$\boxed{\textbf{Tame and wild } C^1\textbf{-dynamics}}$

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(I) MOTIVATIONS

(I) Spectral decomposition of the limit set $L(f)$ for C^1-diffeomorphisms

M^2 a compact and closed surface.

$\text{Diff}^1(M^2)$ the space of C^1-diffeomorphisms (usual topology).

Theorem (Mañé, 82):

For C^1-generic surface diffeomorphisms (diffeos in a residual subset):

A) hyperbolic dynamics:

$L(f) =$ Axiom A $\oplus$ no-cycles,

B) Newhouse coexistence phenomenon:

f has infinitely many sinks or sources.

Hyperbolic case: There is a spectral decomposition of $L(f)$:

$$L(f) = \Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_n$$

Λ_i **basic set** (elementary piece of dynamics)

Properties of basic sets Λ:

- Transitive (i.e. dense orbit)

  indecomposability

- Hyperbolicity: $T_\Lambda M = E^s \oplus E^u$.

- Local maximal (isolation),

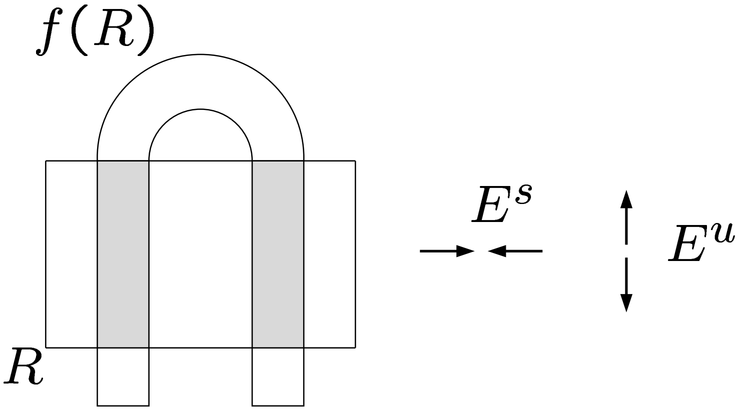
  $\Lambda = \bigcap_{n\in\mathbb{Z}}^{n}(U)$, U nbhood of Λ.

- Homoclinic class,

  $\Lambda = \overline{W^s(P) \pitchfork W^u(P)} = H_P(f)$.

