The end of domination

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Let M be a closed manifold and $\mathrm{Diff}^1(M)$ be the space of C^1-diffeomorphisms defined on M endowed with the usual topology. A property of diffeomorphisms is said C^1*-generic* (or shortly, generic) if it holds for diffeomorphisms in a residual subset of $\mathrm{Diff}^1(M)$. In this note we adopt the generic point of view[1]

The *homoclinic class* of a hyperbolic saddle point p of a diffeomorphism f, denoted by $H(p,f)$, is the closure of the transverse intersections of the invariant manifolds of p. On the one hand, every homoclinic class is f-invariant, *transitive* (there is a dense orbit in the whole class) and the periodic points with the same *Morse index* (dimension of the unstable bundle) as p are dense in $H(p,f)$. On the other hand, in general, homoclinic classes fail to be locally maximal and hyperbolic. For instance, there exist nontrivial homoclinic class accumulated by sinks and sources, thus failing to be locally maximal. In fact, there is a dichotomy for C^1-generic homoclinic classes: *either a homoclinic class has some weak hyperbolicity (existence of a* dominated splitting*) or else it is accumulated by infinitely many sinks or sources,* see [5].

One of the aims of this note is to discuss the dynamics of homoclinic classes without *domination.* Let us recall the definition of *dominated splitting.* Consider an f-invariant set Λ. A Df-invariant splitting over Λ, $T_\Lambda M = E \oplus F$, is dominated if the dimension of the fibers $E(x)$ and $F(x)$ are independent of $x \in \Lambda$ and there is $\ell \in \mathbb{N}$ such that

$$||Df^{\ell}|_{E(x)}|| \cdot ||Df^{-\ell}|_{F(f^{\ell}(x))}|| < \frac{1}{2}$$

for every $x \in \Lambda$. This means that the derivative of f expands the vector in F unifromly more than the vectors in E.

An important feature of the nontrivial homoclinic classes $H(p,f)$ without domination is that there is a perturbation of the diffeomorphism f throughout the orbit of some periodic point $x \in H(p,f)$ which is an homothety, see [5, Proposition 2.1]. For the non-dominated homoclinic classes considered in this

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[1] This approach allows us to use appropriate perturbation lemmas: e.g., the Pugh's closing lemma [17], the ergodic closing lemma [14], the Hayashi's connecting lemma [13], and the Franks' lemma [11] on dynamical perturbations of derivatives.

note (the so called *wild homoclinic classes,* see definition below) this homothety is in fact the identity. This property enables us to find (essentially) any prescribed dynamics after perturbing the initial diffeomorphism f.

A second objective comes from an attemp to understand the *elementary pieces of dynamics* of generic diffeomorphisms. In the hyperbolic setting, Smale's decomposition theorem [18] claims that these elementary pieces are the homoclinic classes, which are then basic sets (hyperbolic and locally maximal sets). This is also the case of generic diffeomorphisms of surfaces having finitely many sinks or sources, see [14].

In the nonhyperbolic setting and in higher dimensions, the first problem is to find a good candidate for the role of an elementary piece of dynamics. In the Smale theory, these elementary pieces are *dynamically undecomposable* (this leads to the notion of transitivity), *maximal* (an elementary piece is not contained in any greater one), and *isolated* (two different pieces are dynamically independent).

The notion of transitivity is fundamental in any attempt to define the concept of elementary piece of dynamics. There are two natural strategies for generalizing the notion of basic set. The notions of transitivity and maximality lead to the concept of *maximal transitive set,* and transitivity together with isolation leads to the concept of *robust transitive set* introduced in [8] [2]. On the other hand, from the mere constructive point of view, the homoclinic relation associates to each hyperbolic saddle p (in a natural way) a transitive set containing it: its homoclinic class. Thus there are three natural candidates for the role of elementary piece of dynamics: maximal transitive sets, robustly transitive sets and homoclinic classes. Recently, in [1] is proved that (in any dimension) for generic diffeomorphisms having finitely many homoclinic classes (the so-called *tame* diffeomorphisms) these three approaches essentially coincide [3].

In this note we focus our attention on the notions of maximal transitive sets and homoclinic classes, as well as on diffeomorphisms with infinitely many homoclinic classes (*wild diffeomorphisms*).

