Heterodimensional Cycles

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(I) HETERODIMENSIONAL CYCLES

$f\colon M \to M$, $\dim M \geq 3$.

P hyperbolic saddle of f, **index of P**,
$\mathrm{ind}(P) =$ dimension of the unstable bundle.

P, Q hyperbolic periodic saddles of f
dimension variability: $\boxed{\mathrm{ind}(P) < \mathrm{ind}(Q)}$

heterodimensional cycle related to P and Q:

$$\underbrace{W^s(P) \cap W^u(Q)}_{\textbf{(a)}} \neq \emptyset \neq \underbrace{W^u(P) \cap W^s(Q)}_{\textbf{(b)}}$$

Typical (generic) situation:

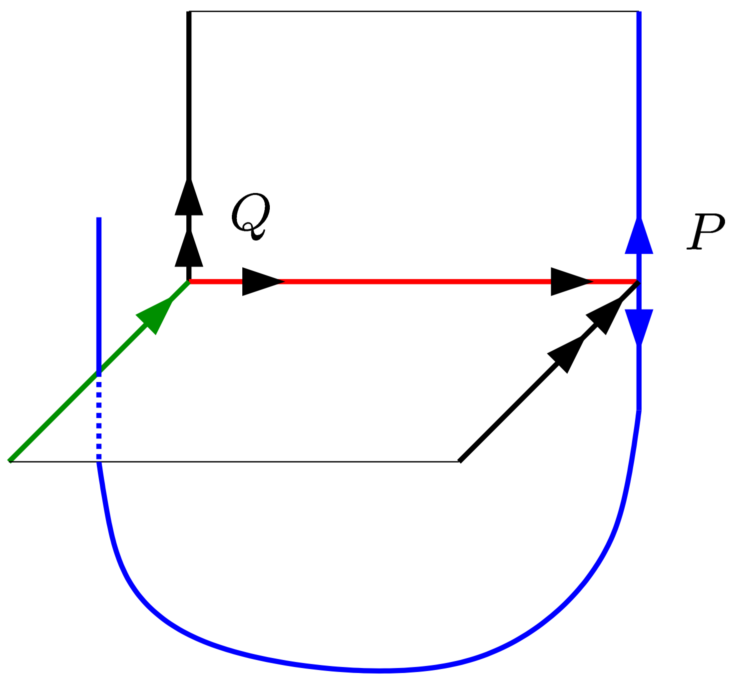
(a) transverse intersection;

(b) quasi-transverse:
for all $x \in W^s(Q) \cap W^u(P)$ heteroclinic point:

$$T_x W^u(P) + T_x W^s(Q) = T_x W^u(P) \oplus T_x W^s(Q)$$

Codimension one-case: $\boxed{\text{ind}(P)+1=\text{ind}(Q)}$

(for simplicity: $\dim(M)=3$, $\text{ind}(Q)=2$).



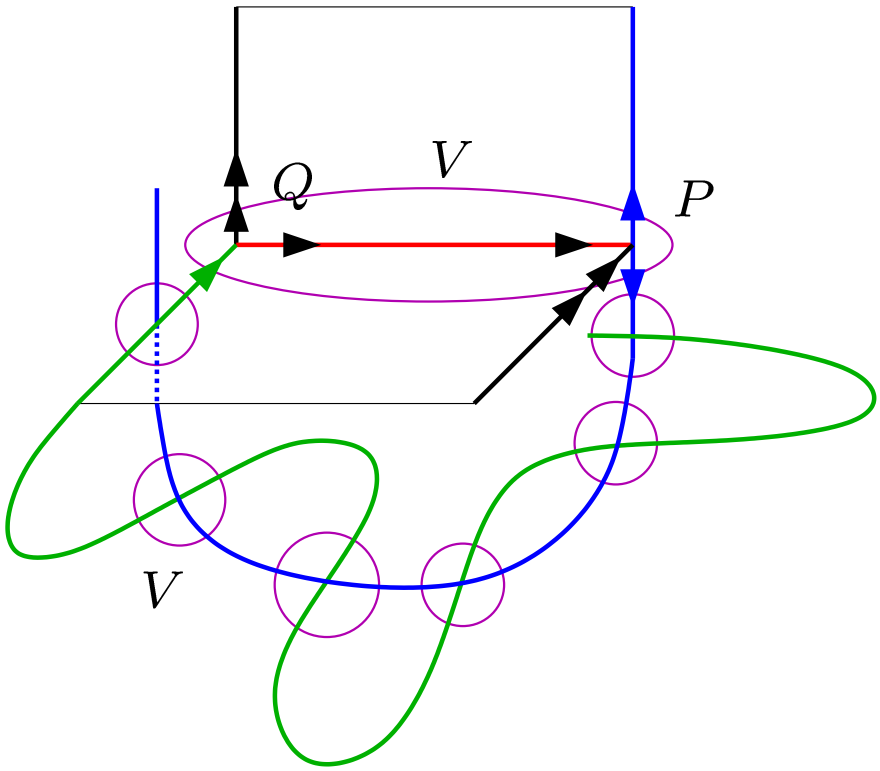
homoclinic class of P:

$$\boxed{H(P(f)=\overline{W^s(P)\pitchfork W^u(P)}}$$

- transitive (dense orbit)
- periodic points of the same index as P are dense (homoclinically related)
- (in general) it may contain points of different indices (non-hyperbolicity)

<u>Goal</u>: Dynamics of nearby diffeomorphisms in a **<u>neighborhood of the cycle</u>** after the unfolding (explosion of the homoclinic classes of P and Q)

neighborhood of the cycle:
(small) open set V containing all elements of the cycle, orbits of the:
- saddles P and Q,
- heteroclinic points of P and Q:
  $W^s(P) \cap W^u(Q)$ and $W^u(P) \cap W^s(Q)$



