Critical saddle-node horseshoes: bifurcations and entropy

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Abstract

In this paper we study one-parameter families $(f_\mu)_{\mu\in[-1,1]}$ of two dimensional diffeomorphisms unfolding critical saddle-node horseshoes (say at $\mu=0$) such that f_μ is hyperbolic for negative μ. We describe the dynamics at some isolated secondary bifurcations which appear in the sequel of the unfolding of the initial saddle-node bifurcation.

We construct two classes of open sets of such arcs. For the first class, we exhibit a collection of parameter intervals I_n, $I_n \subset (0,1]$, converging to the saddle-node parameter, $I_n \to 0$, such that the topological entropy of f_μ is a constant h_n in I_n and h_n is an increasing sequence. So, for parameters in I_n, the topological entropy is upper bounded by the entropy of the initial saddle-node diffeomorphisms. This illustrates the following intuitive principle: a critical cycle of an attracting saddle-node horseshoe is a destroying dynamics bifurcation. In the second class, the entropy of f_μ does not depend monotonely on the parameter μ.

Finally, when the saddle-node horseshoe is not an attractor, we prove that the entropy may increase after the bifurcation.

1 Introduction

1.1 Context

The goal of this paper is to study the *destruction of non-trivial hyperbolic sets* via the loss of hyperbolicity of some periodic point of this set. We consider the case of the so-called *saddle-node horseshoes* introduced by Zeeman in [Z], constructed as follows: consider a two dimensional compact disk D and a diffeomorphism $f\colon D \to \mathrm{int}(D)$ such that the limit set of f in D is the usual Smale's horseshoe Λ (conjugate to a complete shift of two symbols) and a sink S. Then consider the collapse of the sink S and the accessible saddle P of the horseshoe via a saddle-node bifurcation in such a way that at the bifurcation the limit set of the resulting diffeomorphism in D is a non-hyperbolic attractor Λ' conjugate to the initial horseshoe Λ and whose basin of attraction contains the whole disk D. We say that Λ' is a *saddle-node horseshoe.* This bifurcation is depicted in Figure 1.

An important feature of this kind of bifurcations is that the dynamics in the sequel of the bifurcation depends simultaneously on the local dynamics in a neighbourhood of the saddle-node and on some global aspects of the bifurcating diffeomorphism which have no relation with the initial dynamics of the horseshoe Λ. Let us illustrate this fact, let P be the saddle-node of the saddle-node horseshoe Λ', then there is an interval (say) I of the unstable manifold of P such that $f_0(I)$ is

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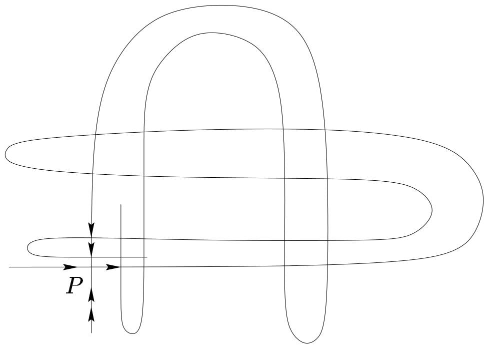

Figure 1: A critical saddle-node horseshoe

contained in the (local) stable manifold of P. So the dynamics of f_0 on I does not affect the limit set of f_0. In the sequel of the bifurcation and after an arbitrarily big number of iterations, points of I returns close to I and are involved in the creation of new limit points which are far from the initial horseshoe and that have no relations with the initial dynamics in the horseshoe (although these points may belong to hyperbolic sets containing the *remainders* of the initial horseshoe).

Consider an one-parameter family of diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ unfolding the previous attracting saddle-node horseshoe (say at $\mu = 0$). Then, after the unfolding of the saddle-node, new bifurcations appear: some periodic points of Λ lose their hyperbolicity and disappear, while homoclinic tangencies associated to the continuations of some hyperbolic periodic points of the initial horseshoe Λ are created, (see [NPT] and [DRoV]). Thus all the complexity of the homoclinic bifurcations appear in this bifurcation (as, for instance, co-existence of infinitely many sinks or sources, [N_2], existence of strange attractors, [MV], persistence of tangencies, [N_2], and cascades of period doubling bifurcations, [YA]). Conversely, Mora has observed that *critical saddle-node bifurcations* do appear as a consequence of the unfolding of homoclinic tangencies, see [PT_3, page 150].

In view of the previous paragraph, secondary bifurcations play a key role in the unfolding of *critical saddle-node horseshoes* (we will give a precise definition later). We analyze some of these secondary bifurcations and study the variation of the topological entropy in the sequel of the bifurcation. This problem is closely related to the one of the creation/destruction of periodic points and hyperbolic sets in the sequel of a bifurcation (see [KKY] for the case of homoclinic tangencies). Here we give a complete description of the dynamics of the limit set Λ_μ of f_μ in D by constructing a Markov partition for Λ_μ for a large set of parameters μ. We also discuss the behavior of the topological entropy in function of the type of saddle-node horseshoe (roughly speaking, if the saddle-node horseshoe is an attractor or not).

As far as we know, the first examples of saddle-node cycles appeared associated to the van der Pol equation as pointed out in [AW]. These cycles were also analyzed in [AL] and [Si]. The systematic study of saddle-node cycles was initiated in [AS, NPT] from the point of view of structural stability. We here use the tools introduced by Newhouse-Palis-Takens in [NPT] to study the local analysis of saddle-node cycles: *adapted vector fields* and *transition maps*, see Section 3.2.

As we have mentioned, saddle-node horseshoes were introduced in [Z] and currently there is a series of works that bring to light part of the richness of phenomena involved in the unfolding of this kind of bifurcation. In the example in [Z] the saddle-node has homoclinic transverse orbits (i.e., the strong stable manifold and the unstable manifold of the saddle-node exhibit transverse intersections, see definitions in Section 1.2). Let us observe that [LS] also considered this type of cycles from the point of view of the occurrence of hyperbolic dynamics following the bifurcation and the existence of non-trivial hyperbolic sets. Latter, [T] studied saddle-node cycles and saddle-node horseshoes as a model for intermittancy.

More recently, in [DRoV] were studied *critical saddle-node cycles* (see definition Section 1.2) from two points of view: occurrence of hyperbolic dynamics in the sequel of the bifurcation and existence of strange attractors. They proved that generic one parameter families of diffeomorphisms unfolding a critical saddle-node cycle always display strange attractors in a *prevalent* way (i. e., the set of parameters corresponding to diffeomorphisms with Hénon-like attractor have positive density at the bifurcation). Moreover, in the case of *one-cycles* (the unstable manifold of the saddle-node is contained in the stable manifold), these strange attractors may have a *global character,* that is, there is an invariant annulus containing the unstable manifold of the saddle-node which is contained in the basin of attraction of the strange attractor. This result was generalized in [Co] in the case of critical saddle-node horseshoes (in this case the strange attractor is the maximal invariant set in the disk D which is contained in its basin of attraction).

Finally, [DU] analyzes the phenomenon of *persistence of tangencies* in the sequel of the unfolding of a critical saddle-node horseshoe and proves that this persistence is essentially independent of the Hausdorff dimension of the saddle-node horseshoe (it is worthy to compare this result with the ones in the case of homoclinic tangencies, where there is a strong relation between frequency of hyperbolicity and fractal dimensions of the hyperbolic set involved in the homoclinic tangency, see the series of papers [N_1], [PT_1], [PT_2], [PY_1], [R] and [M]).

Non-critical saddle-node horseshoes (see definition in the next section) were studied in [CK], where the authors show that a noncritical saddle-node horseshoe may appear as an isolated bifurcation, as it occurs in the case of the derived from Anosov diffeomorphisms, see [W].

In the setting of saddle-node horseshoes (both critical and noncritical) it is known that a wide class of these bifurcation may appear as a first bifurcation, see [LS, Cr].

Finally, for a discussion on global saddle-node bifurcations we refer the reader to [AAIS, Chapter 3.4], and for a survey on saddle-node cycles to [DRiV].

We consider here the destruction of hyperbolic sets via the loss of hyperbolicity of some periodic point of it. Other main mechanism of destruction of hyperbolic sets may be obtained considering *tangencies inside horseshoes*: naively, the periodic points of the hyperbolic set remain hyperbolic but the angles between the stable and unstable manifolds of some periodic points of the horseshoe decreases converging to zero. So, some transverse homoclinic point of the hyperbolic set becomes a non-transverse homoclinic one, obtaining a homoclinic tangency before any hyperbolic point loses its hyperbolicity. This kind of bifurcations were studied in [As], [R] and [K]. See also [BS] and [Ya] for the same type of problems in the context of Hénon maps, and [BDV] and [E] in the case where the hyperbolic set is the whole manifold (Anosov diffeomorphisms) and the tangency is cubic.

1.2 Definitions and Statement of results

Before stating our results let us recall some definitions. A *saddle-node* of a diffeomorphism f is a periodic point P of f such that the derivative of Df^k at P, k is the period of P, has an eigenvalue equal to 1 and all the other eigenvalues outside the unitary circle. Here we consider two dimensional *contracting* saddle-nodes (i.e., all the eigenvalues are in the unitary disk). In such a case, the stable

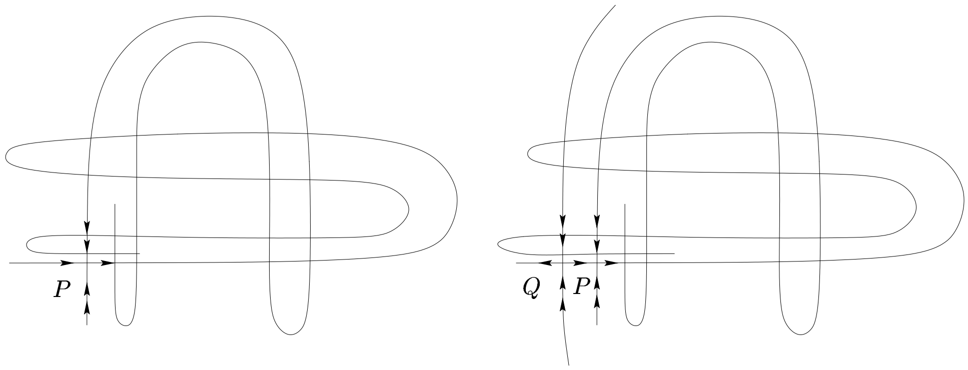

Figure 2: Critical and non critical saddle-node horseshoes

manifold of P has dimension two and its unstable manifold is one-dimensional. We say that the saddle-node is involved in a *saddle-node cycle* if its stable and unstable manifolds have nonempty intersection. This cycle is a *one saddle-node cycle* if the unstable manifold of the saddle-node is contained in the stable one.

In the stable manifold of P it is defined the *strong stable foliation,* the only f-invariant foliation having one dimensional leaves such that the leaf of the saddle-node is tangent to the strong stable bundle (the bundle associated to the contracting eigenvalue of the saddle-node), see [HPS]. The cycle is *critical* if the unstable manifold is tangent to some leaf of this strong stable foliation.

Consider a dissipative saddle-node P, its *homoclinic class,* denoted by $H(P)$, is the closure of the transverse intersections of its unstable and strong stable manifolds. If this homoclinic class is nontrivial we say that it is a *saddle-node horseshoe.* This homoclinic class $H(P)$ satisfies the properties of density of periodic points and transitivity of the usual homoclinic classes. Analogously, the saddle-node P is homoclinically related to a hyperbolic saddle Q if $W^u(Q) \pitchfork W^{ss}(P)$ and $W^u(P) \pitchfork W^s(Q)$ are both nonempty sets. We have that every saddle Q homoclinically related to P belongs to $H(P)$.

We consider here saddle-node horseshoes for which all the periodic points are hyperbolic (except for the saddle-node). A *saddle-node horseshoe cycle* is a saddle-node cycle such that the homoclinic class of the saddle-node is nontrivial. We define *critical* and *noncritical saddle-node horseshoe cycles* in the natural way.

Following the terminology introduced for saddle-node cycles, a *one saddle-node horseshoe cycle* is a saddle-node horseshoe cycle such that the stable manifold of the saddle-node horseshoe Λ contains its unstable manifold (i.e., Λ is an attractor). A *two saddle-node horseshoe cycle* is a saddle-node horseshoe cycle such that there is a saddle Q with $Q \notin \Lambda$, $W^u(P) \pitchfork W^s(Q) \neq \emptyset$ and $W^s(Q) \cap W^{ss}(P) \neq \emptyset$ (P is the saddle-node of Λ). In Figure 2 are depicted different types of saddle-node horseshoe cycles. We will see that, as in the case of one and two saddle-node cycles, the dynamics arising in the sequel of the unfolding of one and a two saddle-node horseshoe cycles may be rather different.

We are now ready to state our results. Let M be a two-dimensional manifold and D a compact disk of M. Consider the set $\mathcal{O}(D)$ of one-parameter families of $\mathcal{C}^\infty$-diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ defined on M unfolding *generically* (see Section 3.1) a critical one saddle-node horseshoe cycle such that:

- Each diffeomorphism f_μ maps D into its interior.
- For $\mu < 0$ the limit set of f_μ in D is the union of a sink S_μ and a hyperbolic saddle set Λ_μ conjugate to a complete shift of two symbols.
- For $\mu = 0$, there is a *saddle-node point* P_0 obtained by collapsing the sink S_μ with some accessible saddle P_μ of Λ_μ. The limit set of f_0 in D is the homoclinic class of P_0, denoted by Λ_0, which is conjugate to Λ_μ ($\mu < 0$).

An arc of diffeomorphisms $(f_\mu)_{\mu \in I}$ has an *isolated saddle-node horseshoe bifurcation at* μ_0 if f_{μ_0} has a saddle-node horseshoe Λ_{μ_0} and the limit set of f_μ is hyperbolic for all $\mu \neq \mu_0$. Observe that an isolated saddle-node horseshoe cycle is necessarily noncritical. See [CK] for an example of such a bifurcation.

We prove that the unfolding of a one critical saddle-node horseshoe cycle can accomplish secondary isolated saddle-node horseshoe cycles where the dynamics is completely controlled:

Theorem A. *Given a two dimensional disk D there is an open subset of $\mathcal{O}(D)$ of arcs $(f_\mu)_{\mu \in [-1,1]}$ unfolding a critical one saddle-node cycle at $\mu = 0$ such that there is a sequence of intervals $H_k \subset (0,1)$ accumulating on 0 and a sequence of parameters $\beta_k \in H_k$ such that*

- *every arc $(f_\mu)_{\mu \in H_k}$ has an isolated saddle-node horseshoe bifurcation at β_k, and*
- *for every $\mu \in H_k$ the topological entropy of f_μ in D, $h(f_\mu|_D)$, is constant (say equal to h_k) and h_k is an strictly increasing sequence converging to $\log(2) = h(f_0|_D)$.*

This theorem gives an explanation to the intuitive idea that a one saddle-node horseshoe bifurcation is a destroying dynamics bifurcation.

Let us observe that the bifurcations at the parameters β_k correspond to isolated bifurcations via non-critical saddle-node cycles, similar to those considered in [CK]. In particular, for each k, the diffeomorphisms $f_\mu \in H_k$ do not exhibit homoclinic tangencies.

Concerning the problem of the variation of the entropy in saddle-node horseshoe cycles, let us observe that, at least in the context of C^1-diffeomorphisms, every diffeomorphism corresponding to a point of entropy variation is approximated by diffeomorphisms with homoclinic tangencies, see [PS]. This result suggests that in the case of non-critical saddle-node horseshoe cycles the entropy is constant throughout the bifurcation. In fact, this is what happens in the case considered in [CK], where the horseshoes before and after the bifurcation are conjugate. In our case, the variation of the entropy is due to the creation of secondary homoclinic tangencies at parameters μ which do not belong to the intervals H_k.

To prove Theorem A we construct a Markov partition for the limit sets of f_μ in D, denoted by Λ_μ, $\mu \in H_k$, (see Section 9) and calculate explicitly its transition matrix (see Section 9.2). Using this matrix we compute the topological entropy of $f_\mu|_D$ in Proposition 9.2.

A key step in our constructions is the one dimensional model for isolated saddle-node horseshoe cycles (analysis of the so-called *ghost dynamics*, see Section 3.3). More precisely, as we have mentioned, there is an interval I in $W^u(P_0) \cap W^s(P_0)$ tangent to the strong stable foliation. The naive idea of our construction is to modify the dynamics of f_0 in I (this modification does not alter the limit set of f_0) in such a way the induced ghost dynamics has an isolated one-dimensional non-critical saddle-node horseshoe, see the example in Section 2. We will see that this bifurcation pattern will be repeated infinitely many times (the same pattern up the period) at every parameter β_k, $\beta_k \to 0^+$, leading to the unfolding of (two-dimensional) isolated saddle-node horseshoes cycles.