First, let us observe that the intersection of two maximal transitive sets may be non-empty. This is an obstruction for considering these sets as elementary

[2] A transitive set Λ is *maximal* if it is a maximal element of the family of all compact transitive sets of f ordered by inclusion. An f-invariant set Λ is robustly transitive for f if there are neighbourhoods U of Λ in M and $\mathcal{N}$ of f in $\mathrm{Diff}^1(M)$ such that, for every $g \in \mathcal{N}$, the set $\Lambda_g = \cap_{i\in\mathbb{Z}} g^i(\bar{U})$ is transitive and $\Lambda_f = \Lambda$.

[3] More precisely, the results in [1] assert that there is a residual subset $\mathcal{T}$ of $\mathrm{Diff}^1(M)$ (the set of *tame diffeomorphisms*) such that, for every $f \in \mathcal{T}$, the number of homoclinic classes is well defined and locally constant in $\mathcal{T}$. Moreover, for a fixed $f \in \mathcal{T}$, there are a neighbourhood $\mathcal{N}$ of f, a *filtration* $M_1 \subset M_2 \subset \cdots \subset M_k$, $f(M_i) \subset \mathrm{int} M_i$, and hyperbolic points $p_1(f), p_2(f), \dots p_k(f)$ whose continuations are defined for all $g \in \mathcal{N}$, such that for each $g \in \mathcal{N} \cap \mathcal{T}$ the maximal invariant set of g in $M_{i+1} \setminus M_i$ is the homoclinic class of $p_{i+1}(g)$. Moreover, $H(p_{i+1}(g))$ admits a dominated splitting.

This result generalizes to higher dimensions the Mañé's generic dichotomy in [14] between hyperbolicity (existence of a spectral decomposition of the limit set) and coexistence of infinitely many sinks or sources for surface diffeomorphisms. Here the dichotomy is between the existence of a spectral decomposition of the limit set whose pieces are weakly hyperbolic and have some topological stability and the coexistence of infinitely many homoclinic classes.

pieces of dynamics. To avoid this problem, one considers the stronger notion of *saturated transitive set,* defined as a transitive set containing any transitive set intersecting it. We observe that there are maximal transitive sets which are not contained in any saturated one, see [10]. The link between homoclinic classes and maximal or saturated transitive sets is stated in recent results by Arnaud and Carballo-Morales-Pacifico:

Theorem 1 (Arnaud, [2]) *Generically in* $\mathrm{Diff}^1(M)$*, every transitive set containing a periodic orbit* p *is contained in the homoclinic class of* p*. In particular, generically, homoclinic classes are maximal transitive sets.*

Theorem 2 (Carballo, Morales, Pacifico, [7]) *Generically in* $\mathrm{Diff}^1(M)$*, the homoclinic classes are saturated transitive sets. In particular, generically, any two homoclinic classes are either equal or disjoint.*

These two results lead naturally to the question of the *existence of saturated transitive sets which are not homoclinic classes with some persistence.* We give a positive answer to this question, showing the existence of persistent saturated transitive sets without periodic orbits. We show that the appearance of such sets is connected with the existence of homoclinic classes which do not admit any dominated splitting.

1 Non-dominated homoclinic classes

In this section we construct homoclinic classes which do not admit any dominated splitting exhibiting pathological dynamics. For simplicity, from now on, we will restrict our attention to diffeomorphisms of 3 manifolds.

Consider a diffeomorphism f having two saddles p and q of Morse indices 1 and 2 involved in a *heterodimensional cycle.* This means that the two dimensional stable manifold of p and unstable manifold of q meet transversely and the one dimensional unstable manifold of p and unstable manifold of q meet quasi-transversely along the orbit of a point x (i.e., $T_xW^u(p,f)+T_xW^s(q,f) = T_xW^u(p,f) \oplus T_xW^s(q,f)$). Under some assumptions on the geometry of the transverse intersection $W^s(p,f) \pitchfork W^u(q,f)$ (including the assumption that the eigenvalues of the derivatives of f at p and q are all real and different in modulus) one gets an open set $\mathcal{W}$ of $\mathrm{Diff}^1(M)$ whose closure contains f such that the homoclinic classes of p and q are equal for every diffeomorphism g nearby f, see [9]. The proof of this result only involves orbits in a neighbourhood of the cycle (the orbits of the transverse intersection $W^s(p,f) \pitchfork W^u(q,f)$ and of the quasi-transverse intersection $W^u(p,f) \cap W^s(q,f)$).

The next step consists in considering points p' and q' homoclinically related to p and q, respectively, such that p' has a non-real contracting eigenvalue and q' has a non-real expanding eigenvalue. Then, by shrinking the open set $\mathcal{W}$ above, one has that, for every $g \in \mathcal{W}$, the homoclinic classes of p' and q' are equal.

