The geometry of the critical set of nonlinear periodic Sturm-Liouville operators

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Abstract

We study the critical set $\mathcal{C}$ of the nonlinear differential operator $F(u) = -u'' + f(u)$ defined on a Sobolev space of periodic functions $H^p(\mathbb{S}^1)$, $p \geq 1$. Let $\mathbb{R}^2_{xy} \subset \mathbb{R}^3$ be the plane $z = 0$ and, for $n > 0$, let $\bowtie_n$ be the cone $x^2 + y^2 = \tan^2 z$, $|z - 2\pi n| < \pi/2$; also set $\Sigma = \mathbb{R}^2_{xy} \cup \bigcup_{n>0} \bowtie_n$. For a generic smooth nonlinearity $f : \mathbb{R} \to \mathbb{R}$ with surjective derivative, we show that there is a diffeomorphism between the pairs $(H^p(\mathbb{S}^1), \mathcal{C})$ and $(\mathbb{R}^3, \Sigma) \times \mathbb{H}$ where $\mathbb{H}$ is a real separable infinite dimensional Hilbert space.

Keywords: Sturm-Liouville, Monodromy, Floquet matrix, Infinite-dimensional manifolds with singularities.

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1 Introduction

A basic object in the study of a smooth nonlinear operator $F : X \to Y$ between Banach spaces is its *critical set* $\mathcal{C} \subset X$, the set of points $u \in X$ for which the derivative $DF(x)$ is not a (bounded) linear isomorphism between X and Y. For instance, knowledge of C and $F(C)$ yields substantial information about the number of solutions of the equation $F(x) = b$, $b \in Y$ ([8], [10]). In this paper we consider a special but relevant example, the nonlinear periodic Sturm-Liouville operator and determine the topology of the pair $(X, \mathcal{C})$ in the generic case.

More precisely, for a smooth nonlinearity $f : \mathbb{R} \to \mathbb{R}$ and $p \geq 1$ denote by F the differential operator

$$F : H^p(\mathbb{S}^1) \to H^{p-2}(\mathbb{S}^1).$$
$$u \mapsto -u'' + f(u)$$

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Here $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ and $H^p(\mathbb{S}^1)$ is the Sobolev space of periodic functions $u(t)$ with square integrable p-th derivative. Clearly, $DF(u) : H^p(\mathbb{S}^1) \to H^{p-2}(\mathbb{S}^1)$ is $DF(u)v = -v'' + f'(u)v$, a Fredholm operator of index 0. Thus, the critical set $\mathcal{C} \subset H^p(\mathbb{S}^1)$ of F is

$$\mathcal{C} = \{u \in H^p(\mathbb{S}^1) \mid DF(u) \text{ has nontrivial kernel}\}.$$

In other words, $u \in \mathcal{C}$ if and only if the equation

$$-v'' + f'(u)v = 0, \quad v(0) = v(2\pi), \quad v'(0) = v'(2\pi)$$

admits a nonzero solution.

This paper continues the project of obtaining geometric understanding for some nonlinear operators $F : X \to Y$. The starting point might be located in a fundamental result of Ambrosetti and Prodi ([2]; see also [1]). As interpreted by Berger and Church ([3]), it states that, for appropriate nonlinearities, $u \mapsto -\Delta u + f(u)$ acting on functions satisfying Dirichlet boundary conditions is a global fold. Subsequently, a number of different operators were shown to be either global folds or global cusps (see, among others, [7],[10]). Ideally, the description of the critical set $\mathcal{C}$ should include its stratification into different kinds of singularities.

In many examples, $\mathcal{C}$ and its strata were shown to be topologically trivial. Differential operators related to nonlinear Sturm-Liouville second order problems $F(u) = -u'' + f(u)$ in finite intervals have very different critical sets depending on the boundary conditions. For generic nonlinearities f, Dirichlet boundary conditions give rise to a critical set $\mathcal{C}$ which is ambient diffeomorphic to a countable (possibly finite) union of parallel hyperplanes ([4], [5]). The situation is very different for periodic boundary conditions.

We now describe the generic nonlinearities f for which our main result holds. A smooth function $g : \mathbb{R} \to \mathbb{R}$ is *nowhere flat* if there exists a positive integer r such that for all $x \in \mathbb{R}$, there exists $r' = r'(x) \in \mathbb{Z}$, $0 < r' \leq r$, such that $g^{(r')}(x) \neq 0$. An *admissible nonlinearity* is a smooth function $f : \mathbb{R} \to \mathbb{R}$ such that f' is surjective and nowhere flat and for all $x_0 \in \mathbb{R}$, if $f'(x_0) = -n^2$, $n \in \mathbb{Z}$, then $(f''(x_0), f'''(x_0)) \neq (0,0)$. A *good nonlinearity* is an admissible nonlinearity for which $f'(x_0) = -n^2$, $n \in \mathbb{Z}$, implies $f''(x_0) \neq 0$. Most polynomials of even degree are good nonlinearities.

Let $\mathbb{R}^2_{xy} \subset \mathbb{R}^3$ be the plane $z = 0$ and $I_n = [2\pi n - \pi/2, 2\pi n + \pi/2]$ for $n \in \mathbb{Z}$, $n > 0$. Let $\bowtie_n$ be the cone $x^2 + y^2 = \tan^2 z$, $z \in I_n$ and $\Sigma = \mathbb{R}^2_{xy} \cup \bigcup_{n>0} \bowtie_n$. The real separable infinite dimensional Hilbert space will be denoted by $\mathbb{H}$.

It turns out that the critical set $\mathcal{C}$ may include (at most countably many) isolated points: let $\mathcal{C}^* \subseteq \mathcal{C}$ be obtained from $\mathcal{C}$ by removing such points.

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be an admissible nonlinearity: the pair $(H^p(\mathbb{S}^1), \mathcal{C}^*)$ is diffeomorphic to the pair $(\mathbb{R}^3, \Sigma) \times \mathbb{H}$; if f is a good nonlinearity then $\mathcal{C}^* = \mathcal{C}$.*

The present paper can also be considered a continuation of [6], where Theorem 1 is proved for the *linear case* $f(x) = x^2/2$: the phrasing is justified by the fact that $f'(x) = x$ and therefore $DF(u)v = -v'' + uv$. In this case we trivially have $\mathcal{C}^* = \mathcal{C}$. Notice that in the periodic case, unlike the Dirichlet case, the critical set $\mathcal{C}$ has singular points and is not a Hilbert manifold. A model of $\mathcal{C}$ at singular points is obtained by the study of the *monodromy map*. We now review the notation and results used in [6] for the linear case, which will be heavily used in this paper.

Let $\Pi : G = \widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$ be the universal cover of $SL(2,\mathbb{R})$. The elements of $SL(2,\mathbb{R})$ and G will be referred to as matrices and lifted matrices, respectively. Recall that G is itself a Lie group diffeomorphic to $\mathbb{R}^3$. For a potential $q \in H^p(\mathbb{S}^1)$, let $v_1, v_2 \in H^{p+2}([0,2\pi])$ be the fundamental solutions

$$v_i''(t) = q(t)v_i(t), \quad v_1(0) = 1, \ v_1'(0) = 0, \ v_2(0) = 0, \ v_2'(0) = 1$$

and define the *lifted fundamental matrix* $\tilde{\Phi} : [0,2\pi] \to G$ by $\tilde{\Phi}(0) = I$ and

$$\Pi(\tilde{\Phi}(t)) = \begin{pmatrix} v_1(t) & v_2(t) \\ v_1'(t) & v_2'(t) \end{pmatrix}.$$

The *monodromy map* $\mu : H^p(\mathbb{S}^1) \to G$ is the lifting $\mu(q) = \tilde{\Phi}(2\pi)$; the projection $\Pi(\mu(q)) \in SL(2,\mathbb{R})$ is the so called Floquet multiplier associated to the potential q. Notice that $L^1([0,2\pi])$ is another legitimate domain for μ.

It is easy to verify that $u \in \mathcal{C}$ if and only if $\mu(u) \in T_2 \subset G$ where T_2 is the set of lifted matrices of trace equal to 2 (or, equivalently, with double eigenvalue 1); here $\mathrm{tr}(g) = \mathrm{tr}(\Pi(g))$ for a lifted matrix $g \in G$. The image of μ is an open set $G_0 \subset G$ diffeomorphic to $\mathbb{R}^3$. Theorem 3 in [6] constructs an explicit smooth diffeomorphism $\Psi : G_0 \times \mathbb{H} \to H^p(\mathbb{S}^1)$ such that $\mu \circ \Psi$ is the projection on the first coordinate. It follows that $(H^p(\mathbb{S}^1), \mathcal{C})$ is diffeomorphic via Ψ^{-1} to $(G_0, T_2 \cap G_0) \times \mathbb{H}$.

In the general case, we define the *nonlinear monodromy map* $\mu_f : H^p(\mathbb{S}^1) \to G_0$ by $\mu_f(u) = \mu(f' \circ u)$. Unlike μ, the map μ_f cannot in general be extended to $L^1([0,2\pi])$ but $L^\infty([0,2\pi])$ will be enough for the purposes of this paper. For admissible nonlinearities f, it turns out that the image of μ_f equals G_0 and we still have that $\mathcal{C} = \mu_f^{-1}(T_2 \cap G_0)$. We shall not construct a counterpart of the diffeomorphism Ψ for the general nonlinearity: instead, we prove the contractibility of fibers of μ_f and show that this information suffices to model $\mathcal{C}$.

The local behavior of μ_f is particularly nasty at constant functions u; on the other hand, such functions form a subspace of infinite codimension and can therefore be excised without changing the homotopy type of the domain. Set $X^* = H^p(\mathbb{S}^1) \setminus \{u \text{ constant}\}$: as we shall see in Theorem 3, the map $\mu_f : X^* \to G_0$ is a surjective submersion with contractible fibers. In order to complete the proof

of Theorem 1 we need some results in infinite dimensional topology. More precisely, we present in Theorem 4 a normal form near $\mathcal{C}$, a Hilbert submanifold with singularities: such singularities arise from the fact that $T_2 \subset G$ is diffeomorphic to a countable union of cones.

As in [5], [9], [10], a key ingredient is the contractibility of level sets of certain functionals defined on infinite dimensional spaces. The reader may see little in common among the several proofs. A unifying feature is that we first construct a fake homotopy and then fix it: it helps that the functional can actually be extended to a larger infinite dimensional space with a weaker topology. Theorem 2 in [5], transcribed below, allows for moving from one space to another without changing the homotopy type of level sets.

Theorem 2 *Let X and Y be separable Banach spaces. Suppose $i : Y \to X$ is a bounded, injective linear map with dense image and $M \subset X$ is a smooth, closed submanifold of finite codimension. Then $N = i^{-1}(M)$ is a smooth closed submanifold of Y and the restrictions $i : Y - N \to X - M$ and $i : (Y, N) \to (X, M)$ are homotopy equivalences.*

Section 2 contains basic facts about the linear monodromy map μ, including a study the effect of adding bumps ℓ_i to a potential q as controlled perturbations of $\mu(q + \sum a_i \ell_i)$. In Section 3 we compute the derivative of the nonlinear monodromy map μ_f and extend the results for bumps to this case. We also verify that under rather general hypothesis on f the image of μ_f is G_0, as in the linear case (Proposition 3.6). The argument runs as follows: for $g \in G_0$, we first obtain a discontinuous $u_0 \in L^\infty$ which is smoothened out yielding u_1 with $\mu_f(u_1) \approx g$; the error is then corrected by adding appropriate bumps to u_1. In Section 4 we prove that level sets of μ_f in X^* are contractible (Theorem 3) by construct homotopies: we first obtain a fake homotopy by composing a homotopy for the linear case with a (discontinuous!) right inverse for f'. Smoothening and correction are then similar to that in the proof of Proposition 3.6. Section 5 contains the necessary results in infinite dimensional topology (including Theorem 4) which take into account the presence of cone-like objects. The ingredients are combined in Section 6 to complete the proof of Theorem 1; we express our thanks to S. Ferry for providing first the arguments and afterwards directing us to the appropriate reference [11] for what we call here Michael's theorem (Fact 2).