We also exhibit an open class of arcs unfolding critical two saddle-node horseshoes increasing the entropy after the bifurcation. More precisely, consider a disk D and the subset $\mathcal{T}(D)$ of saddle-node arcs of $\mathcal{C}^\infty$-diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ such that every f_μ maps the disk D into its interior and

- for $\mu < 0$, the limit set of f_μ in D is the disjoint union of two sinks S^1_μ and S^2_μ, a hyperbolic saddle set Λ_μ conjugate to a complete shift of two symbols, and a saddle Q_μ.
- For $\mu = 0$, there is a *saddle-node point* P_0 obtained by collapsing the sink S^1_μ with some accessible saddle P_μ of Λ_μ. Denote by Λ_0 the homoclinic class of the saddle-node P_0. The limit set of f_0 in the disk D is the disjoint union of Λ_0, a saddle Q_0 and a sink S^2_0. Moreover, at $\mu = 0$ there is a critical two saddle-node horseshoe cycle involving the hyperbolic saddle Q_0 and Λ_0.

Proposition B. *There is an open subset of $\mathcal{T}(D)$ of arcs $(f_\mu)_{\mu\in[-1,1]}$ such that $h(f_\mu|_D) > \log(2) = h(f_0|_D)$ for all $\mu > 0$ small enough.*

Using the construction in the proof of this proposition, in the next result we prove that the dynamics in the sequel of a critical one saddle-node horseshoe cycle is not necessarily destroyed monotonously:

Theorem C. *Given a two dimensional disk D there is an open subset of $\mathcal{O}(D)$ of arcs $(f_\mu)_{\mu\in[-1,1]}$ unfolding a critical one saddle-node horseshoe cycle at $\mu = 0$ such that there is a sequence of intervals $H_k \subset (0,1)$ accumulating on 0 and a sequence of parameters $\beta_k \to 0$, $\beta_k \in H_k$, such that*

- *every arc $(f_\mu)_{\mu\in H_k}$ has a critical two saddle-node horseshoe cycle at β_k, and*
- *let $H_k \setminus \{\beta_k\} = H_k^- \cup H_k^+$ ($\mu^- < \mu^+$ for all $\mu^\pm \in H_k^\pm$), then $h(f_\mu^-|_D) < h(f_\mu^+|_D)$ for all $\mu^\pm \in H_k^\pm$. Moreover, $h(f_\mu|_D)$ is constant for $\mu \in H_k^-$ (say equal to h_k) and h_k is an strictly increasing sequence converging to $\log(2) = h(f_0|_D)$.*

The proof of this theorem is similar to the one of Theorem A, but here we choose the dynamics of f_0 in I in such a way that the induced ghost dynamics exhibits a one dimensional critical two saddle-node horseshoe cycle as the one in Proposition B. In this way, for the parameters β_k the arc unfolds two-dimensional saddle-node cycles exactly as in Proposition B.

In view of the results above let us pose the following problem:

Problem: *Consider a critical one saddle-node horseshoe Λ which is the limit set in a disk D and an arc of diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ unfolding generically the saddle node of Λ at $\mu = 0$.*

Suppose that $\mu = 0$ is a first bifurcation (i.e., f_μ is hyperbolic for every $\mu < 0$). Is $h(f_\mu|_D) < h(f_0|_D)$ for all small $\mu > 0$?

Observe that to solve this problem it is necessary to get estimates for the topological entropy of Hénon-like attractors, for some estimates see [NP].

Let us observe that it is possible to prove similar results to Theorems A and C in the case of critical one saddle-node cycles (to deal with this situation is much more simple than the case of saddle-node horseshoes considered here). In this case, the analogous of Theorem A is the existence of a sequence of intervals H_n, $H_n \to 0$, containing a saddle-node parameter β_n, where the entropy is a constant h_n, $h_n \to 0$ as $n \to \infty$, and such that $(f_\mu)_{\mu \in H_n}$ unfolds a noncritical saddle-node horseshoe at every β_n. The analogous of Theorem C is that the topological entropy does not depend monotonously on the parameter μ.

This paper is organized as follows. We begin by introducing in Section 2 a one-dimensional model for the unfolding of saddle-node horseshoes motivated by the DA-maps in [W]. In Section 3 we introduce the main tools for the local analysis of saddle-nodes (*transition maps* and *reparametrizations*) and the one-dimensional *ghost returns* associated to saddle-node cycles. In Section 4 we study these ghost returns in the case of saddle-node horseshoe cycles. The dynamics of the ghost returns have two parts: a critical part (studied in Sections 5 and 7) and a noncritical part (analyzed in Section 6). The two dimensional ghost returns are introduced in Section 8. In Section 9, using the ghost returns, we construct Markov partitions for a selected family of parameter intervals. These partitions enable us to calculate the topological entropy and to prove Theorem A. In Section 10 we introduce a class of critical two saddle-node horseshoes for which the topological entropy increases after the bifurcation (Proposition B). Using this family we prove Theorem C.

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2 DA-type bifurcations for one dimensional horseshoes

A great part of the constructions in this paper relies on the reduction to the problem to a one-dimensional heuristic model. In order to do our constructions more clear, in this section we give a simple one-dimensional model which is the key to get the hyperbolicity of the limit set in the general two-dimensional setting. This model is motivated by the construction of the Derived from Anosov diffeomorphisms in [W].

Let $(\phi_\mu)_{\mu \in [-1,1]}$ be a one parameter family of interval maps, $\phi_\mu \colon [0,1] \to [0,1]$, inducing a family of differentiable endomorphisms defined on $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ satisfying the following conditions:

a) The map ϕ_μ is smooth for every $\mu \in [-1,1]$, ϕ_0 has a saddle-node $P = P_0$, a sink $S = S_0$ and a source $Q = Q_0$, with $0 < S < Q < P < 1$.

b) For every $\mu \in (-\varepsilon, \varepsilon)$, the map ϕ_μ has exactly two critical points c_1 and c_2 (for simplicity we omit their dependence on μ) whose images are in the basin of attraction of the sink S_μ (the hyperbolic continuation of S) and $0 < S_\mu < c_1 < Q_\mu < P_0 < c_2 < 1$.

c) As a map from the interval $[0,1]$ into itself, ϕ_μ has three discontinuities, say d_1, d_2, d_3 (depending continuously on μ, again such a dependence is omitted), such that $c_1 < P < d_1 < c_2 < d_2 < d_3 < 1$, ϕ_μ is monotone (decreasing) in (d_2, d_3) and surjective in $[d_2, d_3]$.

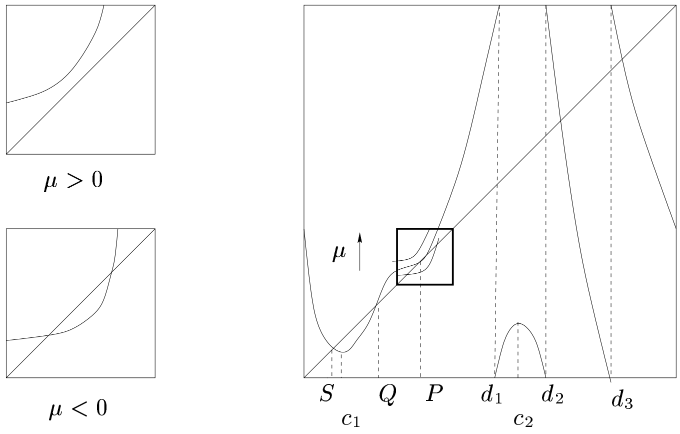

Figure 3: DA-like bifurcation

d) There are small neighborhoods N_1 and N_2 of the criticalities c_1 and c_2, and N_3 and N_4 of the sink S and of the saddle-node P such that, for every x outside $N = \cup_{i=1}^{4} N_i$ and every $\mu \in (-\varepsilon, \varepsilon)$, one has $|\phi'_\mu(x)| > 3$, and $N_1 \cup N_2$ is mapped by ϕ_μ into the basin of attraction of $S_{\{\mu\}}$.

e) The arc (ϕ_μ) unfolds generically the saddle-node P at $\mu = 0$ (see Section 3.1 for the precise definition), having no fixed points inside N_4 for all $\mu > 0$ and two fixed points, a sink S'_μ and a source Q'_μ, $S'_\mu < Q'_\mu$, inside N_4, for $\mu < 0$.

f) For positive values of μ close to zero, there exists a positive integer $N(\mu)$ such that the number $n(\mu, x)$ of iterations by ϕ_μ that a point $x \in N_4$ needs to leave N_4 (i.e., $\phi^j_\mu(x) \in N_4$ for all $0 \leq j \leq n(\mu, x) - 1$, and $\phi_\mu^{n(\mu,x)}(x) \notin N_4$) is bounded from above by $N(\mu)$. Moreover, for every $x \in N_4$, $|D\phi_\mu^{\{}n(\mu, x)\}(x)| > 1/2$.

Condition (c) means that ϕ_0 has a one dimensional saddle-node horseshoe containing P. More precisely, an endomorphism $f : \mathbb{S}^1 \to \mathbb{S}^1$ has a *one-dimensional saddle-node horseshoe* Λ if the set Λ is locally maximal for f, it contains a saddle-node periodic point, the restriction of f to Λ is conjugate to a complete one-sided shift of $k \geq 2$ symbols, and every periodic point of f in Λ is hyperbolic with the exception of a saddle-node. Observe that this saddle-node horseshoe is *noncritical* (the orbits of the critical points do not meet the stable manifold of the saddle-node).

Condition (f) implies that for every small $\mu < 0$ the endomorphism ϕ_μ has a (hyperbolic) one-dimensional horseshoe containing Q_μ.

The existence of non trivial open sets of families verifying (a)–(f) follows from the local form of the saddle node and the construction of the transition maps (see Section 3.2).

Proposition 2.1. *For every family $(\phi_\mu)_{\mu \in (-\varepsilon, \varepsilon)}$ satisfying conditions (a)–(f) above, the limit set of $\{\phi\}_\mu$ is the disjoint union of:*

- *the sink S_μ and a hyperbolic horseshoe Λ_μ containing the source Q_μ, for $\mu > 0$;*

- *the sink S_0, the source Q_0, and a saddle-node horseshoe Λ_0 containing the saddle-node P_0, for $\mu = 0$;*

- *the two sinks S_μ and S'_μ, the source Q_μ and a hyperbolic horseshoe Λ_μ containing the source Q'_μ, for $\mu < 0$.*

Moreover, the horseshoes Λ_μ are conjugate to a complete one-sided shift of three symbols for all small $|\mu|$.

Proof: Since the proposition follows from quite standard arguments we just sketch its proof. Observe that we can assume (without loss of generality) that $\phi_0(0) > P$. Define the points $\bar{P}$, $\bar{Q}'_\mu$ ($\mu < 0$) and $\bar{Q}_\mu$ ($\mu > 0$) as the only points in $(0, c_1)$ with $\phi_0(\bar{P}) = P$, $\phi_\mu(\bar{Q}) = Q_\mu$, and $\phi_\mu(\bar{Q}'_\mu) = Q'_\mu$ (here we are assuming that $|\mu|$ is small enough). Observe that

- for $\mu = 0$, every point in $I_0 = (\bar{P}, P)$ different from Q and $\bar{Q}$ is either in the stable manifold of S or P, ($\mu = 0$),

- for $\mu < 0$, every point in $I_\mu = (\bar{Q}'_\mu, Q'_\mu)$ different from Q_μ and $\bar{Q}_\mu$ is either in the stable manifold of S_μ or S'_μ, and

- for $\mu > 0$, every point in $I_\mu = (\bar{Q}_\mu, Q_\mu)$ is in the stable manifold of S_μ.

For every small μ, define the set Λ_μ as the maximal invariant set of ϕ_μ in $(\mathbb{S}^1 \setminus I_\mu)$. By construction, every point which is not in Λ_μ is either in the stable manifold of the sinks S_μ and S'_μ and of the saddle node P, or it is in the preorbits of Q_μ and Q'_μ (according with the choice of the parameter).

The hyperbolicity of Λ_μ for $\mu > 0$ follows from the fact that there is $M(\mu) > 0$ such that, for all $x \in \Lambda_\mu$, there exists a number $m(\mu, x) \leq M(\mu)$ such that $|D\phi_\mu^{m(\mu,x)}(x)| > 3/2$, where $m(\mu, x) = n(\mu, x) + 1$ if $x \in N_4$ and $m(\mu, x) = 1$ if otherwise, recall (d) and (f) above. The hyperbolicity for $\mu < 0$ follows similarly, and the same holds for the almost hyperbolicity of Λ_0 (see the end of the next paragraph).

To see that the horseshoes Λ_μ are conjugate to a one-sided complete shift of three symbols it is enough to consider the Markov partition defined by the three intervals of $(\mathbb{S}^1 \setminus I_\mu \cup \phi_\mu^{-1}(I_\mu))$. Observe that ϕ_0 is expanding (except at the point P) on each interval of the partition and that for $\mu < 0$ the map ϕ_μ is expanding on each interval of the partition. This also gives the hyperbolicity (or almost hyperbolicity) of Λ_μ, $\mu \geq 0$.

Finally, by construction, the image by ϕ_μ of each interval of the partition is the convex hull (considered as an interval of $[0, 1]$) of the intervals of the partition.

The construction of the Markov partitions implies that the horseshoes Λ_μ are topologically conjugate to the one-sided shift of three symbols.

$\square$

Remark 2.2. *Under natural conditions, the construction above holds in the case of saddle-node arcs (ϕ_μ) having two critical points and an arbitrarily big number of discontinuities d_i such that $\phi_0([d_i, d_{i+1}]) = [0, 1]$ and the restriction of ϕ_0 to $[d_i, d_{i+1}]$ is expanding for every interval $[d_i, d_{i+1}]$ without criticalities. In fact, in this paper we will consider the case where the number of discontinuities and of expanding surjective intervals goes to infinity.*

3 Local dynamics of saddle-nodes: transition maps and ghost dynamics

In this section we define saddle-node bifurcation for diffeomorphisms, state its known properties, and introduce the tools for the analysis of the local dynamics at the saddle-node (adapted vector fields and transitions).

3.1 Saddle-node arcs

The arc $(f_\mu)_{\mu\in[0,1]}$ has a saddle-node bifurcation at $(P_b, b) \in M \times [0,1]$ if there is $k \geq 1$ such that $f_b^k(P_b) = P_b$ and $Df_b^k(P_b)$ has an eigenvalue one and the other eigenvalue has modulus different than one. From now on we take $k = 1$ and $b = 0$. The saddle-node is *attracting* if all the eigenvalues of $Df_0(P_0)$ are contained in the unit circle. Here we consider attracting saddle-nodes. Then there is a Df_0-invariant splitting of the tangent space at P_0, $T_{P_0}M = E^s \oplus E^c$, where $\dim(E^c) = 1$, $Df_0(P_0)|E^c$ is the identity, and $Df_0(P_0)|E^s$ is a contraction.

By the invariant manifold theory, see [HPS], there are one-dimensional center manifolds W_0^c tangent to E^c at P_0, which are locally invariant (i.e., $f_0(W_0^c) \cap W_0^c$ contains a neighbourhood of P_0 in W_0^c). We consider non-degenerate saddle-nodes, meaning that (for a suitable choice of the central manifold) $(f_0|W_0^c)''(P_0) \neq 0$.

By the normal hyperbolicity of W_0^c, the continuation W_μ^c of W_0^c is defined for all small $|\mu|$. In appropriate coordinates, the restriction of f_μ to W_μ^c is of the form

$$f_\mu(x) = x + \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3), \quad \alpha \neq 0.$$

We suppose that $\alpha > 0$. Thus $\{x < 0\} \subset W^s(P_0)$, $\{x > 0\} \subset W^u(P_0)$, f_μ has no fixed points close to P_0 for $\mu > 0$, and, for every small $\mu < 0$, the diffeomorphism f_μ has a sink S_μ and a saddle P_μ, $S_\mu < P_\mu$ (in the natural ordering of W_μ^c), collapsing at $\mu = 0$ to the saddle-node P_0.

By [HPS] there is a strong stable foliation $\mathcal{F}^{ss}$ of $W_{loc}^s(P_0)$, defined as the unique codimension one f_0-invariant foliation of $W_{loc}^s(P_0)$ such that $T_{P_0}(F^{ss}(P_0))$ is E^s, where $F^{ss}(x)$ is the leaf of $\mathcal{F}^{ss}$ containing x.

The leaf of the strong stable foliation containing P_0 is its *strong stable manifold,* denoted by $W^{ss}(P_0)$. Associated to this foliation there is the projection π^s along its leaves from $W_{loc}^s(P_0)$ into an interval of the form (say) $[A, P_0] \subset W_0^c(P_0)$.