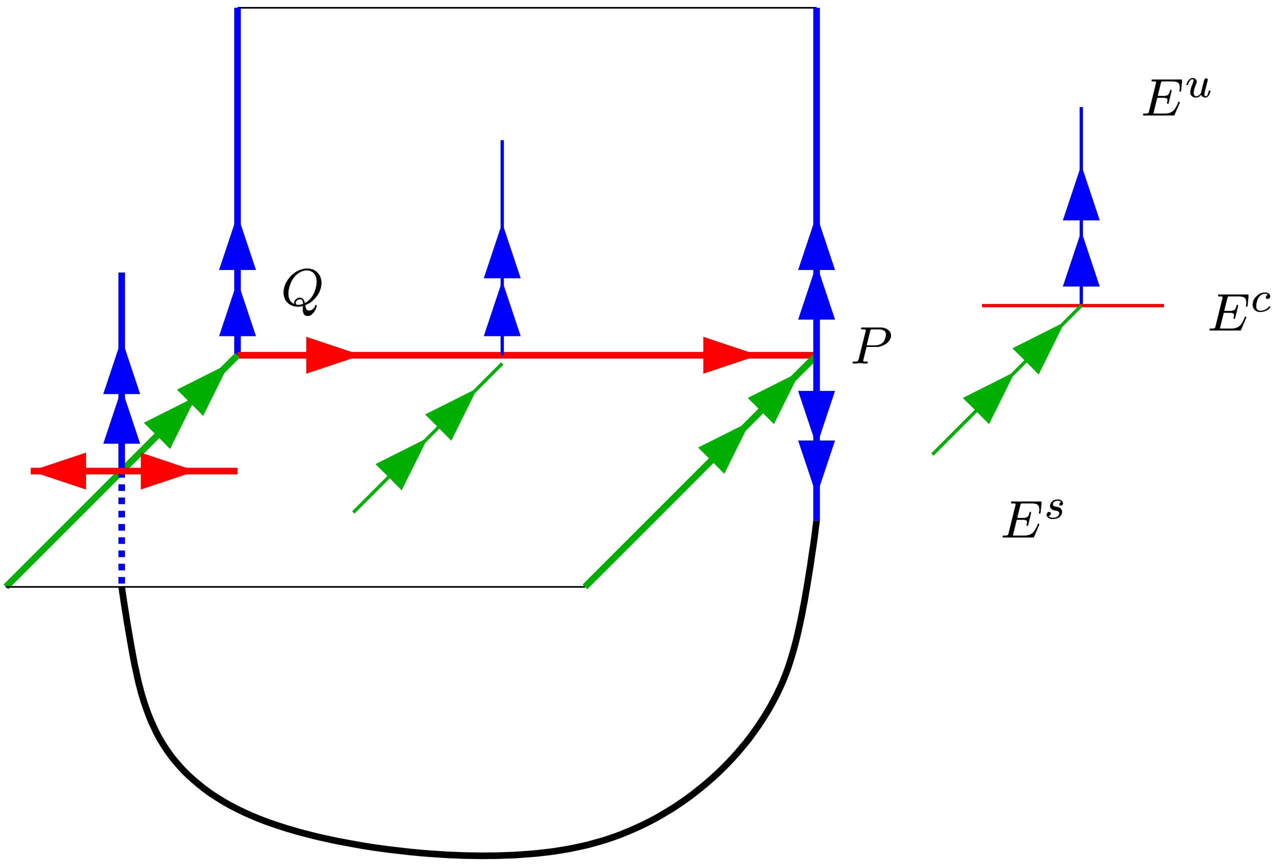
Partially hyperbolic case: g close to f,

$$\Lambda_g = \bigcap_{\mathbb{Z}} g^i(V)$$

Λ_g maximal inv. set of g in V,

$$T(\Lambda_g) = E^s \oplus E^c \oplus E^u$$

E^s, E^u unif. hyper. (stable and unstable),
E^c one-dimensional.

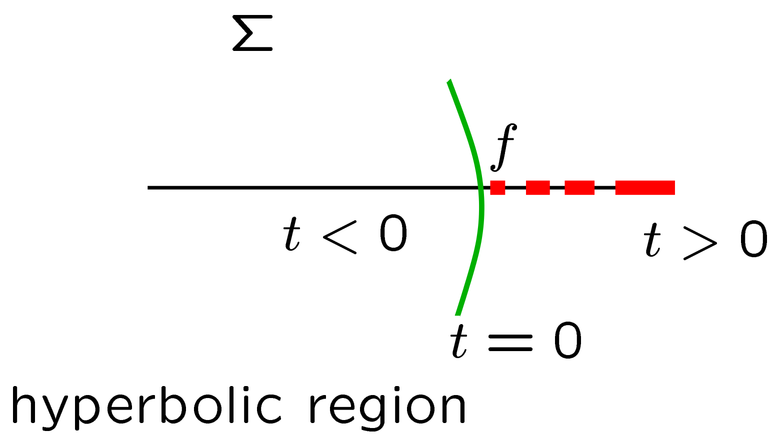


Strategy (Palis-Takens):

• Consider $(f_t)_{t=[-1,1]}$:
– starting in a hyperbolic region (f.i., Morse-Smale relative to neigh. of the cycle)
– crossing transversely its boundary Σ for $t = 0$ at $f_0 = f$
(here: the component of the boundary containing f has codimension one)

• Study the dynamics for $t > 0$ small.

• Interesting dynamics: those displayed by diffeomorphism f_t for a positive measure set of parameters $t > 0$.



(II) DYNAMICS ARISING FROM HETERODIMENSIONAL CYCLES

$\mathcal{P}$ a dynamical property:

$$\mathbb{P} = \{t > 0 \colon f_t \text{ satisfies } \mathcal{P}\}$$

Prevalent: $\liminf_{t\to 0+} \frac{|\mathbb{P}\cap[0,t]|}{t} > 0$.

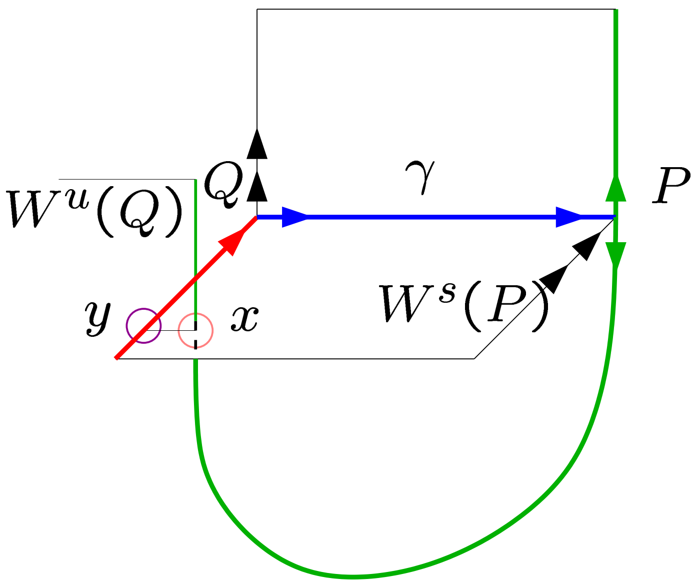
Absolutely Prev.: $\liminf_{t\to 0+} \frac{|\mathbb{P}\cap[0,t]|}{t} = 1$.

Persistent: $|\mathbb{P} \cap [0,t]| > 0$ for all $t > 0$.

(Some) typical features after unfolding a partially hyperbolic heterodimensional cycle:

(1) intermingled homoclinic classes:

$$H(P, f_t) \cap H(Q, f_t) \neq \emptyset$$



Typically: $H(P, f_t) = H(Q, f_t)$, $\gamma \subset H(P, f_t)$

(Carballo et al.) C^1-**Generically:**

$$H(P, f_t) \cap H(Q, f_t) \neq \emptyset \implies H(P, f_t) = H(Q, f_t)$$

(2) hyperbolicity: $\Lambda_{f_t} = H(P, f_t) \cup H(Q, f_t)$ the classes are hyperbolic (indices 1 and 2);

(3) collisions of homoclinic classes (secondary saddle-node biffurcations);

(4) persistence of heterodimensional cycles (thick one-dimensional Cantor sets).