2   The monodromy map μ

We begin with a recollection of elementary facts about the second order ODE $v'' = q(t)v$ and the universal cover $\Pi : G = \widetilde{SL(2,\mathbb{R})} \to SL(2,\mathbb{R})$. For details, including a careful tracking of differentiability classes, see [6].

The *left Iwasawa decomposition* is the diffeomorphism $\phi_L : \mathbb{R} \times (0,\infty) \times \mathbb{R} \to G$ given by $\phi_L(0,1,0) = I$ and

$$(\Pi \circ \phi_L)(\theta, \rho, \nu) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{\rho} & 0 \\ 0 & 1/\sqrt{\rho} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \nu/2 & 1 \end{pmatrix}.$$

The set G_0 is defined by $G_0 = \phi_L((0,+\infty) \times (0,+\infty) \times \mathbb{R}) \subset G$. Equivalently, $g \in G_0$ if and only if the variation in argument from e_2 to ge_2 is negative (the variation in argument is computed along a path $\gamma : [0,1] \to G$ joining $\gamma(0) = I$ to $\gamma(1) = g$). The pair $(G_0, T_2 \cap G_0)$ is shown schematically in Figure 1. Here $T_2 \subset G$ is the set of lifted matrices of trace equal to 2. The thick dashed curve represents ∂G_0, which is diffeomorphic to a plane; the X's formed by crossing curves stand for the connected components of T_2, diffeomorphic to cones with horizontal axis. The dotted vertical lines represent the set of lifted matrices of trace 0 so that vertical regions contain matrices with trace of alternating sign. The pair $(\mathbb{R}^3, \Sigma)$ constructed in the introduction is diffeomorphic to $(G_0, T_2 \cap G_0)$: $\Sigma = \mathbb{R}^2_{xy} \cup \bigcup_{n>0} \bowtie_n$ where $\mathbb{R}^2_{xy} \subset \mathbb{R}^3$ is the plane $z = 0$ and $\bowtie_n$ is the cone $x^2 + y^2 = \tan^2 z$, $|z - 2\pi n| < \pi/2$.

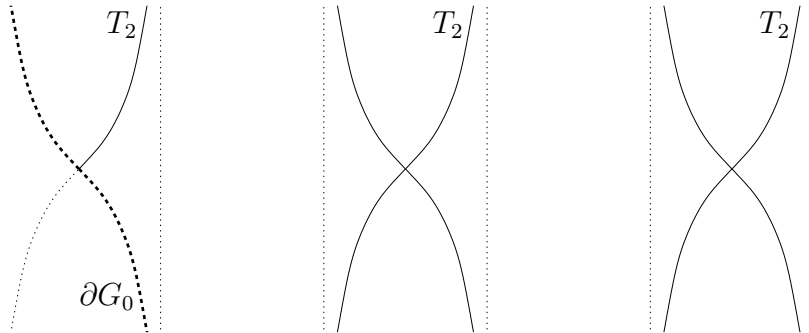
Figure 1: The pair $(G_0, T_2 \cap G_0)$.

Given a potential $q \in L^\infty([0,2\pi])$, let v_1, v_2 be the fundamental solutions

$$v_i''(t) = q(t)v_i(t), \quad v_1(0) = 1,\ v_1'(0) = 0,\ v_2(0) = 0,\ v_2'(0) = 1$$

and define the *lifted fundamental matrix* $\tilde{\Phi} : [0,2\pi] \to G_0$ and its projection $\Phi = \Pi \circ \tilde{\Phi}$ by $\tilde{\Phi}(0) = I$ and

$$\Phi(t) = \Pi(\tilde{\Phi}(t)) = \begin{pmatrix} v_1(t) & v_2(t) \\ v_1'(t) & v_2'(t) \end{pmatrix}.$$

Let $\mathbf{v} : [0, 2\pi] \to \mathbb{R}^2$ be $\mathbf{v}(t) = (v_1(t), v_2(t))$ so that

$$\mathbf{v}''(t) = q(t)\mathbf{v}(t), \quad \mathbf{v}(0) = e_1 = (1, 0), \quad \mathbf{v}'(0) = e_2 = (0, 1).$$

The curve $\mathbf{v}$ never passes through the origin so its argument θ is well defined: it is the only continuous function which satisfies

$$\theta(0) = 0, \quad \frac{\mathbf{v}(t)}{||\mathbf{v}(t)||} = (\cos\theta(t), \sin\theta(t)).$$

Since $\mathbf{v}(t) \wedge \mathbf{v}'(t) = 1$, the function $\theta : [0, 2\pi] \to [0, \theta_M]$ is an increasing bijection. Following [6], the *orbit* $\rho : [0, \theta_M] \to (0, +\infty)$ associated to $\mathbf{v}$ is

$$\rho(\theta(t)) = v_1^2(t) + v_2^2(t) = ||\mathbf{v}(t)||^2.$$

For $\nu(\theta) = \rho'(\theta)/\rho(\theta)$, it turns out that $\phi_L(\theta(t), \rho(\theta(t)), \nu(\theta(t))) = \tilde{\Phi}(t)$.

Conversely, given an orbit, i.e., a number $\theta_M > 0$ and a function $\rho : [0, \theta_M] \to (0, +\infty)$ with $\rho(0) = 1$, $\rho'(0) = 0$ and $\int_0^{\theta_M} \rho(\theta)d\theta = 2\pi$, consider the parametrized curve $\sqrt{\rho(\theta)}(\cos\theta, \sin\theta)$. There is a unique orientation preserving reparametrization $\theta : [0, 2\pi] \to [0, \theta_M]$ of the curve so that equal areas around the origin are swept in equal times. In other words, the reparametrized curve

$$\mathbf{v}(t) = \sqrt{\rho(\theta(t))}(\cos\theta(t), \sin\theta(t))$$

satisfies $\mathbf{v}(t) \wedge \mathbf{v}'(t) = 1$ for all t. Taking derivatives, $\mathbf{v}(t) \wedge \mathbf{v}''(t) = 0$ whence $\mathbf{v}''(t) = q(t)\mathbf{v}(t)$ for some potential $q : [0, 2\pi] \to \mathbb{R}$. The correspondence between potentials and orbits is called the *Kepler transform* in [6].

We now focus on the monodromy map $\mu : L^\infty([0, 2\pi]) \to G_0$ given by $\mu(q) = \tilde{\Phi}(2\pi)$. Recall that $\mathfrak{g} = sl(2, \mathbb{R})$ is the space of 2×2 real matrices $A = (a_{ij})$ with $\operatorname{tr} A = 0$. If $g \in G$ and $M \in T_e(G) = \mathfrak{g}$ we write gM for the element in $T_g(G)$ obtained as the image of M by the differential of the translation g. Alternatively, the natural identification $T_g(G) = T_{\Pi(g)}(SL(2, \mathbb{R}))$ allows us to interpret gM as the matrix product $\Pi(g)M$. For an angle $\omega \in \mathbb{R}$, define $N_\omega \in \mathfrak{g}$ by

$$N_\omega = \begin{pmatrix} -\sin\omega\cos\omega & -\sin^2\omega \\ \cos^2\omega & \sin\omega\cos\omega \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -\sin 2\omega & -1 + \cos 2\omega \\ 1 + \cos 2\omega & \sin 2\omega \end{pmatrix};$$

notice that the matrices N_ω form a circle in the plane $a_{21} - a_{12} = 1$.

Proposition 2.1 *The monodromy map* $\mu : L^\infty([0, 2\pi]) \to G_0$ *is smooth with derivative given by*

$$(D\mu(q))w = \mu(q)\left(\int_0^{2\pi} w(t)\rho(\theta(t))N_{\theta(t)}dt\right).$$

Furthermore, μ is uniformly continuous on bounded subsets of $L^\infty([0, 2\pi])$.

Proof: The variation of the fundamental solutions v_i with respect to the potential is a familiar computation (see [12]), as are expressions for higher derivatives. The formula above for the derivative of the monodromy is then easy.

An alternative way of obtaining this formula is to (temporarily) allow for potentials which are distributions. A straightforward computation yields

$$\rho(\theta(t))N_{\theta(t)} = (\Phi(t))^{-1}N_0\Phi(t) = \begin{pmatrix} -v_1(t)v_2(t) & -v_2^2(t) \\ v_1^2(t) & v_1(t)v_2(t) \end{pmatrix}.$$

Thus, if δ_{t_0} is the Dirac delta centered at t_0, $0 < t_0 < 2\pi$, it is easy to verify that

$$\mu(q + a\delta_{t_0}) = \mu(q)(\tilde{\Phi}(t_0))^{-1}\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}\tilde{\Phi}(t_0)$$

and therefore $(D\mu(q))\delta_{t_0} = \mu(q)\rho(\theta(t))N_{\theta(t)}$.

Clearly, a potential $q \in L^\infty$ induces fundamental solutions, monodromy matrices and functions θ and ρ which are bounded by simple expressions in $|q|_{L^\infty}$. The formula obtains bounds for the derivative and therefore uniform continuity on L^∞-bounded sets. ∎

Assume $p \geq 1$ so that $H^p([0, 2\pi]) \subset C^0([0, 2\pi])$. The formula for $D\mu$ in the proof above implies the known fact that the functions v_1^2, v_1v_2 and v_2^2 are taken by $D\mu$ to a basis of $T_{\mu(q)}G$, whence μ is a submersion. We need a more workable triple of generators, however. A *bump* centered at $t_0 \in \mathbb{S}^1$ is a smooth nonnegative function from $\mathbb{S}^1$ to $\mathbb{R}$ whose support is a small interval centered at t_0; the size of the support is the *width* of the bump. Potentials will be altered by adding bumps in order to adjust the value of μ.

Lemma 2.2 *Let $q \in H^p([0, 2\pi])$ be a potential.*

(a) *Set $\epsilon^{-2} = |q|_{L^\infty}$. If $0 < t^+ - t^- < \epsilon$ then $\theta(t^-) < \theta(t^+) < \theta(t^-) + \pi$.*

(b) *Let ℓ_i, $i = 1, 2, 3$, be bumps with disjoint supports contained in an interval $[t^-, t^+]$ with $\theta(t^+) < \theta(t^-) + \pi$. Then the vectors $D\mu(q)\ell_i$, $i = 1, 2, 3$, are linearly independent. In particular, $\mu : H^p([0, 2\pi]) \to G_0$ and $\mu : H^p(\mathbb{S}^1) \to G_0$ are submersions.*

Proof: With the hypothesis of item (a), let v be the solution of $v(t^-) = 0$, $v'(t^-) = 1$, $v''(t) = q(t)v(t)$; we claim that $v'(t) > 0$ for all $t \in (t^-, t^+)$. Assume by contradiction that $t_{\max} \leq t^+$ is the first local maximum of v (after t^-) so that $v'(t_{\max}) = 0$; set $v_{\max} = v(t_{\max})$. The maximum value of v' in the interval $[t^-, t_{\max}]$ is at least $v_{\max}/\epsilon$ and therefore there exists t in this interval with $v''(t) < -v_{\max}/(\epsilon^2)$ and therefore $v''(t) = q(t)v(t) < -\epsilon^{-2}v(t)$ and $|q(t)|_{L^\infty} > \epsilon^{-2}$, a contradiction.

