How do hyperbolic homoclinic classes collide at heterodimensional cycles?

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Abstract

We present a model illustrating heterodimensional cycles (i.e., cycles associated to saddles having different indices) as a mechanism leading to the *collision* of hyperbolic homoclinic classes (of points of different indices) and thereafter to the *persistence of (heterodimensional) cycles.* The collisions are associated to secondary (saddle-node) bifurcations appearing in the unfolding of the initial cycle.

1 Introduction

1.1 Context

We study partially hyperbolic dynamics obtained when uniformly hyperbolic diffeomorphisms loss their stability by the creation of *heterodimensional cycles,* that is, cycles associated to hyperbolic saddles having different *indices* (dimension of the unstable bundle). We analyze the phenomena of collision of hyperbolic homoclinic classes and persistence of cycles at such bifurcations.

We consider one-parameter families of diffeomorphisms $(f_t)_{t\in[-1,1]}$ starting in a hyperbolic region (Morse-Smale systems) and crossing its boundary at a bifurcation (say $t = 0$) corresponding to a cycle associated to a pair of saddles having different indices (i.e., a heterodimensional cycle). The goal is to describe the dynamical features of the diffeomorphisms f_t having some persistence or prevalence (in terms of the parameter t) immediately after crossing the boundary of the hyperbolicity. Clearly, the resulting dynamics depends on the type of bifurcation responsible for the breakdown of the hyperbolicity (here a heterodimensional cycle). Typically, one considers codimension one bifurcations: the bifurcating diffeomorphism f_0 belongs to a (local) submanifold Σ which is contained in the boundary of the hyperbolic systems and has codimension one in the space of diffeomorphisms. Moreover, the arc $(f_t)_{t\in[-1,1]}$ transversely intersects the submanifold Σ at f_0.

Newhouse-Palis conjectured in [NP$_2$] that the generic way of crossing the boundary of a hyperbolic region is via one of the following two types of bifurcations: (i) loss of hyperbolicity of some periodic orbit (saddle-node, period doubling, and Hopf bifurcations), and (ii) loss of transversality of the intersection between the invariant manifolds of hyperbolic periodic points (creation of cycles). This conjecture holds for surface diffeomorphisms (in this case, the cycle is due to a

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homoclinic or heteroclinic tangency) in the boundary of the Morse-Smale systems, [NP_1]. The first sort of bifurcation is local (the dynamics on a finite orbit) while the second one is semi-global and involves the whole orbit of a non-periodic point (a non-transverse intersection between invariant manifolds). An interesting dynamical feature rises when there is an interplay between these two types of bifurcations. For instance, generic bifurcations via homoclinic tangencies lead to the creation of infinitely many saddle-node, period doubling and Hopf bifurcations immediately after their unfolding, see [YA] and [PT_3, Chapter 3]. For the converse, in general, some semi-global hypotheses on the dynamics are necessary (this is the case, for instance, of the saddle-node cycles, [NPT]). This paper illustrates such an interplay between local (saddle-node) and semi-global (creation of cycles) bifurcations in the context of heterodimensional cycles.

We now focus on the partially hyperbolic (with one-dimensional central direction) case and say a few words about the dynamics after the bifurcation. We first note that in our setting after unfolding the cycle (and for all small $t > 0$) new transverse homoclinic points of P and Q (we omit the dependence of the saddles on the parameter t) are created. Moreover, there are no homoclinic tangencies and all the cycles created throughout the unfolding are heterodimensional. In fact, in the unfolding process infinitely many heterodimensional cycles are created and the parameters corresponding to cycles accumulate the first bifurcation $t = 0$. Finally, by the Smale's homoclinic theorem, there are transitive hyperbolic Cantor sets Λ_P and Λ_Q, containing P and Q and contained in the homoclinic classes of P and Q. In particular, the sets Λ_P and Λ_Q have different indices (dimension of the unstable bundle). In some cases the homoclinic classes of P and Q are both nonhyperbolic and contain properly the sets Λ_P and Λ_Q. But there are examples of arcs $(f_t)_{t\in[-1,1]}$ such that there are parameter intervals corresponding to hyperbolic diffeomorphisms f_t for which these two classes are simultaneously hyperbolic. A goal of this paper is to describe how these diffeomorphisms can bifurcate. So three crucial steps in this context are to determine whether:

(i) the homoclinic classes of P and Q are hyperbolic (so they can be taken equal to Λ_P and Λ_Q);

(ii) the resulting non-wandering set of f_t has elements disjoint from these homoclinic classes; and

(iii) the diffeomorphism f_t exhibits a secondary cycle, associated either to saddles in the homoclinic classes of P and Q or to new saddles, disjoint from the homoclinic classes of P and Q, created throughout the unfolding.

Heuristically, several previous results (see [$\mathrm{Dí}_1$, $\mathrm{Dí}_2$, DRo_1, DRo_3]) suggest that the behavior after the bifurcation is essentially determined by the dynamics of the bifurcating diffeomorphism f_0 in the central direction (we will precise the meaning of this sentence later on). Taking bifurcating diffeomorphisms f_0 with an appropriate central dynamics one gets arcs $(f_t)_{t\in[-1,1]}$ as above such that there are parameter intervals H_n, $H_n \to 0^+$, such that f_t is hyperbolic for all $t \in H_n$. In particular, for every $t \in H_n$ the homoclinic classes of P and Q are hyperbolic and disjoint, see [DRo_1]. But, for the same arc $(f_t)_{t\in[-1,1]}$, there are also other parameter intervals B_n (corresponding to nonhyperbolic diffeomorphisms) for which the homoclinic classes of P and Q coincide, see [DRo_2]. So a natural problem is to understand the transition from hyperbolic to nonhyperbolic dynamics (i.e., the secondary bifurcations of $(f_t)_{t\in[-1,1]}$). This is one of the goals of this article.

We construct arcs $(f_t)_{t\in[-1,1]}$ as before such that there is a sequence of secondary bifurcations t_n, $t_n \to 0^+$, such that for each t_n the hyperbolic homoclinic classes of P and Q collide via a saddle-node bifurcation. This situation exemplifies the interplay between semi-global (heterodimensional cycles) and local (saddle-node) bifurcations, where the (local) saddle-node bifurcations bring substantial global changes of the dynamics. For instance, we see that these collisions lead to *persistence of cycles* and *intermingled homoclinic classes* immediately after each saddle-node.

The phenomenon of persistence of (heterodimensional) cycles here is, in some sense, similar to the one of persistence of tangencies in the setting of homoclinic tangencies, [Ne_1, Ne_2, Mo], and critical saddle-node horseshoes, [DU]: it is also associated to the creation of thick Cantor sets (see Definition 6.1 and the Gap Lemma 7.1, and for an overview [PT_3, Chapter 6.1]). But the underlying dynamical reasons of the creation of these thick Cantor sets are rather different. In the case of homoclinic tangencies, it is related to the critical behavior of the dynamics (proximity to the quadratic family after a renormalization), while in our context the dynamics after the bifurcation is partially hyperbolic and non-critical: the invariant manifolds of the saddles in the cycle do not have folding points and the angles between these manifolds in the intersection points are uniformly bounded away from zero. Thus the mechanism leading to arbitrarily thick hyperbolic Cantor sets in this paper is new. Roughly, the thick Cantor sets are dynamically defined (in the sense in [PT_3, Chapter 4.1]), associated to a system of (infinitely many) iterated functions (whose generators are given by the central dynamics) without criticalities but having discontinuities.

1.2 Homoclinic classes and heterodimensional cycles

1.2.1 Homoclinic classes and their collisions

The *homoclinic class* of a (hyperbolic) saddle P of a diffeomorphism f, denoted by $H(P,f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of P. Homoclinic classes are f-invariant, *transitive* (i.e., there exists a dense orbit), and the subset of the periodic points with the same index of P is dense in $H(P,f)$. An important property of homoclinic classes is their lower semi-continuous dependence on the diffeomorphisms, this property follows from the continuity of the invariant manifolds of hyperbolic saddles and of their transverse intersections on compact parts. Homoclinic classes may fail to be hyperbolic: for example, a homoclinic class may contain saddles of different indices. In fact, this sort of phenomenon typically occurs in the contexts of heterodimensional cycles, see [$\mathrm{Dí}_1$, $\mathrm{Dí}_2$, DRo_2], and (nonhyperbolic) robustly transitive sets and diffeomorphisms, see [BD, BDPR].

Homoclinic classes are natural elementary pieces of dynamics (as in the Axiom A and tame cases, see [Sm, Ab]) playing a key role in the structure of the dynamics. Recently, [Ar, CMP] proved that, for diffeomorphisms f in a residual set of the space of C^1-diffeomorphisms, $\mathrm{Diff}^1(M)$, any two homoclinic classes of f are either disjoint or coincide. Thus, by the C^1 closing lemma [Pu], the homoclinic classes generically define a partition of (a dense part of) the non-wandering set.

Motivated by these results, one aims to understand how homoclinic classes may change from the metric point of view, that is, to understand the discontinuities of the map $(P_f,f)\to H(P_f,f)$, where P_f is a hyperbolic saddle depending continuously on f. By a *(metric) bifurcation of a homoclinic class* $H(P_f,f)$ we mean a discontinuity of the previous map. A well studied way in which homoclinic classes bifurcate is via the creation of cycles. We study (simultaneous) bifurcations of a pair of homoclinic classes via their *collision* and see how this sort of bifurcation can appear related to the unfolding of heterodimensional cycles.

Given a one-parameter family of diffeomorphisms $(f_t)_{t\in[a,c]}$, by a *collision of a pair of homoclinic classes,* say $H(P,f_t)$ and $H(Q,f_t)$, at $t=b$, we mean that $H(P,f_t)\cap H(Q,f_t)=\emptyset$ for every $t\in[a,b)$, and $H(P,f_t)\cap H(Q,f_t)\neq\emptyset$ for $t=b$. In the case studied here, the classes are both nontrivial and hyperbolic before the collision, and they are different for the collision parameter $t=b$. For $t=b$, the intersection of the classes is the orbit of a saddle-node, which is the responsible of the loss of hyperbolicity of the classes. This collision leads to the (simultaneous) explosions of the colliding homoclinic classes as well as to a string of bifurcations (persistence of heterodimensional cycles).

The simplest case of collision of homoclinic classes occurs when two isolated saddles of different indices collide via a saddle-node bifurcation. A much more interesting dynamics rises when the saddle-nodes are non-isolated. In dimension two, the main examples are the *Derived from Anosov (DA)* bifurcations, [Sm], where a repeller and a non-trivial hyperbolic attractor (the homoclinic class of a saddle) collide, and the *saddle-node horseshoes,* [Ze, Ta, Cr], where a horseshoe (the homoclinic class of a saddle) and a sink collide. The DA bifurcations are isolated. Saddle-node horseshoe bifurcations may be isolated (as in [CK]) or lead to a string of bifurcations (including Hénon-like attractors, persistence of tangencies, and variation of entropy, as in [DRoV, Co, DU, DRi]), depending on semi-global aspects of the dynamics. For a survey on this subject see [DRiV].

In the collisions in the previous paragraph, one of the classes is trivial (a sink or a repeller). In our construction the two colliding homoclinic classes are hyperbolic (before the collision) and non-trivial (for that we need to consider an ambient manifold of dimension 3 or higher). Collisions of two non-trivial and nonhyperbolic homoclinic classes were studied in [DS], where the collision leads to (using the terminology in [DS]) the *collapse* of the classes in the sequel of the bifurcation: the two classes explode and each class absorbs the other one, that is, they coincide.

1.2.2 Dynamics at heterodimensional cycles

The *index of a hyperbolic transitive set* Λ of a diffeomorphism f is the dimension of its unstable bundle. A diffeomorphism f has a heterodimensional cycle if there are (transitive) hyperbolic sets Λ and Σ having different indices such that

$$W^s(\Lambda) \cap W^u(\Sigma) \neq \emptyset \neq W^u(\Lambda) \cap W^s(\Sigma).$$

This sort of cycles only can occur in dimension bigger than or equal to three. We consider the simplest case (in fact, codimension-one case): there are saddles $P \in \Lambda$ and $Q \in \Sigma$ such that

- $\mathrm{index}(Q) = \mathrm{index}(P) + 1$;
- $W^u(Q) \cap W^s(P)$ is non-empty and transverse; and
- $W^s(Q) \cap W^u(P)$ is non-empty and quasi-transverse (i.e., for every heteroclinic point $x \in W^s(Q) \cap W^u(P)$ it holds $T_x W^s(Q) \oplus T_x W^u(P) = T_x W^s(Q) + T_x W^u(P)$ and the sum has codimension one in $T_x M$, where M is the ambient manifold).

In many cases, including bifurcations from Morse-Smale systems, heterodimensional cycles lead to persistent nonhyperbolic dynamics. Given an arc $(f_t)_{t\in[-1,1]}$ unfolding a cycle, say at $t=0$ and for positive t, let $H(t) = \{s \in [0,t] \colon f_t \text{ is hyperbolic}\}$ and $B(t) = \{s \in [0,t] \colon H(P, f_s) = H(Q, f_s)\}$. For $t \in B(t)$, we say that the homoclinic classes of P and Q are *intermingled.* There is an open class of families of $(f_t)_{t\in[-1,1]}$ such that $B(t) = (0,t]$ for all small t, see [Dí$_1$, Dí$_2$]. There are also cases where $B(t) \neq (0,t]$, but $B(t)$ contains intervals and verifies $\liminf |B(t)|/t > 0$, ($|\cdot|$ denotes the measure), see [DRo$_2$]. In these cases we say (following [PT$_3$, Chapter 7]) that the dynamics is *robustly* and *prevalently nonhyperbolic* after the bifurcation, respectively.

In some cases both sets $H(t)$ and $B(t)$ contain intervals and satisfy $\liminf |B(t)|/t > 0$ and $\liminf |H(t)|/t > 0$ (using the terminology above hyperbolicity and nonhyperbolicity are both prevalent). Moreover, the arc can be constructed with *density of hyperbolicity arbitrarily close to* 1 *at the bifurcation*: given any $\varepsilon > 0$, there is an open set of arcs (as above) such that $\liminf |H(t)|/t > 1 - \varepsilon$. Following [$\mathrm{PT}_3$] we say that the phenomena of nonhyperbolic dynamics (intermingled classes) and hyperbolic dynamics are both *not totally prevalent* after the bifurcation.

The hyperbolic parameters $t \in H(t)$ in the construction in [DRo_1] are such that the exploding non-wandering set of f_t is the (disjoint) union of the homoclinic classes of P and Q. Thus a natural question is to describe how these hyperbolic diffeomorphisms bifurcate. Two possibilities are the creation of new heterodimensional cycles and the loss of hyperbolicity of a saddle (for instance, and typically, via a saddle-node bifurcation). We construct examples where the unfolding of the cycle leads to a sequence of collisions of (non-trivial) hyperbolic homoclinic classes via saddle-node bifurcations: There is sequence of parameter intervals $(h_n^-, h_n^+) \subset H(t)$, $h_n^{\pm} \to 0^+$, such that each h_n^- is a first (secondary) bifurcation such that there is

1. a collision of the hyperbolic homoclinic classes of P and Q at h_n^-;
2. persistence of heterodimensional cycles in an interval (c_n, h_n^-).

By *persistence of heterodimensional cycles* in a parameter interval (t_0, t_1) we mean that there are two hyperbolic (transitive) sets of different indices Λ_t and Σ_t of f_t (depending continuously on t) such that, for any $t \in (t_0, t_1)$, the diffeomorphism f_t has a cycle associated to Λ_t and Σ_t.

Our construction illustrates the importance of secondary bifurcations in the unfolding of cycles. Such bifurcations also play a key role in the homoclinic setting, think for instance of the secondary tangencies associated to *thick hyperbolic sets,* responsible for the persistence of tangencies and the coexistence of infinitely many sinks, see [Ne_2, Ne_3, PT_3].

1.3 Main result and ingredients of the proof

Let f be a diffeomorphism with a cycle associated to hyperbolic saddles P and Q. A neighbourhood of the cycle is an open set W containing the elements of the cycle: the orbits of the saddles P and Q and the intersections $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$.

Given an arc $(f_t)_{t\in[-1,1]}$ unfolding a cycle at $t = 0$ (for positive t) and a neighbourhood W of the cycle, the *relative nonwandering set* of f_t in W is the intersection of the nonwandering set $\Omega(f_t)$ of f_t and the maximal invariant set $\Lambda_t(W) = \cap_{\mathbb{Z}} f_t^i(W)$ of f_t in W.

Theorem 1. *There are an arc $(f_t)_{t\in[-1,1]}$ unfolding a heterodimensional cycle at $t = 0$ (for positive t) associated to saddles P and Q, of indices r and $r + 1$, a neighbourhood W of the cycle, $n_0 \in \mathbb{N}$, and a sequence of scaled parameter intervals C_n and H_n, $n \geq n_0$,*

$$C_n = [a_n, \bar{t}_n) = \lambda^n \,[1 + \mu^-, 1 + \mu^\star) \quad \textit{and} \quad H_n = [\bar{t}_n, b_n] = \lambda^n \,[1 + \mu^\star, 1 + \mu^+],$$

for some $0 < \lambda < 1$ and $\mu^- < \mu^\star < \mu^+$, such that:

(1) Hyperbolicity and secondary collision bifurcations: *for every $t \in H_n$, the relative nonwandering set of f_t in the neighbourhood W of the cycle is the union of the nontrivial homoclinic classes $H(P, f_t)$ and $H(Q, f_t)$ of P and Q. Moreover,*

(1a) prevalent hyperbolicity: *for every $t \in (H_n \setminus \{\bar{t}_n\})$, $H(P, f_t)$ and $H(Q, f_t)$ are disjoint and hyperbolic;*

(1b) collisions of homoclinic classes via saddle-node bifurcations: *for every $\bar{t}_n$, $H(P, f_{\bar{t}_n}) \cap H(Q, f_{\bar{t}_n})$ is the orbit of a saddle-node. Moreover, every compact invariant set contained in each homoclinic class disjoint from the saddle-node is hyperbolic.*

(2) Persistence of cycles: *the sets C_n are intervals of persistence of heterodimensional cycles. More precisely, for every $t \in C_n$, the diffeomorphism f_t has a cycle associated to a hyperbolic set of index r (contained in the homoclinic class of P) and a hyperbolic set of index $r+1$.*

A key ingredient in our construction is the one-dimensional limit dynamics associated to (partially hyperbolic) heterodimensional cycles introduced in [DRo$_3$]. We will detail this construction (also used in a particular case in [DS]) in Section 3.2. This limit dynamics plays a role similar to the quadratic family in the context of homoclinic tangencies (see, for instance, [PT$_3$, Chapter 6]) or to the limit dynamics associated to saddle-node cycles (see [NPT, DRoV]), for describing the dynamics after the bifurcation. This limit dynamics is non-critical, has discontinuities, and is given by a one-parameter family of systems of iterated functions generated by:

(i) two affine maps, corresponding to the linearization of the dynamics in the weak unstable direction of Q and the weak stable direction of P,

(ii) a transition map given by the iterates from a (scaled) fundamental domain of Q to a (scaled) fundamental domain of P,

(iii) a translation corresponding to the rescaling of the parameter t unfolding the cycle.