Thm A (D-Rocha):

For every $(f_t)_{t\in[-1,1]}$ unfolding a partially hyperbolic heterodimensional cycle there is a sequence of parameter intervals B_n, $B_n \to 0^+$, such that:

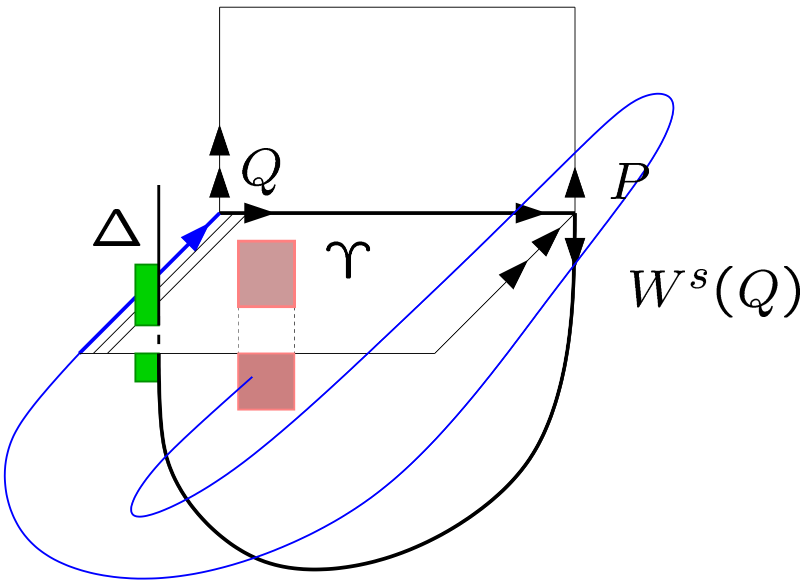
A) $\forall t \in B_n$: $H(P,f_t) = H(Q,f_t)$, and Λ_{f_t} is transitive (containing P and Q).

B) (prevalent non-hyperb., first bifurc.).

$$\liminf_{t\to 0} \frac{|B\cap[-t,t]|}{2\,t} > 0; \quad B = \cup B_n.$$

Main ingredient: **Blender-like construction**

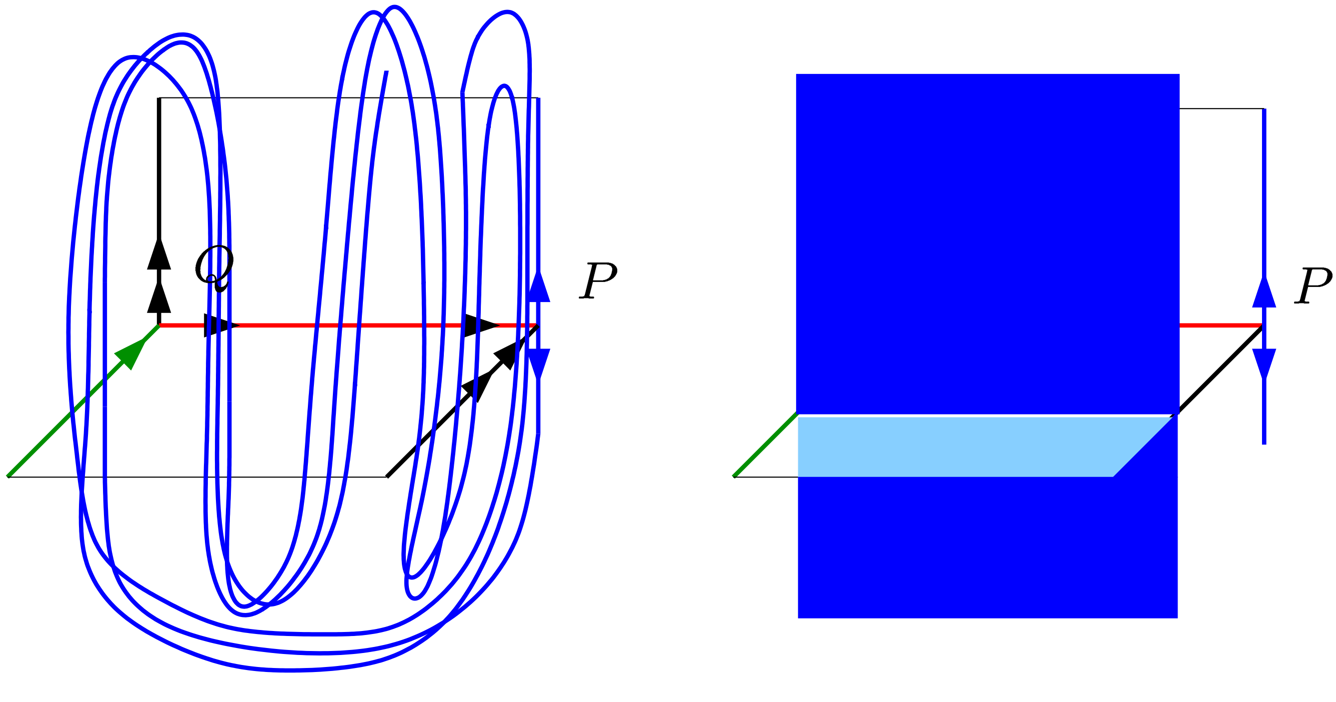
$W^u(Q)$ intersects *every* 2-disk Υ crossing transversely $W^s(P)$.



<u>Consequence</u>:
$H(P, f_t) \subset H(Q, f_t)$
just take disks $\Delta \subset W^u(Q, f_t)$ whose boundary contain a homoclinic point of P).

<u>Heuristically</u>:

the one-dimensional manifold $W^u(Q)$ *is* (topologically) two-dimensional manifold. Similarly for $W^s(P)$.
<u>two-fold behaviour:</u> P and Q *are* simultaneously points of indices one and two.



Applications of blenders (& het. cycl):

• Construction of **robustly transitive** C^1 diffeomorphisms (Bonatti, D):

$\mathcal{T}(M)$: $\{f$: s.t. $\exists$ C^1-nbhood $\mathcal{U}$ of f s.t. any $g \in \mathcal{U}$ is transitive (dense orbit in M)$\}$

• Properties of robustly transitive sets and diffeos (B,D,Pujals,R):

(open and dense) the indices of the periodic points form an interval (in $\mathbb{N}$)

• Additional hypotheses: **(3-bundle) partial hyperbol.** $TM = E^s \oplus E^c \oplus E^u$, with E^c 1-dim, E^s, E^u unif. hyperb.

$$\boxed{\mathcal{S}(M)\colon \{f \in \mathcal{T} \oplus \text{ part. hyp.}\}}$$

$\Longrightarrow$ strong stable $\mathcal{F}^s$ and unstable $\mathcal{F}^u$ foliations (tangent to E^s and E^u) are defined

• Properties of the strong invariant foliations (B,D,Ures):

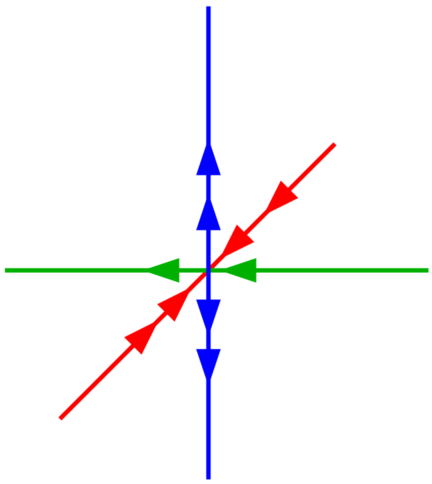
Open and dense in $\mathcal{S}(M)$: Either the strong stable $\mathcal{F}^s$ or the unstable $\mathcal{F}^u$ foliation is <u>minimal</u> (every leaf is dense).