The claim implies the linear independence of the vectors $(v(t^-), v'(t^-))$ and $(v(t^+), v'(t^+))$. Since v is in the linear span of v_1 and v_2, $(v_1(t^-), v_2(t^-))$ and $(v_1(t^+), v_2(t^+))$ are not collinear, concluding the proof of item (a).

Let $[t_i^-, t_i^+]$ be the support of ℓ_i. Without loss of generality,

$$\theta(t^-) \le \theta(t_1^-) < \theta(t_1^+) < \theta(t_2^-) < \theta(t_2^+) < \theta(t_3^-) < \theta(t_3^+) \le \theta(t^+) < \theta(t^-) + \pi.$$

Note that $\theta(t)$ is strictly increasing in t. Write

$$(\mu(q))^{-1}(D\mu(q))\ell_i = \int_{t_i^-}^{t_i^+} \ell_i(t)\rho(\theta(t))N_{\theta(t)}dt = \int_{\theta(t_i^-)}^{\theta(t_i^+)} \frac{\ell_i(\theta^{-1}(\tau))\rho(\tau)}{\theta'(\theta^{-1}(\tau))}N_\tau d\tau.$$

Thus, up to a positive multiplicative factor, $(\mu(q))^{-1}(D\mu(q))\ell_i$ is a convex combination of matrices N_τ in the arc from $N_{\theta(t_i^-)}$ to $N_{\theta(t_i^+)}$ and, in particular, lies in the plane $a_{21} - a_{12} = 1$. Figure 2 illustrates that if we take a point in the convex hull of each arc we necessarily form a non-degenerate triangle on this plane and therefore the vectors $(\mu(q))^{-1}(D\mu(q))\ell_i$ are linearly independent. ∎
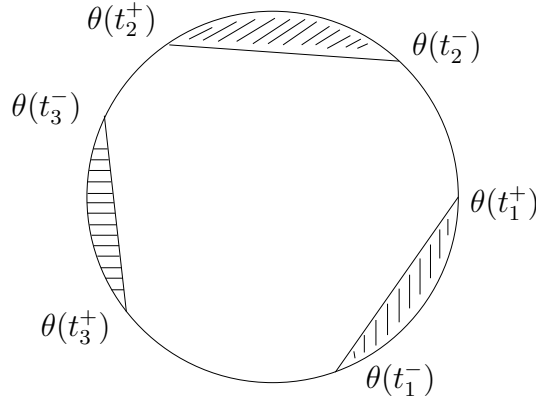

Figure 2: Convex hulls of three arcs

3   The nonlinear monodromy map μ_f

For a smooth nonlinearity $f \in C^\infty(\mathbb{R}, \mathbb{R})$ define the nonlinear monodromy map $\mu_f : L^\infty([0, 2\pi]) \to G_0$ by $\mu_f(u) = \mu(f' \circ u)$. We often consider restrictions of μ_f to smaller spaces such as $\mu_f|_{H^p(\mathbb{S}^1)}$, $p \ge 1$. The critical set $\mathcal{C}$ of the operator $F : H^p(\mathbb{S}^1) \to H^{p-2}(\mathbb{S}^1)$ defined by $F(u) = -u'' + f(u)$ is $\mathcal{C} = (\mu_f|_{H^p(\mathbb{S}^1)})^{-1}(T_2 \cap G_0)$, where $T_2 \subset G$ is the set of lifted matrices with trace 2. More generally, a real separable Banach space X is *smoothing* if the inclusions $C^\infty(\mathbb{S}^1) \subset X \subset L^\infty([0, 2\pi])$ are continuous.

Proposition 3.1 *Let f be a smooth nonlinearity.*

(a) *The map $\mu_f : L^\infty([0, 2\pi]) \to G_0$ is smooth. Given $u \in L^\infty(\mathbb{S}^1)$, define θ and ρ as before for the potential $f' \circ u$. Then, for any $v \in L^\infty([0, 2\pi])$,*

$$(D\mu_f(u))v = \mu_f(u) \left(\int_0^{2\pi} f''(u(t))v(t)\rho(\theta(t))N_{\theta(t)}dt \right).$$

(b) *Let X be a smoothing Banach space; assume f' nowhere flat. If $u \in X$ is not constant or is constant with $f''(u) \neq 0$, then the derivative $D(\mu_f|_X)(u)$ is a surjective linear map and $\mu_f|_X$ is a local submersion at u. If $u \in X$ is constant which is a local extremum of f' then u is an isolated point in the level set $(\mu_f|_X)^{-1}(\mu_f(u))$.*

Proof: Item (a) follows directly from Proposition 2.1. The first case in item (b) follows from item (a). From standard oscillation theory, if $f'(u_1(t)) \leq f'(u_2(t))$, $u_1 \neq u_2$, then the arguments of either column of $\mu_f(u_1)$ are respectively larger than those of $\mu_f(u_2)$. This implies the rest of (b), which alternatively follows from a simple local argument using Proposition 2.1, left to the reader. ∎

If f'' and f''' have no common zeroes then these cover all the possibilities for $u \in X$. For flat f', more complicated scenarios might occur, including accumulations of isolated points in level sets $(\mu_f|_X)^{-1}(g)$. Constant functions u are therefore potentially nasty objects: for a smoothing Banach space X, let $X^* = X - \{u = \text{const.}\}$: recall that X^* is homeomorphic to the separable Hilbert space $\mathbb{H}$ and diffeomorphic to $\mathbb{H}$ if X itself is Hilbert. Denote by λ the Lebesgue measure in $\mathbb{R}$.

Proposition 3.2 *Let X be a smoothing Banach space and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with nowhere flat derivative. Let $u \in X$ and ℓ_i, $i = 1, 2, 3$, be bump functions with disjoint supports contained in an interval $I \subset (0, 2\pi)$ such that $f''(u(t)) \neq 0$ for all $t \in I$. Then, for every $E > 0$ there exists $\epsilon > 0$ such that if $|u|_{L^\infty} < E$ and $\lambda(I) < \epsilon$ then the vectors $D\mu_f(u)\ell_i$, $i = 1, 2, 3$, are linearly independent. Thus, the function $\mu_f : X^* \to G_0$ is a submersion.*

Notice that we are not claiming (yet) that $\mu_f : X^* \to G_0$ is surjective: this is Proposition 3.6.

Proof: Let E' be such that $|x| < E$ implies $|f'(x)| < E'$ and take $\epsilon = \sqrt{1/E'}$. We may assume without loss of generality that $f''(u(t)) > 0$ for all $t \in I$ so that $\tilde{\ell}_i(t) = f''(u(t))\ell_i(t)$, $i = 1, 2, 3$, are bump functions. From lemma 2.2, the three vectors $D\mu(f' \circ u)\tilde{\ell}_i = D\mu_f(u)\ell_i$ are linearly independent and we are done. ∎

The following corollary generalizes the first part of Proposition 3.2; it will be needed in the proof of Theorem 3.

Corollary 3.3 *Let $X \subset C^0([0, 2\pi])$ be a separable Banach space with continuous inclusion such that the closure of X is either $C^0(\mathbb{S}^1)$ or $C^0([0, 2\pi])$. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with nowhere flat derivative. The function $\mu_f : X^* \to G_0$ is a submersion.*

Proof: We must prove that $D\mu_f(u) : X \to T_{\mu_f(u)}G$ is surjective. By Proposition 3.2, $D\mu_f(u)(C^0([0, 2\pi])) = D\mu_f(u)(C^0(\mathbb{S}^1)) = T_{\mu_f(u)}G$ and the result follows by density. ∎

The rest of the section is dedicated to showing in Proposition 3.6 that $\mu_f : X \to G_0$ is surjective for smoothing spaces X. Notice that the image of μ_f clearly equals G_0 if $f' : \mathbb{R} \to \mathbb{R}$ is a diffeomorphism. Indeed, given $g \in G_0$ let $q \in C^\infty(\mathbb{S}^1)$ with $\mu(q) = g$: the function $u = (f')^{-1} \circ q$ satisfies $\mu_f(u) = \mu(f' \circ u) = g$. Similarly, Theorem 1 follows easily from results in [6] if f' is a diffeomorphism. The idea in the proofs of Proposition 3.6 and Theorem 3 is to first operate with a discontinuous right inverse for f', then fix the discontinuities of the potential and add bumps in order not to change the monodromy.

We call a function $h : \mathbb{R} \to \mathbb{R}$ *piecewise smooth* if there exists a discrete set Y_h such that if I is a connected component of $\mathbb{R} - Y_h$ then there exists a continuous function $h_I : \overline{I} \to \mathbb{R}$ such that $(h_I)|_I = h|_I$ is smooth. Notice that an element $x \in Y_h$ may be a discontinuity of h or a point where h is continuous but not smooth, such as $x = 0$ for $h(x) = x^{1/3}$. The proof of the following lemma is left to the reader.

Lemma 3.4 *If $h : \mathbb{R} \to \mathbb{R}$ is smooth, nowhere flat and surjective then there exists a piecewise smooth function $h^\sharp : \mathbb{R} \to \mathbb{R}$ with $(h \circ h^\sharp)(x) = x$ for all x.*

Discontinuities in $u_0 = (f')^\sharp \circ q$ will be handled by considering a smooth function u_1 such that $|u_1 - u_0|_{L^1}$ is small. In particular, $\mu_f(u_1)$ is close to $\mu_f(u_0)$: adding appropriate bumps to u_1 produces a smooth function u_2 with $\mu_f(u_2) = \mu_f(u_0) = g$.

Endow the Lie algebra $\mathfrak{g} = sl(2, \mathbb{R})$ of G with an Euclidean metric and let $\exp_{\mathfrak{g}} : \mathfrak{g} \to G$ be the exponential map: take $\epsilon_{\mathfrak{g}} > 0$ to be such that the restriction of $\exp_{\mathfrak{g}}$ to the ball in $\mathfrak{g}$ of center 0 and radius $\epsilon_{\mathfrak{g}}$ is injective. For $\epsilon \leq \epsilon_{\mathfrak{g}}$, let $B_\epsilon \subset G$ be the image under $\exp_{\mathfrak{g}}$ of the ball of center 0 and radius ϵ. The sets B_ϵ are invariant under inversion ($h \in B_\epsilon$ if and only if $h^{-1} \in B_\epsilon$) but not under conjugation: in general, $B_\epsilon \neq h^{-1}B_\epsilon h$. By continuity, given $\epsilon > 0$, $\epsilon \leq \epsilon_{\mathfrak{g}}$, there exists $\delta > 0$ such that $B_\delta B_\delta \subseteq B_\epsilon$ (i.e., if $h_1, h_2 \in B_\delta$ then $h_1 h_2 \in B_\epsilon$).

