Regular level sets of averages of Nemytskiĭ operators are contractible

Iaci Malta, Nicolau C. Saldanha and Carlos Tomei

Abstract: Let $H^1(S^1)$ be the space of periodic real functions with derivative in L^2 and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with no double roots. Then there is a diffeomorphism of $H^1(S^1)$ taking the set $Z = \{v \in H^1(S^1) | \int_{S^1} f(v(t))dt = 0\}$ to a hyperplane. In this paper we state and prove a general version of this example. We consider a Banach space V of functions from some manifold M to $\mathbb{R}^{n'}$ and a function $f : M \times \mathbb{R}^{n'} \to \mathbb{R}^n$: under suitable hypothesis, there is a homeomorphism of V taking $Z = \{v \in V | \int_M f(m, v(m))dm = 0\}$ to a closed subspace of codimension n.

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Let $H^1(S^1)$ be the space of periodic real functions with derivative in L^2. The set $Z = \{v \in H^1(S^1) | \int_{S^1} v^2(t)dt = 1\}$, at first sight, looks like a sphere: infinite dimensional topology (for which a good reference is [Ku]) tells us that the unit sphere is diffeomorphic to a hyperplane in Hilbert space. Actually, there is a diffeomorphism of $H^1(S^1)$ taking Z to a hyperplane. In this paper, we present a generalization of this example.

Let M be a compact manifold with a smooth Riemannian metric inducing a measure μ with $\mu(M) = 1$. Let $C^\infty(M)$ be the Fréchet ring of smooth real valued functions on M. Set V to be a separable Banach space continuously included in $C^0(M)$ which is also a topological $C^\infty(M)$-module (i.e., multiplication is continuous). Given a continuous function $f_n : M \times \mathbb{R} \to \mathbb{R}^n$, define $F_n : V \to \mathbb{R}^n$ to be the average of the related Nemytskiĭ operator: $F_n(v) = \int_M f_n(m, v(m))d\mu$. We further request that f_n admits continuous partial derivatives of all orders with respect to the second variable, whence F_n is smooth.

Let $\Pi_k : \mathbb{R}^n \to \mathbb{R}^k$ be the projection to the first k coordinates. We say 0 is a *strong regular value* of F_n if it is a regular value of the composition $F_k = \Pi_k \circ F_n$ for all k, $1 \le k \le n$. From now on, assume 0 to be a strong regular value of F_n. Since the ranges of F_k are finite dimensional ([L]), the levels $Z_k = F_k^{-1}(0)$ are nested closed manifolds of codimension k in V.

Theorem: *The levels Z_k are contractible. Furthermore, there is a global homeomorphism Ψ of V taking each Z_k to a closed linear subspace of codimension k; Ψ can be taken to be a diffeomorphism if V is a Hilbert space.*

For a function $g : A_1 \times A_2 \to B$ we write $D_2 g$, say, for the derivative with respect to the second variable; thus, if A_2 and B are vector spaces, $D_2 g$ goes from $A_1 \times A_2$ to $\mathcal{L}(A_2, B)$. The vector spaces $\mathbb{R}^k$ always receive the Euclidean norm and, for a finite matrix A, $\|A\| = \max_{|x|=1} |Ax|$.

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Lemma 1: *Let X be a topological space and $B^k(R)$ be the open ball of radius R around the origin in $\mathbb{R}^k$. Let $g : X \times B^k(R) \to \mathbb{R}^k$ be a continuous function, smooth in the second variable, satisfying $|g(x,0)| < R/2$ and $\|D_2 g - I\| < 1/2$. Then there exists a unique continuous function $h : X \to B^k(R)$ such that $g(x, h(x)) = 0$ for all $x \in X$.*

This result is a variation of the implicit function theorem and its proof is similar.