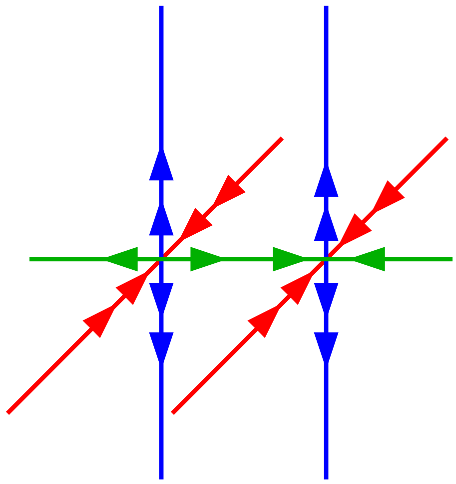
• <u>**Natural habitat**</u> for heterodim. cycles:

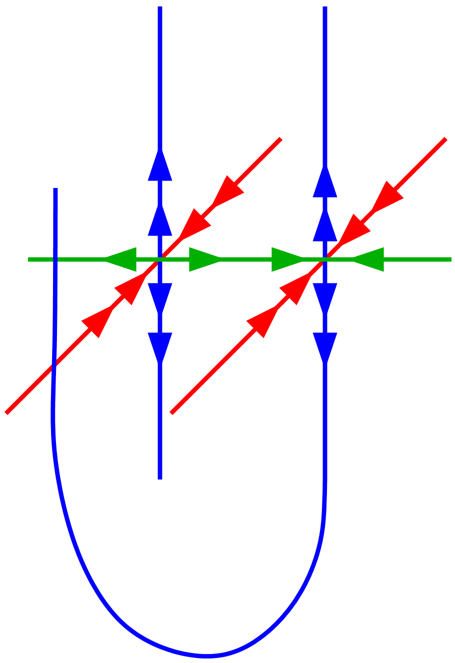
(1) creation of saddle-nodes
(transitivity $\oplus$ non-hyperbolicity $\oplus$ cycles (connecting lemma))

(2) unfolding of the saddle-node bifurcation: saddles with different indices

(3) creation of a (heterodimensional) cycle
(connecting lemma $\oplus$ transitivity))







Given an arc $(f_t)_{t\in[-1,1]}$ unfolding a cycle at $t = 0$:

$$\boxed{H(s) = \{t \in [0, s] : f_s \text{ hyperbolic } \}}$$

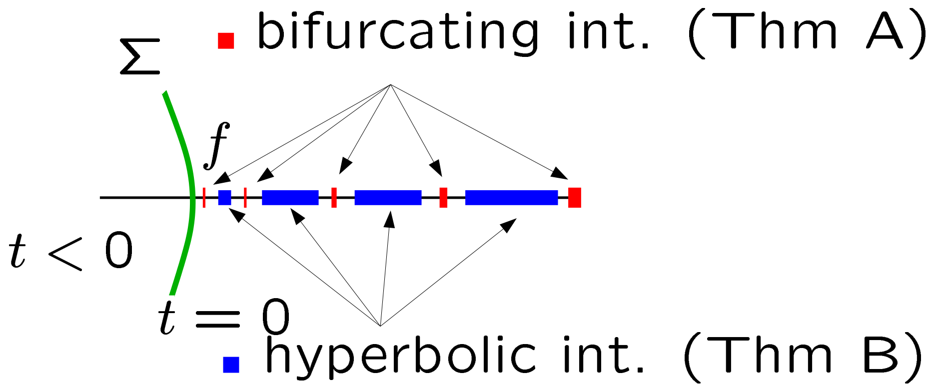
hyperbolic: Axiom A $\oplus$ no-cycles

Thm B (D-Rocha):

For every $\varepsilon > 0$ there is an open set of $\mathcal{A}_\epsilon$ of arcs $(f_t)_{t\in[-1,1]}$ unfolding a partially hyperbolic heterodimensional cycle such that:

$$\liminf_{t\to 0} \frac{|H(t) \cap [0, t]|}{t} > 1 - \varepsilon$$

(by the previous theorem this limit is < 1)
hyperbolicity is prevalent but not absolute

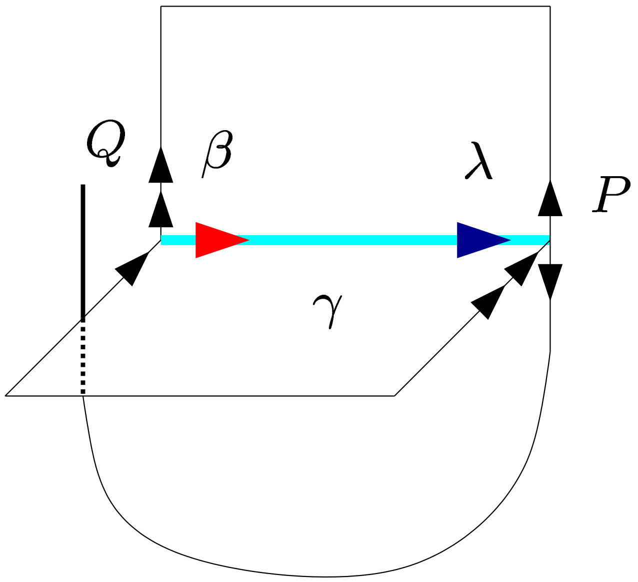


(III) <u>LIMIT DYNAMICS</u>

Dynamics after the bifurcation *depends* on the **<u>central</u>** dynamics of the bifurc. diffeo.

<u>Ingredients</u>:

(a) central eigenvalues of P and Q (corresponding to the central direction E^c)
(b) transition from Q to P throughout the curve $\gamma = W^s(P) \pitchfork W^u(Q)$ (restriction f/γ)



<u>**Mather invariant**</u>:

f/γ close to 0:

time-one of $\boxed{X(y) = (\log\beta)\, y \frac{\partial}{\partial y}}$

flow: $\boxed{y \mapsto \beta^t\, y}$

f/γ close to 8:

time-one of $\boxed{Y(y) = (\log\lambda)\,(y-8)\frac{\partial}{\partial y}}$

flow: $\boxed{y \mapsto \beta^t\, y}$

y near 0^+, big n with $f^n(y)$ near 8^-:

$$\boxed{Df^n(y)(X(y)) = \mu(y)\, Y(f^n(y))}$$

local f-invariance (X, Y near $0, 8$),

- independent of n,
- $\mu(y) = \mu(f(y))$.

μ measures the distortion.

$\mu(y)$: <u>Mather invariant of f/γ</u>

Heuristic principle:
- small distortion: intermingled homoclinic classes, for $t > 0$: $H(P, f_t) = H(Q, f_t)$. (D).
- For Thm B one needs a big distortion (as $\varepsilon \to 0$, distortion $\to \infty$).
- Thm. A: dynamics in *intermediate* cases.

Strategy:
- projection to the central direct. (curve γ),
- fix a (reference) fundam. domain D^+ of γ, consider returns to D^+.

Definition of the limit dynamics:

Simplifying assumptions:
- $\gamma = [0, 8]$,
- f linear in $[0, 1]$, $x \mapsto \beta x$,
- f affine in $[7, 8]$, $x \mapsto \lambda\,(x - 8) + 8$,
- Fix: $D^+ = [\beta^{-1}, 1]$, $D^- = [7, 8 - \lambda]$,
- $f^N(D^+) = D^-$. $F = f^N$ **transition**.

To determine the dynamics one studies the **returns** of points to D^+ by iterations of f_t.