3.2 Transition maps

The main tool for the local analysis of the dynamics of a saddle-node are the *transition maps* introduced in [NPT]. For that one considers *adapted saddle-node vector fields,* i.e., a *saddle-node arc of vector fields* $X(\mu, x)$ of the form

$$X(\mu, x) = \mu + \alpha\, x^2 + \beta\, x\, \mu + \gamma\, \mu^2 + \mathcal{O}(|\mu|^3 + |x|^3), \quad \alpha > 0,$$

and such that $f_\mu(x) = X_1(\mu, x)$ for all $x \in W_\mu^c$, where $X_s(\mu, \cdot)$ is the time-s map of the vector field $X(\mu, \cdot)$, see [DRoV, Section 2.1].

In each component of $(W_0^c \setminus \{P_0\})$ consider fundamental domains of f_0, say $D^- = [A, f_0(A)] \subset W^s(P_0)$ and $D^+ = [B, f_0(B)] \subset W^u(P_0)$, where $A < 0 < B$ and $X(0, x) > 0$ for all $x \in$

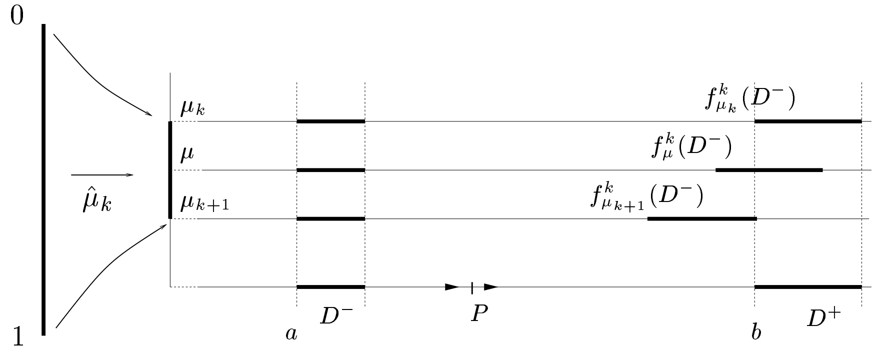

Figure 4: Dynamical normalizations of parameter space

$([f_0^{-2}(A), f_0^2(B)] \setminus \{0\})$. For every big $k \in \mathbb{N}$ define the parameter $\mu_k > 0$ by the relation

$$X_k(\mu_k, A) = B.$$

Then the parameters μ_k are uniquely defined (for big k) and form a decreasing sequence converging to 0^+ as $k \to \infty^+$. Moreover, by [DRoV, Proposition 2.1], for k big enough the maps σ_k defined by

$$\sigma_k : [\mu_{k+1}, \mu_k] \to [0,1], \quad X_{k+\sigma_k(\mu)}(\mu, A) = B. \tag{1}$$

are C^∞ decreasing diffeomorphisms. Denote by $\hat{\mu}_k : [0,1] \to [\mu_{k+1}, \mu_k]$ the inverse of σ_k.

These reparametrizations verify the following distortion property:

Proposition 3.1. ([DRoV, Proposition 2.2]). *Given any $\varepsilon > 0$ there is $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and every Borel set A of $[0,1]$ we have*

$$(1-\varepsilon)|A| \leq \frac{|\hat{\mu}_k(A)|}{|[\mu_{k+1}, \mu_k]|} \leq (1+\varepsilon)|A|,$$

where $|A|$ denotes the Lebesgue measure of the set A.

We define the sequence of transitions $(T_k)_k$ by

$$T_k : [0,1] \times D^- \to \mathbb{R}, \quad T_k(\sigma, x) = f^k_{\hat{\mu}_k(\sigma)}(x).$$

These transitions verify the following properties:

Proposition 3.2. ([DRoV, Theorems 2.3 and 2.6]).

- *The sequence $(T_k)_k$ converges in the C^∞-topology to the* transition map

$$T_\infty : [0,1] \times D^- \to \mathbb{R}, \quad T_\infty(\sigma, x) = X_{t(x)-\sigma}(0, B),$$

where $t(x)$ is given by the relation $X_{t(x)}(0, A) = x$.

- *Consider a small neighbourhood V of 0 in $\mathbb{R}$ and (in local coordinates) the set $D^- \times V \subset W^s_{loc}(P_0)$. Define the maps*

$$\tilde{T}_k,\, \tilde{T}_\infty \colon [0,1] \times D^- \times V \to \mathbb{R} \times V, \qquad \begin{cases} \tilde{T}_k(\sigma,x,y) = f^k_{\bar{\mu}_k(\sigma)}(x,y), \\ \tilde{T}_\infty(\sigma,x,y) = (T_\infty(\sigma,x),0). \end{cases}$$

  Then $(\tilde{T}_k)_k$ converges to $\tilde{T}_\infty$ in the C^∞-topology.

3.3 Ghost dynamics

In [DRoV] (following [NPT]) it is studied (among other things) global aspects of the dynamics of one saddle-node cycles based in the one dimensional *ghost return map* R_∞ defined as a composition of T_∞, an iterate of the bifurcating diffeomorphism, and the projection (on the central manifold of the saddle-node) along the leaves of the strong stable foliation. In that case, the return map is defined in the whole fundamental domain of the center stable manifold of the saddle-node. In the setting of attracting critical one saddle-node horseshoe cycles, it is not possible to define such a return in a complete fundamental domain, so one needs to adapt this construction.

The corresponding step in our setting are the ghost return maps defined from an open (maximal) subinterval J of the fundamental domain D^+ into the whole D^+ as follows.

Observe first that $W^u(P_0)$ is not contained in $W^s(P_0)$. We consider the open subinterval J of D^+ whose extremes are points of transverse intersection between $W^u(P_0)$ and $W^{ss}(P_0)$ and whose interior is contained in $W^s(P_0)$. We can suppose that $f_0^2(J) \subset W^s_{loc}(P_0)$. The one-parameter family $R_\infty(\sigma,\cdot)$, $\sigma \in [0,1]$, of *ghost return maps* is defined by

$$R_\infty(\sigma,\cdot) \colon J \to f_0^{-1}(D^+) \cup D^+, \quad x \mapsto R_\infty(\sigma,x) = T_\infty(\sigma, f_0^{s(x)}(\pi^s f_0^2(x))), \tag{2}$$

where π^s is the projection along the leaves of the strong stable manifold of $W^s(P_0)$ from $W^s_{loc}(P_0)$ into the central stable manifold of P_0 and $s(x)$ is the iterate of $\pi^s(f_0(x))$ in $(A, f_0(A)] \subset D^-$, where $s(x) \to \infty$ as $x \to \partial J$.

Identifying the points of $f_0^{-1}(D^+)$ and D^+ in the same orbit, we get a map from J into the whole D^+. In Figure 5 it is illustrated the case of a ghost return map (in a saddle-node horseshoe) with infinitely many branches.

The relevant property of the ghost return maps is that, for positive parameters μ close to 0^+, an important part of the dynamics of f_μ (up to a reparametrization and up to the period) is close to the one of this family of endomorphisms (this is a consequence of Proposition 3.2).

4 Critical Saddle-node horseshoes

4.1 The saddle-node horseshoe arc

The beginning of our construction is identical to the one in [DRoV, Section 4.2], so we just recall the main steps of this construction, we refer to [DRoV, pages 56-57] for the details.

Consider a rectangle R and affine two-dimensional horseshoe $f \colon R \to R$, $R = R_1 \cup \cdots \cup R_5$, where $R_1, \ldots, R_5$ are compact sub-rectangles of R with pairwise disjoint interiors and such that

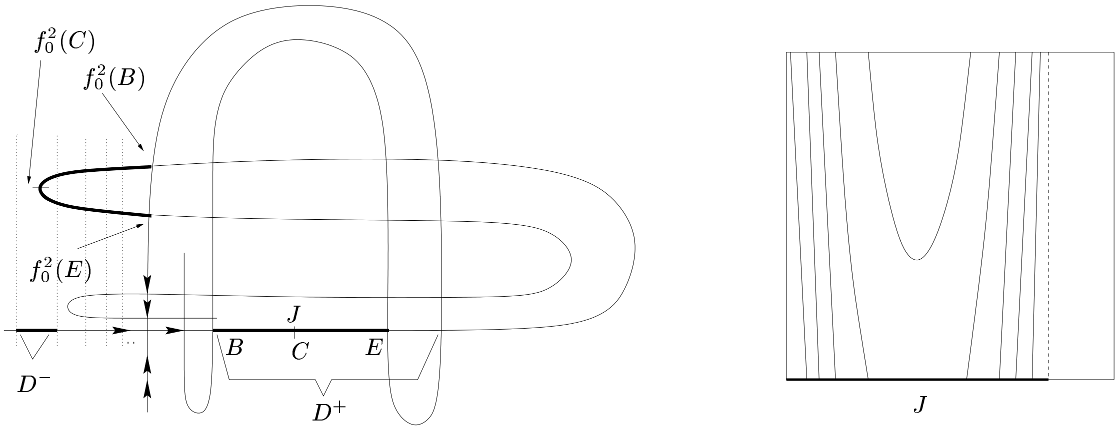

Figure 5: Global dynamics and ghost return maps.

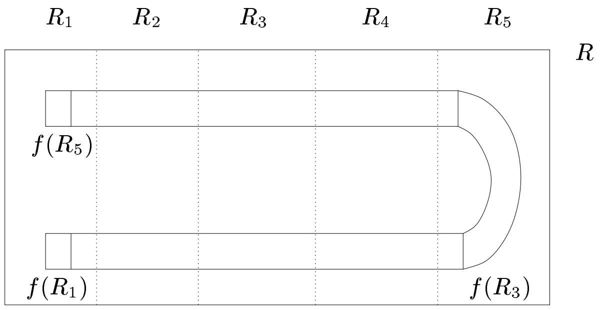

Figure 6: The map f

$f(R_1 \cup R_5) \subset \mathrm{int}(R_1)$, $f(R_3) \subset \mathrm{int}(R_5)$ and

$$Df|_{R_1} = \begin{pmatrix} \phi & 0 \\ 0 & \lambda \end{pmatrix}, \quad Df|_{R_2} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \lambda \end{pmatrix}, \quad Df|_{R_4 \cup R_5} = \begin{pmatrix} -\kappa_2 & 0 \\ 0 & -\lambda \end{pmatrix},$$

where $\kappa_1, \kappa_2 > 1$, $0 < \lambda < 1$, and ϕ is some smooth real map (see Figure 6).

We assume that the fixed point P of f in R_2 is $(s, 0)$, $s > 0$, and that its (local) unstable manifold is contained in $\mathbb{R} \times \{0\}$. Moreover, f has a unique fixed point in R_1, which is a sink $S = (s, 0)$, $s < 0$ such that $R_1 \subset W^s(S)$ (thus $R_5 \subset W^s(S)$). The maximal invariant set of f in R consists of a sink S and a horseshoe Λ (containing P). We now modify the first coordinate of f in order to collapse the saddle P and the sink S *via* a saddle-node bifurcation P_0. In this way we get an arc of diffeomorphisms $(f_\mu)_{\mu \in [-1,1]}$ unfolding the saddle-node at $\mu = 0$ such that for negative μ the maximal invariant set of f_μ in R is the union of a sink S_μ (the continuation of S) and a hyperbolic horseshoe Λ_μ containing P_μ and conjugate to a complete shift of two symbols (Λ_μ and P_μ are the continuation of the horseshoe Λ and the saddle P). For $\mu = 0$ we get a saddle-node horseshoe Λ_0 which is the maximal invariant set of f_0 in R.

We perform the construction above in such a way vertical segments in $R_1 \cup R_2 \cup R_4 \cup R_5$ are mapped by f_μ into vertical segments.

Let us assume that $R = [-2,2] \times [-1,1]$, the saddle-node is $(0,0)$, the vertical direction corresponds to the stable direction, $\Lambda_0 \subset [0,1] \times [0,1]$, $\{0\} \times [0,1] \subset W_0^c(+)$ and $\{0\} \times [-1,0] \subset W_0^c(-)$.

4.2 Ghost dynamics and return maps

Let $W_0^c(+)$ and $W_0^c(-)$ be the two connected components of $(W_0^c \setminus \{P_0\})$, where $W_0^c(-) \subset W^s(P_0)$ corresponds to the points of W_0^c with negative central coordinate. Consider a pair of points B and E in $W_0^c(P_0) \cap W^{ss}(P_0)$ satisfying the following:

- $B \in R_2$, $E \in R_4 \cap (B, f_0(B))$ ($[B, f_0(B)]$ is the segment joining B and $f_0(B)$ in W_0^c), and (B,E) is contained in the interior of $W^s(P_0)$,
- $f_0^2(B), f_0^2(E) \in W_{loc}^{ss}(P_0)$, and $f_0^2([B,E])$ is a parabolic curve contained in $W_{loc}^s(P_0)$ transverse to the strong stable foliation of $W_{loc}^s(P_0)$ except at a point $f_0^2(C)$, $C \in (B,E)$, corresponding to a quadratic tangency. See Figure 5.

We let $\pi^s(f_0^2(C)) = A \in W_0^c(-)$ and consider the fundamental domains $D^- = [A, f_0(A)]$ of $W_0^c(-)$ and $D^+ = [B, f_0(B)]$ of $W_0^c(+)$. As in Section 3.2, associated to D^- and D^+ we define the parameters μ_k, the reparametrizations $\hat{\mu}_k$ and σ_k, and the transitions T_k and T_∞. Since $\pi^s f_0^2((B,E)) = [A, P_0)$ and $f_0^2((B,E))$ is a parabolic curve, given any $x \in (B,E)$ there is a first $s(x)$, $s(x) \leq 0$, such that

$$f_0^{s(x)}(\pi^s(f_0^2(x))) \in D^-,$$

where $s(x) \to -\infty$ as $x \to \{B, E\}$. The *ghost return map* now is defined by

$$R_\infty \colon [0,1] \times (B,E) \to f_0^{-1}(D^+) \cup D^+, \quad R_\infty(\sigma, x) = T_\infty(\sigma, f_0^{s(x)}(\pi^s f_0^2(x))), \tag{3}$$

where T_∞ is defined as in Proposition 3.2. As above, identifying points in $f_0^{-1}(D^+) \cup D^+$ in the same orbit we can think of this return map as a map from (B,E) into D^+. See Figure 5 for the induced one dimensional map.

Observe that $\pi^s(f_0^2((B,E))) = \cup_{i=0}^{\infty} f_0^i(D^-)$. Hence, for each $\sigma \in [0,1]$ and $i \geq 0$ there are exactly two points $d_i^-(\sigma)$ and $d_i^+(\sigma)$ in (B,E) , $d_i^-(\sigma) < d_i^+(\sigma)$, such that

$$t(\pi^s(f_0^2(d_i^\pm(\sigma)))) = i + \sigma, \quad \text{i.e.,} \quad X_{i+\sigma}(0,A) = \pi^s(f_0^2(d_i^\pm(\sigma))).$$

For $i = 0$, we have that $d_0^+(\sigma)$ (resp. $d_0^-(\sigma)$) is decreasing (resp. increasing) function with $d_0^+(0) = d_0^-(0) = C = \pi^s(f_0^2(C))$. Observe that $d_i^+(\sigma)$ (resp. $d_i^-(\sigma)$) is an increasing sequence (resp. decreasing) converging to E (resp. B).

For each $\sigma \in [0,1]$ let $\Delta_0(\sigma) = [d_0^-(\sigma), d_0^+(\sigma)]$ and, for $i > 0$, let $\Delta_i^+(\sigma) = [d_{i-1}^+(\sigma), d_i^+(\sigma)]$ and $\Delta_i^-(\sigma) = [d_i^-(\sigma), d_{i-1}^-(\sigma)]$. The following properties hold for $R_\infty(\sigma, \cdot)$:

(S) Surjective branches: For each $\sigma \in [0,1]$ and $i \geq 1$, the map $R_\infty(\sigma, \cdot) \colon \Delta_i^\pm(\sigma) \to D^+$ is surjective and injective in the interior of $\Delta_i^\pm(\sigma)$. Moreover, $R_\infty(\sigma, \cdot)$ is increasing in $\Delta_i^+(\sigma)$ and decreasing in $\Delta_i^-(\sigma)$.

(C) Criticalities and unimodal branch: For every $\sigma \neq 0$, the restriction of $R_\infty(\sigma, \cdot)$ to $\Delta_0(\sigma)$ is unimodal and C is its critical point, which is quadratic and generic.