Lemma 2: *There are continuous functions $p_i : Z_k \to V$, $i = 1, \ldots, k$ forming a basis of a complement of the tangent space $T_z Z_k$ for all $z \in Z_k$ and such that the derivative $D_2 \mathcal{F}_k$ of the function*

$$\mathcal{F}_k : Z_k \times \mathbb{R}^k \to \mathbb{R}^k$$

$$(z, a) \mapsto F_k(z + \sum_i a_i p_i(z))$$

with respect to the second variable is the $k \times k$ identity matrix at all points of the form $(z, 0)$.

Proof: At any point z of Z_1 there is a vector $q_1(z)$ off the tangent space $T_z Z_1$; $q_1(z)$ can be chosen so that $DF_1(z) \cdot q_1(z) = 1$ (notice that there is no reason for q_1 to be continuous). Since F is smooth, there is a neighbourhood $W_z \subseteq Z_1$ of z such that $1/2 < DF_1(z') \cdot q_1(z) < 2$ for $z' \in W_z$. From the paracompactness of Z_1 ([L]), we can pick a locally finite refinement W_{z_λ}, $\lambda \in \Lambda$, of this covering of Z_1 and an associated continuous partition of unity Ξ_λ. Let $\tilde{q}_1(z) = \sum_{\lambda \in \Lambda} \Xi_\lambda(z) q_1(z_\lambda)$: it satisfies $1/2 < DF_1(z) \cdot \tilde{q}_1(z) < 2$.

We now construct a continuous $\tilde{q}_2 : Z_2 \to V$ such that the vectors $\tilde{q}_1$ and $\tilde{q}_2$ span a complement of $T_z Z_2$. Again, there is a vector $q_2(z)$ at each $z \in Z_2$ such that $\tilde{q}_1$ and q_2 span a complement. We can even pick q_2 so that the determinant of the real 2×2 matrix with columns $DF_2(z) \cdot \tilde{q}_1(z)$ and $DF_2(z) \cdot q_2(z)$ is 1. A similar construction with partitions of unity yields $\tilde{q}_2$ such that $1/2 < \det(DF_2(z) \cdot \tilde{q}_1(z), DF_2(z) \cdot \tilde{q}_2(z)) < 2$; notice that the determinant is linear in the second column, the first being fixed. Inductively, we construct $\tilde{q}_i$, $i = 1, \ldots, k$ and thus have a continuous basis for a complement of $T_z Z_k$. The derivative of F_k at z in this basis restricted to this complement is a continuous $k \times k$ invertible transformation $A(z)$: the pull-back of the canonical basis of $\mathbb{R}^k$ under $A(z)$ gives us the required basis $p_i(z)$, $i = 1, \ldots, k$. ■

Lemma 3: *The sets Z_k are path-connected and the homotopy groups $\pi_r(Z_k)$, $r = 1, 2, \ldots$, are trivial.*

This lemma is the technical core of the paper and an informal description of the proof may be helpful. We must connect $z \in Z_k$ to a fixed base point $z_0 \in Z_k$: first decompose M in a large number L of roughly uniformly distributed sets W_ℓ, $\ell = 1, \ldots, L$. At time steps $1/L, 2/L, \ldots, 1$, substitute the original value of z by the desired value, prescribed by z_0, in the sets $W_1, W_2, \ldots, W_L$. Since the required restrictions defining Z_k are given by integrals, the resulting path should not deviate much from Z_k. The main difficulty lies in controlling the error so that the path can be pulled back to Z_k uniformly on compact families of z's.

Proof: Let $h : S^r \to Z_k$ be a continuous map, $s_0 \in S^r$ be a base point and $z_0 = h(s_0)$. We construct a homotopy $H : S^r \times [0,1] \to Z_k$ of $H(0) = h$ to the constant map

$H(1) = h_1 : S^r \to Z_k$, $h_1(s) = z_0$; the case $r = 0$ corresponds to path-connectedness. Set $f_k = \Pi_k \circ f$.