The series of works [PT$_1$, PT$_2$, Mo, PY, MPV] show that the frequency of hyperbolicity of the dynamics after unfolding a homoclinic tangency is mainly determined by the fractal dimension of the hyperbolic set in the cycle (limit capacity and Hausdorff dimension). For heterodimensional cycles, the role of fractal dimensions is quite enigmatic and until now unexplored. In fact, the dynamics after the bifurcation is essentially determined by the limit dynamics above. Here we describe the creation of *thick one dimensional Cantor sets* (as in [Ne$_2$, Ne$_3$] the sets are dynamically defined but here they are defined by non-critical maps) associated to collisions of hyperbolic homoclinic classes and study how they lead to persistence of heterodimensional cycles.

Let us say a few words about our construction. Denote by $(\varphi_\mu)_{\mu \in [-\delta,\delta]}$ the (non-critical) limit family. As in the case of saddle-node cycles, there are sequences $t_n^\pm$ of parameters, $t_n^\pm \to 0$, and reparametrizations $\varrho_n \colon [-\delta, \delta] \to [t_n^-, t_n^+]$ with nice distortion properties such that the dynamics of $f_{\varrho_n(\mu)}$ is the product of φ_μ (the central part) by a hyperbolic saddle dynamics. Every system φ_μ has two hyperbolic sets Λ'_μ and Σ'_μ (depending continuously on μ) with non-empty intersection. Such an intersection is assured by a condition on the product of their thickness (see the Gap Lemma 7.1). The sets Λ'_μ and Σ'_μ are the projections along the stable and unstable directions of two hyperbolic sets $\Lambda_{\varrho_n(\mu)}$ and $\Sigma_{\varrho_n(\mu)}$ of $f_{\varrho_n(\mu)}$ having different indices. In this way, the intersections between Λ'_μ and Σ'_μ correspond to (heterodimensional) cycles associated to $\Lambda_{\varrho_n(\mu)}$ and $\Sigma_{\varrho_n(\mu)}$.

Some comments about our constructions arise naturally. First, by [Ar, CMP], the collision of homoclinic classes is a non-generic phenomenon (at least in the C^1-topology) and we do not know the flexibility of our construction, roughly a product dynamics where we fix a one-dimensional dynamics and construct a cycle whose central limit dynamics is the fixed one. For instance, we do not know whether our result holds for arcs of diffeomorphisms in a neighbourhood of the arcs constructed in this paper. We point out that the product structure of the example allows us to estimate the thickness of Cantor sets created in the unfolding of the cycle, concluding the persistence of cycles thereafter. But, in general, estimates of the thickness in dimension bigger than two are highly complicated and depend on the regularity of the stable and unstable foliations, see [PV]. Probably, the ideas sketched in [MPV] for studying homoclinic tangencies in higher dimensions could be an ingredient for extending our results to non-product dynamics.

Finally, related to our results, in the context of vector fields of three manifolds, in [BMP] is constructed a vector field with two non-trivial homoclinic classes whose intersection is the closure of the unstable manifold of a hyperbolic singularity. See also [MP], where it is proved that a singular hyperbolic set (see [MPP] for the precise definition) containing a unique singularity is either transitive or the union of two homoclinic classes with non-empty intersection.

This paper is organized as follows. In Section 2, we describe a model arc unfolding a heterodimensional cycle. In Section 3, we introduce a one-parameter system of iterated (one-dimensional) functions describing the central dynamics in the sequel of the bifurcation (limit dynamics). After introducing the basic terminology, we give a sketch of the proof of the theorem in Section 3.3. In Section 4, we determine the central dynamics by considering saddle-node arcs. In Section 5, we construct two isolated hyperbolic sets (of different indices) and describe their collision via a saddle-node bifurcation. In Section 6, we construct two one-dimensional dynamically defined hyperbolic Cantor sets (contracting and expanding), associated to the system of iterated functions, the expanding one with large thickness. Finally, we see that these Cantor sets correspond to two hyperbolic sets of diffeomorphisms of the initial arc (see Sections 6.1 and 6.3) and prove, in Section 7, that these hyperbolic sets have heterodimensional cycles.

2 The model one-parameter family

For notational simplicity, we will carry our constructions in dimension 3, the extension to higher dimensions is straightforward. We consider a one-parameter family of (local) diffeomorphisms $(f_t)_{t\in[-\tau,\tau]}$ defined on the cube $R = [-1,1] \times [-1,5] \times [1,1]$ of $\mathbb{R}^3$. Using local coordinates, this construction can be carried to any manifold of dimension three.

The definition of the arc is done in three steps: (a) semi-local partially hyperbolic dynamics, (b) existence and unfolding of the cycle, and (c) global hypotheses: existence of a filtration and invariant foliations. The filtration will enable us to control the dynamics of the diffeomorphisms f_t after the bifurcation and the invariant foliations will allow us to reduce the study of the dynamics to a one-dimensional family of iterated functions.

The heterodimensional cycle is related to hyperbolic fixed saddles $Q = (0,0,0)$ and $P = (0,4,0)$ of indices 2 and 1, respectively, and the semi-local dynamics satisfies the following properties.

(a) Partially hyperbolic product dynamics. In the cube $R = [-1,1] \times [-1,5] \times [-1,1]$,for every $t \in [-\tau,\tau]$ it holds $f_t = f$, where

$$f(x,y,z) = (\lambda_s\, x, F(y), \lambda_u\, z), \qquad 0 < \lambda_s < 1 < \lambda_u,$$

where F is a strictly increasing function to be fixed in Section 4 such that:

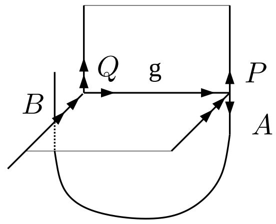
local affinity: $F(y) = \beta\, y$, if $y \in [-1, \beta^{-1}]$, and $F(y) = \lambda\,(y-4)+4$, if $y \in [3,5]$, where $0 < \lambda < 1$ and $\beta = \lambda^{-1}$,

domination: $\lambda_s < F'(y) < \lambda_u$, for every $y \in [-1, 5]$.

The last condition implies that $\lambda_s < \lambda < \beta < \lambda_u$ and that the restriction of f to R has a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ with three non-trivial bundles given by the coordinate directions. Observe also that, by definition of the diffeomorphism f,

$$\begin{array}{lcl} \{0\} \times [-1,4) \times [-1,1] \subset W^u(Q) & \text{and} & [-1,1] \times \{0\} \times \{0\} \subset W^s(Q), \\ \{0\} \times \{4\} \times [-1,1] \subset W^u(P) & \text{and} & [-1,1] \times (0,5] \times \{0\} \subset W^s(P). \end{array}$$

Thus the invariant manifolds $W^s(P)$ and $W^u(Q)$ meet transversely along the curve $\gamma = \{0\} \times (0,4) \times \{0\}$, named *connection.*



$t > 0$

Figure 1: The model cycle and its unfolding

(b) Existence and unfolding of the heterodimensional cycle. Take the points $A = (0, 4, -\frac{1}{2}) \in W^u(P)$ and $B = (-\frac{1}{2}, 0, 0) \in W^s(Q)$. We assume that there is $k_0 \in \mathbb{N}$ such that $f^{k_0}(A) = B$ and $f^j(A) \notin R$ for all $0 < j < k_0$. Thus $W^u(P) \cap W^s(Q) \neq \emptyset$ (i.e., there is a heterodimensional cycle associated to P and Q). Take a small compact neighbourhood U_A of A such that $U_A, \ldots, f^{k_0}(U_A)$ are pairwise disjoint, $U_A, f^{k_0}(U_A) \subset R$, and $f^j(U_A) \cap R = \emptyset$, for all $0 < j < k_0$. We assume that the dynamics of f^{k_0} from U_A to $f^{k_0}(U_A)$ is of the form

affine return: $f^{k_0}(x, y, z) = (x + \frac{1}{2}, y - 4, z + \frac{1}{2})$, for every $(x, y, z) \in U_A$.

This implies that, for some $\delta > 0$, $\{(-1/2, 0)\} \times (-\delta, \delta) \subset W^u(P)$, thus $W^u(P)$ and $W^s(Q)$ meet *quasi-transversely* throughout the orbit of the heteroclinic point A (i.e., $T_A W^u(P) + T_A W^s(Q) = T_A W^u(P) \oplus T_A W^s(Q) = \mathbb{XZ}$ and this sum has dimension two). Hence, since $W^s(P) \pitchfork W^u(Q) \neq \emptyset$, f has a (codimension one) heterodimensional cycle associated to P and Q.

We now define the arc arc $(f_t)_{t \in [-\tau, \tau]}$, small $\tau > 0$, unfolding the cycle above at $t = 0$. For all $(x, y, z) \in R$, $f_t(x, y, z) = f(x, y, z)$ and the restriction of $f_t^{k_0}$ to U_A is the composition of f^{k_0} and a t-translation parallel to the $\mathbb{Y}$ direction:

$$f_t^{k_0}(x, y, z) = f^{k_0}(x, y, z) + (0, t, 0) = (x + \frac{1}{2}, y - 4 + t, z + \frac{1}{2}). \tag{1}$$

By construction, for each $t > 0$ and $s \in (-\delta, \delta)$, one has that $(-1/2, t, s) \in W^u(P, f_t)$. As $[-1, 1] \times (0, 5) \times \{0\} \subset W^s(P, f_t)$ one has that $(\frac{1}{2}, t, 0) = f_t^{k_0}(0, 4, -\frac{1}{2}) = f_t^{k_0}(A)$ is a transverse homoclinic point of P. Similarly, the orbit of B consists of transverse homoclinic point of Q. Thus, for $t > 0$ the homoclinic classes of P and Q are both non-trivial.

(c) Global dynamics: filtrations and invariant foliations We now assume some global properties of the bifurcating diffeomorphism f that allow us to control the dynamics of f in the ambient manifold. First, we say that a pair (M_1, M_2) is a *filtration* of f if M_1 and M_2 are compact manifolds with boundary of the same dimension as M such that $f(M_i) \subset \mathrm{int}(M_i)$, $i = 1, 2$.

($\mathbf{c}_1$) Relative Morse-Smale dynamics. The set $W = R \cup (\cup_{j\in\{1,\dots,k_0-1\}} f^j(U_A))$ is a *filtrating neighborhood of the cycle:* there is a filtration (M_1, M_2) of f such that $W = M_2 \setminus \mathrm{int}(M_1)$. This implies that if $x \in W$ and $f(x) \notin W$ then $f^n(x) \notin W$, for every $n \in \mathbb{N}$. Similarly, if $x \in W$ and $f^{-1}(x) \notin W$ then $f^{-n}(x) \notin W$, for all $n \in \mathbb{N}$. It follows that, for every small $|t|$, the set W is also a filtrating neighbourhood of f_t. Hence, for all small $|t|$,

$$\Omega(f_t) \cap W = \Omega(f_t|_W) \subset \bigcap_{n\in\mathbb{Z}} f_t^n(W), \qquad H(P, f_t) \cup H(Q, f_t) \subset \bigcap_{n\in\mathbb{Z}} f_t^n(W).$$

The semi-local dynamics of f_t implies that, for $t < 0$, the restriction of f_t to W is Morse-Smale (in particular, the homoclinic classes of P and Q are both trivial). Thus at $t = 0$ there is a Ω-*explosion*: for positive t, the classes of $H(P, f_t)$ and $H(Q, f_t)$ are both non-trivial.

($\mathbf{c}_2$) Invariant foliations. Given $X = (x_0, y_0, z_0) \in R$, define its *strong stable, strong unstable* and *central leaves,* denoted by $F^s(X), F^u(X)$, and $F^c(X)$, by

$$\begin{aligned} F^s(X) &= \{(x, y, z) \in R :\ y = y_0, z = z_0\},\\ F^u(X) &= \{(x, y, z) \in R :\ x = x_0, y = y_0\},\\ F^c(X) &= \{(x, y, z) \in R :\ x = x_0, z = z_0\}. \end{aligned}$$

This defines three foliations (strong stable, strong unstable, and central) on R. For each t, we extend them, via f_t, to the whole set W. By construction, these foliations (denoted by $\mathcal{F}_t^\sigma$, $\sigma = s, u, c$) are f_t-invariant in the following sense: if $X \in R$ and $f_t(X) \in R$ then $f_t(F^s(X)) \subset F_t^s(f_t(X))$, $f_t(F^u(X)) \supset F^u(f_t(X))$ and $f_t(F^c(X)) \cap F^c(f_t(X))$ is a neighbourhood of $f_t(X)$ in $F^c(f_t(X))$; if $X \in R$, $f_t(X) \notin R$ and $f_t^{k_0}(X) \in R$ (i.e., $X \in U_A$)) then $f_t(F_t^\sigma(X)) \subset F_t^\sigma(f_t(X))$, $\sigma = s, u, c$.

Let $\Lambda_t = \cap_{i\in\mathbb{Z}} f_t^i(W)$. The dynamics of f_t in Λ_t has two hyperbolic directions (parallel to the axis $\mathbb{X}$ and $\mathbb{Z}$). Observe that if $X \in R$ then $Df_t(X)$ is uniformly contracting in the $\mathbb{X}$-direction and uniformly expanding in the $\mathbb{Z}$-direction, and if $X \in U_A$ then $Df_t^{k_0}(X)$ is the identity. These remarks and the partial hyperbolicity of f_t in R imply that the set Λ_t is partially hyperbolic: the $\mathbb{X}$ and $\mathbb{Z}$-directions are hyperbolic (contracting and expanding) and dominate the (central) $\mathbb{Y}$-direction. This means that the dynamics of f_t is essentially determined by its central dynamics. The study of such central dynamics is done in the next section.

3 Normalized central dynamics, selection of parameters, and outline of the construction

In this section we introduce the key ideas in the proof of the theorem and sketch its proof. We first present the main terminology in our constructions: itineraries (Section 3.1) and normalized central dynamics (Section 3.2). Using these tools we outline the proof in Section 3.3.

3.1 Itineraries

Take fundamental domains $D^+ = [\beta^{-1}, 1]$ and $D^- = [3, 4-\lambda]$ of F and assume that $F^N(D^+) = D^-$ for some $N \in \mathbb{N}$. Consider the reference cubes $\Delta^\pm = [-1,1] \times D^\pm \times [-1,1]$, small $t > 0$, and $X \in \Lambda_t \cap \Delta^+$. There are two possibilities concerning its positive iterates: either there is $n > 1$ such that $f_t^n(X) \in \Delta^+$ or not. In the first case, the smallest $n > 1$ with $f_t^n(X) \in \Delta^+$ is the *forward return time of* X, denoted by $r^+(X)$. In the second case, $X \in W^s(P) \cup W^s(Q)$. Similarly, for negative iterates of X, either there is $n > 1$ with $f_t^{-n}(X) \in \Delta^+$ or not. In the first case, the *backward return time of* X, $r^-(X)$, is defined as above. In the second case, $X \in W^u(P) \cup W^u(Q)$. These facts follow from the existence of the filtration and the geometry of the cycle. For a proof see [DRo$_1$, Lemma 4.1]. As, by the definition of the f_t, the invariant manifolds of P and Q meet transversely or quasi-transversely, according to the case, the previous results may be read as follows:

Lemma 3.1. *Given any point $X \in \Lambda_t \cap \Delta^+$ there are the following possibilities:*

- *X has infinitely many forward or backward returns to Δ^+,*
- *X is a transverse homoclinic point of P or Q,*
- *$X \in W^u(P) \cap W^s(Q)$ (heteroclinic cycle point) and $W^u(P, f_t)$ and $W^s(Q, f_t)$ meet quasi-transversely at X, and*
- *$X \in W^s(P) \cap W^u(Q)$ (heteroclinic connection point) and $W^s(P, f_t)$ and $W^u(Q, f_t)$ meet transversely at X.*

Given $X = (x, y, z)$ let $X_i = (x_i, y_i, z_i) = f_t^i(X)$. Assume that X has a first forward return $r^+(X) = \bar{q} + k_0 + \bar{p} + N$ to Δ^+, where k_0 is as in the definition of the cycle, $F^N(D^+) = D^-$, and

$$\begin{aligned}
&X_N \in \Delta^-,\\
&X_N, \dots, X_{\bar{p}-1+N}, X_{\bar{p}+N} \in R \quad \text{and} \quad X_{\bar{p}+N} \in U_A,\\
&X_{k_0+\bar{p}+N} \in f_t^{k_0}(U_A) \subset R,\\
&X_{i+k_0+\bar{p}+N} \in (R \setminus \Delta^+) \quad \text{for all } i = 0, 1, \dots, \bar{q}-1, \text{ and}\\
&X_{\bar{q}+k_0+\bar{p}+N} \in \Delta^+.
\end{aligned}$$

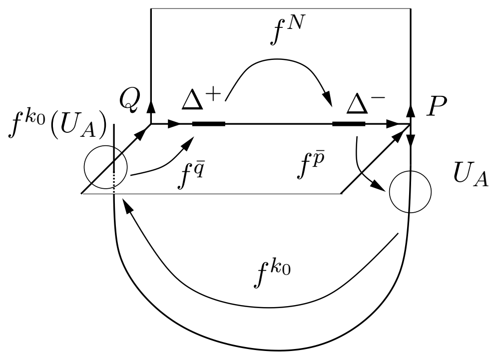


Figure 2: Itineraries

3.2 Choice of parameters, normalized dynamics, and iterated functions

For special parameters, we can estimate $\bar{p}$ and $\bar{q}$, controlling the returns in the central direction.

Lemma 3.2. *Consider the sequences of parameters $t_n = \lambda^n$ and of scaled parameter intervals $B_n = [t_n(1-\delta), t_n(1+\delta)]$, small $\delta > 0$. Then for every point $X \in \Delta^+$ having a positive return to Δ^+ it holds $\bar{p}, \bar{q} \geq (n-1)$. Moreover, if $t = t_n(1+\mu)$, $\mu > 0$, $\bar{p}, \bar{q} \geq n$.*

Proof: Fix $t = t_n(1+\mu)$, $\mu \in (-\delta, \delta)$, and $X = (x, y, z)$. By definition of f_t and since F is affine in $[3, 5]$, $y_{\bar{p}+N} = \lambda^{\bar{p}}(y_N - 4) + 4$. Since X returns to Δ^+, the unfolding condition (1) implies that

$$0 < y_{k_0+\bar{p}+N} = \lambda^{\bar{p}}(y_N - 4) + 4 + (t - 4) = \lambda^{\bar{p}}(y_N - 4) + t.$$

Finally, as $y_N \in D^- = [3, 4-\lambda]$, $-1 \leq (y_N - 4) \leq -\lambda$, it follows

$$-t = -t_n(1+\mu) = -\lambda^n(1+\mu) < \lambda^{\bar{p}}(y_N - 4) \leq -\lambda^{\bar{p}}\lambda = -\lambda^{\bar{p}+1}, \qquad \lambda^n(1+\mu) > \lambda^{\bar{p}+1}.$$

So, if μ is small, $\bar{p} \geq (n-1)$. Moreover, if $\mu > 0$ one has $\bar{p} \geq n$.

