(b)$. We borrow from [DR$_2$, Lemma 4.1] the following lemma, whose proof is straightforward:

Lemma 3.6. *For every $t > 0$, $\mathcal{I}_t^+(b) \subset W^s(P, f_t) \cup W^s(Q, f_t)$ and $\mathcal{I}_t^-(b) \subset W^u(P, f_t) \cup W^u(Q, f_t)$.*

Next result immediately follows by observing that f_t (resp. f_t^{-1}) exponentially expands the vertical (resp. horizontal) segments:

Remark 3.7. *Let $w = (x, y, z) \in \mathcal{I}_t^+(\infty)$ (resp. $w \in \mathcal{I}_t^-(\infty)$). Then $\{(x, y)\} \times [z - \varepsilon, z + \varepsilon] \pitchfork W^s(P, f_t) \neq \emptyset$ (resp. $[x - \varepsilon, x + \varepsilon] \times \{(y, z)\} \pitchfork W^u(Q, f_t) \neq \emptyset$) for every $\varepsilon > 0$.*

To prove the inclusion $\Lambda_t \subset H(Q, f_t)$ in Theorem 1.1 we consider the following four cases.

Case (i): $w = (x, y, z) \in \mathcal{I}_t^-(b) \setminus \mathcal{I}_t^+(b) = \mathcal{I}_t^-(b) \cap \mathcal{I}_t^+(\infty)$**.**

By Remark 3.7, there is a sequence $w_n = (x, y, z_n) \in W^s(P, f_t)$ with $w_n \to w$. We claim that $w_n \in H(Q, f_t)$ for all large n. Thus $w \in H(Q, f_t)$. To prove the claim, note that the distances between the backward iterates of w_n and w exponentially decrease, thus $w_n \in \Lambda_t$. Moreover, since $w \in \mathcal{I}_t^-(b)$ we also have that $w_n \in \mathcal{I}_t^-(b)$. By Lemma 3.6, $w, w_n \in W^u(P, f_t) \cup W^u(Q, f_t)$. If $w_n \in W^u(P, f_t)$, then $w_n \in H(P, f_t) \subset H(Q, f_t)$ (recall the first part of Theorem 1.1 proved above) and we are done. Otherwise, $w_n \in W^u(Q, f_t)$ and for each k large, there is a small vertical strip $\Delta_k = \Delta_k(n)$ of diameter less than $1/k$, whose interior is contained in $W^u(Q, f_t)$ and contains w_n. Since $w_n \in W^s(P, f_t)$, $\Delta_k \pitchfork W^s(P, f_t)$. Thus, by Proposition 3.1, $W^s(Q, f_t)$ intersects transversely the interior of Δ_k. Hence, since the interior of Δ_k is contained in $W^u(Q, f_t)$, Δ_k contains a homoclinic point y_k of Q. From $\mathrm{diam}(\Delta_k) \to 0$, we get $y_k \to w_n$, which implies $w_n \in H(Q, f_t)$.

Case (ii): $w = (x, y, z) \notin \mathcal{I}_t^+(b) \cup \mathcal{I}_t^-(b)$**.**

We claim that w is accumulated by points $w_n \in \mathcal{I}_t^+(\infty) \cap \mathcal{I}_t^-(b)$, and the result follows from Case (i). To prove the claim observe that, by Remark 3.7, there is a sequence $w_n = (x_n, y, z) \in W^u(Q, f_t)$ with $w_n \to w$. Since the distances between the forward iterates of w_n and w exponentially decrease, it is immediate to check that $w_n \in \Lambda_t$. This also implies that $w_n \in \mathcal{I}_t^+(\infty)$. Finally, $w_n \in W^u(Q, f_t)$ implies $w_n \in \mathcal{I}_t^-(b)$, ending the proof of the claim.

Case (iii): $w = (x, y, z) \in \mathcal{I}_t^+(b) \setminus \mathcal{I}_t^-(b) = \mathcal{I}_t^+(b) \cap \mathcal{I}_t^-(\infty)$**.**

By Lemma 3.6, $w \in W^s(P, f_t) \cup W^s(Q, f_t)$ and, by replacing w by a forward iterate, we can assume that $w = (x, y, 0)$, $y \geq 0$. Remark 3.7 gives a sequence $w_n = (x_n, y, 0) \in W^u(Q, f_t)$ with $w_n \to w$. For each n, there is a vertical disk $\Delta_n \subset W^u(Q, f_t)$ centered at w_n, of diameter less than $1/n$. Clearly, Δ_n intersects transversely $W^s(P, f_t)$. Thus, by Proposition 3.1, $\Delta_n \pitchfork W^s(Q, f_t) \neq \emptyset$. As in the previous cases, this implies that $\Delta_n \cap H(Q, f_t) \neq \emptyset$ for all n large, thus $w \in H(P, f_t)$.

Case (iv): $w \in \mathcal{I}_t^{\pm}(b)$**.**

By Lemma 3.6, there are four possibilities: (1) $w \in W^s(Q, f_t) \cap W^u(Q, f_t)$, (2) $w \in W^s(P, f_t) \cap W^u(P, f_t)$, (3) $w \in W^s(P, f_t) \cap W^u(Q, f_t)$, and (4) $w \in W^u(P, f_t) \cap W^s(Q, f_t)$. Recall that the intersections above are transverse or quasi-transverse, depending on the case. Hence, in case (1), $w \in H(Q, f_t)$ and, in case (2), $w \in H(P, f_t) \subset H(Q, f_t)$. In case (3), the same proof of $\{0\} \times [0, 1] \times \{0\} \subset H(Q, f_t)$ implies that $w \in H(Q, f_t)$: just observe that for every disk $\Delta \subset W^u(Q, f_t)$ containing w, $W^s(P, f_t) \pitchfork \Delta \neq \emptyset$, thus $\Delta \cap H(Q, f_t) \neq \emptyset$. It still remains the case $w \in W^s(Q, f_t) \cap W^u(P, f_t)$. By replacing w by a forward iterate, we can assume that $w = (x, 0, 0)$, $x \in [-1, 1]$, and the following lemma and Cases (i) and (ii) easily imply Case (4):

Lemma 3.8. *Consider $w \in \Lambda_t$ of the form $w = (x, 0, 0) \in V_0 \cap (W^s(Q, f_t) \cap W^u(P, f_t))$. Then there is a sequence $w_n \to w$ with $w_n \in \mathcal{I}_t^+(\infty)$.*

Proof: For each $n \in \mathbb{N}$, consider the rectangle $R_n(x) = \{x\} \times [0, 1/n] \times [-1/n, 1/n]$.

Claim 3.9. *There exists $\kappa_n \in R_n(x) \cap \Lambda_t^+$ whose forward orbit returns to V_0 infinitely many times.*

Assuming the claim, we now finish the proof of the lemma. As in Remark 3.7, but now considering points in Λ_t^+, we have that the point $\kappa_n = (x_n, y_n, z_n)$ is accumulated by points $\kappa_n^m \in W^u(Q, f_t)$ of the form (x_n^m, y_n, z_n). Since the distances between the forward iterates of κ_n and κ_n^m decrease, the forward orbit of κ_n^m is contained in the neighbourhood W of the cycle and returns infinitely many times to V_0. On the other hand, since $\kappa_n^m \in W^u(Q, f_t)$, its backward orbit also is in W. Thus the whole orbit of κ_n^m is in W, so $\kappa_n^m \in \Lambda_t$ and $\kappa_n^m \in \mathcal{I}_t^+(\infty)$. By Cases (i) and (ii) above, $\kappa_n^m \in H(Q, f_t)$, thus $\kappa_n \in H(Q, f_t)$. Since $\kappa_n \to w$, $w \in H(Q, f_t)$, ending the proof of Case (iv).

To prove Claim 3.9, we need the following fact:

Fact 3.10. *Let $R = R_n(x)$, $1/n < t$. Then there is $i = i(R) \in \mathbb{N}$ such that, for every $j \geq i$, $f_t^j(R)$ contains a rectangle $\Gamma(R, j)$ of the form $\{a\} \times [0, 1/n] \times [-1/n, 1/n]$, $a \in [-1, 0]$.*

Proof: Let N_t be the smaller $i \in \mathbb{N}$ such that $F^i(1/n) \in (1 - t + 1/n, 1)$ and write

$$e = (1 - t + 1/n + g) = F^{N_t}(1/n), \quad g \in (0, t - 1/n).$$

By the definition of the unfolding of the cycle, for each $j \geq 0$, $f_t^{N_t+j}(R_n(x))$ contains the rectangle

$$\{\lambda_s^{N_t+j}(x) - 1/2\} \times [0, t + \lambda^j(g + 1/n - t)] \times [-1, 1] \supset \{\lambda_s^{N_t+j}(x) - 1/2\} \times [0, 1/n] \times [-1, 1],$$

since $t + \lambda^j(g + 1/n - t) \geq t + \lambda^j(1/n - t) \geq 1/n$. This finishes the proof of the fact. $\square$

To prove Claim 3.9, consider $R_n(x) = R(0)$ and, using Fact 3.10, let $R(1) = \Gamma(R(0), i(R(0))$. Write $i_0 = i(R_0)$ and $R^1 = f_t^{-i_0}(R(1)) \subset R(0)$. Assume inductively defined numbers $i_0, i_1, \dots, i_{k-1}$ and rectangles $R(0), R(1), \dots, R(k)$ and $R^1, \dots, R^k$ as follows:

- $R(k) = \Gamma(R(k-1), i(R(k-1)))$ and $i_{k-1} = i(R(k-1))$, in particular, $R(k)$ satisfies the hypotheses of Fact 3.10,
- $R^k \subset R^{k-1} \subset \dots \subset R^1 \subset R(0) = R_n(x)$ and $R^k = f_t^{-i_0-\dots-i_{k-1}}(R(k))$.

