Pseudo-orbit shadowing in the C^1 topology

Flavio Abdenur and Lorenzo J. Díaz *

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To Carlos Gutierrez and Marco Antonio Teixeira
for their 60^{th} birthday

Abstract

We prove that the shadowing property does not hold for diffeomorphisms in an open and dense subset of the set of C^1-robustly non-hyperbolic transitive diffeomorphisms (i.e., diffeomorphisms with a C^1-neighbourhood consisting of non-hyperbolic transitive diffeomorphisms).

1 Introduction

Numerical simulations of dynamical systems generally produce approximate orbits (or *pseudo-orbits*) rather than actual orbits of the system. In order for such simulations to be relevant, it is important to ensure that sufficiently precise computations produce pseudo-orbits which are followed – or *shadowed* – by true orbits. This property – called the *pseudo-orbit shadowing property* – is therefore of fundamental importance to applications of dynamics.

Moreover, shadowing of pseudo-orbits plays a central role in the general theory of dynamical systems. Every basic set of every hyperbolic dynamical system (in this paper all dynamical systems are given by iterations of diffeomorphisms) displays the shadowing property, as proved by Anosov and Bowen [An, Bo]. The shadowing property in turn is fundamental to the stability of hyperbolic diffeomorphisms. Indeed, shadowing and stability/hyperbolicity are so closely intertwined that it is generally believed that shadowing and hyperbolicity are in some sense equivalent. This equivalence is of course not strictly true: there are examples of systems which are not hyperbolic and nevertheless do exhibit the shadowing property ([YY], for instance, mentions several such examples which are very fragile). But it is reasonable to expect that, for *most* systems in some sense, the shadowing property and hyperbolicity are indeed equivalent. The goal of this paper is to show that, if *most systems* is taken to mean C^1-*generic* or C^1-*open and dense* diffeomorphisms, then the equivalence between shadowing and hyperbolicity does hold in some important contexts.

Let us begin by setting our context and recalling some standard definitions:

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Throughout the paper M denotes a compact boundaryless manifold whose dimension d is greater than or equal to 3. The space of C^1 diffeomorphisms on M, endowed with the usual C^1 topology, is denoted by $\mathrm{Diff}^1(M)$. Given $\delta > 0$ and $f \in \mathrm{Diff}^1(M)$, a sequence $(x_n)_{n\in\mathbb{N}}$ in M is a *δ-pseudo-orbit of f* if the distance between $f(x_n)$ and x_{n+1} is less than δ for all n. Given $\varepsilon > 0$, the pseudo-orbit (x_n) is *ε-shadowed* by a true orbit $(f^n(x))_{n\in\mathbb{N}}$ of f if the distance between x_n and $f^n(x)$ is less than ε for all n.

An invariant set Λ of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ has the *shadowing property* if for every $\varepsilon > 0$ there is some $\delta > 0$ such that every δ-pseudo-orbit of points in Λ is ε-shadowed by some orbit of f in Λ. A diffeomorphism $f \in \mathrm{Diff}^1(M)$ has the shadowing property or *is shadowable* if M has the shadowing property for f. So for a set or diffeomorphism *not* to have the shadowing property means that there is some $\varepsilon > 0$ such that for every $\delta > 0$ there is some δ-pseudo-orbit which is not ε-shadowed by any actual orbit. Finally, given an open set $U \subset M$, then f is *nonshadowable in U* if there is some $\varepsilon > 0$ such that for every $\delta > 0$ there is some δ-pseudo-orbit *in U* which is not ε-shadowed by any actual orbit *in all of M*.

Given $f \in \mathrm{Diff}^1(M)$, an f-invariant compact set Λ is *transitive* if there is some $x \in \Lambda$ such that the ω-limit set $\omega_f(x)$ of x coincides with Λ. The diffeomorphism f is *transitive* if M is a transitive set for f; the diffeomorphism f is *C^1-robustly transitive* if there is a neighborhood $\mathcal{U}_f$ of f in $\mathrm{Diff}^1(M)$ such that every $g \in \mathcal{U}_f$ is transitive. We denote by $\mathcal{T}$ the subset of $\mathrm{Diff}^1(M)$ of C^1-robustly transitive diffeomorphisms. We also consider the open subset $\mathcal{RNT}$ of $\mathcal{T}$ of *C^1-robustly non-hyperbolic transitive diffeomorphisms* of M, that is diffeomorphisms $f \in \mathcal{T}$ having a neighbourhood $\mathcal{V}_f$ consisting of non-hyperbolic diffeomorphisms.

A transitive set Λ is an *attractor* if there is an open neighborhood V of Λ in M such that the closure of $f(V)$ is contained in V and $\bigcap_{n\in\mathbb{N}} f^n(V) = \Lambda$; such a neighborhood V is called an *attracting block* of Λ. The *basin* of Λ is the set $\mathcal{B}(\Lambda) \equiv \{x \in M : \omega(x) \subset \Lambda\}$. It is easy to verify that if V is an attracting block of Λ then $\mathcal{B}(\Lambda) = \bigcup_{n\in\mathbb{N}} f^{-n}(V)$; in particular the basin $\mathcal{B}(\Lambda)$ is an open subset of M. An attractor Λ is *C^1-robust* if there are an attracting block V of Λ and a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ such that if $g \in \mathcal{U}$ then $\Lambda_g \equiv \bigcap_{n\in\mathbb{N}} g^n(V)$ is an attractor of g; in this case Λ_g is said to be the *continuation of Λ relative to g*.

A transitive set Λ is *isolated* if there is an open neighborhood V of Λ, called an *isolating block* of Λ, such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V)$. An isolated transitive set Λ is *C^1-robust* if there are an isolating block V of Λ and a neighborhood $\mathcal{U}$ of f such that given $g \in \mathcal{U}$, then the continuation $\Lambda_g \equiv \bigcap_{n\in\mathbb{Z}} g^n(V)$ of Λ is also a transitive set.

Finally, let us recall some terminology concerning residual sets and genericity. Given an open subset $\mathcal{U}$ of $\mathrm{Diff}^1(M)$, a subset $\mathcal{R}$ of $\mathcal{U}$ is *residual in $\mathcal{U}$* if $\mathcal{R}$ contains the intersection of a countable family of open and dense subsets of $\mathcal{U}$; in this case $\mathcal{R}$ is dense in $\mathcal{U}$. A set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ is said to be *residual* if it is residual in all of $\mathrm{Diff}^1(M)$. A property (P) is *residual* or *generic* in $\mathcal{U}$ if (P) holds for all diffeomorphisms which belong to some residual subset of $\mathcal{U}$; the property (P) is said to be *residual* or *generic* if it is residual in all of $\mathrm{Diff}^1(M)$.

1.1 Statement of the results

Motivated by previous definitions in [Ki, Yo], Yuan and Yorke introduced in [YY] a strong type of non-shadowing property called *absolute non-shadowability*. Let us recall their definition. Fix a diffeomorphism $f \in \mathrm{Diff}^1(M)$ and a point $x \in M$. Consider the set $\Omega_{x,\delta}$ of δ-pseudo-orbits starting at x. Now, given an element x_n of a δ-pseudo-orbit starting at x, the successor x_{n+1} of x_n in the pseudo-orbit can be any one of the points in the open ball $B_\delta(f(x_n))$. Let us assume the uniform probability distribution on $B_\delta(f(x_n))$. In this way, the set $\Omega_{x,\delta}$ constitutes a Markov chain having each δ-pseudo-orbit as a sample sequence. The point x is said to be *absolutely non-shadowable* if there is some $\varepsilon > 0$ such that, for all $\delta > 0$, almost every δ-pseudo trajectory $(x_i)_{i=0}^{\infty}$ (i.e., a set of full measure of the space of δ-pseudo trajectories) is ε-shadowed by no true orbit.

For the sake of concision, we will say that an attractor Λ is *absolutely non-shadowable* if every point in the basin $\mathcal{B}(\Lambda)$ of Λ is absolutely non-shadowable. In [YY] the intersection of the following three conditions is shown to be sufficient for an attractor Λ of a map f to be absolutely non-shadowable:

- y1) *dimension variability*: the attractor Λ contains saddles P and Q having different *indices* (i.e., dimension of the unstable manifold), say $\text{index}(P) < \text{index}(Q)$;

- y2) the attractor Λ is the closure of the unstable manifold of the saddle P of lower index; and

- y3) there is a dominated splitting $E \oplus F$ (see Definition 2.7) over Λ such that the dimension of F coincides with $\text{index}(P)$.

We observe that, a priori, these properties are quite fragile: in general, they can be destroyed after arbitrarily small perturbations of the map. Yorke and Yuan [YY], however, construct some examples for which the hypotheses (y1), (y2), and (y3) hold robustly: these examples include some non-invertible maps on the two-dimensional torus (following [KKGOY]) and some diffeomorphisms on $\mathbb{S}^3 \times \mathbb{S}^1$ (following [Sm] and modifying [KKGOY]). It follows that these examples are absolutely non-shadowable in a robust way.

Our first result follows from verifying that the three hypotheses from [YY] – and hence the absolute non-shadowability condition – hold for generic robustly transitive non-hyperbolic diffeomorphisms (in this case, the attractor is the whole manifold):

Theorem 1. *Generically in the set $\mathcal{RNT}$ of robustly non-hyperbolic and transitive diffeomorphisms, the whole ambient manifold is absolutely non-shadowable.*

At this point we note that, as far as the authors know, all of the known robustly non-hyperbolic transitive diffeomorphisms (see [Sh, $\text{M}\tilde{\text{n}}_1$, Ca, BD_1, BV]) satisfy the aforementioned conditions (y1) and (y3) given in [YY]. Condition (y2) always holds for generic robustly transitive diffeomorphisms (this is a consequence of the Connecting Lemma of Hayashi, [Ha], see Lemma 2.5). Indeed, it seems possible that *all* robustly non-hyperbolic transitive diffeomorphisms – not just C^1-generic ones – are absolutely non-shadowable. Unfortunately, we do not know how to prove this at this point.