We claim that the transitive set $\Lambda_g = H(p', g) = H(q', g)$ does not admit any dominated splitting. Otherwise, $T_{\Lambda_g}(M) = E \oplus F$, where F dominates E. Suppose, for instance, that E has dimension 2. Then $E(p') = E^s(p')$ and $E(q') \cap E^u(q')$ is an one dimensional g-invariant bundle, which contradicts the fact that the derivative of g at q has non-real expanding eigenvalues (which in turn implies that $E^u(q')$ does not contain nontrivial invariant sub-bundles).

Theorem 3 (Bonatti, D., [4]) *Let $\mathcal{W}$ be an open set as above. Then there is a residual subset $\mathcal{R}$ (in the C^1-topology) of $\mathcal{W}$ of diffeomorphisms g such that $H(p', g) = H(q', g)$ is contained in the closure of the (infinite) set consisting of sinks and sources of g.*

Moreover, if the jacobian of p is greater than one and the jacobian of q is less than one, then $H(p', g) = H(q', g)$ is simultaneously contained in the closure of the set of sinks and in the closure of the set of sources for every $g \in \mathcal{R}$.

This result shows the occurrence of the Newhouse phenomenon (coexistence of infinitely many sinks or sources) in the C^1-topology in dimension bigger than or equal to three. Recall that for surface diffeomorphism this phenomenon is associated to the unfolding of homoclinic tangencies of C^2-diffeomorphisms, [15, 16]. Theorem 3 shows how the Newhouse phenomenon is related with the failure of domination of a nontrivial homoclinic class. In fact, the previous result illustrates a much more general principle:

Theorem 4 (Bonatti, D., Pujals [5]) *There is a residual subset of diffeomorphisms $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ such that, for every diffeomorphism $f \in \mathcal{R}$ and every saddle p of f, the homoclinic class $H(p, f)$ verifies one of the following properties:*

- *either $H(p, f)$ is contained in the closure of infinitely many sources or sinks,*
- *or $H(p, f)$ admits a dominated splitting.*

If the dimension of the ambient manifold is two, the dominated splittings in the theorem are hyperbolic, and if the dimension of the ambient manifold is three the splittings are (at least) *partially hyperbolic* (at least one bundle of the dominated splitting is hyperbolic). In higher dimensions, there are generic homoclinic classes having a dominated splitting without any hyperbolic direction, see [6].

Our objective now is to describe another stronger form of the Newhouse phenomenon in homoclinic classes that do not admit any dominated splitting.

2 Wild dynamics

The construction in [4] motivates the following definition. A homoclinic class Λ_f of a diffeomorphism f (defined on a 3-manifold) is *wild* if it contains saddles p_f, q_f and r_f and there is a neighbourhood $\mathcal{V}$ of f where the continuations p_g, q_g and r_g of p_f, q_f and r_f are defined such that, for every $g \in \mathcal{V}$:

1. the saddles p_g and r_g have Morse index one and are homoclinically related, the saddle p_g has a pair of contracting non-real eigenvalues, and the saddle q_g has Morse index two and a pair of expanding non-real eigenvalues,

2. $q_g \in H(p_g, g)$,

3. $\mathrm{Jac}_g \mathrm{per}_{(p_g)}(p_g) > 1 > \mathrm{Jac}_g \mathrm{per}_{(r_g)}(r_g)$, where $\mathrm{Jac}_g(r)$ denotes the Jacobian of g at r.

Denote by $\mathcal{W}(M)$ the set of diffeomorphisms of $\mathrm{Diff}^1(M)$ having a wild homoclinic class. By definition, $\mathcal{W}(M)$ is open in $\mathrm{Diff}^1(M)$. Theorem 3 guarantees the existence of wild homoclinic classes.

Before stating the next theorem, let us recall some definitions. An f-invariant set Σ is *minimal* if the orbit of any point of Σ is dense in Σ (i.e., Σ has no proper f-invariant sets). Observe that every infinite minimal set Σ does not contain periodic orbits. An f-invariant set Σ is *uniquely ergodic* if it supports only one f-invariant probability measure (which is necessarily ergodic). Observe that there are minimal sets where the restriction of f is not uniquely ergodic (see [12]). Finally, a compact set Σ is *Lyapunov stable* for f if for every neighbourhood U of Σ there is a neighbourhood V of Σ such that $f^i(V) \subset U$ for all $i \geq 0$. Observe that every Lyapunov stable transitive set Λ is saturated: any transitive set intersecting Λ is contained in any neighbourhood of Λ (so it is contained in Λ).

