Spectra of semi-regular polytopes

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Abstract: We compute the spectra of the adjacency matrices of the semi-regular polytopes. A few different techniques are employed: the most sophisticated, which relates the 1-skeleton of the polytope to a Cayley graph, is based on methods akin to those of Lovász and Babai ([L], [B]). It turns out that the algebraic degree of the eigenvalues is at most 5, achieved at two 3-dimensional solids.

Introduction

Important information about a graph is conveyed by its spectrum, i.e., the set of eigenvalues of its adjacency matrix; an extensive bibliography can be found in [CDS]. Symmetries of a graph are helpful in computing the spectrum ([PS]). If the group of symmetries acts transitively on vertices, Lovász ([L]) and Babai ([B]) show how to apply techniques from representation theory of groups to reduce significantly the algebraic degree of the problem.

In [ST], we computed the spectra of all regular polytopes (i.e., of the graph formed by their vertices and edges); somewhat surprisingly, all eigenvalues have algebraic degree at most 3. The complete list of regular polytopes is known since Schläfli ([Sc], [C]). *Semi-regular polytopes* are polytopes with regular faces whose isometry group acts transitively on vertices. Trivial examples are regular polytopes and, in three dimensions, prisms and antiprisms. Nontrivial 3-dimensional semi-regular polytopes are known as *Archimedean solids*, despite the fact that the writings by Archimedes on the subject are lost ([C]). Kepler ([K]) wrote the first available list of Archimedean solids with a proof its completeness. The semi-regular polytopes in higher dimensions were known to Gosset ([Go]) and were extensively studied by Coxeter ([C2], [C3]), but only recently Blind and Blind ([BB]) showed that Gosset's list is complete.

In this paper, we compute the spectra of all semi-regular polytopes. Section 1 contains our results: characteristic polynomials of the adjacency matrices are completely factored in $\mathbb{Z}[\tau]$, $\tau = (1+\sqrt{5})/2$. It turns out that the algebraic degree of the eigenvalues is at most 5, achieved at two 3-dimensional solids. Section 2 is devoted to the study of some Archimedean solids whose spectra can be obtained by taking into account a few planes of symmetry. Gosset polytopes are described in section 3: since they have very large groups of symmetry, the technique used in [ST] for regular polytopes is convenient here. In section 4, we show how to apply group representations to compute the spectrum of a Cayley graph by Lovász's, Babai's and our modified version of their methods. Chung and Sternberg ([CS]) used similar ideas to compute the spectra of some regular graphs and of the buckyball molecule, which corresponds to a weighted version of $\langle 5,6,6\rangle$ (in the notation of the next section). Group representation techniques are used in section 5 to

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compute the spectra of the remaining Archimedean solids. In section 6 we consider three discrete subgroups of S^3, the unit quaternionic sphere. These come in handy in sections 7 and 8 where we compute the spectra of the two missing semi-regular polytopes. We tried to make this paper accessible to readers with a scant knowledge of the semi-regular polytopes; most difficulties should be resolved by consulting the excellent book [C].

1. Statement of results

The semi-regular polytopes in dimension three are the five Platonic and the thirteen Archimedean solids, besides prisms and antiprisms. We shall denote the Archimedean solids by the types of polygons surrounding each vertex. Thus, $\langle 3,6,6\rangle$ is the solid with two hexagons and a triangle at each vertex, which is obtained by removing small tetrahedra of edge 1 from each vertex of a regular tetrahedron of edge 3. In this notation, the Archimedean solids are $\langle 3,6,6\rangle$, $\langle 3,8,8\rangle$, $\langle 3,4,3,4\rangle$, $\langle 3,4,4,4\rangle$, $\langle 3,3,3,3,4\rangle$, $\langle 4,6,8\rangle$, $\langle 4,6,6\rangle$, $\langle 5,6,6\rangle$, $\langle 3,10,10\rangle$, $\langle 3,5,3,5\rangle$, $\langle 3,4,5,4\rangle$, $\langle 3,3,3,3,5\rangle$, $\langle 4,6,10\rangle$. In higher dimensions, the non-trivial semi-regular polytopes are the Gosset family $G_4,\ldots,G_8$ (subscripts indicate dimension) and two 4-dimensional polytopes P_{96} and P_{720} (subscripts now indicate the number of vertices). Coxeter ([C]) uses