We have that the maps $R_\infty(\sigma, \cdot)$ are very expansive close to E and B:

Lemma 4.1. *The derivative of $R_\infty(\sigma,\cdot)$ verifies $|D_x(R_\infty(\sigma,\cdot))| \to \infty$ as $x \to E$ or $x \to B$ for every $\sigma \in [0,1]$.*

Proof: Since the size of the fundamental domains $[x, f(x)]$ of the restriction of f to the central stable manifold of P goes to zero as x goes to P, we have that $|\Delta_i^{\pm}(\sigma)| \to 0$ as $i \to \infty$. By the mean value theorem, for each $\Delta_i^{\pm}(\sigma)$, $i \geq 1$ and $\sigma \in [0,1]$, there is $x_i^{\pm}(\sigma) \in \Delta_i^{\pm}(\sigma)$, with

$$|D_{x_i^{\pm}}(R_\infty(\sigma,\cdot))| = \frac{|D^+|}{|\Delta_i^{\pm}(\sigma)|}.$$

Since $|\Delta_i^{\pm}(\sigma)| \to 0$ as $i \to \infty$ this implies that $|D_{x_i^{\pm}(\sigma)}(R_\infty(\sigma,\cdot))| \to \infty$ as $i \to \infty$.

The bounded distortion of the saddle-node now implies that there is $K > 0$ such that

$$\frac{|D_x(R_\infty(\sigma,\cdot))|}{|D_y(R_\infty(\sigma,\cdot))|} > K \quad \text{for all } \sigma \in [0,1],\ x, y \in \Delta_i^{\pm}(\sigma), \text{ and } i \geq 2,$$

see Claim 5.5 for details. This ends the proof of the lemma. □

A subinterval J of (B,E) *is mapped into k fundamental domains of W_0^c* if there is $x \in W_0^c(-)$ such that $[x, f_0^k(x)] \subset \pi^2(f_0^2(J))$. Observe that given any $\kappa \in \mathbb{N}$ there is $\varepsilon_\kappa > 0$, $\varepsilon_\kappa \to 0$ as $\kappa \to \infty$, such that the interval $K(\varepsilon_\kappa) = [B + \varepsilon_\kappa, E - \varepsilon_\kappa] \subset (B,E)$ is mapped by f_0 into κ fundamental domains of W_0^c. Fix now a big κ and consider ε_κ and $K(\varepsilon_\kappa)$ as above. Take small $\varepsilon > 0$ such that $[C-\varepsilon, C+\varepsilon] \subset K(\varepsilon_\kappa)$ if κ is big enough (and then ε_κ is small).

By construction, for a fixed small ε_κ, given any $\sigma \in [0,1]$ and $y \in D^+$, the set of points $x \in (K(\varepsilon_\kappa) \setminus (C-\varepsilon, C+\varepsilon))$ with $R_\infty(\sigma,x) = y$ has (at least) $(2\kappa - 3)$ elements. Thus, fixed a big κ, we can modify f_0 in $f_0(K(\varepsilon_\kappa))$ to have that $|D_x(R_\infty(\sigma,x))| > 4$ for all $x \in K(\varepsilon_\kappa) \setminus (C-\varepsilon, C+\varepsilon)$ and $\sigma \in [0,1]$. Observing that $f_0(K(\varepsilon_\kappa))$ is contained in the stable manifold of the saddle-node, we have that the previous perturbation can be done without modifying the limit set of f_0 in R. The previous comments and Lemma 4.1 are summarized in the following condition:

(E) Uniform expansion: $|D_x(R_\infty(\sigma,x))| > 4$ for all $x \in (B,E) \setminus (C-\varepsilon, C+\varepsilon)$ and $\sigma \in I$.

4.3 Return maps. Approximation by the ghost dynamics

4.3.1 One dimensional return maps

For every small $|\mu|$ there are defined the continuations E_μ, B_μ and C_μ of E, B and C as follows: E_μ and B_μ are the points of W_μ^c close to B and E with $f_\mu^2(E_\mu), f_\mu^2(B_\mu) \in F^{ss}(0)$ and $f_\mu([B_\mu, E_\mu])$ is tangent to the strong stable foliation at $f_\mu^2(C_\mu)$. For simplicity, let us omit the dependence on μ of the points B_μ, E_μ and C_μ.

For small $\mu > 0$ define the *return map* R_μ as follows,

$$R_\mu\colon [B,E] \to D^+, \quad R_\mu(x) = f_\mu^{s_\mu(x)}(\pi^s(f_\mu^2(x))),$$

where $s_\mu(x)$ is the first forward iterate of $\pi^s(f_\mu^2(x))$ in D^+.

Observe that, with the notation in Section 4.2, for $\mu > 0$,the map R_μ has only finitely many surjective branches. More precisely, for each $\mu > 0$, the function $s_\mu(x)$ is upper bounded by $N(\mu) = s_\mu(C)$, where $N(\mu) \to \infty$ as $\mu \to 0^+$.

Associated to the maps R_μ consider the sequence of one-parameter families of returns defined by

$$R_k\colon [0,1]\times[B,E]\to D^+,\quad R_k(\sigma,x)=R_{\hat\mu_k(\sigma)}(x), \tag{4}$$

where $\hat\mu_k\colon[0,1]\to[\mu_{k+1},\mu_k]$ are the reparametrizations in Section 3.2.

Observe that the numbers $s(x)$ in the definition R_∞ (see equation (3)) and $s_\mu(x)$ above are uniformly bounded from below and above (independently of small $\mu\geq 0$) in each compact subset of (B,E) (bounds depend only on the choice of the compact subset). This fact and Proposition 3.2 imply the following:

Lemma 4.2. *Let K be a compact subset of (B,E). Then the restriction of R_k to $[0,1]\times K$ converges in the C^∞-topology to the restriction of R_∞ to $[0,1]\times K$.*

Since $R_\mu(\cdot)=R_k(\sigma,\cdot)$ if $\mu=\sigma_k(\sigma)$, Lemma 4.2 and the uniform expansivity property (E) immediately give the following:

Lemma 4.3. *Let $\varepsilon>0$ as in property (E). Then there is $k_0\in\mathbb{N}$ such that $|D_x(R_\nu(\cdot))|>3$ for all $x\notin(C-\varepsilon,C+\varepsilon))$ and $\nu\in\sigma_k([0,1])$, $k\geq k_0$.*

In what follows, the compact set $K(\varepsilon_\kappa)$ containing $(C-\varepsilon,C+\varepsilon)$ will remain fixed. In the next section we will analyze the dynamics of $R_\infty(\sigma,\cdot)$ in the critical set $(C-\varepsilon,C+\varepsilon)$ for certain sets of parameters. In Section 6. we will study the dynamics in $[B,E]\setminus K(\varepsilon_\kappa)$ (this dynamics will be strongly expansive). Observe that to study the dynamics in $D^+\setminus K(\varepsilon_\kappa)$ it is not possible to use (directly) the results in [DRoV] (Propositions 3.2).

5 The dynamics of the critical region

Lemma 5.1. *There are a strictly increasing sequence of natural numbers (r_k) and a decreasing sequence of parameters $(\nu_k)_k$, $\nu_k\in(\mu_{r_k+1},\mu_{r_k}]$, $\nu_k\to 0^+$ as $k\to\infty$, such that*

$$R_{\nu_k}(B)=R_{\nu_k}(E)=B. \tag{5}$$

Moreover, the parameters ν_k are uniquely defined if k is sufficiently big.

Proof: By definition of E, B and of the returns R_ν, to prove the lemma it is enough to see that there is sequence of parameters $(\nu_k)_k$ such that

$$f^k_{\nu_k}(0)=f^k_{\nu_k}(\pi^s(f^2_{\nu_k}(B)))=f^k_{\nu_k}(\pi^s(f^2_{\nu_k}(E)))=f^k_{\nu_k}(B). \tag{6}$$

By construction, the parameters μ_k in Section 3.2 verify $f^k_{\mu_k}(A)=B$. Take a big $k>0$, by continuity of the map $\mu\mapsto f_\mu^{-k-1}(B)$ and since $f_{\mu_{k+1}}^{-k-1}(B)=A<0<f_0^{-k-1}(B)$, there is a first $r_k\in\mathbb{N}$ such that

$$f_{\mu_{r_k}}^{-k-1}(B)\leq 0<f_{\mu_{r_k}}^{-k}(B),\qquad B\leq f^{k+1}_{\mu_{r_k}}(0)<f_{\mu_{r_k}}(B).$$

Consider now the continuous map

$$h_k : [\mu_{r_k+1}, \mu_{r_k}] \to \mathbb{R}, \quad h_k(\mu) = f_\mu^{k+1}(0).$$

Observe that, due to the local form of the saddle-node, the map $h_k(\cdot)$ is strictly increasing (if k is big enough). Thus, by definition of r_k,

$$h_k(\mu_{r_k+1}) < B \le h_k(\mu_{r_k}).$$

Thus, there is $\nu_k \in (\mu_{r_k+1}, \mu_{r_k}]$ such that $h_k(\nu_k) = f_{\nu_k}^{k+1}(0) = B$. Since $\pi^s(f_{\nu_k}^2(B)) = \pi^s(f_{\nu_k}^2(E)) = 0$, this ends the proof of (6), and hence, of the first part of the lemma.

Since (for big k) the maps h_k are increasing the parameter ν_k is unique. By construction, the sequence (ν_k) is decreasing and $\nu_k \to 0^+$ as $k \to \infty$. The proof of the lemma is now complete. $\square$

We associate to each parameter ν_k in Lemma 5.1 the number m_k defined by

$$f_{\nu_k}^{-m_k}(0) \in (A, f_0(A)].$$

This choice and the definition of μ_{r_k} and r_k imply that

$$r_k = m_k + k. \tag{7}$$

Given $\nu_k \in (\mu_{r_k+1}, \mu_{r_k}]$, let $\sigma_k^* = \sigma_{r_k}(\nu_k) \in [0,1)$. The sequence (σ_k^*) has a convergent subsequence, converging to some $\sigma^* \in [0,1]$. For notational simplicity, let us assume that the initial sequence (σ_k^*) is convergent.

We now modify the dynamics of f_0 in the interior of the rectangle R_5, without modifying the forward orbits of B and E, to make C a fixed point of $R_\infty(\sigma^*, \cdot)$. This perturbation preserves the limit set of f_0 in R.

Define $\sigma_1 \in (0,1)$ by $X_{\sigma_1}(0,B) = C$. If $\sigma^* = \sigma_1$, then the critical point C is a fixed point of $R_\infty(\sigma_1, \cdot)$, and there is nothing to do. Otherwise, since $f_0(C) \in R_5$, after modifying the dynamics of f_0 in R_5, we can assume that

$$\begin{aligned} \pi^s(f_0^2(C)) &= X_\tau(0,A), \quad \text{where } \tau = \sigma_1 + \sigma^*, \text{ if } \sigma_1 + \sigma^* < 1, \text{ or} \\ \pi^s(f_0^2(C)) &= X_\tau(0,A), \quad \text{where } \tau = \sigma_1 + \sigma^* - 1, \text{ if } \sigma_1 + \sigma^* > 1. \end{aligned}$$

In both cases, $\tau \in (0,1)$ and $\pi^s(f_0^2(C)) \in [A, f_0(A)) \subset D^-$. In the first one, we have

$$T_\infty(\sigma^*, \pi^s(f_0^2(C))) = T_\infty(\sigma^*, X_\tau(0,A)) = X_{\tau-\sigma^*}(0,B) = X_{\sigma_1}(0,B) = C.$$

In the second case, we have

$$T_\infty(\sigma^*, \pi^s(f_0^2(C))) = T_\infty(\sigma^*, X_\tau(0,A)) = X_{\tau-\sigma^*}(0,B) = X_{\sigma_1-1}(0,B) = X_1(0,C) \simeq C,$$

where, in the last identity, we identify points of $D^+ \cup f_0^{-1}(D^+)$ in the same orbit. The definition of R_∞ in Section 3.3 implies that $R_\infty(\sigma^*, C) = C$.

Observe that the perturbation above can be done in such a way that the resulting ghost returns $R_\infty(\sigma, \cdot)$ verify conditions (S), (C), and (E) in Section 4.2.

Since C is a quadratic critical point of $R_\infty(\sigma^*, \cdot)$, we can assume that the constant $\varepsilon > 0$ in (E) is such that

$$R_\infty(\sigma^*, [C-\varepsilon, C+\varepsilon]) \subset (C-\varepsilon/6, C+\varepsilon/6).$$

By the continuous dependence of $R_\infty(\sigma, \cdot)$ on σ, there is $\delta > 0$ such that

$$R_\infty(\sigma, [C-\varepsilon, C+\varepsilon]) \subset (C-\varepsilon/5, C+\varepsilon/5), \quad \text{for all } \sigma \in I^* = (\sigma^* - \delta, \sigma^* + \delta). \tag{8}$$

Since $\sigma_k^* \to \sigma^*$, there is $\xi > 0$ with $[\sigma_k^* - \xi, \sigma_k^* + \xi] \subset (\sigma^* - \delta, \sigma^* + \delta) \subset (0,1)$, for every k big enough. Let

$$J_k = \hat{\mu}_{r_k+1}([\sigma_k^* - \xi, \sigma_k^* + \xi]) \subset I_k^* = \hat{\mu}_{r_k+1}([\sigma_0 - \delta, \sigma_0 + \delta]) \subset [\mu_{r_k+1}, \mu_{r_k}].$$

By (8), the definition of R_ν, and Proposition 3.2, we get

$$R_\nu([C-\varepsilon, C+\varepsilon]) \subset (C-\varepsilon/4, C+\varepsilon/4), \quad \text{for all } \nu \in J_k \text{ and } k \text{ big enough.} \tag{9}$$

Proposition 5.2. (Choice of parameters) *There are ε and $\xi > 0$ and sequences of natural numbers r_k, of parameters $\nu_k = \hat{\mu}_{r_k}(\sigma_k^*)$, and of intervals $J_k = \hat{\mu}_{r_k}(\sigma_k^* - \xi, \sigma_k^* + \xi)$ such that for every $\nu \in J_k$ it holds*

- *$R_\nu(C-\varepsilon, C+\varepsilon) \subset (C-\varepsilon/3, C+\varepsilon/3)$,*
- *$|D_x R_\nu| > 3$ for all $x \in ([B,E] \setminus (C-\varepsilon, C+\varepsilon))$.*

The first part of the proposition is stated in (9). The second part follows from Lemma 4.3 and the lemma below:

Lemma 5.3. *Given any $M > 2$ there are $\kappa = \kappa(M) > 0$ and $k_0 = k_0(M) \in \mathbb{N}$ such that*

$$|R'_\nu(x)| > M,$$

for every $x \in [B, B+\kappa] \cup [E-\kappa, E]$, ν in J_k, and $k \geq k_0$.

Remark 5.4. *Lemma 5.3 implies that B is a hyperbolic fixed point of R_{ν_k}, so it has a continuation $\tilde{B}_\nu$, which is fixed and hyperbolic for R_ν, for every ν in a small vicinity of ν_k. Similarly, in this parameter neighbourhood, it is defined the hyperbolic continuation $\tilde{E}_\nu$ of E, with $R_\nu(\tilde{E}_\nu) = \tilde{B}_\nu$.*

Proof: The main ingredient of the proof of the lemma is the following bounded distortion property. Consider $\nu > 0$, $x \in [f_\nu^2(A), f_\nu(B)] \subset W_\nu^c$, and $y \in [f_\nu^{-1}(x), x]$. Let $x^* = f_\nu^{-\ell(x)}(x)$, $\ell(x) > 0$, be the first element of the backward f_ν-orbit of x in D^- and write $y^*(x) = f_\nu^{-\ell(x)}(y)$.