From Lemma 2, $D_2\mathcal{F}_k(z, 0) = I$ for $z \in Z_k$. Set $\epsilon > 0$, $\epsilon < 1/8$ such that $\|D_2\mathcal{F}_k(h(s), a) - I\| < 1/4$ for all $s \in S^r$ and all $a \in \mathbb{R}^k$, $|a| < 4\epsilon$. Let C be such that

$$|h(s)(m)| < \frac{C}{2},\ |p_i(h(s))(m)| < \frac{C}{2k},$$

for all $s \in S^r$, $m \in M$ and $i = 1, \dots, k$. Let $L > 8$ be an integer satisfying

$$|f_k(m, c)| < \frac{L\epsilon}{4},\ |D_2 f_k(m, c)| < \frac{L\epsilon}{4C},$$

for all $c \in \mathbb{R}$, $|c| < C$ and $m \in M$. By uniform continuity, take $\epsilon_1, \delta_1 > 0$ such that

$$d(m, m') < \delta_1,\ |c - c'| < \epsilon_1,\ |c|, |c'| < C \Rightarrow \begin{cases} |f_k(m, c) - f_k(m', c')| < \epsilon/8, \\ |D_2 f_k(m, c) - D_2 f_k(m', c')| < \epsilon/8C. \end{cases}$$

Take $\delta > 0$, $\delta < \delta_1$, such that

$$d(m, m') < \delta \Rightarrow \begin{cases} |h(s)(m) - h(s)(m')| < \epsilon_1/2, \\ |p_i(h(s))(m) - p_i(h(s))(m')| < \min\{\epsilon_1/2k, C/4kL\}, \end{cases}$$

for all $s \in S^r$, $m, m' \in M$.

Decompose $M = \bigcup_{j=1,\dots,J} \overline{U_j}$ into disjoint open sets U_j of diameter less than $\delta/2$: for example, take a finite set $\{m_1, \dots, m_J\}$ whose complement contains no balls of radius $\delta/4$, and define $U_j = \{m \in M | j \neq j' \Rightarrow d(m, m_j) < d(m, m_{j'})\}$ (i.e., the Voronoi cells associated to $\{m_1, \dots, m_J\}$). Split $\overline{U_j} = \bigcup_{\ell=1,\dots,L} \overline{U_{j\ell}}$ into disjoint open sets $U_{j\ell}$ of equal measure. Roughly, the homotopy H replaces h by h_1 inside $W_\ell = \bigcup_{j=1,\dots,J} U_{j\ell}$ in the time interval $[(\ell - 1)/L, \ell/L]$.

Choose $\zeta > 0$ such that

(a) for all j and ℓ, $\mu(U_{j\ell}^{+\zeta}) < (1 + \frac{1}{8L})\mu(U_{j\ell})$, where $U_{j\ell}^{+\zeta} = \{m \,|\, d(m, U_{j\ell}) < \zeta\}$,

(b) for all j and ℓ, $\mu(U_{j\ell}^{-\zeta}) > (1 - \frac{1}{8L})\mu(U_{j\ell})$, where $U_{j\ell}^{-\zeta} = U_{j\ell} - \bigcup_{j' \neq j \text{ or } \ell' \neq \ell} U_{j'\ell'}^{+\zeta}$.

Denote by $\phi_{j\ell}$ a smooth partition of unity associated to the finite covering $\{U_{j\ell}^{+\zeta}\}$ so that $\phi_{j\ell}$ is 0 outside $U_{j\ell}^{+\zeta}$ and 1 in $U_{j\ell}^{-\zeta}$. We now construct a path $\tilde{\phi}$ of smooth functions from M to $[0, 1]$ joining the constant functions 0 and 1, for which intermediate functions are equal to 0 or 1 on most of M and their average inside large open sets is roughly t at time t:

$$\tilde{\phi}(t) = \sum_{\substack{j=1,\dots,J \\ \ell=1,\dots,\tilde{\ell}-1}} \phi_{j\ell} + \tilde{t} \sum_{j=1,\dots,J} \phi_{j\tilde{\ell}},$$