Lemma 3.5 *Consider the monodromy $\mu_f : L^\infty([0, 2\pi]) \to G_0$ for a smooth non-linearity $f : \mathbb{R} \to \mathbb{R}$. Then for all $M \in \mathbb{R}$ and for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $u_1, u_2 \in L^\infty([0, 2\pi])$*

$$|u_1|_{L^\infty}, |u_2|_{L^\infty} < M, \quad \lambda(\{t | u_1(t) \neq u_2(t)\}) < \delta \quad \Longrightarrow \quad \mu_f(u_2)(\mu_f(u_1))^{-1} \in B_\epsilon.$$

Proof: Let $\tilde{M} = \sup(\{|f'(x)|, |x| < M\})$. Then $|f' \circ u_i|_{L^\infty} < \tilde{M}$ and the condition $\lambda(\{t | u_1(t) \neq u_2(t)\}) < \delta$ implies $|f' \circ u_2 - f' \circ u_1|_{L^1} < 2\tilde{M}\delta$. From Proposition 2.1, the monodromy map $\mu : L^1([0, 2\pi]) \to G_0$ is uniformly continuous with respect to the L^1 norm in the set $\{q \in L^1([0, 2\pi]), |q|_{L^\infty} < \tilde{M}\}$, completing the proof. ∎

It will be convenient to restrict the monodromy to intervals. For $T = [t_-, t_+] \subset [0, 2\pi]$ and $u \in L^\infty(T)$, let $\tilde{\Phi} : T \to G$ be the only solution of

$$\tilde{\Phi}(t_-) = I, \quad \tilde{\Phi}'(t) = \begin{pmatrix} 0 & 1 \\ f'(u(t)) & 0 \end{pmatrix} \tilde{\Phi}(t)$$

and define the *T-variation* $\mu_{f,T}(u) = \tilde{\Phi}(t_+)$. Variations juxtapose, and the order is important: if $t_0 < t_1 < t_2$ and $u \in L^\infty([t_0, t_2])$ then $\mu_{f,[t_0,t_2]}(u) = \mu_{f,[t_1,t_2]}(u)\mu_{f,[t_0,t_1]}(u)$. In particular, $\mu_f(u) = \mu_{f,T_2}(u)\mu_{f,T_1}(u)$.

Proposition 3.6 *Let X be a smoothing Banach space and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that f' is nowhere flat and surjective. Then $\mu_f : X^* \to G_0$ is surjective.*

Proof: Take $g \in G_0$. For y_0 a regular point of $(f')^\sharp$, set $x_0 = (f')^\sharp(y_0)$ so that x_0 is a regular point of f' and $f'(x_0) = y_0$. From the Kepler transform construction, outlined in the previous section, there exists $\epsilon_1 > 0$ and $q \in C^\infty([0, 2\pi])$ such that $\mu(q) = g$, q is nonflat in $(\epsilon_1, 2\pi - \epsilon_1)$ and q is constant equal to y_0 in $[0, \epsilon_1] \cup [\epsilon_1, 2\pi]$. Take $u_0 = (f')^\sharp \circ q$: clearly $u_0 \in L^\infty([0, 2\pi])$, $\mu_f(u_0) = g$. The function u_0 is piecewise smooth: let $T_1 = [0, 2\pi - \epsilon_1]$, $T_2 = [2\pi - \epsilon_1, 2\pi]$ and $Y \subset T_1$ be the (discrete) set of discontinuities of u_0. We need to alter u_0 in an open neighborhood $\breve{Y} \supset Y$, $\breve{Y} \subset T_1$, and add bumps in T_2 so as to obtain $u_2 \in X$ with $\mu_f(u_2) = g$.

Let ℓ_i, $i = 1, 2, 3$, be bumps with disjoint supports contained in T_2. From Lemma 2.2 we may assume the linear independence of the vectors $D\mu(q)\ell_i$ and therefore of the vectors $D\mu_{f,T_2}(u_0)\ell_i$. From the inverse function theorem, there exists an open neighborhood $B \subset G_0$ of $\mu_{f,T_1}(u_0)$ such that for any $h \in B$ there exist a_i, $i = 1, 2, 3$, which adjust the monodromy:

$$\mu_{f,T_2}\left(u_0 + \sum_{i=1,2,3} a_i \ell_i\right) h = g.$$

Let $E = 2|u_0|_{L^\infty}$. Use Lemma 3.5 to obtain $\epsilon_2 > 0$ be such that

$$|u_1|_{L^\infty} < 2E, \quad \lambda(\{t \in T_1 | u_0(t) \neq u_1(t)\}) < \epsilon_2 \quad \Rightarrow \quad \mu_{f,T_1}(u_1) \in B.$$

Choose an open neighborhood $\check{Y} \supset Y$ with $\check{Y} \subset T_1$ and $\lambda(\check{Y}) < \epsilon_2$. Let u_1 be an arbitrary smooth function coinciding with u_0 in $[0, 2\pi] \smallsetminus \check{Y}$, $|u_1|_{L^\infty} < 2E$. We have $h = \mu_{f,T_1}(u_1) \in B$ and therefore there exist a_i, $i = 1, 2, 3$, such that $u_2 = u_1 + \sum a_i \ell_i$ satisfies $\mu_f(u_2) = g$. ∎

4   Levels of generic μ_f are contractible

For the linear case, the levels sets of the monodromy map μ were explicitly parametrized in [6]. Indeed, for a smoothing Banach space X, let $\mathrm{Lev}^X(g) \subset X$ be the level set $\mu^{-1}(\{g\})$, where $\mu : X \to G_0 \subset G$ is the monodromy map. Theorem 3 in [6] gives a diffeomorphism between $\mathrm{Lev}^X(g)$ and $\mathbb{H}$ for $X = H^p([0, 2\pi])$, $p \geq 0$: in particular, $\mathrm{Lev}^X(g)$ is contractible. In this section we generalize this last result: let $\mathrm{Lev}_f^X(g)$ be the level set $\mu_f^{-1}(\{g\}) \cap X^*$ where $X^* = X - \{u = \text{const.}\}$.

Theorem 3 *Let X be a smoothing Banach (resp. Hilbert) space and $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that f' is nowhere flat and surjective. Then for any $g \in G_0$ the level $\mathrm{Lev}_f^X(g) \subset X^*$ is a nonempty contractible Banach (resp. Hilbert) submanifold of codimension 3.*

The fact that $\mathrm{Lev}_f^X(g)$ is a submanifold follows from Proposition 3.2. As discussed in Proposition 3.1, the more natural level set $\mu_f^{-1}(\{g\}) \subset X$ may not be a Banach manifold at constants x for which $f''(x) = 0$. Removing such points from $\mu_f^{-1}(\{g\})$ obtains a Banach manifold Lev_0. Notice that in the definition of $\mathrm{Lev}_f^X(g)$ we remove from $\mu_f^{-1}(\{g\})$ all constant functions so that $\mathrm{Lev}_f^X(g) \subseteq \mathrm{Lev}_0 \subseteq \mu_f^{-1}(\{g\})$. The inclusion $\mathrm{Lev}_f^X(g) \subseteq \mathrm{Lev}_0$ is a homotopy equivalence since the two Banach manifolds differ by the excision of a subset of infinite codimension; the set $\mu_f^{-1}(\{g\})$, on the other hand, may contain additional isolated points.

It might seem that different choices of X would demand different arguments. Fortunately, this is not so: Theorem 2 from [5], transcribed in the introduction, implies the homotopy equivalence of the spaces $\mathrm{Lev}_f^X(g)$ for different choices of X. It is then sufficient to prove that all homotopy groups of $\mathrm{Lev}_f^X(g)$ are trivial. For the standard monodromy map μ, i.e., for the linear case $f(x) = x^2/2$, this result is proved in [6]. Now, for $X = H^1(\mathbb{S}^1)$, given a loop $\gamma : \mathbb{S}^k \to \mathrm{Lev}_f^X(g)$, define $\gamma_{\mathrm{Lin}} : \mathbb{S}^k \to H^1(\mathbb{S}^1)$ by $\gamma_{\mathrm{Lin}}(s) = f' \circ \gamma(s)$ so that $\mu(\gamma_{\mathrm{Lin}}(s)) = g$ for all $s \in \mathbb{S}^k$. From [6], the loop γ_{Lin} admits an extension $\Gamma_{\mathrm{Lin}} : \mathbb{B}^{k+1} \to H^1(\mathbb{S}^1)$ with $\mu(\Gamma_{\mathrm{Lin}}(s)) = g$ for all $s \in \mathbb{B}^{k+1}$. In order to obtain $\Gamma : \mathbb{B}^{k+1} \to \mathrm{Lev}_f^{H^1}(g)$ such that $\mu_f(\Gamma(s)) = g$, we might want to define $\Gamma_{\mathrm{Lin}}(s) = f' \circ \Gamma(s)$: such a

construction will not respect $H^1(\mathbb{S}^1)$ if f' is not invertible. As in Proposition 3.6, the idea of the proof is to use the piecewise smooth right inverse $(f')^\sharp$ of f': we must now regularize a family of discontinuous potentials without changing their monodromies.

Proof of Theorem 3: For $\epsilon > 0$, let $R_\epsilon \subset \mathbb{C}$ be the closed rectangle of complex numbers $z = a + bi$, $a \in [-\epsilon, 2\pi + \epsilon]$, $b \in [-\epsilon, \epsilon]$ and let A_ϵ be the set of continuous functions $f : R_\epsilon \to \mathbb{C}$ which are holomorphic in the interior of R_ϵ and which satisfy $f(t) \in \mathbb{R}$ for $t \in [-\epsilon, 2\pi + \epsilon]$ and $f(z + 2\pi) = f(z)$ whenever z and $z + 2\pi$ are in R_ϵ. Clearly, the inclusion $i : A_\epsilon \to H^p(\mathbb{S}^1)$ is bounded and, from Theorem 2 (with $X = H^p(\mathbb{S}^1)$ and $Y = A_\epsilon$) and Corollary 3.3, we may assume $\gamma : \mathbb{S}^k \to \mathrm{Lev}_f^{A_\epsilon}(g)$. From the same theorem, it suffices to extend γ to $\Gamma : \mathbb{B}^{k+1} \to \mathrm{Lev}_f^{C^0(\mathbb{S}^1)}(g)$.

Since $\gamma(s)$ is analytic for all s, each $\gamma(s)$ is nowhere flat. For a positive integer r, a smooth function $g : \mathbb{R} \to \mathbb{R}$ is *nowhere r-flat* if, for all $x \in \mathbb{R}$, there exists $r' = r'(x) \in \mathbb{Z}$, $0 < r' \leq r$, such that $g^{(r')}(x) \neq 0$. Continuity in the A_ϵ norm yields a uniform bound: there exists r_γ such that each $\gamma(s)$ is nowhere r_γ-flat (this is the reason we introduced the space A_ϵ). Assume f' to be nowhere r_f-flat; set $r = r_\gamma r_f$. Let $\gamma_{\mathrm{Lin}} : \mathbb{S}^k \to \mathrm{Lev}^{C^r(\mathbb{S}^1)}(g)$ be defined by $(\gamma_{\mathrm{Lin}}(s))(t) = f'((\gamma(s))(t))$; clearly, $\gamma_{\mathrm{Lin}}(s)$ is nowhere r-flat for all $s \in \mathbb{S}^k$. Again by Theorem 2, now setting $X = C^r(\mathbb{S}^1)$ and $Y = A_\epsilon$, there exists $\gamma_{\mathrm{An}} : \mathbb{S}^k \to \mathrm{Lev}^{A_\epsilon}(g)$ arbitrarily close to γ_{Lin} and a homotopy $\Gamma_{\mathrm{Lin,An}} : [0,1] \times \mathbb{S}^k \to \mathrm{Lev}^{C^r(\mathbb{S}^1)}$ joining γ_{Lin} and γ_{An}. We may furthermore assume that $\Gamma_{\mathrm{Lin,An}}(\tau, s)$ is nowhere r-flat for all $(\tau, s) \in [0,1] \times \mathbb{S}^k$. Since $\mathrm{Lev}^{A_\epsilon}(g)$ is contractible there exists $\Gamma_{\mathrm{An}} : \mathbb{B}^{k+1} \to \mathrm{Lev}^{A_\epsilon}(g)$ extending γ_{An}. Juxtapose $\Gamma_{\mathrm{Lin,An}}$ and Γ_{An} to define $\Gamma_{\mathrm{Lin}} : \mathbb{B}^{k+1} \to \mathrm{Lev}^{C^r(\mathbb{S}^1)}(g)$ extending γ_{Lin}:

$$\Gamma_{\mathrm{Lin}}(s) = \begin{cases} \Gamma_{\mathrm{An}}(2s), & ||s|| \leq 1/2, \\ \Gamma_{\mathrm{Lin,An}}(2 - 2||s||, s/||s||), & ||s|| \geq 1/2; \end{cases}$$

notice that $\Gamma_{\mathrm{Lin}}(s)$ is nowhere flat for all $s \in \mathbb{B}^{k+1}$. Define $\tilde{\Gamma}_{\mathrm{Lin}} : \mathbb{B}^{k+1} \times \mathbb{S}^1 \to \mathbb{R}$ by $\tilde{\Gamma}_{\mathrm{Lin}}(s, t) = (\Gamma_{\mathrm{Lin}}(s))(t)$ (there will be similar correspondences between Γ_1, Γ_2, Γ_3 below and their counterparts $\tilde{\Gamma}_1$, $\tilde{\Gamma}_2$, $\tilde{\Gamma}_3$; deformations are often easier to describe from this second point of view).