$$\left\{\begin{matrix} 3 \\ 3,3 \end{matrix}\right\}, s\{3,4,3\}, \left\{\begin{matrix} 3 \\ 3,5 \end{matrix}\right\}, h\gamma_5, 2_{21}, 3_{21} \text{ and } 4_{21}$$

instead of our G_4, P_{96}, P_{720}, G_5, G_6, G_7 and G_8, respectively.

The spectrum of a prism of n-gonal basis ([CDS]) is

$$\pm 1 + 2\cos(2\pi i/n), \quad i = 0,1,\ldots,n-1$$

and the spectrum of an antiprism of n-gonal basis is

$$2\left(\cos(2\pi i/n) + \cos(\pi i/n)\right), \quad i = 0,1,\ldots,2n-1,$$

as we will see in Section 2. Below, we list the characteristic polynomials of the non-trivial semi-regular polytopes (i.e., the characteristic polynomials of the adjacency matrices of their graphs). The polynomials are factored in $\mathbb{Z}[\tau]$, where $\tau = (1+\sqrt{5})/2$ and $\overline{\tau} = (1-\sqrt{5})/2$; the numerical values of non-integer eigenvalues are listed in the same order as the factors, with semi-colons separating roots of different factors. By the Perron-Frobenius theorem ([Ga]), the eigenvalue with largest module is simple and equals the number of neighbours of a vertex, since adjacency matrices have non-negative entries and are irreducible.

$\langle 3,6,6\rangle$
$(X-3)(X-2)^3X^2(X+1)^3(X+2)^3$

$\langle 3,4,3,4\rangle$
$(X-4)(X-2)^3X^3(X+2)^5$

$\langle 4,6,6\rangle$
$(X-3)(X-1)^3(X+1)^3(X+3)(X^2-3)^2(X^2-2X-1)^3(X^2+2X-1)^3$
$(1.732051, -1.732051; 2.414214, -0.414214; -2.414214, 0.414214)$

$\langle 3,8,8\rangle$
$(X-3)(X-2)^3(X-1)X^5(X+1)^3(X+2)^5(X^2-X-4)^3$
$(2.561553, -1.561553)$

$\langle 3,4,4,4\rangle$
$(X-4)(X-3)^3(X-1)^2X^4(X+1)^6(X+3)^2(X^2+X-4)^3$
$(1.561553, -2.561553)$

$\langle 3,3,3,3,4\rangle$
$(X-5)(X+1)^4(X^2+2X-2)^2(X^2-2X-6)^3(X^3+X^2-4X-2)^3$
$(0.7320508, -2.7320508; 3.645751, -1.645751; 1.81361, -0.470683, -2.34292)$

$\langle 4,6,8\rangle$
$(X-3)(X-2)^2(X-1)^4X^4(X+1)^4(X+2)^2(X+3)(X^2-2X-2)^3(X^2+2X-2)^3$
$(X^3+X^2-4X-2)^3, (X^3-X^2-4X+2)^3$
$(2.732051, -0.732051; 0.732051, -2.732051;$
$1.81361, -0.470683, -2.34292; 2.34292, 0.470683, -1.81361)$

$\langle 3,5,3,5\rangle$
$(X-4)(X-2)^5(X-1)^4(X+1)^4(X+2)^{10}(X-2\tau)^3(X-2\overline{\tau})^3$
$(3.236068; -1.236068)$

