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Two results on tilings of quadriculated annuli

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Abstract

We provide a more informal explanation of two results in our manuscript *Tilings of quadriculated annuli.* Tilings of a quadriculated annulus A are counted according to *volume* (in the formal variable q) and *flux* (in p). The generating function $\Phi_A(p,q)$ is such that, for $q=-1$, the non-zero roots in p are roots of unity and for $q>0$, real negative.

There is an unexpected rigidity for the -1-counting of tilings of quadriculated disks, as described in [1]:

Theorem 1 *Let D be a quadriculated disk. Then the determinant of the adjacency matrix of the squares of D equals -1, 0 or 1.*

In [4] we prove Theorem 3 below which, in a sense, extends this rigidity to annuli. In this shorter text, we present a more informal proof of this result.

1 Connectivity

We assume that the squares in a quadriculated annulus A are colored in a checkerboard pattern and that the numbers of black and white squares are equal. Without loss, A is embedded in the plane.

Let T_A be the set of domino tilings of A. Two tilings are *adjacent* if they differ by a *flip*, a 90° rotation of two dominoes filling a 2×2 square. Assign the counterclockwise (resp. clockwise) orientation to the boundary of white (resp. black) squares. A flip is *positive* (resp. *negative*) if in the original 2×2 square the two sides of squares from center to boundary not trespassing dominoes point outwards (resp. inwards), as in Figure 2. The attribution of signs to flips is *exact* in the sense any closed sequence of flips contains the same number of positive and negative flips (this follows easily from properties of height functions and sections as discussed in [5]; height functions were introduced by Thurston in [6]).

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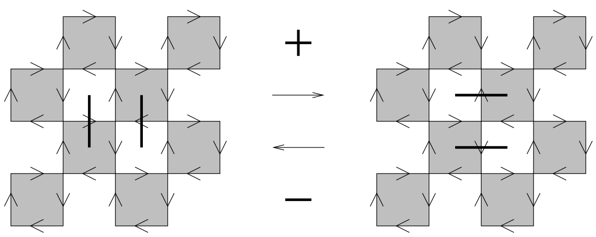

Figure 1: Positive and negative flips

When are two tilings joined by a sequence of flips? They clearly must have the same *flux* across a *cut*: in Figure 2, the two tilings have different fluxes. Indeed, flips don't alter flux.

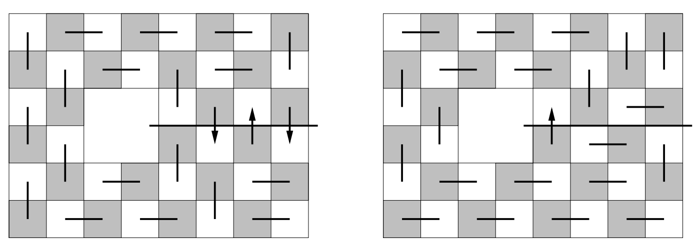

Figure 2: Tilings with counterclockwise fluxes -1 and 1

Something else may happen to prevent connectivity: the presence of *ladders* as in Figure 3, where annuli have been sliced open. Flips don't change ladders.

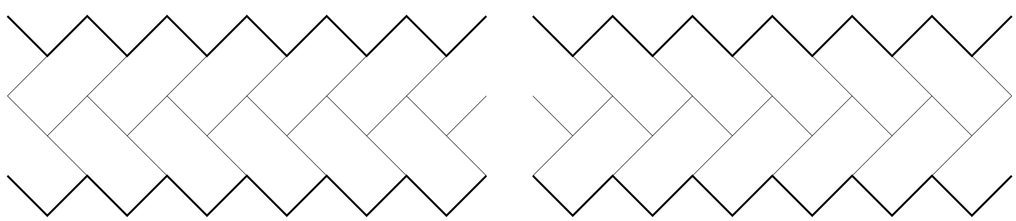

Figure 3: Ladders

Boundaries of ladders are *walls*. Tilings don't trespass walls. Thus, for the intent of studying tilings, walls decompose annuli into independent disks and narrow annuli (as in Figure 4). There is no real loss in considering, for the purposes of this paper, *wall-free annuli*. Fortunately, there are no other obstructions to connectivity:

Theorem 2 ([5]) *Let A be a wall-free quadriculated annulus. Two tilings of A can be joined by flips if and only if they have the same flux.*