$\boxed{\text{(nontrivial) } \textbf{Models} \text{ of basic sets:}}$

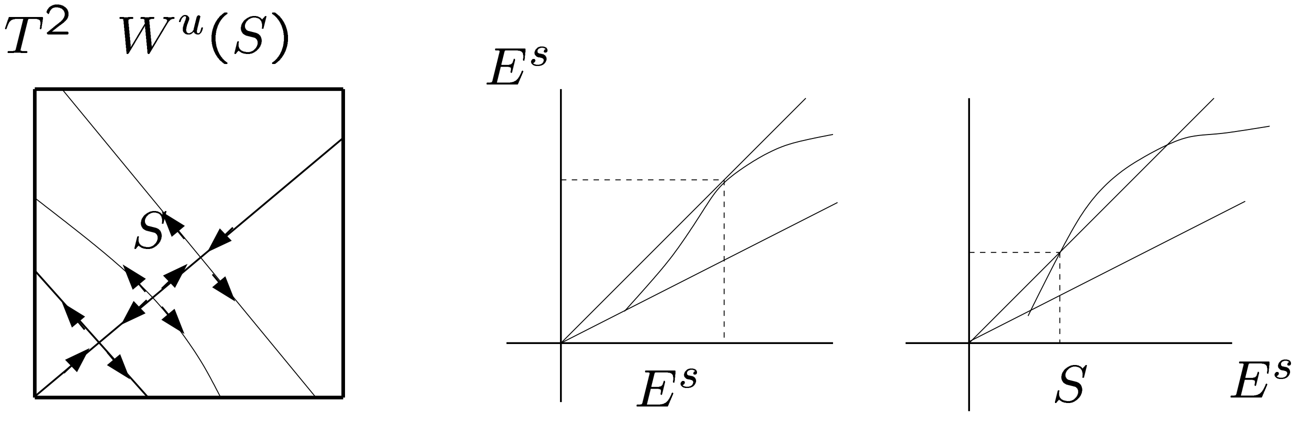
Smale's horseshoes,



Anosov diffeomorphisms,

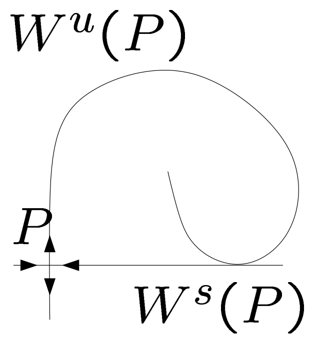
$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Derived from Anosov attractors,



Remarks and goals:

- Is it possible to eliminate (B) in the Thm.?. The only known mechanism for persistent tangencies for surface diffeos is the unfolding of C^2 homoclinic tangencies (Distortion properties).



- Reformulate the theorem in higher dimensions (false) (see nonhyperbolic robustly transitive diffeos)

- Give examples leading to persistent coexistence of infinitely many sinks for C^1 diffeos (higher dimensions)

Remarks and restictions: Why C^1-top.?

Good perturbation results are avaliable: Short list with hints.

- **Pugh closing lemma:** (closing orbits of recurrent points) this allows to estructurate the dynamics over the pdic. pts.

- **Hayashi's connecting lemmas:** creation of cycles

- **Mañé's ergodic closing lemma:** useful to prove hyperbolic-like properties for invariant splittings.

- **Frank's lemma:** dynamical realization of the derivative (it relates co-cycles and dynamics).

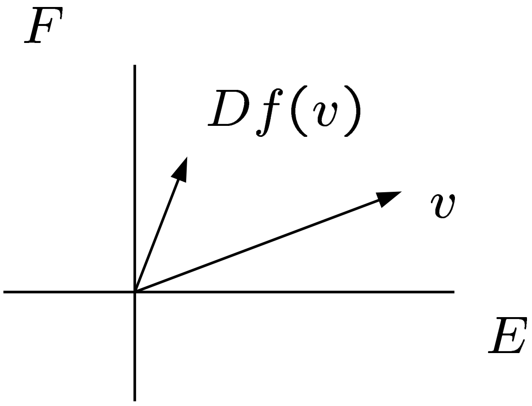
$\boxed{\textbf{Forms of hyperbolicity}}$

Hyperbolicity.

Dominated splitting:

$T_\Lambda M = E \oplus F$, Df-invariant $\oplus$ const. dim.

$$\exists \ell \in \mathbb{N} \quad \text{s.t } \|Df^\ell|_{E(x)}\| \cdot \|Df^{-\ell}|_{F(f^\ell(x))}\| < \frac{1}{2}$$



Partial hyperbolicity:

(dominated) $\oplus$ (E or F hyperbolic)
Strong part. hyp splt, $E \oplus F \oplus G$:
E and G hyperbolic
$(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ dominated

Robustly transitive diffeos

Transitive diffeos: f has a dense orbit in the whole manifold'

Models:

- irrational rotations (non-persistent)
- Anosov diffeos (persistent)

Robustly transitive C^1-diffeos: any C^1-pertb. of f is transitive.

There are non-hyperbolic robustly transitive diffeos in dimension ≥ 3

$\Longrightarrow$

Mañé's Thm is false in higher dimensions!: transitive diffeos have not neither sinks nor sources.