where $t \in [(\tilde{\ell} - 1)/L, \tilde{\ell}/L]$ and $\tilde{t}$ is a linear interpolation going from 0 to 1 as t goes from $(\tilde{\ell} - 1)/L$ to $\tilde{\ell}/L$. Define $\tilde{H}(t)(s) = h(s) + \tilde{\phi}(t)(h_1(s) - h(s))$: this gives a smooth deformation from h to h_1 which at intermediate times take us only slightly away from Z_k. Notice that, by convexity, $|\tilde{H}(t)(s)(m)| < C/2$ for all t, s and m.

Claim: $|F_k(\tilde{H}(t)(s))| < \epsilon$.

In an obvious notation,

$$\begin{aligned}
F_k(\tilde{H}(t)(s)) &= \left(\int_{\bigcup_{\ell<\tilde{\ell}} W_\ell} + \int_{W_{\tilde{\ell}}} + \int_{\bigcup_{\ell>\tilde{\ell}} W_\ell}\right)\left(f_k(m, \tilde{H}(t)(s)(m))d\mu\right) \\
&= A_< + A_= + A_>.
\end{aligned}$$

Clearly $|A_=| < \epsilon/4$ since the domain of integration has measure $1/L$ and $|f_k| < L\epsilon/4$. Also,

$$\begin{aligned}
A_< &= \sum_{j=1,\ldots,J}\left(\int_{\bigcup_{\ell<\tilde{\ell}} U_{j\ell}} f_k(m, \tilde{H}(t)(s)(m))d\mu\right) \\
&= \sum_{j=1,\ldots,J}\left(\int_{\bigcup_{\ell<\tilde{\ell}} U_{j\ell}} f_k(m, h_1(s)(m))d\mu + E_{1j}\right) \\
&= \sum_{j=1,\ldots,J}\left(\frac{(\tilde{\ell}-1)}{L}\int_{U_j} f_k(m, h_1(s)(m))d\mu + E_{2j} + E_{1j}\right) \\
&= \sum_{j=1,\ldots,J}(E_{2j} + E_{1j}),
\end{aligned}$$

where the last integrals vanish since $h_1(s) \in Z_k$ and we are left with estimating the errors E_{1j} and E_{2j}. The functions $\tilde{H}(t)(s)$ and $h_1(s)$ coincide in $\bigcup_{\ell<\tilde{\ell}} U_{j\ell}^{-\zeta}$ and we therefore have

$$|E_{1j}| < 2\frac{L\epsilon}{4}\frac{1}{8L}\mu(U_j),$$

since $\mu(\bigcup_{\ell<\tilde{\ell}} U_{j\ell}^{-\zeta}) > (1 - 1/8L)\mu(\bigcup_{\ell<\tilde{\ell}} U_{j\ell})$ (by (b)) and $|f_k| < L\epsilon/4$; adding in j, $\sum_j |E_{1j}| < \epsilon/16$. For $m \in U_j$, $f_k(m, h_1(s)(m))$ differs by at most $\epsilon/8$ from $f_k(m_j, h_1(s)(m_j))$, for a fixed but arbitrary $m_j \in U_j$. Thus,

$$\begin{aligned}
|E_{2j}| &= \left|\int_{\bigcup_{\ell<\tilde{\ell}} U_{j\ell}} f_k(m, h_1(s)(m))d\mu - \frac{(\tilde{\ell}-1)}{L}\int_{U_j} f_k(m, h_1(s)(m))d\mu\right| \\
&\leq \left|\int_{\bigcup_{\ell<\tilde{\ell}} U_{j\ell}} (f_k(m, h_1(s)(m)) - f_k(m_j, h_1(s)(m_j))d\mu\right| \\
&\quad + \frac{(\tilde{\ell}-1)}{L}\left|\int_{U_j} (f_k(m, h_1(s)(m)) - f_k(m_j, h_1(s)(m_j))d\mu\right| \\
&< \frac{\epsilon}{4}|U_j|
\end{aligned}$$

and $\sum_j |E_{2j}| < \epsilon/4$. Summing up, $|A_<| < 5\epsilon/16$.