Let $(f')^\sharp$ be as in lemma 3.4. Let $\tilde{\Gamma}_1 : \mathbb{B}^{k+1} \times \mathbb{S}^1 \to \mathbb{R}$ be the bounded (possibly discontinuous) function $(f')^\sharp \circ \tilde{\Gamma}_{\mathrm{Lin}}$: define $\Gamma_1 : \mathbb{B}^{k+1} \to L^\infty([0, 2\pi])$ by $(\Gamma_1(s))(t) = \tilde{\Gamma}_1(s, t)$. Notice that $\mu_f(\Gamma_1(s)) = g$ for all $s \in \mathbb{B}^{k+1}$ since

$$\mu_f(\Gamma_1(s)) = \mu(f' \circ \Gamma_1(s)) = \mu(\Gamma_{\mathrm{Lin}}(s)) = g.$$

Let $Y_1 \subset \mathbb{B}^{k+1} \times \mathbb{S}^1$ be the set of discontinuities of $\tilde{\Gamma}_1$. From nowhere-flatness, Y_1 intersects circles $(s, \cdot)$ in sets of measure zero; call this set $Y_1(s) \subset \mathbb{S}^1$:

$$Y_1(s) = \Pi_{\mathbb{S}^1}(Y_1 \cap (\{s\} \times \mathbb{S}^1)).$$

Recall that there may exist points where $(f')^\sharp$ is continuous but not smooth: at such points, $\tilde{\Gamma}_1$ loses some differentiability but since our aim is to construct $\Gamma : \mathbb{B}^{k+1} \to \mathrm{Lev}_f^{C^0(\mathbb{S}^1)}(g)$, this is not an issue and such points require no special treatment. We modify Γ_1 by adding a thick shell to its domain so that discontinuities will stay away from the boundary of the domain. With the identification $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$, set $t_0 = 0 \in \mathbb{S}^1$; define

$$\tilde{\Gamma}_2(s,t) = \begin{cases} \tilde{\Gamma}_1(2s,t), & \text{if } ||s|| \leq 1/2, \\ \tilde{\Gamma}_1(s/||s||,t), & \text{if } ||s|| \leq \frac{1}{2} + \frac{1}{10}d(t,(Y_1(s/||s||) \cup \{t_0\})), \\ \gamma(s/||s||,t), & \text{otherwise.} \end{cases}$$

The three cases are indicated by I, II and III in the right side of Figure 3. The function $\tilde{\Gamma}_2$ is also discontinuous: let $Y_2 \subset \mathbb{B}^{k+1} \times \mathbb{S}^1$ be its set of discontinuities. As for Y_1, define $Y_2(s) \subset \mathbb{S}^1$ to be the projection on $\mathbb{S}^1$ of the intersection of Y_2 with the circle $(s,\cdot)$; clearly, the measure of $Y_2(s)$ is zero for all $s \in \mathbb{B}^{k+1}$. The closed set Y_2 keeps away from the boundary (it is contained in a ball of radius 0.85) and $\tilde{\Gamma}_2$ is a (discontinuous) extension of $\tilde{\gamma}$. Notice that $\mu_f(\Gamma_2(s)) = g$ for all s since, for $s \leq 1/2$, $\Gamma_2(s) = \Gamma_1(2s)$ and, for $s \geq 1/2$,

$$\mu_f(\Gamma_2(s)) = \mu(f' \circ \Gamma_2(s)) = \mu(\gamma_{\mathrm{Lin}}(s/||s||)) = g.$$

Assume $|\tilde{\Gamma}_2(s,t)| < E$ for all $(s,t) \in \mathbb{B}^{k+1} \times \mathbb{S}^1$. We must regularize $\tilde{\Gamma}_2$ without changing the monodromy: at this point a sketch of what comes ahead is appropriate.
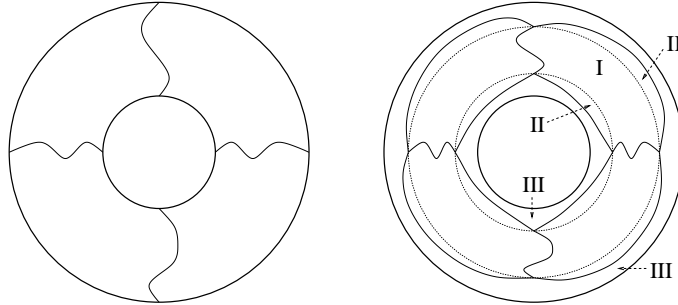

Figure 3: The sets of discontinuities Y_1 and Y_2 of Γ_1 and Γ_2 (here, $k = 0$).

We will take an open set $\check{Y}_2 \subset \mathbb{B}^{k+1} \times \mathbb{S}^1$, $Y_2 \subset \check{Y}_2$ far from the boundary (i.e., $(s,t) \in \check{Y}_2$ implies $||s|| < 9/10$) and within it $\tilde{\Gamma}_2$ will be altered to obtain a continuous function $\tilde{\Gamma}_3$ with $|\tilde{\Gamma}_3(s,t)| < 2E$ for all $(s,t) \in \mathbb{B}^{k+1} \times \mathbb{S}^1$. Notice that Γ_3 extends the original loop γ continuously but has slightly wrong values for the monodromy in the interior.

The monodromy of Γ_3 will be corrected with bumps roughly as done in Proposition 3.6: in this parametrized version, however, we have to find places to support the bumps (the shaded boxes in Figure 4) and coordinate their effect. More precisely, set $Z_t = \{s \in \mathbb{B}^{k+1} \,|\, (s,t) \notin Y_2, f''(\tilde{\Gamma}_2(s,t)) \neq 0\}$ for $t \in (0, 2\pi)$. The sets Z_t form an open cover of the compact ball $\mathbb{B}^{k+1}$. Let $J = \{1, 2, \ldots, j_{\max}\}$ be a finite index set such that the sets Z_{t_j} cover $\mathbb{B}^{k+1}$. Let $\check{K}_j \subset K_j \subset Z_{t_j}$ be such that K_j is compact and the sets $\check{K}_j$ form an open cover of $\mathbb{B}^{k+1}$. Let $\epsilon_0 > 0$ be such that the sets $K_j \times [t_j - \epsilon_0, t_j + \epsilon_0] \subset \mathbb{B}^{k+1} \times \mathbb{S}^1$, $j \in J$, are pairwise disjoint, disjoint from Y_2 and disjoint from the set of pairs (s,t) for which $f''(\tilde{\Gamma}_2(s,t)) = 0$. Assume furthermore that $\epsilon_0 < \epsilon$, where ϵ is given by Proposition 3.2. The sets $K_j \times T_j$, $T_j = [t_j - \epsilon_0, t_j + \epsilon_0]$, are shown schematically in Figure 4. In the figure, the dotted radii are the sets $\mathbb{B}^{k+1} \times \{t_j\}$, the two thick circles form the boundary $\mathbb{S}^k \times \mathbb{S}^1$ of $\mathbb{B}^{k+1} \times \mathbb{S}^1$, the wiggly curve is Y_2 and the shaded boxes are $K_j \times T_j$: notice that the interiors $\check{K}_j$ indeed form an open cover in the figure.
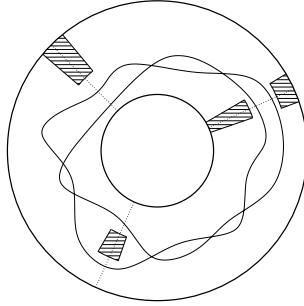

Figure 4: The sets $K_j \times T_j \subset \mathbb{B}^{k+1} \times \mathbb{S}^1$ avoid Y_2 (here, $k = 0$).

We must first choose the bumps ℓ_i^j: we will have

$$\Gamma(s) = \Gamma_3(s) + \sum_{i=1,2,3; \, j \in J} a_i^j(s) \ell_i^j$$

where $a_i^j : \mathbb{B}^{k+1} \to \mathbb{R}$ are continuous functions with support contained in K_j. We next measure their capacity of correcting the monodromy. Only then we choose $\check{Y}_2$ so thin that Lemma 3.5 implies that the monodromy of $\Gamma_3(s)$ is so close to g that it can be fixed to obtain Γ with the formula above.

Before proceeding we must clarify how the implicit function theorem accounts for the capacity of a given triple of bumps to adjust monodromy. As in Proposition 3.6, we use the T-variation $\mu_{f,T}$. For $u \in L^\infty([0, 2\pi])$ and $0 < t_- < t_+ < 2\pi$, set $T_- = [0, t_-]$, $T_0 = [t_-, t_+]$, $T_+ = [t_+, 2\pi]$, $h_- = \mu_{f,T_-}(u)$, $h_0 = \mu_{f,T_0}(u)$ and $h_+ = \mu_{f,T_+}(u)$: we have $\mu_f(u) = \mu_{f,[0,2\pi]}(u) = h_+ h_0 h_-$. Now suppose that ℓ_i, $i = 1, 2, 3$, are bumps supported in T_0 and consider $\tilde{u} = u + \sum_i a_i \ell_i$: then $\mu_f(\tilde{u}) = h_+ \mu_{f,T_0}(\tilde{u}) h_-$. We compare monodromies of u and $\tilde{u}$ by writing $\mu_{f,T_0}(\tilde{u}) = \psi \mu_{f,T_0}(u)$, which is equivalent to $\psi = (h_+)^{-1} \mu_f(\tilde{u})(\mu_f(u))^{-1} h_+$.