We define $i_k = i(R(k))$, $R(k+1) = \Gamma(R(k), i_k)$ and $R^{k+1} = f_t^{-i_0-\dots-i_k}(R(k+1))$, completing the inductive process. Now it suffices to take any point in the non-empty intersection $\cap_{k\in\mathbb{N}} R^k$. $\square$

The proof of Theorem 1.1 is now complete.

4 Saddle-node heterodimensional cycles

In this section, we consider *saddle-node heterodimensional cycles.* For that, in the model heterodimensional cycle in Section 1, we replace the function F (defining the central dynamics) by a one-parameter family of C^2-maps $\Phi_s \colon [-1, 2] \to \mathbb{R}$ such that:

- For every s, the point 1 is an attracting hyperbolic point of Φ_s and Φ_s is affine in a neighborhood of 1 (independent of s). We denote by $0 < \lambda < 1$ the eigenvalue of Φ_s at 1.
- Locally in 0, the map Φ_s is of the form $\Phi_s(x) = x + x^2 - s$. Thus, for $s > 0$, Φ_s has two hyperbolic fixed points $\pm\sqrt{s}$ (an attractor and a repellor) collapsing at $s = 0$. Moreover, for every $s < 0$, Φ_s has no fixed points close to 0.

- Every Φ_s is strictly increasing and has no fixed points different from 1 and $\pm\sqrt{s}$.

In this way, as in Section 1, one has a two-parameter family of diffeomorphisms $f_{t,s}$, the parameters t and s describing the motion of the unstable manifold of P and the unfolding of the saddle-node, respectively. Note that $P = (0,1,0)$ and $S_s^{\pm} = (0, \pm\sqrt{s}, 0)$ $(s \geq 0)$ are fixed points of $f_{t,s}$.

We let $f_t = f_{t,0}$. For the saddle-node $S = (0,0,0)$ of f_t there are defined the stable and unstable manifolds (denoted $W^s(S, f_t)$ and $W^u(S, f_t)$) and the strong stable and unstable manifolds (denoted by $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$). Observe that $W^s(S, f_t)$ and $W^u(S, f_t)$ are two-manifolds with boundary and $W^{ss}(S, f_t)$ and $W^{uu}(S, f_t)$ have both dimension one. Note that, by construction,

$$\{0\} \times [0,1) \times [-1,1] \subset W^u(S, f_t), \quad [-1,1] \times [-1,0] \times \{0\} \subset W^s(S, f_t),$$
$$[-1,1] \times \{(0,0)\} \subset W^{ss}(S, f_t), \quad \{(0,0)\} \times [-1,1] \subset W^{uu}(S, f_t).$$

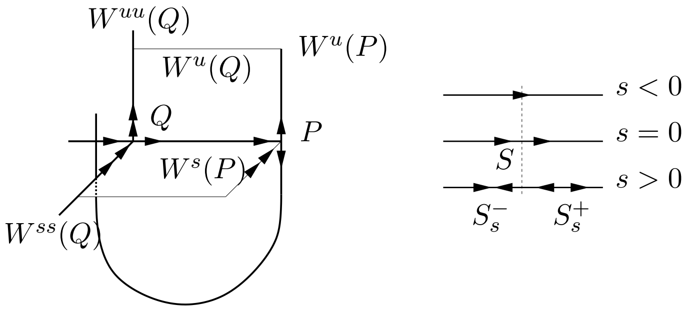

Figure 3: A saddle-node heterodimensional cycle

Keeping in mind these relations, we have that,

- for all t, $W^u(S, f_t)$ meets transversely $W^s(P, f_t)$ throughout the segment $\{0\} \times (0,1) \times \{0\}$,
- for $t = 0$, $W^u(P, f_0)$ meets quasi-transversely $W^{ss}(S, f_0)$ along the orbit of $(-1/2, 0, 0)$,
- for $t > 0$, the point $(-1/2, t, 0)$ is a transverse homoclinic point of P and $(-1/2, 0, 0)$ is a point of transverse intersection between $W^{ss}(S, f_t)$ and $W^u(S, f_t)$.

In this case, we say that the arc $f_t = f_{t,0}$ has a *saddle-node heterodimensional cycle* associated to P and S at $t = 0$. This cycle can be thought as a limit case of the heterodimensional cycles in Section 1, where the derivative of the point of index two Q is 1^+.

The two-fold behavior of the saddle-node S, as a point of index two and one simultaneously, leads us to consider, for small positive t, the *lateral homoclinic classes of* S defined by

$$H^+(S, f_t) = \overline{W^u(S, f_t) \pitchfork W^{ss}(S, f_t)} \quad \text{and} \quad H^-(S, f_t) = \overline{W^s(S, f_t) \pitchfork W^{uu}(S, f_t)}.$$

As in the case of the usual homoclinic classes, we have that:

Proposition 4.1. *For every small $t > 0$, $H^+(S, f_t)$ (resp. $H^-(S, f_t)$) is transitive and the periodic points of index two (resp. one) form a dense subset of it.*

Consider a neighborhood W of the saddle-node heterodimensional cycle defined as in Section 3 and denote by Υ_t the maximal invariant set of f_t in W.

Theorem 4.2. *For every small $t > 0$, one has that $H(P, f_t) \subset H^+(S, f_t)$ and $\Upsilon_t \subset H(S^+, f_t)$.*

The proof of Theorem 4.2 follows as the one of Theorem 1.1, the only difficulty being to redefine appropriately the one-dimensional dynamics associated to the cycle (recall Section 2). This will be briefly done in the next section. To get the inclusion $H^+(S, f_t) \subset H(P, f_t)$ we need the following distortion property for the saddle-node map $\Phi = \Phi_0$.

(SN) Let $K = \max\{|\Phi''(x)|/|\Phi'(x)|,\ x \in [0, 1]\} > 0$. Then $\dfrac{4\, e^K\, (1-\lambda)}{\lambda^6} < \dfrac{1}{2}$, where $\lambda \in (2/3, 1)$.

Theorem 4.3. *Under the assumption (SN), $H^+(S, f_t) \subset H(P, f_t)$ holds for all small positive $t > 0$.*

To prove this theorem we need new ingredients that will be introduced in Section 4.3. Theorems 4.2 and 4.3 imply $H^+(S, f_t) = H(P, f_t)$ for all small $t > 0$.

4.1 One-dimensional dynamics for the saddle-node cycle

We now adapt the definitions of scaled fundamental domains, transitions and returns for saddle-node cycles. As in Section 2, for each $t > 0$, define the fundamental domains $D_t^- = [1-t, 1-\lambda\, t]$ and $D_t^+ = [a_t, b_t]$, $a_t = \Phi^{-1}(b_t)$, where D_t^+ is the first backward iterate of D_t^- by Φ contained in $[0, t]$. We have $\Phi^{k_t}(D_t^+) = D_t^-$, for some $k_t \in \mathbb{N}$. Observe that $|D_t^-| = t\,(1-\lambda)$ and, since $b_t \in (0, t]$, $|D_t^+| \le t^2$. For small $t > 0$, define the *transition* $\mathfrak{T}_t$ and the map $\mathfrak{G}_t$ by

$$\mathfrak{T}_t \colon D_t^+ \to D_t^-,\ \ x \mapsto \mathfrak{T}_t(x) = \Phi^{k_t}(x) \quad \text{and} \quad \mathfrak{G}_t \colon D_t^+ \to [0, t(1-\lambda)],\ \ x \mapsto \mathfrak{G}_t(x) = \mathfrak{T}_t(x) + (t-1).$$

Lemma 4.4. *The maps $\mathfrak{T}_t$ and $\mathfrak{G}_t$ are uniformly expanding for all small $t > 0$.*

Proof: It suffices to see that $(\Phi^{k_t})'(z) > 1$ for all $z \in D_t^+$. We use the following standard lemma (whose proof is omitted here):

Bounded Distortion Lemma 4.5. *Let $K > 0$ be as in condition (SN). Then, for every pair of points $z, y \in D_t^+$ and every small $t > 0$,*

$$e^{-K} \le \frac{(\Phi^{k_t})'(z)}{(\Phi^{k_t})'(y)} \le e^K.$$

The lemma now follows by the mean value theorem, taking $y \in D_t^+$ with

$$(\Phi^{k_t})'(y) = |D_t^-|/|D_t^+| \ge (1-\lambda)/t.$$

Thus, if t is small, $(\Phi^{k_t})'(z) \ge (e^{-K}\,(1-\lambda))/t > 1$, for all $z \in D_t^+$. $\square$

As in Section 2, given $x \in (a_t, b_t] = \tilde{D}_t^+$, let $i(x) \in \mathbb{Z}$ be the first i with $\Phi^i(\mathfrak{G}_t(x)) \in D_t^+$. The *return map* $\mathfrak{R}_t$ is now defined by

$$\mathfrak{R}_t \colon \tilde{D}_t^+ \to D_t^+, \quad \mathfrak{R}_t(x) = \Phi^{i(x)}(\mathfrak{G}_t(x)) = \Phi^{i(x)}(\mathfrak{T}_t(x) + (t-1)).$$

Lemma 4.6. *There exists $i_0 > 0$ such that $i(x) \ge i_0$ for all $x \in \tilde{D}_t^+$.*

Proof: To prove the lemma it is enough to see that $\mathfrak{G}_t(D_t^+) \subset (0, a_t)$. By definition, $\mathfrak{G}_t(D_t^+) = [0, (1-\lambda)\, t]$. Observe that, if t is small enough,

$$\Phi^2((1-\lambda)\, t) = \Phi((1-\lambda)\, t + (1-\lambda)^2\, t^2) = (1-\lambda)\, t + 2\,(1-\lambda)^2\, t^2 + \text{h.o.t} < t.$$

Thus, the right extreme $\Phi^2((1-\lambda)t)$ of $\Phi^2(\mathfrak{G}_t(D_t^+))$ is less than t. In particular, the right extreme of $\mathfrak{G}_t(D_t^+)$ is less than $\Phi^{-2}(t)$, and the the lemma follows from $D_t^+ \subset (\Phi^{-2}(t), t]$. □

As in the case of the return map R_t in Section 2, for each $i \geq i_0$, there is $\delta_i \in \tilde{D}_t^+$ with $\mathfrak{G}_t(\delta_i) = \Phi^{-i}(a_t)$. The points δ_i are the discontinuities of $\mathfrak{R}_t$. In this way, we get a decreasing sequence $(\delta_i)_{i \geq i_0}$ with $\delta_i \to a_t$, and intervals $J_i = [\delta_{i+1}, \delta_i]$, $i > i_0$, and $J_{i_0} = [\delta_{i_0}, b_t]$ such that $\mathfrak{R}_t$ is continuous and increasing in the interior of each J_i. Extending $\mathfrak{R}_t$ continuously to the whole J_i we get a bi-valuated map with $\mathfrak{R}_t(\delta_i) = \{a_t, b_t\}$ for all $i > i_0$.