On the other hand, since in most of the domain of integration of $A_>$ the functions $\tilde{H}(t)(s)$ and $h(s)$ coincide, similar estimates yield $|A_>| < 5\epsilon/16$. □

We now show how to correct $\tilde{H}$ to obtain the desired homotopy H with values in Z_k.

Let $\tilde{p}(t,s)$ be the k-tuple of functions $\tilde{p}_i(t,s) = p_i(h(s)) + \tilde{\phi}(t)(p_i(h_1(s)) - p_i(h(s)))$ so that $|\tilde{p}_i(t,s)(m)| < C/2k$ for all t, s and m; similarly, we denote by p the k-tuple of functions p_i. Define

$$\begin{aligned}\tilde{\mathcal{F}}_k : [0,1] \times S^r \times B^k(4\epsilon) &\to \mathbb{R}^k. \\ (t,s,a) &\mapsto F_k(\tilde{H}(t)(s) + \sum_i a_i \tilde{p}_i(t,s))\end{aligned}$$

Claim: $\|D_3\tilde{\mathcal{F}}_k(t,s,a) - I\| < 1/2$, *for all* $t \in [0,1]$, $s \in S^r$, $a \in B^k(4\epsilon)$.

For convenience, set

$$\begin{aligned}\alpha(m, v_0, w) &= D_2 f_k \left(m, v_0(m) + \sum_i a_i w_i(m) \right) \\ \beta(m, w) &= \sum_i b_i w_i(m),\end{aligned}$$

where b is an arbitrary vector. Again, split M as in the previous claim to get

$$\begin{aligned}D_3\tilde{\mathcal{F}}_k(t,s,a) \cdot b &= \int_M \alpha(m, \tilde{H}(t)(s), \tilde{p}(t,s)) \beta(m, \tilde{p}(t,s)) d\mu \\ &= (A'_< + A'_= + A'_>) \cdot b.\end{aligned}$$

Recall that $t \in [(\tilde{\ell}-1)/L, \tilde{\ell}/L]$. The domain of integration of $A'_=$ has measure $1/L$ and, for b of norm 1, the integrand is bounded by $\frac{L\epsilon}{4C}\frac{C}{2}$, yielding $\|A'_=\| < \epsilon/8$.

Also,

$$\begin{aligned}A'_< \cdot b &= \sum_{j=1,\ldots,J} \int_{\bigcup_{\ell<\tilde{\ell}} U_{j\ell}} \alpha(m, H(t)(s), \tilde{p}(t,s)) \beta(m, \tilde{p}(t,s)) d\mu \\ &= \sum_{j=1,\ldots,J} \left(\int_{\bigcup_{\ell<\tilde{\ell}} U_{j\ell}} \alpha(m, h_1(s), p(h_1(s))) \beta(m, p(h_1(s)) d\mu + E'_{1j} \cdot b \right) \\ &= \sum_{j=1,\ldots,J} \left(\frac{(\tilde{\ell}-1)}{L} \int_{U_j} \alpha(m, h_1(s), p(h_1(s))) \beta(m, p(h_1(s)) d\mu + E'_{2j} \cdot b + E'_{1j} \cdot b \right) \\ &= \frac{(\tilde{\ell}-1)}{L} D_2\mathcal{F}_k(h_1(s), a) \cdot b + \sum_{j=1,\ldots,J} (E'_{2j} \cdot b + E'_{1j} \cdot b),\end{aligned}$$

and thus

$$\|A'_< - \frac{\tilde\ell-1}{L}I\| \le \frac{\tilde\ell-1}{L}\|D_2\mathcal{F}_k(h_1(s),a)-I\| + \sum_j \left(\|E'_{1j}\|+\|E'_{2j}\|\right).$$