Claim: Let $K \subset \mathbb{B}^{k+1}$ be a compact set and $T = [t_-, t_+] \subset (0, 2\pi)$. Assume $K \times T$ to be disjoint from Y_2 and that $(s, t) \in K \times T$ implies $f''(\tilde{\Gamma}_2(s, t)) \neq 0$. Assume also that $t_+ - t_- < \epsilon$, where ϵ is given in Proposition 3.2. Let ℓ_i, $i = 1, 2, 3$, be bumps with disjoint supports contained in T with $|\ell_i|_{L^\infty} < 1$. Then there exists $\tilde{\epsilon} > 0$ such that, for all continuous functions $\psi : K \to B_{\tilde{\epsilon}} \subset G$ with $\psi(s) = I$ for $s \in \partial K$ there are continuous functions $a_i : K \to [-E, E]$, $i = 1, 2, 3$, with $a_i(s) = 0$ for $s \in \partial K$ for which

$$\mu_{f,T}\left(\Gamma_2(s) + \sum_{i=1,2,3} a_i(s)\ell_i\right) = \psi(s)\mu_{f,T}(\Gamma_2(s)).$$

Proof of claim: Define $\zeta : K \times \mathbb{R}^3 \to G$ by

$$\zeta(s, a_1, a_2, a_3) = \mu_{f,T}\left(\Gamma_2(s) + \sum_i a_i\ell_i\right)(\mu_{f,T}(\Gamma_2(s)))^{-1}.$$

Clearly $\zeta(s, 0, 0, 0) = I$ for all $s \in K$. From Proposition 3.2, the three vectors $\partial\zeta_j/\partial a_i$ are linearly independent. From the implicit function theorem, there exists $\tilde{\epsilon} > 0$ and a continuous function $\alpha : K \times B_{\tilde{\epsilon}} \to \mathbb{R}^3$ with $\alpha(s, I) = 0$ and $\zeta(s, \alpha(s, h)) = h$ for all $s \in K_j$ and $h \in B_{\tilde{\epsilon}}$. We can furthermore assume without loss of generality that $\|\alpha(s, h)\| < E$ for all $s \in K$ and $h \in B_{\tilde{\epsilon}}$. Now take $(a_1(s), a_2(s), a_3(s)) = \alpha(s, \psi(s))$, completing the proof of the claim. $\quad\square$

We now put the partial results to work: for each $j \in J$, construct bumps $\ell_1^j, \ell_2^j, \ell_3^j$ with support contained in T_j and $|\ell_i^j|_{L^\infty} < 1$. Apply the claim to K_j, T_j and ℓ_i^j to obtain $\tilde{\epsilon}_j$; let ϵ_1 be the minimum among all $\tilde{\epsilon}_j$, $j \in J$. The set of all $\tilde{\Phi}(t)$ where

$$\tilde{\Phi}(0) = I, \quad \tilde{\Phi}'(t) = \begin{pmatrix} 0 & 1 \\ f'(u(t)) & 0 \end{pmatrix} \tilde{\Phi}(t), \quad t \in [0, 2\pi], \quad |u|_{L^\infty(\mathbb{S}^1)} < 2E,$$

is contained in a compact set $K' \subset G$. Let $\epsilon_2' > 0$ be such that

$$B_{\epsilon_2'} \subseteq \bigcap_{h \in K'} h^{-1} B_{\epsilon_1} h$$

and $\epsilon_2 > 0$ such that $B_{\epsilon_2} B_{\epsilon_2} \subseteq B_{\epsilon_2'}$. From Lemma 3.5, let $\epsilon_3 > 0$ be such that

$$|u_0|_{L^\infty}, |u_1|_{L^\infty} < 2E, \quad \lambda(\{t | u_0(t) \neq u_1(t)\}) < \epsilon_3 \quad \Rightarrow \quad \mu_f(u_0)(\mu_f(u_1))^{-1} \in B_{\epsilon_2}.$$

Select an open set $\check{Y}_2 \subset \mathbb{B}^{k+1} \times \mathbb{S}^1$, $Y_2 \subset \check{Y}_2$, which is removed from the boundary and satisfies $\lambda(\check{Y}_2(s)) < \epsilon_3$ for all $s \in \mathbb{B}^{k+1}$ where, as before,

$$\check{Y}_2(s) = \Pi_{\mathbb{S}^1}\left(\check{Y}_2 \cap (\{s\} \times \mathbb{S}^1)\right).$$

Define a continuous function $\tilde{\Gamma}_3$ with $|\tilde{\Gamma}_3(s,t)| < 2E$ for all s,t and a corresponding $\Gamma_3 : \mathbb{B}^{k+1} \to C^0(\mathbb{S}^1)$ coinciding with $\tilde{\Gamma}_2$ outside $\check{Y}_2$ (and otherwise arbitrary in $\check{Y}_2$). From the construction of ϵ_3, we have $\mu_f(\Gamma_3(s))g^{-1} \in B_{\epsilon_2}$ for all s (recall that $\mu_f(\Gamma_2(s)) = g$).

We will define inductively in j the functions a_i^j and

$$\Gamma_{4,j}(s) = \Gamma_{4,j-1}(s) + \sum_{i=1,2,3} a_i^j(s)\ell_i^j, \quad \Gamma_{4,0} = \Gamma_3;$$

notice that $\Gamma_{4,j_{\max}} = \Gamma$. The idea is that we activate one $K_j \times T_j$ box at a time.

Recall that $\exp_{\mathfrak{g}} : \mathfrak{g} \to G$ is the exponential map. Let $v_0 : \mathbb{B}^{k+1} \to \mathfrak{g}$ be defined by $\mu_f(\Gamma_3(s))g^{-1} = \exp_{\mathfrak{g}}(v_0(s))$; the function v_0 provides a linear measure for the error in the monodromy of Γ_3: we now construct intermediate functions $v_j : \mathbb{B}^{k+1} \to \mathfrak{g}$ with $v_{j_{\max}} = 0$ and functions $\Gamma_{4,j}$ with $\mu_f(\Gamma_{4,j}(s))g^{-1} = \exp_{\mathfrak{g}}(v_j(s))$. Let $r_j : \mathbb{B}^{k+1} \to [0,1]$ be a smooth partition of unity associated to the partition $\check{K}_j$, so that $\sum_j r_j = 1$ and the support of r_j is contained in $\check{K}_j$. Set

$$v_j(s) = \left(\sum_{j'>j} r_{j'}(s) \right) v_0(s);$$

notice that $v_j(s) = v_{j-1}(s)$ for $s \notin \check{K}_j$. Define inductively in j the functions a_i^j so that

$$\Gamma_{4,j}(s) = \Gamma_{4,j-1}(s) + \sum_{i=1,2,3} a_i^j(s)\ell_i^j, \quad \Gamma_{4,0} = \Gamma_3, \quad \mu_f(\Gamma_{4,j}(s))g^{-1} = \exp_{\mathfrak{g}}(v_j(s)) :$$

then set $\Gamma_{4,j_{\max}} = \Gamma$. We are left with showing that this is indeed possible, i.e., that the functions a_i^j are continuous and well defined.

Assume that $\Gamma_{4,j-1}$ has been constructed, in other words, the functions $a_i^{j'}$ have been obtained for $j' < j$ and we have $\mu_f(\Gamma_{4,j-1}(s))g^{-1} = \exp_{\mathfrak{g}}(v_{j-1}(s))$. We need functions a_i^j such that

$$\mu_f \left(\Gamma_{4,j-1}(s) + \sum_{i=1,2,3} a_i^j(s)\ell_i^j \right) g^{-1} = \exp_{\mathfrak{g}}(v_j(s)).$$

Notice that $a_i^j(s) = 0$ for $s \notin K_j$, as required. For $s \in K_j$ and $T_j = [t_j^-, t_j^+]$, set $h_-(s) = \mu_{f,[0,t_j^-]}(\Gamma_{4,j-1}(s))$, $h_0(s) = \mu_{f,T_j}(\Gamma_{4,j-1}(s))$, $h_+(s) = \mu_{f,[t_j^+,2\pi]}(\Gamma_{4,j-1}(s))$ so that

$$h_+(s)h_0(s)h_-(s) = \mu_f(\Gamma_{4,j-1}) = \exp_{\mathfrak{g}}(v_{j-1}(s))g.$$

Define $\psi_j : \mathbb{B}^{k+1} \to G$ so that $h_+(s)\psi_j(s)h_0(s)h_-(s) = \exp_{\mathfrak{g}}(v_j(s))g$; in other words,

$$\psi_j(s) = (h_+(s))^{-1} \exp_{\mathfrak{g}}(v_j(s)) \exp_{\mathfrak{g}}(-v_{j-1}(s)) h_+(s).$$

By construction, $h_+(s) \in K' \subset G$ and $\exp_{\mathfrak{g}}(v_j(s)), \exp_{\mathfrak{g}}(-v_{j-1}(s)) \in B_{\epsilon_2}$ whence $\exp_{\mathfrak{g}}(v_j(s)) \exp_{\mathfrak{g}}(-v_{j-1}(s)) \in B_{\epsilon_2'}$, so that $\psi_j(s) \in B_{\epsilon_1}$. Apply the claim to K_j, T_j, ℓ_i^j and ψ_j to obtain a_i^j and we are done. ∎

5   Hilbert manifolds with cone-like singularities

In this section we obtain normal forms near the critical set $\mathcal{C}$. We first consider a simple scenario. Recall that $\bowtie \subset \mathbb{R}^3$ is the cone $x^2 + y^2 - z^2 = 0$ with vertex $0 \in \mathbb{R}^3$.

Theorem 4 *Let $\mathcal{H}$ be an infinite dimensional smooth Hilbert manifold and $\sigma :$ $\mathcal{H} \to \mathbb{R}^3$ a smooth surjective submersion with contractible fibers. Set $\mathcal{C}_1 = \sigma^{-1}(\bowtie)$ and $\mathcal{C}_2 = \sigma^{-1}(\{0\})$. Then there exists a diffeomorphism*

$$\theta : (\mathcal{H}, \mathcal{C}_1, \mathcal{C}_2) \to (\mathbb{R}^3, \bowtie, \{0\}) \times \mathbb{H}.$$

The proof will be organized in two steps which are stated now and justified later. For a manifold with boundary X (of finite or infinite dimension), we write int X for its interior and ∂X for its boundary.

Step 1 *There is a closed tubular neighborhood $\mathcal{D} \subset \mathcal{H}$ of the smooth submanifold $\mathcal{C}_2$ and a diffeomorphism $\theta_1 : (\mathcal{D}, \partial\mathcal{D}) \to (D_1, \partial D_1) \times \mathcal{C}_2$ satisfying $\theta_1(\mathcal{C}_2) = 0 \times \mathcal{C}_2$ and $\theta_1(\mathcal{D} \cap \mathcal{C}_1) = (D_1 \cap \bowtie) \times \mathcal{C}_2$.*

Here D_r denotes the closed disk (or ball) of radius r in $\mathbb{R}^3$ centered at the origin. Recall that a tubular neighborhood $\mathcal{D}$ is the image by an embedding of the *unit disk bundle*, a subset of the normal bundle of $\mathcal{C}_2$ in $\mathcal{H}$, $\nu : \mathcal{D} \to \mathcal{C}_2$, which, in this case, is trivial since $\mathcal{C}_2$ is contractible. The fibers of $\nu : \mathcal{D} \to \mathcal{C}_2$ can be identified to the unit disk in $\mathbb{R}^3$, and therefore ν can be identified to $\Pi_1 : \mathcal{C}_2 \times D_1 \to \mathcal{C}_2$, where Π_1 is the projection on the first coordinate. Note that $\mathcal{D}$ is a (codimension zero) smooth submanifold with boundary, $\partial\mathcal{D}$ being diffeomorphic to $\mathcal{C}_2 \times \mathbb{S}^2$.

Since σ is a submersion and, for $\mathcal{D}$ satisfying Step 1, $\bowtie$ is transversal to ∂D_1, we have that $\mathcal{C}_1$ is also transversal to $\partial\mathcal{D}$. We then consider the smooth Hilbert manifold with boundary $\mathcal{V} = \mathcal{H} \smallsetminus \text{int}\,\mathcal{D}$. The subset $\mathcal{K} = \mathcal{C}_1 \smallsetminus \text{int}\,\mathcal{D}$ is a codimension 1 submanifold with boundary which intersects $\partial\mathcal{V} = \partial\mathcal{D}$ transversally.