Examps of non-hyp robst trans diffeos

- $\boxed{\text{Skew products in } T^4 = T^2 \times T^2}$ (Shub)

  (Anosov $\times$ DAnosov),

  strong part. hyper., $TM = E^s \oplus E^c \oplus E^u$,

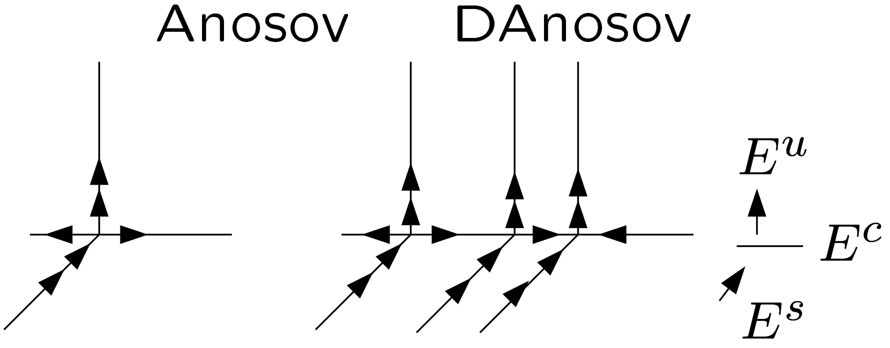
  1-dim central direction,

- $\boxed{\text{DAnosov in } T^3}$ (Mañé):

  a transitive set contains saddles of indices 2 and 1,

  strong part. hyperb., $TM = E^s \oplus E^c \oplus E^u$,

  1-dim central direction.



saddle-node bif.

(saddle-node bif. does not creates sinks/sources)

- $\boxed{\text{Skew products in any dimension}}$ (B-D)

  (pertb. of Anosov $\times$ identity)

  strong part. hyper., $TM = E^s \oplus E^c \oplus E^u$,

  k-dim central direction,

- $\boxed{\text{pertb. of time-1 of Anosov v. f.}}$ (B-D)

  (isotopic to the identity)

- $\boxed{\text{DAnosov in } T^3}$ (Bonatti)

  partial hyperbolicity, $TM = E^s \oplus E^c$,

  2-dim central direction,

  only one hyperbolic direction.

- $\boxed{\text{DAnosov in } T^4}$ (Bonatti-Viana)

  no hyperbolic directions (no partially hyperbolic),

  $TM = E \oplus F$, dominated splitting.

Remark: In a reformulation of Mañé's Thm.: one needs to consider forms of hyperbolicity weaker than partial hyperbolcity.

(II) Persistence of sinks and sources for C^1-diffeomorphisms

Theorem (Bonatti-D, 99):

Diff$^1(M^3)$.