$\langle 5,6,6\rangle$
$(X-3)(X-1)^9(X+2)^4(X+2-\tau)^3(X+2-\overline{\tau})^3(X+\tau)^5(X+\overline{\tau})^5$
$(X^2-(1+\tau)X-2+\tau)^3(X^2-(1+\overline{\tau})X-2+\overline{\tau})^3(X^2+X-4)^4(X^2-X-3)^5$
$(-0.381966; -2.618034; -1.618034; 0.618034; 2.756598, -0.138564; 1.820249, -1.438283;$
$1.561553, -2.561553; 2.302776, -1.302776)$

$\langle 3,10,10\rangle$
$(X-3)X^{10}(X+2)^{11}(X-\tau)^4(X-\overline{\tau})^4(X^2-X-2-2\tau)^3(X^2-X-2-2\overline{\tau})^3$
$(X^2-X-3)^4(X^2-X-4)^5$
$(1.618034; -0.618034; 2.842236, -1.842236; 1.506942, -0.506942; 2.302776, -1.302776;$
$1.561553, -2.561553)$

$\langle 3,4,5,4\rangle$
$(X-4)(X-1)^4X^6(X+1)^4(X-2-\tau)^3(X-2-\overline{\tau})^3(X+2-\tau)^8(X+2-\overline{\tau})^8$
$(X+1-2\tau)^4(X+1-2\overline{\tau})^4(X^3-X^2-7X+4)^5$
$(3.618034; 1.381966; -0.381966; -2.618034; 2.236068; -2.236068;$
$2.92542, 0.551929, -2.47735)$

$\langle 3,3,3,3,5\rangle$
$(X-5)(X+1)^6(X^2-2\tau X-4-\tau)^3(X^2-2\overline{\tau}X-4-\overline{\tau})^3(X^4-8X^2-2X+10)^4$
$(X^5+X^4-11X^3-19X^2-X+1)^5$
$(4.48789, -1.25182; 1.32205, -2.55812; 2.71687, 1.07082, -1.50739, -2.2803;$
$3.5766, 0.195279, -0.285153, -2.1357, -2.35102)$

$\langle 4,6,10\rangle$
$(X-3)(X-1)^6(X+1)^6(X+3)(X^2+2X-1-\tau)^3(X^2-2X-1-\tau)^3$
$(X^2+2X-1-\overline{\tau})^3(X^2-2X-1-\overline{\tau})^3(X^4-6X^2-2X+2)^4(X^4-6X^2+2X+2)^4$
$(X^5-3X^4-3X^3+11X^2-X-3)^5(X^5+3X^4-3X^3-11X^2-X+3)^5$
$(0.902113, -2.902113; 2.902113, -0.902113; 0.175571, -2.175571; 2.175571, -0.175571;$
$2.54501, 0.439406, -0.830209, -2.15421;$
$2.15421, 0.830209, -0.439406, -2.54501; 2.72142, 1.88838, 0.684645, -0.466437, -1.82801;$
$1.82801, 0.466437, -0.684645, 1.88838, 2.72142)$

G_4
$(X-6)(X-1)^4(X+2)^5$

P_{96}
$(X-9)(X-3)^8(X-1)^8X^{14}(X+2)^{24}(X+3)^6(X^2-4X-24)^4(X^3-X^2-16X-16)^9$
$(7.291503, -3.291503; 4.91638, -1.19656, -2.71982)$

P_{720}
$(X-10)(X-3)^{16}X^{16}(X+1)^{60}(X+2)^{28}$
$(X-\sqrt{5})^{24}(X+\sqrt{5})^{24}(X-5-2\sqrt{5})^4(X-5+2\sqrt{5})^4(X+1-3\tau)^{24}(X+1-3\overline{\tau})^{24}$
$(X+2+\tau)^{16}(X+2+\overline{\tau})^{16}(X+1+\tau)^{48}(X+1+\overline{\tau})^{48}(X^2+3X-3)^{40}$
$(X^2-7X-4)^{16}(X^2+(-3+2\sqrt{5})X-10)^9(X^2+(-3-2\sqrt{5})X-10)^9$
$(X^2-X-10+4\sqrt{5})^{36}(X^2-X-10-4\sqrt{5})^{36}(X^3-4X^2-15X+6)^{25}$
$(2.23607; -2.23607; 9.47214; 0.527864; 3.8541; -2.8541; -3.61803; -1.38197;$
$-2.61803; -0.38197; 0.79129, -3.79129; 7.53113, -0.53113;$
$2.51075, -3.98288; 8.63078, -1.15864; 1.642685, -0.642685; 4.88113, -3.88113;$
$6.2473, 0.36732, -2.61463)$

