Routes to wild dynamics

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ABSTRACT

We present dynamical scenarios leading to wild dynamics. We first discuss the C^2 persistent coexistence of infinitely many sinks associated to homoclinic tangencies of surface diffeomorphisms (Newhouse coexistence phenomenon). We next study heterodimensional cycles in higher dimensions and apply these results to get the C^1 coexistence phenomenon. Finally, we introduce the notion of dominated splitting and explain how the lack of domination of the non-wandering set may lead to stronger forms of the C^1 coexistence phenomenon.

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1. Introduction

Throughout this paper M denotes a finite dimensional closed Riemannian manifold and $\mathrm{Diff}^k(M)$, $k \geq 1$, the space of C^k diffeomorphisms of M endowed with the usual

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[*] Enrique Outerelo was my M. Sc. advisor in 1984. Before that he was my instructor in the courses of General and Differential Topology at the Universidad Complutense of Madrid. The course of General Topology was a true eye-opener for me. Back in 1984, good times, Juan Manuel López de Sá and Roberto Moriyón organized an enthusiastic Dynamical Systems Seminar in the Universidad Complutense held every Saturday morning (it seems to me that surviving these long sessions was some sort of a miracle, maybe being a lot younger helped). Enrique attended these seminars and I asked him whether he would be my thesis advisor. Following a crazy (but good) idea of López de Sá, we studied a paper on wild dynamics, the so-called Newhouse phenomenon, the subject of the present article. Although Enrique was not familiar with this subject he agreed to supervise my thesis. The M. Sc. thesis was the first step in my research on wild dynamics, my main and favorite topic of research since then. It is a great pleasure for me to thank warmly Enrique for his advice and support during those years.

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topology. By a C^k *generic diffeomorphism* we mean a diffeomorphism in a residual subset of $\mathrm{Diff}^k(M)$ (i.e., a set containing a countable intersection of open and dense subsets).

Recently, Abdenur stated in [1] a dichotomy for generic C^1 diffeomorphisms: those diffeomorphisms having finitely many *elementary pieces of dynamics* in the ambient manifold, the *tame* diffeomorphisms, and those with infinitely many elementary pieces (including an uncountable number), the *wild* ones. Our goal is to describe dynamical scenarios for the occurrence of wild dynamics. We will see how the wild dynamics is related to *non-dominated dynamics,* the *persistence of cycles,* and *homoclinic tangencies* (we give precise definitions later on). We will not discuss the notion of *elementary piece of dynamics* (for a broad discussion on this topic see [7, Chapter 10] and [2]). Naively speaking and avoiding technicalities, one aims to split the part of the ambient manifold supporting the non-trivial dynamics (recurrences) into invariant, independent, indecomposable, and maximal pieces of dynamics (which will be the elementary pieces). Archetypical examples of such pieces are the *basic sets* in the Smale hyperbolic theory.

We present in Section 3 the first (chronologically) example of wild dynamics: the persistent coexistence of infinitely many sinks for surface C^2 diffeomorphisms, associated to homoclinic tangencies. This phenomenon (the so-called Newhouse coexistence phenomenon) is typically C^2 and relies on the notion of *thickness* (a kind of fractal dimension) of a hyperbolic set. In Section 4.1, we study *heterodimensional cycles* and how persistently non-hyperbolic sets arise from these cycles. In Section 4.2, we apply these results to get the coexistence phenomenon in the C^1 topology in higher dimensions. Finally, in Section 5 we introduce the notion of dominated splitting and present dynamical configurations (non-dominated homoclinic classes) leading to specially interesting cases of wild dynamics: diffeomorphisms displaying persistently and simultaneously infinitely many non-trivial homoclinic classes, minimal sets, non-hyperbolic attractors, and sinks (for instance).

2. Definitions and background

Let us recall some notions of dynamics. Given a diffeomorphism $f\colon M \to M$, an f-invariant closed set Λ (i.e., $f(\Lambda) = \Lambda$) is *hyperbolic* if the tangent bundle of M over Λ has a (continuous) Df-invariant splitting $E^s \oplus E^u$ such that there are constants $0 < \lambda < 1$ and $c > 0$ with

$$|D_x f^n(v^s)| \le c\,\lambda^n\,|v^s| \quad \text{and} \quad |D_x f^{-n}(v^u)| \le c\,\lambda^n\,|v^u|,$$

for all $n \in \mathbb{N}$, every point $x \in \Lambda$, and any pair of vectors $v^s \in E^s_x$ and $v^u \in E^u_x$, where E^i_x is the fiber of the bundle E^i at x ($i = s, u$) and $|\cdot|$ is the Riemannian metric. The bundles E^s and E^u are the *stable* and *unstable* directions of Λ. Any hyperbolic set Λ of a diffeomorphism f verifies the following two properties: (1) its continuation is defined (i.e., diffeomorphisms g close to f have a hyperbolic set Λ_g close to Λ), and (2) the angles between the stable and unstable bundles over Λ are (uniformly) bounded away from zero. Prototypes of hyperbolic sets, besides the simplest case of orbits of hyperbolic periodic points, are the Smale horseshoe (see Figure 1 below) and the whole two dimensional torus for the Anosov maps, a model example being the map induced on $\mathbb{T}^2$ by the linear map of $\mathbb{R}^2$

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

The *non-wandering set* of a diffeomorphism f, denoted by $\Omega(f)$, is the set of points $x \in M$ such that, for every neighbourhood U of x, there is $n = n(U) > 0$ with $f^n(U) \cap U \neq \emptyset$. The non-wandering set $\Omega(f)$ is f-invariant and closed. Moreover, it contains the sets of periodic and limit points. In fact, C^1 generically, $\Omega(f)$ is the closure of the periodic points of f, [31]. A diffeomorphism f verifies the *Axiom A* if its non-wandering set is hyperbolic and equal to the closure of its periodic points. In the Axiom A case, the non-wandering set $\Omega(f)$ is the union of finitely many pairwise disjoint *homoclinic classes,* which are the *basic sets* of the Smale spectral decomposition of $\Omega(f)$, see [35]. Axiom A diffeomorphisms and basic sets are archetypes of tame diffeomorphisms and elementary pieces of dynamics, respectively.