Next theorem gives a sample of some pathological dynamics associated to the appearance of a wild homoclinic class:

Theorem 5 (Bonatti, D., [3]) *There is a residual subset $\mathcal{R}(M)$ of the set $\mathcal{W}(M)$ of diffeomorphisms with wild homoclinic classes consisting of diffeomorphisms f having simultaneously infinitely many:*

a) *Sinks and sources.*

b) *Saddles with* trivial homoclinic classes *(i.e., the classes consist only of the periodic saddle itself).*

c) *Non-trivial uniformly hyperbolic attractors and repellers.*

d) *Non-trivial partially hyperbolic attractors and repellers.*

e) *Non-trivial (non-hyperbolic) homoclinic classes Λ_i containing points of different indices. Moreover, infinitely many of these homoclinic classes are wild ones.*

f) *Maximal transitive Cantor set $\mathcal{F}_i$ where f is minimal. Moreover, the sets $\mathcal{F}_i$ are simultaneously Lyapunov stable for f and f^{-1} and uniquely ergodic.*

Moreover, the wild homoclinic classes Λ_i and the minimal transitive sets $\mathcal{F}_i$ can be chosen such that,

g) *each wild homoclinic class* Λ_i *is contained in the closure of the union of infinitely many maximal transitive sets* $\mathcal{F}_j$ *with minimal dynamics,*

h) *each wild maximal transitive sets* $\mathcal{F}_i$ *with minimal dynamics is contained in the closure of the union of infinitely many wild homoclinic classes* Λ_j.

The main naive idea in the proof of this theorem is that, after perturbing the dynamics, one gets an open set where the dynamics (up to the period) is the identity. This allows us, after a new perturbation, to get new wild homoclinic classes. Finally, the existence of such a new wild homoclinic class allows us to repeat the previous procedure infinitely many times. The main technical step of the proof of the theorem is the following:

Proposition 2.1 *Let* p_f *be a point of Morse index one such that, for every* g *in a neighbourhood* $\mathcal{N}$ *of* f, *the homoclinic class* $H(p_g, g)$ *is wild. Then, for every* $g \in \mathcal{N}$ *and every* $\varepsilon > 0$, *there are a saddle* x_g *homoclinically related to* p_g *whose orbit is* ε*-dense in* $H(p_g, g)$ *and a neighbourhood* V_g *of* x_g *such that*

- $V_g, g(V_g), \dots, g^{n-1}(V_g)$ *are pairwise disjoint, where* n *is the period of* x_g,
- *the derivative of* g^n *at* x_g *is* ε*-close to the identity.*

This proposition is a fundamental technical step in the proofs in [5]. Using this result one perturbs the derivative of g^n at x_g to get the identity. A lemma by Franks in [11] enables us to perform dynamically such a perturbation of the derivative, getting a diffeomorphism h arbitrarily close to g whose restriction to V is the identity.

Let us briefly explain how, using the previous result, one constructs a transitive set with minimal dynamics. The next step is, after perturbation, to get a new diffeomorphism f_1 and disk $D_1 \subset V_g$ such that D_1, $f_1(D_1), \dots, f_1^{n_1-1}(D_1)$ are pairwise disjoint, $f_1^{n_1}(D_1)$ is contained in the interior of D_1, and $f_1^{n_1}$ has a wild homoclinic class in D_1. Arguing inductively, one constructs a nested sequence of open sets $\mathcal{U}_k$ and of periodic disks D_k of period n_k, $n_k \to \infty$ and $n_{k+1} = n_k\, m_k$, $m_k > 1$, such that

- $\mathcal{U}_{k+1}$ is dense in $\mathcal{U}_k$,
- for every $f \in \mathcal{U}_k$, D_k, $f(D_k), \dots, f^{n_k-1}(D_k)$ are pairwise disjoint, $f^{n_k}(D_k)$ is contained in the interior of D_k, and f^{n_k} has a wild homoclinic class in D_k,
- the diameters of the disks D_k go to zero as $k \to \infty$.

It is now immediate that, for each f in the locally residual set $\mathcal{R} = \cap_{k\in\mathbb{N}} \mathcal{U}_k$, the set

$$\Upsilon_f = \bigcap_{k\in\mathbb{N}} (\cup_{i=0}^{n_k-1} f^i(D_k))$$

is a minimal Cantor set which is Lyapunov stable.

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