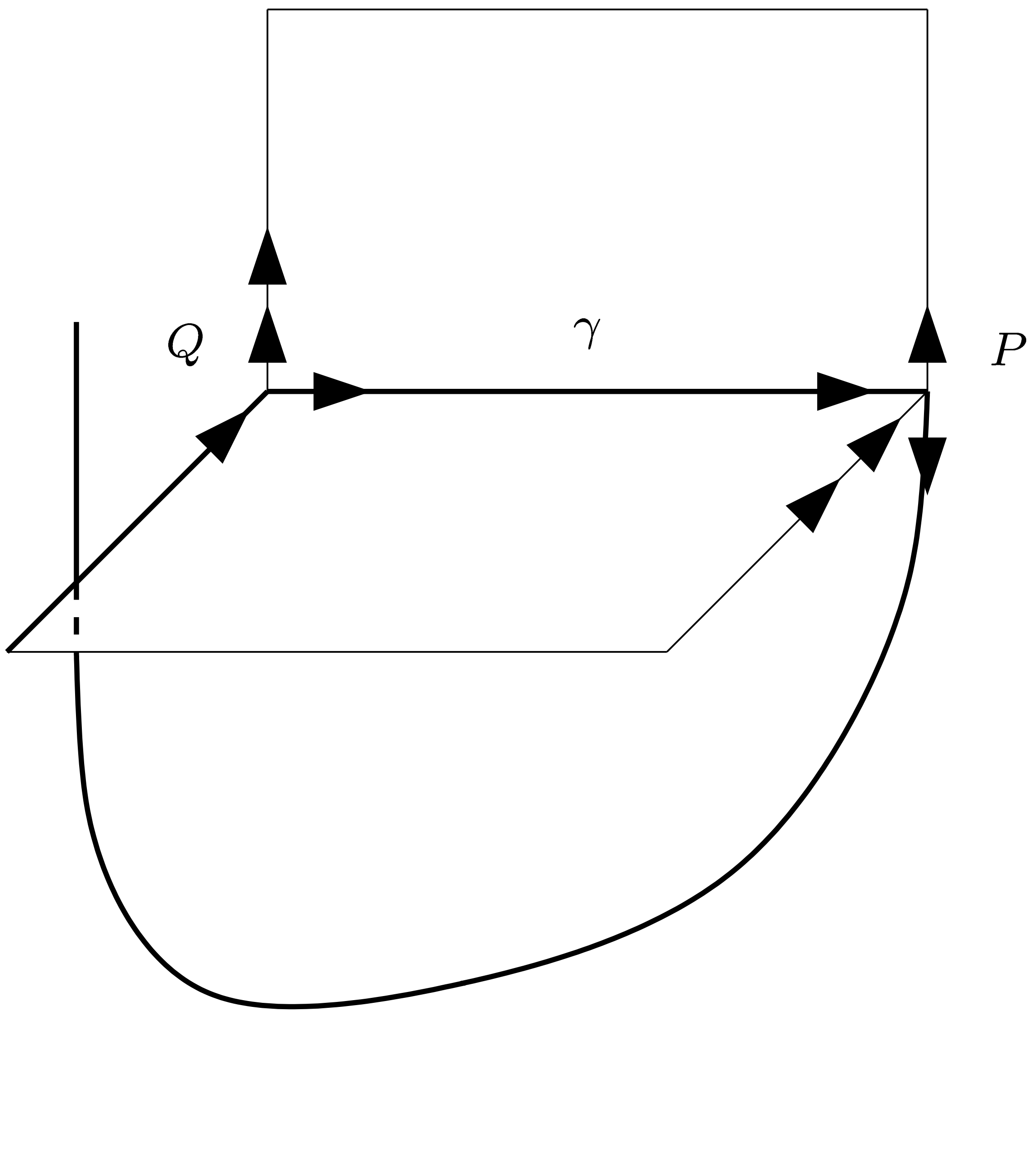
There are U open and $R \subset U$ residual in U s.t. every f in R has simultaneously infinitely many sinks and sources.

Ingredients:

- **heterodimensional cycles**,
- **non-dominated homoclinic classes.**

Bifurc. via heterodim. cycles (dim 3)

Take saddles P and Q of indices 1 and 2
$W^s(P) \pitchfork W^u(Q) \neq \emptyset$,
$W^u(P)$ and $W^s(Q)$ have a quasi-transverse intersection.



Theorem (D-Rocha, 00):

$\text{Diff}^1(M^n)$, $n \geq 3$.

f has a heterodim. cycle as in the figure

There is U open in $\text{Diff}^1(M^n)$ s.t.

- the closure the U contains f,
- for every $g \in U$, $H_P(g) = H_Q(g)$

  (non-hyperb. homoclinic classes).

The proof only involves orbits contained in a neighborhood V of the cycle (semi-local argument).

The dynamics after the bifurcation *essentially* given by the dynamics in the curve γ.

For bifurcations from hyperbolic dynamics (f.i. Morse-Smale)

$$\Lambda_g = H_P(g) = H_Q(g) = \cap_{i \in \mathbb{Z}} g^i(V)$$

V is a nbhood of the cycle.

Key for New Ph in surfaces: persistence of tangencies.

Key for persist. of cycles in heterodim bifs:
$W^u(Q)$ (2-dim), $W^u(Q) \subset \overline{W^u(P)}$.
naive implication: $W^u(P)$ is 2-dim
(i.e. P has indices 2 and 1).