Claim 5.5. (Bounded distortion property) *There is $K_0 > 0$ such that for every small $\nu > 0$ and every pair of points x and y with $x \in [f_\nu^2(A), f_\nu(B)]$ and $f_\nu^{-1}(x) \leq y < x$, the following holds:*

$$|[y^*(x), x^*]| \geq K_0\, |[y, x]|.$$

Proof of the claim: Write

$$I^* = [y^*(x), x^*] \subset \Delta^* = [f_\nu^{-1}(x^*), x^*],\ I = f_\nu^{\ell(x)}(I^*) = [y, x] \subset \Delta = f_\nu^{\ell(x)}(\Delta^*) = [f_\nu^{-1}(x), x].$$

Then, there are points $z \in I^*$, $w \in \Delta^*$ such that

$$\frac{|I|}{|\Delta|} = \frac{|D_z f_\nu^{\ell(x)}|}{|D_w f_\nu^{\ell(x)}|}\,\frac{|I^*|}{|\Delta^*|} \leq \exp(K_1 \sum_{i=0}^{\ell(x)-1} |f_\nu^i(z) - f_\nu^i(w)|)\,\frac{|I^*|}{|\Delta^*|},$$

where K_1 is an uniform upper bound of the derivative of $\log|(D_z f_\nu)|$ in the segment $[f_\nu^{-1}(A), f_\nu(B)]$ (for all small $|\nu|$). Observe that the fundamental domain Δ^* contains the interval J bounded by z and w. Thus,

$$\sum_{i=0}^{\ell(x)-1} |f_\nu^i(J)| \leq |[f_\nu^{-1}(A), f_\nu(B)]| \leq K_2,$$

where K_2 is independent of ν. Finally,

$$\frac{|I|}{|\Delta|} \leq \exp(K_1 \sum_{i=0}^{\ell(x)-1} |f_\nu^i(J)|)\,\frac{|I^*|}{|\Delta^*|} \leq \exp(K_1\, K_2)\,\frac{|I^*|}{|\Delta^*|} \leq K_3\,\frac{|I^*|}{|\Delta^*|}.$$

The claim follows observing that $|\Delta|$ is upper bounded and $|\Delta^*|$ is lower bounded away from 0. $\square$

To end the proof of the lemma, fix small $\nu > 0$ and take big k and small $\kappa > 0$ such that $C \notin [B, B+\kappa] \cup [E-\kappa, E] = \Xi_\kappa$. Write $\Gamma_\kappa = \pi^s(f_\nu^2(\Xi_\kappa))$. Then there are constants $K_1, \dots, K_5 > 0$, independent of small $\nu > 0$, satisfying the following:

- **e1)** $K_1 < |D_x(\pi^s(f_\nu^2))| < K_2$ for every $x \in \Xi_\kappa$ (just observe that $C \notin \Xi_\kappa$),
- **e2)** $\pi^s(f_\nu^2(\Xi_\kappa)) \subset [-K_3\,\kappa, K_3\,\kappa] = I(\kappa)$ (just recall that $\pi^s(f_{\nu_k}^2(B)) = \pi^s(f_{\nu_k}^2(E)) = 0$),
- **e3)** $|f_\nu(z) - z| < K_4\,(\kappa^2 + \nu)$ for all $z \in I(\kappa)$ and small $\nu > 0$ (just use the local coordinates of the saddle-node), and
- **e4)** Given any $z \in I(\kappa)$, define $z^* = f_\nu^{-\ell(z)}(z) \in D^-$ as the first element of the backward orbit of z in D^-, then $K_5^{-1}\,|D^-| < |[z^*, f_\nu(z^*)]| < K_5\,|D^-|$.

Consider now $z \in I(\kappa)$. Comparing the sizes of the fundamental domains $[z^*, f_\nu(z^*)]$ and $[z, f_\nu(z)]$ (see (e3) and (e4)) and using Claim 5.5, we get $K_6 > 0$ (independent of small $\nu > 0$) such that

$$|D_z(f_\nu^{-\ell(z)})| \geq \frac{K_6}{\kappa^2 + \nu}, \quad \text{for every } z \in I(\kappa). \tag{10}$$

On the other hand, $f_\nu^{r_k}(D^-) \cap D^+ \neq \emptyset$ for every $\nu \in J_k$. Again, by using Claim 5.5, we get $K_7 > 0$ (independent of small $\nu > 0$) such that

$$K_7 < |D_{z^*} f_\nu^{r_k}| \quad \text{for every } z \in I(\kappa). \tag{11}$$

Given $x \in [B,E]$, let $y(x) = \pi^s(f_\nu^2(x))$, and observe that $R_\nu(x) = f_\nu^{r_k-\ell(y(x))}(y(x)^*)$. Thus, from (e1), (10) and (11), there is $K_8 > 0$ (independent of small $\nu > 0$) such that

$$|D_x R_\nu| \geq \frac{K_8}{\kappa^2 + \nu}, \quad \text{for every } x \in \Xi_\kappa \text{ and } \nu \in J_k.$$

The lemma now follows by taking κ and ν_k small enough. $\square$

6 Dynamics in the noncritical region

In this section, we prove that, for every parameter $\nu \in J_k$, we have an appropriate control of the dynamics of the returns to the fundamental domains $D^+(\nu) = [\tilde{B}_\nu, f_\nu(\tilde{B}_\nu)]$, $\tilde{B}_\nu \to B_{\nu_k}$ as $\nu \to \nu_k$, of the central manifold, where $\tilde{B}_\nu$ is the continuation of the hyperbolic periodic point B_{ν_k} of R_{ν_k}, (see Remark 5.4). Similarly, the point $\tilde{E}_\nu$ is the point close to E_{ν_k} with $R_\nu(\tilde{E}_\nu) = R_\nu(\tilde{B}_\nu) = \tilde{B}_\nu$. We claim that the continuations of these points are defined for every parameter in the interval J_k (provided that ξ is small). This follows from the next lemma (see Remark 6.2).

Lemma 6.1. *Fixed $\varepsilon > 0$ the parameter $\nu(\varepsilon)$ close to ν_k defined by $R_{\nu(\varepsilon)}(B_{\nu_k}) = B_{\nu_k} + \varepsilon$ is of the form $\nu(\varepsilon) = \nu_k + \ell\,\varepsilon\,\nu_k^{3/2}$, for some constant ℓ uniformly bounded from below and above (independently of k). Moreover, the parameter $\nu(\varepsilon)$ is uniquely defined.*

Proof: For each $\nu \in J_k$ define the map

$$\Psi(\nu) = \int_0^{B_{\nu_k}} \frac{dx}{X(\nu,x)}.$$

The parameters $\nu(\varepsilon)$ and ν_k verify the following relations:

$$\int_0^{B_{\nu_k}+\varepsilon} \frac{dx}{X(\nu(\varepsilon),x)} = \Psi(\nu(\varepsilon)) + \int_{B_{\nu_k}}^{B_{\nu_k}+\varepsilon} \frac{dx}{X(\nu(\varepsilon),x)} = k, \qquad \int_0^{B_{\nu_k}} \frac{dx}{X(\nu_k,x)} = \Psi(\nu_k) = k.$$

Thus,

$$|\Psi(\nu(\varepsilon)) - \Psi(\nu_k)| = \int_{B_{\nu_k}}^{B_{\nu_k}+\varepsilon} \frac{dx}{X(\nu(\varepsilon),x)} \in [k_1\,\varepsilon, k_2\,\varepsilon], \tag{12}$$

for some constants k_1 and $k_2 > 0$ independent of small ν (observe that $|X(\nu,\cdot)|$ is uniformly bounded away from zero in the interval $[f_0^{-2}(B), f_0^2(B)]$ for every small $|\nu|$).

By [DRoV, Lemma 2.4], $\Psi'(\nu)$ is of order of $\nu^{-3/2}$ (small ν). Therefore, $|\Psi(\nu(\varepsilon)) - \Psi(\nu_k)|$ is of order of $\nu_k^{-3/2}|\nu(\varepsilon) - \nu_k|$. The first part of the lemma now follows from (12). Observe that the estimates on the derivative of Ψ also gives the uniqueness of $\nu(\varepsilon) \in J_k$. $\square$

Remark 6.2. *The proof of Lemma 6.1 implies that the parameters μ_k are of order of $\mu_{k+1}+\mu_{k+1}^{3/2}$. Thus there is a subinterval L_k of $J_k \subset [\mu_{r_k+1}, \mu_{r_k}]$ where the continuations $\tilde{B}_\nu$ and $\tilde{E}_\nu$ of B_{ν_k} and E_{ν_k} are defined. and constants k_1 and $k_2 > 0$ (independent of k) such that*

$$k_1 \leq \frac{|L_k|}{\mu_{r_k} - \mu_{r_k+1}} \leq k_2.$$

In particular, by Proposition 3.1, there is $\tau > 0$ such that $\sigma_{r_k}([\sigma_k^* - \tau, \sigma_k^* + \tau]) \subset L_k$. By shrinking τ (if necessary), we can assume that $\tau = \xi$, where ξ is as in Proposition 5.2 (i.e., we can take $L_k = J_k$).

Given a parameter $\nu \in J_k$ and $i \in \{0, \ldots, m_k\}$, $r_k = k + m_k$, let $\mathcal{D}_\nu^i$ be the set of points in $\Delta(\nu) = [\tilde{B}_\nu, \tilde{E}_\nu]$ such that

$$\mathcal{D}_\nu^i = \{x \in \Delta(\nu) \quad \text{such that} \quad f_\nu^{k+m_k-i}(\pi^s f_\nu^2(x)) = f_\nu(\tilde{B}_\nu)\}, \text{ i.e. } R_\nu(x) = \tilde{B}_\nu\}.$$

Since $f_\nu^2([\tilde{B}_\nu, \tilde{E}_\nu])$ is a parabolic curve transverse to the strong stable foliation except at the point $f_\nu^2(C)$, by the definition of m_k (see (7)), we have that

- **($\mathbf{r}_1$)** $\mathcal{D}_\nu^i = \emptyset$ for all $i > m_k$,
- **($\mathbf{r}_2$)** $\mathcal{D}_\nu^i = \{d_i^-(\nu), d_i^+(\nu)\}$ where $d_i^-(\nu) < d_i^+(\nu)$, for every $i \in \{0, 1, \ldots, m_k\}$,
- **($\mathbf{r}_3$)** $[d_i^-(\nu), d_i^+(\nu)] \subset (d_{i+1}^-(\nu), d_{i+1}^+(\nu))$, $d_{m_k}^-(\nu) = \tilde{B}_\nu$ and $d_{m_k}^+(\nu) = \tilde{E}_\nu$,
- **($\mathbf{r}_4$)** for $i > 0$, let $\Delta_i^- = [d_i^-, d_{i-1}^-]$ and $\Delta_i^+ = [d_{i-1}^+, d_i^+]$, then $R_\nu(\Delta_i^\pm(\nu)) = [\tilde{B}_\nu, f_\nu(\tilde{B}_\nu)] = D^+(\nu)$ for all $i \in \{1, 2, \ldots, m_k\}$.

Remark 6.3. *The points $d_i^\pm(\nu)$ are the continuations, for R_ν, of the discontinuities of R_∞. They are defined for every big k and every $\nu \in J_k$ (see Lemma 4.2).*

Next proposition summarizes the main properties of the return maps R_ν:

Proposition 6.4. *For every k sufficiently big and $\nu \in J_k$ consider the subinterval $\Delta(\nu) = [\tilde{B}_\nu, \tilde{E}_\nu]$ of $D^+(\nu)$ and the return map R_ν defined by*

$$R_\nu \colon \Delta(\nu) \to D^+(\nu), \quad x \mapsto f_\nu^{s_\nu(x)}(\pi^s f_\nu^2(x)),$$

where $s_\nu(x)$ is the first forward iterate of $\pi^s f_\nu^2(x)$ in $D^+(\nu)$. The return map *R_ν satisfies the following properties for every $\nu \in J_k$:*

- **a) invariance of the critical region:** $R_\nu([C - \varepsilon, C + \varepsilon]) \subset (C - \varepsilon, C + \varepsilon)$,
- **b) surjective expansive branches:** *There is a finite set of points, the* discontinuities of R_ν, $\mathcal{D}_\nu = \{d_i^-(\nu), d_i^+(\nu)\}_{i=0}^{m_k}$, *with*

  $$d_{m_k}^-(\nu) < \cdots < d_1^-(\nu) < d_0^-(\nu) < C < d_0^+(\nu) < d_1^-(\nu) < \cdots < d_{m_k}^+(\nu)$$

  such that

b$_1$) $[C-\varepsilon, C+\varepsilon] \subset [d_0^-(\nu), d_0^+(\nu)] = \Delta_0(\nu)$, $\Delta_0(\nu)$ *is the* critical branch,

b$_2$) *let* $\Delta_i^+(\nu) = [d_{i-1}^+(\nu), d_i^+(\nu)]$ *and* $\Delta_i^-(\nu) = [d_i^-(\nu), d_{i-1}^-(\nu)]$, *then* $R_\nu(\Delta_i^\pm(\nu)) = D^+(\nu)$, *and* R_ν *is increasing in* $\Delta_i^+(\nu)$ *and decreasing in* $\Delta_i^-(\nu)$,

b$_3$) $|D_x R_\nu| > 3$ *for all* $x \in (\Delta(\nu) \setminus (C-\varepsilon, C+\varepsilon)$.

Remark 6.5. *In this proposition, we take the map* R_ν *being bi-valuated in the set* $\mathcal{D}_\nu$, *considering its extension with* $R_\nu|_{\Delta_i^\pm}(d_i^\pm(\nu)) = \{\tilde{B}_\nu, f_\nu(\tilde{B}_\nu)\}$. *This is the same that to consider* $D^+(\nu)$ *as a circle identifying its two extremes.*

Proof: Bearing in mind properties (r$_1$)-(r$_4$), we have that items (a), (b$_1$) and (b_3) of Proposition 6.4 were proved Proposition 5.2. To get (b$_2$), observe that, since $C \notin \Delta_i^\pm(\nu)$, $i \neq 0$, the restrictions of R_ν to $\Delta_i^\pm(\nu)$ are monotone maps. The fact that these maps are increasing in $\Delta_i^+(\nu)$ and decreasing in $\Delta_i^-(\nu)$ follows observing that $\pi^s f_\nu$ preserves the orientation in $(C, \tilde{E}_\nu]$ and reverses the orientation in $[\tilde{B}_\nu, C)$. $\square$

In the next section we describe the dynamics of R_ν, $\nu \in J_k$, in the critical region $(C-\varepsilon, C+\varepsilon)$.

7 The dynamics of the return maps R_ν in the critical region

We now perturb the initial saddle-node diffeomorphism f_0 in a small neighbourhood of $f_0(C)$ (without modifying the limit set of f_0) in such a way that the restriction of $R_\infty(\sigma^*, \cdot)$ to the neighbourhood $(C-\varepsilon, C+\varepsilon)$ of C, (σ^* and ε as in equation (8) and Proposition 5.2), satisfies the following conditions:

(c$_1$) the point C is the (unique) critical point of $R_\infty(\sigma, \cdot)$ for all $\sigma \in [0,1]$ and $R_\infty(\sigma^*, C) < C$,

(c$_2$) there are $\delta \in (0, \varepsilon)$ and $\tau > 0$ such that

$$R_\infty(\sigma^*, [C-\delta, C+\delta]) = [R_\infty(\sigma^*, C), C+\tau], \quad R_\infty(\sigma^*, C \pm \delta) = C+\tau,$$

(c$_3$) $|R'_\infty(\sigma^*, x)| > 2$ for all $x \in ((C-\varepsilon, C+\varepsilon) \setminus (C-\delta, C+\delta))$, (this is possible due to condition (b$_3$) in Proposition 6.4),

(c$_4$) the restriction of $R_\infty(\sigma^*, \cdot)$ to $[C-\delta, C+\delta]$ is a unimodal map having (exactly) three fixed points: a sink S, a repellor Q, and a saddle-node P, where $S < C < Q < P$, such that

- C is in the basin of attraction of the sink S,
- let Q' and P' be the points in $(C-\delta, C)$ with $R_\infty(\sigma^*, Q') = Q$ and $R_\infty(\sigma^*, P') = P$, then $(P', Q') \cup (Q, P] \subset W^s(P)$ and $(Q', Q) \subset W^s(S)$.

For σ close to σ^*, we can define the continuations $\delta^\pm(\sigma)$ of $\pm\delta$ by $R_\infty(\sigma, C \pm \delta^\pm(\sigma)) = C+\tau$. Observe that, by definition, the maps $R_\infty(\sigma, \cdot)$ are *($\sigma - \sigma^*$)-translations* of $R_\infty(\sigma^*, \cdot)$. Thus for parameters σ close to σ^* one has the following:

(c$_5$) For $\sigma > \sigma^*$, the restriction of $R_\infty(\sigma,\cdot)$ to $[C-\delta^-(\sigma), C+\delta^+(\sigma)]$ has (exactly) to fixed points, a sink S_σ (the continuation of S), whose basin of attraction contains the critical point C, and a repellor Q_σ (the continuation of Q). Moreover, if Q'_σ is the point in $(C-\delta^-(\sigma), C)$ with $R_\infty(\sigma, Q'_\sigma) = Q_\sigma$, then $(Q'_\sigma, Q_\sigma) \subset W^s(S_\sigma)$.

(c$_6$) For $\sigma < \sigma^*$, the saddle-node P of $R_\infty(\sigma^*,\cdot)$ splits into a sink P^ℓ_σ and a source P^r_σ, $P^\ell_\sigma < P^r_\sigma$. The restriction of $R_\infty(\sigma,\cdot)$ to $[C-\delta^-(\sigma), C+\delta^+(\sigma)]$ has (exactly) four fixed points, two sinks S_σ and P^ℓ_σ, and two repellors Q_σ and P^r_σ, where $S_\sigma < C < Q_\sigma < P^\ell_\sigma < P^r_\sigma$. As before, define $(P^{\ell,r}_\sigma)'$ by $R_\infty(\sigma, (P^{\ell,r}_\sigma)') = P^{\ell,r}_\sigma$. Then $((P^r_\sigma)', Q'_\sigma) \cup (Q_\sigma, P^r_\sigma) \subset W^s(P^\ell_\sigma)$ and $(Q'_\sigma, Q_\sigma) \subset W^s(S_\sigma)$.