Recall that the *homoclinic class* of a (hyperbolic) saddle point p, $H(p, f)$, is the closure of the transverse intersections of the orbits of the stable and unstable manifolds of p. For instance, the horseshoe and the whole two torus are the homoclinic class of any saddle of them. Homoclinic classes are always *transitive sets* (i.e., f-invariant sets being the closure of the orbit of some point). The stable manifold of x, $W^s(x, f)$, is the set of points w such that $\mathrm{dist}(f^n(x), f^n(w)) \to 0$ as $n \to \infty$ (here, d is the distance induced by the Riemannian metric). The unstable manifold of x, $W^u(x, f)$, is defined by $W^u(x, f) = W^s(x, f^{-1})$.

In dimension greater than or equal to three there are manifolds M and open sets $\mathcal{N} = \mathcal{N}(M)$ of $\mathrm{Diff}^1(M)$ consisting of *transitive diffeomorphisms* f (the whole manifold is the closure of the orbit of some point, in particular, $\Omega(f) = M$) having saddles of different indices (the *index* of a saddle is the dimension of its unstable bundle), see [34, 19, 3, 8]. Since, for every $f \in \mathcal{N}$ it holds $\Omega(f) = M$, it follows that $\Omega(f)$ is not hyperbolic (just note that in a transitive hyperbolic set the dimension of the unstable bundle is constant). In particular, every $f \in \mathcal{N}$ does not verify the Axiom A. Therefore, Axiom A diffeomorphisms fail to be dense in $\mathrm{Diff}^1(M)$ in these cases. In fact, this holds in $\mathrm{Diff}^k(M)$, for any M of dimension $n \geq 3$ and any $k \geq 1$. In Section 3, we outline the Newhouse construction showing that Axiom A diffeomorphisms are not dense in the space of C^2 diffeomorphisms. The density of Axiom A maps in the C^1 topology (for surface diffeomorphisms) remains an open problem.

An important difference between dynamics in dimension two and in higher dimensions arises from the following simple fact. First, note that in dimension two any non-trivial transitive set (i.e., the set is not a periodic orbit) cannot contain either sinks or sources. Hence, for surface diffeomorphisms, all the (hyperbolic) periodic points of a transitive set are saddles, in particular, every periodic point has index one. In higher dimensions, for instance in dimension three, the saddles may have either index two or one, so, a priori, a transitive set may contain saddles with different indices. In fact, in the construction of non-hyperbolic transitive diffeomorphisms in [19], one starts with a hyperbolic transitive diffeomorphism (Anosov) whose saddles have index two, thereafter performing a saddle-node bifurcation, then saddles of index one are created without breaking the transitivity, obtaining transitive diffeomorphisms with saddles of indices one and two. The previous argument gives the following heuristic principle: in a persistently transitive set of a surface diffeomorphism the saddles cannot bifurcate. In Section 4.1, we describe another semi-local mechanism (heterodimensional cycles) for obtaining transitive sets containing saddles with different indices (in dimension three or higher).

3. Two dimensional C^2 wild dynamics: persistence of tangencies and infinitely many sinks

The constructions above suggest that is simpler to construct open sets of diffeomorphisms which do not verify the Axiom A seems to be simpler in higher dimensions than in surfaces. We begin by presenting the Newhouse construction of an open set $\mathcal{T}$ of C^2 diffeomorphisms which do not verify the Axiom A, [23]. In this construction (which is local and can be done in any surface) the C^2 regularity is essential. The set $\mathcal{T}$ consists of diffeomorphisms having tangencies: for every $g \in \mathcal{T}$, there are points x and y in some hyperbolic subset of $\Omega(g)$ (a horseshoe) such that the stable manifold of x and the unstable one of y are tangent throughout the orbit of a point $z \in \Omega(g)$. By definition of invariant manifolds, $W^s(x,g) = W^s(z,g)$ and $W^u(y,g) = W^u(z,g)$. Since in the hyperbolic case $T_w W^j(w,g) = E^j_w$, $j = s,u$, it follows that any $g \in \mathcal{T}$ does not verify the Axiom A: at the tangency point $z \in \Omega(g)$ it is not possible to exhibit a hyperbolic splitting (the stable and unstable directions should coincide).

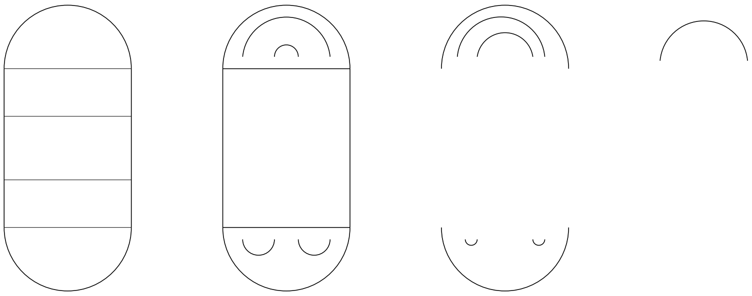

Figure 1: A linear horseshoe and its local perturbation

We outline the construction in the simplest situation. Consider a diffeomorphism f having a linear Smale horseshoe Λ as depicted in Figure 1, where (local) stable manifolds are horizontal segments and (local) unstable manifolds are vertical ones. We next modify the dynamics of f far from the horseshoe to create a point z of tangency between the stable and unstable manifolds of a saddle p of the horseshoe. In this construction, corresponding to the creation of a *one-cycle* via a *homoclinic tangency,* (we will discuss these notions later on), the point z is wandering, i.e. $z \notin \Omega(f)$.

