Errata to
Morin singularities and global geometry in a class of ordinary differential operators

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October 9, 2007

Theorem 1.2, page 142, is wrong as stated. We give a counterexample and present a convenient hypothesis on the nonlinearity f under which the theorem and its proof are correct. The hypothesis is satisfied by all examples in the rest of the paper.

The proof of the last sentence of Lemma 3.5 is missing: "Also, $\hat{S}_k \neq \emptyset$ implies $S_k \neq \emptyset$.".

About Theorem 1.2

We begin with a counterexample. Let $f(t, u) = 2\pi \cos(2\pi t) \cosh^2(u)$. Then there are no periodic functions u for which $u'(t) + f(t, u(t))$ is constant. In particular, the point $0 \in B^0$ is not in the image of the map Ψ constructed in Theorem 1.2.

Proof: The solutions of the equation $u'(t) + f(t, u(t)) = 0$ are

$$u = -\text{arctanh}\,(\sin(2\pi t) + C)\,, \quad C \in (-2, 2).$$

For $C = 0$, consider the solutions u_- and u_+ on disjoint domains $(-1/4, 1/4)$ and $(1/4, 3/4)$. Notice that u_- (resp. u_+) is strictly decreasing (resp. increasing) with absolute value tending to infinity at the endpoints of the domain.

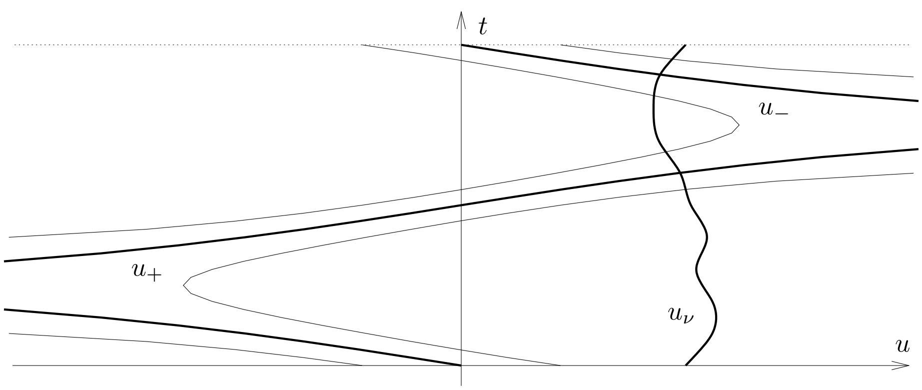

Figure 1: Solutions for $C = 0$ and $C = \pm 0.3$

The graph of any periodic function u_ν must cross the graphs of both u_- and u_+ at times t_- and t_+, respectively, for which $u'_\nu(t_-) + f(t_-, u_\nu(t_-)) \geq 0$ and $u'_\nu(t_+) + f(t_+, u_\nu(t_+)) \leq 0$. If $u'_\nu(t) + f(t, u_\nu(t)) = \nu$ for all t then, from the conditions above, $\nu = 0$. This, however, implies that u_ν must equal both u_- and u_+, a contradiction. ∎

A function $f:\mathbb{S}^1\times\mathbb{R}\to\mathbb{R}$ is *wild at* $+\infty$ (resp. $-\infty$) if

$$\int_I \frac{ds}{\max(1,\sup_{t\in\mathbb{S}^1} f(t,s))}<+\infty,\quad \int_I \frac{ds}{\max(1,\sup_{t\in\mathbb{S}^1}(-f(t,s)))}<+\infty$$

for $I=[0,+\infty)$ (resp. $I=(-\infty,0]$). We show that if f is *tame* (i.e., not wild at $\pm\infty$) then Theorem 1.2 and Lemma 1.3 hold, with the same proof. Furthermore, autonomous functions (i.e., which do not depend on t), proper functions and functions f which are non-decreasing in the second variable are all tame. This makes all other arguments and statements in the paper correct as written. Loosely, f being tame implies that a solution u can not go very far and come back in bounded time.

The offending sentence in the proof of Theorem 1.2 is "By continuous dependence on parameters, both [A^+ and A^-] are closed" (page 143, lines 18 and 19): in the above example A^+ is an open half-line. Let $f:\mathbb{S}^1\times\mathbb{R}\to\mathbb{R}$, $\tilde{v}\in B^0\subset L^1$ and u_ν be the maximal solution of $u_\nu'(t)+f(t,u_\nu(t))=\tilde{v}(t)+\nu$, $u_\nu(0)=c$. Set $\nu_0=\sup A^-=\inf A^+$: if $\nu_0\in A^-\cap A^+$ then u_{ν_0} is periodic. We show that $\nu_0\notin A^\mp$ implies that f is wild at $\pm\infty$. We consider the case $\nu_0\notin A^-$.