G_5
$(X-10)(X-2)^5(X+2)^{10}$

G_6
$(X-16)(X-4)^6(X+2)^{20}$

G_7
$(X-27)(X-9)^7(X+1)^{27}(X+3)^{21}$

G_8
$(X-56)(X-28)^8(X-8)^{35}(X+2)^{112}(X+4)^{84}$

2. Archimedean solids I

In this section, we compute the spectra of seven Archimedean solids: $\langle 3,6,6\rangle$, $\langle 3,4,3,4\rangle$, $\langle 4,6,6\rangle$, $\langle 3,8,8\rangle$, $\langle 3,4,4,4\rangle$, $\langle 4,6,8\rangle$ and $\langle 3,5,3,5\rangle$ as well as the spectra of the antiprisms, by using mirrors ([PS]) and relations between adjacency matrices. We illustrate this method by computing the spectrum of $\langle 3,4,4,4\rangle$.

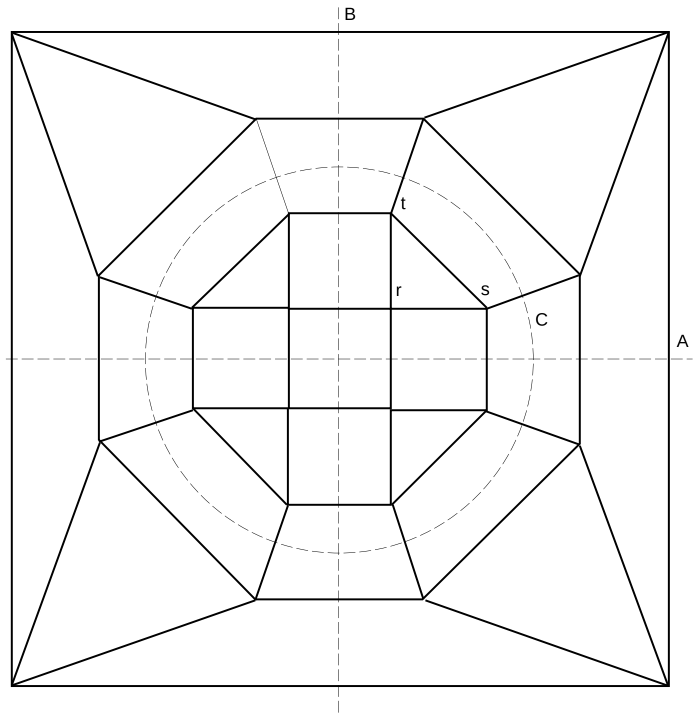

Figure 2.1

In Figure 2.1, full lines form the Schlegel diagram ([C]) of $\langle 3,4,4,4\rangle$, i.e., the stereographic projection of the polyhedron to the plane. Dotted horizontal, vertical and round lines indicate three planes of symmetry associated to three commuting involutions A, B and C. Let V be the vector space of complex valued functions on the set of vertices of $\langle 3,4,4,4\rangle$. Let X be the 24×24 adjacency matrix of the polyhedron (i.e., of its graph). The involutions A, B and C, as well as the matrix X, can be interpreted as commuting linear transformations from V to V. Thus, the eigenspaces V_+^A and V_-^A associated to the eigenvalues 1 and -1 of A are invariant under B, C and X. In a similar fashion, V splits as a direct sum of eight subspaces (which a priori could be trivial) of the form $V_{s_A,s_B,s_C} = V_{s_A}^A \cap V_{s_B}^B \cap V_{s_C}^C$, where subscripts denote signs. These subspaces are invariant under X and the problem of computing the spectrum of X reduces to the same problem for the restriction X_{s_A,s_B,s_C} of X to V_{s_A,s_B,s_C}.