(c$_7$) The dynamics in a neighbourhood of the saddle-node is expanding: There is an interval N containing the saddle node P in its interior, $Q \notin N$, such that

- $R'_\infty(\sigma^*, x) > 3/2$, for all $x \in ([Q, C+\delta] \setminus N)$,
- For values of $\sigma > \sigma^*$ close to σ^*, there exists a positive integer $N(\sigma)$ such that the number $n(\sigma,x)$ of iterations by $R_\infty(\sigma,\cdot)$ that a point $x \in N$ needs to leave N (i.e., $R^j_\infty(\sigma,x) \in N$ for all $0 \le j \le n(\sigma,x)-1$, and $R^{n(\sigma,x)}_\infty(\sigma,x) \notin N$) is bounded from above by $N(\sigma)$. Moreover, for very $x \in N$, $|(R^{n(\mu,x)}_\infty(\sigma,x))'| > 1/2$.

Observe that this condition is similar to hypothesis (f) in Section 2.

We now fix small $\epsilon > 0$ with $(\sigma^*-\epsilon, \sigma^*+\epsilon) \subset (\sigma^*-\xi, \sigma^*+\xi)$ and let $H_k = \sigma_k(\sigma^*-\epsilon, \sigma^*+\epsilon)$.

Since the set $[C-\delta, C+\delta]$ is a compact subset of (B,E) we can apply Lemma 4.2 to the sequence of return maps $R_k(\sigma,\cdot) = R_{\hat{\mu}_k(\sigma)}(\cdot)$, for parameters $\sigma \in (\sigma^*-\epsilon, \sigma^*+\epsilon)$. Since $R_k(\sigma,\cdot)$ converges to $R_\infty(\sigma,\cdot)$ and the saddle-node is a codimension one bifurcation we have the following lemma:

Lemma 7.1. *For every k sufficiently big, there is $\beta_k \in H_k$, $\beta_k = \sigma_k(\eta_k)$, $\eta_k \to \sigma^*$ as $k \to \infty$, such that R_{β_k} has a saddle-node at β_k whose local dynamics satisfies conditions (c$_5$) and (c$_6$) above.*

8 Two dimensional return maps

In this section, we introduce the two dimensional return maps. For small $\mu > 0$, consider the points $\tilde{B}_\mu$ and $\tilde{E}_\mu$ introduced in Remark 5.4 and the critical point C_μ in Section 4.3.1 (from now on, we omit the dependence on μ) and consider the sub-rectangles $[\tilde{B},\tilde{E}]^s$ and $(D^+)^s$ of R defined as follows. Define $F^v(\tilde{B})$, $F^v(\tilde{E})$, and $F^v(f_\mu(\tilde{B}))$ as the connected components of $(F^{ss}(\tilde{B}_\mu) \cap R)$ containing the points $\tilde{B}$, $\tilde{E}$ and $f_\mu(\tilde{B})$. The rectangles $[\tilde{B},\tilde{E}]^s$ and $(D^+)^s$ are the sub-rectangles of R whose vertical sides are $F^v(\tilde{B})$ and $F^v(\tilde{E})$, and $F^v(\tilde{B})$ and $F^v(f_\mu(\tilde{B}))$, respectively.

The *return map* $\tilde{R}_\mu$ is defined by

$$\tilde{R}_\mu \colon [\tilde{B},\tilde{E}]^s \to (D^+)^s, \quad \tilde{R}_\mu(x,y) = f_\mu^{s_\mu(x,y)}(f^2_\mu(x,y)),$$

where $s_\mu(x,y)$ is the first forward iterate of (x,y) in $(D^+)^s$.

As in the case of the one-dimensional returns, consider the sequence of one-parameter families of two dimensional returns defined by

$$\tilde{R}_k\colon [0,1]\times[\tilde{B},\tilde{E}]^s \to (D^+)^s, \quad \tilde{R}_k(\sigma,x)=\tilde{R}_{\hat{\mu}_k(\sigma)}(x). \tag{13}$$

Also define

$$\tilde{R}_\infty\colon [0,1]\times[\tilde{B},\tilde{E}]^s \to (D^+), \quad \tilde{R}_\infty(\sigma,x)=T_\infty(\pi_s(f_\mu^{s_\mu(x,y)}(f_\mu^2(x,y)))). \tag{14}$$

Using Proposition 3.2, we have the following:

Lemma 8.1. *Let K be a compact subset contained in the interior of $[\tilde{B},\tilde{E}]^s$. Then the restriction of $\tilde{R}_k$ to $[0,1]\times K$ converges in the C^∞-topology to the restriction of $\tilde{R}_\infty$ to $[0,1]\times K$.*

Using Lemma 4.3, we get:

Lemma 8.2. *There are κ and $k_0\in\mathbb{N}$ such that $|D_z(\tilde{R}_\nu((v_1,0))|\geq 3|v_1|$, for all $z\notin[C-\varepsilon,C+\varepsilon]^s$, every vector $(v_1,0)$, and $\nu\in\sigma_k([0,1])$, $k\geq k_0$, where the set $[C-\varepsilon,C+\varepsilon]^s$ is defined in the natural way.*

As in Lemma 7.1, we get a sequence of parameters β_k corresponding to saddle-node bifurcations of $\tilde{R}(\mu,\cdot)$ (for simplicity we use the same notation). Recall that $H_k=\sigma_k(\sigma^*-\epsilon,\sigma^*+\epsilon)$ and denote by H_k^+ and H_k^- the two connected components of $(H_k\setminus\{\beta_k\})$, where $\mu^-<\mu^+$ for all $\mu^\pm\in H_k^\pm$. By Lemma 8.1, we have:

Remark 8.3. *Let $\mu=\sigma_k(\sigma)\in H_k$. Denote by Q_μ the hyperbolic saddle of $\tilde{R}_\mu=\tilde{R}_k(\sigma,\cdot)$ corresponding to the continuation of the repeller Q_σ of $\tilde{R}_\infty(\sigma,\cdot)$. Observe that Q_μ is a saddle of period r_k+2 of the diffeomorphism f_μ.*

- *For $\mu\in H_k^+$, let Q'_μ be the continuation for $\tilde{R}_\mu=\tilde{R}_k(\sigma,\cdot)$ of the preperiodic point Q'_σ of $\tilde{R}_\infty(\sigma,\cdot)$. Such a continuation is the first intersection (starting from Q_μ) of the unstable manifold of Q_μ (for f_μ) and the stable manifold of Q_μ (for f_μ) given by the connected component of $W^s(Q_\mu)\cap R$ containing Q_μ. Thus, Q'_μ is a transverse homoclinic point of Q_μ (for f_μ).*
- *For the saddle-node parameter $\mu=\beta_k$, let P_{β_k} be the saddle-node of $\tilde{R}_{\beta_k}=\tilde{R}_\infty(\eta_k,\cdot)$. Observe that P_{β_k} is a periodic saddle-node (of period r_k+2) of f_{β_k}. Define P'_{β_k} as the continuation of P'_{σ^*} obtained as the first intersection (starting from P_{β_k}) of the strong unstable manifold of P_{β_k} (for f_μ) and strong stable manifold of P_{β_k} (for f_μ) given by the connected component of $W^{ss}(P_{\beta_k})\cap R$ containing P_{β_k}.*
- *For $\mu\in H_k^-$, we define as above the continuations $P_\mu^{r,\ell}$ and $(P_\mu^r)'$ of the hyperbolic periodic points $P_\sigma^{r,\ell}$ and of the preperiodic points $(P_\sigma^{r,\ell})'$ of $\tilde{R}_\infty(\sigma,\cdot)$, recall ($c_6$) in Section 7. The continuation $(P_\mu^r)'$ is a homoclinic point of P_μ.*

By shrinking the size of the parameter interval J containing σ^*, we can assume that there are intervals $\tilde{I}\subset D^+$ and $\hat{I}\subset D^-$ such that, for all $\sigma\in J$, one has that

- the interval $\tilde{I}$ contains C and S_σ and it is contained in $W^s(S_\sigma)$,

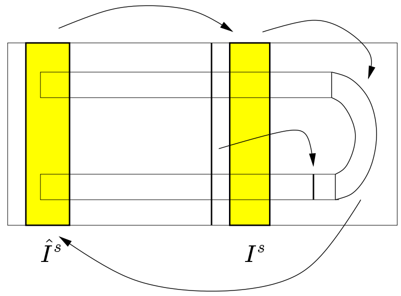

Figure 7: The critical region

- $R_\infty(\sigma, \tilde{I}) \subset \tilde{I}$ and C and S_σ belong to the interior of $T_\infty(\sigma, \hat{I})$.

Consider the sub-rectangles of R defined by

$$\tilde{I}^s = \tilde{I} \times [-1, 1] \quad \text{and} \quad \hat{I}^s = \hat{I} \times [-1, 1],$$

and the vertical and horizontal foliations $\mathcal{F}^h$ and $\mathcal{F}^v$ of R whose leaves are defined by

$$F^h_{y_0} = \{(x, y_0) \in R\} \quad \text{and} \quad F^v_{x_0} = \{(x_0, y) \in R\}.$$

Observe that, by construction, $f^2_\mu(C)$ belongs to the interior of $\hat{I}^s$ for all $|\mu|$ small enough. Thus, if the initial saddle-node horseshoe is sufficiently contracting in the vertical direction, we can assume that $(f_\mu(F^h_{y_0}) \setminus \hat{I}^s)$ is transverse to $\mathcal{F}^v$ for every y_0, and that the angle between the image of $\mathcal{F}^h$ by f_μ and $\mathcal{F}^v$ outside $\hat{I}^s$ is bigger than (for instance) $\pi/6$, see Figure 7.

Since, for every $\sigma \in J$, the rectangle $\tilde{I}^s$ is contained in the stable manifold of S_σ for $\tilde{R}_\infty(\sigma, \cdot)$, Lemma 8.1 implies that:

Remark 8.4. *Consider any $\mu \in H_k$, big k. Then, $(\hat{I}^s \cup \tilde{I}^s) \subset W^s(S_\mu)$.*

In view of this remark, for parameters $\mu \in H_k$, we can focus our attention on the maximal invariant set of f_μ in $(R \setminus \tilde{I}^s)$. Observe that in this region we have a dynamics *without criticalities.* Let G_0 be the unique hyperbolic fixed point of the initial saddle-node horseshoe Λ_0. Denote by G_μ the continuation of G_0, for small μ, and by Σ_μ the homoclinic class of G_μ. Observe that $\Sigma_0 = \Lambda_0$.

Proposition 8.5. *The limit set in the rectangle R of f_μ is hyperbolic, for every $\mu \in (H_k \setminus \beta_k)$, k big enough. More precisely,*

- *for every $\mu \in H^-_k$, the limit set of f_μ in R is the disjoint union of Σ_μ, and the orbits of the sinks S_μ and P^ℓ_μ and of the saddle Q_μ,*

- *for every $\mu \in H^+_k$, the limit set of f_μ in R is the disjoint union of Σ_μ and the orbit of the sink S_μ.*

For $\mu = \beta_k$, the limit set of f_{β_k} in R is the union of Σ_{β_k} and the orbits of the sink S_{β_k} and of the saddle Q_{β_k}. Moreover, Σ_{β_k} is a saddle-node horseshoe containing the saddle-node P_{β_k}.

Finally, the restrictions of the diffeomorphisms f_μ to the sets Σ_μ are topologically conjugate for all $\mu \in H_k$.

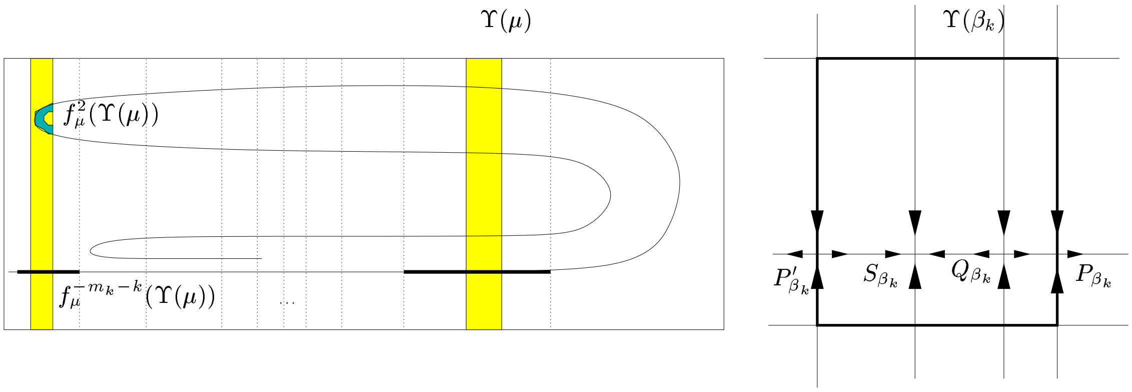

Figure 8: The rectangles $\Upsilon(\mu)$

The first part of the proof of the proposition follows using the arguments in [DRoV, Section 4.2], where a standard cone-field construction is used to prove the hyperbolicity of invariant sets in the unfolding of saddle-node horseshoes. Since these are folklore constructions in bifurcation theory, we just sketch the main ideas of the proof of the proposition. To prove the topological part of the proposition, we will construct Markov partitions for the sets Σ_μ. In rough terms, we have *essentially* the same partition for all the parameters μ in H_k, and the partition for parameters in H_{k+1} is obtained by adding a new rectangle to the partition for $\mu \in H_k$. These partitions will allow us to compute *exactly* the topological entropy of f_μ in R ($\mu \in H_k$). We begin by constructing the Markov partitions. This is done in the next section.

9 Markov Partitions for the sets Σ_μ

9.1 Rectangles of the partition

Fix a big k and, for each $\mu \in H_k$, consider the sub-rectangle $\Upsilon(\mu)$ of R defined as follows and depicted in Figure 8. Given a set A and a point $x \in A$, denote by $C(x, A)$ the connected component of A containing x.

- For $\mu \in H_k^+$, the rectangle $\Upsilon(\mu)$ is the closure of $C(S_\mu, W^s(S_\mu) \cap R)$.
- For $\mu = \beta_k$, the rectangle $\Upsilon(\beta_k)$ is the closure of

$$C(S_{\beta_k}, W^s(S_{\beta_k}) \cap R) \cup C(P_{\beta_k}, W^s(P_{\beta_k}) \cap R) \cup C(P'_{\beta_k}, W^s(P_{\beta_k}) \cap R).$$

- For $\mu \in H_k^-$, the rectangle $\Upsilon(\mu)$ is the closure of the union of

$$C(S_\mu, W^s(S_\mu) \cap R) \cup C(P_\mu^\ell, W^s(P_\mu^\ell) \cap R) \cup C((P_\mu^\ell)', W^s(P_\mu^\ell) \cap R).$$

The *left and right vertical sides* $\partial_\ell \Upsilon(\mu)$ and $\partial_r \Upsilon(\mu)$ of $\Upsilon(\mu)$ are defined as follows,

- $\partial_\ell \Upsilon(\mu) = C(Q'_\mu, W^s(Q_\mu) \cap R)$ and $\partial_r \Upsilon(\mu) = C(Q_\mu, W^s(Q_\mu) \cap R)$, for $\mu \in H_k^+$,

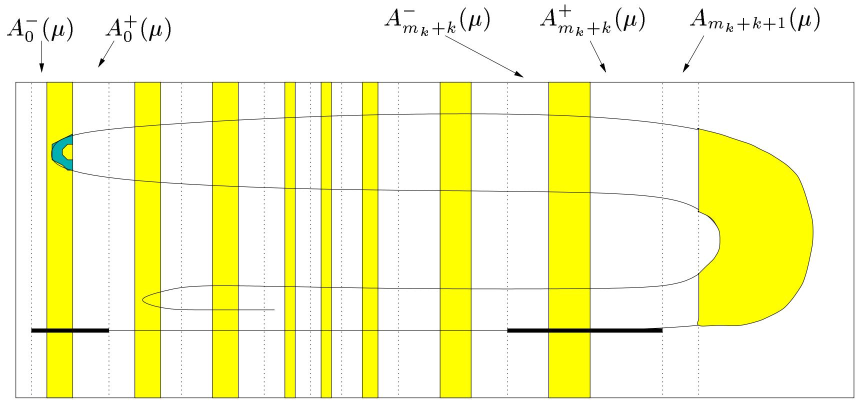

Figure 9: The recatangles $A_j^{\pm}(\mu)$

- $\partial_\ell \Upsilon(\beta_k) = C(P'_{\beta_k}, W^{ss}(P_{\beta_k}) \cap R)$ and $\partial_r \Upsilon(\mu) = C(P_{\beta_k}, W^{ss}(P_{\beta_k}) \cap R)$, for $\mu = \beta_k$,
- $\partial_\ell \Upsilon(\mu) = C((P^r_\mu)', W^s(P^r_\mu) \cap R)$ and $\partial_r \Upsilon(\mu) = C(P^r_\mu, W^s(P^r_\mu) \cap R)$, for $\mu \in H_k^-$.