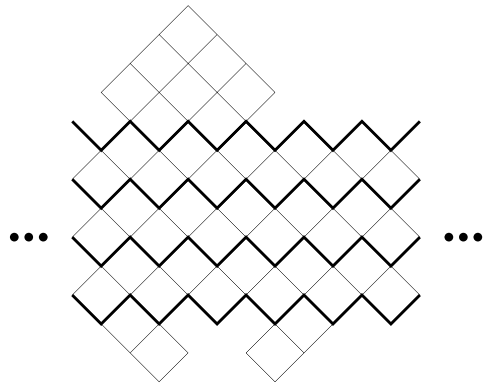

Figure 4: Walls

2 The q-flux polynomial

We will define the *volume* $\nu(t)$ of a tiling $t \in T_A$; volume increases by one when applying a positive flip. The q-flux polynomial $\Phi_A(p,q)$ will (p,q)-count tilings with respect to flux and volume.

Open the annulus A by a *cut* ξ in order to obtain a *track segment* Δ as in Figure 5; the annulus is obtained by identifying the vertical sides, which both correspond to the cut ξ. Draw a graph G_A whose vertices are centers of squares of A and whose edges join vertices of squares sharing a side. In $G_\Delta \subset G_A$ choose a maximal tree and assign weight 1 to its edges (solid segments in the figure). Assign weight p to an arbitrarily chosen edge in $G_A \setminus G_\Delta$ (i.e., an edge crossing the cut). Now consider a 2×2 square contained in A: say a positive flip on the square takes edges e_0, e_1 to edges e_2, e_3. The weights w_0, w_1, w_2, w_3 assigned to these four edges must satisfy $w_2 w_3 = -q w_0 w_1$: these requirements uniquely determine weights on all edges of G_A.

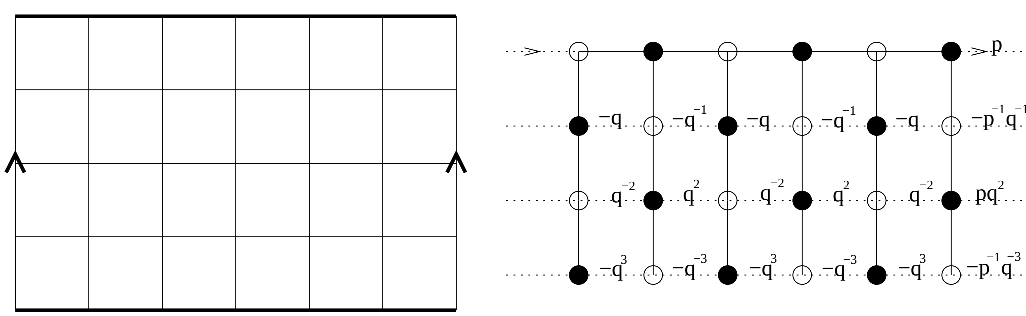

Figure 5: Annulus and Kasteleyn weights

For each tiling $t \in T_A$, the *Kasteleyn weights* above obtain a signed monomial $\pm p^{\phi(t)} q^{\nu(t)}$ by multiplying the weights of the edges asociated to dominoes in t. By construction, a positive flip preserves the *flux* ϕ and increases the *volume* ν by one. This also shows how to define the volume of a tiling of a quadriculated disk (there is no flux variable).

Why are the weights signed? So that we can construct a *Kasteleyn matrix* M_A whose determinant, the *q-flux polynomial*, (p, q)-counts tilings, i.e.,

$$\Phi_A(p,q) = \sum_{t \in T_A} p^{\phi(t)} q^{\nu(t)} = \pm \det(M_A).$$

Rows and colums of M_A correspond to black and white squares in A and entries are given by weights. The reader should recognize the construction above as an extension of Kasteleyn's ([3]). For the example in Figure 5,