Finally, the estimate for $\bar{q}$ follows similarly noting that $y_{k_0+\bar{p}+N} < t$. □

Fix $t = t_n(1+\mu) \in \lambda^n(1-\delta, 1+\delta)$. Using Lemma 3.2, we write

$$r^+(X) = q(X) + n + k_0 + p(X) + n + N, \qquad q(X), p(X) \geq -1,$$

and we say that X has a *return $r^+(X)$ of type $(q(X), p(X))$*. Since $\beta^n = t_n^{-1}$ and $\lambda^n = t_n$, the y-coordinate of $X_{r^+(X)}$ is

$$\begin{aligned} y_{r^+(X)} &= \beta^{q(X)+n}\left[(t_n(1+\mu) - 4) + (\lambda^{n+p(X)}(F^N(y) - 4) + 4)\right] = \\ &= \beta^{q(X)+n}\left[t_n(1+\mu) + (\lambda^{n+p(X)}(F^N(y) - 4))\right] = \\ &= \beta^{q(X)}\left[1 + \mu + \lambda^{p(X)}(F^N(y) - 4)\right]. \end{aligned} \tag{2}$$

Equation (2) leads to the following one-parameter family of maps of the interval describing the central dynamics of returns to Δ^+:

$$\Phi_\mu^{q,p}\colon D_\mu^{q,p} \to D^+, \qquad \Phi_\mu^{q,p}(y) = \beta^q\left[1 + \mu + \lambda^p(F^N(y) - 4)\right], \quad \mu \in [-\delta, \delta], \quad p, q \geq -1, \tag{3}$$

where $D_\mu^{q,p}$ is the maximal subset of D^+ consisting of points y with $\Phi_\mu^{q,p}(y) \in D^+$ (see Figure 3). We call this family the *normalized central dynamics of f_t*. Next remark justifies this name and implies that $\Phi_\mu^{q,p}$ gives the y-coordinate of any (q,p)-return of f_t, $t = t_n(1+\mu)$, *for any n*.

Remark 3.3. (Central dynamics). *Let $t = \lambda^n(1+\mu) \in B_n$.*

- *By the monotonicity of F, $D_\mu^{q,p}$ is either empty or a closed interval.*
- *Suppose that $X = (x_0, y_0, z_0) \in \Delta^+$ has a (q,p)-return. Then $y_0 \in D_\mu^{q,p}$ and the y-coordinate of $f_t^{q+n+k_0+p+n+N}(X)$ is $\Phi_\mu^{q,p}(y_0)$. Conversely, given any $y_0 \in D_\mu^{q,p}$, there is at least one point of the form $X = (x, y_0, z) \in \Delta^+$ with a (q,p)-return.*
- **(consecutive returns)** *Suppose that $X_0 = (x_0, y_0, z_0) \in \Delta^+$ has a (q_0, p_0)-return X^1 and X^1 has a (q_1, p_1)-return X^2. Then X^2 is the* second return *of X_0 whose y-coordinate is $\Phi_\mu^{q_1,p_1} \circ \Phi_\mu^{q_0,p_0}(y_0)$.*

The dynamics of the maps $\Phi_\mu^{q,p}$ depend *only* on F^N, λ, and μ. In the next section, we will choose F^N and λ accordingly. We first consider returns of type $(0,0)$, given by $F^N(y) - 3 + \mu$.

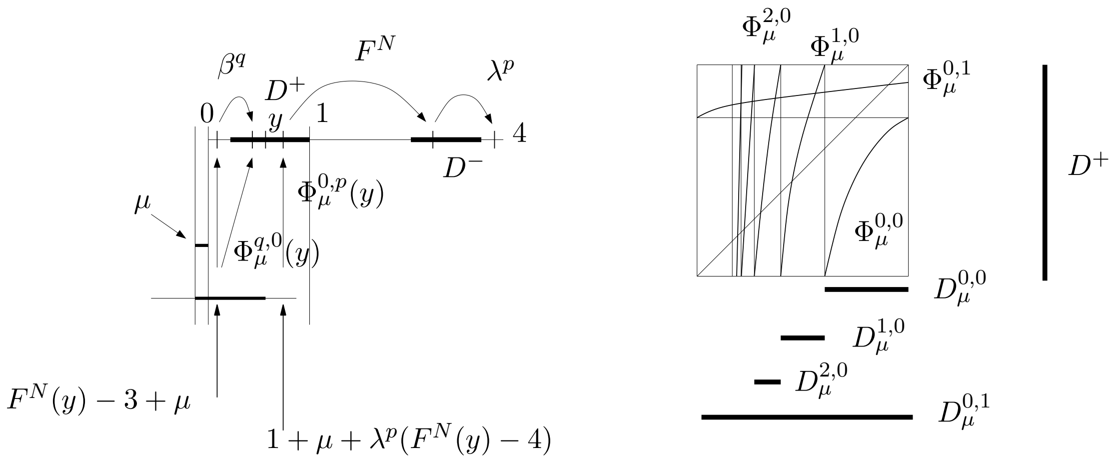


Figure 3: The maps $\Phi_\mu^{q,p}$

3.3 Outline of the constructions. Sketch of the proof

In this section, we present the main steps of our constructions and sketch the proof of the theorem.

Selection of the central dynamics: the map F. To define F, we consider the auxiliary saddle-node arc g_α

$$g_\alpha(y) = y - \varepsilon(y - 1/2)^2 + \alpha\,(1 - y),$$

with a saddle-node at the point 1/2 for the parameter $\alpha = 0$ (here $\varepsilon > 0$ is small). The saddle-node is unfolded for negative α: for $\alpha > 0$, the map g_α has two hyperbolic fixed points close to 1/2, collapsing to 1/2 for $\alpha = 0$ and disappearing thereafter. We select an interval L containing the saddle-node 1/2 in the reference fundamental domain $D^+ = [\beta^{-1}, 1]$. To each parameter α (close to 0), we associate a map F_α defining a possible central dynamics of f: in the cube R, $f(x, y, z) = (\lambda_s\, x, F_\alpha(y), \lambda_u\, z)$. The key property of F_α, see condition (F3) in Section 4.1, is

$$F_\alpha^N : L \to D^- = [3, 4 - \lambda], \quad F_\alpha^N(y) = g_\alpha(y) + 3. \tag{4}$$

We select α_0 and let $F = F_{\alpha_0}$. We will explain latter on this choice, which is done for guaranteeing the creation of one dimensional Cantor sets with large thickness after the unfolding the saddle-node (see the last paragraph in this section). This is a key step for obtaining the persistence of cycles.

Dynamics of the returns. Fixed α_0, so selected the central dynamics F and defined the arc, we consider the new saddle-node arc $h_\mu(y) = g_{\alpha_0}(y) + \mu$. Then, for each parameter of the form $t = t_n\,(1+\mu)$ (any n), the map h_μ defines the central y-coordinate for the points in $[-1,1]\times L\times[-1,1]$ having a return of type $(0,0)$: by equations (2) and (4) and Remark 3.3 it holds:

$$\begin{aligned} y_{r^+(X)} &= \beta^{0+n}\left[(t_n\,(1+\mu) - 4) + (\lambda^{n+0}\,(F^N(y) - 4) + 4)\right] = \\ &= 1 + \mu + (F^N(y) - 4) = \mu + F^N(y) - 3 = g_\alpha(y) + \mu = h_\mu(y). \end{aligned}$$

Secondary (rescaled) saddle-node bifurcations. The arc (h_μ) has a saddle-node for the parameter $\mu^\star$ (close to 0) at some point $s_{\mu^\star}$ (close to 1/2). This saddle-node is also unfolded for negative μ. The construction of the arc (f_t) and the comments above imply that, for every $t_n^\star = t_n\,(1+\mu^\star)$, the diffeomorphism $f_{t_n^\star}$ has a saddle-node at some point $S_{t_n^\star}$ in Δ^+ whose y-coordinate is $s_{\mu^\star}$. For every $t_n^\star$ the saddle-node is *essentially* the same, the only difference being that the periods p_n of the saddle-nodes $S_{t_n^\star}$ increase as n, but the local central dynamics of the $f_{t_n^\star}^{p_n}$ at the $S_{t_n^\star}$ are always the same (and given by $h_{\mu^\star}$).

Hyperbolicity before the unfolding of the saddle-node. By Remark 3.3, for $t = t_n\,(1+\mu)$ small μ, the system $(\Phi_\mu^{p,q})$ determines the central dynamics of the returns to the reference cube Δ^+. We prove that there is μ^+, $\mu^+ > \mu^\star$, such that the system $(\Phi_\mu^{p,q})$ is *hyperbolic* for all $\mu \in (\mu^\star, \mu^+]$. For the precise meaning of this assertion see Section 5.1.

For $\mu > \mu^\star$, the map $h_\mu = \Phi_\mu^{0,0}$ has two fixed points s_μ^- (repelling) and s_μ^+ (contracting) collapsing to the saddle-node $s_{\mu^\star}$ at $\mu^\star$. As above, for $t = t_n(1+\mu)$, these points correspond to saddles S_t^- and S_t^+ of f_t, of indices 2 and 1, whose y-coordinate are s_μ^- and s_μ^+.

Using Remark 3.3, the product structure of the dynamics, and noting that the diffeomorphisms f_t uniformly contract in the $\mathbb{X}$-direction and uniformly expand in the $\mathbb{Z}$-direction, we translate the *hyperbolicity* of $(\Phi_\mu^{p,q})$ to f_t for every $t = t_n\,(1+\mu)$ with $\mu \in (\mu^\star, \mu^+]$ (any n). The next step is to prove, using the system $(\Phi_\mu^{p,q})$, that for such parameters one has that

$$H(P, f_t) = H(S_t^+, f_t) \qquad \text{and} \qquad H(Q, f_t) = H(S_t^-, f_t),$$

these classes are both hyperbolic, and their union is the relative non-wandering set in the filtrating neighbourhood W. Finally, we see that these hyperbolic classes collide for $t_n^\star$.

Persistence of cycles at the collisions. To get the intervals of persistence of cycles (after the unfolding of the saddle-nodes) we need to choose carefully the parameter α_0 in the definition of the map F. In fact, in the previous steps the choice of α_0 is essentially irrelevant.

For $\mu < \mu^\star$ and close to $\mu^\star$, the system $(\Phi_\mu^{p,q})$ defines (using an inducing map method) a family of (uniformly) expanding maps from the subset $[\beta^{-1} + \varepsilon(\mu), 1]$ of D^+ to the whole D^+, where $\varepsilon(\mu)$ is small. The choice of α_0 is such that $\varepsilon(\mu) \simeq \ell(\alpha_0)$, where $\ell(\alpha_0)$ is very small.

We consider an invariant Cantor set Σ_μ of these functions in $[\beta^{-1}+\epsilon(\mu), 1]$ (roughly, the maximal invariant set). Since $\epsilon(\mu)$ is very small (here the choice of α_0 is crucial), we get that the set Σ_μ has large thickness (roughly of order of $1/\ell(\alpha_0)$, see Proposition 6.6). Essentially, the *thickness* of a Cantor set is the quotient length(bridge)/length(gap), where in each step of the construction of the Cantor set a *gap* is a removed open interval and its two *bridges* are the adjacent resulting connected components, see Definition 6.1. Roughly, here the gaps of Σ_μ are the pre-images of $[\beta^{-1}, \beta^{-1}+\epsilon(\mu)]$ and the corresponding bridges pre-images of $[\beta^{-1} + \epsilon(\mu), 1]$, using bounded distortion properties, we see that these quotients are of order of $(1 - \beta^{-1} - \epsilon(\mu))/\epsilon(\mu)$. As $\varepsilon(\mu)$ is very small, this ratio (and thus the thickness of the Cantor set) is very big. Finally, Remark 3.3 implies that Σ_μ is the projection in the central direction of a hyperbolic set Γ_t, $t = t_n\,(1+\mu)$, of index two.

We also consider a (hyperbolic) part Υ_t of the homoclinic class of P having thickness uniformly bounded from zero for all t, see Section 6.1.

The Gap Lemma (Lemma 7.1) asserts that two Cantor sets whose convex hulls are non-disjoint and whose product of their thickness is larger than one have non-empty intersection. The previous construction (taking $\ell(\alpha_0)$ small) implies that Cantor sets defined as projections of Γ_t and Υ_t verify the hypotheses of the Gap Lemma. The product structure gives now the announced persistence of heterodimensional cycles (associated to the sets Γ_t and Υ_t).

4 Definition of the map F. Saddle-node arcs

In this section we define the map F giving the central dynamics of the diffeomorphisms f_t.

4.1 A saddle-node family

For small positive ε (further restrictions on the size of ε will appear throughout the text) consider the saddle-node arc (g_α), small $|\alpha|$,

$$g_\alpha(y) = y - \varepsilon(y - 1/2)^2 + \alpha\,(1 - y).$$

The arc g_α unfolds, at $\alpha = 0$, a saddle-node at $1/2$ as follows. For $\alpha > 0$, the map g_α has two hyperbolic points. These points collapse to the saddle-node at $\alpha = 0$ and disappear for negative α.

For α and ε small, let d_α be the critical point of g'_α, $d_\alpha = 1/2+(1-\alpha)/(2\,\varepsilon) > 1$. Since the maps g_α are increasing in $(-\infty, 1]$, $g_0(1/2) = 1/2 > \beta^{-1}$, and $g_0(\beta^{-1}) < \beta^{-1}$ (small ε), for every α close to 0 there is a unique $c_\alpha \in (\beta^{-1}, 1/2)$ with $g_\alpha(c_\alpha) = \beta^{-1}$. For each α, write $D^+ = [\beta^{-1}, c_\alpha] \cup [c_\alpha, 1]$.

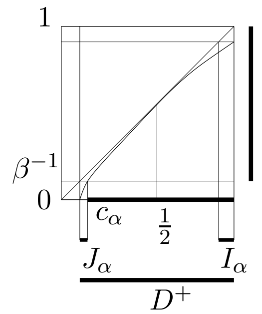


Figure 4: The saddle-node arc g_α

We take $\lambda = \beta^{-1} = \varepsilon/4$ and, for small α, choose a C^2-map $F_\alpha \colon [-1, 5] \to \mathbb{R}$ verifying conditions (F1)–(F5) below:

(F1) F_α is strictly increasing and $F_\alpha^N(D^+) = D^- = [3, 4 - \lambda]$,

(F2) $F_\alpha(y) = \beta\, y$ in $[-1, \beta^{-1}]$, $F_\alpha(y) = \lambda\,(y - 4) + 4$ in $[3, 5]$,

(F3) the restriction of F_α^N to the interval $[c_\alpha, 1]$ is equal to $g_\alpha + 3$,

(F4) $(F_\alpha^N)'(y) > 1$ for all $y \in [\beta^{-1}, c_\alpha)$, and

(F5) $\lambda\,(F_\alpha^N)'(\beta^{-1}) = \beta\,(F_\alpha^N)'(1)$.

We will take the map F giving the central dynamics, see Section 2, equal to some F_α, for a convenient α chosen in Proposition 6.6.

For the existence of F_α, note first that (F1) and (F3) are compatible. On the one hand, by definition, $g_\alpha(1) = 1 - \varepsilon/4 = 1 - \lambda$, so $F_\alpha^N(1) = 4 - \lambda = g_\alpha(1) + 3$. On the other hand, and that $g_\alpha(c_\alpha) + 3 = \beta^{-1} + 3 > 3$. So there is space for extending F_α^N to the subinterval $[\beta^{-1}, c_\alpha]$ of D^+ in such a way $F_\alpha^N([\beta^{-1}, c_\alpha]) = [3, 3 + \beta^{-1}]$.

By continuity, it suffices to check (F4) for $\alpha = 0$. Since $F_0^N([\beta^{-1}, c_0]) = [3, 3+\beta^{-1}]$, it is enough to see that $c_0 - \beta^{-1} < \beta^{-1}$, that is, $c_0 < 2\,\beta^{-1}$. Since g_0 is strictly increasing in $[0,1]$, $c_0 > 0$, and $2\,\beta^{-1} < 1$, this condition is equivalent to

$$g_0(c_0) = \beta^{-1} < g_0(2\,\beta^{-1}). \tag{5}$$

Since $\beta^{-1} = \varepsilon/4$,

$$g_0(2\,\beta^{-1}) = g_0(\varepsilon/2) = \varepsilon/2 - \varepsilon\,(\varepsilon/2 - 1/2)^2 = \varepsilon(1/2 - 1/4\,(\varepsilon - 1)^2).$$

Thus, $g_0(2\,\beta^{-1}) > \beta^{-1}$ is equivalent to

$$\varepsilon(1/2 - 1/4(\varepsilon-1)^2) > \varepsilon/4, \quad \text{or} \quad 1/4 > 1/4(\varepsilon - 1)^2, \quad \text{or} \quad 1 > (\varepsilon - 1)^2,$$

which is automatically verified for small positive ε.

Finally, to check condition (F5), observe that by (F3), $(F_\alpha^N)'(1) = g'_\alpha(1)$, so (F5) is equivalent to $(F_\alpha^N)'(\beta^{-1}) = 16\,(1 - \varepsilon - \alpha)/\varepsilon^2$, which is clearly compatible with the previous conditions.

4.2 Estimates on the derivative of the transition map

For α close to 0, consider the fundamental domains of g_α and the constants

$$\begin{array}{l} J_\alpha = [c_\alpha, g_\alpha^{-1}(c_\alpha)] \quad \text{and} \quad I_\alpha = [g_\alpha(1), 1] = [1-\lambda, 1], \\ d_m(\alpha) = \min\{g'_\alpha(y),\, y \in [c_\alpha, 1]\} \quad \text{and} \quad d_M(\alpha) = \max\{g'_\alpha(y),\, y \in [c_\alpha, 1]\}. \end{array}$$

Note that for $\alpha < 0$, given any $w \in (g_\alpha^{-1}(c_\alpha), 1]$ there is a first $n_\alpha(w) \in \mathbb{N}$ such that $g_\alpha^{n_\alpha(\omega)}(w) \in J_\alpha$.