Locally at $z = (z_1, z_2)$ the stable manifold of p is a horizontal segment γ_s and the unstable one a parabola γ_u. The next step is to unfold the tangency, considering an arc of diffeomorphisms f_t, $t \geq 0$ and $f_0 = f$, preserving the horseshoe Λ and the local stable manifold of p containing z, and translating the parabolic segment γ_u in the vertical direction by the vector $(0, t)$. In the unfolding new homoclinic points of the saddle p are created (this corresponds to a Ω-explosion). We now explain how the persistence of tangencies arises, for that a new hypothesis on the horseshoe is necessary.

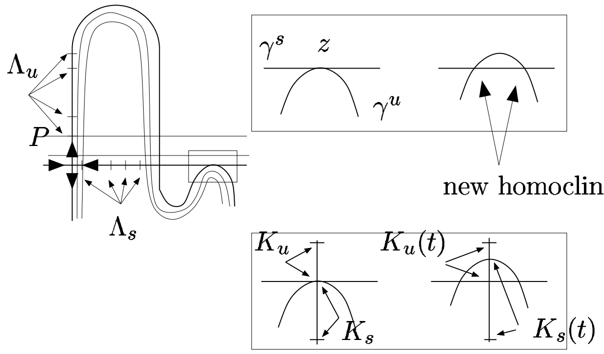

Figure 2: The homoclinic tangency

Suppose that the linear horseshoe Λ is constructed in such a way that $p = (0, 0)$, $\{0\} \times [0, 1] \subset W^u(p, f)$, $[0, 1] \times \{0\} \subset W^s(p, f)$ (these segments are the local invariant manifolds of p, $W^s_{loc}(p, f)$ and $W^u_{loc}(p, f)$), and $z = (z_1, 0)$. Moreover, the horseshoe Λ is the product of the Cantor sets $\Lambda_s = \Lambda \cap W^s_{loc}(p, f)$ and $\Lambda_u = \Lambda \cap W^u_{loc}(p, f)$. Consider the curve $\Upsilon = \{z_1\} \times [-\epsilon, +\epsilon]$ containing z and the (Cantor) sets $K_s = W^u(\Lambda_s, f) \cap \Upsilon$ and $K_u = W^s(\Lambda_u, f) \cap \Upsilon$. Then $K_u \subset \Upsilon^+ = \{z_1\} \times [0, +\epsilon]$ and $K_s \subset \Upsilon^- = \{z_1\} \times [-\epsilon, 0]$. The same construction for parameters $t > 0$ gives

$$\begin{aligned} (z_1, 0) &\in K_u(t) = W^s(\Lambda_u, f_t) \cap \Upsilon \subset \{z_1\} \times [0, +\epsilon]; \\ (z_1, t) &\in K_s(t) = W^u(\Lambda_u, f_t) \cap \Upsilon \subset \{z_1\} \times [-\epsilon, t]. \end{aligned}$$

Moreover, the sets $K_s(t)$ and $K_u(t)$ are *linked,* meaning that their convex hulls have non-empty intersection ($\{z_1\} \times [0, t] \subset K_s(t) \cap K_u(t)$). Take the set $B = \{t \geq$

$0 \colon K_u(t) \cap K_s(t) \neq \emptyset\}$. This set corresponds to parameters t such that the stable and the unstable manifolds of the horseshoe Λ of f_t have a tangency. The key point is the following: there is $t_0 > 0$ such that $[0, t_0] \subset B$ (i.e., $[0, t_0]$ is an interval of persistent tangencies associated to Λ). Moreover, for $t > 0$, the tangency points belong to the non-wandering set of f_t. To perform this construction, that gives an interval of tangencies, we use the notion of *thickness* of a Cantor set of $\mathbb{R}$.

The thickness of a Cantor set of $\mathbb{R}$ is a fractal dimension, defined in the same spirit as the more usual notions of Hausdorff dimension and limit capacity. It corresponds to the relation between the lengths of the intervals we remove in the inductive construction of the set and the lengths of the two remaining adjacent connected components. Consider a Cantor set $K \subset \mathbb{R}$ whose convex hull is (to say) the interval $[0,1]$ and a presentation of it, that is an enumeration of the connected components $\{G_i\}_{i \in \mathbb{N}}$ of $[0,1] \setminus K$ (the *gaps* of K). Let $F_0 = [0,1]$ and, for $i \geq 1$, $F_i = F_{i-i} \setminus G_i$, where each F_i is the union of $(i+1)$ pairwise disjoint closed intervals. Let B_i^r and B_i^ℓ the components of F_i intersecting the boundary of G_i, called *bridges* of G_i. The *thickness of the presentation* $\{G_i\}$ of K is

$$\tau(K, \{G_i\}) = \inf_{i \geq 1} \left\{ \min \frac{|B_i^r|}{|G_i|}, \frac{|B_i^\ell|}{|G_i|} \right\}.$$

The thickness of the Cantor set K, $\tau(K)$, is the supremum of $\tau(K, \{G_i\})$ taken over all the presentations $\{G_i\}$ of K. In our context, the importance of thickness is given by the next lemma.