We can, however, prove that C^1*-open and densely* every robustly non-hyperbolic transitive diffeomorphism does not have the shadowing property.

Theorem 2. *There is an open and dense subset $\mathcal{N}$ of the set $\mathcal{RNT}$ of robustly non-hyperbolic transitive diffeomorphisms consisting of non-shadowable diffeomorphisms.*

This theorem was proved in [BDT] for diffeomorphisms of dimension three under the additional hypothesis on the global dynamics of the existence of a dominated splitting having three non-trivial directions. Naively speaking, the proof of Theorem 2 consists in showing that this global hypothesis can be obtained locally. This fact and an argument we import from [DR] about invariant manifolds of saddles involved in partially hyperbolic heterodimensional cycles allow us to get an analogue of condition (y2) in [YY]. Our proof also involves the transitions associated to heterodimensional cycles introduced in [BDPR] following [BDP] (see the sketch of the proof at the end of this introduction).

Theorem 2 is in fact a corollary of the following more general result (whose proof is essentially the same as above, with the addition of certain generic arguments):

Theorem 3. *Given an isolated transitive set Λ of a generic diffeomorphism f, then either:*

a) Λ is a hyperbolic set; or

b) there are a neighborhood $\mathcal{U}$ of f in $\mathrm{Diff}^1(M)$ and arbitrarily small isolating blocks V of Λ such that every $g \in \mathcal{U}$ is non-shadowable in the neighborhood V.

This result easily yields the following corollary, of which Theorem 2 is a special case:

Corollary 1.1. *Let Λ be a C^1-robust attractor of a diffeomorphism f, with continuations Λ_g defined for all diffeomorphisms g in some open neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f. Assume that Λ_g is non-hyperbolic for every $g \in \mathcal{U}$. Then there is an open and dense subset $\mathcal{U}_0$ of $\mathcal{U}$ such that for every $g \in \mathcal{U}_0$ there is an arbitrarily small isolating block V_g of Λ_g in which g is non-shadowable.*

From Theorem 3, which is a "semilocal" result, we immediately obtain not just Theorem 2 but also an interesting "global" result. It was explained in [Ab1] that generic diffeomorphisms come in one of two types: *tame* diffeomorphisms, which have a finite number of homoclinic classes and whose nonwandering sets admit partitions into a finite number of disjoint transitive sets; and *wild* diffeomorphisms, which have an infinite number of (disjoint and different) homoclinic classes and whose nonwandering sets admit no such partitions.

Theorem 4. *There is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that if $f \in \mathcal{R}$ is tame, then the following two conditions are equivalent:*

a) f is hyperbolic (i.e., Axiom A without cycles)

b) f is shadowable

The result above motivates the following conjecture:

Conjecture 1. *Generically hyperbolicity and shadowability are equivalent. That is, there is a residual set $\mathcal{R} \subset \mathrm{Diff}^1(M)$ such that $f \in \mathcal{R}$ is shadowable if and only if it is Axiom A without cycles.*

In view of Theorem 4 above, which settles the conjecture in the tame case, Conjecture 1 above is reduced to the following:

Conjecture 2. *Every wild generic diffeomorphism is non-shadowable.*

We remark that this last conjecture at first seems easy to prove, by constructing pseudo-orbits which "jump" from one homoclinic class to the other, but in fact this naive idea does not work by itself, so that this is a much more subtle problem than might appear at first sight.

1.2 Further comments

We also stress that while our results above show that "typically" (i.e. C^1-generically or C^1-open and densely, at least in some contexts) hyperbolicity and shadowability are equivalent, it may very well be that in a statistical sense shadowability is quite abundant: that is, it may be that many nonhyperbolic systems exhibit the pseudo-orbit property for most (but not all) pseudo-orbits. In the appendix to [KH] Katok and Mendoza have shown that given a $C^{1+\alpha}$-diffemorphism f which preserves a hyperbolic measure μ, then "most" pseudo-orbits near the support of μ are shadowed by actual orbits. The work in progress [ABC] shows that C^1-generic transitive difeomorphisms exhibit hyperbolic measures whose supports coincide with the whole ambient manifold. If Katok's techniques (which rely on a weak form of Pesin theory) can be adapted to the C^1-generic context, then this would imply that generic (including generic nonhyperbolic) transitive diffeomorphisms do have the shadowing property, *modulo sets of arbitrarily small measure.*

It is worthwhile to compare the previous results with [PP], which claims that shadowing is generic in the C^0-topology. See also the papers [AHK, Mz, Od].

We close this part of the introduction with some comments concerning shadowing and hyperbolicity. Different notions of weak shadowing have been introduced in recent years (for the various types of shadowing and their properties we refer the reader to the books [Pa, Pi]). One of them is the *first weak shadowing property, (f.w.s.p),* see [CP, Sa_2], meaning that, for every $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo trajectory is contained in an ε-neighbourhood of a true orbit (in our context this definition automatically is verified: the diffeomorphisms are transitive and thus the ε-neighbourhood of a dense orbit is the whole manifold). [Sa_2] states that the C^1-interior of the surfaces diffeomorphisms verifying the f.s.w.p are structurally stable (so hyperbolic).

For results in the same spirit of [Sa_2], relating shadowing and hyperbolicity, see [Sa_1] and [PRS]. We observe that the existence of robustly transitive diffeomorphisms, see [Sh, Mñ_1, Ca, BD_1, BV], in dimensions greater than or equal to three shows that this result does not hold in higher dimensions. In fact, the main difference between dimension two and higher dimensions arises from the dimension variability, which is forbidden in dimension two (in the context of diffeomorphisms). Finally, recently [Cr, Corollary 1.10] states that there is a generic subset of C^1-diffeomorphisms satisfying a stronger formulation of the f.w.s.p. formulated above.

1.3 Ingredients of the proofs

Let us describe the broad outlines of the proofs of our main results. These proofs have two main ingredients: the properties of the dominated splittings of homoclinic classes and robustly transitive diffeomorphisms obtained in [DPU, BDP, BDPR] and the analysis of the dynamics arising from partially hyperbolic heterodimensional cycles, see [DR].

Theorem 1, as was previously mentioned, follows from checking that conditions (y1), (y2), and (y3) of [YY] hold for any generic non-hyperbolic robustly transitive diffeomorphism. Condition (y1), *dimension variability,* follows from combining the Connecting Lemma of Hayashi [Ha] (which extends the previous work of Mañé [Mñ_2] on the stability conjecture) with some standard C^1-generic arguments. Not being hyperbolic implies that non-hyperbolic periodic points may be created via small C^1-perturbations. This implies that saddles having different indices may be created. Since hyperbolic saddles are persistent, it follows that the existence of points of different indices is an open and dense property in $\mathcal{RNT}$.

Condition (y2), *existence of a periodic point (of appropriate index) whose unstable manifold is* *dense in the manifold,* is guaranteed generically by combining Pugh's General Density Theorem [Pu] with a result from [BD_2] which in turn is a consequence of the aforementioned Connecting Lemma. Indeed, this combination proves that generically the unstable manifold of *every* periodic point in the attractor is dense in the manifold (see Lemma 2.5).

Condition (y3), *existence of a dominated splitting $E \oplus F$ such that the dimension of F is the index of a saddle in the manifold whose index is not maximal,* is the toughest of the three to check. The existence of a dominated splitting is a straightforward consequence of the results in [BDP]. Showing that there is a dominated splitting with an appropriate dimension, however, requires more subtlety. We employ an argument by contradiction using techniques from [BDPR], a paper which extends some of the ideas in [BDP].

For the sake of clarity we first prove Theorem 2 and later (in Subsection 3.3.4) show how this proof can be modified in order to obtain the more general Theorem 3. In fact, the proof of Theorem 3 only differs from that of Theorem 2 in that it requires some extra, "soft", generic arguments; the core of both results are the arguments found in the proof of Theorem 2.

The proof of Theorem 2 relies on the C^1-generic machinery generated by the Connecting Lemma and on the analysis of the dynamics arising from the unfolding of partially hyperbolic heterodimensional cycles. In our context, the creation of a heterodimensional cycle follows immediately from the Connecting Lemma and the robust transitivity (see Remark 2.4). But here it is essential to have some local domination around the cycle: we need to create partially hyperbolic heterodimensional cycles. To get these cycles we use the machinery developed in [BDP, BDPR] on dominated splittings of homoclinic classes.

Heterodimensional cycles, introduced by Newhouse-Palis in [NP] in the seventies and recently developed in the series of papers of the second author and Rocha, see for instance [D_1, D_2, DR], consist of cycles between two (or more) hyperbolic periodic saddles which have different indices. That is, there are two saddles P and Q, with $\mathrm{index}(P) \neq \mathrm{index}(Q)$, such that the stable manifold of P intersects the unstable manifold of Q and vice-versa.

In case the indices of P and Q are consecutive integers, however, the use of the *blenders* studied in [BD_2, DR] does allow the creation of C^1-robust *quasi-heterodimensional cycles*: here the invariant manifolds of P and Q are not required to intersect each other, but rather to (C^1-robustly) accumulate (in a dominated way) on each other, see Theorem 3.9. This is the main step to construct arbitrarily fine pseudo-orbits which cannot be shadowed by actual orbits. That is, the presence of a dominated quasi-heterodimensional cycle is an obstruction to the shadowing property, and so the presence of a robust dominated quasi-heterodimensional cycle is a robust obstruction to the shadowing property.