Proposition 4.1. *Fix $\varepsilon \in (0, \frac{1}{10})$. There is $\alpha_\varepsilon > 0$ such that*

$$(g_\alpha^{n_\alpha(w)})'(w) > 1/2, \qquad \textit{for all } \alpha \in [-\alpha_\varepsilon, 0) \textit{ and every } w \in (g_\alpha^{-1}(c_\alpha), 1].$$

The main step in the proof of Proposition 4.1 is the next estimate of the derivative of $g_\alpha^{n_\alpha(w)}$:

Lemma 4.2. *There is $\alpha_\epsilon > 0$ such that, for any $\alpha \in [-\alpha_\varepsilon, 0)$ and any $w \in I_\alpha = [g_\alpha(1), 1]$, it holds*

$$(g_\alpha^{n_\alpha(w)})'(w) \geq \frac{d_m(\alpha)}{(d_M(\alpha))^2}\,\frac{|J_\alpha|}{|I_\alpha|} > \frac{1}{2}.$$

Proof of the lemma: To prove the second inequality it is enough to check it for $\alpha = 0$. The general case follows from the continuous dependence of $d_m(\alpha)$, $d_M(\alpha)$, $|J_\alpha|$, and $|I_\alpha|$ on the parameter α. Since g'_0 is decreasing in $(-\infty, d_0)$, I_0 and J_0 are contained in $(-\infty, d_0)$, and $c_0 > \beta^{-1}$,

$$\begin{aligned} d_m(0) &= g'_0(1) = (1-\varepsilon), \\ d_M(0) &= g'_0(c_0) < g'_0(\beta^{-1}) = g'_0(\varepsilon/4) = 1 - 2\,\varepsilon\,(\varepsilon/4 - 1/2) = (1 + \varepsilon - \varepsilon^2/2) < (1+\varepsilon). \end{aligned}$$

By definition, $|I_0| = \varepsilon/4$. It remains to estimate $|J_0|$. By equation (5), $g_0(\varepsilon/2) = g_0(2\,\beta^{-1}) > \beta^{-1}$, thus, since g_0 is increasing in $(-\infty, d_0)$, $\varepsilon/2 > g_0^{-1}(\beta^{-1}) = c_0$. There are two cases, either $\varepsilon/2 > g_0^{-1}(c_0)$ or $\varepsilon/2 \in [c_0, g_0^{-1}(c_0)]$. In the first case, since g_0' is decreasing in $(-\infty, d_0)$, $\varepsilon/2, c_0, g_0^{-1}(c_0) < 1/2$, and $g_0'(1/2) = 1$,

$$|J_0| \geq \varepsilon/2 - g_0(\varepsilon/2) = \varepsilon/2 - \varepsilon/2 + \varepsilon\,(\varepsilon/2 - 1/2)^2 = (\varepsilon/4)\,(1-\varepsilon)^2.$$

In the second case, write $[c_0, g_0^{-1}(c_0)] = g^{-1}([\beta^{-1}, c_0])$ and observe that

$$|J_0| = \frac{c_0 - \beta^{-1}}{g_0'(\xi)}, \quad \text{for some } \xi \in [\beta^{-1}, c_0].$$

Thus, since (by hypothesis) $g_0^{-1}(\beta^{-1}) < \varepsilon/2 < g_0^{-1}(c_0)$ and the derivative of g_0 is decreasing,

$$|J_0| \geq \frac{\varepsilon/2 - g_0(\varepsilon/2)}{g_0'(\beta^{-1})} > \frac{(\varepsilon/4)\,(1-\varepsilon)^2}{1+\varepsilon}.$$

In both cases one has that

$$|J_0| \geq \frac{\varepsilon\,(1-\varepsilon)^2}{4\,(1+\varepsilon)}.$$

Putting all the previous inequalities together and noting that $\varepsilon < 1/10$, we get the second inequality:

$$\frac{|J_0|}{|I_0|}\,\frac{d_m(0)}{(d_M(0))^2} \geq \frac{\varepsilon\,(1-\varepsilon)^2}{4\,(1+\varepsilon)\,(\varepsilon/4)}\,\frac{(1-\varepsilon)}{(1+\varepsilon)^2} \geq \frac{(1-\varepsilon)^3}{(1+\varepsilon)^3} > \frac{1}{2}.$$

We now prove the first inequality in the lemma. Given $\alpha < 0$, let N_α be the smallest natural number such that $g_\alpha^{N_\alpha}(I_\alpha) \cap J_\alpha \neq \emptyset$. By the choice of N_α, since g_α' is decreasing, and J_α and $g_\alpha^{N_\alpha}(I_\alpha)$ are at the left of the point where the derivative of g_α is 1 (if α is small),

$$|J_\alpha| \geq |g_\alpha^{N_\alpha}(I_\alpha)| \geq |g_\alpha^{-1}(J_\alpha)| \geq \frac{|J_\alpha|}{d_M(\alpha)}.$$

Therefore, taking $z_\alpha \in I_\alpha$ such that $(g_\alpha^{N_\alpha})'(z_\alpha) = |g_\alpha^{N_\alpha}(I_\alpha)|/|I_\alpha|$, one has

$$(g_\alpha^{N_\alpha})'(z_\alpha) \geq \frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}. \tag{6}$$

Define $b_\alpha \in I_\alpha$ by $g_\alpha^{N_\alpha}(b_\alpha) = g_\alpha^{-1}(c_\alpha)$. For a given $w \in I_\alpha$ there are three possibilities:

- $w \in [g_\alpha(1), z_\alpha]$. Note that $n_\alpha(w) = N_\alpha$ or $n_\alpha(w) = N_\alpha + 1$. Observe that if $n_\alpha(w) = N_\alpha + 1$ then $0 < g_\alpha'(g_\alpha^{N_\alpha}(w)) < 1$. Thus using the monotonicity of g_α', one gets

$$(g_\alpha^{n_\alpha(w)})'(w) \geq (g_\alpha^{N_\alpha})'(w) \geq (g_\alpha^{N_\alpha})'(z_\alpha) \geq \frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}.$$

- $w \in [b_\alpha, 1]$. In this case, $n_\alpha(w) = N_\alpha + 1$. Since $g_\alpha(w) < z_\alpha$, from the monotonicity of g_α', $(g_\alpha^{N_\alpha})'(g_\alpha(w)) \geq (g_\alpha^{N_\alpha})'(z_\alpha)$. Thus from equation (6) and $g_\alpha'(w) \geq d_m(\alpha)$,

$$(g_\alpha^{N_\alpha+1})'(w) = (g_\alpha^{N_\alpha})'(g_\alpha(w))\,g_\alpha'(w) \geq (g_\alpha^{N_\alpha})'(z_\alpha)\,g_\alpha'(w) \geq \frac{|J_\alpha|}{|I_\alpha|\,d_M(\alpha)}\,d_m(\alpha).$$

- $w \in [z_\alpha, b_\alpha]$ (if $z_\alpha < b_\alpha$). In this case $n_\alpha(w) = N_\alpha$. Arguing as in the previous cases,

$$(g_\alpha^{N_\alpha})'(w) \geq (g_\alpha^{N_\alpha})'(1) = \frac{(g_\alpha^{N_\alpha+1})'(1)}{(g_\alpha)'(g_\alpha^{N_\alpha}(1))} \geq \frac{1}{d_M(\alpha)} (g_\alpha^{N_\alpha+1})'(1) \geq \frac{1}{d_M(\alpha)} \frac{|J_\alpha|}{|I_\alpha|\, d_M(\alpha)}\, d_m(\alpha).$$

The proof of the inequality (and of the lemma) now follows noting that $d_m(\alpha) < 1 < d_M(\alpha)$. □

Proof of the proposition: Let s_α be the (unique) point with $g'_\alpha(s_\alpha) = 1$. Take any $w \in (g_\alpha^{-1}(c_\alpha), g_\alpha(1)]$. First, if $w \leq s_\alpha$ the result is obvious: all the derivatives throughout the orbit of w are bigger or equal than 1. If $w \geq s_\alpha$ take $k \geq 0$ with $g_\alpha^{-k}(w) = \bar{w} \in I_\alpha$. The result follows from the case $w \in I_\alpha$ (Lemma 4.2) noting that

$$(g_\alpha^{n_\alpha(w)})'(w) = (g_\alpha^{n_\alpha(w)+k})'(\bar{w})\, (g_\alpha^{-k})'(w), \quad n_\alpha(w) + k = n_\alpha(\bar{w}), \quad \text{and} \quad (g_\alpha^{-k})'(w) \geq 1.$$

This completes the proof of Proposition 4.1. □

5 Hyperbolic dynamics

In this section, we prove the parts of the theorem concerning hyperbolic dynamics and collisions of homoclinic classes. In Section 5.1, we state a hyperbolicity result for the family $(\Phi_\mu^{p,q})$, for parameters $\mu \in (\mu^\star, \mu^+]$, where $\mu^\star$ is the saddle-node parameter of the family (h_μ) in Section 3.3. In Section 5.2, using Remark 3.3 and the product structure, we will translate the hyperbolicity of the system $(\Phi_\mu^{p,q})$ to the diffeomorphisms f_t, $t \in t_n\,(1+\mu^\star, 1+\mu^+]$. Finally, in Section 5.3, we state the collision of homoclinic classes, related to the saddle-node bifurcations $\mu^\star$ of $h_\mu = \Phi_\mu^{0,0}$. Since we will follow closely [DRo$_1$, Section 4] we will omit most of the details of the proofs.

5.1 Hyperbolicity of the normalized family

Fix small $\mu^\star < 0$ and let $\alpha^\star = \alpha(\mu^\star) > 0$ be the saddle-node parameter of the family $g_\alpha + \mu^\star$. Define now the auxiliary saddle-node arc

$$h_\mu = h_\mu^{\alpha^\star} = g_{\alpha^\star} + \mu$$

having a saddle-node close to 1/2 for the parameter $\mu^\star$. For small $\mu > \mu^\star$, the map h_μ has (exactly) a pair of (hyperbolic) fixed points in D^+, s_μ^- and s_μ^+, $s_\mu^- < s_\mu^+$, where s_μ^+ is contracting and s_μ^- is expanding. These points collapse to the saddle-node $s_{\mu^\star}$. For small $\mu \geq \bar{\mu}$, consider the partition of D^+ given by the intervals

$$L_u(\mu) = [\beta^{-1}, s_\mu^-], \quad L_c(\mu) = (s_\mu^-, s_\mu^+), \quad \text{and} \quad L_s(\mu) = [s_\mu^+, 1].$$

As in Section 4.1, let $h_{\mu^\star}(c_{\mu^\star}) = \beta^{-1}$. Take the system of functions $\Phi_\mu^{q,p}$, defined on D^+, induced by $F = F_{\alpha^\star}$ associated to $h_{\mu^\star}$, i.e., the restriction of F^N to $[c_{\mu^\star}, 1]$ is $h_{\mu^\star} + 3$, see condition (F3) in Section 4. Equation (3) now implies that, for all $y \in [c_{\mu^\star}, 1]$ and μ close to $\mu^\star$, one has

$$\Phi_\mu^{0,0} = \beta^0(1 + \mu + \lambda^0(F^N - 4)) = (1 + \mu + (g_{\alpha^\star} + 3) - 4) = g_{\alpha^\star} + \mu = h_\mu.$$

Thus, for $\mu > \mu^\star$, $s_\mu^\pm$ are (hyperbolic) fixed points of $\Phi_\mu^{0,0}$ in $[c_{\mu^\star}, 1]$.

We need to state two definitions. We say that y is a *ω-limit point of* $\{\Phi_\mu^{p,q}\}$ if there are a point x_0 and sequences of non-negative numbers (p_n) and (q_n) such that $x_n \to y$, where $x_{n+1} = \Phi_\mu^{p_n,q_n}(x_n)$. The *(forward) limit set of* $\{\Phi_\mu^{p,q}\}$, $L^+(\Phi_\mu^{p,q})$, is the closure of all the ω-limit points of $\{\Phi_\mu^{p,q}\}$.

Next lemmas mean that there is μ^+, $0 > \mu^+ > \mu^\star$, such that, for every $\mu \in H = [\mu^\star, \mu^+]$, the forward limit set $L^+(\Phi_\mu^{q,p})$ is contained in $L_s(\mu) \cup L_u(\mu)$. Moreover, if $\mu \in (\mu^\star, \mu^+]$ this set is hyperbolic (the part contained in $L_s(\mu)$ is contracting and the part in $L_u(\mu)$ is expanding). By Remark 3.3, these properties have a direct translation (in terms of hyperbolicity) to the arc of diffeomorphisms $(f_t)_t$ (defined in terms of F above) for $t \in [t_n\,(1+\mu^\star), t_n(1+\mu^+)]$.

Lemma 5.1. *There is μ^+, $0 > \mu^+ > \mu^\star$, such that, for every $\mu \in H = [\mu^\star, \mu^+]$, it holds:*

- $\Phi_\mu^{0,p}(L_s(\mu)) \subset L_s(\mu)$ *for all $p \geq 0$. If $\mu \neq \mu^\star$, these applications are uniformly contracting.*
- *All the returns of points of $L_s(\mu)$ to D^+ are of the form $(0,p)$ with $p \geq 0$, i.e., there are no returns of type (q,p) with $q > 0$.*

Proof: To prove the first item, recall that, by definition of F, $F^N(D^+) = D^-$, so $F^N(1) = (4-\lambda)$. The definition of $\Phi_\mu^{p,q}$, see (3), now implies that $\Phi_\mu^{0,0}([s_\mu^+, 1]) = [s_\mu^+, 1-\lambda+\mu] \subset [s_\mu^+, 1]$ and that $\Phi_\mu^{0,p+1}([s_\mu^+, 1])$ is at the right of $\Phi_\mu^{0,p}([s_\mu^+, 1])$ for all $p \geq 0$. As $\beta^{-1} = \lambda = \varepsilon/4$, small ε, it holds

$$(1-\lambda) + \mu > \beta^{-1}, \qquad \text{for every small } |\mu|. \tag{7}$$

Observe that $F^N(y) \in [3, 4-\lambda]$, thus $(F^N(y) - 4) \in (-1, -\lambda)$, for all if $y \in D^+$. Hence

$$\Phi_\mu^{0,p}(y) \in (1+\mu-\lambda^p, 1+\mu-\lambda^{p+1}) \subset (\beta^{-1}, 1) \subset D^+. \tag{8}$$

Thus, $\Phi_\mu^{0,p}([s_\mu^+, 1]) \subset [s_\mu^+, 1)$ for all $p \geq 0$, proving the inclusions. The hyperbolicity follows from the definition of the maps $\Phi_\mu^{0,p}$ and $0 < h'_\mu < 1$ in $(s_\mu^+, 1]$, $\mu \in (\mu^\star, \mu^+]$.

The second part of the lemma follows directly from $\Phi_\mu^{0,p}(L_s(\mu)) \subset (\beta^{-1}, 1)$ (equation (8)), hence $\Phi_\mu^{q,p}(y) = \beta^q\, \Phi_\mu^{0,p}(y) > \beta\,\beta^{-1} = 1$ if $y \in L_s(\mu)$ and $q \geq 1$. $\square$

The proof of the next lemma follows arguing as in the previous one, so it will be omitted.

Lemma 5.2. *For every $\mu \in H$, $\mu \neq \mu^\star$, it holds*

- $\Phi_\mu^{0,0}(L_c(\mu)) = L_c(\mu)$ *and the restriction of $\Phi_\mu^{0,0}$ to $L_c(\mu)$ is strictly increasing.*
- $\Phi_\mu^{0,p}(L_c(\mu)) \subset L_s(\mu)$ *for all $p \geq 1$.*
- *All the returns of points of $L_c(\mu)$ to D^+ are of the form $(0,p)$ with $p \geq 0$. Moreover, the returns of $L_c(\mu)$ to $L_c(\mu)$ are of the form $(0,0)$.*

Lemma 5.3. *For every $\mu \in H$ it holds*

- $\Phi_\mu^{0,p}(L_u(\mu)) \subset L_s(\mu)$ *for every $p \geq 1$.*
- *All the returns of points of $L_u(\mu)$ to D^+ are of the form $(0,p)$ or $(q,0)$, with $p, q \geq 0$.*
- *If $\mu \neq \mu^\star$, the restriction of $\Phi_\mu^{q,0}$ to $L_u(\mu)$ is uniformly expanding for all $q \geq 0$.*

Proof: For the first claim just note that, if $p \geq 1$, then $\Phi_\mu^{0,p}(L_u(\mu))$ is at the right of $\Phi_\mu^{0,0}(L_s(\mu)) \subset L_s(\mu)$. The second assertion follows from the first one and the proof of Lemma 5.1. The expansion of $\Phi_\mu^{0,0}$ in $D_\mu^{0,0} \cap L_u(\mu)$ ($D_\mu^{0,0}$ is the interval of definition of $\Phi_\mu^{0,0}$) follows noting that $D_\mu^{0,0} \cap L_u(\mu) = [c_\mu, s_\mu^-]$ and using the expansivity of $g_{\alpha(\bar\mu)}$ in $[c_\mu, s_\mu^-]$, where (as above) c_μ is defined by $h_\mu(c_\mu) = \beta^{-1}$. The expansivity of $\Phi_\mu^{q,0}$ in $D_\mu^{q,0} \subset [\beta^{-1}, c_\mu)$ follows from its definition and condition (F4). $\square$

5.2 Prevalent hyperbolicity for the arc (f_t)

In this section we translate the hyperbolicity of the system $(\Phi_\mu^{p,q})$, $\mu \in (\mu^\star, \mu^+]$, to the diffeomorphisms f_t for $t \in (t_n\,(1+\mu^\star), t_n\,(1+\mu^+)]$. Remark 3.3 allows us to perform this translation.