Proof: If u_{ν_0} is defined in $[0,1]$ with $u_{\nu_0}(1)>c$ then continuous dependence implies that some open neighborhood of ν_0 is contained in A^+, a contradiction. Define $t_c\in(0,1]$ by

$$\lim_{t\to t_c} u_{\nu_0}(t)=+\infty.$$

If $t_c=1$, continuous dependence again implies that ν_0 is in the interior of A^+; we therefore have $t_c<1$. Thus, for every $M\in\mathbb{R}$ there exists $\nu<\nu_0$ and $t_\nu>t_c$ such that $u_\nu(t_c)>M$, $u_\nu(t_\nu)<c$. Set

$$I_{\tilde{v}}=\{t\in[0,t_c]\mid -f(t,u_\nu(t))\le\tilde{v}(t)+\nu\}.$$

For $t\in I_{\tilde{v}}$ we have $u_\nu'(t)\le 2\tilde{v}(t)+2\nu$ and the Lebesgue measure $\mu(u_\nu(I_v))$ is bounded above by $2|\nu|+2||\tilde{v}||_{L^1}$. Define $h:[c,M]\to[0,t_c]$ by $h(s)=\inf\{t\in[0,t_c]\mid u_\nu(t)=s\}$. Even though h may have discontinuities, it is strictly increasing and then, for almost all s, h is differentiable with $h'(s)=1/u_\nu'(h(s))$. Let $J_f=[c,M]\smallsetminus u_\nu(I_v)$: for $s\in J_f$, we have $h(s)\notin I_{\tilde{v}}$ and $h'(s)\ge -1/(2f(h(s),s))$. Thus,

$$t_c\ge\mu(h(J_f))\ge\int_{J_f}h'(s)ds\ge\frac{1}{2}\int_{J_f}\frac{ds}{\max(1,\sup_{t\in\mathbb{S}^1}(-f(t,s)))}$$

and therefore

$$\int_c^M\frac{ds}{\max(1,\sup_{t\in\mathbb{S}^1}(-f(t,s)))}\le 2t_c+2|\nu|+2||\tilde{v}||_{L^1}.$$

Since this estimate holds for arbitrarily large M,

$$\int_0^{+\infty}\frac{ds}{\max(1,\sup_{t\in\mathbb{S}^1}(-f(t,s)))}<+\infty.$$

A similar argument for the interval $[t_c,t_\nu]$ yields

$$\int_0^{+\infty}\frac{ds}{\max(1,\sup_{t\in\mathbb{S}^1}(f(t,s)))}<+\infty,$$

implying that f is wild at $+\infty$. ■

About Lemma 3.5

Proof: Assume $\hat{S}_k \neq \emptyset$. Following the notation of the proof of Lemma 3.5, use the space V and the function ϕ to obtain $r > 0$ and a function $H : \mathbb{B}^k \to B^1$ with

$$\int \hat{\gamma}_k(H(s)(t))dt = rs$$

for $s \in \mathbb{B}^k$ where $\mathbb{B}^k \subset \mathbb{R}^k$ is the unit ball.

Define the N-*replicator* to be the isomorphism $R_N : B^i \to R_N(B^i) \subset B^i$, $(R_N(u))(t) = u(Nt)$, $i = 0, 1$. Clearly, $(R_N(u))' = NR_N(u')$. We claim that given $\epsilon > 0$ there exists N such that

$$\left|\left(w\Phi(R_N(H(s))), \ldots, w^k\Phi(R_N(H(s)))\right) - rs\right| < \epsilon$$

for all $s \in \mathbb{B}^k$ and the proof is completed by a standard degree theory argument.

At this point it is convenient to make explicit the dependence of $w = w(u)$ in terms of u. From Proposition 1.1, $\lambda = \int f'(u(t))dt$ is the same for u and $R_N(u)$ and we have

$$(w(R_N(u)))'(t) + f'(R_N(u(t)))w(R_N(u))(t) = \lambda w(R_N(u))(t).$$

Define $w_N(u)$ by $R_N(w_N(u)) = w(R_N(u))$ so that

$$(w_N(u))'(t) + \frac{f'(u(t))}{N} w_N(u)(t) = \frac{\lambda}{N} w_N(u)(t)$$

and, from the formula for w in Proposition 1.1, $w_N(u)(t) = (w(u)(t))^{(1/N)}$. Recall that $\Phi(u) = \int f(u(t))dt$ and therefore $\Phi(R_N(u)) = \Phi(u)$ and

$$\left(w\Phi(R_N(u)), \ldots, w^k\Phi(R_N(u))\right) = \left(w_N\Phi(u), \ldots, w_N^k\Phi(u)\right).$$

The sequence (w_N) of vector fields tends to the constant vector field $\mathbf{1}$ (i.e., the constant function 1 at every point u) in the C^n-metric (for any n). Also,

$$\left(\mathbf{1}\Phi(R_N(u)), \ldots, \mathbf{1}^k\Phi(R_N(u))\right) = \int \hat{\gamma}_k(u(t))dt,$$

proving the claim. ∎

It also follows from the above argument that $D((w\Phi, \ldots, w^k\Phi) \circ R_N \circ H)$ tends to the identity matrix when N goes to infinity, establishing condition (c) in Proposition 2.1. Condition (b) follows from the additional hypothesis of f being $(k+1)$-good, thus proving the existence of Morin singularities of order k.

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