Let us now explain how such a partially hyperbolic heterodimensioanl cycle is created. As in the proof of Theorem 1, the lack of hyperbolicity of the attractor generates (modulo C^1-small perturbations) *dimension variability,* that is, two periodic saddles of different indices. The arguments from [BDPR] show that the indices of the periodic points form a non-trivial interval in $\mathbb{N}$; in particular, the manifold contains two periodic points P and Q with consecutive indices. Moreover, using [BDP] (see also Lemma 3.4) we see that this cycle can be associated to saddles P and Q whose multipliers are all real, positive and have multiplicity one. Unfolding this cycle we obtain a partially hyperbolic saddle-node (Proposition 3.5). Unfolding this saddle-node one gets two new saddles of different indices bounding a segment where the dynamics is locally partially hyperbolic. Finally, the Connecting Lemma allows us to create a partially hyperbolic heterodimensional cycle associated to these saddles.

Finally, we note that the proof of [YY, Theorem 2.2] shows that their conditions (y1), (y2), and (y3) essentially imply the existence of a quasi-heterodimensuonal cycle (actually, this type of quasi-cycle condition is considered in [CK] in the context of homeomorphisms). The main point is how to obtain such quasi-cycles. So, despite the substantial differences in terms of vocabulary and technique between this paper and [YY], we in fact rely on the same type of mechanism, which can be summed up by the phrase *dimensional variability, local domination and transitivity imply no shadowing.*

This paper is organized as follows. In Section 2, we recall the C^1-generic machinery that allows us to create cycles after perturbations and state the main facts about dominated splittings of robustly transitive diffeomorphisms and homoclinic classes. Using these facts, we prove Theorem 1. In Section 3, we prove Theorems 2 and 3 (in Section 3.3.3 and 3.3.4, respectively). In Section 3.1, we explain how partially hyperbolic heterodimensional cycles are produced in the robustly transitive setting. This construction relies on the creation of partially hyperbolic saddle-nodes. In Section 3.2, we prove that the diffeomorphisms having partially hyperbolic heterodimensional cycles are dense in $\mathcal{RNT}$. Finally, in Section 3.3, we explain how the constructions in [DR] implies the persistence of dominated quasi-cycles in the set of robustly transitive diffeomorphisms. Using this fact, we construct non-shadowable pseudo-orbits.

2 Generic non-hyperbolic robustly transitive diffeomorphisms

In this section we prove Theorem 1. Recall that $\mathcal{RNT}$ denotes the set of C^1-robustly non-hyperbolic transitive diffeomorphisms of M, that is, diffeomorphisms f having a neighbourhood $\mathcal{U}_f$ in $\mathrm{Diff}^1(M)$ such that every $g \in \mathcal{U}_f$ is non-hyperbolic and transitive. Denote by $\mathcal{Y}$ the subset of $\mathcal{RNT}$ consisting of diffeomorphisms f such that:

(y1) there are (hyperbolic) saddles P_f and Q_f of f such that $\mathrm{index}(P_f, f) < \mathrm{index}(Q_f, f)$;

(y2) the ambient manifold M is the closure of the unstable manifold of the saddle P_f;

(y3) there is a (non-trivial) dominated splitting $E \oplus F$ of f (see Definition 2.7) defined on the whole manifold such that the dimension of F is equal to the dimension of the unstable bundle of P_f.

Denote by $\mathcal{G}$ the set of diffeomorphisms $f \in \mathcal{RNT}$ such that either f or f^{-1} belongs to $\mathcal{Y}$.

Proposition 2.1. *The set $\mathcal{G}$ contains a residual subset of $\mathcal{RNT}$.*

For proving this proposition we need some preliminary results and definitions.

2.1 Perturbation lemmas

We need the following perturbation lemma about the creation of heteroclinic intersections:

Lemma 2.2 (Connecting Lemma, [Ha]). *Let f be a C^1 diffeomorphism and Λ a transitive set of f containing a pair of hyperbolic saddles A_f and B_f. Then there is g arbitrarily C^1-close to f such that $W^u(A_g, g) \cap W^s(B_g, g) \neq \emptyset$, where A_g and B_g are the continuations of the saddles A_f and B_f of f for g. This lemma holds when $A_f = B_f$.*

We state two consequences from Lemma 2.2 we use repeatedly throughout the paper. Next remark is an immediate consequence from the lemma in the case $A_f = B_f$:

Remark 2.3. *Let f be a C^1 diffeomorphism and Λ a transitive set of f containing saddle A_f. Then there is g arbitrarily C^1-close to f such that $W^u(A_g, g) \pitchfork W^s(A_g, g) \neq \emptyset$.*

Remark 2.4. *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(\mathrm{M})$ of diffeomorphisms f having a transitive set Λ_f containing a pair of (hyperbolic) saddles A_f and B_f depending continuously on f and having different indices (there is no hypotheses on the variation of the transitive set Λ_f). There is a dense subset $\mathcal{C}$ of $\mathcal{U}$ of diffeomorphisms g having a heterodimensional cycle associated to A_g and B_g.*

Proof: Suppose that the $p = \mathrm{index}(A_f, f) < \mathrm{index}(B_f, f) = q$. Applying Lemma 2.2 to $W^s(A_f, f)$ and $W^u(B_f, f)$, we get g close to f such that $W^s(A_g, g) \cap W^u(B_g, g) \neq \emptyset$. As the sum of the dimensions of these manifolds is greater than the dimension n of the ambient manifold M (this sum is $(n-p)+q > n-q+q = n$), after a new perturbation we can assume that $W^s(A_g, g) \pitchfork W^u(A_g, g) \neq \emptyset$. This implies that the set $\mathcal{C}_0$ of diffeomorphisms $g \in \mathcal{U}$ such that $W^s(A_g, g) \pitchfork W^u(B_g, g) \neq \emptyset$ is open and dense in $\mathcal{U}$.

To finish the proof of the remark just take $g \in \mathcal{C}_0$ and apply Lemma 2.2 to $W^u(A_f, f)$ and $W^s(B_f, f)$, obtaining $h \in \mathcal{C}_0$ close to g with $W^u(A_h, h) \cap W^s(B_h, h) \neq \emptyset$. As $h \in \mathcal{C}_0$, $W^s(A_h, h) \cap W^u(B_h, h) \neq \emptyset$, thus h has a heterodimensional cycle associated to A_h and B_h. $\square$

Next Lemma 2.5 is a reformulation of [BD$_2$, Théorème 1.4] to the context of robustly transitive diffeomorphisms and it is obtained from Lemma 2.2 using a standard argument of genericity. Let us recall that the *homoclinic class* of a saddle P_f of a diffeomorphism f, denoted by $H(P_f, f)$, is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P_f. A homoclinic class can be equivalentely defined as the closure of the set of saddles Q *homoclinically related* to P_f (that is, the stable manifold of the orbit of P_f transversely meets the unstable manifold of the orbit of Q and vice-versa). Note that two saddles homoclinicially related have the same index. This implies that the periodic points of the same index as P_f are dense in the class $H(P_f, f)$. Finally, a homoclinic class is always transitive (for the proof of these properties see, for instance, [Ne]).

Lemma 2.5. *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(M)$ of transitive diffeomorphisms f such that every f has a saddle A_f depending continuously on $f \in \mathcal{U}$. There is a residual subset $\mathcal{J}_A$ of $\mathcal{U}$ such that, for all $f \in \mathcal{J}_A$, the homoclinic class of A_f is the whole ambient manifold M. In particular, the stable and unstable manifolds of A_f are both dense in M.*

Given a hyperbolic saddle A_f of a diffeomorphism f, denote by Σ_{A_f} the subset of the homoclinic class $H(A_f, f)$ of A_f of saddles of the same index as A_f and homoclinically related to A_f. The set Σ_{A_f} always is dense in $H(A_f, f)$. The previous lemma and an argument of transversality implies the following.

Remark 2.6. *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(M)$ of transitive diffeomorphisms f such that every f has a saddle A_f depending continuously on $f \in \mathcal{U}$. There is a residual subset $\mathcal{L}$ of $\mathcal{U}$ such that, for all $f \in \mathcal{L}$, the set Σ_{A_f} is dense in M.*

2.2 Dominated splittings

In this section we state the main facts about dominated splittings of robust transitive diffeomorphisms and prove Proposition 2.1. For properties of dominated splittings see, for example, [BDV, Appendix B].

Definitions and properties 2.7. *Consider a diffeomorphism f and an f-invariant set Λ.*

- **Dominated splitting:** *A Df-invariant splitting $E \oplus F$ of TM over Λ is* dominated *if the fibers of the bundles have constant dimension and there are a metric $||\cdot||$ and a natural number $n \in \mathbb{N}$ such that*
$$||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2} \quad \textit{for all } x \in \Lambda.$$
*In this case, we say that the splitting is n-*dominated.

- **Extension to the closure,** [BDV, Section B.1.1]**:** *Every n-dominated splitting defined over a set Λ can be extended to an n-dominated splitting defined over the closure of Λ.*

- **Extension and persistence of dominated splittings,** [BDV, Section B.1.1]**:** *Suppose that Λ has a dominated splitting. Then this splitting can be extended in a dominated way to the maximal invariant set of f in a neigbourhood of Λ. Moreover, every n-dominated splitting persists under C^1 perturbations: for any $\varepsilon > 0$ there are neighbourhoods U of Λ and $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that for every $g \in \mathcal{U}$ the maximal invariant set in the closure of U has an (n-ε)-dominated splitting*[1] *having the same dimensions as the initial dominated splitting over Λ.*

- **Finest dominated splitting:** *The splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is* dominated *if, for all $i = 1, \dots, (k-1)$, the splitting $(\oplus_{j=1}^{i} E_j) \oplus (\oplus_{j=i+1}^{k} E_j)$ is dominated. The splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is the* finest dominated splitting *of Λ if any bundle of the splitting is* indecomposable, *that is no E_i admits any dominated splitting.*

- **Clustering property of dominated splittings,** [BDV, Proposition B.2]**:** *Let $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ be the finest dominated splitting of Λ and $E \oplus F$ a dominated splitting over Λ, then there is i such that $E = \oplus_{j=1}^{i} E_j$ and $F = \oplus_{j=i+1}^{k} E_j$.*

- **Volume hyperbolicity:** *The dominated splitting $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is* volume hyperbolic *if there is n such that the derivative of f^n uniformly contracts the (induced) volume in E_1 and uniformly expands the (induced) volume in E_k.*

Theorem 2.8. [BDP, Theorems 2 and 4] *Every robustly transitive diffeomorphism f has a dominated splitting. Moreover, the finest dominated splitting of f (which is well and uniquely defined) is volume hyperbolic.*

Given $f \in \mathcal{RNT}$, denote by $s^-(f)$ and $s^+(f)$ the minimum and the maximum of the dimensions of the stable bundles of the (hyperbolic) saddles of f (if f has no hyperbolic saddles, these numbers are not defined). Consider also the finest dominated splitting $E_1(f) \oplus \cdots \oplus E_{m(f)}(f)$ of f defined on the whole M. Denote by $d_i(f)$ the dimension of the bundle $E_i(f)$.