Step 2 *There is a diffeomorphism $\theta_2 : (\mathcal{V}, \partial\mathcal{V}) \to (\mathbb{R}^3 \smallsetminus \text{int}\,D_1, \partial D_1) \times \mathcal{C}_2$ with $\theta_2(\mathcal{K}) = (\bowtie \smallsetminus \text{int}\,D_1) \times \mathcal{C}_2$.*

We may furthermore assume that θ_1 and θ_2, given by Steps 1 and 2, coincide on $\partial\mathcal{V} = \partial\mathcal{D}$. Indeed, start with θ_1 given by Step 1; in order to define θ_2 we first set $\theta_2|_{\partial\mathcal{V}} = \theta_1|_{\partial\mathcal{V}}$, $\theta_2|_{\partial\mathcal{V}} : (\partial\mathcal{V}, \partial\mathcal{V} \cap \mathcal{C}_1) \to (\partial D_1, \partial D_1 \cap \bowtie) \times \mathcal{C}_2$.

From Step 2 and the identification between $(\mathbb{R}^3 \smallsetminus \text{int}\,D_1, \bowtie \smallsetminus \text{int}\,D_1)$ and $(\partial D_1, \bowtie \cap \partial D_1) \times [1, \infty)$, extend this restriction to obtain the desired diffeomorphism θ_2. Combining such θ_1 and θ_2, and in view of the fact that $\mathcal{C}_2$ is diffeomorphic to $\mathbb{H}$, one obtains Theorem 4.

We need a few additional notations and observations. For a positive smooth function $\epsilon : \mathcal{C}_2 \to \mathbb{R}_+$, we denote by $(\mathcal{C}_2 \times D)_\epsilon = \{(p, v) \mid ||v|| \le \epsilon(p)\}$. Let

$\Pi_1 : (\mathcal{C}_2 \times D)_\epsilon \to \mathcal{C}_2$ be the first factor projection, whose fiber above $p \in \mathcal{C}_2$ is $p \times D_{\epsilon(p)}$. Note that if ϵ is a smooth function then $(\mathcal{C}_2 \times D)_\epsilon$ is a smooth Hilbert manifold with boundary and one can produce a fiber diffeomorphism $\theta_\epsilon : (\mathcal{C}_2 \times D)_\epsilon \to (\mathcal{C}_2 \times D)_{\mathbf{1}}$ where $\mathbf{1}$ denotes the constant function and the diffeomorphism keeps the first coordinate fixed.

To accomplish Step 1, we use a smooth closed tubular neighborhood $\omega : (\mathcal{C}_2 \times D)_\epsilon \to \mathcal{H}$, where $\epsilon : \mathcal{C}_2 \to \mathbb{R}_+$ is a smooth function so that the composition $\sigma \circ \omega$, when restricted to the fibers $p \times D_{\epsilon(p)}$ is a diffeomorphism onto its image. Clearly, for such ω, $(\sigma \circ \omega, \Pi_1) : (\mathcal{C}_2 \times D)_\epsilon \to \mathbb{R}^3 \times \mathcal{C}_2$ is a diffeomorphism onto the image. To construct such pair (ω, ϵ), we choose $\omega_0 : \mathcal{C}_2 \times \mathbb{R}^3 \to T(\mathcal{H})|_{\mathcal{C}_2}$, a splitting of the surjective bundle map $D\sigma : T(\mathcal{H})|_{\mathcal{C}_2} \to \mathcal{C}_2^*(T(\mathbb{R}^3))|_{\mathcal{C}_2} = \mathcal{C}_2 \times \mathbb{R}^3$, fixing base point, for which $D\sigma \circ \omega_0 = \mathrm{id}$. Such splitting exists since σ is a submersion. We choose a complete Riemannian metric on $\mathcal{H}$ and use the exponential map with respect to this metric to define the smooth map $e : \mathcal{C}_2 \times \mathbb{R}^3 \to \mathcal{H}$ (we regard $\mathcal{C}_2 \times \mathbb{R}^3$ as the total space of the normal bundle of $\mathcal{C}_2$ in $\mathcal{H}$). The differential of e at any point of $\mathcal{C}_2 \times 0$ (the zero section of the trivial bundle $\mathcal{C}_2 \times \mathbb{R}^3 \to \mathcal{C}_2$) is an isomorphism and therefore there exists $\epsilon_1 : \mathcal{C}_2 \to \mathbb{R}_+$ so that e restricted to $(\mathcal{C}_2 \times D)(\epsilon_1)$ is a diffeomorphism onto its image. Since the differential at zero of $\sigma \circ e$ when restricted to any fiber of $p \times \mathbb{R}^3 \to \mathcal{C}_2$ is an isomorphism, one obtains $\epsilon_2 : \mathcal{C}_2 \to \mathbb{R}_+$ so that $\sigma \circ e$, when restricted to the disk $D_{\epsilon_2(p)}$, is a diffeomorphism on the image. Choose $\epsilon \leq \min(\epsilon_1, \epsilon_2)$ but still smooth and positive, and take ω the restriction of e to $(\mathcal{C}_2 \times D)_\epsilon$, thus completing the proof of Step 1.

For Step 2, we need a few preliminary results. We begin with an easy consequence of item 1 of Proposition 3.1 in [5].

Lemma 5.1 *Suppose $(\mathcal{V}, \partial\mathcal{V})$ is a Hilbert manifold with boundary and that the inclusion $\partial\mathcal{V} \hookrightarrow \mathcal{V}$ is a homotopy equivalence. Then there exists a diffeomorphism $\theta : (\mathcal{V}, \partial\mathcal{V}) \to (\partial\mathcal{V} \times [1, \infty), \partial\mathcal{V} \times \{1\})$.*

We now use this lemma to prove an amplification (the lemma is the degenerate case $\mathcal{K} = \emptyset$).

Proposition 5.2 *Suppose $(\mathcal{V}, \partial\mathcal{V})$ is a Hilbert manifold with boundary, $\mathcal{K} \subset \mathcal{V}$ is a finite codimension submanifold such that $\mathcal{K}$ is transversal to $\partial\mathcal{V}$ and $\partial\mathcal{K} = \mathcal{K} \cup \partial\mathcal{V}$. Suppose also that the following inclusions are homotopy equivalences: $\partial\mathcal{V} \hookrightarrow \mathcal{V}$, $\partial\mathcal{K} \hookrightarrow \mathcal{K}$, $\partial\mathcal{V} \smallsetminus \partial\mathcal{K} \hookrightarrow \mathcal{V} \smallsetminus \mathcal{K}$. Then there exists a diffeomorphism $\theta : (\mathcal{V}, \partial\mathcal{V}) \to (\partial\mathcal{V} \times [1, \infty), \partial\mathcal{V} \times \{1\})$ so that $\theta(\mathcal{K}) = \partial\mathcal{K} \times [1, \infty)$.*

Proof: Construct a relative collar neighborhood ϕ of $(\partial\mathcal{V}, \partial\mathcal{K})$ in $(\mathcal{V}, \mathcal{K})$, i.e., a closed embedding $\phi : (\partial\mathcal{V} \times [1, 2], \partial\mathcal{V} \times \{1\}) \to (\mathcal{V}, \partial\mathcal{V})$ such that its restriction to $\partial\mathcal{V} \times \{1\}$ is the projection on the first coordinate and $\phi(\partial\mathcal{V} \times [1, 2]) \cap \mathcal{K} = \phi(\partial\mathcal{K} \times [1, 2])$. It follows from lemma 5.1 and from the homotopy equivalence $\partial\mathcal{K} \hookrightarrow \mathcal{K}$ that the pair $(\partial\mathcal{K} \times [1, \infty), \partial\mathcal{K} \times \{1\})$ is diffeomorphic to $(\mathcal{K}, \partial\mathcal{K})$:

we may assume that this diffeomorphism coincides with ϕ on $\partial\mathcal{K} \times [1,2]$ and we also call it ϕ. Let $\mathcal{K}^1 = \phi(\partial\mathcal{K} \times [1,2))$ and $\mathcal{K}^2 = \phi(\partial\mathcal{K} \times [2,\infty))$. Standard arguments ("pushing to infinity") imply that the triple $(\mathcal{V}, \partial\mathcal{V}, \mathcal{K})$ is diffeomorphic to $(\mathcal{V} \smallsetminus \mathcal{K}^2, \partial\mathcal{V}, \mathcal{K}^1)$. The set $\mathcal{W} = (\mathcal{V} \smallsetminus \mathcal{K}^2) \smallsetminus \phi(\partial\mathcal{V} \times [1,2))$ is a smooth manifold with boundary, $\partial\mathcal{W}$ being diffeomorphic to $(\partial\mathcal{V} \smallsetminus \mathcal{K}) \times \{2\}$. The homotopy equivalences in the statement imply that $\partial\mathcal{W} \to \mathcal{W}$ is a homotopy equivalence and then, by lemma 5.1, we have that $(\mathcal{W}, \partial\mathcal{W})$ and $(\partial\mathcal{W} \times [2,\infty), \partial\mathcal{W} \times \{2\})$ are diffeomorphic. We conclude that there exists a diffeomorphism from $(\mathcal{V} \smallsetminus \mathcal{K}^2, \partial\mathcal{V})$ to $(\partial\mathcal{V} \times [1,\infty) \smallsetminus \partial\mathcal{K} \times [2,\infty), \partial\mathcal{V} \times \{1\})$ which sends $\mathcal{K}^1$ into $\partial\mathcal{K} \times [1,2)$. By the same trick of pushing to infinity, there is a diffeomorphism from the last pair to $(\partial\mathcal{V} \times [1,\infty), \partial\mathcal{V} \times \{1\})$ which sends $\partial\mathcal{K} \times [1,2)$ into $\partial\mathcal{K} \times [1,\infty)$, concluding the proof. ∎

Step 2 is now accomplished and so is the proof of Theorem 4. In order to prove Theorem 1 in the next section, we need to strengthen Theorem 4 somewhat: this is done in Propositions 5.3 and 5.5 below. A *closed disk* $\mathcal{B}$ in a Hilbert manifold is a closed tubular neighborhood of a point, i.e., a closed set which is a codimension zero submanifold with contractible boundary.

Proposition 5.3 *Let $\mathcal{H}$ be an infinite dimensional smooth Hilbert manifold and $\sigma : \mathcal{H} \to \mathbb{R}^3$ a smooth surjective submersion with contractible fibers. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ (resp. $\mathcal{B}'_1$ and $\mathcal{B}'_2$) be two closed disks contained in the two contractible components of $\mathcal{H} \smallsetminus \mathcal{C}_1$ (resp. $(\mathbb{R}^3 \smallsetminus \bowtie) \times \mathbb{H}$). Then there exists a diffeomorphism $\theta : (\mathcal{H}, \mathcal{C}_1, \mathcal{C}_2) \to (\mathbb{R}^3, \bowtie, 0) \times \mathbb{H}$ so that $\theta(\mathcal{B}_i) = \mathcal{B}'_i$, $i = 1, 2$.*

Proof: First construct $\theta_1 : (\mathcal{H}, \mathcal{C}_1, \mathcal{C}_2) \to (\mathbb{R}^3, \bowtie, 0) \times \mathbb{H}$ using Theorem 4. The diffeomorphism θ_1 sends $\mathcal{C}_1$ into $\bowtie \times \mathbb{H}$ and can be easily modified away from a neighborhood of $\mathcal{C}_1$ to obtain the desired diffeomorphism θ in view of Fact 1 below, usually referred to as the Cerf Lemma. ∎

Fact 1 (Cerf lemma) *Let $\mathcal{B}$ be the closed disk in $\mathbb{H}$, $\phi_1, \phi_2 : \mathcal{B} \to \mathcal{H}$ be smooth codimension zero closed embeddings and $\mathcal{U} \subset \mathcal{H}$ an open set with $\phi_i(\mathcal{B}) \subset \mathcal{U}$. Then there is an isotopy $h_t : \mathcal{H} \to \mathcal{H}$, $t \in [0,1]$, with $h_t|_{\mathcal{H} \smallsetminus \mathcal{U}} = id$, $h_0 = id$ and $h_1 \circ \phi_2 = \phi_1$.*

The following lemma is another easy consequence of Proposition 3.1 in [5].