Quotient dynamics: After identifying the stable and unstable directions

<u>**(Quotient) Family of returns to D^+**</u>:
a one-parameter family of system of iterated functions (∞-many maps) generated by:

- two affine maps (dynamics at Q and P),

$$\boxed{x \mapsto \beta x} \qquad \boxed{x \mapsto \lambda (x-8)+8}$$

- a transition $\boxed{\text{the map } F}$,

- a translation (induced by the parameter t after reparametrization).

- **Step 1:** Transition
- **Step 2:** Affine iterates

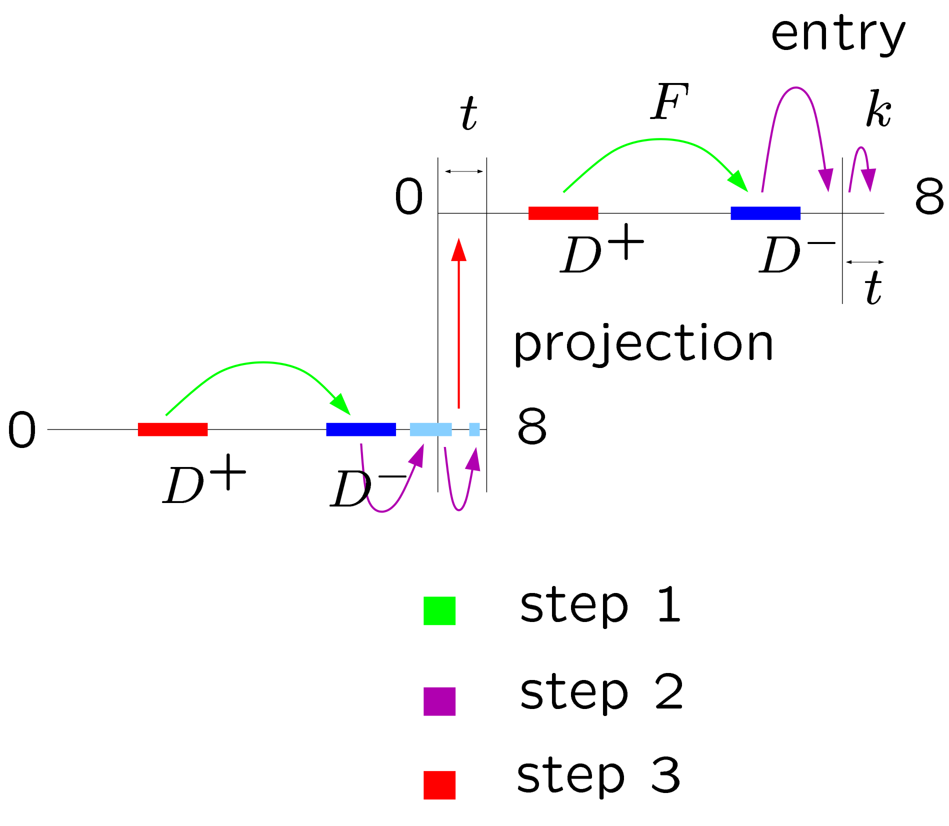
Part 1: approx. to $P=8$ at distance t (entry iterates $\lambda^{\text{entry}} \simeq t$),
Part 2: further iterates ($k \geq 0$).

Steps 1–2: $\boxed{\lambda^{\text{entry}+k}(F(x)-8)+8}$

- **Step 3:** effect of the parameter t (acting as a translation) and projection

Steps 1–3: $\boxed{\boxed{(\lambda^{\text{entry}+k}(F(x)-8)+8)}\ +t-8}$

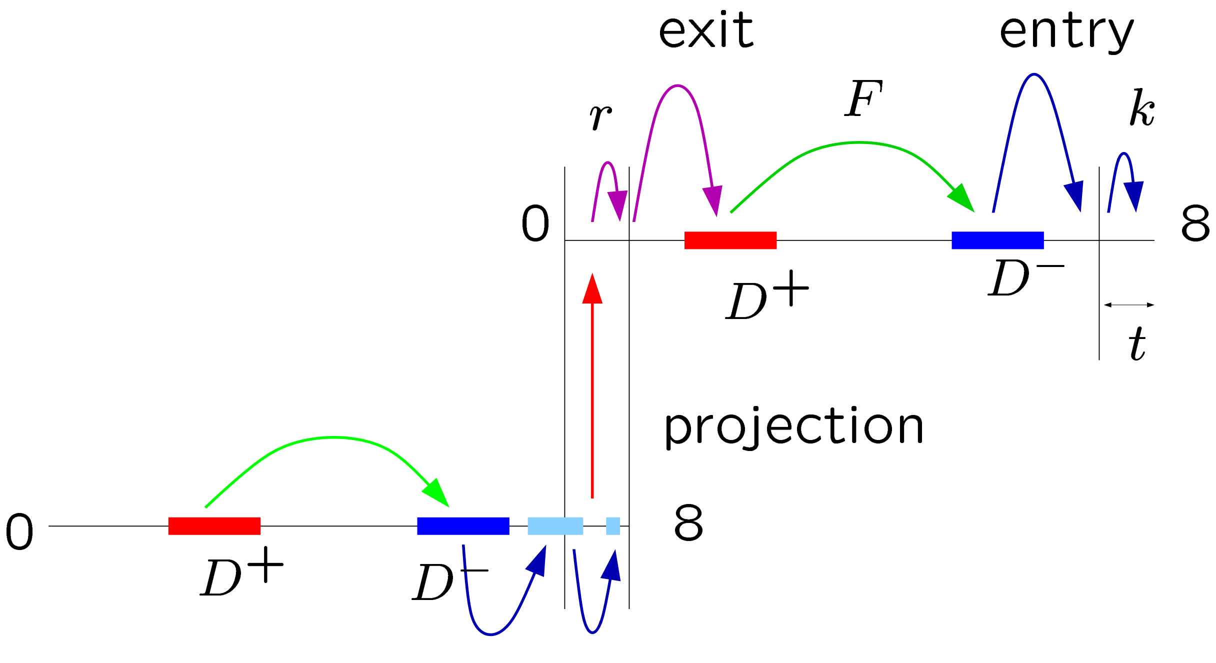
Steps 1–3: $\boxed{\boxed{\lambda^{\text{entry}+k}(F(x)-8)}\ +t}$