Lemma 4.7. *The restriction of $\mathfrak{R}_t$ to each interval $J_i, i > i_0$, is onto. Moreover, there is $\ell > 1$ such that $\mathfrak{R}_t'(x) > \ell > 1$ for all $x \in (a_t, b_t]$ (if $x = \delta_i$ this means that the lateral derivatives of $\mathfrak{R}_t$ at x are greater than ℓ). Finally, $0 \in \mathfrak{G}_t(\mathfrak{R}_t(\delta_i))$ for all $i \geq i_0$.*

Proof: The lemma follows as Lemma 2.4 observing that $i_0 > 0$ (Lemma 4.6), $\mathfrak{G}_t$ is expanding (Lemma 4.4), and that the derivative of Φ in $(0, t]$ is bigger than one. □

Arguing as in Section 2, one gets the following lemma (corresponding to Lemma 2.5):

Lemma 4.8. *Given any subinterval I of D_t^+ there are $x \in I$ and $i \geq 0$ with $\mathfrak{G}_t(\mathfrak{R}_t^i(x)) = 0$.*

4.2 Lateral Homoclinic classes. Proof of Theorem 4.2

To prove Theorem 4.2 we proceed as in Section 3. After redefining vertical strips and segments and using Lemma 4.8, one gets that, for any small $t > 0$ and any disk χ with $W^s(P, f_t) \pitchfork \chi \neq \emptyset$, $W^{ss}(S, f_t) \pitchfork \chi \neq \emptyset$ (recall Proposition 3.1). The inclusion $(H(P, f_t) \cup \Upsilon_t) \subset H^+(S, f_t)$ follows exactly as $(H(P, f_t) \cup \Lambda_t) \subset H(Q, f_t)$ in the case of heterodimensional cycles. So we omit the details of the proofs of these inclusions.

4.3 Proof of Theorem 4.3: the inclusion $H^+(S, f_t) \subset H(P, f_t)$

Consider the homoclinic point $x_t = (-1/2, t, 0)$ of P for f_t and the fundamental domains of Φ $\Delta_t^+(i) = \Phi^{-i}(\Delta_t^+(0))$, $i \geq 0$, where $\Delta_t^+(0) = [\Phi^{-1}(t), t]$. We now construct a family $\mathcal{H}_t$ of homoclinic points of P for f_t of the form $(x, y, 0)$ such that the set $\{y\colon (x, y, 0) \in \mathcal{H}_t\}$ is dense in $\Delta_t^+(0)$:

Proposition 4.9. *For every small $t > 0$ there are sequences of homoclinic points of P of the form $(b_{i_1,i_2,\dots,i_m,k}, x_{i_1,i_2,\dots,i_m,k}, 0)_{k \in \mathbb{N}^*}$, $b_{i_1,i_2,\dots,i_m,k} \in [-1, 0]$, $i_j \in \mathbb{N}^*$, such that*

- **(H1)** $x_{i_1,i_2,\dots,i_m,k} \in \bigcup_{i=0}^4 \Delta_t^+(i) = \Delta_t$,
- **(H2)** $x_{i_1,i_2,\dots,i_m,k} \to x_{i_1,i_2,\dots,i_m}$ *as* $k \to \infty$,
- **(H3)** $x_{i_1,i_2,\dots,i_m,0} < x_{i_1,i_2,\dots,(i_m-1)}$ *for every* $i_m \geq 1$,
- **(H4)** $\mathrm{diam}((x_{i_1,i_2,\dots,i_m,k})_k) \to 0$ *as* $m \to \infty$,
- **(H5)** (x_k) *is increasing and* $x_k \to t^-$ *as* $k \to \infty$,

(H6) *$x_0 \in \Delta_t^+(1)$ and $x_0 \notin \Delta_t^+(0)$.*

This proposition will be proved in Section 4.3.2. From the proposition one gets the following:

Corollary 4.10. *The set $\mathcal{H}_t = \bigcup_{n,k\in\mathbb{N}^*}(x_{i_1,i_2,\dots,i_k,n})$ contains a dense subset of $\Delta_t^+(0)$.*

Proof: The proof of is identical to [D$_2$, Lemma 4.1], but we repeat it here for completeness. Take any point x in the interior of $\Delta_t^+(0)$. If $x \in \mathcal{H}_t$ we are done. Otherwise, by (H5) and (H6), there is $i_1 > 1$ with $x_{(i_1-1)} < x < x_{i_1}$. Analogously, by (H2) and (H3), there is $i_2 > 1$ with $x_{i_1,(i_2-1)} < x < x_{i_1,i_2}$. Inductively, using (H2) and (H3) as above, we get a sequence $\{i_k\}$, $i_k > 1$, such that, for all k, $x_{i_1,\dots,(i_k-1)} < x < x_{i_1,\dots,i_k}$. Finally, from (H4), $\lim_{k\to\infty} x_{i_1,\dots,i_k} = x$. □

4.3.1 Proof of Theorem 4.3

The deduction of Theorem 4.3 from Corollary 4.10 follows as in [D$_2$, Section 5]. For completeness, we sketch here this proof. Consider any $w \in W^u(S, f_t) \pitchfork W^{ss}(S, f_t)$. By replacing w by some iterate of it, we can assume that $w = (x, 0, 0)$, $|x|$ small. We prove that, for every $\varepsilon > 0$, the square $S(\varepsilon) = (x-\varepsilon, x+\varepsilon) \times (0, \varepsilon) \times \{0\} \subset W^s(P, f_t)$ intersects transversely $W^u(P, f_t)$. This implies that $w \in H(P, f_t)$. The configuration of the cycle and the λ-lemma imply that there is $n(\varepsilon) > 0$ such that $f_t^{-n(\varepsilon)}(S(\varepsilon))$ contains a disk $S'(\varepsilon)$ of the form

$$S'(\varepsilon) = [-1,1] \times (\bar{y}-\xi, \bar{y}+\xi) \times \{\bar{z}\}, \quad \bar{y} \in (1-t, 1),\ \bar{z} \in [-1,1] \text{ and small } \xi > 0.$$

Let $m \in \mathbb{N}$ be such that $\Phi^{-m}(\bar{y}) \in \Delta_t^+(0)$. Thus $f_t^{-m}(S'(\varepsilon))$ contains the horizontal strip

$$\hat{S}(\varepsilon) = [-1,1] \times (\Phi^{-m}(\bar{y}-\xi), \Phi^{-m}(\bar{y}+\xi)) \times \{\lambda_u^{-m}\,\bar{z}\} \subset W^s(P, f_t).$$

Since $\Phi^{-m}(\bar{y})$ belongs to $\Delta_t^+(0)$, Corollary 4.10 implies that $\hat{S}(\varepsilon)$ meets $W^u(P, f_t)$. Thus $\hat{S}(\varepsilon)$ contains a homoclinic point of P and the same holds for $S(\varepsilon)$. □

4.3.2 Proof of Proposition 4.9: Sequences of homoclinic points:

Let κ_t be the first $k \in \mathbb{N}$ such that $\Phi^k(\Delta_t^+(0)) \subset [1-t, 1]$, $\Delta_t^+(0) = [\Phi^{-1}(t), t]$. Since, for small $t > 0$, $|\Phi^{\kappa_t}(\Delta_t^+(0))| \le t\,(1-\lambda)$ and $|\Delta_t^+(0)| \ge \lambda\, t^2$, the Bounded Distortion Lemma 4.5 implies that

$$(\Phi^{\kappa_t})'(x) < \frac{1-\lambda}{\lambda\, t}\, e^K, \quad \text{for all } x \in \Delta_t^+(0). \tag{2}$$

Denote by δ_t^i the length of $\Delta_t^+(i)$. Since the derivative of Φ near 0 is close to 1 and strictly bigger than 1 in $(0, t]$, for small t, we have that

$$\delta_t^0 \ge \delta_t^i \ge \frac{9\,\delta_t^0}{10}, \quad i = 1, \dots, 4. \quad \text{In particular,} \quad \sum_{i=0}^{4} \delta_t^i \in [4\delta_t^0, 5\delta_t^0]. \tag{3}$$

Consider the interval $[1-\eta_t, 1]$, $\eta_t = \delta_t^1 + \delta_t^0$, and let α_t be the first natural number α with

$$\Phi^{\kappa_t+\alpha}(\Delta_t^+(0)) \subset [1-\eta_t, 1].$$

Observe that $|\Delta_t^+(0)| = \delta_t^0 < t^2$ and $\delta_t^1 < \delta_t^0$. Thus, for small $t > 0$, $\eta_t < 2\,\delta_t^0 < 2\,t^2 < \lambda\, t$. Since, $\Phi^{\kappa_t}(\Phi^{-1}(t)) \in [1-t, 1-\lambda\, t]$ and $(1-\lambda\, t) < (1-\eta_t)$, we get that $\alpha_t \ge 1$ for all small t. Observe also that $t^2 < \eta_t < 2\,t^2$, where the first inequality follows from (3) and $\delta_t^0 > 3t^2/4$ if t is small enough.