$$\begin{aligned}\Phi_A(p,q) = q^{-18}\,p^{-2}\,\big(\,&q^{36}\,p^4 + \\ &(q^{36} + 3\,q^{35} + 3\,q^{34} + 4\,q^{33} + 6\,q^{32} + 6\,q^{31} + 7\,q^{30} + 6\,q^{29} + 6\,q^{28} + 7\,q^{27} + \\ &\quad 6\,q^{26} + 6\,q^{25} + 7\,q^{24} + 6\,q^{23} + 6\,q^{22} + 4\,q^{21} + 3\,q^{20} + 3\,q^{19} + q^{18})\,p^3 + \\ &(q^{30} + 3\,q^{29} + 3\,q^{28} + 4\,q^{27} + 9\,q^{26} + 12\,q^{25} + 16\,q^{24} + 24\,q^{23} + 33\,q^{22} + \\ &\quad 41\,q^{21} + 45\,q^{20} + 51\,q^{19} + 57\,q^{18} + 51\,q^{17} + 45\,q^{16} + 41\,q^{15} + \\ &\quad 33\,q^{14} + 24\,q^{13} + 16\,q^{12} + 12\,q^{11} + 9\,q^{10} + 4\,q^{9} + 3\,q^{8} + 3\,q^{7} + q^{6})\,p^2 + \\ &(q^{18} + 3\,q^{17} + 3\,q^{16} + 4\,q^{15} + 6\,q^{14} + 6\,q^{13} + 7\,q^{12} + 6\,q^{11} + 6\,q^{10} + 7\,q^{9} + \\ &\quad 6\,q^{8} + 6\,q^{7} + 7\,q^{6} + 6\,q^{5} + 6\,q^{4} + 4\,q^{3} + 3\,q^{2} + 3\,q + 1)\,p + 1\big)\end{aligned}$$

We are ready to state our main result.

Theorem 3 *Let A be a balanced bicolored wall-free quadriculated annulus.*

(a) All non-zero roots of the polynomial $\Phi_A(p, -1)$ are roots of unity.

(b) Let $q > 0$ be fixed: all non-zero roots of the q-flux polynomial $\Phi_A(p, q)$ are distinct, negative numbers.

In the example above, $\Phi_A(p, -1) = p^{-2}(p^4 + p^3 + p^2 + p + 1)$ and $\Phi_A(p, 1) = p^{-2}(p^4 + 91\,p^3 + 541\,p^2 + 91\,p + 1)$ (with roots approximately equal to -84.619, -6.2077, $-.16109$ and $-.011818$).

3 The connection matrix

A cut ξ transforms an annulus A into a track segment Δ with *left* and *right attachments*. The n-th cover A^n of A can be obtained by juxtaposing n copies $\Delta_0, \dots, \Delta_{n-1}$ of Δ and then *closing up*, i.e., identifying extreme attachments. There are copies $\xi_{i+0.5}$ of the cut between Δ_i and Δ_{i+1}, $i = 0, \dots, n-1$ where

$\Delta_n = \Delta_0$. A tiling of A^n restricts to Δ_i yielding a *tiling of the track segment* Δ_i: notice that such tilings include half-dominoes belonging to dominoes trespassing the cuts $\xi_{i\pm0.5}$ across a certain set of sides of squares.

A *shape* at an attachment is a set of sides contained in the attachment. Tilings of a track segment Δ thus induce shapes on both attachments. In particular, shape determines flux. A pair of shapes on the attachments of Δ describes a *pruning* of Δ (a smaller disk or a union of disks) by removing the squares with sides belonging to either shape, as in Figure 6. If two sides of the same square are selected, pruning is not defined. Tilings of the track segment with prescribed shapes are in natural bijection with tilings of the pruned segment. Pairs of shapes for which pruning is not defined are not induced by any tilings of Δ.

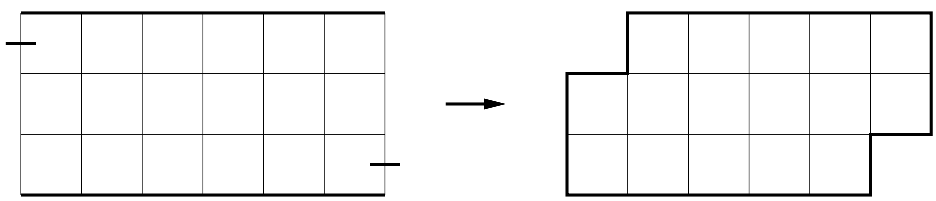

Figure 6: Pruning a track segment to obtain a disk

We now construct the connection matrix C_Δ. Order shapes so that flux is non-decreasing. Rows and columns of C_Δ correspond to shapes at the left and right attachments. Given a pair of shapes, the associated entry q-counts tilings of the pruned segment (the entry is 0 if pruning is not defined). We do not discuss how to compare volumes of tilings on different prunings—this introduces a certain ambiguity, given by possible multiplication of entries of the matrix by different powers of q; the issue is resolved via height functions in [4]. Thus, the matrix C_Δ is block diagonal with blocks $C_{\Delta,f}$ labeled by f, the value of the flux. Also,

$$\Phi_A(p,q) = p^* q^* \sum_f (\operatorname{tr} C_{\Delta,f}) p^f.$$

Let Δ^n be the juxtaposition of n copies of Δ, i.e., Δ^n is obtained by cutting A^n across any $\xi_{i+0.5}$. The connection matrix of C_{Δ^n} is $(C_\Delta)^n$: essentially, entries of the n-th power of the adjacency matrix of a graph count paths of length n between two vertices.