Non-dominated homoclinic classes

Complete the figure as follows:

Suppose that there are saddles P' and Q'

(i) P' of index 1 homoclinic. related to P,

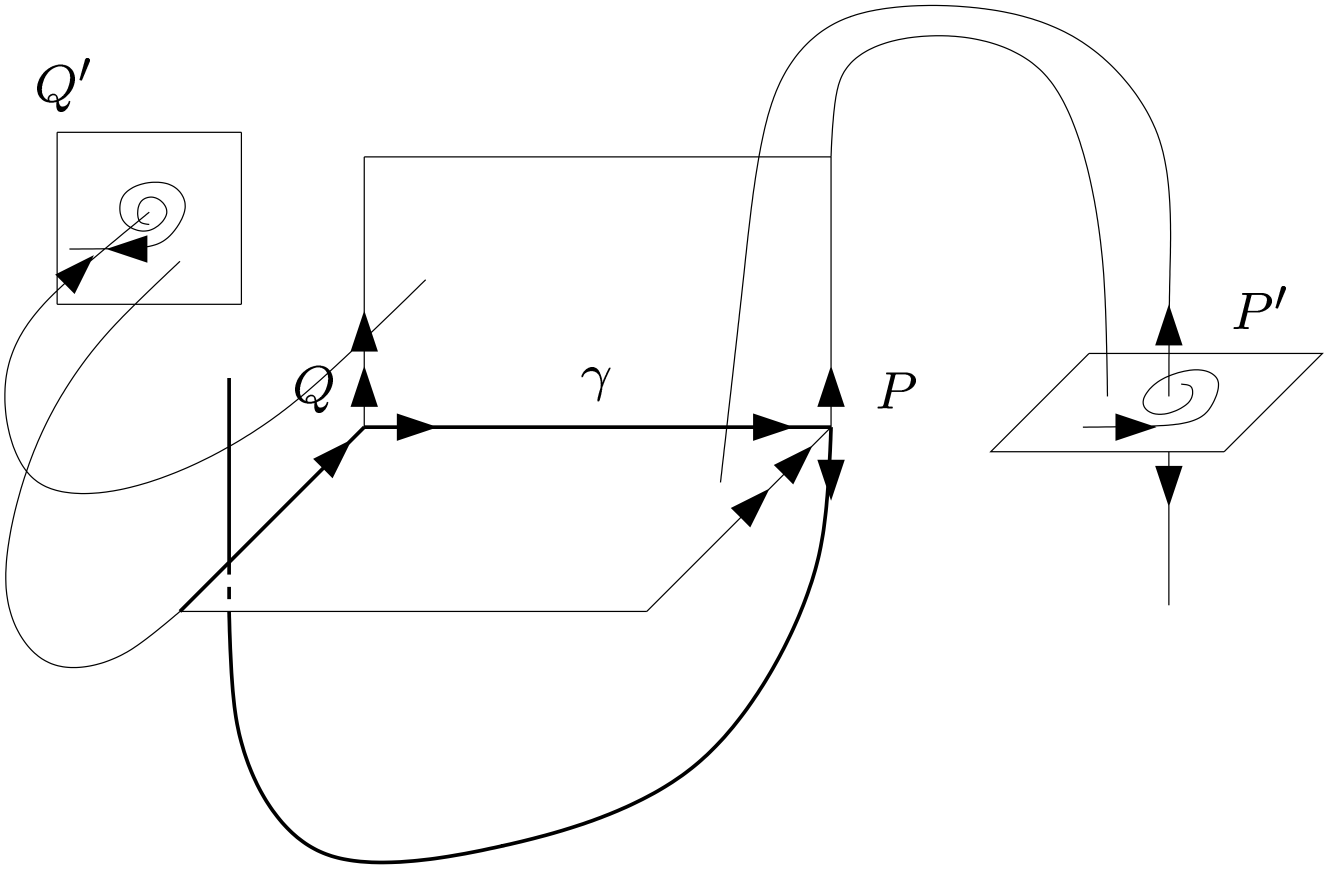
thus $H_P(f) = H_{P'}(f)$

(ii) P' has a non-rela contrac. eigenv.,

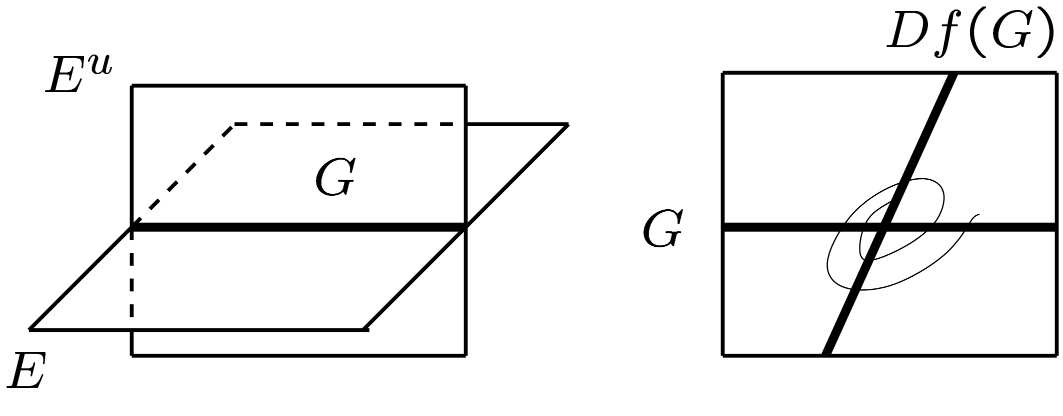
(iii) Q' of index 2 homoclinic. relat. to Q,

thus $H_Q(f) = H_{Q'}(f)$,

(iv) Q' has a non-real expand. eigenv.,



(i) $\oplus$ (iii) $\Longrightarrow$ $\Lambda_g = H_{P'}(g) = H_P(g) = H_Q(g) = H_{Q'}(g)$ for all $g \in U$.



(ii) $\oplus$ (iv) $\Longrightarrow$ the transitive set Λ_g does not admit any dominated splitting $T_{\Lambda_g}M = E \oplus F$.

$\dim(E) = 2 \Longrightarrow F(Q') \subset E^u(Q')$, $E^u(Q')$ has an invariant direction, contradicting complex eigenvalues.

$\dim(F) = 2 \Longrightarrow E(P') \subset E^s(Q')$...

Proposition:

(P, P', Q, Q', and U as above) $\oplus$

jacobian$(P') > 1 >$ jacobian(Q')

There is $R \subset U$, R residual in U s.t.