Each invariant subspace can be coordinatized by the values r, s and t of each vector on the three vertices indicated in Figure 2.1: mirroring by A, B and C prescribes the values at the remaining vertices. In particular, in this example, the eight invariant subspaces are of dimension 3. The adjacency matrices for V_{++-}, V_{+-+} and V_{-++} are trivially conjugate (by renaming the dotted lines), and therefore have the same spectrum; the same happens to V_{+--}, V_{-+-} and V_{--+}. We have

$$X_{+++} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad X_{++-} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$X_{+--} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & -2 \end{pmatrix}, \qquad X_{---} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix},$$

and the spectrum of X is thus immediately computed.

For $\langle 3,6,6\rangle$, there are two commuting involutions corresponding to reflections with respect to the two orthogonal planes indicated by dotted lines in the Schlegel diagram in Figure 2.2. The values r, s, t and u in the vertices marked in the figure determine a unique vector in V_{++}. Similarly, r, s, t provide a basis for V_{+-} and s, t a basis for V_{--} (V_{-+} is equivalent to V_{+-} by renaming planes).

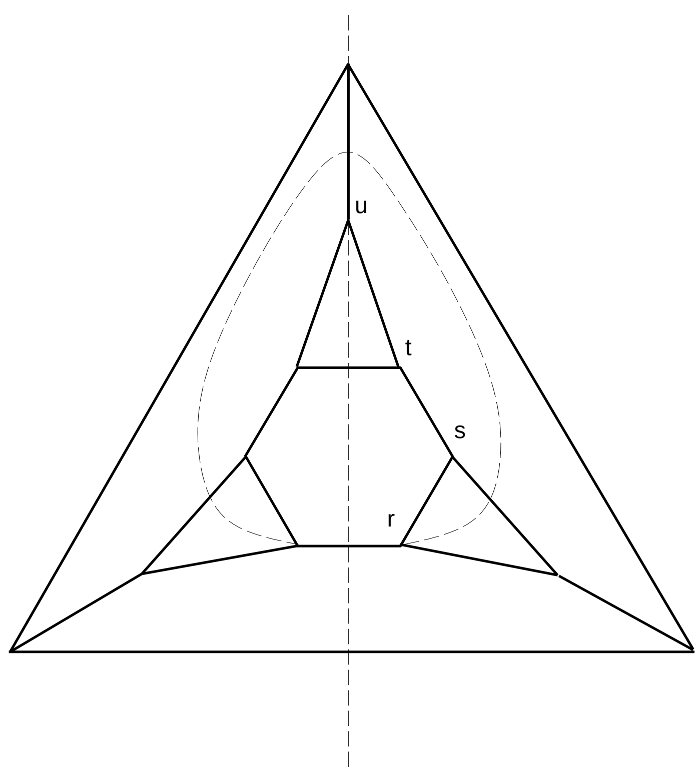

Figure 2.2

The same three mirrors used for $\langle 3,4,4,4\rangle$ work for $\langle 3,4,3,4\rangle$, reducing the problem to that of finding the spectra of a 3×3 matrix in V_{+++}, a 2×2 matrix in V_{++-} and a 1×1 matrix in V_{+--} (V_{---} is 0-dimensional). Again, the mirrors used for $\langle 3,4,4,4\rangle$

convert the computation of the spectrum of $\langle 3,8,8\rangle$ into the study of the restriction of the adjacency matrices to invariant subspaces of dimension 3. For $\langle 4,6,8\rangle$, however, these mirrors produce 6×6 matrices. A fourth mirror, indicated in Figure 2.3, can be used to further split the invariant subspaces. The reflection D on the fourth mirror does not commute with either A or B. Still, $V_{+++}, V_{++-}, V_{--+}$ and V_{---} are invariant under D and can thus be decomposed into the two eigenspaces for the restrictions of D. Notice that the four remaining subspaces are not invariant under D. This is no serious problem, however: for example, the restriction of the adjacency matrix to V_{-++} has the same spectrum as the restriction to V_{++-}, and this last space can be split by D. The values r, s, t in the figure again provide a basis for each of the eight relevant invariant subspaces and the spectrum of $\langle 4,6,8\rangle$ is now easily obtained. The solid $\langle 4,6,6\rangle$ can be handled in the same fashion; in section 5 we compute its spectrum using other methods.