[1] meaning $||Df^n(x)_E|| \cdot ||Df^{-n}(x)_F|| < \frac{1}{2} + \varepsilon$ for all x.

Theorem 2.9. ([BDPR, Theorem A and Lemma 4.1]) *There is an open and dense subset $\mathcal{I}$ of $\mathcal{RNT}$ consisting of diffeomorphisms f such that*

- *the numbers $s^+(f)$ and $s^-(f)$ are well defined, locally constant, and $s^-(f) < s^+(f)$,*
- *for every $i \in [s^-(f), s^+(f)] \cap \mathbb{N}$ there is a (hyperbolic) saddle of f of index i,*
- *the number $m(f)$ of bundles of the finest dominated splitting and the dimensions $d_i(f)$ of the bundles of the splitting are locally constant.*

The last assertion in Theorem 2.9 follows from the fact that any dominated splitting of f defined on the whole M has a continuation for any g close to f (see, for instance, [BDP, Lemma 1.4]) and the uniqueness of a dominated splitting fixed the number of sub-bundles, their dimensions, and the ordering of such dimensions, (see [BDPR, Lemma 4.1]). In fact, this assertion follows essentially from the clustering property in Definition 2.7.

We now pick $f \in \mathcal{I}$ and a small neighbourhood $\mathcal{U}_f$ of it where the numbers above are constant in $\mathcal{U}_f$, so we omit the dependence on g of $s^\pm(g)$, $m(g)$ and $d_i(g)$ on $g \in \mathcal{U}_f$. Finally, write $r_j = d_1 + \cdots + d_j$, $j = 1, \ldots, m$.

Let $s^+ - s^- = k$ and consider saddles $Q_f^0, Q_f^1, \ldots, Q_f^k$ such that the dimensions of the stable bundles are $s^-, (s^- + 1), \ldots, s^+ = s^- + k$. Since the continuations of these saddles are defined on a neighbourhood of f, we also omit the dependence on f of them.

Lemma 2.10. *Let $f \in \mathcal{I}$ and $\mathcal{U}_f$ be a neighbourhood of f as above (i.e., $s^\pm(g)$, $m(g)$, and $d_i(g)$ are constant in $\mathcal{U}_f$). There is an open and dense subset $\mathcal{V}_f$ of $\mathcal{U}_f$ such that for every $g \in \mathcal{V}_f$ there is $i \in [s^-, s^+]$ such that there is g-invariant splitting $E \oplus F$ such that* $\dim(F)$ *is the index of Q_g^i.*

We postpone the proof of the lemma and prove Proposition 2.1.

Proof of Proposition 2.1: Lemma 2.10 implies that the diffeomorphisms g such that either g of g^{-1} verifies (simultaneously) (y1) and (y3) form an open and dense subset of $\mathcal{I}$. To see how this assertion follows note that if $i \in [s^-, s^+)$ then it is enough to take in (y1) the saddles $P_g = Q_i^g$ and $Q_g = Q_{i+1}^g$ and in (y3) the dominated splitting $E \oplus F$ given by Lemma 2.10. If $i = s^+$ consider g^{-1}, the saddles $Q_{s^+}^g$ and $Q_{s^+-1}^g$, and the dominated splitting $F \oplus E$ of g^{-1} (we need to reverse the ordering of the bundles to get a dominated splitting of g^{-1}). It is now enough to take in (y1) the saddles $P_{g^{-1}} = Q_{s^+}^g$ and $Q_{g^{-1}} = Q_{s^+-1}^g$ and in (y3) the dominated splitting $F \oplus E$. Noting that the index of $P_{g^{-1}}$ (for g^{-1}) is

$$n - \operatorname{index}(Q_g^{s^+}, g) = n - \dim(F) = \dim(E)$$

we get that $F \oplus E$ verifies the dimension condition (y3).

Finally, the density condition (y2) follows from Lemma 2.5, which assures that (locally) generically the invariant manifolds $W^s(Q_g^i, g)$ and $W^u(Q_g^i, g)$ are both dense in M. This completes the proof of the proposition. □

Proof of the lemma: As the properties in Lemma 2.10 are open, it suffices to prove the density of the diffeomorphisms verifying the lemma. By Lemma 2.5, there is residual subset $\mathcal{J}_f$ of $\mathcal{U}_f$ such that every $g \in \mathcal{J}_f$ has a saddle P_g whose homoclinic class $H(P_g, g)$ is the whole manifold M. Thus the finest dominated splittings of Df over M and over $H(P_g, g)$ coincide. Moreover, by Remark 2.6, we also can assume the set Σ_{P_g} is dense in M. Since every dominated splitting defined on a set can be extended (in a dominated way) to its closure (recall Definition 2.7), the restriction of the finest dominated splitting of TM to Σ_{P_g} consists also of indecomposable sub-bundles. In other words, $T_{\Sigma_g}M = E_1(g) \oplus \cdots \oplus E_m(g)$ is a dominated splitting consisting of indecomposable sub-bundles.

For a given $g \in \mathcal{J}_f$ there are two possibilities:

Case (a): There are $j < m$ and $i = 0, \ldots, (k-1)$ such that $s^- + i = r_j$.

Case (b): for every j and every $i = 0, \ldots, (k-1)$, one has $r_j \neq s^- + i$.

In Case (a) the proof is trivial: just take the saddle Q^i and the g-invariant dominated splitting $E \oplus F$, $E = E_1 \oplus \cdots \oplus E_j$ and $F = E_{j+1} \oplus \cdots \oplus E_m$. By construction, $E^s(Q^i) = E$ and $E^u(Q^i) = F$, thus $\operatorname{index}(Q_g, g) = \dim(F)$.

Note that $s^+ < r_m = \dim(M)$ (otherwise, g has a sink, which prevents the transitivity of g). In Case (b) we have that for every diffeomorphism close to g:

$$\text{there is } 0 \leq j \leq (m-1) \text{ such that } r_j < s^- < \cdots < s^+ < r_{j+1} \text{ (here we let } r_0 = 0). \tag{2.1}$$

We now need the following results we export from [BDP, Propositions 2.4 and 2.5] (we use here the formulation in [BDPR] which is more convenient in our context) and [Fr].

Lemma 2.11. [BDPR, Lemma 5.3] . *Let P_f be a saddle of a diffeomorphism f whose homoclinic class $H(P_f, f)$ is non-trivial. Let $E_1 \oplus \cdots \oplus E_m$ be the restriction to Σ_{P_f} of the finest dominated splitting of $T_{H(P_f,f)}M$ of Df. Then, for every $\varepsilon > 0$, there are a dense subset Σ_ε of Σ_{P_f} and an ε-perturbation A_ε of Df preserving the splitting $E_1 \oplus \cdots \oplus E_m$ such that, for every saddle $R \in \Sigma_\varepsilon$, the restriction of the linear map*

$$M_{A_\varepsilon}(R) = A_\varepsilon(f^{n(R)-1}(R)) \circ \cdots \circ A_\varepsilon(f(R)) \circ A_\varepsilon(R)$$

to each bundle $E_i(R)$ is a homothety, where $n(R)$ is the period of the saddle R.

Moreover, if there are $i \in \{1, \ldots, m\}$ and $Q \in \Sigma_{P_f}$ such that the modulus of the Jacobian of the restriction of $f^{n(Q)}$ to $E_i(Q)$ is one then $R \in \Sigma_\varepsilon$ can be taken such that the restriction of $M_{A_\varepsilon}(R)$ to $E_i(R)$ is identity.

Lemma 2.12. ([Fr], [Mñ$_2$]) *Consider a $\mathcal{C}^1$-diffeomorphism f and an f-invariant finite set Σ. Let A be an ε-perturbation of the derivative Df of f along Σ (i.e., the linear maps $A(x)$ and $Df(x)$ are ε-close for all $x \in \Sigma$). Then for every neighbourhood U of Σ there is a diffeomorphism g C^1-ε-close to f such that*

- *$g(x) = f(x)$ if $x \in \Sigma$ or if $x \notin U$,*
- *$Dg(x) = A(x)$ for all $x \in \Sigma$.*

We are now ready to finish the proof of the lemma. Consider the bundle E_{j+1}, by Lemma 2.11, there are a saddle R of g and an arbitrarily small perturbation A of the derivative of g along the orbit of R such that the restriction of $M_A(R)$ to E_{j+1} is a homothety. After a new perturbation, we can take the ratio of this homothety different from 1. Applying Lemma 2.12, taking Σ equal to the orbit of R and considering the perturbation A, we get h close to g such that R is a periodic orbit of h with $Dh^{n(R)}(R) = M_A(R)$.