Lemma 3.1 (Gap Lemma, [25, 28]) *Let K_1 and K_2 be Cantor sets with $\tau(K_1) \cdot \tau(K_2) > 1$ and such that their convex hulls are non disjoint. Then either $K_1 \cap K_2 \neq \emptyset$ or K_1 is contained in a gap K_2 or vice-versa.*

In the constructions in [23, 24] one starts considering a *thick horseshoe* Λ, which means with the notation above that $\tau(\Lambda_s) \cdot \tau(\Lambda_u) > 1$. Using bounded distortion arguments and that the holonomies along the stable and the unstable manifolds of Λ are C^1 (this is a deep assertion typical of two-dimensional dynamics), one gets $\tau(K_i(t)) = \tau(\Lambda_i)$, $i = s, u$, hence $\tau(K_s(t)) \cdot \tau(K_u(t)) > 1$. Since, by construction, the Cantor sets $K_s(t)$ and $K_u(t)$ are linked (in particular, no gap of $K_u(t)$ contains $K_s(t)$ and vice-versa) Lemma 3.1 implies that, for all $t \geq 0$, $K_s(t) \cap K_u(t) \neq \emptyset$, obtaining a parameter interval of persistence of tangencies, thus ending the proof in the model.

The proof in the general case follows along similar ideas, involving the fact that the thickness of a hyperbolic set Λ_f of a C^2 surface diffeomorphism f (defined as the sum of the the thickness of the sets $\Lambda_f \cap W^s_{loc}(p, f)$ and $\Lambda_f \cap W^u_{loc}(p, f)$, where p is any saddle of Λ_f) continuously depend on the dynamics (for g close to f consider the continuation Λ_g of Λ_f). Hence, for perturbations of the linear model the condition on the product of the thickness persists, and we can repeat the construction.

The previous constructions can be summarized as follows. An open set $\mathcal{U}$ of $\mathrm{Diff}^k(M)$, $k \geq 1$, has *persistence of homoclinic tangencies* if there is a continuous map Λ defined on a dense subset $\mathcal{D}$ of $\mathcal{U}$ associating to each $f \in \mathcal{D}$ a transitive hyperbolic set Λ_f containing points x and y such that $W^s(x,f)$ and $W^u(y,f)$ have a tangency. A hyperbolic set Λ_f is called *thick* if there is a saddle $p \in \Lambda_f$ such that $\tau(\Lambda_f \cap W^s_{loc}(p,f)) \cdot \tau(\Lambda_f \cap W^u_{loc}(p,f)) > 1$ (this definition does not depend on the choice of p).

Theorem 3.1 (Newhouse, [23]) *Let M be a closed surface and $f \in \mathrm{Diff}^2(M)$ a diffeomorphism having a thick hyperbolic set Λ_f with a homoclinic tangency associated to a saddle $p \in \Lambda_f$. Then there is an open set $\mathcal{U} \subset \mathrm{Diff}^2(M)$ with persistence of homoclinic tangencies whose closure contains f.*

Theorem 3.1 is the key step in the proof of the following result:

Theorem 3.2 (Newhouse, [24]) *Given any surface M there are an open set $\mathcal{U}$ of $\mathrm{Diff}^2(M)$ and a residual subset $\mathcal{S}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks.*

The proof of this theorem has two steps. For the first one, recall that a saddle p of a surface diffeomorphism f is *dissipative* if the absolute value of the product of the eigenvalues of D_pf^n (n the period of p) is less than one. Let f be a diffeomorphism with a (quadratic) homoclinic tangency associated to a dissipative saddle p. Then there is a diffeomorphimsm g arbitrarily close to f having a sink (a periodic attracting point). By a *quadratic homoclinic tangency* associated to p we mean that the stable and unstable manifolds of p have a non-transverse intersection at some point, where the invariant manifold have a quadratic contact. The second step (the most difficult one) is to prove that the unfolding of a homoclinic tangency leads to persistence of tangencies.

Consider an open set $\mathcal{U}$ of persistence of tangencies and let $\mathcal{D}$ be a dense subset of $\mathcal{U}$ of diffeomorphisms with homoclinic tangencies (we assume that the tangencies are associated to dissipative saddles). To get a residual subset $\mathcal{S}$ of $\mathcal{U}$ of diffeomorphisms with infinitely many sinks, note first that to have a sink is an open property. Thus the set $\mathcal{S}_k$ of diffeomorphisms with k (different) sinks is open. To prove Theorem 3.2 we see that each $\mathcal{S}_k$ is dense in $\mathcal{U}$ and let $\mathcal{S} = \cap \mathcal{S}_k$ (by construction, $\mathcal{S}$ is residual in $\mathcal{U}$). The density follows inductively, it is enough to see that by the first step every diffeomorphism in the set $\mathcal{D}$ can be approximated by a diffeomorphism with a sink. The density of $\mathcal{D}$ in $\mathcal{U}$ gives the density of $\mathcal{S}_1$. The inductive pattern is now clear: to construct $\mathcal{S}_2$ one produces new homoclinic tangencies (this is possible by the density of $\mathcal{D}$), and by unfolding the tangency one gets a new sink preserving the previous one.

The constructions in this section rely on the notion of thickness and are genuinely C^2. A result of Ures [36], claims that hyperbolic sets of C^1 generic diffeomorphisms have zero thickness, thus the Newhouse construction of persistence of tangencies cannot be carried out in the C^1 topology. Thence to construct (if possible) open sets of C^1 surface diffeomorphisms not satisfying the Axiom A new ingredients are necessary.

In fact, the density of C^1 surface diffeomorphisms verifying the Axiom A is an open question (such a density is false in higher dimensions). This is a central problem in two dimensional dynamics. In this context, recently Pujals and Sambarino proved that, for C^1 surface diffeomorphisms, the union of those with homoclinic tangencies and the ones verifying the Axiom A form a dense subset of $\mathrm{Diff}^1(M)$, see [32].