• **Step 4:** Linear iterates

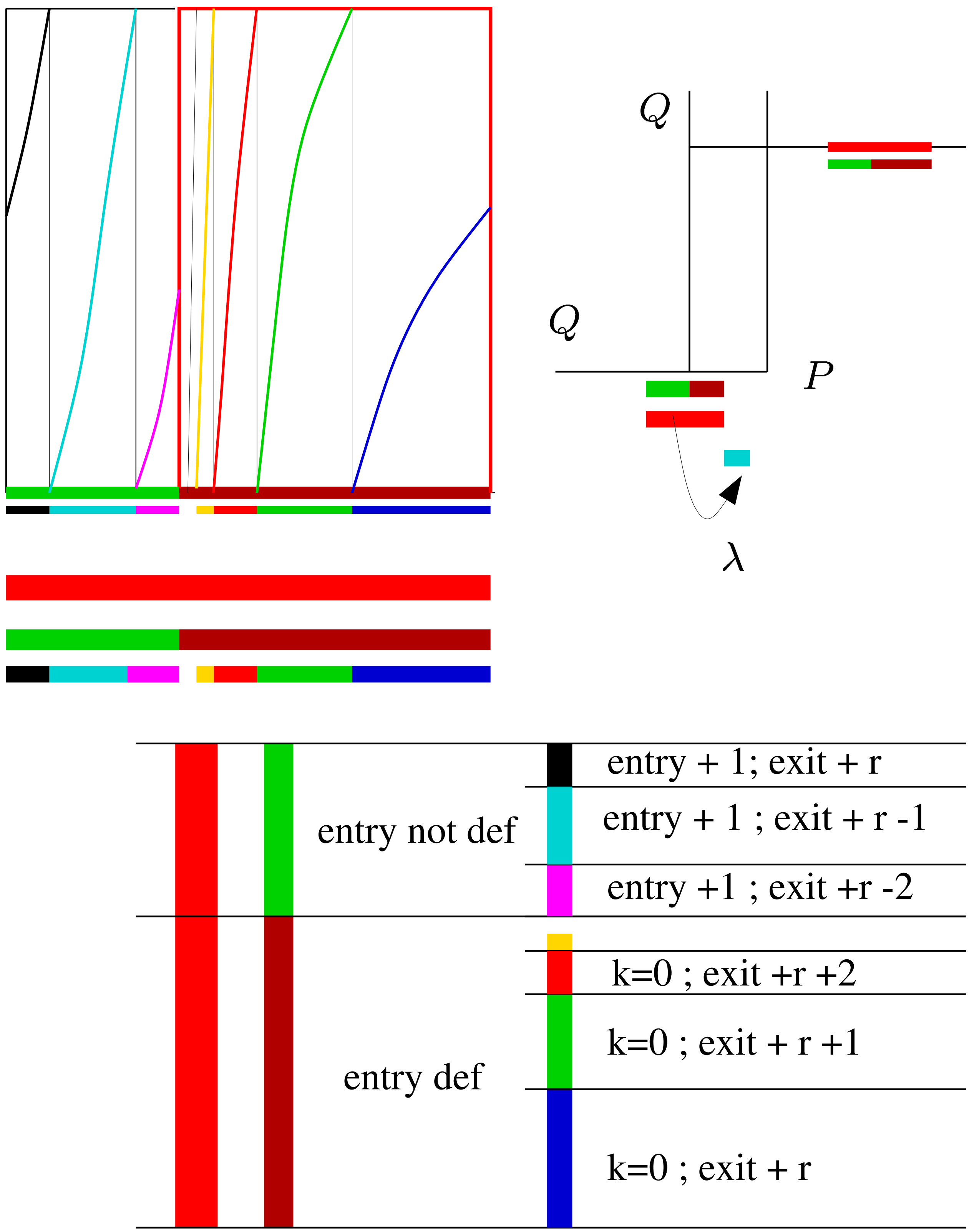
Part 1: Separation at distance t from $Q = 0$

Part 2: Return to D^+, entry iterates $\beta^{\text{entry}} \simeq 1/t$.



Steps 1–4:

$$\boxed{\beta^{\text{exit}+r}\left(\ \boxed{\lambda^{\text{entry}+k}(F(x)-8)+t}\ \right)}$$



Q
Q
P
λ
entry + 1; exit + r
entry not def
entry + 1 ; exit + r -1
entry +1 ; exit +r -2
k=0 ; exit +r +2
k=0 ; exit + r +1
entry def
k=0 ; exit + r

Special simplified case:

$$\lambda^{\text{entry}} = t, \qquad \beta^{\text{exit}} = 1/t$$

Returns for the parameter t (of type k, r):

$$\boxed{\beta^r(\lambda^k(F(x) - 8)) + 1}$$

Re-scaling effect: $s = t - \mu$, small $\mu > 0$.

One-param. family iterated functions $R_\mu^{k,r}$

$$\boxed{R_\mu^{k,r}(x) = \beta^r\,(\ \lambda^k(F(x) - 8)) + 1 + \beta^r \mu}$$

Remark: k is free and $r = r(n, x)$.
Typically: $k \to \infty$, $r(n, x) = 0$.
$k = 0$, there are x with $r(n, x)$ arbitrarily big.

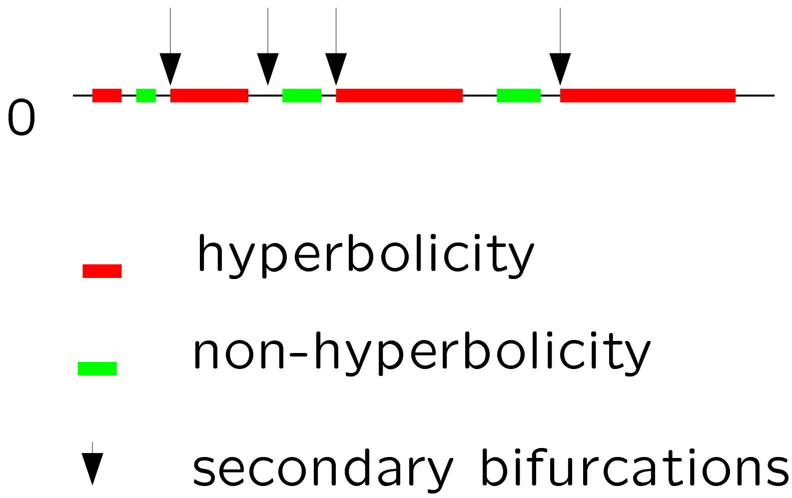
To get $H(P, f_t) = H(Q, f_t)$:

$$\boxed{R_\mu^{0,r},\ R_\mu^{1,r} \text{ uniformly expanding}}$$

Thm B: F has a big distortion.

(IV) COLLISIONS OF HOMOCLINIC CLASSES

Next step: to understand the transition from hyperbolic to bifurcating intervals (Theorems A and B)

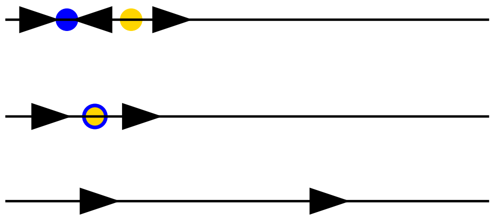


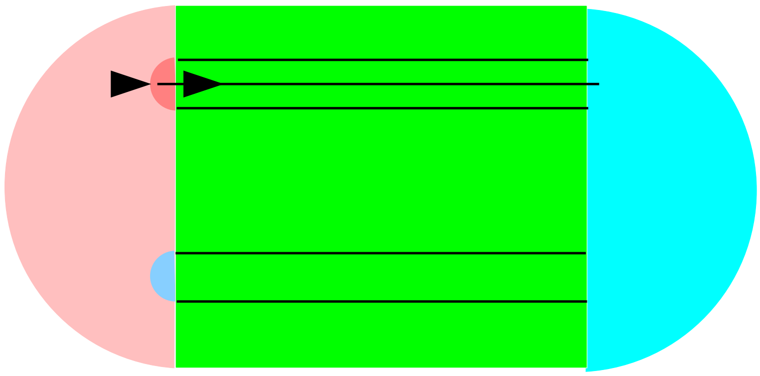
Secondary bifurcations:

- new heterodimensional cycles;
- loss of hyperbolicity of some saddle (<u>saddle-node bifurcations</u>)
- (others ?)