Lemma 4.11. *For every small $t > 0$ it holds $\lambda^{\alpha_t} \leq (2\,t)/\lambda$.*

Proof: By definitions of κ_t and α_t, $\Phi^{\kappa_t}(\Delta_t^+(0)) = [1 - e_t^-, 1 - e_t^+]$, where $e_t^- \in [\lambda\, t, t]$, and $\Phi^{\alpha_t}(1 - e_t^-) \in [1 - \eta_t, 1 - \lambda\, \eta_t]$. Thus, since Φ is affine near 1, $\lambda^{\alpha_t}(e_t^-) \in (0, \eta_t]$. Hence, from $t^2 < \eta_t < 2\,t^2$ and $\lambda\, t \leq e_t^- \leq t$,

$$\lambda^{\alpha_t} \leq \frac{\eta_t}{e_t^-} \leq \frac{2\,t^2}{\lambda\, t} = \frac{2\,t}{\lambda},$$

ending the proof of the lemma. □

Next lemma is necessary for getting (H4) and along the inductive definition of $(x_{i_1,i_2,\dots,i_m,k})_k$.

Lemma 4.12. $L = \max\{(\Phi^{\kappa_t+\alpha_t+j})'(x);\ x \in \cup_{i=0}^4 \Delta_t^+(i)$ *and* $j \geq 0\} < \frac{1}{2}$.

Proof: Since Φ is a contraction near 1, it is enough to compute the estimate when $j = 0$. We split the trajectory of a point $x \in \Delta_t^+(i)$ going from $\Delta_t^+(i)$ to $[1 - \eta_t, 1)$ as follows: **(i)** i iterates, $i \leq 4$, for x going from $\Delta_t^+(i)$ to $\Delta_t^+(0)$; **(ii)** κ_t iterates for $\Phi^i(x)$ going from $\Delta_t^+(0)$ to $\Phi_t^{k_t}(\Delta_t^+(0))$; and **(iii)** α_t iterates for $\Phi^{\kappa_t+i}(x)$ going from $\Phi_t^{k_t}(\Delta_t^+(0))$ to $[1 - \eta_t, 1]$. This construction involves $(i + \kappa_t + \alpha_t)$ iterations of x by Φ, so we need to remove the last i iterations, corresponding to a contraction by λ^i. We claim that

$$L \leq \underbrace{((2\,t+1)^4)}_{(\mathbf{a})}\ \underbrace{\frac{e^K\,(1-\lambda)}{\lambda\, t}\,\lambda^{\alpha_t}}_{(\mathbf{b})}\ \underbrace{\frac{1}{\lambda^4}}_{(\mathbf{c})}, \tag{4}$$

corresponding $(\mathbf{a})$ to the expansion of the first i iterates by Φ (just observe that in $[0, t]$ the derivative of Φ is upper bounded by $(2\,t + 1)$ and that $i \leq 4$), $(\mathbf{b})$ to an upper bound of the derivative of $\Phi^{\kappa_t+\alpha_t}$, recall (2), and $(\mathbf{c})$ to the i ($i \leq 4$) negative iterates of Φ close to 1. By (SN), Lemma 4.11, and the fact that $(2\,t + 1)^4 < 2$ if $t > 0$ is small, we get

$$L \leq ((2\,t+1)^4)\,\frac{e^K\,(1-\lambda)}{\lambda\, t}\,\frac{2\,t}{\lambda}\,\frac{1}{\lambda^4} = (2\,t+1)^4\,\frac{2\,(1-\lambda)\,e^K}{\lambda^6} \leq 4\,e^K\,\frac{(1-\lambda)}{\lambda^6} < \frac{1}{2},$$

which ends the proof of the lemma. □

Construction of the sequences $(x_{i_1,i_2,\dots,i_m,k})$**.** To construct the sequences $(x_{i_1,i_2,\dots,i_m,k})$, we need the following algorithm about the creation of homoclinic points, which is a consequence of the definition of the unfolding of the heterodimensional cycle.

Algorithm 4.13. *Let $(x, y, 0)$, $x \in [-1, 1]$ and $y \in [0, t]$, be a homoclinic point of P (for f_t) such that $\{(x, y)\} \times [-1, 1] \subset W^u(P, f_t)$. Then, for every m with $\Phi^m(y) \in (1-t, 1)$, there is a homoclinic point of P of the form $(\bar{x}, \Phi^m(y) + t - 1, 0)$ such that $\{(\bar{x}, \Phi^m(y) + t - 1)\} \times [-1, 1] \subset W^u(P, f_t)$.*

Take the homoclinic point $(-1/2, t, 0)$ of P and the sequences $(y_i)_{i\in\mathbb{N}^*}$ and $(x_i)_{i\in\mathbb{N}^*}$ defined by

$$y_i = \Phi^{\kappa_t+\alpha_t+i}(t) \quad \text{and} \quad x_i = (t-1) + y_i, \quad y_i \to 1 \quad \text{and} \quad x_i \to t.$$

Observe that, for each $i \geq 0$, there is a homoclinic point $(b_i, x_i, 0)$ of P verifying Algorithm 4.13. Also, by the definitions of α_t and κ_t, $y_i \in [1 - \eta_t, 1]$ for all $i \geq 0$. Thus, since $\eta_t = \delta_t^0 + \delta_t^1$, one has

$$x_i \in [t - \eta_t, t] = [t - (\delta_t^1 + \delta_t^0), t] = \Delta_t^+(1) \cup \Delta_t^+(0). \tag{5}$$

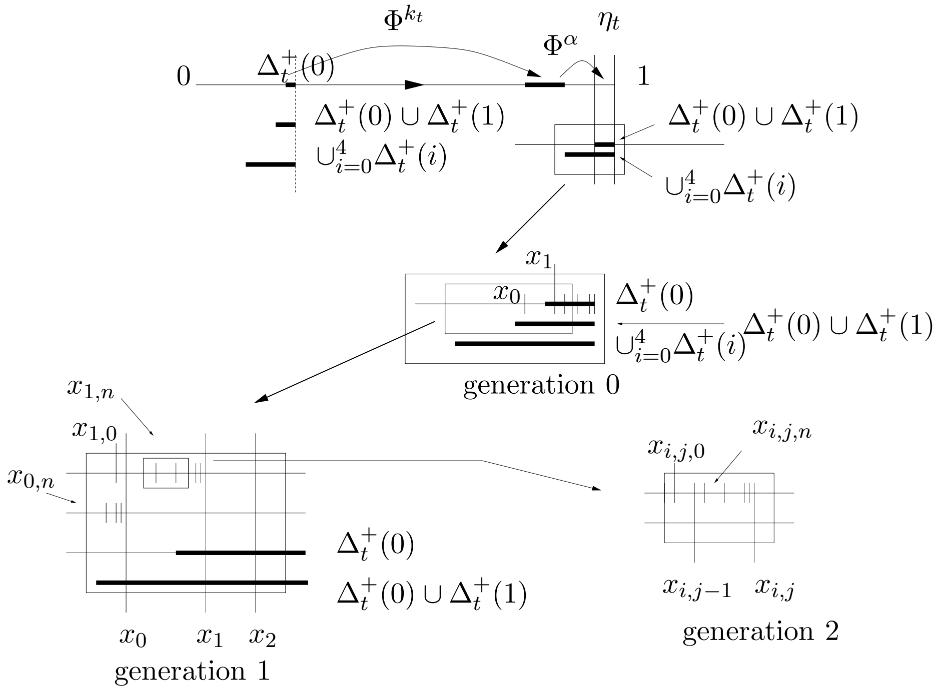

Figure 4: The sequences $(x_{i_1,i_2,\ldots,i_m,n})_n$

Lemma 4.14. *The sequence $(x_i)_{i\in\mathbb{N}^*}$ verifies (H5) and (H6).*

Proof: Condition (H5) follows by definition. To get (H6), i.e.$x_0 \in (\Delta_t^+(1) \setminus \Delta_t^+(0))$, note that, by construction, $x_0 \in [t-\eta_t, t-\lambda\,\eta_t]$ and $\Delta_t^+(1) = [t-\eta_t, t-\delta_t^0]$. Thus, by (5), it is enough to check that $\lambda\,\eta_t = \lambda\,(\delta_t^0+\delta_t^1) > \delta_t^0$. This inequality follows from $\delta_t^0 > \delta_t^1$, (3) and (SN), observing that

$$\frac{\delta_t^0}{\eta_t} = \frac{\delta_t^0}{\delta_t^0+\delta_t^1} < \frac{\delta_t^0}{2\,\delta_t^1} < \frac{\delta_t^0}{2\,(9/10)\,\delta_t^0} = \frac{10}{18} < \frac{2}{3} < \lambda.$$

The proof of the lemma is now complete. □

We now proceed with the construction of the sequences in Proposition 4.9. For each $j \in \mathbb{N}^*$, define the sequences $(y_{j,i})_{i\in\mathbb{N}^*}$ and $(x_{j,i})_{i\in\mathbb{N}^*}$ as follows,

$$y_{j,i} = \Phi^{\kappa_t+\alpha_t+j}(x_i) \quad \text{and} \quad x_{j,i} = (t-1) + y_{j,i}.$$

We claim that $y_{j,i} \to y_j$ and, consequently, $x_{j,i} \to x_j$, as $i \to \infty$. For that just observe that $\lim_{i\to\infty} x_i = t$, thus, by continuity, $\lim_{i\to\infty} y_{j,i} = \lim_{i\to\infty} \Phi^{\kappa_t+\alpha_t+j}(x_i) = \Phi^{\kappa_t+\alpha_t+j}(t) = y_j$.