Note that, for any $\mu \in H_k$, the positive limit set of any point visiting the interior of $\Upsilon(\mu)$ is the orbit of one of the following periodic points: S_μ if $\mu \in H_k^+$, S_{β_k} or Q_{β_k} for $\mu = \beta_k$, and S_μ or Q_μ or P^ℓ_μ if $\mu \in H_k^-$. Thus, the nontrivial part of the dynamics of f_μ is given by points whose orbit avoid $\Upsilon(\mu)$. In particular,

$$\Sigma_\mu \subset \bigcap_{i \in \mathbb{Z}} f^i_\mu(R \setminus \Upsilon(\mu)). \tag{15}$$

In fact, we will get the equality in the previous inclusion. Consider the closed sub-rectangle $R(\mu)$ of R whose left and right vertical sides are the segments $C(\tilde{B}_\mu, W^s(\tilde{B}_\mu) \cap R)$ and $C(f_\mu(\tilde{B}_\mu), W^s(\tilde{B}_\mu) \cap R)$, respectively. Let

$$\Upsilon^-(\mu) = C(\tilde{B}_\mu, R(\mu) \setminus \Upsilon(\mu)) \quad \text{and} \quad \Upsilon^+(\mu) = C(f_\mu(\tilde{B}_\mu), R(\mu) \setminus \Upsilon(\mu)).$$

We are now ready to construct the Markov partition of Σ_μ. For $j \in \{0, 1, \ldots, m_k + k\}$, we let

$$\begin{aligned} A_j^-(\mu) &= C(f_\mu^{-(m_k+k)+j}(\tilde{B}_\mu), f_\mu^{-(m_k+k)+j}(\Upsilon^-(\mu)) \cap R), \text{ and} \\ A_j^+(\mu) &= C(f_\mu^{-(m_k+k)+j}(f_\mu(\tilde{B}_\mu)), f_\mu^{-(m_k+k)+j}(\Upsilon^+(\mu)) \cap R). \end{aligned}$$

To complete the Markov partition of Σ_μ, we need to consider the rectangle $A_{m_k+k+1}(\mu)$ defined as follows: the left vertical side of $A_{m_k+k+1}(\mu)$ is $C(f_\mu(\tilde{B}_\mu), W^s(f_\mu(\tilde{B}_\mu)) \cap R)$. Before defining the right side of $A_{m_k+k+1}(\mu)$, let $W^s_\mu = W^s(Y_\mu)$, ($Y_\mu \in \{Q_\mu, P_{\beta_k}, P^r_\mu\}$) and observe that the images by f_μ of the vertical sides of $\Upsilon(\mu)$ are contained in the same connected component of $W^s_\mu \cap R$. Finally, the other vertical side of $A_{m_k+k+1}(\mu)$ is $C(f_\mu(Y_\mu), W^s_\mu \cap R)$. The family of rectangles $A_j^{\pm}(\mu)$ is depicted in Figure 9.

Lemma 9.1. *The family of rectangles*

$$\mathbb{M}_k(\mu) = \{A_0^+(\mu), A_1^\pm(\mu), \ldots, A_{m_k+k}^\pm(\mu), A_{m_k+k+1}(\mu)\}$$

is a Markov partition for Σ_μ.

Proof: By (15), the family of rectangles $\mathbb{M}'_k(\mu) = \{A_0^\pm(\mu), A_1^\pm(\mu), \ldots, A_{m_k+k}^\pm(\mu), A_{m_k+k+1}(\mu)\}$ cover Σ_μ (by now, we also consider the set $A_1^-(\mu)$, we will discard this set later). Thus it is enough to check that:

- The two vertical sides of each rectangle of $\mathbb{M}'_k(\mu)$ are strong stable leaves (in fact, stable manifolds of hyperbolic periodic points).
- The image of each vertical side of any rectangle of $\mathbb{M}'_k(\mu)$ is contained in the interior of the vertical side of a rectangle of $\mathbb{M}'_k(\mu)$, and the image of any horizontal side of any rectangle of $\mathbb{M}'_k(\mu)$ intersects the left and the right vertical sides of any rectangle of $\mathbb{M}'_k(\mu)$ whose interior it intersects, and this image is transverse to the vertical lines.

By construction, these two assertions are obvious for the sets $A_i^\pm(\mu)$, $i = 0, \ldots, m_k + k$. To prove these assertions for $A_{m_k+k+1}(\mu)$, observe first that its left vertical side is mapped into $F^s(0)$, which is, by construction, the left vertical side of $A_{m_k}^-(\mu)$. On the other hand, the image by f_μ of right vertical side of $A_{m_k+k+1}(\mu)$ is contained in the left vertical side of $A_0^+(\mu)$. To see that just recall that, by construction, the right and the left vertical sides of $A_{m_k+k+1}(\mu)$ and the left vertical side of $A_1^+(\mu)$ are contained in the stable manifold of the saddle Y_μ of period $m_k + k + 2$.

Finally, the rectangle $A_0^-(\mu)$ can be discarded in order to get a Markov partition of Σ_μ: the set $f_\mu(A_0^-(\mu))$ is contained in $A_1^-(\mu)$ and that there are no points of Σ_μ whose orbits meet $A_0^-(\mu)$. The proof of the lemma is now complete. $\square$

9.2 Transition matrices of the Markov partitions

Take any $\mu \in H_k$ and let $\ell_k = 2(m_k + k + 1)$ be the number of rectangles of the Markov partition $\mathbb{M}_k(\mu)$. The transition matrix $T_k = (a_{i,j})$ associated to $\mathbb{M}_k(\mu)$ is a $\ell_k \times \ell_k$ matrix $(a_{i,j})$ such that $a_{i,j} = 1$ if the image of the j-th rectangle of $\mathbb{M}_k(\mu)$ intersects the interior of the i-th rectangle (in this case, the transition from the j-th to the i-th rectangle is allowed), and $a_{i,j} = 0$ otherwise.

Observe that, by construction and the arguments in the proof of Lemma 9.1, the only transitions allowed are the following;

- for every $0 \leq j \leq m_k + k - 1$, from $A_j^+(\mu)$ to $A_{j+1}^+(\mu)$ and from $A_j^-(\mu)$ to $A_{j+1}^-(\mu)$,
- from $A_{m_k+k}^-(\mu)$ to $A_{m_k+k+1}(\mu)$,
- $A_{m_k+k}^+(\mu)$ to the rectangles $A_{m_k+i}^\pm(\mu)$, for every $0 \leq i \leq k$, and to A_{m_k+k+1},
- from $A_{m_k+k+1}(\mu)$ to $A_0^+(\mu)$ and $A_i^\pm(\mu)$, for every $1 \leq i \leq m_k$.

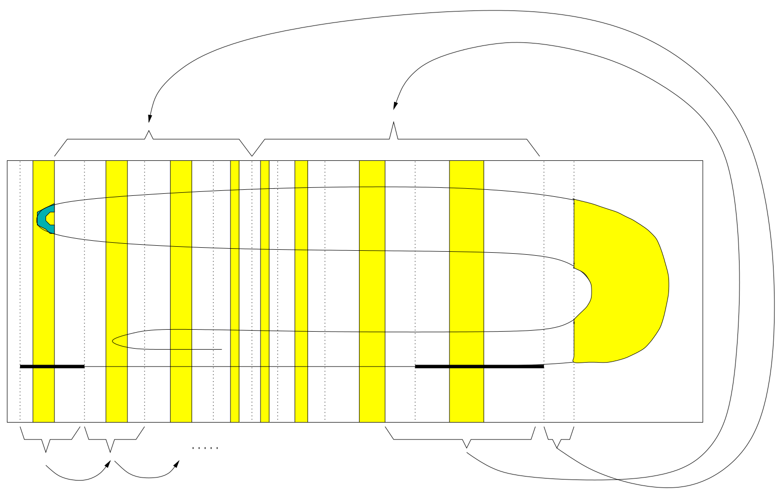

Figure 10: Transitions in $\mathbb{M}_k(\mu)$

So, the matrix $T_k = (a_{i,j})$ has the form:

$$T_k = (a_{i,j}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 1 & 0 \end{pmatrix} \quad \begin{bmatrix} A_0^+(\mu) \\ A_1^-(\mu) \\ A_1^+(\mu) \\ A_2^-(\mu) \\ A_2^+(\mu) \\ \vdots \\ A_{m_k}^+(\mu) \\ A_{m_k+1}^-(\mu) \\ \vdots \\ A_{m_k+k}^+(\mu) \\ A_{m_k+k+1}(\mu) \end{bmatrix} \tag{16}$$

Note that this matrix does not depend on $\mu \in H_k$. Also note, for future uses, that there are $(2m_k - 1)$ non-zero elements in the ℓ_k-th column, and $(2k+1)$ in the $(\ell_k - 1)$-th one.

9.3 Hyperbolicity of the set Σ_μ

As we have mentioned, the proof of the hyperbolicity of Σ_μ (the homoclinic class of G_μ), $\mu \neq \beta_k$, follows by using quite standard cone field arguments (see also [DRoV, Section 4.2]). So, we just sketch this proof.

Let us first consider parameters $\mu \in H_k^+$. Observe that in the vertical direction we have an uniform contraction. Using this contraction and a cone field argument, we get the stable direction.

To get the unstable direction, we will see that there are $\sigma > 1$ and $N_\mu \geq 1$ such that, for every $z \in \Lambda_\mu$ and every vector $v = (1,0)$, there is $m \in \{1, \dots N_\mu\}$ such that $Df_\mu^m(1,0) = (h,v)$, where $|h| \geq \sigma$. To prove this claim, the only difficulty to bypass is the presence of criticalities (which are far from Σ_μ, by construction, recall Remark 8.4). In fact, the only points where we may have delicate estimates to get expansion are those whose orbits meet $A_{m_k+k+1}(\mu)$. Observe that such orbits also meet either $A^+_{m_k+k}(\mu)$ or $A^-_{m_k+k}(\mu)$. Moreover, every $x \in A^-_{m_k+k}(\mu)$ returns to $A^\pm_{m_k+k}(\mu)$ after i iterates, $i \in [k, m_k + k + 2]$. Thinking of the map R_μ as the horizontal component of f^i_μ, Proposition 8.2 gives the announced expansion. Finally, for a point in $A^+_{m_k+k}(\mu)$ its first iterate is expanding. This ends the sketch of the proof of the hyperbolicity of Σ_μ, for $\mu \in H^+_k$.

For parameters in H^-_k the proof follows similarly, but we need to think on the iterates of the orbits close to the saddle-node as a unique iterate, which is expanding, recall the construction for the one dimensional model family in Section 2. This standard construction is also done in [CK]. We omit the details.

Finally, the fact that Λ_{β_k} has only a nonhyperbolic periodic point (the saddle-node P_{β_k}) follows similarly.

Observe that our constructions also imply that the stable and unstable manifolds of periodic points points of Σ_μ intersects the horizontal and the vertical sides, respectively, of any rectangle of the partition $\mathbb{M}_k(\mu)$. So, every pair of periodic points of Σ_μ are homoclinically related. Finally, using the arguments above, one can prove that $\Sigma_\mu = \cap_{i\in\mathbb{Z}} f^i_\mu(R \setminus \Upsilon(\mu))$ (recall equation (15)). This equality implies that:

- for $\mu \in H^-_k$, the limit set of f_μ in R is the disjoint union of the hyperbolic sets Σ_μ, S_μ, Q_μ and P^ℓ_μ,
- for $\mu \in H^+_k$, the limit set of f_μ in R is the disjoint union of the hyperbolic sets Σ_μ and S_μ,
- for $\mu = \beta_k$, the limit set of f_μ in R is the disjoint union of the saddle-node horseshoe Σ_{β_k} and S_{β_k}.

9.4 Topological Entropy

In this section, we compute the topological entropy of the restriction of f_μ to Σ_μ (which is equal to the topological entropy of the restriction of f_μ to R).

Proposition 9.2. *The topological entropy $h(f_\mu)$ of the restriction of f_μ to Σ_μ is equal to a constant h_k, for every $\mu \in H_k$ (big k). Moreover, the sequence (h_k) is increasing and verifies*

$$\log\left(\frac{2\,\ell_k}{\ell_k+2}\right) < h_k < \log(2),$$

where $\ell_k = 2(m_k + k)$ is the number of rectangles of the Markov partition $\mathbb{M}_k(\mu)$.

Proof: Let σ_ℓ be the eigenvalue of the transition matrix T_k in equation (16) with biggest modulus. Recall that $h(f_\mu)$ is $\log(|\sigma_\ell|)$, see for instance [M, Corollary 7.7]. We will show that

$$\frac{2\,\ell_k}{\ell_k+2} < |\sigma_\ell| < 2.$$

Since the set Σ_μ contains horseshoes (just observe that the continuation of the hyperbolic fixed point G_0 of Λ_0 has transverse homoclinic points), $h(f_\mu)$ is strictly positive. Hence, the matrix T_k has an eigenvalue of modulus bigger than one. So, to find σ_ℓ, we can focus on those eigenvalues of T_k having modulus bigger than one.

Lemma 9.3. *Let λ be an eigenvalue of T_k such that $|\lambda| > 1$. Then every eigenvector $X = (x_1, x_2, \dots, x_\ell)$ associated to λ satisfies $x_{2i} = x_{2i-1}$ for every $i = 1, \dots, \ell/2$.*

Proof: Observe that, for every X and $\lambda \neq 0$ with $(a_{i,j})X^t = \lambda X^t$, the equations below hold:

$$\begin{array}{rcl}
-\lambda\, x_1 + x_\ell & = & 0 \\
-\lambda x_2 + x_\ell & = & 0 \\
x_1 - \lambda\, x_3 + x_\ell & = & 0 \\
x_2 - \lambda\, x_4 + x_\ell & = & 0 \\
\vdots & \vdots & \vdots \\
x_{2m_k-2} - \lambda\, x_{2m_k} + x_\ell & = & 0 \\
x_{2m_k-1} - \lambda\, x_{2m_k+1} + x_{\ell-1} & = & 0 \\
\vdots & \vdots & \vdots \\
x_{2m_k+2k-2} - \lambda\, x_{2m_k+2k} + x_{\ell-1} & = & 0.
\end{array} \tag{17}$$

In particular, we have the following:

$$\begin{array}{rcl}
x_2 - x_1 & = & 0 \\
x_4 - x_3 & = & 0 \\
\vdots & \vdots & \vdots \\
x_{2m_k} - x_{2m_k-1} & = & 0 \\
x_{2m_k+1} - x_{2m_k} & = & \dfrac{x_{\ell-1} - x_\ell}{\lambda} \\
x_{2m_k+2} - x_{2m_k+1} & = & \dfrac{x_{\ell-1} - x_\ell}{\lambda^2} \\
\vdots & \vdots & \vdots \\
x_\ell - x_{\ell-1} & = & \dfrac{x_{\ell-1} - x_\ell}{\lambda^k}.
\end{array} \tag{18}$$

The last equation implies that if $(x_{\ell-1} - x_\ell) \neq 0$, then $|\lambda| = 1$, which is a contradiction. So, we can assume that $x_\ell = x_{\ell-1} = 1$ (the option $x_\ell = x_{\ell-1} = 0$ leads to the zero vector, which is not an eigenvector). The lemma follows arguing recursively. $\square$

Returning to the original system in (17), putting $x_\ell = x_{\ell-1} = 1$, and recalling that $\ell = 2m_k + 2$, we find:

$$\begin{array}{rcl}
x_2 & = & \lambda^{-1} \\
x_4 & = & \lambda^{-1} + \lambda^{-2} \\
\vdots & \vdots & \vdots \\
x_{\ell-2} & = & \sum_{i=1}^{\ell/2-1} \lambda^{-i} \\
x_{\ell-2} + 1 & = & \lambda.
\end{array} \tag{19}$$

By the last two equations above, it follows that

$$-\lambda^{-(\ell/2)} - \lambda + 2 = 0 \quad \Longrightarrow \quad \lambda^{(\ell/2)+1} - 2\,\lambda^{\ell/2} + 1 = 0.$$

Consider now the polynomial

$$\Pi_\ell(\lambda) = \lambda^{(\ell/2)+1} - 2\,\lambda^{\ell/2} + 1.$$

Observe that $\lambda = 1$ is a root of Π_ℓ and that

$$\Pi'_\ell(\lambda) = \lambda^{(\ell/2)-1}((\ell/2)+1)\,\lambda - \ell)$$

has only one non-zero root $\lambda_0 = (2\,\ell)/(\ell+2) > 0$. Since $\Pi_\ell(x) \to +\infty$ as $x \to +\infty$, it is immediate to check that λ_0 is the only minimum of Π_ℓ. More precisely, if ℓ is large enough, then λ_0 is close to 2^- and has negative image by Π_ℓ (observe that $\Pi_\ell(1) = 0$, $\lambda_0 > 1$ and that Π_ℓ is decreasing in the interval $(1, \lambda_0)$). These properties, together with the expression of Π_ℓ, imply that Π_ℓ has a unique root λ_1 bigger that λ_0. Since $\Pi_\ell(2) = 1$, we have

$$\lambda_1 \in (\lambda_0, 2) \subset (\frac{2\,\ell}{\ell+2}, 2).$$

Observe that $\Pi_\ell(0) = 1$ and that 0 and λ_0 are the only critical points of Π_ℓ. Thus, if the degree of Π_ℓ is even, there are no negative roots for Π_ℓ and λ_1 is the biggest eigenvalue, in modulus, of Π_ℓ. On the other hand, if the degree of Π_ℓ is odd, the polynomial Π_ℓ has (exactly) one negative root, say λ_2. Since $\Pi_\ell(-1) = -2$ and $\Pi_\ell(0) = 1$ we have that $\lambda_2 \in (-1, 0)$. Thus, $|\lambda_2| < \lambda_1$ and λ_1 is biggest eigenvalue in modulus of Π_ℓ.