$$\Lambda_g = H_{P'}(g) = \cdots = H_{Q'}(g) \subset \\ \subset \{\text{sinks and sources of } g\}$$

(Λ_g ∞) $\Longrightarrow$ ∞-many sinks and sources)

Motivated by the previous result **Generalization:**

Theorem (Bonatti-D.-Pujals, 99):

Generically in $\text{Diff}^1(M^k)$ there is a **dichotomy:**

(d) $H_P(f)$ has a dominated splitting,

- $\dim M = 2$ hyp.,
- $\dim M = 3$ part. hyp.

(∞) $H_P(f)$ is contained in the closure of an infinite set of sinks and sources.

Problem: To study the dynamics of homoclinic classes withou domination (pathological dynamics associated to such a configuration)

(III) Notion of elementary piece of dynamics (non-hyperbolic setting). Chasing a sepectral decomposition

Theorem ? (Naive reform. of Mañés Th):

For C^1-generic diffeomorphisms

A) $L(f) = \Lambda_1 \cup \cdots \cup \Lambda_k$, $\Lambda_i \cap \Lambda_j = \emptyset$,

Λ_i elementary piece of dynamics with some weak hyperbolicity.

B) pathological dynamics

Newh. coexist. phenom. and worse,

(related to homoclinic classes without domination).