Lemma 4.15. *The points $(x_{j,i})_i$ belong to $\cup_{i=0}^4 \Delta_t^+(i)$ for all $i, j \in \mathbb{N}^*$.*

Proof: Since, by construction, the sequences $(x_{j,i})_i$ are increasing, it is enough to see that $x_{0,0} \in \cup_{i=0}^4 \Delta_t^+(i)$. Consider the diameter $d_0 = (t - x_0)$ of $(x_i)_{i\in\mathbb{N}^*}$. By (5), $d_0 < \delta_t^0 + \delta_t^1$. Let $d_1 = |x_0 - x_{0,0}|$ be the diameter of $(x_{0,i} = \Phi^{\kappa_t+\alpha_t}(x_i) + (t-1))_i$, which is equal to the diameter of $(\Phi^{\kappa_t+\alpha_t}(x_i))_i$. Thus, since $(x_i)_i \subset \Delta_t^+(0) \cup \Delta_t^+(1)$, by Lemma 4.12, the diameter d_1 is bounded by

$$d_1 \le L\,d_0 < d_0/2 < (\delta_t^0+\delta_t^1)/2 < (2\,\delta_t^0)/2 < \delta_t^0. \tag{6}$$

Since $x_0 \in \Delta_t^+(1)$ (Lemma 4.14), to prove that $x_{0,0} \in \cup_{i=0}^4 \Delta_t^+(i)$, it is enough to see that

$$x_{0,0} = x_0 - d_1 > x_0 - (\delta_t^2 + \delta_t^3 + \delta_t^4) \Longleftrightarrow d_1 < (\delta_t^2 + \delta_t^3 + \delta_t^4),$$

which immediately follows from $\delta_t^2 + \delta_t^3 + \delta_t^4 > \delta_t^0 > d_1$, the first inequality being consequence of $\delta_t^i > (9\,\delta_t^0)/(10)$, see (3), and the last from (6). This ends the proof of the lemma. □

Suppose now inductively defined sequences $(y_{i_i,i_2,\ldots,i_m,i})_{i\in\mathbb{N}^*}$ and $(x_{i_i,i_2,\ldots,i_m,i})_{i\in\mathbb{N}^*}$ by

$$y_{i_1,i_2,\ldots,i_m,i} = \Phi^{\kappa_t+\alpha_t+i_1}(x_{i_2,\ldots,i_m,i}) \quad \text{and} \quad x_{i_1,i_2,\ldots,i_m,i} = (t-1) + y_{i_1,i_2,\ldots,i_m,i},$$

satisfying conditions (H1), (H2), (H3) and

(H4b) Let d_m, $m \geq 0$, be the diameter of the sequence $(x_{\underbrace{0,\ldots,0}_{m\,0's},i})_i$. Then $d_m \leq (d_{m-1})/2$.

Observe that (H2) and (H3) and are equivalent to (H2b) and (H3b) below, respectively,

(H2b) $(y_{i_1,i_2,\ldots,i_m,i})_i \to y_{i_i,i_2,\ldots,i_m}$ as $i \to \infty$,

(H3b) $y_{i_1,i_2,\ldots,i_m,0} < y_{i_i,i_2,\ldots,(i_m-1)}$ for all $i_m \geq 1$.

Notice that, for $m = 1$, (H1) follows from Lemma 4.15, (H2) (or (H2b)) from definition, and (H4b) from the estimates in (6). To check (H3b), $y_{i,0} < y_{i-1}$, for every $i \geq 1$, recall that, by Lemma 4.14, $x_0 < \Phi^{-1}(t) < t$, thus

$$y_{i-1} = \Phi^{\kappa_t+\alpha_t+i-1}(t) = \Phi^{\kappa_t+\alpha_t+i}(\Phi^{-1}(t)) > \Phi^{\kappa_t+\alpha_t+i}(x_0) = y_{i,0}.$$

For simplicity, we say that the sequences $(z_{i_i,i_2,\ldots,i_m,i})_{i\in\mathbb{N}^*}$, $z = x, y$, are *of generation m*.

Lemma 4.16. *Property (H4b) implies (H4) in Proposition 4.9.*

Proof: By construction, the diameter of any sequence $(x_{i_1,\ldots,i_m,k})$ of generation m is bounded by the diameter d_m of $(x_{\underbrace{0,\ldots,0}_{m\,0's},i}))_{i\in\mathbb{N}^*}$. Thus, inductively, $d_m \leq (1/2)\,d_{m-1}$, so $d_m \to 0$. □

Keeping in mind Lemmas 4.14 and 4.16, to prove Proposition 4.9 it suffices to see that the sequences verify (H1), (H2b), (H3b) and (H4b). We argue inductively on the generation of the sequences and assume satisfied these conditions for sequences of generation less than or equal to m. To verify (H2b) for the sequences of generation $m+1$ note that, by induction, $(y_{i_1,i_2,\ldots,i_m,i})_i \to y_{i_1,i_2,\ldots,i_m}$. Thus, by continuity of Φ and by definition,

$$(y_{j,i_1,i_2,\ldots,i_m,i})_i = (\Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\ldots,i_m,i}))_i \to \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\ldots,i_m}) = y_{j,i_1,i_2,\ldots,i_m}.$$

To prove (H3b) observe that, by induction, $x_{i_1,i_2,\ldots,i_m,0} < x_{i_1,i_2,\ldots,i_m-1}$. Thus, since Φ is increasing,

$$y_{j,i_1,i_2,\ldots,i_m,0} = \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\ldots,i_m,0}) < \Phi^{\kappa_t+\alpha_t+j}(x_{i_1,i_2,\ldots,i_m-1}) = y_{j,i_1,i_2,\ldots,i_m-1}.$$

To check (H4b) observe that, by the induction hypotheses (H1), $x_{0,\ldots,0,i} \in \cup_{i=0}^4 \Delta_t^+(i)$, Lemma 4.12 and the fact that the sequences $(y_{0,\ldots,0,i})$ and $(x_{0,\ldots,0,i})$ have the same diameter imply that

$$d_{m+1} = \operatorname{diam}(\underbrace{(y_{0,\ldots,0,i})_i}_{(m+1)\,0's}) = \operatorname{diam}((\Phi^{\kappa_t+\alpha_t}(\underbrace{(x_{0,\ldots,0,i})_i}_{m\,0's})) = L\operatorname{diam}((x_{0,\ldots,0,i}))_i) = L\,d_m \leq \frac{d_m}{2}.$$

Finally, to get (H1) for the generation $(m+1)$ it is enough to see that, for every m, $\sum_{i=0}^{m} d_m < 4\delta_t^0 < \sum_{i=0}^{4} \delta_t^i$, recall (3). By induction and Lemma 4.14, which implies that $d_0 \leq \delta_t^0 + \delta_t^1$, we have

$$\sum_{i=0}^{m} d_m \leq \sum_{i=0}^{m-1} (1/2)^i\, d_0 \leq \sum_{i=0}^{m-1} (1/2)^i\, (\delta_t^0 + \delta_t^1) \leq \frac{1}{1-2}\, (\delta_t^0 + \delta_t^1) < 2(\delta_t^0 + \delta_t^1) < 4\delta_t^0,$$

finishing the proof of our claim, and the construction of the sequences in Proposition 4.9.

5 Homoclinic classes before collapsing the saddles S_s^+ and S_s^-

We now return to the family $f_{t,s}$ in Section 4. Note that $S_s^+ = (0, \sqrt{s}, 0)$, $s > 0$, is a fixed point of index two of $f_{t,s}$ (any $t > 0$) and that $f_{\sqrt{s},s}$ has a heterodimensional cycle associated to S_s^+ and P:

- $W^u(S_s^+, f_{\sqrt{s},s})$ meets transversely $W^s(P, f_{\sqrt{s},s})$ throughout the segment $\{0\} \times (\sqrt{s}, 1) \times \{0\}$,
- $W^u(P, f_{\sqrt{s},s})$ meets quasi-transversely $W^s(S_s^+, f_{\sqrt{s},s})$ along the orbit of $(-1/2, \sqrt{s}, 0)$ (just observe that $[-1,1] \times \{(\sqrt{s}, 0)\} \subset W^s(S_s^+, f_{\sqrt{s},s})$ and that $(-1/2, \sqrt{s}, 0) \in W^u(P, f_{\sqrt{s},s})$).

In what follows we assume that the saddle-node arc Φ_s verifies condition (SN) in Section 4.