The Newhouse coexistence result was generalized to higher dimensions by Palis and Viana, [29], and Romero, [33]. The result in [29] state the coexistence of infinitely many sinks for homoclinic tangencies associated to *sectionally dissipative* saddles p (i.e., the product of any pair of eigenvalues of $D_p f^n$ (n the period of p) has modulus less than one, in particular the unstable manifold of p is one-dimensional). The paper [33] proves persistence of tangencies without any restriction on the eigenvalues of the saddle, exhibiting normally hyperbolic surfaces where the restricted dynamics have homoclinic tangencies (i.e., the problem is reduced to a two-dimensional one).

Finally, an important phenomenon associated to homoclinic tangencies is the existence of Hénon-like strange attractors with *some persistence* (contrary to the case of sinks, the continuation of a non-hyperbolic attractor is not well defined). See the paper by Mora and Viana [22] for precise statements. Combining the (weak) persistence of Hénon-like attractors and the persistence of tangencies at homoclinic bifurcations, Colli obtained (a version of) the coexistence phenomenon for Hénon-like attractors at homoclinic tangencies, see [9] for details and precise statements.

For a complete proof of the main results in this section and a broad discussion of homoclinic tangencies, besides the original papers by Newhouse, we refer the reader to the book by Palis and Takens [28, Chapters 3 and 6].

4. C^1 coexistence phenomenon in three manifolds

4.1. Heterodimensional cycles. Intermingled homoclinic classes

Let f be an Axiom A diffeomorphism and $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ (different) basic sets of the spectral decomposition of the non-wandering set of f. The diffeomorphism f has a *k-cycle* associated to $\Lambda_1, \Lambda_2, \dots, \Lambda_k$ if the unstable manifold of Λ_i (the union of the unstable manifolds of the points of Λ_i) meets the stable one of Λ_{i+1} for all $i = 1, \dots, k$, where $i_{k+1} = 1$. The cycle is *equidimensional* if all the basic sets in the cycle have the same index (the dimension of the unstable bundle, which coincides with the index of any saddle of the set). Otherwise the cycle is *heteredimensional.* Heterodimensional cycles can only occur in dimension greater than or equal to three. The horseshoe in Section 3 with a tangency is an example of equidimensional one-cycle.

We now discuss the dynamics at heterodimensional cycles in the simplest situation. Consider a diffeomorphism f on a three manifold with a cycle associated to fixed points p and q of indices one and two such that the eigenvalues of $D_p f$ and $D_q f$ verify $0 < \lambda_s < \lambda_c < 1 < \lambda_u$ and $0 < \beta_s < 1 < \beta_c < \beta_u$, respectively. Moreover, the transverse intersection between the two dimensional manifolds $W^s(p, f)$ and $W^u(q, f)$ contains a curve γ (the *connection*) with endpoints p and q. The curve γ is simultaneously transverse to the strong stable foliation of $W^s(p,f)$ and to the strong unstable foliation of $W^u(q,f)$ (the unique f-invariant one-dimensional foliations of $W^s(p,f)$ and $W^u(q,f)$ whose leaves through p and q are tangent to the eigenspaces of λ_s and β_u). Finally, the intersection between the one-dimensional manifolds $W^u(p,f)$ and $W^s(q,f)$ is quasi-transverse and occurs throughout the orbit of a *heteroclinic point* x: $W^u(p,f) \cap W^s(p,f) = \{f^k(x)\}_{k\in\mathbb{Z}}$ and $T_xW^u(p,f) + T_xW^s(q,f)$ is a direct sum. See Figure 3. The cycle in Figure 3 is an example of a *connected* and *non-critical* one.

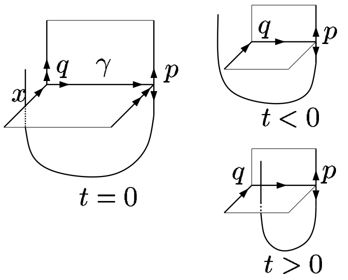

Figure 3: A heterodimensional cycle and its unfolding

In the context of equidimensional cycles (homoclinic tangencies) in surfaces, the dynamics after unfolding the cycle mainly depends on the fractal dimensions (thickness, Hausdorff dimension, and limit capacity) of the hyperbolic sets in the cycle (see the series of papers [23, 26, 27, 30]). Moreover, the occurrence of wild dynamics is related to the persistence of cycles (homoclinic tangencies in this setting). In the case of heterodimensional cycles, as the one described above, the situation is rather different and the dynamics after the unfolding of the cycle is essentially determined by the restriction of the bifurcating diffeomorphism f to the connection curve γ. This claim is suggested by the series of papers [10, 13, 15], where the dynamics in the sequel of the unfolding is reduced to the analysis of a system of iterated functions generated by f and a translation. For an expository explanation of the dynamics at heterodimensional cycles see [7, Chapter 6] and [16].

When the distortion of the restriction of f to the connection γ is small, after unfolding the cycle the homoclinic classes of p and q are persistently intermingled and non-hyperbolic transitive sets containing simultaneously p and q are created throughout the bifurcation. The definition of an arc $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ unfolding a heterodimensional cycle at $t=0$, $f_0 = f$, is similar to the one of a homoclinic tangency: there are compact parts $K^u(t)$ of $W^u(p,f_t)$ and $K^s(t)$ of $W^s(q,f_t)$ depending continuously with t and intersecting at the heteroclinic point x for $t=0$, moving with positive velocity one with respect the other one (for instance and in local coordinates, $x=(0,0,0)$, $K^s(t) = [-1,1] \times \{(0,0)\}$ and $K^u(t) = \{(0,t)\} \times [-1,1]$), see Figure 3.