To each point $X_0 = (x_0, y_0, z_0) \in \Delta^+ \cap \Lambda_t$ (Λ_t is the maximal invariant set of f_t in W) we associate a sequence $i(X_0) = (i_k(X_0))_{k \in E(X_0)}$ of symbols s, c and u, as follows. We let $i_0(X_0) = j$ if $y_0 \in L_j(\mu)$. Let $X_1 = (x_1, y_1, z_1)$ and $X_{-1} = (x_{-1}, y_{-1}, z_{-1})$ be the first forward and backward returns of X_0 to Δ^+ by f_t (if they are defined); let $i_{\pm 1}(X_0) = j$ if $y_{\pm 1} \in L_j(\mu)$. If possible, we inductively define first forward and backward returns $X_{\pm(n+1)}$ of $X_{\pm n}$ and let $i_{n+1}(X_0) = i_1(X_n) = i_n(X_1)$ and $i_{-(n+1)}(X_0) = i_{-1}(X_{-n}) = i_{-n}(X_{-1})$. The set $E(X_0) \subset \mathbb{Z}$ is the maximal interval where these indices are defined. The sequence $i(X_0) = (i_k(X_0))_{k \in E(X_0)}$ is the *itinerary of X.*

Using the cycle configuration, it is easy to see that every $X \in \Lambda_t$, $X \neq P, Q$, has some (backward or forward) iterate in Δ^+. So we can associate to each $X \in \Lambda_t \setminus \{P, Q\}$ the sequence $i(Y)$, where Y is some iterate of X in Δ^+. This sequence is well defined, up to a shift, for points whose returns do not have central coordinates equal to β^{-1} or 1 (in fact, for the parameters that we select, we will see that this situation corresponds to wandering points). Define now the sets:

$$
\begin{aligned}
\Omega_s(t) &= \{X \in \Omega(f_t)\colon E(X) \neq \emptyset \text{ and } i_k(X) = s \text{ for all } k \in E(X)\} \cup \{P\},\\
\Omega_u(t) &= \{X \in \Omega(f_t)\colon E(X) \neq \emptyset \text{ and } i_k(X) = u \text{ for all } k \in E(X)\} \cup \{Q\},\\
\Omega_c(t) &= \{X \in \Omega(f_t)\colon E(X) \neq \emptyset \text{ and } i_k(X) = c \text{ for all } k \in E(X)\}.
\end{aligned}
$$

For each n, write $H_n = \{t = t_n\,(1+\mu),\ \mu \in H\}$, $H = [\mu^\star, \mu^+]$ as in Lemma 5.1, and fix $t = t_n(1+\mu) \in H_n$. Take $X = (x, y, z) \in \Delta^+$ with a first return $X_1 = (x_1, y_1, z_1)$ to Δ^+ of type (q, p). By Remark 3.3, $y_1 = \Phi_\mu^{q,p}(y)$. This fact and Lemmas 5.1, 5.2 and 5.3 imply that *the itinerary of a non-wandering point of Λ_t is constant (equal to s, c or u).* Moreover:

- If $i_k(X) = s$ for all $k \in E(X)$ then all the returns are of type $(0, p)$, $p \geq 0$. Thus, if $\mu \neq \mu^\star$, the returns are uniformly contracting in the $\mathbb{Y}$-direction.
- If $i_k(X) = u$ for all $k \in E(X)$ then all the returns are of type $(q, 0)$, $q \geq 0$. Thus, if $\mu \neq \mu^\star$, the returns are uniformly expanding in the $\mathbb{Y}$-direction.
- If $i_k(X) = c$ for all $k \in E(X)$ then all the returns are of type $(0, 0)$.

Actually, the last possibility does not hold for a non-wandering point: just note that $\Phi_\mu^{0,0}$ restricted to $L_c(\mu)$ is a monotone map without fixed points. Similar arguments imply that a point having a return with central coordinate equal to β^{-1} or 1 is wandering. These facts are the ingredients of the proof of the next lemma (for details see [DRo$_1$, Section 4]) which implies (1a) in Theorem 1.

Lemma 5.4. ([DRo$_1$, Steps B-G]) *For every $t \in H_n$,*

- *$\Omega(f_t) \cap W = \Omega_s(t) \cup \Omega_u(t)$. In particular, $\Omega_c(t) = \emptyset$.*
- *$\Omega_s(t) = H(P, f_t)$ and $\Omega_u(t) = H(Q, f_t)$.*
- *If $t \neq \lambda^n\,(1+\mu^\star)$, the homoclinic classes of P and Q are hyperbolic.*
- *There are no cycles associated to the homoclinic classes of P and Q.*

5.3 Sequence of saddle-node bifurcations

For $t = \lambda^n (1+\mu)$, $\mu \in (\mu^\star, \mu^+]$, let $S_t^\pm$ be the periodic point of f_t in Δ^+ whose central coordinate is $s_\mu^\pm$. The points S_t^+ and S_t^- are hyperbolic (of indices 1 and 2, respectively) with period $n+k_0+n+N$. By, Lemma 5.4, $S_t^+ \in H(P, f_t)$ and $S_t^- \in H(Q, f_t)$. For $t_n^\star = t_n (1+\mu^\star)$, one has a saddle-node $S_{t_n^\star}^+ = S_{t_n^\star}^- = S_{t_n^\star}$ whose central coordinate is $s_{\mu^\star}$. Next lemma implies (1b) in Theorem 1:

Lemma 5.5. *The intersection $H(P, f_{t_n^\star}) \cap H(Q, f_{t_n^\star})$ is the orbit of the saddle-node $S_{t_n^\star}$. Moreover, every compact $f_{t_n^\star}$-invariant subset of $H(P, f_{t_n^\star}) \cup H(Q, f_{t_n^\star})$ disjoint from $S_{t_n^\star}$ is hyperbolic.*

This lemma follows as Lemma 5.4 using Remark 3.3 and Lemmas 5.1, 5.2, and 5.3. First, note that $L_c(\mu^\star) = \emptyset$. For $p \geq 0$, the map $\Phi_{\mu^\star}^{0,p}$ is uniformly contracting in any compact subset of $L_s(\mu^\star)$ that does not contain $s_{\mu^\star}$. For $q \geq 0$, the map $\Phi_{\mu^\star}^{q,0}$ is uniformly expanding in any compact subset of $L_u(\mu^\star)$ disjoint from $s_{\mu^\star}$. These properties can be translated to $f_{t_n^\star}$. Finally, the fact that $H(P, f_{t_n^\star}) \cap H(Q, f_{t_n^\star})$ is the orbit of $S_{t_n^\star}$ follows from $s_{\mu^\star} = L_s(\mu^\star) \cap L_u(\mu^\star)$, Lemmas 5.1, 5.2, and 5.3, and that $s_{\mu^\star}$ is a saddle-node of $\Phi_{\mu^\star}^{0,0}$.

6 Dynamically defined Cantor sets

In this section we construct dynamically defined Cantor sets for the family $(\Phi_\mu^{p,q})$ having arbitrarily big thickness. In this step, the selection of the parameter α in the definition of F (so in the definition of $(\Phi_\mu^{p,q})$) is crucial. Therefore, for the sake of clearness, we write explicitly the two parameters μ and α involved in our constructions. In this way, we consider a two parameter family of maps $\Phi_{\mu,\alpha}^{q,p}$ defined as $\Phi_\mu^{q,p}$ just taking the central map F equal to F_α. Similarly, we consider the two-parameter family of diffeomorphisms $f_{t,\alpha}$.

In what follows, we systematically use *dynamically defined Cantor sets* and their *thickness.* For further information on this topic we refer to [PT$_3$, Chapter 4].

Definition 6.1. *A* presentation *of a Cantor set $\Upsilon \subset \mathbb{R}$ is an enumeration $\mathcal{G} = \{G_n\}$ of the* bounded gaps *of Υ (i.e., the bounded connected components of $\mathbb{R} \setminus \Upsilon$). The* thickness of the presentation $\mathcal{G}$, *$\tau(\Upsilon, \mathcal{G})$, is defined as follows. For each n and each point g in the boundary of a gap G_n, let B_g be the connected component of $I \setminus (G_1 \cup \cdots \cup G_n)$ containing g (here I is the convex hull of Υ). The interval B_g is the* left *or* right bridge *of G_n, according to its relative position. Then*

$$\tau(\Upsilon, \mathcal{G}) = \inf_{n \in \mathbb{N}, g \in \partial G_n} \tau(\Upsilon, \mathcal{G}, g), \quad \tau(\Upsilon, \mathcal{G}, g) = |B_g|/|G_n|.$$

Finally, the thickness *of Υ is $\tau(\Upsilon) = \sup \tau(\Upsilon, \mathcal{G})$, where the supremum is taken over all the presentations of Υ. Thus, for any presentation $\mathcal{G}$ of Υ, $\tau(\Upsilon, \mathcal{G})$ is a lower bound of $\tau(\Upsilon)$.*

We define the *central localization* of an $f_{t,\alpha}$-invariant set Λ in the neighbourhood W by

$$\Lambda^c = \{y_0 \in D^+ : \text{ there is } X \in \Lambda \text{ of the form } (x, y_0, z)\}. \tag{9}$$

In this section, for appropriate t and α, we build two (transitive) hyperbolic sets $\Lambda_{t,\alpha}$ and $\Omega_{t,\alpha}$ of $f_{t,\alpha}$, $\Lambda_{t,\alpha}$ of index one contained in the homoclinic class of P and $\Omega_{t,\alpha}$ of index two, such that the product of the thickness of their central localizations is bigger than one. This fact (and Lemma 7.1) is the key for getting a t-parameter interval with persistence of cycles, (part (2) of the theorem).

6.1 Construction of the hyperbolic set $\Lambda_{t,\alpha}$ of index one

Lemma 6.2. *There are parameters μ_0 and $\alpha_0 > 0$ and strictly positive constants d^c and τ^c such that, for every $t \in B_n = [t_n(1-\mu_0), t_n(1+\mu_0)]$, $t_n = \lambda^n$, and $\alpha \in [-\alpha_0, \alpha_0]$, there exists a (transitive) hyperbolic set $\Lambda_{t,\alpha}$ of $f_{t,\alpha}$ of index one, contained in the homoclinic class of P, such that the diameter and the thickness of $(\Lambda_{t,\alpha})^c$ are lower bounded by d^c and τ^c, respectively.*

Proof: The proof of this lemma has two parts. We first construct a dynamically defined Cantor set for the system $\Phi^{0,p}_{\mu,\alpha}$, for (μ,α) close to $(0,0)$. We next see, using Remark 3.3 that such a set is the central localization of a hyperbolic set of $f_{t_n(1+\mu),\alpha}$ in the homoclinic class of P.

We claim that there is p_1 such that, for all $p \geq p_1$, **(i)** $\Phi^{0,p}_{\mu,\alpha}$ is contracting, **(ii)** $D^{0,p}_{\mu,\alpha} = D^+$, and **(iii)** $\Phi^{0,p}_{\mu,\alpha}(D^+)$ is in the interior of D^+. To get the contraction property (i), just take p_1 big enough in order that the factor λ^{p_1} compensates any expansion introduced by F^N_α, any small $|\alpha|$, see the definition of $\Phi^{0,p}_{\mu,\alpha}$ in (3). Equation (8) implies that $\Phi^{0,p}_{\mu,\alpha}(D^+) \subset \mathrm{int}(D^+)$, for $p \geq 1$, which (by continuity) implies (ii) and (iii), for small $|\mu|$ and $|\alpha|$ and $p \geq 1$.

Define $K^{p_1,p_1+1}_{\mu,\alpha}$ as the maximal invariant set in D^+ of $\Phi^{0,p_1}_{\mu,\alpha}$ and $\Phi^{0,p_1+1}_{\mu,\alpha}$, that is,

$$K^{p_1,p_1+1}_{\mu,\alpha} = \bigcap_{j\geq 0} D^+_{\mu,\alpha}(j), \quad \text{where} \quad D^+_{\mu,\alpha}(0) = D^+ \quad \text{and} \quad D^+_{\mu,\alpha}(i+1) = \bigcup_{p=p_1}^{p_1+1} \Phi^{0,p}_{\mu,\alpha}(D^+(i)).$$

By construction, $K^{p_1,p_1+1}_{\mu,\alpha}$ is a hyperbolic dynamically defined Cantor set, which is transitive and has a dense subset of periodic points. Let $d^c(0,0) > 0$ and $\tau^c(0,0) > 0$ be the diameter and the thickness of $K^{p_1,p_1+1}_{0,0}$. As the thickness and the diameters of $K^{p_1,p_1+1}_{\mu,\alpha}$ depend continuously on $\Phi^{0,p_1}_{\mu,\alpha}$ and $\Phi^{0,p_1+1}_{\mu,\alpha}$, thus on μ and α, there are small μ_0 and α_0 such that, for all $(\mu,\alpha) \in [-\mu_0,\mu_0]\times[-\alpha_0,\alpha_0]$, it holds

$$\mathrm{diam}(K^{p_1,p_1+1}_{\mu,\alpha}) > d^c(0,0)/2 = d^c, \quad \text{and} \quad \tau(K^{p_1,p_1+1}_{\mu,\alpha}) > \tau^c(0,0)/2 = \tau^c.$$

To prove the lemma, it suffices to find a set $\Lambda_{t,\alpha}$ (contained in $H(P, f_{t,\alpha})$) whose central localization is $K^{p_1,p_1+1}_{\mu,\alpha}$. Let us explain this in details.

Fix an interval $B_n = [t_n(1-\mu_0), t_n(1+\mu_0)]$. By Remark 3.3, for each $t(\mu) = t_n(1+\mu) \in B_n$, the central coordinate of a return of type $(0,p)$ by $f_{t(\mu),\alpha}$ is given by $\Phi^{0,p}_{\mu,\alpha}$. Define $\Lambda_{t(\mu),\alpha}$ as the $f_{t(\mu),\alpha}$-invariant set of points with infinitely many consecutive returns to $\Delta^+ = [-1,1]\times D^+ \times [-1,1]$ of types $(0,p_1)$ or $(0,p_1+1)$. By construction, the central localization of $\Lambda_{t(\mu),\alpha}$ is $K^{p_1,p_1+1}_{\mu,\alpha}$. Since the directions parallel to the axis $\mathbb{X}$ and $\mathbb{Z}$ are contracting and expanding and, by the previous step, the $\mathbb{Y}$-direction is contracting for the restriction of $f_{t(\mu),\alpha}$ to $\Lambda_{t(\mu),\alpha}$, this set is hyperbolic of index one. The claims for the diameter and the thickness now are obvious.

It remains to prove that $\Lambda_{t(\mu),\alpha}$ is a transitive subset of the homoclinic class of P. As a consequence of the contraction in the $\mathbb{X}$-direction and the expansion in the $\mathbb{Z}$-direction and the $f_{t,\alpha}$-invariance of the planes $\mathbb{XZ}$, the set $\Lambda_{t(\mu),\alpha}$ intersects every plane parallel to the $\mathbb{XZ}$-plane in at most one point. Thus there is a one-to-one correspondence between $\Lambda_{t(\mu),\alpha} \cap \Delta^+$ and $(\Lambda_{t(\mu),\alpha})^c$, so the dynamics (via $f_{t(\mu),\alpha}$) of the returns of the points of $\Lambda_{t(\mu),\alpha} \cap \Delta^+$ to Δ^+ and the dynamics of $(\Lambda_{t(\mu),\alpha})^c = K^{p_1,p_1+1}_{\mu,\alpha}$ by the system $\Phi^{0,p_1}_{\mu,\alpha}$ and $\Phi^{0,p_1+1}_{\mu,\alpha}$ are conjugate. Hence, the periodic points are dense in $\Lambda_{t(\mu),\alpha}$ and this set is transitive.

To see that any periodic point $A \in \Lambda_{t(\mu),\alpha}$ is homoclinically related to P note that, by construction, the orbit of the unstable manifold of A meets the rectangle $[-1,1]\times D^+\times\{0\} \subset W^s(P, f_{t(\mu),\alpha})$. Thus $W^u(A, f_{t(\mu),\alpha}) \pitchfork W^s(P, f_{t(\mu),\alpha}) \neq \emptyset$. To see that $W^u(P, f_{t(\mu),\alpha}) \pitchfork W^s(A, f_{t(\mu),\alpha}) \neq \emptyset$ note that, by construction and considering negative iterations by $f_{t(\mu),\alpha}$, the stable manifold of A contains a disk of the form $[-1,1]\times D^+\times\{z_0\}$ and that $W^u(P,f_{t(\mu),\alpha})$ intersects any disk of this form (any $z_0\in[-1,1]$). This completes the proof of the lemma. $\square$

6.2 Construction of the thick hyperbolic set $\Omega_{t,\alpha}$ of index two

As in Section 6.1, we first construct a dynamically defined Cantor set $\Sigma_{\mu,\alpha}$ associated to the family $\Phi^{q,0}_{\mu,\alpha}$, $q\geq 0$, with large thickness (Proposition 6.6). This set corresponds to the central localization of a transitive hyperbolic set of index two (the announced set $\Omega_{t,\alpha}$).

Since $\Phi^{0,0}_{\mu,\alpha}=F^N_\alpha-3+\mu$, see (3), by condition (F1) it holds $\Phi^{0,0}_{\mu,\alpha}(\beta^{-1})=\mu$. So, for small $\mu<0$, there are $c_{\mu,\alpha}$ (the continuation of c_α in Section 4) and $\ell_{\mu,\alpha}\in D^+$, $\beta^{-1}<\ell_{\mu,\alpha}<c_{\mu,\alpha}$, with

$$\Phi^{0,0}_{\mu,\alpha}(\ell_{\mu,\alpha})=0 \quad\text{and}\quad \Phi^{0,0}_{\mu,\alpha}(c_{\mu,\alpha})=\beta^{-1}.$$

Lemma 6.3. (Bounded distortion lemma) *There is a constant L, independent of small $|\alpha|$ and $|\mu|$, such that*

$$L^{-1}\,\frac{|I|}{|J|}\leq\frac{|(\Phi^{0,0}_{\mu,\alpha})^k(I)|}{|(\Phi^{0,0}_{\mu,\alpha})^k(J)|}\leq L\,\frac{|I|}{|J|},$$

for every pair of intervals I and J in the same fundamental domain of $\Phi^{0,0}_{\mu,\alpha}$ in $[c_{\mu,\alpha},1]$ with $(\Phi^{0,0})^j_{\mu,\alpha}(I\cup J)\subset[c_{\mu,\alpha},1]$ for all $j=1,\dots,k-1$.

The proof of this lemma is quite standard and uses that the map $\Phi^{0,0}_{\mu,\alpha}$ is C^2 and monotone in $[c_{\mu,\alpha},1]$, which follows from condition (F3) and equation (3). For details see [PT3, pages 58-59].