Theorem 5.1. *There exist a small $s_0 > 0$ and a strictly positive map τ defined on $(0, s_0)$ such that, for every $s \in (0, s_0)$ and $t \in (\sqrt{s}, \sqrt{s} + \tau(s))$,*

- *$H(P, f_{t,s}) = H(S_s^+, f_{t,s})$, and*
- *there is a neighborhood W_s of the cycle of $f_{\sqrt{s},s}$ (associated to P and S_s^+) such that the maximal invariant set $\Lambda_{t,s}$ of $f_{t,s}$ in W_s is contained to $H(S_s^+, f_{t,s})$.*

The the inclusion $H(P, f_{t,s}) \subset H(S_s, f_{t,s})$ and the second part of the theorem follow as in Theorem 1.1. So we just sketch these proofs. For a fixed $s > 0$ and $t > \sqrt{s}$, t close to $\sqrt{s}$, $t = \sqrt{s} + \tau$, consider the scaled fundamental domains $D_{t,s}^{\pm}$ of Φ_s,

$$D_{t,s}^- = [1 - (t - \sqrt{s}), 1 - \lambda\,(t - \sqrt{s})] = [1 - \tau, 1 - \lambda\,\tau]$$

and $D_{t,s}^+$ defined as the first backward iterate of $D_{t,s}^-$ in $[\sqrt{s}, \sqrt{s} + \tau]$. Let $\Phi_s^{k_{t,s}}(D_{t,s}^+) = D_{t,s}^-$, where $k_{t,s} \in \mathbb{N}$. These domains play the role of $D_t^{\pm}$ in Section 2. Observe that

$$\ell(t,s) = \frac{|D_{t,s}^-|}{|D_{t,s}^+|} \geq \frac{\tau\,(1-\lambda)}{(\sqrt{s}+\tau)^2 - s} = \frac{\tau\,(1-\lambda)}{\tau\,(2\sqrt{s}+\tau)} = \frac{1-\lambda}{2\,\sqrt{s}+\tau}.$$

By shrinking s, we can assume that $|\Phi_s''(x)|/|\Phi_s'(x)| < 2\,K$ for all $x \in [-1, 2]$ (K as in condition (SN)). Thus, there is $s_0 > 0$ and a map $\tau\colon (0, s_0) \to \mathbb{R}^+$ such that

$$\ell(t,s)\, e^{-2\,K} > 2, \quad \text{for all } s \in (0, s_0) \text{ and } t \in (\sqrt{s}, \sqrt{s} + \tau(s)). \tag{7}$$

Exactly as in Section 2, for $s \in (0, s_0)$ and $t \in (\sqrt{s}, \sqrt{s} + \tau(s))$, we define maps

$$\begin{aligned}
&T_{t,s}\colon D_{t,s}^+ \to D_{t,s}^-, \quad T_{t,s}(x) = \Phi_s^{k_{t,s}}(x),\\
&G_{t,s}\colon D_{t,s}^- \to [\sqrt{s}, \sqrt{s} + \tau\,(1-\lambda)], \quad G_{t,s}(x) = T_{t,s}(x) + (t-1),\\
&R_{t,s}\colon D_{t,s}^+ \to D_{t,s}^+, \quad R_{t,s}(x) = \Phi_s^{i(x)}(G_{t,s}(x)),
\end{aligned}$$

where, as in Section 2, $i(x)$ is the first forward iterate of $G_{t,s}(x)$ by Φ_s in $D^+_{t,s}$. As in Lemma 4.4, the Bounded Distortion Lemma 4.5 and equation (7) imply that $T_{t,s}$, $G_{t,s}$ and $R_{t,s}$ are uniformly expanding. The inclusion $(H(P, f_{t,s}) \cup \Lambda_{t,s}) \subset H(S^+_s, f_{t,s})$ now follows as in Theorem 1.1.

To get $H(S^+_s, f_{t,s}) \subset H(P, f_{t,s})$ recall first that Theorem 4.3 gives small $\bar{t} > 0$ with $H^+(S, f_{t,0}) \subset H(P, f_{t,0})$ for all $t \in (0, \bar{t})$. This inclusion follows by constructing the sequences of homoclinic points of P in Proposition 4.9. The proof Proposition 4.9 only involves distortion control of the saddle-node map in $[0, 1]$ and the contracting itineraries in Lemma 4.12. Clearly, these properties hold after replacing, for small positive s, the saddle-node S by the hyperbolic point S^+_s and considering the restriction of the saddle-node map to $[\sqrt{s}, 1]$. This ends the sketch of the proof of Theorem 5.1.

6 Collision, explosion and collapse of homoclinic classes

In this section we prove Theorems A and B. Consider a two parameter family of diffeomorphism $f_{t,s}$ locally defined as follows:

Partially hyperbolic local dynamics:

- In the set $C = [-1, 1] \times [-2, 2] \times [-1, 1]$, $f_{t,s}(x, y, z) = (\lambda^s x, \Psi_s(y), \lambda^u z)$, where $0 < \lambda^s < 1 < \lambda^u$ and $\Psi_s \colon [-2, 2] \to (-3, 2)$ is a strictly increasing C^2-map such that $\lambda^s < d_m < \Psi_s'(y) < d_M < \lambda^u$ for all $y \in [-2, 2]$ and small $|s|$.

- $\Psi_s(1) = 1$ and $\Psi_s(-1) = -1$ for all s and Ψ_s is affine and independent of s in $[-1-\delta, -1+\delta]$ and $[1-\delta, 1+\delta]$, for some small $\delta > 0$. Furthermore, $\Psi_s'(-1) = \beta > 1 > \Psi_s'(1) = \lambda > 0$.

- There is $\delta > 0$ such that the restriction of Ψ_s to $[-\delta, \delta]$ is of the form $\Psi_s(x) = x + x^2 - s$.

- For $s < 0$, Ψ_s has (exactly) two fixed points (± 1), for $s = 0$, Ψ_0 has three fixed points (± 1 and 0), and, for $s > 0$, Φ_s has four fixed points (± 1 and $\pm\sqrt{s}$). Let $P = (0, 1, 0)$, $Q = (0, -1, 0)$, $S = (0, 0, 0)$ and, for positive s, $S^{\pm}_s = (0, \pm\sqrt{s}, 0)$, the fixed points of $f_{t,s}$ in C.

Existence and unfolding of cycles:

- There are $k_0 \in \mathbb{N}$ and small neighborhoods of $(0, -1, -1/2)$, $(0, 0, -1/2)$ and $(0, 1, -1/2)$ such that, for each small $|s|$, in such neighborhoods, each $f^{k_0}_{t,s}$ is the translation $f^{k_0}_{t,s}(x, y, z) = (x - 1/2, y - 1 + t, z + 1/2)$, recall the definition of f_t in Section 1.

As in previous sections, we have:

- for $(t, s) = (0, 0)$, $f_{0,0}$ has a pair of saddle-node heterodimensional cycles, associated to P and S, $W^u(P, f_{0,0})$ meets quasi-transversely $W^{ss}(S, f_{0,0})$, and to Q and S, $W^s(Q, f_{0,0})$ intersects $W^{uu}(S, f_{0,0})$, (note that $[-1, 1] \times \{(0, 0)\} \subset W^{ss}(S, f_{0,0})$ and $\{(0, 0)\} \times [-1, 1] \subset W^{uu}(S, f_{0,0})$).

- For small $|t|$ and $s < 0$, the homoclinic classes of P and Q are both nontrivial: notice that, for negative s, $[-1, 1] \times (-1, 2) \times \{0\} \subset W^s(P, f_{t,s})$ and $\{0\} \times (-1, 2) \times [-1, 1] \subset W^u(Q, f_{t,s})$, thus $(-1/2, t, 0)$ and $(-1/2, -1, 0)$ are homoclinic points of P and Q, respectively.

- $f_{\sqrt{s},s}$ has a pair of heterodimensional cycles associated to P and S^+_s and to Q and S^-_s (this is obtained exactly as in Section 5).

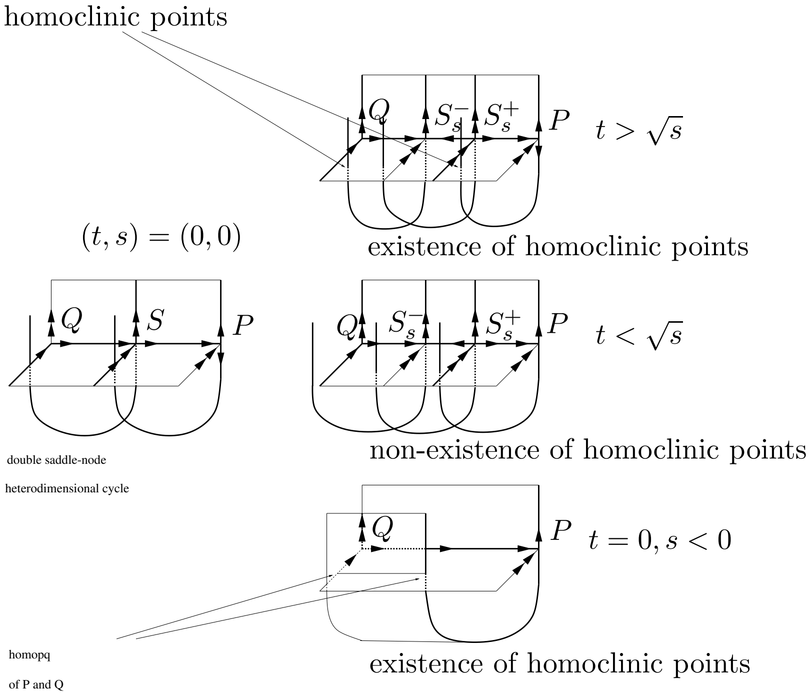

Figure 5: The two parameter family $f_{t,s}$

As in Section 4, we assume the following distortion property (similar to (SN)). Let K be an upper bound for $|\Psi_s''(x)|/|\Psi_s'(x)|$, for small $|s|$ and $x \in [-1, 1]$,

$$\textbf{(DS)}\ \max\left\{\frac{4\,e^K\,(1-\lambda)}{\lambda^6}, \frac{4\,e^K\,(1-\beta^{-1})}{\beta^{-6}}\right\} < \frac{1}{2}, \text{ where } 3/4 < \lambda < 1 < \beta < 4/3.$$

6.1 Dynamics before collapsing the saddles S_s^+ and S_s^-

Theorems 4.2 and 4.3 give a small $\bar{t}_0$ such that $H(P, f_{\bar{t},0}) = H^+(S, f_{\bar{t},0})$ and $H(Q, f_{\bar{t},0}) = H^-(S, f_{\bar{t},0})$ (for this inclusion consider $f_{\bar{t},0}^{-1}$) for all $\bar{t} \in (0, \bar{t}_0]$. These proofs only involve the following ingredients:

- The inclusions $H(P, f_{t_0,0}) \subset H^+(S, f_{t,0})$ and $H(Q, f_{t,0}) \subset H^-(S, f_{t,0})$ are obtained considering the ratio between the lengths of the scaled fundamental domains at the hyperbolic point and at the saddle-node and using that such a ratio is arbitrarily large.