Since $\|D_2\mathcal{F}_k(h_1(s),a)-I\| < 1/4$ we are again left with estimating the errors E'_{1j} and E'_{2j}. The integrands in the first and second lines coincide in $\bigcup_{\ell<\tilde\ell} U_{j\ell}^{-\zeta}$ and therefore

$$|E'_{1j}\cdot b| < 2\frac{L\epsilon}{8}\frac{1}{8L}\mu(U_j),$$

the bound on the integrand being as above for $A'_=$; adding in j, $\sum_j \|E'_{1j}\| < \epsilon/32$. For $m, m' \in U_j$ and $|b|=1$, the integrand for m differs by at most $\epsilon/8$ from the integrand for m'. Indeed,

$$\begin{aligned}
&|\alpha(m,h_1(s),p(h_1(s)))\beta(m,p(h_1(s)) - \alpha(m',h_1(s),p(h_1(s)))\beta(m',p(h_1(s))| \le \\
&\le |\alpha(m,h_1(s),p(h_1(s))) - \alpha(m',h_1(s),p(h_1(s)))|\,|\beta(m,p(h_1(s)))| + \\
&\qquad + |\alpha(m',h_1(s),p(h_1(s)))|\,|\beta(m,p(h_1(s))) - \beta(m',p(h_1(s)))| < \\
&< \frac{\epsilon}{8C}\frac{C}{2} + \frac{L\epsilon}{4C}k\frac{C}{4Lk} = \epsilon/8.
\end{aligned}$$

Hence, $\|E_{2j}\| < \frac{\epsilon}{4}|U_j|$ and the rest of the proof proceeds as for the first claim. □

From Lemma 1, we can solve the equation $\tilde{\mathcal{F}}_k(t,s,a)=0$ in a, uniquely and continuously in t and s. For such $a(t,s)$, set $H(t)(s) = \tilde H(t)(s) + \sum_i a_i(t,s)\tilde p_i(t,s)$; this completes the proof of the lemma. ■

In order to prove our main theorem, we need a couple of known results.

Proposition 1: *Given a contractible connected smooth submanifold H' of codimension 1 of a separable Hilbert space H of infinite dimension, there is a diffeomorphism of H to itself taking H' to a closed subspace of codimension 1.*

The proof of this proposition is entirely similar to the one given in [S] for the finite dimensional situation, $dim(H) \ge 4$.

Following [BH], we define a *topological slicing submanifold* of a separable infinite-dimensional Hilbert space H to be a closed bicollared topological submanifold Z of codimension 1 in H such that the complement of Z has two connected components. We now state Proposition 1.7 in [BH] suitably restricted to our needs.

Proposition 2: *Let H be a separable infinite-dimensional Hilbert space and let Z be a topological slicing manifold in H. Then there exists a homeomorphism of H taking Z to a smooth slicing manifold.*

Proof of the theorem: If V is a Hilbert space, the triviality of $\pi_r(Z_k)$ implies that Z_k is contractible ([Ku]). By induction, assume (after composing with a diffeomorphism) $Z_{k'}$, $k' < k$, to be closed subspaces (set $Z_0 = V$). By Proposition 1, there is a diffeomorphism of Z_{k-1} taking Z_k to a closed subspace of codimension 1 which extends by a cartesian product to a diffeomorphism of V taking $Z_{k'}$, $k' \leq k$, to closed spaces, thus proving the theorem in this case.