Define, for $\mu<0$, the interval $G_0(\mu,\alpha)=(\beta^{-1},\ell_{\mu,\alpha})$. Since $|\mu|=|(\Phi^{0,0}_{\mu,\alpha})'(\xi)|\,|G_0(\mu,\alpha)|$ for some $\xi\in[\beta^{-1},\ell_{\mu,\alpha}]\subset[\beta^{-1},1]$, there is a constant K (independent of small α and μ) such that

$$\frac{|\mu|}{K}<|G_0(\mu,\alpha)|=(\ell_{\mu,\alpha}-\beta^{-1})<K\,|\mu|. \tag{10}$$

We take small $\mu<0$ with

$$K\,|\mu|<<\min\{d^c,\tau^c\},\qquad \text{where } d^c \text{ and } \tau^c \text{ are as in Lemma 6.2.}$$

6.2.1 The induced map

Fix small $\mu<0$ and let $\alpha(\mu)$ be the parameter corresponding to the saddle-node bifurcation of the family $(g_\alpha+\mu)_\alpha$. Take $\alpha<\alpha(\mu)$ and note that $\Phi^{0,0}_{\mu,\alpha}=g_\alpha+\mu$ has no fixed points in $[c_{\mu,\alpha},1]$. Define

$$I_0(\mu,\alpha)=[\ell_{\mu,\alpha},c_{\mu,\alpha}]\quad\text{and}\quad H_0(\mu,\alpha)=[\beta^{-1},c_{\mu,\alpha}]=\{\beta^{-1}\}\cup G_0(\mu,\alpha)\cup I_0(\mu,\alpha).$$

Since $H_0(\mu,\alpha)$ is a fundamental domain of $\Phi^{0,0}_{\mu,\alpha}$ and the graph of $\Phi^{0,0}_{\mu,\alpha}$ is under the diagonal, there is a first $k_0\in\mathbb{N}$ such that $(\Phi^{0,0}_{\mu,\alpha})^{k_0}(1)\in H_0(\mu,\alpha)$. For each $k\in\{1,\dots,k_0\}$, let

$$L_k(\mu,\alpha)=(\Phi^{0,0}_{\mu,\alpha})^{-k}(L_0(\mu,\alpha))\cap[\beta^{-1},1],\qquad L=G,H,I. \tag{11}$$

Note that $I_{k_0}(\mu,\alpha)=\emptyset$ if $1\in G_{k_0}(\mu,\alpha)$ and $\Phi^{0,0}_{\mu,\alpha}((\ell_{\mu,\alpha},c_{\mu,\alpha}))=(0,\beta^{-1})$. Take the following partition $J_i(\mu,\alpha)$ of $(\ell_{\mu,\alpha},c_{\mu,\alpha}]$ by intervals accumulating to $\ell_{\mu,\alpha}$ whose interiors are pairwise disjoint:

$$(\ell_{\mu,\alpha},c_{\mu,\alpha}]=\bigcup_{i\geq 1} J_i(\mu,\alpha),\quad \text{where}\quad J_i(\mu,\alpha)=(\Phi^{0,0}_{\mu,\alpha})^{-1}(\beta^{-i}([\beta^{-1},1])). \tag{12}$$

Using this partition, we define the following induced map (depicted in Figure 5)

$$\Psi_{\mu,\alpha}\colon(\ell_{\mu,\alpha},c_{\mu,\alpha}]\to[\beta^{-1},1],\quad x\in J_i(\mu,\alpha)\mapsto\Psi_{\mu,\alpha}(x)=\beta^i(\Phi^{0,0}_{\mu,\alpha}(x))=\Phi^{i,0}_{\mu,\alpha}(x). \tag{13}$$

The map $\Psi_{\mu,\alpha}$ is bi-valuated in boundary of $J_i(\mu,\alpha)$ and maps the interior of $J_i(\mu,\alpha)$ onto $(\beta^{-1},1)$.

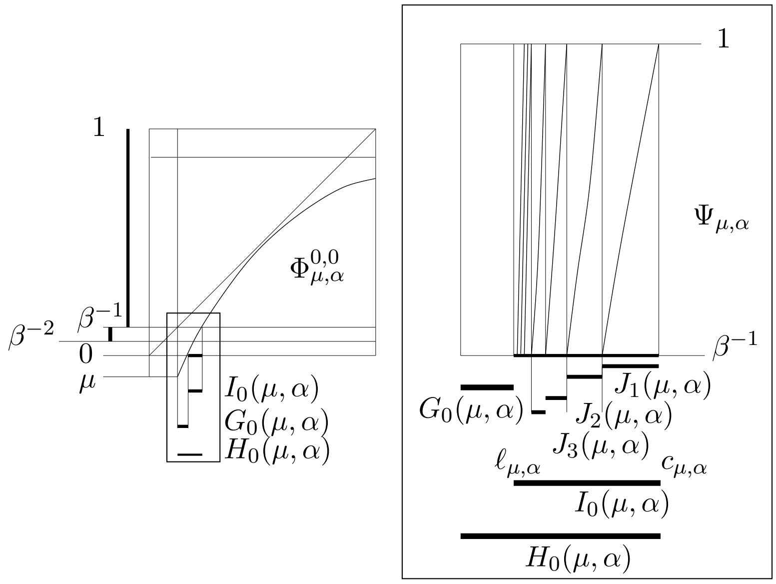


Figure 5: The induced map $\Psi_{\mu,\alpha}$

Define the map $\Gamma_{\mu,\alpha}$ (which is bi-valuated at infinitely many points) as follows (see Figure 6):

$$\Gamma_{\mu,\alpha}\colon I(\mu,\alpha)=\bigcup_{i=0}^{k_0} I_i(\mu,\alpha)\to[\beta^{-1},1],\quad x\in I_i(\mu,\alpha)\mapsto\Gamma_{\mu,\alpha}(x)=\Psi_{\mu,\alpha}\circ(\Phi^{0,0}_{\mu,\alpha})^i(x). \tag{14}$$

Remark 6.4. *Let m^+ and $m^->0$ be upper and lower bounds of the derivative of $\Phi^{0,0}_{\mu,\alpha}$ in $[\beta^{-1},1]$ independent of small μ and α. Then $\beta^i\,m^-\leq\Psi'_{\mu,\alpha}(x)\leq\beta^i\,m^+$ for all $x\in J_i(\mu,\alpha)$.*

Lemma 6.5. *The map $\Gamma_{\mu,\alpha}$ verifies the following:*

- *its lateral derivatives are defined at every point in the interior of $I(\mu,\alpha)$;*
- *it is uniformly expanding (i.e., there is $\rho>1$ such that every lateral derivative is strictly bigger than ρ); and*

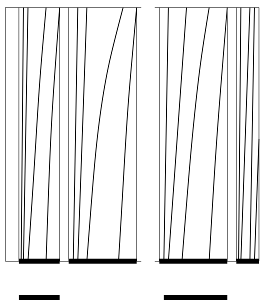


Figure 6: The map $\Gamma_{\mu,\alpha}$

- $\Gamma'_{\mu,\alpha}(x) \to +\infty$ *as* $x \to (\Phi^{0,0}_{\mu,\alpha})^{-i}(\ell_{\mu,\alpha})$, $i = 0, \ldots, k_0$.

Proof: The first and the third items follow immediately from the definition of $\Gamma_{\mu,\alpha}$. For the second assertion, we begin by estimating the derivative of $\Psi_{\mu,\alpha}$ in $I_0(\mu,\alpha)$. Take any $x \in I_0(\mu,\alpha)$, $x \in J_k(\mu,\alpha)$ for some $k \geq 1$, by definition,

$$\Psi_{\mu,\alpha}(x) = \beta^k\,(\Phi^{0,0}_{\mu,\alpha}(x)) = \beta^k\,(F^N_\alpha(x) + 3 + \mu) = \beta^k\,(g_\alpha(x) + \mu).$$

Since $J_k(\mu,\alpha) \subset [\beta^{-1}, c_{\mu,\alpha}]$, condition (F4) implies that $\Psi'_{\mu,\alpha}(y) > \beta$.

Suppose now that $x \in I_j(\mu,\alpha)$, $j \geq 1$, then $x, (\Phi^{0,0}_{\mu,\alpha})(x), \ldots, (\Phi^{0,0}_{\mu,\alpha})^{j-1}(x)$ belong to $[c_{\mu,\alpha}, 1]$, thus $\Phi^{0,0}_{\mu,\alpha}$ coincides with $g_\alpha + \mu$ along the first j iterates of x. Therefore,

$$\Gamma_{\mu,\alpha}(x) = \Psi_{\mu,\alpha} \circ (g_\alpha + \mu)^j(x).$$

We claim that $((g_\alpha + \mu)^{j-1})'((g_\alpha + \mu)(x)) > 1/2$, which implies the result: Since $(g_\alpha + \mu)^j(x) \in I_0(\mu,\alpha)$, $\Psi'_{\mu,\alpha}((g_\alpha + \mu)^j(x)) > \beta$, now it is enough to note that $\beta/4 > 1$ and $g'_\alpha(x) > 1/2$ for all x.

To prove the claim, note that, fixed small $\mu < 0$, $(g_\alpha + \mu)_\alpha$ is a saddle-node arc bifurcating at $\alpha(\mu) = \varepsilon\,(-1 + \sqrt{1 - 4\mu})$ close to 0. Repeating the proof of Proposition 4.1 and noting that, with the notation of the proposition, $j - 1 = n_\alpha((g_\alpha + \mu)(x))$, one gets, for small $\alpha < \alpha(\mu)$,

$$((g_\alpha + \mu)^{j-1})'((g_\alpha + \mu)(x)) > 1/2,$$

ending the proof of the claim and of the lemma. □

We are now ready to construct a dynamically defined Cantor set $\Sigma_{\mu,\alpha}$, associated to $\Gamma_{\mu,\alpha}$, with large thickness and small gaps, whose central localization is a hyperbolic set $\Omega_{\mu,\alpha}$ of index 2. The key step in this construction is Proposition 6.6, which together with Lemma 6.2 and (Gap) Lemma 7.1 are the ingredients of the persistence of cycles in the theorem, see Section 7.

Archetypally, the set $\Sigma_{\mu,\alpha}$ consists of the points in D^+ whose orbits by $\Gamma_{\mu,\alpha}$ do not meet $G_0(\mu,\alpha)$. In general, this set fails to be dynamically defined (and thus we do not know how to calculate its thickness). To bypass this difficulty, we consider a dynamically defined subset of it. This is done by constructing a Markov partition for $\Gamma_{\mu,\alpha}$ and defining $\Sigma_{\mu,\alpha}$ as the maximal invariant set associated to such a partition (this corresponds to exhibit a good presentation of $\Sigma_{\mu,\alpha}$).

Recall that, fixed small $\mu^\star < 0$, $\alpha(\mu^\star)$ is the parameter corresponding to the saddle-node bifurcation of the family $(g_\alpha + \mu^\star)_\alpha$. Thus the arc $(g_{\alpha(\mu^\star)} + \mu)_\mu$ unfolds a saddle-node at $\mu^\star$.

Proposition 6.6. *There are constants τ_0, $\tau_1 > 0$, and $0 < \eta \leq 1$, independent of small $\mu^\star < 0$, such that, for all $\mu < \mu^\star$, the map $\Gamma_{\mu,\alpha(\mu^\star)}$ has a dynamically defined Cantor set $\Sigma_{\mu,\alpha(\mu^\star)}$ such that:*

(a) *$\tau(\Sigma_{\mu,\alpha(\mu^\star)}) \geq \tau_0/|\mu|^\eta$, where τ denotes the thickness of the Cantor set,*

(b) *every (bounded) gap of the Cantor set $\Sigma_{\mu,\alpha(\mu^\star)}$ has length less than $\tau_1\,|\mu|^\eta$, and*

(c) *the diameter of $\Sigma_{\mu,\alpha(\mu^\star)}$ is bigger than $(1-\beta^{-1})/2$.*

Notational remark: *From now on, the parameter α is fixed and equal to $\alpha(\mu^\star)$, the saddle-node parameter of the arc $(g_\alpha + \mu^\star)_\alpha$, for some small negative $\mu^\star$. Thus for notational simplicity, the dependence on α of the maps and the associated points and sets will be omitted.*

For $\mu < \mu^\star$ and close to μ, to construct the set Σ_μ we consider a Markov partition. There are two cases, according to the position of $G_{k_0}(\mu)$.

6.2.2 Construction of the Markov partition. Case 1: $1 \in G_{k_0}(\mu)$ (i.e., $I_{k_0}(\mu) = \emptyset$)

Let $q_\mu \in I_{k_0-1}(\mu)$ be the closest point to 1 with $\Gamma_\mu(q_\mu) = q_\mu$ (the existence of q_μ is guaranteed by $\Gamma_\mu(I_{k_0-1}(\mu)) = [0,1]$). By construction, there is a sequence $x_n \in (\ell_\mu, 1)$ of fixed points of Γ_μ converging to ℓ_μ. In particular, there is a fixed point p_μ of Γ_μ arbitrarily close to ℓ_μ satisfying

$$\Gamma_\mu(p_\mu) = p_\mu, \quad p_\mu - \ell_\mu < |\mu|. \tag{15}$$

Lemma 6.7. *There are constants K_1, K_2, and K_3, independent of small $\mu^\star < 0$, such that, for all $\mu < \mu^\star$ close to $\mu^\star$,*

1. *$|G_{k_0}(\mu)| < K_1\,|G_0(\mu)| < K_2\,|\mu|$,*
2. *$(1-q_\mu) \leq \dfrac{K_1\,\rho}{\rho-1}\,|G_0(\mu)| \leq \dfrac{K_2\,\rho}{\rho-1}\,|\mu| \leq K_3\,|\mu|$, where $\rho > 1$ is as in Lemma 6.5.*

Proof: The inequalities in item (1) follow from Proposition 4.1, the first one, and equation (10), the second one. To prove item (2), let $G_{k_0}(\mu) = (1-v_\mu, 1)$. By construction, considering the branch of Γ_μ in $I_{k_0-1}(\mu)$, one has $\Gamma_\mu([q_\mu, 1-v_\mu]) = [q_\mu, 1]$. Since Γ_μ has no discontinuities in $[q_\mu, 1-v_\mu]$, by Lemma 6.5, there is $x \in (q_\mu, 1-v_\mu)$ such that

$$(1-q_\mu) = \Gamma'_\mu(x)(1-v_\mu-q_\mu) \geq \rho(1-v_\mu-q_\mu).$$

Since $v_\mu = |G_{k_0}(\mu)|$, using the first part of the lemma, one has that

$$(1-q_\mu) \leq \frac{\rho}{\rho-1}\,v_\mu \leq \frac{\rho}{\rho-1}\,K_1\,|G_0(\mu)| \leq \frac{\rho}{\rho-1}\,K_2\,|\mu| \leq K_3\,|\mu|,$$

ending the proof of the inequalities in the second item of the lemma. □

Let $A_0(\mu) = (\beta^{-1}, p_\mu)$ and $A_{k_0}(\mu) = (q_\mu, 1)$. Equations (10) and (15) and Lemma 6.7 imply:

Lemma 6.8. *There is a constant $K_4 > 0$, independent of small $\mu^\star$, such that*

$$\max\{|A_0(\mu)|, |A_{k_0}(\mu)|\} \le K_4\,|\mu|, \qquad \textit{for all } \mu < \mu^\star \textit{ close to } \mu^\star.$$

We now define the intervals corresponding to the *initial* gaps and bridges of the Cantor set Σ_μ:

- For $j \in \{1, \dots, (k_0-1)\}$, $A_j(\mu) = (a_j, b_j)$ is an interval containing $G_j(\mu)$ such that $\Gamma_\mu(a_j) = q_\mu$ and $\Gamma_\mu(b_j) = p_\mu$. More precisely, a_j is the biggest point in $I_{j-1}(\mu)$ with $\Gamma_\mu(a_j) = q_\mu$ and $b_j \in I_j(\mu)$ is chosen close to the right extreme of $G_j(\mu)$. A priori, there are infinitely many possibilities for the choice of b_j, we will explicit this choice below. In Figure 7 are depicted the gaps $A_0(\mu)$ and $A_1(\mu)$.
- For $j = 0, \dots, k_0 - 1$, $B_j(\mu) = [b_j, a_{j+1}]$, where $a_{k_0} = q_\mu$ and $b_0 = p_\mu$.

Remark 6.9. *By construction, $(\Phi_\mu^{0,0})^k(B_k(\mu)) = B_0(\mu)$.*

In $[p_\mu, c_\mu]$, the map $\Psi_\mu = \Gamma_\mu$ has finitely many discontinuities, say r. The number r increases as p_μ approaches to ℓ_μ, so r can be taken arbitrarily large after an appropriate choice of p_μ. We choose the points b_j above such that the restriction of Γ_μ to each $B_j(\mu)$ also has r discontinuity points $d_j^1 < d_j^2 < \cdots < d_j^r$, where Γ_μ is bi-valuated and takes the values β^{-1} and 1. Thus the restriction of Γ_μ to each $B_j(\mu)$ has $(r+1)$ (injective) branches, where

- $\Gamma_\mu([b_j, d_j^1]) = [p_\mu, 1]$,
- $\Gamma_\mu([d_j^i, d_j^{i+1}]) = [\beta^{-1}, 1]$, for $i = 1, \dots, r-1$, and
- $\Gamma_\mu([d_j^r, a_{j+1}]) = [\beta^{-1}, q_\mu]$.

To get a Markov partition of Γ_μ, we refine the bridges $B_j(\mu)$ by adding new gaps, see Figure 7.

- For each $j \in \{0, \dots, k_0-1\}$ and $i \in \{1, \dots, r\}$, $A_j^i(\mu)$ is the smallest interval (a_j^i, b_j^i) containing the discontinuity d_j^i such that $\Gamma_\mu(a_j^i) = q_\mu$ and $\Gamma_\mu(b_j^i) = p_\mu$.
- The intervals $B_j^0(\mu)$, $B_j^1(\mu), \dots, B_j^r(\mu)$ are the connected components of $B_j(\mu) \setminus \cup_{i=1}^r A_j^i(\mu)$, ordered in such a way $B_j^i(\mu)$ is at the left of $B_j^{i+1}(\mu)$. By construction, $\Gamma_\mu(B_j^i(\mu)) = [p_\mu, q_\mu]$. Note that $B_j^i(\mu) = [b_j^i, a_j^{i+1}]$ and that $b_0^0 = p_\mu$.

The previous construction and Lemma 6.5 imply the following:

Proposition 6.10. *Let $\mathcal{B}_\mu = \{B_j^i(\mu)\}_{j=0,\dots,(k_0-1)}^{i=1,\dots,r}$, $\mathbb{B}_\mu = \cup_{i,j} B_j^i(\mu)$, and Σ_μ be the maximal invariant set of Γ_μ in $\mathbb{B}_\mu$. The set Σ_μ is a (expanding) dynamically defined Cantor set associated to the Markov partition $\mathcal{B}_\mu$.*

Throughout this section, in our estimates of the thickness, we use repeatedly the next result:

Lemma 6.11. (Bounded distortion property, [PT$_3$, Theorem 1 page 58]) *Let Σ be a dynamically defined Cantor set of a $C^{1+\varepsilon}$-expanding map Γ. Given any $\delta > 0$, there is $\kappa(\delta) > 0$ such that: for every pair of points x and y and $n \ge 1$ such that $|\Gamma^n(x) - \Gamma^n(y)| \le \delta$ and the interval of extremes $\Gamma^i(x)$ and $\Gamma^i(y)$ is contained in the domain of definition of Γ, for all $i = 0, 1, \dots, (n-1)$, it holds*

$$|\log|(\Gamma^n)'(x)| - \log|(\Gamma^n)'(y)|| \le \kappa(\delta).$$

Moreover, $\kappa(\delta)$ converges to 0 as $\delta \to 0$.