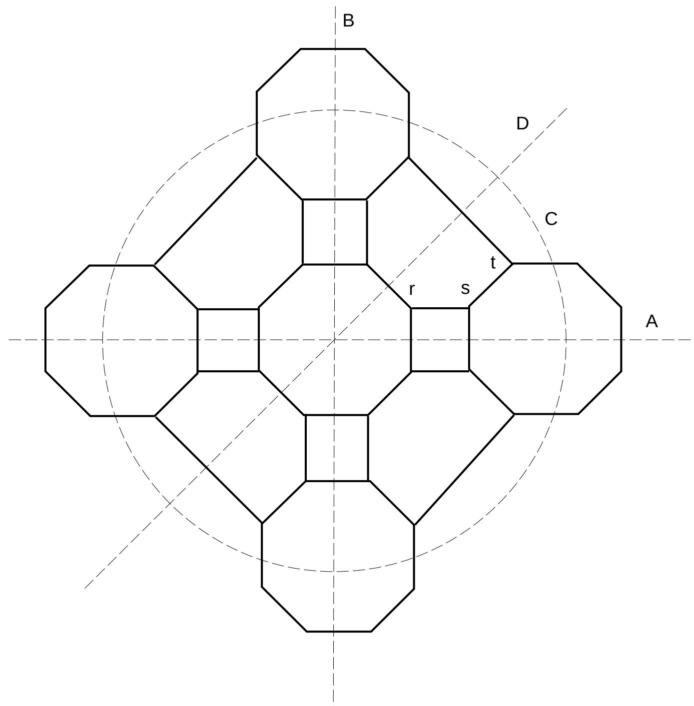

Figure 2.3

We finish this section with three examples of computations of spectra based on algebraic relations between adjacency matrices. Let Y be the adjacency matrix of a $2n$-gon: then $X = Y^2 + Y - 2I$ is the adjacency matrix of the antiprism with n-gonal basis and an eigenvalue λ of Y corresponds to an eigenvalue $\lambda^2 + \lambda - 2$ of X. As another example, the spectrum of X_{3434}, the adjacency matrix of $\langle 3,4,3,4\rangle$, can be computed in terms of the spectrum of X_{444}, the adjacency matrix of the cube. Indeed, let $Y_{v,e}$ be a 8×12 incidence matrix obtained by numbering vertices and edges of the cube and setting $y_{i,j} = 1$ if the i-th vertex belongs to the j-th edge. Dually, let $Y_{e,v} = (Y_{v,e})^T$. Notice that $X_{444}+3I = Y_{v,e}Y_{e,v}$ and that $X_{3434} + 2I = Y_{e,v}Y_{v,e}$: thus, the spectra of the left hand sides are equal except for 4 extra zero eigenvalues in the second one. The same method yields the spectrum of $\langle 3,5,3,5\rangle$ from the spectrum of the dodecahedron. The spectrum of a regular polygon, the cube and the dodecahedron are given in [CDS].

3. Gosset polytopes

In this section, we compute the spectra of the *Gosset polytopes* G_n. The solid G_3 is the triangular prism in dimension 3 and the vertex figure of G_n is G_{n-1}. Coordinates for the vertices of the Gosset polytopes are given in [G], [C] and [C3].