Theorem 4.1 ([10, 11]) *There is an open set $\mathcal{O}$ of arcs of diffeomorphisms $(f_t)_{t\in[-\varepsilon,\varepsilon]}$ unfolding a heterodimensional cycle at $t=0$, associated to saddles p and q of indices r and $(r+1)$, such that the homoclinic classes of p and q coincide for all positive t, i.e., $H(p,f_t)=H(q,f_t)$, $t\in(0,\varepsilon]$.*

The cycles in Theorem 4.1 are similar to those in Figure 3 (non-critical and connected). The open set $\mathcal{O}$ includes arcs of diffeomorphisms being *Morse-Smale* at the bifurcation (i.e., the non-wandering set of f_0 is finite and hyperbolic, thus the basic sets containing p and q are both trivial). It is interesting to compare Theorem 4.1 with the results about homoclinic tangencies, where to get persistent non-hyperbolic dynamics after the unfolding of the cycle (i.e., f_t is non-hyperbolic for every $t>0$) one needs to consider tangencies associated to thick hyperbolic sets.

An important point in Theorem 4.1 is that the only assumption on the dynamics of the bifurcating diffeomorphism f_0 involved in its proof (besides the type of geometry of the curve $\gamma\subset W^s(p,f_0)\cap W^u(q,f_0)$) is that the restriction of f_0 to the connection γ has small distortion. These conditions are compatible with other hypotheses on the global dynamics of f_0, for instance, the homoclinic classes of p and q may contain other saddles (we will use this fact latter).

The main step of the proof of Theorem 4.1 (in three manifolds) is the following property of the one-dimensional unstable manifold of p: *for every $t>0$, the closure of $W^u(p,f_t)$ contains the two-dimensional unstable manifold $W^u(q,f_t)$ of q.* Heuristically, this means that the stable and unstable manifolds of p have both (topological) dimension two, thus the saddle p behaves simultaneously as a point of indices one and two. Similarly, the two dimensional manifold $W^s(p,f_t)$ is contained in the closure of the one dimensional manifold $W^s(q,f_t)$, so the saddle q also *has* indices two and one. This is the main step for proving that the homoclinic classes of p and q coincide.

Roughly, the property $W^u(q,f_t)\subset \text{closure}(W^u(p,f_t))$ persistently (for all $t>0$) plays a role analogous to the thick horseshoe in the proof of the persistence of tangencies. The key point here (and the main difference) is that the property $W^u(q,f_t)\subset \text{closure}(W^u(p,f_t))$ persistently only requires C^1 regularity, while the construction of depends heavily on the C^2 differentiability of the dynamics.

The construction in [10] of homoclinic classes containing saddles having different indices persistently was generalized and systematized in [3] in order to construct open sets of non-hyperbolic transitive diffeomorphisms. In [3] is introduced the notion of *blender,* a topological *plug* depending only on semi-local properties, guaranteeing that the closure of the unstable manifold of a saddle of index r contains (C^1 persistently) the unstable manifold of a point of index $(r+1)$. For an expository construction of blenders see [7, Chapter 6.2]. See also a much more elementary discussion in [12].

Finally, recall that the constructions in Section 3 of wild diffeomorphisms had two steps. First, the unfolding of a tangency associated to a thick horseshoe leads to persistence of tangencies, and using such a persistence of tangencies one gets coexistence of infinitely many sinks. Second, by unfolding any homoclinic tangency thick horseshoes with tangencies are created. Combining these two steps one has that persistence of tangencies and coexistence of infinitely many sinks are inherent to the unfolding of a tangency. There is a somewhat parallel situation in the heterodimensional setting. First, we can think of property $W^u(q, f_t) \subset \text{closure}(W^u(p, f_t))$ (where the indices of p and q are 1 and 2, respectively) of the diffeomorphisms in Theorem 4.1 as playing the role of the thick horseshoes. Second, for cycles *far from homoclinic tangencies* (i.e., the bifurcating diffeomorphism cannot be perturbed to get a tangency associated to the saddles in the cycle), after unfolding the cycle (and arbitrarily close to the bifurcating diffeomorphism) one gets new diffeomorphisms with $W^u(q, f_t) \subset \text{closure}(W^u(p, f_t))$. Thus persistence of intermingled homoclinic classes of different indices is a phenomenon inherent to the unfolding of heterodimensional cycles. See [14] for details and precise statements.

4.2. C^1 coexistence phenomenon on three manifolds

We next discuss the C^1 coexistence phenomenon in dimension 3 (the arguments can be adapted to higher dimensions after minor changes). The main result is:

Theorem 4.2 ([4]) *Let M be a manifold of dimension three. There are an open set $\mathcal{N}$ of $\text{Diff}^1(M)$ and a residual subset $\mathcal{S}$ of $\mathcal{N}$ such that every $f \in \mathcal{S}$ has infinitely many sinks.*

The key point is to get C^1 persistence of tangencies (associated to sectionally dissipative saddles). Observe that, in dimension three, the unfolding of a tangency does not necessarily lead to the creation of sinks: consider a diffeomorphism F defined as the product of a surface diffeomorphism $f: S \to S$ with a tangency associated to a dissipative saddle and a strong (linear) expansion. Thus the tangency of F is not sectionally dissipative. The unfolding of this tangency leads to diffeomorphisms with saddles of index one: the saddles have a two dimensional stable bundle corresponding to a sink of the restriction to S and a one-dimensional unstable direction given by the transverse expansion.