Chasing candidates for of elementary piece of dynamics (those must satisfy most of the properties of the basic sets)

$\boxed{\textbf{Models:}}$

- non-hyperbolic transitive sets in the unfolding of heterod. cycles.
- Nonhyperbolic transitive sets.

$\boxed{\textbf{Properties:}}$

- Transitivity: **Maximal transitive sets**
- Homoclinic classes,
- Local maximality: **Robust transitive sets**
- Weak hyperbolicity: **Hyperbolicity, partial hyperbolicity, dominated splittings**.

Thm (Arnaud, 99, Car-Mor-Pac, 01)

Generically in $\mathrm{Diff}^1(M)$ every homoclinic class is a maximal **saturated** transitive set.

Λ **is saturated:** Σ transitive and $\Sigma \cap \Lambda \neq \emptyset$ $\Longrightarrow \Sigma \subset \Lambda$.

Corollary Generically in $\text{Diff}^1(M)$:,

- either $H_P(f) \cap H_Q(f) = \emptyset$,
- or $H_P(f) = H_Q(f)$

Good news: For generic diffeomorphisms, the homoclinic classes induce a partition on (a priori) a part of the limit set.

Natural and naive (related) questions (with negative answers):

Generically, is $L(f)$ the union of (infinitely many) homoclinic classes?

Are the saturated sets of generic diffeomorphisms homoclinic classes?

Theorem (Negative answers) (B.-D., 01)

There are U open in Diff$^1(M)$

R residual in U

s. t. every f in R has a

(nontrivial) maxi. satur. trans. set Λ_f,

Λ_f is a Cantor (∞) set,

the dynamics of f over Λ_f is **minimal**

Minimal: Every orbit of Λ_f is dense in Λ_f. There are not periodic points.

The set Λ_f is obtained via a (infinite) renormalization process.

The set Λ_f is an adding machine like set.

(IV) The retrieved spectral decomposition (non-hyperbolic setting)

R residual subset of Diff$^1(M)$ where

$$\boxed{H_P(f) \cap H_Q(f) = \emptyset \text{ or } H_P(f) = H_Q(f).}$$

Define:

$\# R \to \mathbb{N} \cup \infty, \quad f \to \#(\{$ hmclnc classes$\}$

The map $\#$ is locally constant in R.

(T) Tame diffeos: $\#(f) \in \mathbb{N}$,

(W) Wild diffeos: $\#(f) = \infty$ (for instance, Newh. Phen.)

Theorem (Retrieved spectral decomposition) (Abdenur, 01)

For tame diffeos f:

$$L(f) = H_{P_1}(f) \cup \cdots \cup H_{P_m}(f),$$

every $H_{P_i}(f)$ has a dominated splitting.

There is a **filtration** $\{M_i\}_i$

($M_i \subset M_{i+1}$ and $f(M_i) \subset \text{int}(M_i)$)

s.t. for every $g \in T$ close to f,

$$H_{P_i}(g) = \bigcap_{k \in \mathbb{Z}} g^k(M_{i+1} \setminus M_i).$$

Corollary: The naive reformulation of the Mañé's Thm is correct.

(II) WILD DYNAMICS. HMCLNC CLSS WITHOUT DOMINATION

$\dim M = 3$, higher dimensions similar, but much more complicated

Wild homoclinic classes (def. motivated by the constructions leading to Newh. Ph. in the C^1-top.)

The class $H_P(f)$ is **wild** if it has hyp. prdc. pts. A and B of indices 1 and 2 s.t. • A has a contracting non-real eigenvalue, • B has an expanding non-real eigenvalue, • jacobian $(A) > 1 >$ jacobian (B). For every g close to f, $A, B \in H_P(g)$. (Special class of non-dominated hmclnc class)

This condition is much more stronger than to be wild (f.i. infinitely many independent saddles in a normally hyperbolic manifold).

B set of diffeos with a wild homocl. class (Rmk: B is open and $B \cap R \subset W$).