The polytope G_n has for faces simplices α_{n-1} and cross polytopes β_{n-1}. Recall that a cross polytope is a generalized octahedron, or $\{3,\ldots,3,4\}$ in Schläfli's notation; the vertices of β_n consist of a north and south pole together with the vertices of an equatorial β_{n-1}. We make extensive use of Table 3.1 ([C], section 11.8).

	G_4	G_5	G_6	G_7	G_8
Number of vertices	10	16	27	56	240
Number of β_{n-1} faces	5	10	27	126	2160

Table 3.1

A more combinatorial description of the graphs G_i, $i=4,\ldots,8$, is given in [BCN] where the notation $E_i(1)$ (reminiscent of the Lie algebras E_n) is employed.

Let $\Gamma(G_n)$ be the group of isometries of G_n and, for an arbitrary choice of a vertex p_1 of G_n, call $\Gamma_1(G_n)$ the subgroup of $\Gamma(G_n)$ fixing p_1. As explained in [C3], $\Gamma_1(G_n)=\Gamma(G_{n-1})$. Also, $\Gamma(G_n)$ acts transitively both on the set of simplicial faces and on the set of cross polytope faces. The orbits of the action of $\Gamma_1(G_n)$ on the set of vertices of G_n are called *suborbits* in the terminology of association schemes and *levels* in [ST]. The vertex p_1 forms a suborbit by itself; the next suborbit is a G_{n-1}. As we shall see, the number of suborbits of G_n is 3, 3, 3, 4 and 5 for $n=4$, 5, 6, 7 and 8. Thus, the method employed in [ST] for regular polytopes is well suited for the Gosset family. Actually, this method is a special case of the technique of *regular partitions* ([BCN]) for the computation of the spectrum of an association scheme.

We briefly describe this special method. Let $v\neq 0$ satisfy $Xv=\lambda v$ for the adjacency matrix X of a semi-regular polytope. Without loss, the value of v at p_1 is nonzero. Let S be the space of vectors which are constant on suborbits: S is invariant under X. The (nonzero) average $s\in S$ of v under the action of Γ_1 is another eigenvector of X associated to the same eigenvalue λ. Thus, up to multiplicities, the spectrum of X equals the spectrum of its restriction B to S. For the basis of S consisting of vectors s_k taking the value 1 at the k-th suborbit and 0 elsewhere, the entry b_{ij} of B is the number of neighbours in suborbit j of a vertex in suborbit i. We now obtain the matrices B_n for G_n, $n\geq 4$.

The 16 vertices of G_5 are $(\pm 1,\ldots,\pm 1)$ with an even number of minus signs ([C]). Taking $p_1=(1,\ldots,1)$, different sums of coordinates clearly imply different suborbits. The

3 possible values of this sum, 5, 1 and -3, correspond to sets of 1, 10 and 5 vertices which might, in principle, be the union of more than one suborbit. The intermediate set is a G_4, on which $\Gamma_1(G_5) = \Gamma(G_4)$ is transitive: this set is therefore a suborbit. Each of the five β_3 faces of the second suborbit is an equator of a β_4 face of G_5 with north pole p_1 and south pole in the third set. Since $\Gamma(G_4)$ acts transitively on the cross polytope faces of G_4, this bottom set is a suborbit. The 3×3 matrix B_5 is

$$B_5 = \begin{pmatrix} 0 & 10 & 0 \\ 1 & 6 & 3 \\ 0 & 6 & 4 \end{pmatrix}$$

with eigenvalues 10, 2 and -2. Let their multiplicities in X be 1, a and b (recall that the largest eigenvalue is always simple). We must then have $1 + a + b = 16$ and $1 \cdot 10 + a \cdot 2 + b \cdot (-2) = \mathrm{tr}(X) = 0$; a and b are then 5 and 10.

The central suborbit of G_5 gives coordinates (in $\mathbb{R}^5$) for G_4 and, as above, it is easy to obtain the decomposition of G_4 into suborbits of 1, 6 and 3 elements, yielding

$$B_4 = \begin{pmatrix} 0 & 6 & 0 \\ 1 & 3 & 2 \\ 0 & 4 & 2 \end{pmatrix},$$

from which the spectrum of G_4 follows.