The polynomial $\Phi_{A^n}(p,q)$ admits two useful descriptions. From the previous paragraph,

$$\Phi_{A^n}(p,q) = p^* q^* \sum_f (\operatorname{tr}(C_{\Delta,f})^n) p^f.$$

The second relates the roots λ_i of $\Phi_A(p,q)$ (in p for fixed q) with the roots $\lambda_{i,n}$ of $\Phi_{A^n}(p,q)$:

$$\lambda_{i,n} = (-1)^{n+1}\lambda_i^n.$$

This follows from constructing a Kasteleyn matrix M_{A^n} from M_A and diagonalizing both matrices by exploiting the obvious action of $\mathbb{Z}/(n)$ over A^n.

4 The proof

We begin with item (a). Suppose that Δ is a track segment obtained from cutting A. Consider, for a fixed value f of the flux and $q = -1$, the blocks $C_{\Delta^n,f} = (C_{\Delta,f})^n$ of the connection matrix C_{Δ^n} of the juxtaposition Δ^n. From the left hand side of the equality, the block entries -1-count the numbers of tilings of (unions of) disks obtained by pruning, or are equal to zero in case pruning is not defined. From Theorem 1, these entries, for all n, take very few values.

Thus, for all n, there are only finitely many possible values for the matrices $C_{\Delta^n,f}$. In particular, there are powers n_0 and n_1 such that, for all values f of the flux, we have $C_{\Delta^{n_0},f} = C_{\Delta^{n_1},f}$, implying in turn the equality of the polynomials $\Phi_{A^{n_0}}(p,-1) = \Phi_{A^{n_1}}(p,-1)$. Without loss, n_0 and n_1 can be taken to be powers of 2 so that $n_1 = n'n_0$ for a natural number $n' > 1$. Combine this fact with the relationship $\lambda_{i,n} = (-1)^{n+1}\lambda_i^n$ to learn that, up to signs, raising to n' induces a permutation on the set of roots of $\Phi_{A^{n_0}}(p,-1)$. For a sufficiently high power n'' of n', raising to n'' keeps all roots of $\Phi_{A^{n_0}}(p,-1)$ fixed: the nonzero roots are therefore roots of unity, and item (a) is proved.

The proof of (b) takes a different route. Take $q > 0$ fixed throughout the proof. The first step is to prove that each block $C_{\Delta,f}$ has a simple eigenvalue $\Lambda_f > 0$ of absolute value larger than that of any other eigenvalue.

A shape is *bi-active* if a tiling of the bi-infinite band $\Delta^\infty = \cdots\Delta_{-1}\Delta_0\Delta_1\cdots$ exists with the prescribed shape at the cut $\xi_{0.5}$. The *bi-active submatrix* $C_{*,\Delta,f}$ is the intersection of rows and columns of $C_{\Delta,f}$ associated with bi-active shapes. We leave it to the reader to check that the corresponding submatrix of $(C_{\Delta,f})^n$ equals $(C_{*,\Delta,f})^n$ and that the spectra of $C_{\Delta,f}$ and $C_{*,\Delta,f}$ coincide, up to null eigenvalues; details are given in [4].

We now show that sufficiently large powers of $C_{*,\Delta,f}$ have only positive entries, i.e, that for a given track segment Δ and a value f of the flux, there exists an integer N such that for all $n > N$ and any two bi-active shapes ℓ and r, there exists a tiling of Δ^n with these prescribed shapes at the left and right attachments. We proceed to join ℓ and r by a tiling of a long track segment Δ^n. By hypothesis, ℓ extends as a tiling to the right of $\xi_{0.5}$ in Δ^∞: since there are only finitely many shapes, two cuts $\xi_{I_L+0.5}$ and $\xi_{I_L+P_L+0.5}$ see the same shape. By repeating this chunk of tiling, ℓ also extends as an eventually periodic tiling with period P_L after an initial stretch of length I_L. Similarly, r also extends as an eventually periodic tiling with period P_R and final stretch of length F_R, say, to the left of $\xi_{n+0.5}$.