Let $T_R M = E^s(R) \oplus E^u(R)$ and note that for every vector $v \notin E^u$ one has $Dh^m(v) \to E^s(R)$ as $m \to \infty$ (similarly, if $v \notin E^s$ then $Dh^{-m}(v) \to E^u(R)$). The invariance of the bundles $E_i(R)$ and the fact that the restrictions of $Dh^{n(R)}$ to the bundles $E_i(R)$ are homotheties imply that either $E_i(R) \subset E^s(R)$ or $E_i(R) \subset E^u(R)$.

If the homothety $M_A(R)|_{E_{j+1}(R)}$ is a contraction then $E_{j+1}(R)$ is a stable bundle of R. Thus, by the domination, every $E_i(Q)$, $i = 1, \dots, (j+1)$, is also contracting. Hence $E_1(R) \oplus \cdots \oplus E_{j+1}(R) \subset E^s(R)$. Therefore the dimension of the stable bundle of R is at least r_{j+1}. Thus $r_{j+1} \le s^+$, contradicting (2.1).

Similarly, if the homothety $M_A(R)|_{E_{j+1}(R)}$ is an expansion then $E_{j+1}(R)$ is a unstable bundle of R. Thus, by the domination, every $E_i(Q)$, $i = (j+1), \dots, m$, is also expanding. Hence $E_{j+1}(R) \oplus \cdots \oplus E_m(R) \subset E^u(R)$. Therefore $E^s(R) \subset E_1(R) \oplus \cdots \oplus E_j(R)$. Thus $\dim(E^s(R)) \le r_j$. Hence $s^- \le r_j$, contradicting (2.1). This ends the proof of the lemma. $\square$

3 Partially hyperbolic heterodimensional cycles

In this section we prove Theorems 2 and 3.

3.1 Creation of partially hyperbolic heterodimensional cycles

Denote by $\mathcal{P}$ the subset of $\mathcal{RNT}$ of diffeomorphisms g satisfying the following conditions:

(p1) There are saddles A_g and B_g of the same period k of indices m_s and $m_s + 1$, respectively, and an open curve γ_g (called *connection*) whose extremes are A_g and B_g such that $\gamma_g \subset W^s(A_g) \pitchfork W^u(B_g)$ and $g^k(\gamma_g) = \gamma_g$.

(p2) There is a small neigbourhood U of the orbit of the closure of γ_g, $\overline{\gamma_g} = \{A_g, B_g\} \cup \gamma_g$, and a g-invariant dominated splitting defined of U

$$E^s_1(g) \oplus \cdots \oplus E^s_{m_s}(g) \oplus E^c(g) \oplus E^u_1(g) \oplus \cdots \oplus E^u_{m_u}(g)$$

such that the bundles $E^s_i(g)$ ($i = 1, \dots, m_s$), $E^c(g)$, and $E^u_j(g)$ ($j = 1, \dots, m_u$) are one dimensional and

$$\begin{aligned} &E^s(A_g) = E^s_1(g) \oplus \cdots \oplus E^s_{m_s}(g) \oplus E^c(g), \\ &E^u(B_g) = E^c(g) \oplus E^u_1(g) \oplus \cdots \oplus E^u_{m_u}(g), \\ &T_{\gamma_g} M = E^c(g). \end{aligned}$$

Theorem 3.1. *The set $\mathcal{P}$ is open and dense in $\mathcal{RNT}$.*

First note that the set $\mathcal{P}$ is open (transverse intersections and dominated splittings are persistent). Thus the point is the density of $\mathcal{P}$. The proof of the theorem has three steps:

1. Consider the open subset $\mathcal{O}$ of $\mathcal{RNT}$ consisting of diffeomorphisms f having a pair of (hyperbolic) periodic saddles P_f and Q_f such that:

   (o1) $\mathrm{index}(P_f, f) + 1 = \mathrm{index}(Q_f, f)$,

(o2) all the eigenvalues of $Df^{n(P_f)}(P_f)$ and $Df^{n(Q_f)}(Q_f)$ ($n(R_f)$ is the period of R_f) are real, have multiplicity one, and their moduli are different.

We prove that set $\mathcal{O}$ is dense in $\mathcal{RNT}$ (Proposition 3.2).

2. Lemma 3.5 claims that every f in $\mathcal{O}$ can be C^1-approximated by some g in $\mathcal{RNT}$ having a saddle-node S_g such that all the eigenvalues of $Dg^{n(S_g)}(S_g)$ are different and positive and have multiplicity one. Thus the splitting given by the sum of the eigenspaces of S_g is dominated.

3. Theorem 3.1 follows by unfolding the saddle-node S_g to get a pair of hyperbolic saddles A_g and B_g as in the theorem, considering the segment $\gamma_g = (A_g, B_g)$ tangent to the central direction of the saddle-node, and using that a dominated splitting is persistent (recall Definition 2.7).

Proposition 3.2. *The set $\mathcal{O}$ is open and dense in $\mathcal{RNT}$.*

Proof: Since the conditions in the definition of the set $\mathcal{O}$ are open, it is enough to prove the density of the set $\mathcal{O}$ in $\mathcal{RNT}$.

Lemma 3.3. *There is an open and dense subset $\mathcal{K}$ of $\mathcal{RNT}$ of diffeomorphisms f with the following property: there are $p \in \mathbb{N}$ and saddles P_f and Q_f of f of indices p and $p+1$ such that $H(P_f, f)$ and $H(Q_f, f)$ are non-trivial.*

Proof: Since the existence of a pair of saddles P_f and Q_f of f of indices p and $p+1$ such that $H(P_f, f)$ and $H(Q_f, f)$ are both non-trivial is an open property (this follows from the persistence of hyperbolic saddles and of transverse intersections of invariant manifolds), it suffices to prove the density of the set $\mathcal{K}$ in $\mathcal{RNT}$.

Consider the open and dense subset $\mathcal{I}$ of $\mathcal{RNT}$ of diffeomorphisms f such that the indices of the saddles of f form an interval in $\mathbb{N}$ given by Theorem 2.9. By the definition of the set $\mathcal{RNT}$, we can assume that f has saddles P_f and Q_f of indices p and $p+1$ for some p. Consider a small neighbourhood $\mathcal{U}$ of f where the continuations of these saddles are defined and the residual subset $\mathcal{J} = \mathcal{J}_P \cap \mathcal{J}_Q$ of $\mathcal{U}$, where $\mathcal{J}_P$ and $\mathcal{J}_Q$ are the residual subsets of $\mathcal{U}$ given by Lemma 2.5, i.e., $H(R_g, g) = M$ if $g \in \mathcal{J}_R$. Then every $g \in \mathcal{J}$ verifies Lemma 3.3. $\square$

We need the following result:

Lemma 3.4 ([BDP]). *Let $\mathcal{U}$ be an open set in $\mathrm{Diff}^1(M)$ such that every f in $\mathcal{U}$ has a saddle P_f whose homoclinic class is non-trivial (here the saddle P_f depends continuously on f).*

There is a residual subset $\mathcal{D}$ of $\mathcal{U}$ of diffeomorphisms g such that the homoclinic class of P_g contains a saddle R_g of the same index as P_g such that every eigenvalue of $Dg^{n(R_g)}(R_g)$ is real and positive and has multiplicity one.

This lemma is just a reformulation of the results in [BDP], where Lemma 3.4 is formulated for *periodic linear systems (cocyles)* $(\Sigma, f, \mathcal{E}, A)$ *with transitions* (here f is a diffeomorphisms, Σ a set of periodic points of f, $\mathcal{E}$ an Euclidean bundle over Σ, and $A \in \mathrm{GL}(\Sigma, f, \mathcal{E})$ so that for each $x \in \Sigma$ the map $A(x)$ is a linear isomorphism $A(x)\colon \mathcal{E}_x \to \mathcal{E}_{f(x)}$). See [BDP, Section 1] for the precise definition.

[BDP, Lemma 4.16] asserts that if $(\Sigma, f, \mathcal{E}, A)$ is a periodic linear system with transitions then, for every $\varepsilon > 0$, there is a *diagonalizable* ε-perturbation A' of A defined on a dense subset of Σ' of Σ. By a diagonalizable system we mean that if x is a periodic orbit of period k then all the eigenvalues of the linear map $M_A(x) = A(f^{k-1}(x)) \circ \cdots \circ A(f(x)) \circ A(x)\colon \mathcal{E}_x \to \mathcal{E}_x$ are real and positive and have multiplicity one. Lemma 3.4 now follows from the next two remarks:

- Let $H(P_f, f)$ be the homoclinic class of a saddle P_f of index k. Then the derivative of f induces a periodic linear system with transitions in Σ_{P_f} (recall that $\Sigma_{P_f} \subset H(P_f, f)$ is the set of saddles of index k homoclinically related to P_f). See [BDP, Lemma 1.9].
- By the previous item, we can apply [BDP, Lemma 4.16] to the dense subset Σ_{P_f} of $H(P_f, f)$ to obtain a dense subset Σ' of Σ_{P_f} such that for each $R_f \in \Sigma'$ there is an ε-perturbation A of Df along the finite orbit of R_f such that every eigenvalue of $M_A(R_f) = A(f^{n(R_f)-1}(R_f)) \circ \cdots \circ A(f(R_f)) \circ A(R_f)$ is real and positive and has multiplicity one. Using Lemma 2.12, we can perform the previous perturbation dynamically: there is g close to f such that $R_f = R_g$ is a periodic point of g of period $n(R_g) = n(R_f)$ with $Dg^{n(R_g)}(R_g) = M_A(R_f)$. Finally, using that R_f and P_f are homoclinically related, one has that the same holds for P_g and $R_g = R_f$. Note that in our case the fact that R_g is in the homoclinic class of P_g immediately follows from the robust transitivity.

Lemmas 3.3 and 3.4 imply immediately Proposition 3.2. □

Let $\mathcal{D}$ be the subset of $\mathcal{RNT}$ of diffeomorphisms g having a saddle-node S_g such that:

(d1) every eigenvalue λ of $Df^{n(S_g)}(S_g)$ is real and positive, and has multiplicity one,

(d2) 1 is an eigenvalue of $Df^{n(S_g)}(S_g)$,

(d3) there is a pair of eigenvalues λ and β of $Df^{n(S_g)}(S_g)$ with $0 < \lambda < 1 < \beta$.