The first step in the proof of Theorem 4.2 is to get C^1 persistence of (heterodimensional) cycles. Theorem 4.1 gives an open set $\mathcal{H}$ of $\text{Diff}^1(M)$ of diffeomorphisms f having saddles p and q of indices one and two such that their homoclinic classes are equal. On the other hand, the Hayashi's connecting lemma implies the following:

Lemma 4.1 (Hayashi, [17]) *Let Σ be a transitive set of a diffeomorphisms f containing a pair of saddles a and b. Then there is g arbitrarily C^1 close to f such that $W^s(a, g) \cap W^u(b, g) \neq \emptyset$.*

We apply Lemma 4.1 to the diffeomorphisms in $\mathcal{H}$. Let $f \in \mathcal{H}$ and consider the transitive set $\Sigma_f = H(p, f) = H(q, f)$ and any pair of saddles $p', q' \in \Sigma_f$ of indices one and two. Then there is g arbitrarily C^1 close to f with a heterodimensional cycle associated to p' and q'. To prove this claim we apply Lemma 4.1 twice. First, taking $a = p'$ and $b = q'$, we get g close to f such that $W^s(p', g)$ meets $W^u(q', g)$. As both invariant manifolds have dimension two, we can assume that the intersection is transverse and thus persistent: $W^s(p',h) \cap W^u(q',h) \neq \emptyset$, for all h close to g. Applying now Lemma 4.1 to the g, taking $a = q'$ and $b = p'$, we get h close to g (thus to f) with $W^u(p',h) \cap W^s(q',h) \neq \emptyset$. As, by construction, $W^s(p',h) \cap W^u(q',h) \neq \emptyset$, the diffeomorphism h has a heterodimensional cycle. In other words, the diffeomorphisms of $\mathcal{H}$ having heterodimensional cycles form a dense subset of it.

To get persistence of tangencies in $\mathcal{H}$ we need an extra hypotheses on the dynamics. As we mentioned above, the semi-local hypotheses in Theorem 4.1 are compatible with other features of the dynamics. Suppose that the saddle q is *homoclinically related* to a saddle r having a pair of non-real expanding eigenvalues. This means that $W^s(q,f)$ and $W^u(r,f)$ meet transversely, and similarly for $W^u(q,f)$ and $W^s(r,f)$. The homoclinic relation implies that the homoclinic classes of q and r are equal. Thus $H(p,f) = H(q,f) = H(r,f)$. We now apply Lemma 4.1 to p and r, obtaining a heterodimensional cycle associated to p and r. The facts that r has a expanding non-real eigenvalue and that $W^s(p,f)$ meets transversely $W^u(q,f)$ imply that the stable manifold of p spirals around the stable manifold of r. This property together with the heterodimensional cycle condition allows us to get (after a perturbation) a tangency associated to p. In other words, if there is a saddle with a pair of non-real expanding eigenvalues homoclinically related to q, then $\mathcal{H}$ is an open set of C^1 persistence of tangencies. Taking the saddle p to be sectionally dissipative we can repeat (essentially) the construction in Section 3, obtaining Theorem 4.2.

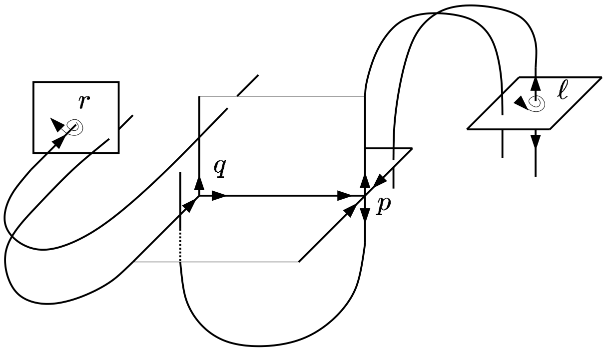

Figure 4: Non-dominated homoclinic classes

One can perform the construction above assuming also the existence of a saddle ℓ homoclinically related to p with a pair of non-real contracting eigenvalues (see Figure 4). This leads to persistence of homoclinic tangencies associated to q. Thus, when q is sectionally expansive (i.e., sectionally dissipative for f^{-1}) we also get a residual subset of $\mathcal{H}$ of diffeomorphisms having infinitely many sources. Hence, under this assumption, intersecting the two residual sets we have obtained, we get a new residual subset of $\mathcal{H}$ of diffeomorphisms simultaneously having infinitely many sinks and sources. In fact, the last construction is a particular case of a more general result we are going to discuss in the next section: the dynamics at non-dominated homoclinic classes.

5. Non-dominated dynamics: a sample of wild dynamics

In the previous paragraph, we exhibited an open set of $\mathrm{Diff}^1(M)$ consisting of diffeomorphisms f having homoclinic classes with saddles r and ℓ of different indices and having non-real eigenvalues. This type of homoclinic class is the prototype of *non-dominated dynamics* and leads to the C^1 Newhouse coexistence phenomenon and other stronger forms of wild dynamics.

We first introduce the definition of *dominated splitting,* an extension of the notion of hyperbolicity, introduced by Liao [18] and Mañé [20] that played a fundamental role in the proof of the stability conjecture, [21]. Consider a diffeomorphism $f\colon M \to M$ and an f-invariant closed set Λ of it. A Df-invariant splitting $T_\Lambda M = E \oplus F$ of the tangent bundle of M over Λ is dominated if there is $\ell \in \mathbb{N}$ such that

$$\frac{|D_x f^\ell(v)|}{|D_x f^\ell(w)|} \leq \frac{1}{2},$$

for every point $x \in \Lambda$ and any pair of unitary vectors $v \in E$ and $w \in F$. Moreover, the dimensions of the fibers E_x and F_x are independent of the point x of Λ.