Interesting: the saddle is <u>involved in the dynamics</u>

- saddle-node cycles and horseshoes;





- derived from Anosov;

- <u>collisions of homoclinic classes</u>

dimension two: derived from Anosov, saddle-node horeseshoes, always a class is trivial

$\boxed{\text{Goal}}$ collisions of hyperb. homoclinic classes (at heterodimensional cycles)

__**Model (product structure)**__ (D. Rocha):

the arc $(f_t)_{t\in[-1,1]}$ unfolds a (partially hyperbolic) heterodimensional cycle at $t = 0$ such that there are sequences $(t_n), (t_n^+), (t_n^-)$, $t_n, t_n^{\pm} \to 0^+$, with

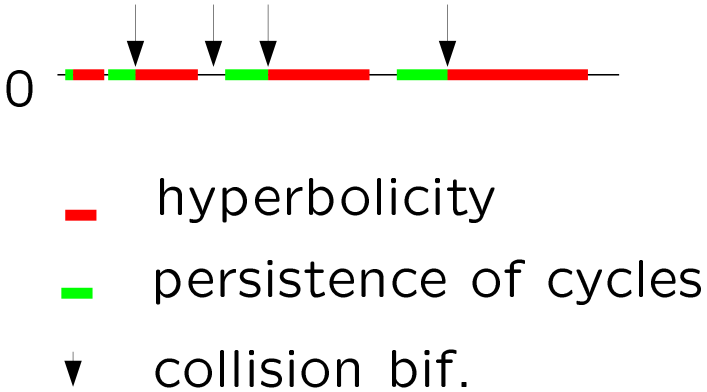
• f_t is hyperbolic if $t \in (t_n, t_n^+)$, (in particular $H(P, f_t)$ and $H(Q, f_t)$ are hyperbolic (thus disjoint)),

• for $t = t_n$ there is a **collision of the homoclinic classes** of P and Q:
$H(P, f_{t_n}) \cap H(Q, f_{t_n}) \neq \emptyset$ (the orbit of a saddle-node)

• there is **persistence of heterodimensional cycles** in (t_n^-, t_n^+)

__Persistence of (heterodimensional) cycles__ (in the open interval I): there is a dense subset of I such that every f_t has a heterodimensional cycle.

• the intervals (t_n^-, t_n) and (t_n, t_n^+) are both prevalent are the bifurcation.



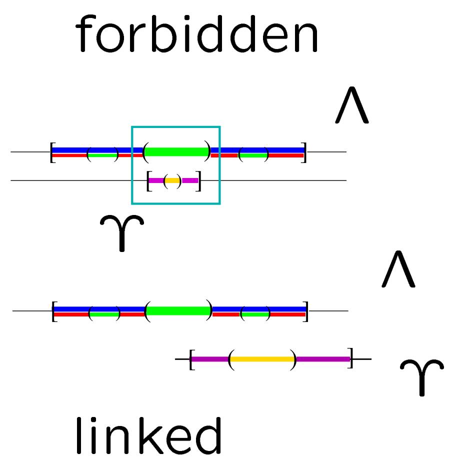
<u>Ingredients</u>:

• construct F such that $R_0^{0,0}$ unfolds a saddle-node;

• creation of arbitrarily thick hyperbolic sets Υ (of index 2) immediately before the unfolding of the saddle-node (the unfolding is for decreasing t):
Υ is dynamically defined, associated to the system of iterated functions
(non-critical dynamics)

• the homoclinic class of P contains hyperbolic sets Λ with thicknes uniformly bounded from zero.

• **Gap Lemma**:
$\tau(\Lambda) \cdot \tau(\Upsilon) > 1$ and each set is not contained in a gap of the other one, then $\Lambda \cap \Upsilon \neq \emptyset$.



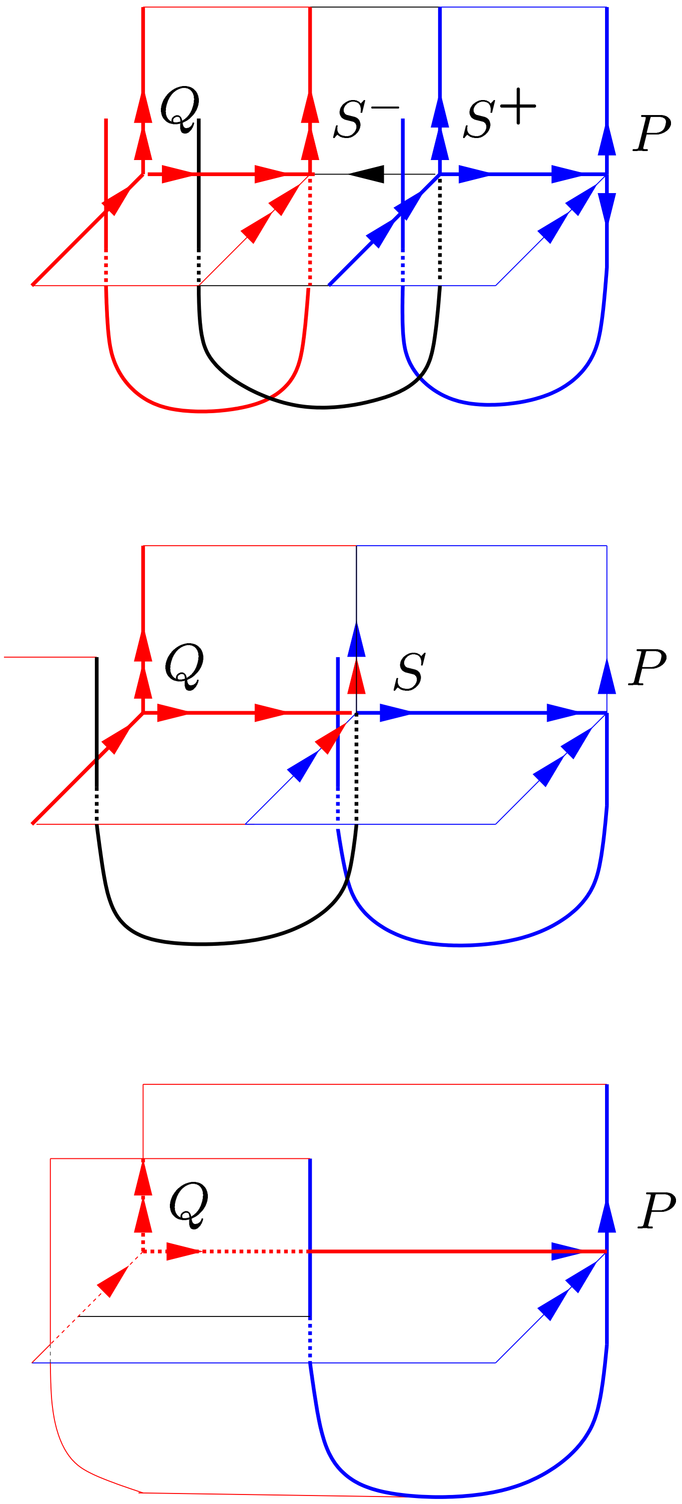
• the product dynamics gives a cycle associated to Υ and Λ.