Proposition 3.5. *The set $\mathcal{D}$ is dense in $\mathcal{RNT}$.*

Proof: For a given $f \in \mathcal{O}$ (the open and dense subset of $\mathcal{RNT}$ in Proposition 3.2), let A_f and B_f be the saddles in the definition of $\mathcal{O}$ satisfying (o1) and (o2). Let $0 < \lambda_1 < \cdots < \lambda_{p+1} < 1 < \lambda_{p+2} < \cdots < \lambda_n$ be the eigenvalues of $Df^{n(P_f)}(P_f)$ and consider the dominated splitting $T_{P_f}M = E_1(P_f) \oplus \cdots \oplus E_n(P_f)$ given by the one-dimensional eigenspaces of $Df^{n(P_f)}(P_f)$. Similarly, $0 < \beta_1 < \cdots < \beta_p < 1 < \beta_{p+1} < \cdots < \beta_n$ are the eigenvalues of $Df^{n(Q_f)}(Q_f)$ and $T_{Q_f}M = E_1(Q_f) \oplus \cdots \oplus E_n(Q_f)$ is the splitting given by the eigenspaces of $Df^{n(Q_f)}(Q_f)$. By Remark 2.4, after an arbitrarily small C^1-perturbation, we can assume that f has heterodimensional cycle associated to A_f and B_f. This cycle has a dominated structure in the saddles exactly as in [BDPR, Theorem 3.1] (in fact, here we have even more: all the bundles of the dominated splittings of the saddles in the cycle are one-dimensional). In fact, [BDPR, Theorem 3.1] can be read as follows (see also the proof of [BDPR, Corollary 3.6]):

Lemma 3.6. *Let $f \in \mathcal{O}$ such that the saddles A_f and B_f has a heterodimensional cycle. Then there are linear maps T_1 and T_2 preserving the dominated splittings of A_f and B_f*

$$T_1(E_i(P_f)) = E_i(Q_f) \quad \text{and} \quad T_2(E_i(Q_f)) = E_i(P_f), \qquad \text{for all } i = 1, \dots, n,$$

such that for every m and n large enough there is a diffeomorphism g close to f having a saddle S_g of period $n(S_g) \geq n \cdot n(P_f) + m \cdot n(Q_f)$ such that $Dg^{n(S_g)}(S_g)$ is arbitrarily close[2] to

$$T_1 \circ (Df^{n(P_f)}(P_f))^n \circ T_2 \circ (Df^{n(Q_f)}(Q_f))^m.$$

[2]In this statement there is an abuse of notation: we can take the saddle S_g nearby Q_g in such a way that a continuation of the dominated splitting of Q_g is defined in S_g and thus the previous expression makes sense.

As a consequence, the central eigenvalue of $Dg^{n(S_g)}(S_g)$ (the $(p+1)$-th eigenvalue λ_c) is arbitrarily close to

$$C \cdot \lambda_{p+1}^n \cdot \beta_{p+1}^m,$$

for some constant C independent of n and m.

The last equation in Lemma 3.6 means that we can select large n and m such that $C_1 < \log(\lambda_c) < C_2$, for some constants C_1 and C_2 independent of n and m. Taking large n and m, this implies that S_g can be chosen such that its $(p+1)$-th Lyapunov exponent is arbitrarily close to zero. Thus, after a new perturbation, we can assume that $\lambda_c = 1$, thus S_g is a saddle node. For details, see the proof of [BDPR, Corollary 3.6], specially the equation in [BDPR, page 203] and observe that we are considering cycles whose saddles only have positive eigenvalues. This completes the proof of Proposition 3.5. □

Remark 3.7. *Proposition 3.5 implies a weaker version of Theorem 2: there is a dense subset of $\mathcal{RNT}$ of non-shadowable diffeomorphisms. It is enough to take $f \in \mathcal{D}$ having a partially hyperbolic saddle-node S with one-dimensional center-stable $W^{cs}(S)$ and center-unstable $W^{cu}(S)$ manifolds (these manifods are tangent to the central direction and have the saddle-node as an extreme). Consider $x \in W^{cs}(S)$ and a one-jump pseudo-orbit (z_n) of the form $z_n = f^n(x)$ for $n = 0, \dots, k$, $z_{k+1} = y$ for some $y \in W^{cu}(S)$, and $z_{k+1+m} = f^m(y)$. Fixed any $\varepsilon > 0$, one gets an ε-pseudo orbit by taking big k and y close to the saddle-node. Due to the partial hyperbolicity of the saddle-node, this pseudo-orbit can not be shadowed.*

3.1.1 End of the proof of Theorem 3.1

Given $g \in \mathcal{D}$ consider the saddle-node S_g satisfying conditions (d1)–(d3). Let $0 < \lambda_1 < \cdots < \lambda_{m_s} < 1 = \lambda_c < \beta_1 < \cdots < \beta_{m_u}$ be the eigenvalues of $Dg^{n(S_g)}(S_g)$. Consider the dominated splitting along the orbit $O(S_g)$ of S_g given by

$$T_{O(S_g)}M = (E_1^s(g) \oplus \cdots \oplus E_{m_s}^s(g)) \oplus E^c(g) \oplus (E_1^u(g) \oplus \cdots \oplus E_{m_u}^u(g)),$$

where $E_i^s(g)$, $E^c(g)$, and $E_i^u(g)$ are the (one-dimensional) eigenspaces associated to λ_i, 1, and β_i. Note that $E^s(g) = E_1^s(g) \oplus \cdots \oplus E_{m_s}^s(g)$ is the stable bundle of S_f, $E^u(g) = E_1^u(g) \oplus \cdots \oplus E_{m_u}^u(g)$ is the unstable bundle of S_g, and $E^c(g)$ is the central direction.

Using the dominated splitting of the saddle-node S_g and the fact that a dominated splitting admits an extension (see, Definition 2.7), we can perform a local perturbation of g throughout the finite orbit of S_g to get a diffeomorphism f verifying (p1) and (p2): it suffices to obtain the saddles A_f and B_f in a normally hyperbolic local central manifold of S_g in such a way γ_f is a center stable manifold of A_f and a center unstable manifold of B_f tangent to the central direction (see Figure 1). This ends the proof of the theorem. □

3.2 Density of partially hyperbolic heterodimensional cycles in $\mathcal{RNT}$

Denote by $\mathcal{H}$ the subset of $\mathcal{P}$ such that the saddles A_f and B_f verifying (p1) and (p2) are related by a heterodimensional cycle as follows:

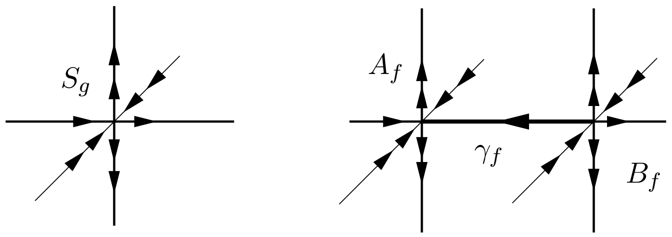

Figure 1: Unfolding a saddle-node

(p3) The unstable manifold of A_f and the stable one of B_f meet throughout the orbit of a heteroclinic point x_f:

$$W^u(A_f) \cap W^s(B_f) \supset \bigcup_{i\in\mathbb{Z}} f^i(x_f).$$

Considering some forward iterate of x_f, we can assume that the point x_f belongs to open set U where is defined the dominated splitting in the definition of $\mathcal{P}$. Moreover, the intersection between $W^u(A_f)$ and $W^s(B_f)$ at x_f is quasi-transverse, meaning that

$$T_{x_f}W^u(A_f) + T_{x_f}W^s(B_f) = T_{x_f}W^u(A_f) \oplus T_{x_f}W^s(B_f)$$

and

$$T_{x_f}M = T_{x_f}W^u(A_f) \oplus T_{x_f}W^s(B_f) \oplus E^c(f)(x_f).$$

Lemma 3.8. *The set $\mathcal{H}$ is dense in $\mathcal{RNT}$.*

Proof: By Theorem 3.1, it is enough to prove the density of $\mathcal{H}$ in $\mathcal{P}$. Let $f \in \mathcal{P}$, since f is transitive and $\mathcal{P}$ is open, by Lemma 2.2, there is $h \in \mathcal{P}$ arbitrarily C^1-close to f such that $W^u(A_h)\cap W^s(B_h) \neq \emptyset$. Thus, since $W^s(A_h) \pitchfork W^u(B_h) \neq \emptyset$, h has a heterodimensional cycle associated to A_h and B_h. By a transversality argument, we can assume (after a new perturbation if necessary) that this intersection occurs along the orbit of a point x_h and that $T_{x_h}W^u(A_h)$ and $T_{x_h}W^s(B_h)$ are in general position. A new transversality argument assures that we can assume that $T_{x_h}W^u(A_h)\oplus T_{x_h}W^s(A_h)$ and $E^c(h)(x_h)$ are in general position, thus $T_{h_f}M = T_{h_f}W^u(A_h) \oplus T_{x_h}W^s(B_h) \oplus E^c(h)(x_h)$. $\square$

3.3 Unfolding partially hyperbolic heterodimensional cycles

Consider $f \in \mathcal{H}$, the saddles A_f and B_f (say of indices p and $p+1$), the connection γ_f, and the heteroclinic point x_f in the definitions of the sets $\mathcal{P}$ and $\mathcal{H}$. Take a small neigbourhood W of the cycle as follows: consider the open set U containing the orbits of A_f, B_f, and γ_f where the dominated splitting $E^s(f) \oplus E^c(f) \oplus E^u(f)$ in (p2) is defined and a neighbourhood U_0 of x_f, then

$$W = U \cup (\cup_{i=-N}^{N} f^i(U_0),$$

where $N \in \mathbb{N}$ is such that $f^i(x_f) \in U$ for all $|i| \geq N$ (in fact, without loss of generality we can assume that $x_f \in U$ and its whole forward orbit is contained in U).