In general, if V is a Banach space, use the fact that all separable infinite dimensional Banach spaces are homeomorphic ([Ka]) to identify under a homeomorphism the space V with some Hilbert space H. Again by induction, after composition with a homeomorphism, $Z_{k'}$, $k' < k$, are closed subspaces of H (set $Z_0 = H$). Now, Z_k is a topological slicing submanifold of the Hilbert space Z_{k-1}. Indeed, strong regularity of F_n at 0 implies that Z_k is a closed bicollared topological submanifold of codimension 1. Also, the complement of Z_k in Z_{k-1} has two components: the sign of the k-th coordinate of F_n indicates a splitting of the complement in two open sets. These are connected: we can always join an arbitrary point in the complement to a tubular neighbourhood of Z_k by some path contained in the complement and Z_k, as well as its tubular neighbourhood, are known to be path-connected. By Proposition 2, we can assume Z_k to be smooth after a homeomorphism in Z_{k-1}, which again extends to a homeomorphism in H. We now see that Z_k is contractible and Proposition 1 gives us a diffeomorphism of Z_{k-1} (and thus of H) taking Z_k to a linear subspace, completing the proof. ■

Remarks:

1. A similar theorem holds for functions v from M to $\mathbb{R}^{n'}$: such an apparently more general result can be reduced to our case by substituting M for the cartesian product $M \times \{1, 2, \ldots, n'\}$, where each connected component will take care of a coordinate of v. Notice that there is no requirement that M be connected.

2. Very little of the manifold structure of M is used. Manifolds with boundary, for instance, can be handled with minor alterations of the proof. More generally, the hypothesis could be weakened at the price of more cumbersome statement and proof.

3. It is essential in this construction that μ must have no atoms. In particular, the theorem fails rather trivially if M is replaced by a finite set and the degenerate case of manifolds of dimension zero must be excluded.

4. Strong regularity is a necessary hypothesis. Consider $M = S^1$, $V = H^1(S^1)$, $n = 1$ and $f(m, x) = x^2 - x^3$; the reader may easily check that the constant function 0 is an isolated point of Z_1 which is therefore disconnected.

5. Recently, Church, Dancer and Timourian [CDT] made use of contractibility arguments to show that a differentiable operator is equivalent by change of variable to a global cusp in infinite dimensional space. The result in the present paper was motivated by our interest in proving similar global normal forms for other operators [MST].

References:

[BH] Burghelea, D. and Henderson, D., *Smoothings and homeomorphisms for Hilbert manifolds*, Bull. Amer. Math. Soc., 76, 1261–1265 (1970).

[CDT] Church, P. T., Dancer, E. N. and Timourian, J. G., *The structure of a nonlinear elliptic operator*, Trans. Amer. Math. Soc. 338, no. 1, 1–42 (1993).

[Ka] Kadec, M. I., *A proof of the topological equivalence of all separable infinite-dimensional Banach spaces*, Funkcional Anal. i Prilozen, 1, 61–70 (1967).

[Ku] Kuiper, N. H., *Variétés Hilbertiennes — Aspects Géométriques*, Publications du Séminaire de Mathématiques Supérieures, No. 38, Les Presses de l'Université de Montreal, 1971.

[L] Lang, S., *Differentiable Manifolds*, Addison-Wesley, Reading, MA, 1972.

[MST] Malta, I., Saldanha, N. C. and Tomei, C., *Morin Singularities and Global Geometry in a Class of Ordinary Differential Operators*, to appear.

[S] Stallings, J. R., *On Infinite Processes Leading to Differentiability in the Complement of a Point*, 245–254, in *Differentiable and Combinatorial Topology, a symposium in honor of Marston Morse*, ed. Cairns, S. S., Princeton University Press, Princeton, NJ, 1965.

Iaci Malta and Carlos Tomei, Departamento de Matemática, PUC-Rio
R. Marquês de S. Vicente 225, Rio de Janeiro, RJ 22453-900, Brazil

Nicolau C. Saldanha, IMPA
Estr. D. Castorina 110, Rio de Janeiro, RJ 22460-320, Brazil

e-mail: malta@mat.puc-rio.br, nicolau@impa.br, tomei@mat.puc-rio.br