Remark: (D. Santoro). There is a model family for the collision of partially hyperbolic (non-hyp.) homoclinic classes $H(P)$ and $H(Q)$ leading to the collapse of the classes:

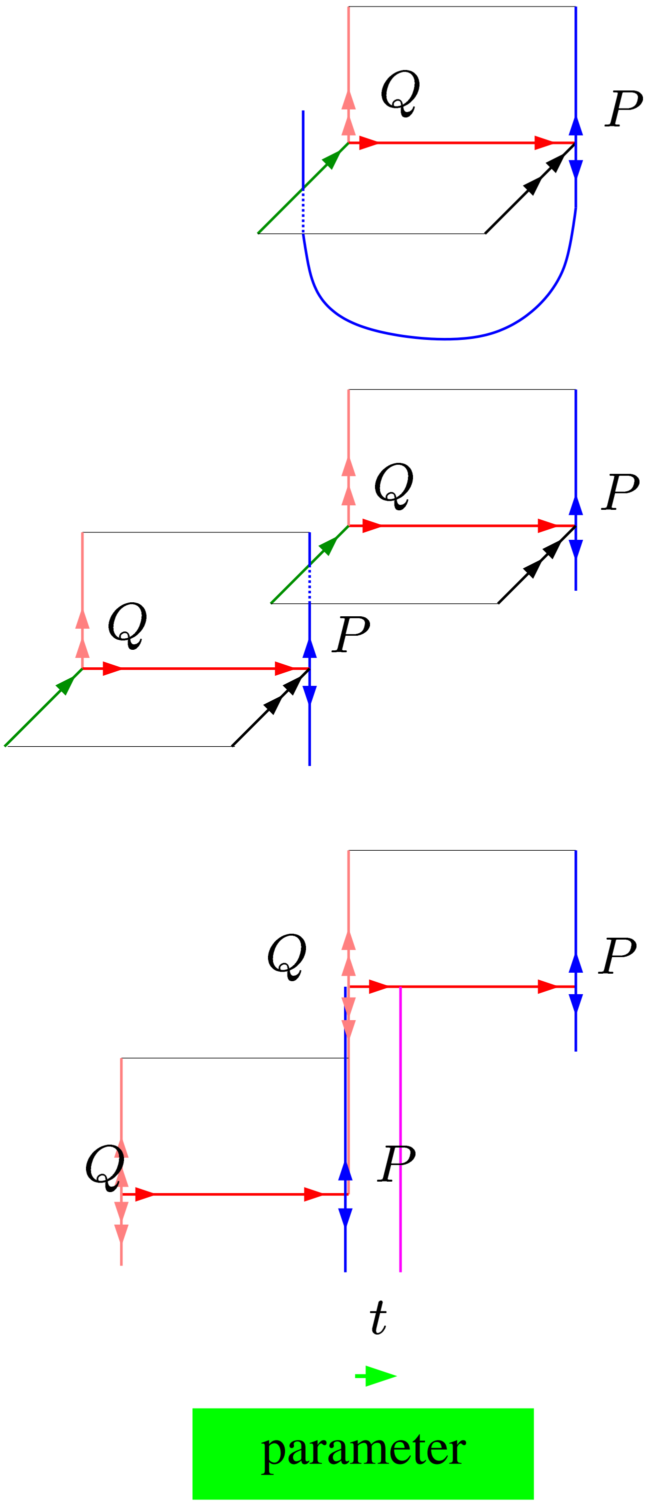
$$\boxed{H(P) = H(Q) \text{ after the collision}}$$

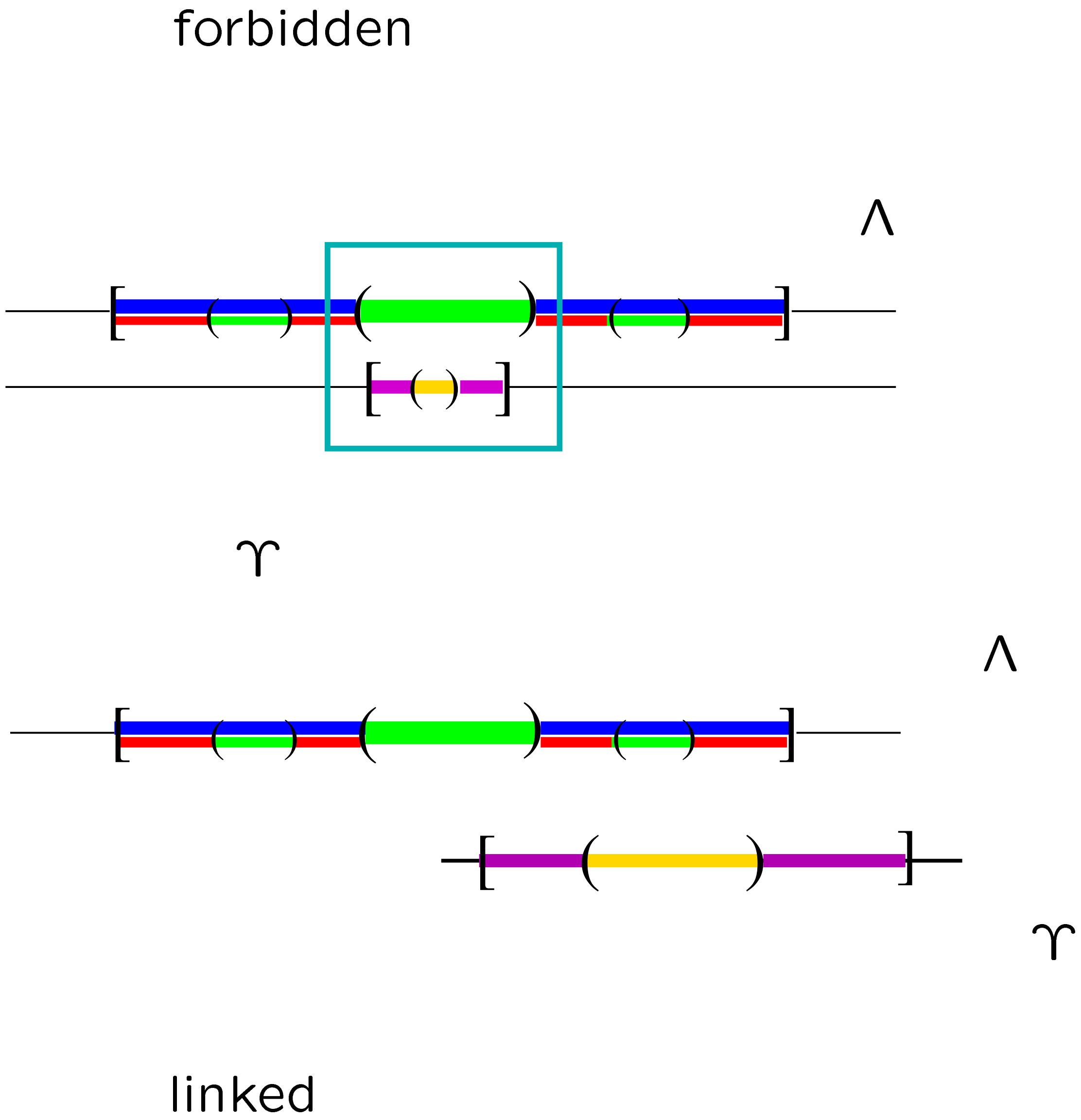
Methods for the collapse $\neq$.

collision of non-hyperbolic homoclinic classes



model cycle (one-dim reduction)

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