Next theorem is a reformulation (in fact, a weaker version stated in a more suitable form for our goals) of the results in [DR] on bifurcations of partially hyperbolic heterodimensional cycles.

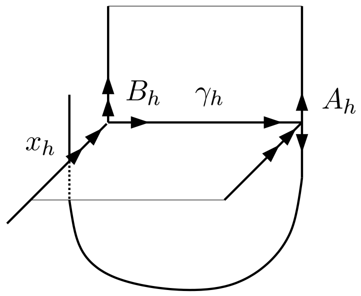

Figure 2: A partially hyperbolic heterodimensional cycle

Theorem 3.9 ([DR]). *Given any $f \in \mathcal{H}$ there is an open set $\mathcal{B}$ in $\mathrm{Diff}^1(M)$ whose closure contains f such that, for every $g \in \mathcal{B}$, the unstable manifold $W^u(A_g, g)$ of A_g accumulates the local stable manifold $W^s_{loc}(B_g, g)$ of B_g as follows: there are $r > 0$ and sequences $(x_n)_n$ of transverse homoclinic points of A_g, $x_n \in W^u(A_g, g) \pitchfork W^s(A_g, g)$, and $(\Delta^u(x_n))_n$ of disks $\Delta^u(x_n) \subset W^u(A_g, g)$ (of the same dimension as $W^u(A_g, g)$) centered at x_n such that:*

- *(x_n) converges to some point $x_\infty \in W^s_{loc}(B_g, g)$, $x_\infty \neq B_g$,*
- *the radius of each disk $\Delta^u(x_n)$ is greater than r,*
- *the tangent direction of $\Delta^u(x_n)$ is ϵ-close to $E^u(g)$ (the number ε can be taken arbitrarily small by reducing the sizes of $\mathcal{B}$ and U),*
- *$\Delta^u(x_n) \cap W^s_{loc}(B_g, g) = \emptyset$,*
- *$\Delta^u(x_n) = g^{k_n}(\delta^u(n))$, where $k_n > 0$, $\delta^u(n)$ is a disk contained in $W^u_{loc}(A_g) \subset U$ such that $\delta(n), g(\delta(n)), \dots, g^{k_n}(\delta(n))$ are contained in the neighbourhood W of the cycle.*

[DR, Theorem A] states that if f has a (partially hyperbolic) heterodimensional cycle as the ones above, then there is a C^1-open set $\mathcal{U}_f$ of diffeomorphisms, whose closure contains f, such that for every $g \in \mathcal{U}_f$ the homoclinic classes of A_g and B_g coincide. The main step to prove this theorem is to see that the there is an open set $\mathcal{V}_f$ of diffeomorphisms, $f \in \overline{\mathcal{V}_f}$, such that for every $g \in \mathcal{V}_f$ the closure of $W^u(A_g, g)$ (of dimension p) contains the whole $W^u(B_g, g)$ (of dimension $(p+1)$). The proof of this result involves a blender argument (see [BD$_1$, DR] and also [BDV, Chapter 6.2] for an expository explanation of blenders). The proof of this fact consists in showing the existence of a cube $C \subset U$ around a fundamental domain of the curve γ_g (the continuation of the connection γ_f for g close to f) endowed with a cone-field structure $\mathcal{C}^u$, $\mathcal{C}^{cu}$, $\mathcal{C}^{cs}$, and $\mathcal{C}^s$, corresponding to the bundles $E^u(g)$, $E^c(g) \oplus E^u(g)$, $E^c(g) \oplus E^s(g)$, and $E^s(g)$, such that every $(p+1)$-dimensional strip S, tangent to the cone-field $\mathcal{C}^{cs}$ and crossing the two sides of the cube C parallel to $\mathcal{C}^{cu}$, transversely intersects a disk Δ^u ($\Delta^u \pitchfork S \neq \emptyset$) contained $W^u(A_g, g) \cap C$ and tangent to $\mathcal{C}^u$. Moreover, this construction only involves the semi-global dynamics of g in the neighbourhood W of the cycle. The previous statement essentially corresponds to [DR, Proposition 3.6].

We next detail the construction of the points x_n and the disks $\Delta^u(x_n)$ in Theorem 3.9.

3.3.1 Itineraries in the neighbourhood of the cycle

To define exactly the disks in Theorem 3.9, we need to choose carefully the neighbourhood of the cycle. For the sake of simplicity, suppose that A_f and B_f are fixed points We assume that $W^u_{loc}(B_f, f)$ and $W^s_{loc}(A_f, f)$ are contained in the neighbourhood U of $\overline{\gamma_f}$. There is a first k_0 such that $f^{k_0}(W^u_{loc}(A_f, f)) \cap W^s_{loc}(B_f, f) \neq \emptyset$ and this intersection is just the heteroclinic point x_f in condition (p3) in the definition of $\mathcal{H}$ (we assume that x_f is in the interior of $f^{k_0}(W^u_{loc}(A_f, f))$). We take a small neighbourhood U_0 of x_f such that $U_0 \subset U$ and, for every g close to f, $g^{-k_0}(U_0) \subset U$ and the compact sets

$$K_0(g) = \overline{U}, \quad K_1(g) = g^{-1}(\overline{U_0}), \quad K_2(g) = g^{-2}(\overline{U_0}), \quad \ldots, \quad K_{k_0-1}(g) = g^{-k_0+1}(\overline{U_0})$$

are pairwise disjoint. There is $\varepsilon_1 > 0$ such that

$$d(K_i(g), K_j(g)) > \varepsilon_1 \quad \text{for every } g \text{ close to } f \text{ and } i \neq j. \tag{3.2}$$

We now consider the following neighbourhood W of the cycle,

$$W = U \cup f^{-1}(U_0) \cup f^{-2}(U_0) \cup \cdots \cup f^{-k_0+1}(U_0).$$

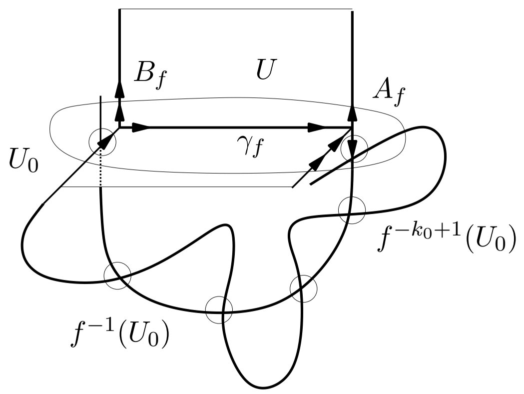

Figure 3: Neighbourhood of the cycle

Next step is to unfold the cycle associated to A_f and B_f, considering g close to f such that there is a continuation x_g of the heteroclinic point x_f which is a transverse homoclinic point of A_g (see Figure 2). Take a point $x \in W^u_{loc}(A_g) \subset U$ and a segment of its orbit $x, g(x), \ldots, g^m(x)$ in W such that $g^m(x) \in U$. We split this segment of orbit as follows: there are $m_1(x), \ldots, m_\ell(x)$ such that

- $x, g(x), \ldots, g^{m_1(x)}(x) \in U$,
- $g^{m_1(x)+1}(x) \in g^{-k_0+1}(U_0), \quad \ldots, \quad g^{m_1(x)+k_0-1}(x) \in g^{-1}(U_0), \quad g^{m_1(x)+k_0}(x) \in U_0 \subset U$,
- $g^j(x) \in U$ for all $j \in [m_1(x) + k_0, m_2(x)]$.

Inductively, for $i = 2, \ldots, \ell$,

- $g^{m_i(x)+1}(x) \in g^{-k_0+1}(U_0), \quad \ldots, \quad g^{m_i(x)+k_0-1}(x) \in g^{-1}(U_0), \quad g^{m_i(x)+k_0}(x) \in U_0 \subset U,$
- $g^j(x) \in U$ for all $j \in [m_i(x)+k_0, m_{i+1}]$ if $i < \ell$, if $i = \ell$ it holds $g^j(x) \in U$ for all $j \in [m_\ell(x)+k_0, m]$.

Take points $x, y \in W^u_{loc}(A_g, g)$ and $m > 0$ such that the segments of orbits $x, g(x), \ldots, g^m(x)$ and $y, g(y), \ldots, g^m(y)$ are contained in W but not in U, and $g^m(x), g^m(y) \in U$. Consider the numbers $m_1(x), \ldots, m_{\ell(x)}(x)$ and $m_1(y), \ldots, m_{\ell(y)}(y)$ defined as above. The choice of x and y (i.e., the segments of orbits are not contained in U) implies that $\ell(x), \ell(y) \geq 1$ and $m_1(x) \neq m \neq m_1(y)$.