- The inclusion $H^+(S, f_{t,0}) \subset H(P, f_{t_0,0})$ (resp. $H^-(S, f_{t,0}) \subset H(Q, f_{t_0,0})$) is obtained by constructing sequences of homoclinic points of P (resp. Q) verifying Proposition 4.9. The proof of such a proposition only involves distortion control of the saddle-node map in $[0, 1]$ (resp. $[-1, 0]$) and contracting itineraries (Lemma 4.12).

These properties hold after replacing, for small $s > 0$, the saddle-node S by the saddle S_s^+ (considering the restriction of Ψ_s to $[\sqrt{s}, 1]$) and S by the saddle S_s^- (considering the restriction of Ψ_s to $[-1, -\sqrt{s}]$). In this way, we get:

Theorem 6.1. *If $t > 0$ is small, there is a small $s(t) > 0$ such that $H(P, f_{t,s}) = H(S_s^+, f_{t,s})$ and $H(Q, f_{t,s}) = H(S_s^-, f_{t,s})$ for all $s \in (0, s(t))$.*

6.2 Dynamics after collapsing the saddles S_s^+ and S_s^-

Theorem 6.2. *For every small $t > 0$ there is $s(t) < 0$ such that $H(P, f_{t,s}) = H(Q, f_{t,s})$ for all $s \in [s(t), 0)$.*

This theorem follows adapting the constructions in Theorem 1.1. We prove the inclusion $H(P, f_{t,s}) \subset H(Q, f_{t,s})$ ($H(Q, f_{t,s}) \subset H(P, f_{t,s})$ follows by taking $f_{t,s}^{-1}$). As in Sections 2 and 4.1, define transitions $\mathcal{T}_{t,s}$ and returns $\mathcal{R}_{t,s}$ as follows: take the fundamental domain $D_t^- = [1-4\,t, 1-4\,\lambda\,t]$ of Ψ_s and let $k_{t,s}$ be the first $k \in \mathbb{N}$ with $-2\,t \in \Psi_s^{-k}(D_t^-)$. Let $D_{t,s}^+ = \Psi_s^{-k_{t,s}}(D_t^-)$ and define

$$\mathcal{T}_{t,s} \colon D_{t,s}^+ \to D_t^-, \ \mathcal{T}_{t,s}(x) = \Psi_s^{k_{t,s}}(x), \qquad \mathcal{G}_{t,s} \colon D_{t,s}^+ \to [-3\,t, -2\,t], \ \mathcal{G}_{t,s}(x) = \mathcal{T}_{t,s}(x) + (t-1).$$

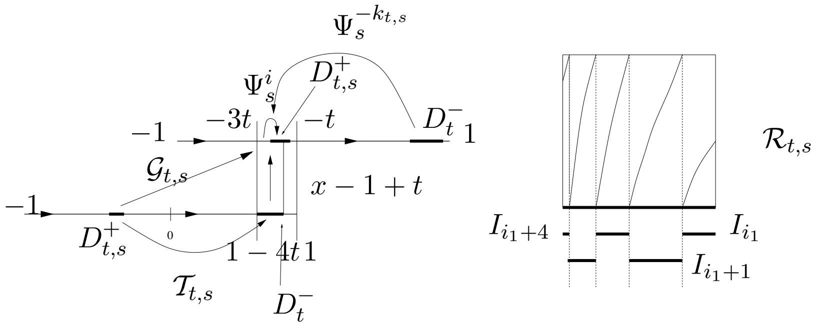

Figure 6: The expanding maps $\mathcal{T}_t$, $\mathcal{G}_t$ and $\mathcal{R}_t$

By definition, $\mathcal{G}_{t,s}(D_{t,s}^+) \subset [-3\,t, t\,(1-4\,\lambda)] \subset [-3\,t, -2\,t]$ (recall that $\lambda > 3/4$). Since, $|D_{t,s}^+| \le 9\,t^2 + s$, the Bounded Distortion Lemma 4.5 and $|D_t^-| = 4\,\lambda\,(1-t)$ immediately give

$$\mathcal{T}_{t,s}'(x) \ge (e^{-2\,K})\,\frac{|D_t^-|}{|D_{t,s}^+|} \ge (e^{-2\,K})\,\frac{4\,\lambda(1-t)}{9\,t^2 + s}.$$

This inequality immediately implies the following:

Lemma 6.3. *For every small $t > 0$ there is $s(t) < 0$ such that $\mathcal{T}_{t,s}$ and $\mathcal{G}_{t,s}$ are 63-expanding for all $s \in [s(t), 0)$.*

Since, by definition, the right extreme of $\mathcal{G}_{t,s}(D_{t,s}^+)$ is less than the right extreme of $D_{t,s}^+$, for each $x \in D_{t,s}^+$ there is a first $i(x) \ge 0$ with $\Psi_s^{i(x)}(\mathcal{G}_{t,s}(x)) \in D_{t,s}^+$. The return map $\mathcal{R}_{t,s}$ is defined by

$$\mathcal{R}_{t,s} \colon D_{t,s}^+ \to D_{t,s}^+, \quad \mathcal{R}_{t,s}(x) = \Psi_s^{i(x)} \mathcal{G}_{t,s}(x).$$

Lemma 6.4. *The map $\mathcal{R}_{t,s}$ is 3-expanding for all small $t > 0$ and $s \in [s(t), 0)$.*

Proof: Note that the expansion for $\mathcal{R}_{t,s}$ does not follow immediately from the expansion of $\mathcal{T}_{t,s}$: the $i(x)$ iterates by Ψ_s at the left of 0 introduce a contraction. By Lemma 6.3, it is enough to see that this contraction is at most 1/21. Recall that $\mathcal{G}_{t,s}(D^+_{t,s}) \subset [-3\,t, -2\,t]$ and observe that $0 \leq i(x) \leq i_0$, where $\Psi^{i_0}(-3\,t) \in D^+_{t,s}$. Write $D^+_{t,s}(j) = \Psi_s^{-j}(D^+_{t,s})$ and note that, for every $x \in D^+_{t,s}$, $|D^+_{t,s}(i(x))| \leq |D^+_{t,s}(i_0)| < 10\,t^2 + s$. For each $i = i(x)$ there is $z_i \in D^+_{t,s}(i)$ such that

$$(\Psi^i_s)'(z_i) = \frac{|D^+_{t,s}|}{|D^+_{t,s}(i)|} \geq \frac{|D^+_{t,s}|}{|D^+_{t,s}(i_0)|} \geq \frac{|D^+_{t,s}|}{10\,t^2 + s}.$$

Using the arguments in the Bounded Distortion Lemma 4.5, we have that, for every $x \in D^+_{t,s}(i)$,

$$(\Psi^i_s)'(x) \geq \frac{e^{-\,K\,L_t}\,|D^+_{t,s}|}{10\,t^2 + s},$$

where $L_t = 2\,t$ is the length of the interval $[-3\,t, -t]$ where the iterations by Ψ_s are considered. Finally, since $|D^+_{t,s}| > (t^2)/2$,

$$(\Psi^{i(x)}_s)'(x) \geq \frac{e^{-K\,2\,t}\,|D^+_{t,s}|}{10\,t^2 + s} > \frac{e^{-K\,2\,t}\,t^2}{20\,t^2 + 2s} > \frac{1}{21},$$

for every small t and $s \in [s(t), 0)$, ending the proof of the lemma. $\square$

Let $D^+_{t,s} = [e^-_{t,s}, e^+_{t,s}]$. Note that there are $i_1, i_2 \in \mathbb{N}$ such that $i(x) \in [i_1, i_2]$ for all $x \in D^+_{t,s}$ and, for each $i \in [i_1, i_2 - 1]$, there is $d_i \in D^+_{t,s}$ with $\Psi_s(\mathcal{G}_{t,s}(d_i)) = e^-_{t,s}$. As in Section 2, $d_{i_2-1} < d_{i_2-2} < \cdots < d_{i_1}$ and the d_i are the discontinuities of $\mathcal{R}_{t,s}$. Moreover, $\mathcal{G}_{t,s}((d_{i+1}, d_i)) = \mathrm{int}(D^+_{t,s})$ and $\mathcal{G}_{t,s}$ is increasing in (d_{i+1}, d_i). Write $I_{i_1} = [d_{i_1}, e^+_{t,s}]$, $I_i = [d_i, d_{i-1}]$, $i_1 < i \leq i_2 - 2$, and $I_{i_2} = [e^-_{t,s}, d_{i_2-1}]$. We now continuously extend $\mathcal{R}_{t,s}$ to the closed intervals I_i (so $\mathcal{R}_{t,s}$ is bivaluated at any d_i).