By construction, the root λ_1 of Π_ℓ corresponds to an (the biggest in modulus) eigenvalue of T_k. Thus, $\lambda_1 = \sigma_\ell$, ending the proof of the first part of the proposition.

To end the proof of the proposition, we need to see that the sequence (h_k) is increasing. Consider $\ell = \ell_k$ and ℓ_{k+1} and observe that $\ell_{k+1} = \ell + 2n$ for some $n \geq 1$. Then, since $\sigma_\ell > 1$,

$$\Pi_{\ell+2n}(\sigma_\ell) = (\sigma_\ell)^{(\ell/2)+n+1} - 2\,(\sigma_\ell)^{(\ell/2)+n} + 1 < (\sigma_\ell)^{(\ell/2)+n+1} - 2\,(\sigma_\ell)^{(\ell/2)+n} + (\sigma_\ell)^n < (\sigma_\ell)^n\,\Pi_\ell(\sigma_\ell) = 0.$$

Thus, since $\lim_{x\to\infty} \Pi_{\ell+2n}(x) = \infty$ and $\Pi_{\ell+2n}$ is increasing in $(\sigma_{\ell+2n}, \infty)$, $\sigma_{\ell+2n} > \sigma_\ell$, ending the proof of the proposition. □

10 Increasing the entropy after unfolding saddle-node horseshoes

In this section, we consider critical two saddle-node horseshoe cycles and give an informal explanation (we just sketch the constructions omitting the technical details) of how the topological entropy may increase after the unfolding of this class of cycles (Proposition B). Using this construction, we show how the entropy may be non-monotone in the unfolding of one critical saddle-node horseshoes (Theorem C).

10.1 Entropy and the unfolding of critical two saddle-node horseshoe cycles

Consider a rectangle R as in Section 4.1 and a diffeomorphism f such that $f(R)$ is mapped in the interior of R and the limit set of f in R is the disjoint union of:

- a fixed sink S, having two real positive multipliers of different modulus,

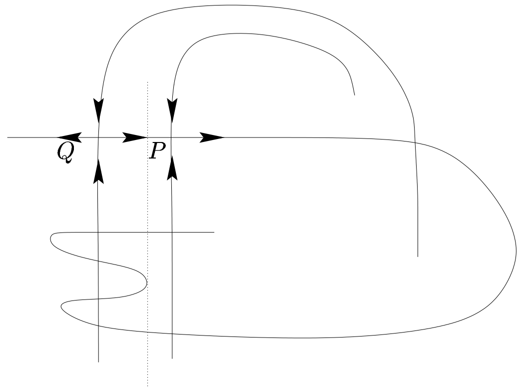

Figure 11: Critical two saddle-node horseshoe

- a saddle-node horseshoe Λ conjugate to a complete shift of two symbols which is the homoclinic class of the saddle node P of Λ,
- a fixed saddle Q having a trivial homoclinic class such that one branch of $W^u(Q)$ is contained in $W^s(P)$ and the other branch contained in $W^s(P)$,
- the unstable manifold of the saddle-node P verifies the following properties:
  - $W^u(P)$ intersects transversely $W^s(Q)$ throughout the orbit of four points,
  - $W^u(P)$ intersects stable manifold of the saddle-node and is tangent to the strong stable foliation of $W^s(S)$ along the orbit of two points,
  - $W^u(P)$ is tangent to the strong stable foliation of $W^s(P)$ along the orbit of a point.

The geometry of this class of two cycles is depicted in Figure 11.

The cycle above can be constructed in such a way the following holds:

Proposition 10.1. *Consider an arc of diffeomorphisms $(f_\mu)_{\mu\in[-1,1]}$ unfolding generically the two saddle-node cycle above at $\mu = 0$ in such a way f_μ is hyperbolic for negative μ.*

Then, for every $\mu > 0$ small enough, there is m_μ, $m_\mu \to \infty$ as $\mu \to 0^+$, such that

$$h(f_\mu|_R) \geq \frac{\log(2^{m_\mu} + 2)}{m_\mu}.$$

In particular, $h(f_\mu|_R) > \log(2) = h(f_0|_R)$.

Observe that this result is just a reformulation of Proposition B.

Proof: To prove the proposition, given a small $\mu > 0$, we consider disjoint rectangles R_1, R_2, R_3, and R_4 as in Figure 12, where the dynamics in R_1 and R_2 contains a hyperbolic horseshoe Λ_μ conjugate to a complete shift of two symbols. The sets R_3 and R_4 are small rectangles having one vertical side contained in the stable manifold of the saddle Q, the horizontal sides are small segments contained in the unstable manifold of Q. By the λ-lemma, there exists m such that f^m has horseshoes in R_3 and in R_4, see Figure 12.

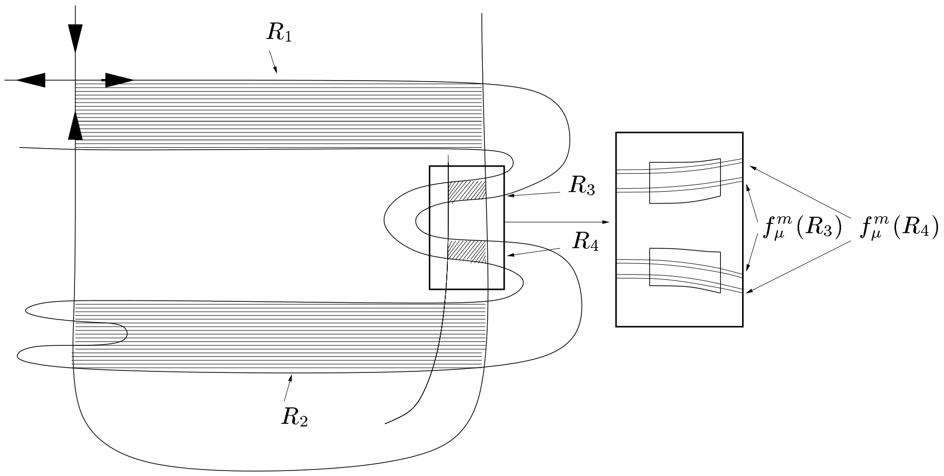

Figure 12: The rectangles R_i and their images

Moreover, one can take the rectangles above in such a way the maximal invariant set of f_μ in $\hat{R} = R_1 \cup R_2 \cup R_3 \cup R_4$ is hyperbolic. The construction above imply that, for every k, the diffeomorphism $(f_\mu^m)^k$ has at least $(2^m + 2)^k$ different hyperbolic fixed points in $\hat{R}$. Thus

$$h(f_\mu^m) \geq \log(2^m + 2).$$

The proposition follows from the fact that $m\, h(f_\mu) = h(f_\mu^m)$. $\square$

10.2 Non-monotonicity of the topological entropy for critical one saddle-node horseshoes

In this section, we sketch the proof of Theorem C. Let us first recall that, in the construction for critical one saddle-node horseshoes cycles, we consider the interval $(B, E) \subset W^u(P)$, P the saddle-node, and E and B two points of transverse intersections between $W^u(P)$ and $W^{ss}(P)$ with $(B, E) \subset W^s(P)$. With these ingredients we induce a ghost return dynamics such that the unidimensional map $R_\infty(\sigma, \cdot)$ unfolds a (one-dimensional) noncritical two saddle-node horseshoe cycle at some parameter σ^*. Following this idea, consider a subinterval I of (B, E) and modify the dynamics of f in $f^2(I)$ in such a way the new ghost return $R_\infty(\sigma, \cdot)$ unfolds a (one-dimensional) two critical saddle-node horseshoe cycle at σ^*. In Figure 13 is depicted this type of one dimensional cycle.

Using this one dimensional cycle, we get that for the parameters β_k (defined exactly as in Lemma 7.1) the arc f_μ unfolds (two-dimensional) critical two saddle-node horseshoe cycles similar to the one in the previous section.

Define H_k^- and H_k^+ as in Remark 8.3. The calculations for the topological entropy for parameters in H_k^- are exactly the same as in Section 9.4, and they lead to the same result: $h(f_{\nu_k}|_R) < h(f_{\nu_{k+1}}|_R) < \log(2)$ for all $\nu_k \in H_k^-$ and $\nu_{k+1} \in H_{k+1}^-$.

On the other hand, the calculations in the previous section (Proposition 10.1) imply that $h(f_\mu|_R) > h(f_\nu|_R)$ for all $\mu \in H_k^+$ and $\nu \in H_k^-$ (big k). This construction ends the sketch of the proof of Theorem C.

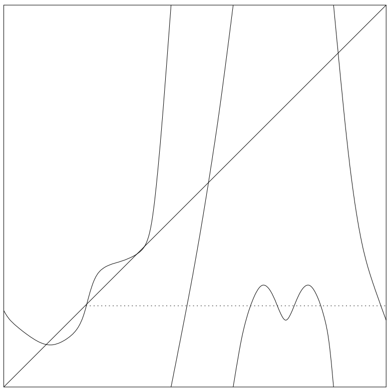

Figure 13: Two critical saddle-node horseshoe cycle for $R_\infty(\sigma^*, \cdot)$

References

[AAIS] V.S. Afraimovich, V.I. Arnol'd, Yu.S. Il'yasenko, and L.P. Sil'nikov, *Bifurcation theory,* Dynamical Systems V, Encyclopedia of Math. Sciences, vol 5, (1989).

[AS] V.S. Afraimovich and L.P. Sil'nikov, *On some global bifurcations connected with the disappearance of a fixed point of saddle-node type,* Soviet Math. Dokl., **15**(6), 1761-1765, (1974).

[AL] A.A. Andronov and E.A. Leontovic, *Some cases of dependence of limit cycles on a parameter,* Ucen. Zap. Gorkov. Gos. Univ. **6**, 3-24, (1939) (Russian).

[AW] A.A. Andronov and A.A. Witt, *Zur Theorie des Mitnehmens von van der Pol,* Arch. Elektrotechnick, **24**, 99-110, (1930).

[As] M. Asaoka, *A natural horseshoe-breaking family which has a period doubling bifurcation as the first bifurcation,* J. Math. Kioto Univ., **37**, 493-511, (1997).

[BS] E. Bedford and J. Smillie, *Real polynomial diffeomorphisms with maximal entropy: tangencies,* preprint.

[BDV] C. Bonatti, L. J. Díaz, F. Viullemin, *Cubic tangencies and hyperbolic diffeomorphisms,* Bol. Soc. Bras. Mat., **29**, 99-144, (1998).

[CK] Y. Cao and S. Kiriki, *An isolated saddle-node bifurcation ocurring inside a horseshoe,* Dynamics and Stability of Systems, **15**(1), 11-22, (2000).

[Co] M.J. Costa, *Saddle-node horseshoes giving rise to global Hénon-like attractors,* An. Acad. Bras. Ci, **70**, 393-400, (1998).

[Cr] S. Crovisier, *Saddle-node bifurcations for hyperbolic sets,* Ergod. Th. & Dynam. Sys., **22**, 1079-1115, (2002).

[DRiV] L.J. Díaz, I. Rios, and M. Viana, *The intermittency route to chaotic dynamics,* Global Analysis of Dynamical Systems, Festschrift dedicated to F. Takens for his 60th birthday, Institute of Physics Publishing (IOP), 310-328, (2001).

[DRoV] L.J. Díaz, J. Rocha, and M. Viana, *Strange attractors in saddle-node cycles: prevalence and globality,* Inventiones Math., **125**, 37-74, (1996).

[DU] L.J. Díaz and R. Ures, *Critical saddle-node cycles: Hausdorff dimension and persistence of tangencies,* Ergod. Th. & Dynam. Sys., **22**, 1117-1140, (2002).

[E] H. Enrich, *A heteroclinic bifurcation of Anosov diffeomorphisms,* Ergod. Th. & and Dynam. Sys., **18**(3), 567-608, (1998).

[HPS] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds,* Lecture Notes in Math., **583**, (1997), Springer Verlag.

[KKY] I. Kan, H. Koçak, and J. Yorke, *Antiminotonicity: Concurrent creation and annihilation of periodic orbits,* Annals of Math., **136**, 219-252, (1992).

[K] S. Kiriki, *The Palis-Takens problem on the first homoclinic tangency inside a horseshoe,* Int. J. Bifurc. Chaos Appl. Sci. Eng., **6**, 737-744, (1996).

[LS] V.I. Luk'janov and L.P. Sil'nikov, *On some bifurcations of dynamicas systems with homoclinic structures,* Soviet Math. Dokl., **19**(6), 1314-1318, (1978).

[M] R. Mañé, *Ergodic theroy and differentiable dynamics,* Springer Verlag, (1987).

[MV] L. Mora and M. Viana, *Abundance of strange attractors,* Acta Mathematica, **171**, 125-150, (1993).

[M] C.G.T.A Moreira, *Stable intersections of Cantor sets and homoclinic bifurcations,* Ann. Inst. H. Poincaré Anal. Non Linéaire, **13**(6), 741-781, (1996).

[N_1] S. Newhouse, *Diffeomorphisms with infinitely many sinks,* Topology, **13**, 9-18, (1974).

[N_2] S. Newhouse, *The abundance of wild hyperbolic sets and nonsmmoth stable set for diffeomorphisms,* Publ. Math. I.H.E.S., **50**, 101-151, (1979).

[NP] S. Newhouse and T. Pignataro, *On the estimation of topological entropy,* Journal Stat. Phys., **79**(5-6), 1331-51, (1993).

[NPT] S. Newhouse, J. Palis, and F. Takens, *Bifurcations and stability of families of diffeomorphisms,* Publ. Math. I.H.E.S., **57**, 7-71, (1983).

[PT_1] J. Palis and F. Takens, *Cycles and measure of bifurcation sets for two-dimensional diffeomorphisms,* Inventiones Math., **82**, 397-422, (1985).

[PT_2] J. Palis and F. Takens, *Hyperbolicity and the creation of homoclinic orbits,* Annals of Math., **125**, 337-374, (1987).

[PT_3] J. Palis and F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations,* Cambridge Univ. Press, (1993).

[PY_1] J. Palis and J.C. Yoccoz, *Homoclinic tangencies for hyperbolic sets of large Hausdorff dimension,* Acta Math., **172**(1), 91-136, (1994).

[PM] Y. Pomeau and P. Manneville, *Intermittant transitions to turbulence in dissipative dynamical systems,* Comm. Math. Phys., **74**, 189-197, (1980).

[PS] E.R. Pujals and M. Sambarino, *On homoclinic tangencies, hyperbolicity, creation of homoclinic orbits and variation of the entropy,* Nonlinearity, **13**(3), 921-926, (2000).

[R] I.L. Rios, *Unfolding homoclinic tangencies inside horseshoes: hyperbolicity, fractal dimensions and persistence of tangencies,* Nonlinearity, **14**, 431-462, (2001).

[Sa] P. Sabini, *Non-periodic bifurcations at the boundary of hyperbolic systems,* IMPA Thesis 2001.

[Si] L.P. Sil'nikov, *Some cases of generation of periodic motions from singular trajectories,* Mat. Sb. **61 (103)**, 443-466, (1963) (Russian).

[T] F. Takens, *Intermittancy: global aspects*, Lecture Notes in Math., **1331**, 213-239, (1988).

[W] R. Williams, *The DA maps of Smale and structural stability,* Global Analysis, Proc. Symp. Pure Math., **14**, 329-334, (1970).

[YA] J. Yorke and K.T. Alligood, *Cascades of period doubling bifurcations: a prerequisite for horsheshoes,* Bull. A.M.S., **9**, 319-322, (1983).

[Ya] Z. Yang, *Hénon-like maps before or at the first tangency,* Ph. Thesis.

[Z] C. Zeeman, *Bifurcations, catastrophe and turbulence,* Proc. Cent. Conf., New directions in Appl. Math., Case Western Reserve Univ., Cleveland, Springer Verlag, (1981).

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