Lemma 5.4 *Let $\mathcal{V}_0$ be a contractible Hilbert manifold. Let $\mathcal{V}_1 = \mathcal{V}_0 \times [0,1]$, $\mathcal{V}_2 = \mathcal{V}_0 \times [0,1) \smallsetminus \text{int}\,\mathcal{B}$, where $\mathcal{B}$ is a closed disk contained in $\mathcal{V}_0 \times (0,1)$. Also, $\partial\mathcal{V}_1 = \mathcal{V}_0 \times \{0,1\}$ and $\partial\mathcal{V}_2 = \mathcal{V}_0 \times \{0\} \cup \partial\mathcal{B}$. Then there exists a diffeomorphism $\theta : (\mathcal{V}_1, \partial\mathcal{V}_1) \to (\mathcal{V}_2, \partial\mathcal{V}_2)$ such that $\theta|_{\mathcal{V}_0 \times \{0\}} = id$.*

Proposition 5.5 *Let $(\mathcal{V}, \partial\mathcal{V})$ be a Hilbert manifold with boundary consisting of two contractible components $\partial_-\mathcal{V}$ and $\partial_+\mathcal{V}$. Let $\phi_\pm : \partial_\pm\mathcal{V} \times [0,1] \to \mathcal{V}$ be two disjoint closed collar neighborhoods of $\partial_\pm\mathcal{V}$, say $\mathcal{D}_\pm = \phi_\pm(\partial_\pm\mathcal{V} \times [0,1])$ and let*

$\mathcal{B}_\pm$ *be two closed disks contained in* $\phi_\pm(\partial_\pm \mathcal{V} \times (0,1))$. *Then there exists a diffeomorphism* $\theta : (\mathcal{V}, \partial \mathcal{V}) \to (\mathcal{V} \smallsetminus (\partial \mathcal{V} \cup \text{int}(\mathcal{B}_+ \cup \mathcal{B}_-)), \partial(\mathcal{B}_- \cup \mathcal{B}_+))$ *which restricts to the identity on* $\mathcal{V} \smallsetminus (\mathcal{D}_+ \cup \mathcal{D}_-)$.

Proof: Apply lemma 5.4 twice: in the first instance, take $\mathcal{V}_1 = \phi_+(\partial_+ \mathcal{V} \times [0,1])$, $\partial \mathcal{V}_1 = \phi_+(\partial_+ \mathcal{V} \times \{0,1\})$, $\mathcal{V}_2 = \mathcal{V}_1 \smallsetminus (\phi_+(\partial_+ \mathcal{V} \times \{0\}) \cup \text{int}\,\mathcal{B}_+)$, $\partial \mathcal{V}_2 = \phi_+(\partial_+ \mathcal{V} \times \{1\}) \cup \partial \mathcal{B}_+$. In the second, replace the $+$ signs by $-$. ∎

6   Proof of the main theorem

The idea of the proof of Theorem 1 is to consider the restriction of μ_f to the preimage of the set of lifted matrices with nonnegative trace (the regions between dotted vertical lines containing components of T_2 in Figure 1) and use Proposition 5.5. Extending the diffeomorphism to the complement is then easy. To insure that the hypothesis of Proposition 5.5 hold, however, we need Michael's continuous selection theorem (the main result in [11]), which we state below in the special situation we use.

Recall that a surjective continuous map $f : X \to Y$ of arbitrary topological spaces is a *topological submersion* if for each $x \in X$ there exist a neighborhood $U \subset Y$ of $f(x)$, a neighborhood $V \subset f^{-1}(f(x))$ of x and an open embedding $h : U \times V \to X$ so that $f \circ h$ is the projection on the first coordinate. Clearly a smooth surjective submersion of Hilbert manifolds is a topological submersion and the pullback of a topological submersion by a continuous map is a topological submersion.

Fact 2 (Michael's theorem) *Let X and Y be (possibly infinite dimensional) manifolds and $f : X \to Y$ be a topological submersion with k-connected fibers. Then f induces an isomorphism on homotopy groups in dimension smaller than k and an epimorphism in dimension k. In particular, if the fibers are contractible then f is a homotopy equivalence.*

The proof of this result is based on the observation that for a topological submersion f the assignment $y \in Y \rightsquigarrow f^{-1}(y)$ is lower semi-continuous in the sense of Michael (since f is open) and equi-LC^m (locally m-connected) for any m (see [11] for definitions). In view of the main theorem in [11], if Y has dimension n and any fiber is n-connected then any continuous section $\varphi : A \to X$, where A is a closed subset of Y, has an extension to a section $\tilde{\varphi} : Y \to X$. This clearly implies the first part of the statement and therefore the second, since manifolds have the homotopy type of ANR's.

The proposition below gives a normal form for a submersion with contractible fibers onto a closed neighborhood of a cone.

Proposition 6.1 *Let $n \in \mathbb{Z}$, $n > 0$, set $I_n = [2\pi n - \pi/2, 2\pi n + \pi/2]$ and $\bowtie_n = \{(x, y, z) \in \mathbb{R}^2 \times \text{int } I_n \mid x^2 + y^2 = \tan^2 z\}$. Let $(\mathcal{H}, \partial\mathcal{H})$ be a Hilbert manifold with boundary and $\sigma : \mathcal{H} \to \mathbb{R}^2 \times I_n$ be a surjective submersion with contractible fibers so that $\partial\mathcal{H} = \sigma^{-1}(\mathbb{R}^2 \times \partial I_n)$. Let $\mathcal{C} = \sigma^{-1}(\bowtie_n)$. Then there exists a diffeomorphism $\theta : (\mathcal{H}, \partial\mathcal{H}) \to \mathbb{R}^2 \times (I_n, \partial I_n) \times \mathbb{H}$ so that $\theta(\mathcal{C}) = \bowtie_n \times \mathbb{H}$.*

Proof: Set $\mathcal{V} = \mathcal{H}$, $\partial_\pm \mathcal{V} = \sigma^{-1}(\mathbb{R}^2 \times \{2\pi n \pm \pi/2\})$. Fact 2 implies that each space $\partial_\pm \mathcal{V}$ is contractible. Choose two disjoint closed collar neighborhoods $\mathcal{D}_\pm$ of $\partial_\pm \mathcal{V}$ which are disjoint from $\mathcal{C}$ and two disjoint closed collar neighborhoods $\mathcal{D}'_\pm$ of $\mathbb{R}^2 \times \{2\pi n \pm \pi/2\} \times \mathbb{H}$ which are disjoint from $\bowtie_n \times \mathbb{H}$. Now, choose closed disks $\mathcal{B}_+$, $\mathcal{B}_-$, $\mathcal{B}'_+$ and $\mathcal{B}'_-$ in the interior of the closed tubular neighborhoods $\mathcal{D}_+$, $\mathcal{D}_-$, $\mathcal{D}'_+$ and $\mathcal{D}'_-$, respectively. Set $\mathcal{B}_\pm = \mathcal{B}_- \cup \mathcal{B}_+$ and $\mathcal{B}'_\pm = \mathcal{B}'_- \cup \mathcal{B}'_+$. Use Proposition 5.5 to construct the diffeomorphisms

$$\theta_1 : (\mathcal{V}, \partial\mathcal{V}) \to (\mathcal{V} \smallsetminus (\partial\mathcal{V} \cup \text{int}(\mathcal{B}_\pm)), \partial(\mathcal{B}_\pm)),$$
$$\theta_2 : (I \times \mathbb{R}^2 \times \mathbb{H}, \partial I \times \mathbb{R}^2 \times \mathbb{H}) \to ((\text{int } I \times \mathbb{R}^2 \times \mathbb{H}) \smallsetminus (\text{int}(\mathcal{B}'_\pm)), \partial(\mathcal{B}'_\pm))$$

and Proposition 5.3 to construct the diffeomorphism

$$\theta : (\mathcal{V} \smallsetminus (\partial\mathcal{V} \cup \text{int}(\mathcal{B}_\pm)), \partial(\mathcal{B}_\pm)) \to (\text{int } I \times \mathbb{R}^2 \times \mathbb{H} \smallsetminus \text{int}(\mathcal{B}'_\pm), \partial(\mathcal{B}'_\pm)).$$

The desired diffeomorphism is $\theta_2^{-1} \circ \theta \circ \theta_1$.   ∎

We now return to the nonlinear monodromy $\mu_f : X^* \to G_0$ where $X = H^p(\mathbb{S}^1)$. From Proposition 3.2, μ_f is a submersion provided that f' is nowhere flat. If additionally f' is surjective, Proposition 3.6 implies the surjectivity of μ_f and Theorem 3 proves that the fibers of μ_f are contractible. Let $Z_2 = \Pi^{-1}(I) \subset T_2 \subset G$ be the set of vertices of cones in T_2: notice that Z_2 is a group isomorphic to $\mathbb{Z}$.

Proposition 6.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth nonlinearity. Assume that f' is surjective and nowhere flat. Let $X = H^p(\mathbb{S}^1)$, $p \geq 1$, $X^* = X \smallsetminus \{constant\ functions\}$ and $\mu_f : X^* \to G_0$ be the monodromy map. Let $\mathcal{C}^* = \mathcal{C} \cap X^* = \mu_f^{-1}(G_0 \cap T_2)$ and $\mathcal{C}_2^* = \mu_f^{-1}(G_0 \cap Z_2)$. Then the triple $(X^*, \mathcal{C}^*, \mathcal{C}_2^*)$ is diffeomorphic to the triple $(G_0, G_0 \cap T_2, G_0 \cap Z_2) \times \mathbb{H}$.*

Proof: Let $\Sigma \subset \mathbb{R}^3$ be as defined in the introduction and $\Sigma_2 = \{(0, 0, 2\pi n), n \in \mathbb{Z}, n > 0\} \subset \Sigma$ the set of vertices of cones in Σ. From [6] there exists a diffeomorphism $\psi : (G_0, G_0 \cap T_2, G_0 \cap Z_2) \to (\mathbb{R}^3, \Sigma, \Sigma_2)$. Apply Proposition 6.1 for $\sigma = \psi \circ \mu_f$, $\mathcal{H} = \sigma^{-1}(\mathbb{R}^2 \times I_n)$; attaching the pieces presents no difficulty.   ∎

The attentive reader will notice that this proposition also holds (with the same proof) if X is a smoothing Hilbert space or if it is a separable Hilbert space satisfying the hypothesis of corollary 3.3. Also, if X is a smoothing Banach space (or a Banach space satisfying the hypothesis of of corollary 3.3) then a similar but weaker conclusion holds: the triples are homeomorphic.

The use of Michael's Theorem can be avoided if we are willing to prove directly that if $P \subset G_0$ is diffeomorphic to a plane then the sets $\mu_f^{-1}(P) \cap X^*$ are contractible. This would be achieved by mimicking the proof of Theorem 3.

Theorem 1 is slightly different from Proposition 6.2: the Proposition is stated for the triple $(X^*, \mathcal{C}^*, \mathcal{C}_2^*)$ while the theorem is stated for the pair $(X, \mathcal{C}^*)$. Proposition 3.1 guarantees that if f is good then $\mathcal{C}^* = \mathcal{C}$. Also, f is admissible then $\mathcal{C}$ is the disjoint union of $\mathcal{C}^*$ and isolated points.

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