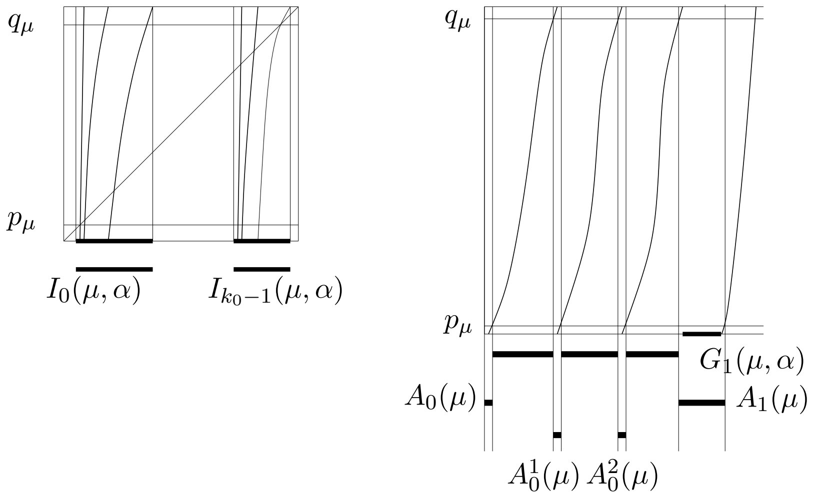


Figure 7: Initial gaps and bridges

We apply this lemma to the set Σ_μ associated to the expanding map Γ_μ (restricted to the Markov partition $\mathbb{B}_\mu$). In our context, it is enough to fix $\delta = (1 - \beta^{-1})$, which is an upper bound for the length of any interval of the partition, and let $\kappa_\mu = \kappa_\mu(\delta)$, the constant given by the lemma. Using the definition of Γ_μ, see (13) and (14), and that it is C^2, one gets

$$\kappa_\mu \leq \frac{\max |\Psi''_\mu(x)|}{(\min |\Psi'_\mu(y)|)^2} \, (1 - \beta^{-1}) \, \frac{1}{\rho - 1},$$

where $\rho > 1$ is as in Lemma 6.5. Thus $\kappa_\mu = \kappa$ can be taken independent of small μ. This inequality (whose proof we omit here) is obtained adapting straightforwardly the proof of Lemma 6.11 to the C^2-case (see [PT$_3$, page 59]).

To estimate the thickness of Σ_μ we choose the following presentation $\mathcal{A}_\mu$ of it:

Presentation of $\mathcal{A}_\mu$: *The generation* $(0, 1)$ *of gaps is* $A_1(\mu), A_2(\mu), \ldots, A_{k_0}(\mu)$. *The generation* $(0, 2)$ *of gaps is defined by*

$$A_0^1(\mu), A_0^2(\mu), \ldots, A_0^r(\mu), A_1^1(\mu), A_1^2(\mu), \ldots, A_1^r(\mu), \ldots, A_{k_0}^1(\mu), A_{k_0}^2(\mu), \ldots, A_{k_0}^r(\mu).$$

Inductively, for each $i = 1$ *or* 2, *we define the* $(k + 1, i)$*-generation of gaps as the pre-images by* Γ_μ *of the gaps of the* (k, i)*-generation, enumerated according to the ordering in* $\mathbb{R}$. *Moreover, any gap of generation* $(k + 1, 2)$ *is posterior to any gap of generation* $(k + 1, 1)$.

Take a gap G of (n, i)-generation and a bridge B of G. Then the number n and any pair of points $x \in G$ and $y \in B$ verify the hypotheses of Lemma 6.11. Moreover, $\Gamma_\mu^n(G)$ is a gap of $(0, i)$-generation and $\Gamma_\mu^n(B)$ is a bridge of it. Using Lemma 6.11 one gets (this is a standard argument)

$$e^{-\kappa} \frac{|B|}{|G|} \leq \frac{|\Gamma_\mu^n(B)|}{|\Gamma_\mu^n(G)|} \leq e^{\kappa} \frac{|B|}{|G|}. \tag{16}$$

Remark 6.12. *Equation* (16) *implies that if* $\tau > 0$ *is such that* $|B|/|G| > \tau$, *for every gap* G *of generation* $(0, 1)$ *or* $(0, 2)$ *and every bridge* B *of* G, *then the thickness of* Σ_μ *is bigger than* $\tau \, e^{-\kappa}$.

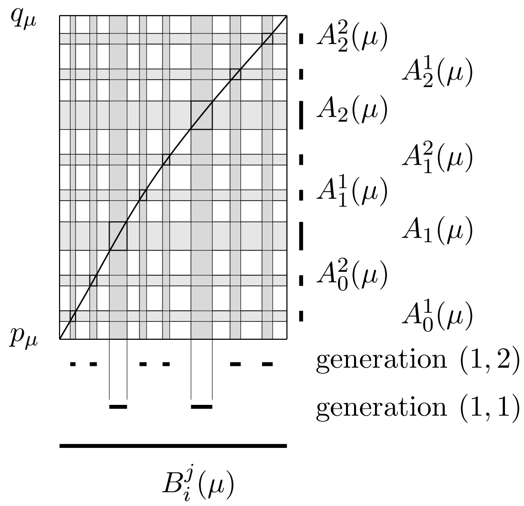


Figure 8: New generation of gaps

By Remark 6.12, to get a lower bound for $\tau(\Sigma_\mu)$ (i.e. to prove Proposition 6.6 in Case 1) it suffices to estimate the ratios between bridges and gaps of generations $(0,1)$ and $(0,2)$. This is done in the next lemma.

Lemma 6.13. *There is K_0, independent of small $\mu^\star < 0$, such that, for every gap G of generation $(0,1)$ or $(0,2)$ and any bridge B of G, it holds $|B|/|G| > K_0/|\mu|$, for every $\mu < \mu^\star$ close to $\mu^\star$.*

Proof: We first estimate $|B_0(\mu)|/|A_1(\mu)|$ and $|B_1(\mu)|/|A_1(\mu)|$. Note that $B_0(\mu)$ is the left bridge of $A_1(\mu)$ and $B_1(\mu)$ is contained in the right bridge of $A_1(\mu)$. To estimate the length of $A_1(\mu) = (a_1, b_1)$, write $G_1(\mu) = (c_\mu, g_1^r) \subset A_1(\mu)$ (see equation (11)). Then

$$|A_1(\mu)| = b_1 - a_1 = (b_1 - g_1^r) + |G_1(\mu)| + (c_\mu - a_1).$$

As c_μ and a_1 are in the same branch of Γ_μ, $\Gamma_\mu(c_\mu) = 1$ and $\Gamma_\mu(a_1) = q_\mu$, Lemmas 6.5 and 6.7 imply

$$(c_\mu - a_1) \le \frac{1 - q_\mu}{\rho} \le \frac{K_3\,|\mu|}{\rho}. \tag{17}$$

Also, since by construction, $\Gamma_\mu(G_1(\mu)) = G_0(\mu)$, from Lemmas 6.5 and 6.7,

$$|G_1(\mu)| \le \frac{|G_0(\mu)|}{\rho} \le \frac{K_2\,|\mu|}{K_1\,\rho} = \frac{K_5\,|\mu|}{\rho}.$$

By construction, $\Phi_\mu^{0,0}(g_1^r) = \ell_\mu$ and $\Phi_\mu^{0,0}(b_1) = p_\mu$, Proposition 4.1 and equation (15) imply that

$$b_1 - g_1^r \le 2\,(p_\mu - \ell_\mu) \le 2\,|\mu|.$$

Putting these inequalities together, one gets

$$|A_1(\mu)| \le \frac{K_3 + K_5 + 2\,\rho}{\rho}\,|\mu| = K_6\,|\mu|. \tag{18}$$

To estimate $B_0(\mu) = [p_\mu, a_1]$ write

$$|B_0(a_\mu)| = (c_\mu - \ell_\mu) - (p_\mu - \ell_\mu) - (c_\mu - a_1).$$

Note first that the size of $[\ell_\mu, c_\mu]$ is uniformly bounded away from zero (independent of small $\mu^\star$). By (15) and (17), $(p_\mu - \ell_\mu) + (c_\mu - a_1) \leq (1 + (K_3/\rho))\,|\mu|$. Thus there is a constant K_7 such that

$$|B_0(a_\mu)| > K_7 > 0.$$

Finally, by Remark 6.9 and Lemma 6.3, the previous inequality implies that

$$|B_1(\mu)| > K_8 > 0.$$

All the estimates above lead to

$$\min\left\{\frac{|B_1(\mu)|}{|A_1(\mu)|}, \frac{|B_0(\mu)|}{|A_1(\mu)|},\right\} \geq \frac{K_9}{|\mu|}. \tag{19}$$

Similarly, using Lemma 6.3, Remark 6.9, and equation (19), one gets

$$\min\left\{\frac{|B_i(\mu)|}{|A_i(\mu)|}, \frac{|B_{i-1}(\mu)|}{|A_i(\mu)|},\right\} \geq \frac{L\,K_9}{|\mu|} = \frac{K_{10}}{|\mu|}. \tag{20}$$

Observing that, for each i, $B_{i-1}(\mu)$ is the left bridge of $A_i(\mu)$ and $B_i(\mu)$ is contained in the right bridge of $A_i(\mu)$, this ends the estimates for the generation $(0,1)$.

We now estimate the ratio between bridges and gaps of generation $(0,2)$. In this step, each interval $B_j(\mu)$ is subdivided into $(r+1)$ new bridges $B_j^i(\mu)$, $i = 0,\ldots,r$, obtained from $B_j(\mu)$ by removing the intervals $A_j^i(\mu)$ around the discontinuities d_j^i, $i = 1,\ldots,r$, where r is the number of of discontinuities of Ψ_μ in $[p_\mu, c_\mu]$. As before, bearing in mind Lemma 6.3, we first estimate these ratios for the bridges $B_0^i(\mu)$.

Observe that, by definition, $B_0^i(\mu) = [b_0^i, a_0^{i+1}] \subset J_{r+1-i}(\mu)$ (recall equation (12)) and

$$\Gamma_\mu((d_0^i, b_0^i)) = (\beta^{-1}, p_\mu) \quad \text{and} \quad \Gamma_\mu((a_0^i, d_0^i)) = (q_\mu, 1).$$

Recall that $|(\beta^{-1}, p_\mu)| < (K+1)\,|\mu|$ (see (10) and (15)) and $|(q_\mu, 1)| < K_3\,|\mu|$ (Lemma 6.7). Note that $(a_0^i, d_0^i) \cup B_0^i(\mu) \subset J_{r+1-i}(\mu)$ and that $(d_0^i, b_0^i) \subset J_{r+1-i-1}(\mu)$. Remark 6.4 now implies that

$$|A_0^i(\mu)| = (b_0^i - a_0^i) = (b_0^i - d_0^i) + (d_0^i - a_0^i) \leq \frac{(K+1+K_3)\,\mu}{\beta^{r-i}\,m^-} = \frac{K_{11}\,|\mu|}{\beta^{r-i}\,m^-}. \tag{21}$$

Analogously, as $\Psi_\mu(B_0^i(\mu)) = [p_\mu, q_\mu]$, and the size of $[p_\mu, q_\mu]$ is uniformly lower bounded, one has

$$|B_0^i(\mu)| \geq \frac{K_{12}}{\beta^{r-i+1}\,m^+}. \tag{22}$$

Now, from inequalities (21) and (22), it follows that, for $i = 1,\ldots,r$,

$$\min\left\{\frac{|B_0^{i-1}(\mu)|}{|A_0^i(\mu)|}, \frac{|B_0^i(\mu)|}{|A_0^i(\mu)|},\right\} \geq \frac{K_{13}}{|\mu|}. \tag{23}$$

Arguing exactly as in the gaps of generation $(0,1)$, by using Lemma 6.3, for all $i = 1,\dots,r$ and $j = 0,\dots,k_0$, one gets

$$\min\left\{\frac{|B_j^{i-1}(\mu)|}{|A_j^i(\mu)|}, \frac{|B_j^i(\mu)|}{|A_j^i(\mu)|},\right\} \geq \frac{K_{14}}{|\mu|}. \tag{24}$$

Since, for each i, $B_j^{i-1}(\mu)$ is the left bridge of $A_j^i(\mu)$ and $B_j^i(\mu)$ is contained in the right bridge of $A_j^i(\mu)$, this ends the estimates for the generation $(0,2)$ and the proof of the lemma. ☐

Part (b) of Proposition 6.6 follows from Lemma 6.5, noting that the previous estimates show that the lengths of the gaps of $(0,1)$ and $(0,2)$-generations are (at most) of order of $|\mu|$. Finally, to get (c), just note that, by Lemmas 6.7 and 6.8,

$$\text{diameter}(\Sigma_\mu) = q_\mu - p_\mu = (1-\beta^{-1}) - (1-q_\mu) - (p_\mu - \beta^{-1}) \geq (1-\beta^{-1}) - K_3\,|\mu| - K_4\,|\mu|.$$

The proof of Proposition 6.6 in Case 1 is now complete.

6.2.3 Construction of the Markov partition. Case 2. $1 \notin G_{k_0}(\mu)$ (i.e., $1 \in I_{k_0}(\mu)$)

Let $\tau > 2/\tau^c$ (τ^c as in Lemma 6.2). We will modify the Markov partition in Case 1 in order to construct a dynamically defined Cantor set Σ_μ (associated to such a partition) such that the ratios bridges/gaps of the *zero generation* are (at least) of order of $1/|\mu|^\eta$, for some $\eta \in (0,1)$. By Remark 6.12 and Proposition 6.6 the thickness of the set Σ_μ is larger than τ. We consider two cases, according to the size of $I_{k_0}(\mu)$. Let

$$\xi = 1/|\mu^\star|^{3/5}. \tag{25}$$

Case 2.1. $|I_{k_0}(\mu)| \leq \xi\,|\mu|$**.** We modify the construction of Case 1 replacing $G_{k_0}(\mu)$ by

$$G'_{k_0}(\mu) = G_{k_0}(\mu) \cup I_{k_0}(\mu)$$

and defining p_μ and q_μ exactly as before (i.e., we add the relatively small interval $I_{k_0}(\mu)$ to the relatively big gap $G_{k_0}(\mu)$). The estimate for $p_\mu - \ell_\mu$ (equation (15)) is exactly the same and the one for $(1-q_\mu)$ follows as in Lemma 6.7, but in this case one gets

$$(1-q_\mu) \leq (K_3 + \xi)\,|\mu|.$$

We consider the dynamically defined Cantor set Σ_μ associated to the map Γ_μ in $[p_\mu, q_\mu]$ (the Markov partition and the gaps of generations $(0,1)$ and $(0,2)$ are defined as before). In this way, the Cantor set Σ_μ verifies Proposition 6.6 (taking η equal to $\frac{2}{5}$).

Case 2.2. $|I_{k_0}(\mu)| > \xi\,|\mu|$**.** We split $I_{k_0}(\mu)$ into two intervals $I_{k_0}^\ell(\mu)$ and $I_{k_0}^r(\mu)$ with disjoint interiors, where $I_{k_0}^\ell(\mu)$ is the domain of all (infinitely many) surjective branches of the restriction of Γ_μ to $I_{k_0}(\mu)$ and $I_{k_0}^r(\mu)$ is its complement (corresponding to the only non surjective branch of Γ_μ). See Figure 9. Write $\Gamma_\mu(I_{k_0}^r(\mu)) = [\beta^{-1}, r]$, where $r < 1$. By Lemma 6.5,

$$|I_{k_0}^r(\mu)| \leq \frac{(r-\beta^{-1})}{\rho}. \tag{26}$$

There are two sub-cases, according to the ratios between the sizes of $I_{k_0}^r(\mu)$ and $I_{k_0}^\ell(\mu)$: either $|I_{k_0}^r(\mu)|/|I_{k_0}^\ell(\mu)| \leq \xi\,|\mu|$ (Case 2.2.1) or $|I_{k_0}^r(\mu)|/|I_{k_0}^\ell(\mu)| > \xi\,|\mu|$ (Case 2.2.2).

Case 2.2.1. $|I^r_{k_0}(\mu)|/|I^\ell_{k_0}(\mu)| \le \xi\,|\mu|$**.** In this case $I^r_{k_0}(\mu)$ is considered as an extremal gap and plays a role similar to the gap $G_{k_0}(\mu)$ in Case 1. Let q'_μ be the fixed point of Γ_μ in $I^\ell_{k_0}(\mu)$ closest to 1. Consider the dynamically defined Cantor set Σ_μ in $[p_\mu, q'_\mu]$ associated to Γ_μ and the Markov partition obtained by modifying the one in Case 1 as follows:

- We consider k_0 gaps $A_i(\mu)$ ($i = 1, \dots, k_0$) of generation $(0,1)$ (instead of $k_0 - 1$ as in Case 1), where each gap $A_i(\mu)$ contains $G_i(\mu)$ and is defined as in Case 1 replacing q_μ by q'_μ.

- The gaps of generation $(0,2)$ are obtained as the ones in Case 1, but now adding the discontinuities of Γ_μ in $I^\ell_{k_0}(\mu)$.

By Remark 6.12, it is enough to estimate the ratios bridge/gap for the generations $(0,1)$ and $(0,2)$. We first consider the generation $(0,1)$. Since $I^r_{k_0}(\mu)$ is (at most) of order of $\xi\,|\mu| \simeq |\mu|^{2/5}$, arguing as in (2) in Lemma 6.7, it follows that $|1-q'_\mu|$ is of the same order as $|I^r_{k_0}(\mu)|$, hence $|1-q'_\mu|$ is at most of order of $|\mu|^{2/5}$. This implies that the ratios bridge/gap of generation $(0,1)$, excluded $|I^\ell_{k_0}(\mu)|/|A_{k_0}(\mu)|$, are of order of (at least) $1/|\mu|^{2/5}$. As in Case 1, to estimate $|I^\ell_{k_0}(\mu)|/|A_{k_0}(\mu)|$ it suffices to estimate $|I^\ell_{k_0}(\mu)|/|G_{k_0}(\mu)|$ (these quotients are of the same order). Note that

$$\xi\,|\mu| \le |I_{k_0}(\mu)| = |I^r_{k_0}(\mu)| + |I^\ell_{k_0}(\mu)| = |I^\ell_{k_0}(\mu)| \left(\frac{|I^r_{k_0}(\mu)|}{|I^\ell_{k_0}(\mu)|} + 1 \right) \le |I^\ell_{k_0}(\mu)|\,(\xi\,|\mu| + 1).$$

Thus,

$$|I^\ell_{k_0}(\mu)| \ge \frac{\xi\,|\mu|}{1+\xi\,|\mu|}.$$

Arguing as in Lemma 6.7, one gets $|G_{k_0}(\mu)| < K_2\,|\mu|$. Finally, recalling the definition of ξ in (25),

$$\frac{|I^\ell_{k_0}(\mu)|}{|G_{k_0}(\mu)|} \ge \frac{\xi}{K_2\,(1+\xi\,|\mu|)} = K'_2\,\frac{1}{|\mu|^{3/5}} \ge K'_2\,\frac{1}{|\mu|^{2/5}},$$

ending the estimates for the generation $(0,1)$.