In this definition, the derivative of f in the bundle E (similarly in F) may exhibit contractions and expansions (according to the point). Assume, for simplicity, that $\ell = 1$, the key point in the definition is that any expansion of Df in E is weaker than the corresponding expansion in F (similarly, contractions in F are weaker than the ones in E). The definition also implies that positive Df-iterations of vectors v, $v \notin E$, converge to the F direction.

The next result means that non-dominated homoclinic classes are a natural habitat for wild dynamics:

Theorem 5.1 (C^1 generic dichotomy for homoclinic classes, [6]) *Let M be a closed manifold. There is a residual subset $\mathcal{R}$ of $\mathrm{Diff}^1(M)$ of diffeomorphisms f such that, for every saddle p of f, the homoclinic class $H(p,f)$ of p satisfies the following;*

- *either $H(p,f)$ has a dominated splitting (weak hyperbolicity),*
- *or $H(p,f)$ is contained in the closure of an infinite set of sinks or sources of f (coexistence phenomenon).*

In this theorem and in the weak hyperbolic case, if M is a surface then the homoclinic class is hyperbolic (this was proved by Mañé in [20]). When the dimension of the manifold is three, the splitting is *partially hyperbolic* (i.e., its one-dimensional bundle is either uniformly contracting or expanding). Finally, it is an open question whether for C^1 surface diffeomorphisms the coexistence phenomenon can be eliminated.

Theorem 5.1 gives a new path leading to the coexistence phenomenon. Take a three dimensional diffeomorphism f with (fixed) saddles $\bar{p}$ and $\bar{q}$ of such that:

- the index of $\bar{p}$ is one and $D_{\bar{p}}f$ has a pair of non-real contracting eigenvalues,
- the index of $\bar{q}$ is two and $D_{\bar{p}}f$ has a pair of non-real expanding eigenvalues,
- there is a C^1-neighborhood $\mathcal{V}$ of f such that $H(\bar{p}, g) = H(\bar{q}, g)$ for all $g \in \mathcal{V}$.

Then there is a residual subset of $\mathcal{V}$ of diffeomorphisms with infinitely many sinks or sources (in fact, the closure of such points contains the infinite set $H(\bar{p}, g) = H(\bar{q}, g)$). Note that here we do not make any assumption on the sectionally dissipativiness or expansiviness of the saddles $\bar{p}$ and $\bar{q}$ as in Section 4.2. Let us explain this claim.

By construction, the transitive set $\Sigma_g = H(\bar{p}, g) = H(\bar{q}, g)$ does not admit any dominated splitting for all $g \in \mathcal{V}$. Otherwise, assume by contradiction that $E \oplus F$ is a dominated splitting of Σ_g such that E is one dimensional. Consider the stable bundle $E^s_{\bar{p}}$ of $\bar{p}$ (the eigenspace associated to the contracting eigenvalue). Then, necessarily (this follows from the domination of the splitting) $E \subset E^s_{\bar{p}}$, thus $D_{\bar{p}}g$ leaves invariant a one dimensional direction of $E^s_{\bar{p}}$, which is incompatible with the fact that $D_{\bar{p}}g$ has non-real contracting eigenvalues. The contradiction follows similarly when F is one dimensional by considering $\bar{q}$ and its unstable direction. Theorem 5.1 now implies that there is a residual subset of $\mathcal{V}$ consisting of diffeomorphisms with infinitely many sinks or sources. In fact, if the set Σ_f contains a saddle a (the index of a now is irrelevant) such that the Jacobian of $D_a f^n$ (n the period of a) is less (resp. bigger) than one, then one gets infinitely many sinks (resp. sources). Again, we may have both situations simultaneously.

The sets $H(\bar{p}, f) = H(\bar{q}, f)$ in the previous construction are examples of non-dominated homoclinic classes. We end this section explaining how the dynamics at a non-dominated class may be extremely rich when an extra hypothesis is assumed:

- for every $f \in \mathcal{V}$, the set $\Sigma_f = H(\bar{p}, f) = H(\bar{q}, f)$ contains a pair of saddles with Jacobians greater and less than one (where such saddles are homoclinically related to $\bar{p}$ or $\bar{q}$).

A class verifying this hypotheses is named *wild homoclinic class.* For this type of classes there is the following coexistence phenomenon.

Theorem 5.2 ([5]) *Let $\mathcal{W}$ be an open set of (three dimensional) C^1 diffeomorphisms f having a wild homoclinic class $H(p, f)$. Then there is a residual subset $\mathcal{R}$ of $\mathcal{W}$ such that, for every $g \in \mathcal{R}$, the set $H(p, g)$ is simultaneously contained in the closure of infinitely many pairwise disjoint:*

- *saturated transitive sets with minimal dynamics,*

- *non-trivial uniformly hyperbolic attractors and repellers,*
- *non-trivial partially hyperbolic attractors and repellers,*
- *wild homoclinic classes.*
- *infinitely many sinks and sources.*

Let us recall that an (infinite) f-invariant closed set Λ is *minimal* if every orbit of it is dense in the whole Λ (in particular, Λ does not contain periodic points). We say that a transitive set Υ is *saturated* if it contains any transitive set intersecting it (thus these sets are maximal transitive).

The main idea in the proof of Theorem 5.2 is to observe that there are diffeomorphisms g close to the initial f whose dynamics in small balls are the identity (the precise definition of this property corresponds to the notion of *universal dynamics* in [5]). The proof also involves a renormalization-like inductive argument: (roughly) using the universal dynamics one recovers, in small scale and up to the period, the initial wild dynamics.

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