Theorem (Strong forms of Newhouse phenomenon) (B.-D., 01)

There is a residual subset S of B s.t. every $f \in S$ simultaneously has ∞-many:

- sinks and sources,
- saddles with trivial hmclnc clss.,
- wild hmclnc clss.,
- partially hyp. attracators and repellors,
- minimal saturated transitive sets.

(I) Step (BDP arguments)

$H_P(f)$ does not have a domin. split., then it is possible to create a sink/source.

Homothety lemma:

$H_P(f)$ does not have a dominated splitting,

For all $\varepsilon > 0$ there are:

a prdc. pt. $x \in H_P(f)$ with a ε-dense orbit in $H_P(f)$

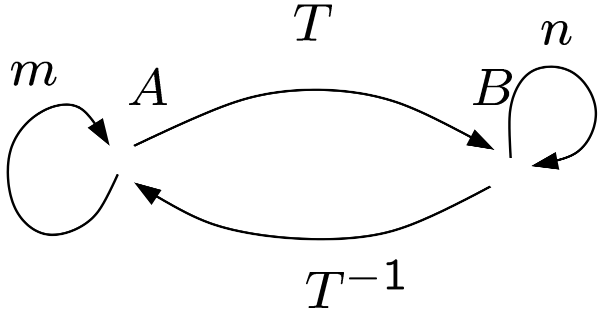
g C^1-close to f and a nbhood V of x s. t.

$$g^m/V = \text{homothety}, \quad m = \text{the period of } x.$$

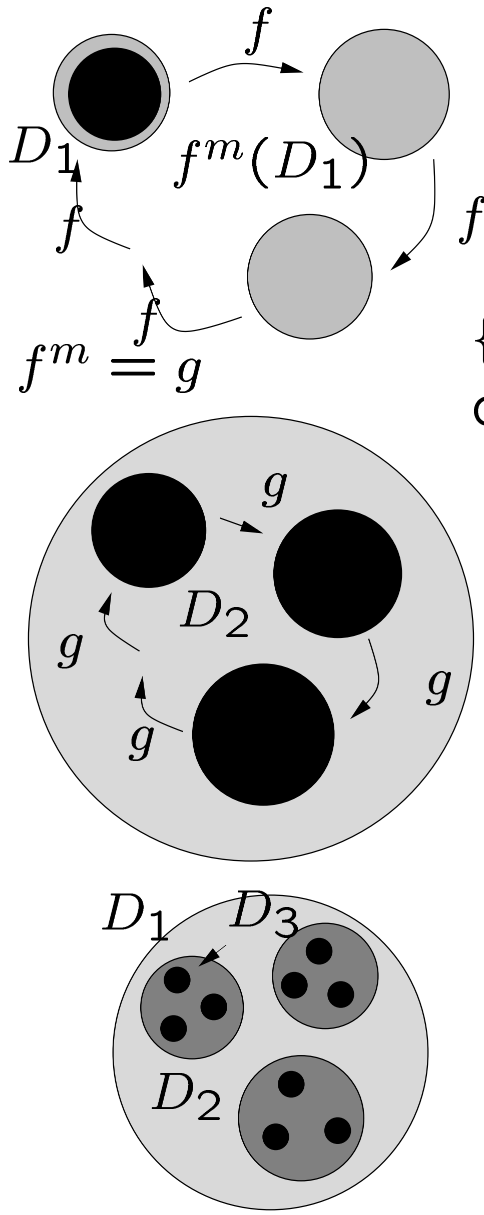
The ratio of the homothety depends on the jacobians of the prdc. pts. in $H_P(f)$.

Using pts. A and B with jacobians greater and less than 1 one gets the identity.

Construction involves **transitions** (main technical tool in BDP)



(II) Inductive and renormalization-like proc



$\{D_i\}$ nested fam.
diam$(D_i) \to 0$

$\Lambda = \bigcap_{i=1}^{\infty} \Delta_i$

Λ is a Cantor set
Every orbit is dense in Λ