The estimates for the ratios bridge/gap of generation $(0,2)$ follow as in Case 1, but now these ratios are of order of (at least) $1/|\mu|^{2/5}$. The previous calculations and the choice of ξ in (25) imply that, for small $\mu^\star$ and $\mu < \mu^\star$ close to $\mu^\star$, the ratios bridge/gap of generations $(0,1)$ and $(0,2)$ are at least of order of $1/|\mu|^{2/5}$. Thus, by Remark 6.12, the set Σ_μ verifies Proposition 6.6 taking $\eta = 2/5$. Finally, the lower bound for the diameter of Σ_μ is obtained similarly replacing q_μ by q'_μ.

Case 2.2.2. $|I^r_{k_0}(\mu)|/|I^\ell_{k_0}(\mu)| > \xi\,|\mu|$**.** We claim that, in this case, $\Gamma_\mu(I^r_{k_0}(\mu))$ contains (at least) two surjective branches of Γ_μ. Note first that we can choose the fixed point p_μ of Γ_μ arbitrarily close to ℓ_μ such that there is $s_\mu > p_\mu$ such that $[p_\mu, s_\mu]$ contains two consecutive branches of Γ_μ and $(s_\mu - \ell_\mu) < |\mu|$. Since $\Gamma_\mu(I^r_{k_0}(\mu)) = [\beta^{-1}, r]$ and $G_0(\mu) = (\beta^{-1}, \ell_\mu)$, to get $(p_\mu, s_\mu) \subset \Gamma_\mu(I^r_{k_0}(\mu))$ it is enough to see that $(\beta^{-1}, s_\mu) \subset \Gamma_\mu(I^r_{k_0}(\mu))$. By equation (10),

$$(s_\mu - \beta^{-1}) \le |G_0(\mu)| + |\mu| \le K\,|\mu| + |\mu| < |\Gamma_\mu(I^r_{k_0}(\mu))|.$$

We argue by contradiction, if the last inequality does not hold, then, by Lemma 6.5,

$$|I^r_{k_0}(\mu)| \le \frac{(1+K)\,|\mu|}{\rho}.$$

Thus, by hypothesis,

$$\xi\,|\mu| < |I_{k_0}(\mu)| = |I^r_{k_0}(\mu)| + |I^\ell_{k_0}(\mu)| = |I^r_{k_0}(\mu)| \left(1 + \frac{|I^\ell_{k_0}(\mu)|}{|I^r_{k_0}(\mu)|}\right) \le \frac{(K+1)\,|\mu|}{\rho} \left(1 + \frac{1}{\xi\,|\mu|}\right).$$

Hence,

$$\xi \le \frac{(K+1)}{\rho} \left(1 + \frac{1}{\xi\,|\mu|}\right),$$

which, by the definition of ξ in (25), is not verified for μ close to small $\mu^\star$. This proves the claim.

By the claim, there is $h \ge 3$ such that $I^r_{k_0}(\mu)$ is of the form

$$I^r_{k_0}(\mu) = \bigcup_{j=1}^{h} (G_{k_0+j}(\mu) \cup I_{k_0+j}(\mu)), \qquad \text{where}$$

- the interval $I_{k_0+h}(\mu)$ may be empty,
- $\Gamma_\mu(G_{k_0+j}(\mu)) = G_{j-1}(\mu)$ and $\Gamma_\mu(I_{k_0+j}(\mu)) = I_{j-1}(\mu)$, if $j \ne h$,
- if $j = h$ and $I_{k_0+h}(\mu) \ne \emptyset$, then $\Gamma_\mu(G_{k_0+h}(\mu)) = G_{h-1}(\mu)$ and $\Gamma_\mu(I_{k_0+h}(\mu)) \subset I_{h-1}(\mu)$,
- if $j = h$ and $I_{k_0+h}(\mu) = \emptyset$, then $\Gamma_\mu(G_{k_0+h}(\mu)) \subset G_{h-1}(\mu)$.

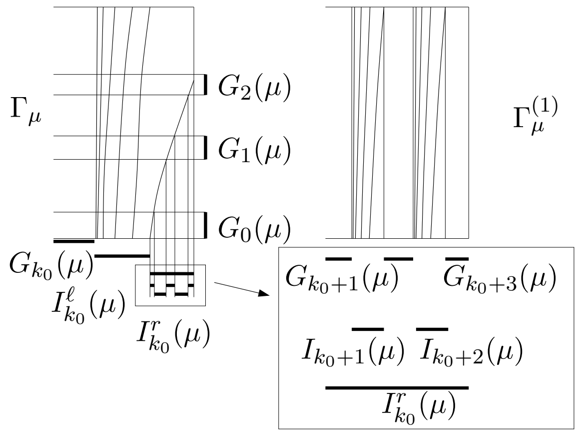


Figure 9: New gaps and the map $\Gamma^{(1)}_\mu$

Consider now the map (see Figure 9)

$$\Gamma^{(1)}_\mu \colon [\beta^{-1},1] \setminus (\cup_{j=0}^{k_0+h} G_j(\mu)) \to [\beta^{-1},1], \qquad \begin{cases} \Gamma^{(1)}_\mu(y) = \Gamma_\mu(y), & \text{if } y \notin I^r_{k_0}(\mu), \\ \Gamma^{(1)}_\mu(y) = \Gamma_\mu \circ \Gamma_\mu(y), & \text{if } y \in I^r_{k_0}(\mu). \end{cases}$$

Remark 6.14. *By construction and Lemma 6.5, the map $\Gamma^{(1)}_\mu$ is ρ-expanding. Moreover, its restriction to $I_{k_0+1}(\mu), \ldots, I_{k_0+h}(\mu)$ is ρ^2-expanding.*

As in the case of the map Γ_μ, we consider two cases according to the position of 1.

Case (a): the interval $I_{k_0+h}(\mu)$ is empty, or, equivalently, $1 \in G_{k_0+h}(\mu)$.

Case (b): $I_{k_0+h}(\mu) \ne \emptyset$.

Case (a). In this case, we argue as in Case 1 (Section 6.2.2) with the following modifications.

- Take the fixed points p_μ (the same as above) and q_μ^1 (the fixed point of $\Gamma_\mu^{(1)}$ closest to 1). The gaps of generation $(0,1)$ are the intervals $A_0(\mu), \dots, A_{k_0+h}(\mu)$, each of them containing the corresponding $G_j(\mu)$ and whose extremes are mapped by $\Gamma_\mu^{(1)}$ into p_μ and q_μ^1.
- To define the gaps of generation $(0,2)$, consider the discontinuities d_i^j, $d_i^j \in I_i(\mu)$, of $\Gamma_\mu^{(1)}$ and, for each d_i^j, the interval $A_i^j(\mu)$ containing d_i^j and mapped onto $[\beta^{-1}, p_\mu) \cup (q_\mu^1, 1]$ by $\Gamma_\mu^{(1)}$.

As in Case 1, the key step is to estimate the ratios bridges/gaps of generations $(0,1)$ and $(0,2)$. As before, it is enough to estimate the ratios $|I_j(\mu)|/|G_j(\mu)|$ and $|I_{j-1}(\mu)|/|G_j(\mu)|$ instead of the ratios of the bridges and gaps (for notational simplicity, we let $I_{k_0}^\ell(\mu) = I_{k_0}(\mu)$). First, as in Case 1, one has $|1 - q_\mu^1| < (K_{15}\,|\mu|)/\rho^2)$, where K_{15} is independent of small μ. We have the following:

- For $j = 1, \dots, k_0 - 1$, the estimates for the quotients
$$\frac{|I_j(\mu)|}{|G_j(\mu)|} \quad \text{and} \quad \frac{|I_{j-1}(\mu)|}{|G_j(\mu)|}$$
are the similar to the ones made in Case 1. This also holds for $|I_{k_0-1}(\mu)|/|G_{k_0}(\mu)|$
- Since the *old* interval $I_{k_0}(\mu)$ was subdivided, we need to estimate the ratios
$$\frac{|I_{k_0}(\mu)|}{|G_{k_0}(\mu)|} \quad \text{and} \quad \frac{|I_{k_0}(\mu)|}{|G_{k_0+1}(\mu)|},$$
considering the *new* $I_{k_0}(\mu) = I_{k_0}^\ell(\mu)$. This is done using Claim 6.15 below.
- The ratios $|I_{k_0+j}(\mu)|/|G_{k_0+j}(\mu)|$ and $|I_{k_0+j}(\mu)|/|G_{k_0+j+1}(\mu)|$, $j \geq 1$, are calculated exactly as the ratios of gaps of generation $(1,1)$ in Case 1.

Claim 6.15. $|I_{k_0}^r(\mu)| < K_{16}\,|I_{k_0}^\ell(\mu)|$, K_{16} *is independent of* μ *close to* $\mu^\star$.

The claim follows immediately from equation (16), taking $B = I_{k_0}^\ell(\mu)$ and $A = I_{k_0}^r(\mu)$ and noting that $I_{k_0}^\ell(\mu) \cup I_{k_0}^\ell(\mu)$ is contained in one interval (branch) of definition of Γ_μ, see Lemma 6.11.

By Lemma 6.7, $|G_{k_0}(\mu)| < K_2\,|\mu|$, by the claim and hypothesis (Case 2.2), one has
$$\frac{\xi}{K_2} = \frac{\xi\,|\mu|}{K_2\,|\mu|} \leq \frac{|I_{k_0}(\mu)|}{|G_{k_0}(\mu)|} = \frac{|I_{k_0}^\ell(\mu)| + |I_{k_0}^r(\mu)|}{|G_{k_0}(\mu)|} \leq \frac{(K_{16}+1)\,|I_{k_0}^\ell(\mu)|}{|G_{k_0}(\mu)|}.$$
Thus
$$\frac{|I_{k_0}^\ell(\mu)|}{|G_{k_0}(\mu)|} > \frac{\xi}{K_2\,(1+K_{16})}.$$

We now estimate $|I_{k_0}^\ell(\mu)|/|G_{k_0+1}(\mu)|$. From the definition of $G_{k_0+1}(\mu)$, inequality (10), and Lemma 6.5,
$$|G_{k_0+1}(\mu)| \leq \frac{|G_0(\mu)|}{\rho} \leq \frac{K\,|\mu|}{\rho}.$$

By hypothesis (Case 2.2) and the claim,

$$\xi\,|\mu| \leq |I_{k_0}(\mu)| = |I^{\ell}_{k_0}(\mu)| + |I^{r}_{k_0}(\mu)| \leq (1+K_{16})\,|I^{\ell}_{k_0}(\mu)|.$$

Putting these inequalities together we have,

$$\frac{|I^{\ell}_{k_0}(\mu)|}{|G_{k_0+1}(\mu)|} > \frac{\xi\,\rho}{(1+K_{16})\,K}.$$

The estimates for the ratios bridge/gap of generation $(0,2)$ follow as in Case 1, using Lemma 6.11 and noting that the sizes of $[\beta^{-1}, p_\mu]$ and $[q^1_\mu, 1]$ are of order of (at most) $|\mu|$. Arguing as in the previous case, this implies that all the ratios $|I|/|G|$ of generations $(0,1)$ and $(0,2)$ are of order of (at least) $1/|\mu|^{2/5}$. This ends the estimate of the thickness of Σ_μ in Case (a).

Case (b). In this the situation similar to Case 2, now considering the map $\Gamma^{(1)}_\mu$ and $1 \in I_{k_0+h}(\mu) = I_{k_1}(\mu)$. We proceed exactly as before, if we fall in Cases (similar to) 1, 2.1, 2.2, 2.2.1 or 2.2.2(a) then the construction goes as above ends and the proof of Proposition 6.6 is finished.

If we are in Case 2.2.2(b), we define a new map $\Gamma^{(2)}_\mu$, obtained from $\Gamma^{(1)}_\mu$ exactly as $\Gamma^{(1)}_\mu$ from Γ_μ. There is a new interval $I_{k_2}(\mu)$ containing the point 1. Since the definition of $I_{k_2}(\mu)$ involves two backwards iterations by Γ_μ, by Lemma 6.5, $|I_{k_2}(\mu)| \leq 1/\rho^2$.

Lemma 6.11 and equation (16) guarantee that all the constants in Case 2.2.2(b) for $\Gamma^{(1)}_\mu$ are the same as the ones obtained in the first inductive step. We now proceed inductively: in each step n either the construction is finished (i.e., we fall in *new* Cases 1, 2.1, 2.2, 2.2.1 or 2.2.2(a)) or we define a new map $\Gamma^{(n)}_\mu$ and an interval $I_{k_n}(\mu)$ of size less than $1/\rho^n$ containing the point 1. Therefore, there is a first n such that $|I_{k_n}(\mu)| < 1/\rho^n < \xi\,|\mu|$, so we are in a situation similar to Case 2.1 and the recurrent procedure ends.

The proof of item (a) (lower bound for the thickness) of Proposition 6.6 is now complete. By construction, all the gaps of Σ_μ (in all sub-cases) are at most of order of $|\mu|^{2/5}$, implying (b). The claim on the diameter follows noting that the convex hull of Σ_μ always contains $[p_\mu, q_\mu]$.

6.3 The thick hyperbolic unstable set Ω_t

Consider $t \in B_n$, $t = \lambda^n(1+\mu)$, where $\mu < \mu^\star$ and close to $\mu^\star$. Recall that α is fixed and equal to $\alpha(\mu^\star)$ and the transition map F associated to the arc f_t is $F_{\alpha(\mu^\star)}$. The (transitive) hyperbolic set Ω_t of index two is the (minimal) f_t-invariant set such that:

- **(a)** The central localization of Ω_t (see (9) for the definition) is $(\Omega_t)^c = \Sigma_\mu$.
- **(b)** Every $X = (x,y,z) \in \Omega_t \cap \Delta^+$ has *consecutive* returns (by f_t) to Δ^+ such that the central dynamics of the composition of the returns is $\Gamma^{(n)}_\mu$ (n is the last step in the inductive process of the construction of Σ_μ),

$$f^r_t(X) = (x', y', z') \quad \text{and} \quad y' = \Gamma^n_\mu(y),$$

where r is the number of iterations for obtaining such a final return.

Let us explain a bit more condition (b) and why the set Ω_t is hyperbolic (of index two). Let us assume, for simplicity, that $\Sigma_\mu = (\Omega_t)^c$ is obtained in the first step of the process (i.e., associated to Γ_μ). Note that $y \in \cup_{j=0}^{k_0} I_j(\mu)$, see (11). First, if $y \in I_0(\mu)$ then $y \in J_i(\mu)$ and thus it has a return of type $(i, 0)$, recall the definitions of the intervals $J_i(\mu)$ in (12) and of return of type (p, q) in Section 3.2 (just before equation (2)). Using Remark 3.3 one has

$$f_t^{i+n+k_0+0+n+N}(x, y, z) = (x', y', z'), \qquad y' = \Psi_\mu(y) = \Phi_\mu^{i,0}(y) = \Gamma_\mu(y).$$

By Lemma 6.5, $Df_t^{i+n+k_0+n+N}(X)$ expands in the central direction (at least) by a factor $\rho > 1$.

Second, if $y \in I_k(\mu)$ then $(\Phi_\mu^{0,0})^k$ belongs to some subinterval $J_i(\mu)$ of $I_0(\mu)$. Therefore, using Remark 3.3, the point $X = (x, y, z) \in \Delta^+$ has k consecutive returns to Δ^+ of type $(0, 0)$ (each one corresponding to $(0 + n + k_0 + 0 + n + N)$ iterations of f_t) followed by a return of type $(i, 0)$ (corresponding to $(i + n + k_0 + 0 + n + N)$ iterations). Thus

$$f_t^{(i+n+k_0+n+N)+k\,(n+k_0+n+N)}(x, y, z) = (x', y', z'), \qquad y' = \Gamma_\mu(y).$$

By Lemma 6.5, the central direction is (at least) ρ-expanding, $\rho > 1$. As the number of iterations involved in those returns is upper bounded (the bound depending on the localization of p_μ and the number k_0 of initial bridges of the Markov partition), one gets the hyperbolicity of the set Ω_t.

If the set Σ_μ is obtained in the step n of the process (associated to $\Gamma_\mu^{(n)}$), one needs to consider other types of returns (defined similarly). The hyperbolicity follows from the fact that $\Gamma_\mu^{(n)}$ is ρ-expanding and that the returns involve a uniformly upper bounded number of iterates of f_t.

7 Linked sets. Persistence of heterodimensional cycles

We prove that, for $\mu^- < \mu^\star$ close to $\mu^\star$, for every $t \in C_n = [\lambda^n\,(1+\mu^-), \lambda^n(1+\mu^\star))$, the hyperbolic sets $\Lambda_t = \Lambda_{t,\alpha(\mu^\star)}$ in Lemma 6.2 (of index one and contained in the homoclinic class of P) and Ω_t in Section 6.3 (of index two) are heteroclinically related. This will end the proof of the theorem.

Lemma 7.1. (Gap Lemma, [Ne$_2$] and [PT$_3$, page 63]) *Let K_1 and K_2 be Cantor sets of $\mathbb{R}$ with thickness τ_1 and τ_2, respectively. If $\tau_1 \cdot \tau_2 > 1$ then one of the following alternatives occur: K_1 is contained in a gap of K_2; K_2 is contained in a gap of K_1; $K_1 \cap K_2 \neq \emptyset$.*

Write $t = \lambda^n(1 + \mu)$ and recall that $\Lambda_\mu = (\Lambda_t)^c$ and $\Sigma_\mu = (\Omega_t)^c$. By Lemma 6.2 and Proposition 6.6, the product of the thickness of Λ_μ and Σ_μ is bigger than one. Moreover, by construction, the convex hull of Λ_μ and Σ_μ are non-disjoint. Finally, Lemma 6.2 and Proposition 6.6 imply that each Cantor set is not contained in a gap of the other one (recall the conditions on the size of the gaps of Σ_μ and on the diameters of these two sets). Thus, by Lemma 7.1, $\Sigma_\mu \cap \Lambda_\mu \neq \emptyset$.

By construction, it is immediate that $W^s(\Lambda_t) \cap W^u(\Omega_t) \neq \emptyset$. To see that $W^u(\Lambda_t) \cap W^s(\Omega_t) \neq \emptyset$ take any $y_0 \in \Lambda_\mu \cap \Sigma_\mu$. By Remark 3.3, there are points $X_1 = (x_1, y_0, z_1) \in \Lambda_t$ and $X_2 = (x_2, y_0, z_2) \in \Omega_t$. By construction,

$$\{(x_1, y_0)\} \times [-1, 1] \subset W^u(X_1) \quad \text{and} \quad [-1, 1] \times \{(y_0, z_2)\} \subset W^s(X_2).$$

So $(x_1, y_0, z_2) \in W^u(\Lambda_t) \cap W^s(\Omega_t)$ and there is a (heterodimensional) cycle related to Λ_t and Ω_t.

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