Lemma 6.5. *Given any subinterval J of $D^+_{t,s}$, there is $m \geq 0$ such that $\mathcal{R}^m_{t,s}(J) = D^+_{t,s}$.*

Proof: The proof follows as in Lemma 2.5. It is enough to see $\mathcal{R}^m_{t,s}(J)$ contains an interval I_i, $i_1 < i < i_2$, for some $m \in \mathbb{N}^*$. Write $J = J_0$. If $\mathcal{R}_{t,s}(J_0)$ contains two discontinuities we are done. Otherwise, $\mathcal{R}_{t,s}(J_0) \subset I_i \cup I_{i-1}$ for some i. Write $J_1^- = \mathcal{R}_{t,s}(J_0) \cap I_{i-1}$ and $J_1^+ = \mathcal{R}_{t,s}(J_0) \cap I_i$, and let J_1 be the biggest $J_1^\pm$. By Lemma 6.4, $|J_1| > (3/2)\,|J_0|$. Inductively, one gets intervals J_i contained in the orbit of J_0 such that either J_{i+1} contains two discontinuities or $|J_{i+1}| \geq (3/2)^i |J_0|$. Since the size of $D^+_{t,s}$ is finite, this ends the proof of the lemma. $\square$

Lemma 6.5 is the main step to prove Theorem 6.2, whose proof follows arguing as in the proof of Theorem 1.1 after proving the following:

Proposition 6.6. *Let $\chi = \{x\} \times A \subset [-1,1] \times [-1,2] \times [-1,1]$, where $x \in [-1,1]$ and A is a disk of $\mathbb{R}^2$ whose interior contains a point $(y, 0)$ with $y \in (-1, 2)$. Then $\chi \pitchfork W^s(Q, f_{t,s}) \neq \emptyset$.*

This result corresponds to Proposition 3.4. After proving it, Theorem 6.2 is deduced as follows. As in Section 3, Proposition 6.6 implies that $W^s(Q, f_{t,s})$ meets transversely every two-disk χ with $W^s(P, f_{t,s}) \pitchfork \chi \neq \emptyset$. Now, arguing as in Section 3.1, we get $H(P, f_{t,s}) \subset H(Q, f_{t,s})$. We now prove Proposition 6.6. The first step is the next lemma, corresponding to Lemma 3.3 in the proof of Propositions 3.1 and 3.4 (its proof follows exactly as Lemma 3.3, so it will be omitted).

Lemma 6.7. *There are $x \in [-1,1]$ and $y \in int(D^+_{t,s})$ such that $\{(x,y)\} \times [-1,1] \subset W^u(P, f_{t,s})$.*

Lemma 6.8. *For every $x \in [-1,1]$, $W^s(Q, f_{t,s})$ meets transversely $\{x\} \times D^+_{t,s} \times [-1,1]$.*

Proof: Observe that, by construction, the point $(0, -t, -1/2)$ belongs to $W^s(Q, f_{t,s}) \cap W^u(Q, f_{t,s})$. Define $j > 0$ by $\Psi_s^{-j}(-t) \in D^+_{t,s}$. It is now immediate that

$$H = [-1,1] \times \{(\Psi_s^{-j}(0), -\lambda_u^{-j}(1/2))\} \subset W^s(Q, f_{t,s}) \quad \text{and} \quad H \pitchfork (\{x\} \times D^+_{t,s} \times [-1,1]) \neq \emptyset,$$

ending the proof of the lemma. □

We are now ready to prove Proposition 6.6. By the λ-lemma and Lemma 6.7, the forward orbit of the disk χ contains a strip Δ of the form $\{x\} \times [a,b] \times [-1,1]$, where $x \in [-1,1]$, and $\alpha_0 = [a,b] \subset D^+_{t,s}$. Using the map $\mathcal{R}_{t,s}$ and arguing as in Section 3.1, we inductively define strips $\Delta_k = \{x_k\} \times \alpha_k \times [-1,1]$, $x_k \in [-1,1]$ and $\alpha_k \subset D^+_{t,s}$, such that $\Delta_{k+1} \subset f^{n_k}_{t,s}(\Delta_k)$ and $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$. By Lemma 6.5, there is a first $k \in \mathbb{N}$ such that $\alpha_{k+1} = \mathcal{R}_{t,s}(\alpha_k)$ contains $D^+_{t,s}$. By Lemma 6.8, $W^s(Q, f_{t,s}) \pitchfork \Delta_{k+1} \neq \emptyset$, thus $W^s(Q, f_{t,s}) \pitchfork \Delta \neq \emptyset$, ending the proof of the proposition.

6.3 End of the proof of Theorem A

We now construct a one-parameter family of diffeomorphisms (g_s) satisfying Theorem A. For that consider the arc $f_{t,s}$ defined as in the beginning of Section 6. We fix small $\bar{t} > 0$ and consider the arc $g_s = f_{\bar{t},-s}$. The results in the previous section imply that

- for every $s < 0$, $H(P, g_s)$ and $H(Q, g_s)$ are non-hyperbolic and disjoint (Theorem 6.1),
- $s = 0$, $H(P, g_0) = H^+(S, g_0)$ and $H(Q, f_0) = H^-(S, g_0)$, (Theorems 4.2 and 4.3),
- for every $s > 0$, $H(P, g_s) = H(Q, g_s)$, (Theorem 6.2).

To finish the proof of Theorem A we need to see that $\{S\} \in H(P, g_s) \cap H(Q, g_s)$ and to describe the maximal invariant set of g_s in the neighborhood W of the cycle. We assume that W is a *level of a fitration* of $f_{0,0}$ (thus, by continuity and compacity, it is also a level of a filtration for $f_{t,s}$ for every small $|s|$ and $|t|$): there are compact sets M_2 and M_1, $M_1 \subset \mathrm{int}(M_2)$, such that $M_2 \setminus M_1 = W$ and $f_{0,0}(M_i) \subset \mathrm{int}(M_i)$, $i = 1, 2$. Hence, if $x \in W$ and $f^i_{t,s}(x) \notin W$ for some i, then x is wandering: suppose, for instance, that $f^{i_0}_{t,s}(x) \in \mathrm{int}(M_1)$, where $i_0 > 0$. Then, there is a neighborhood $U_x \subset W$ of x with $f^{i_0}_{t,s}(U_x) \subset M_1$. By the definition of the filtration, $f^{i_0+j}_{t,s}(U_x) \subset M_1$ for all $j \geq 0$. Thus $f^{i_0+j}_{t,s}(U_x) \cap U_x = \emptyset$ for all $j \geq 0$. By shrinking U_x, we have that $f^j_{t,s}(U_x) \cap U_x = \emptyset$ for all $j > 0$, and x is wandering.

Using the definition of the arc $f_{t,s}$, it is immediate to check the following:

Remark 6.9. *Let $\Lambda_{t,s}$ be the maximal invariant set of $f_{t,s}$ in W. Then, for $s > 0$ small,*

- *Every point $(x, y, s) \in C = [-1,1] \times [-2,2] \times [-1,1]$ with $y \in (-\sqrt{s}, \sqrt{s})$ (resp, $y \in [-2,-1)$ or $y \in (1,2)$) is wandering.*
- *Consider $w = (x, y, z) \in C \cap \Lambda_{t,s}$ and, for $i \in \mathbb{Z}$, let $w_i = g^i_s(w) = f^i_{\bar{t},-s}(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in [\sqrt{s}, 1]$ and $y_j \in [-2, \sqrt{s})$ for some $j > 0$. Then, for every $n \geq j$ with $w_n \in C$, $y_n \in [-2, \sqrt{s})$.*

Consider now $s = 0$.

- *Every point $(x, y, s) \in C$ with $y \in [-2, -1)$ or $y \in (1, 2)$ is wandering.*
- *Consider $w = (x, y, z) \in C \cap \Lambda_{t,0}$ and, for $i \in \mathbb{Z}$, let $w_i = g_0^i(w) = f_{\bar{t},0}^i(w)$. If $w_i \in C$ we let $w_i = (x_i, y_i, z_i)$. Suppose that $y_i \in (0, 1]$ and $y_j \in [-2, 0)$ for some $j > 0$. Then, for every $n \geq j$ with $w_n \in C$, $y_n \in [-2, 0)$.*

For $s \geq 0$ let $\Lambda^+_{\bar{t},s}$ (resp. $\Lambda^-_{\bar{t},s}$) be the set of points $w \in \Lambda_{\bar{t},s} \cap \Omega(f_{\bar{t},s})$ such that $y_i \in [\sqrt{s}, 1]$ (resp. $y_i \in [-1, \sqrt{s}]$) for all $i \in \mathbb{Z}$ with $w_i = (x_i, y_i, z_i) \in C$. Remark 6.9 implies that, for every small $s \geq 0$,

$$\Lambda_{\bar{t},s} \cap \Omega(f_{\bar{t},s}) = \Lambda^+_{\bar{t},s} \cup \Lambda^-_{\bar{t},s}.$$

Arguing as in Section 3, one gets $H(P, f_{\bar{t},s}) = \Lambda^+_{\bar{t},s}$ and $H(Q, f_{\bar{t},s}) = \Lambda^-_{\bar{t},s}$. Observe that, for positive s, one needs to exclude the segment $\{0\} \times (-\sqrt{s}, \sqrt{s}) \times \{0\} \subset \Lambda_{\bar{t},s}$ consisting of wandering points.

Finally, for the saddle-node parameter $s = 0$, it is immediate that $\Lambda^+_{\bar{t},0} \cap \Lambda^-_{\bar{t},0} = \{S\}$.

For parameters $s < 0$ the result follows similarly (but now the situation is much more simple).

6.4 Proof of Theorem B

Clearly, the homoclinic classes $H(P, f_{\bar{t},0})$ and $H(Q, f_{\bar{t},0})$ are not saturated. We claim that there is not any transitive saturated set Σ containing $H(P, f_{\bar{t},0})$. Otherwise, the set Σ must also contain $H(Q, f_{\bar{t},0})$. Thus Σ contains the whole $\Lambda_{\bar{t},0}$. Using the filtration, one has $\Sigma = \Lambda_{\bar{t},0}$. But this set is not transitive: Remark 6.9 implies that there is no orbit going from a small neighborhood U_Q of Q to a small neighborhood U_P of P and thereafter returning to U_Q. This contradiction ends the proof of the theorem.

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