Lemma 3.10. *Consider x and y as above. Suppose that there is j such that $m_j(x) \neq m_j(y)$. Then $d(g^k(x), g^k(y)) > \varepsilon_1$ for some $k \in \{0, \ldots, m\}$, where ε_1 is as in equation (3.2).*

Proof: Take a first j with $m_j(x) \neq m_j(y)$ and suppose that $m_j(x) > m_j(y)$. By definition,

$$g^{m_j(x)+1}(x) \in g^{-k_0+1}(U_0) \subset K_{k_0-1}(g).$$

Since $m_{j-1}(y) = m_{j-1}(x) < m_j(x) + 1 \leq m_j(y)$ (if $j = 1$ we let $m_0(y) = m_0(x) = 0$),

$$g^{m_j(x)+1}(y) \in U \subset K_0(g).$$

The lemma follows from equation (3.2). □

3.3.2 The disks $\Delta^u(x_n)$

We are now ready to explain the construction of the transverse homoclinic points x_n and the disks $\Delta^u(x_n)$. The homoclinic points x_n in Theorem 3.9 have their full orbit contained in the neighbourhood W of the cycle. One can take these points in U_0 and for each x_n consider its first backward iterate $\bar{x}_n$ in $W^u_{loc}(A_g, g)$, $\bar{x}_n = g^{-k_n}(x_n) \in W^u_{loc}(A_g, g)$, and the numbers $m_1(\bar{x}_n), \ldots, m_\ell(\bar{x}_n)$ corresponding to the segment of orbit of the first k_n iterates of $\bar{x}_n$. The set $\delta^u(n)$ is the set of points $z \in W^u_{loc}(A_g, g)$ such that

- $g^i(z) \in W$ for all $i = 0, \ldots, k_n$,
- $\ell(z) = \ell(\bar{x}_n)$ and $m_i(\bar{x}_n) = m_i(z)$ for all $i = 1, \ldots, \ell(\bar{x}_n)$.

Then

$$\Delta^u(x_n) = g^{k_n}(\delta^u(n)).$$

3.3.3 Proof of Theorem 2

For notational simplicity, we will explain the proof of Theorem 2 when A_g and B_g are fixed points (for the general case it suffices to take a power of g). Next remark is an immediate consequence from the hyperbolicity of A_g and B_g. Define $W^{s,u}_{loc}(C_g, g)$, $C = A, B$, as the connected component of $W^{s,u}(C_g, g) \cap D_\delta(C_g)$ containing C_g, here $D_\delta(C_g)$ is the ball of radius small δ centered at C_g.

Remark 3.11. *Let $C = A, B$. There is $\varepsilon_0 > 0$ such that:*

- *if $y \in D_\delta(C_g)$ and $y \notin W^s_{loc}(C_g, g)$ then there is $n \geq 0$ such that $d(g^n(y), W^s_{loc}(C_g, g)) > \varepsilon_0$;*

- *if $y \in D_\delta(C_g)$ and $y \notin W^u_{loc}(B_g, g)$ then there is $n \geq 0$ such that $d(g^{-n}(y), W^u_{loc}(C_g, g)) > \varepsilon_0$.*

Take x_n as in Theorem 3.9 close to x_∞ and the one-jump pseudo-orbit $(z_n)_n$ defined by:

$$z_k = g^k(x_n), \quad k \leq 0, \quad z_k = g^k(x_\infty), \quad k \geq 0.$$

Lemma 3.12. *The pseudo-orbit $(z_n)_n$ can not be shadowed by a true orbit.*

Proof: The proof is by contradiction, suppose that there is a true orbit $(g^k(y))_k$ that ε-shadows $(z_k)_k$ for some small ε. Remark 3.11 immediately implies that (if ε is small enough)

$$y \in W^s_{loc}(B_g, g) \quad \text{and} \quad g^{-k_n}(y) \in W^u_{loc}(A_g, g). \tag{3.3}$$

Therefore, $y \in W^s(B_g, g) \cap W^u(A_g, g)$.

Remark 3.13. *The fact $y \in W^s(B_g, g) \cap W^u(A_g, g)$ implies that if g is Kupka-Smale, then the pseudo-orbit $(z_n)_n$ can not be shadowed. The genericity of the Kupka-Smale diffeomorphisms implies a weaker version of Theorem 2: there is a residual subset of $\mathcal{RNT}$ of non-shadowable diffeomorphisms.*

We are now ready to finish the proof of Lemma 3.12, which is by contradiction. Fix $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ and take a homoclinic point x_n at distance less than ε from x_∞. Then $(z_k)_k$ is an ε-pseudo-orbit. Assume that the g-orbit of y δ-shadows z_n if $\delta < \min\{\varepsilon_0, \varepsilon_1\}$. By equation (3.3) and the definition of k_n,

$$g^{-k_n}(y) = \bar{y} \in W^u_{loc}(A_g, g) \quad \text{and} \quad g^{-k_n}(x_n) = \bar{x}_n \in W^u_{loc}(A_g, g).$$

Consider the segments of orbits of $\bar{y}$ and $\bar{x}_n$ corresponding to the first k_n iterates of $\bar{y}$ and $\bar{x}_n$ and the numbers $m_1(\bar{x}_n), \ldots, m_{\ell(\bar{x}_n)}(\bar{x}_n)$ and $m_1(\bar{y}), \ldots, m_{\ell(\bar{y})}(\bar{y})$ defined as above. Lemma 3.10 implies that $\ell = \ell(\bar{x}) = \ell(\bar{y})$ and $m_i(\bar{x}) = m_i(\bar{y})$ for all $i = 0, \ldots, \ell$ (otherwise, $d(f^j(y), z_j) > \varepsilon_1 > \varepsilon$ for some j). Therefore, by the definition of $\delta^u(n)$, $\bar{y} \in \delta^u(n)$. Thus $g^{k_n(\bar{y})} = y \in \Delta^u(x_n)$. Since $y \in W^s_{loc}(B_g, g)$ and $\Delta^u(x_n) \cap W^s_{loc}(B_g, g) = \emptyset$, we obtain a contradiction. $\square$

3.3.4 Proof of Theorem 3

The proof of this theorem involves the ingredients used to prove Theorem 2 plus some generic machinery. We now explain how the proof of Theorem 2 may be modified so as to yield Theorem 3.

Let U be an open set in M and f be a generic diffeomorphism[3] such that the set $\Lambda_f = \Lambda_f(U) \equiv \cap_{i \in \mathbb{Z}} f^i(\overline{U}) \subset U$ is transitive and non-hyperbolic. (If Λ_f were hyperbolic, one would of course have the shaowing property in a sufficiently small neighborhood of Λ_f.)

- By Theorem B of [Ab2] there is a neighborhood $\mathcal{V}$ of f in $\mathrm{Diff}^1(M)$, an arbitrarily small neighborhood (which we still call U) of Λ_f in M, and a residual subset $\mathcal{R}$ of $\mathcal{V}$ such that if $g \in \mathcal{R}$ then the set $\Lambda_g(U) = \cap_{i \in \mathbb{Z}} g^i(\overline{U}) \subset U$ is transitive and non-hyperbolic[4].

[3]When we say that f is a "generic diffeomorphism", we mean that f simultaneously satisfies all of the generic properties listed in the remainder of the proof. So we are doing the proof backwards: a more formal proof would begin with "Let $\mathcal{R}$ be the residual subset of $\mathrm{Diff}^1(M)$ obtained by intersecting the residual sets $\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s$ of diffeomorphisms having respectively the following properties...".

[4]Theorem B of [Ab2] actually deals with attracting sets, but the same arguments yield the result in the case of locally maximal transitive sets

- Now, the transitive set $\Lambda_g(U)$ of the generic diffeomorphism g is a *relative homoclinic class* in U of some periodic point $P_g \in U$. By the *relative homoclinic class* $H(P_g, g, U)$ in U we mean the set of points of the homoclinic class $H(P_g, g)$ whose full orbits are contained in U. This result is stated in [BD$_2$, Proposition 2.4] for robustly transitive sets, but the arguments can be carried out in the same way for locally maximal transitive sets of generic diffeomorphisms.

- We also have that for generic g the set $\Lambda_g(U)$ contains two saddles P_g and Q_g of consecutive indices and having only real positive multipliers of multiplicity one. This is just a reformulation of [BDPR, Theorem A] for the generic case (in fact, the proof is the same as in the robust case). Theorem 2.9 formulates this result in the special case of robustly transitive diffeomorphisms.

- Now, using the Connecting Lemma (Lemma 2.2), we create a cycle associated to these saddles. Unfolding the cycle and using Lemma 3.6, we get $h \in \mathcal{V}$ arbitrarily close to g having a saddle-node in $\Lambda_h(U)$ (a priori h might not belong to $\mathcal{R}$).

- Exactly as in the proof of Theorem 2, we unfold the saddle-node to obtain (after an arbitrarily small C^1 perturbation) $\ell \in \mathcal{V}$ having two saddles (with different indices) A_ℓ and B_ℓ bounding a (normally hyperbolic) periodic segment γ_ℓ such that its (finite) orbit is contained in $\Lambda_\ell(U)$. Now, since this configuration is robust under C^1 perturbations, we can after another small perturbation assume that $\ell \in \mathcal{R}$.

- We now use the transitivity of $\Lambda_\ell(U)$ and the Connecting Lemma to get a dominated heterodimensional cycle associated to A_h and B_h. We then proceed exactly as in the proof of Theorem 2.

- Once this is done, we have achieved the following: arbitrarily near the generic diffeomorphism f we have created an open subset $\mathcal{W}$ of $\mathrm{Diff}^1(M)$ such that every $g \in \mathcal{W}$ exhibits arbitrarily fine pseudo-orbits *inside* U (because the pseudo-orbits occur arbitrarily near the dominated heterodimensional cycle, which is contained in U) which cannot be shadowed by actual orbits. Simple generic arguments (see for instance the proof of Theorem A in [Ab2]) now allow us to show that given a generic diffeomorphism f with a non-hyperbolic locally maximal transitive set Λ, then f is *contained* in some neighborhood $\mathcal{W}$ as above, that is, in a neighborhood consisting of diffeomorphisms g which exhibit non-shadowable pseudo-orbits arbitrarily near Λ.

This completes the sketch of the proof of Theorem 3.

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Flavio Abdenur (flavio@impa.br)
Instituto Nacional de Matemática Pura e Aplicada (IMPA)
Estrada Dona Castorina 110
2246-320 Rio de Janeiro RJ
Brazil

Lorenzo J. Díaz (lodiaz@mat.puc-rio.br)
Departamento de Matemática, PUC-Rio
Marquês de São Vicente 225
22453-900 Rio de Janeiro RJ
Brazil