Assume without loss that P_L and P_R are both even, $I_L \equiv 0 \pmod{P_L P_R}$ and $F_R \equiv n \pmod{P_L P_R}$; call t_I and t_F the restrictions of the infinite tilings above to Δ^{I_L} and Δ^{F_R}. Both infinite tilings restricted to periodic stretches (i.e., tilings of Δ^{P_L} and Δ^{P_R}) by replication give rise to tilings t_L and t_R of the annulus $A^{P_L P_R}$ (and thus of $\Delta^{P_L P_R}$). It is not too hard to see that if A is wall-free than so is $A^{P_L P_R}$ (see [4] for details). From Theorem 2, there exists a sequence of flips connecting these tilings. Call the tilings in this sequence $t_L = t_0, t_1, \ldots, t_M = t_R$.

We would like to join shapes ℓ and r by a tiling obtained by juxtaposing $t_I, t_0, t_1, \ldots, t_M, t_F$: this is not quite correct since t_i and t_{i+1} may differ at the common cut. This difficulty is circumvented by constructing tilings $t_{i+0.5}$ of $\Delta^{P_L P_R}$ with the same left shape as t_i and the same right shape as t_{i+1} and then juxtaposing $t_I, t_{0.5}, t_{1.5}, \ldots, t_{M-0.5}, t_F$. More precisely, if t_i and t_{i+1} have the same left shape take $t_{i+0.5} = t_i$; otherwise, take $t_{i+0.5}$ to coincide with t_i on the left $\Delta^{P_L P_R/2}$ subsegment and with t_{i+1} on the right subsegment. This obtains a tiling in $\Delta^{I_L+(M-1)P_L P_R+F_R}$: since this number is congruent to n modulo $P_L P_R$, the desired tiling can be obtained, for sufficiently large n, by inserting copies of t_M before t_F.

The Perron-Frobenius theorem ([2]) applied to $C_{*,\Delta,f}$ then completes the proof of the first step.

Label the nonzero roots of $\Phi_A(p,q)$ as $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_m| > 0$. Assume by induction that $\lambda_1, \ldots, \lambda_{k-1}$ are real negative and that

$$|\lambda_1| > \cdots > |\lambda_{k-1}| > |\lambda_k| = \cdots = |\lambda_{k'}| > |\lambda_{k'+1}|,$$

with $k' \geq k$. We must prove that $k = k'$ and that λ_k is real negative. Consider the usual symmetric function

$$\sigma_k(x_1, \ldots, x_m) = \sum_{1 \leq i_1 < \cdots < i_k \leq m} x_{i_1} \cdots x_{i_k}.$$

From what we have seen above,

$$\operatorname{tr} C_{\Delta, f_{\max}-k} = a^n \sigma_k((-\lambda_1)^n, \ldots, (-\lambda_m)^n) = (1+o(1))\Lambda^n_{f_{\max}-k}$$

when n goes to infinity. Here $a\,p^{f_{\max}}$ is the leading monomial of $\Phi_A(p,q)$.

The expression $\sigma_k((-\lambda_1)^n, \ldots, (-\lambda_m)^n)$ is the sum of $k'-k+1$ terms of the form $((-1)^k \lambda_1 \lambda_2 \cdots \lambda_{k-1} \lambda_\ell)^n$, $\ell = k, \ldots, k'$ and other terms which grow at exponentially smaller rates. Thus

$$(-1)^{kn} \lambda_1^n \lambda_2^n \cdots \lambda_{k-1}^n (\lambda_k^n + \cdots + \lambda_{k'}^n) = (1+o(1))(\Lambda_k/a)^n$$

whence

$$b_k^n + \cdots + b_{k'}^n = 1 + o(1)$$

where $b_\ell = -\lambda_\ell/|\lambda_\ell|$. The upshot is that $b_k, \ldots, b_{k'}$ belong to the unit circle and we can take arbitrarily large n such that $2(k'-k+1)|b_\ell^n - 1| < 1$ for all ℓ and therefore

$$|(k'-k+1) - (b_k^n + \cdots + b_{k'}^n)| < 1/2,$$

a contradiction unless $k = k'$. Finally, $b_k = 1$ and λ_k is real negative. This concludes the proof of item (b).

As an application of (b), we state the result below. A sequence a_k of non-negative real numbers is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all k. In particular, log-concave sequences are either monotone or unimodal.

Corollary 4 *Let q be a fixed positive real number and let a_f be the coefficient of p^f in $\Phi_A(p,q)$. Then the sequence a_f is log-concave.*

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