From coordinates for G_5, one obtains coordinates for a G_6 inscribed in S^6 for which suborbits are indicated by the first entry. Start with $p_1 = (1, 0, \ldots, 0)$ and add the next suborbit of vertices, of the form $(a, b\,G_5)$, where $a = 1/4$ and $b = (\sqrt{3}/4)$; the coefficients are taken so that the second suborbit lies in the unit sphere and all edges have equal length. The vertices of a β_5 face of G_6 containing p_1 are, besides p_1, the vertices of a β_4 face of the second suborbit (a G_5) and the reflection of p_1 on the hyperplane containing this β_4. We thus obtain 10 new vertices of G_6, with first coordinate $-1/2$, associated to the 10 β_4 faces of the second suborbit; $\Gamma_1(G_6)$ is transitive on the set of β_4 faces of the second suborbit and these 10 points form a third suborbit of G_6. Since G_6 has 27 vertices, these are the only suborbits; adjacencies among suborbits are given by

$$B_6 = \begin{pmatrix} 0 & 16 & 0 \\ 1 & 10 & 5 \\ 0 & 8 & 8 \end{pmatrix}$$

and we are done with G_6.

In a similar fashion, coordinates for G_7 in which the first entry indicates the suborbit are

$$(1, 0, \ldots, 0),\ (1/3, (2\sqrt{2}/3)\,G_6),\ (-1/3, -(2\sqrt{2}/3)\,G_6),\ (-1, 0, \ldots, 0).$$

The polytope G_7 then admits an involution taking v to $-v$ splitting V_{G_7} into two 28-dimensional subspaces V_+ and V_-, invariant under X, and S into 2-dimensional subspaces S_+ and S_-, invariant under B_7. The two restrictions of B_7 are

$$B_+ = \begin{pmatrix} 0 & 27 \\ 1 & 26 \end{pmatrix}, \quad B_- = \begin{pmatrix} 0 & 27 \\ 1 & 6 \end{pmatrix},$$

with spectra 27, −1 and 9, −3. Considered as eigenvalues of X, 27 and −1 have multiplicities 1 and 27 since their eigenspaces are contained in V_+. The two remaining multiplicities are now computed taking into account that $\mathrm{tr}(X) = 0$.

Finally, analogous coordinates for G_8 are

$$(1, 0, \ldots, 0),\ (1/2, (\sqrt{3}/2)\, G_7),\ (0, *),\ (-1/2, (\sqrt{3}/2)\, G_7),\ (-1, 0, \ldots, 0),$$

where the vertices of the central suborbit correspond to the 126 β_6 faces of the second suborbit (a G_7). Again, G_8 admits an antipodal involution and B_8 splits as

$$B_+ = \begin{pmatrix} 0 & 56 & 0 \\ 1 & 28 & 27 \\ 0 & 24 & 32 \end{pmatrix}, \quad B_- = \begin{pmatrix} 0 & 56 \\ 1 & 26 \end{pmatrix}.$$

To compute the multiplicities, we only need the additional remark that, since antipodal points are never neighbours, the restrictions of X to the 120-dimensional spaces V_+ and V_- have trace zero.

4. Groups, Cayley graphs and representations

By using representations of groups, Lovász ([L]) described a method to compute the spectrum of a Cayley graph, or, more generally, of a graph with a transitive group of isomorphisms. Babai ([B]) modified Lovász's method (and corrected a minor mistake in [L]) and obtained families of Cayley graphs with the same spectrum. We present yet another technique based on representations of groups which is more convenient for our purposes.

For an undirected graph $\mathcal{G}$, the *doubling* $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is the directed graph obtained by substituting each edge of $\mathcal{G}$ by two edges with opposite orientations. A *Cayley structure* for an undirected graph $\mathcal{G}$ consists of a choice of a vertex (to be the identity) and a colouring of the edges of the doubling $\tilde{\mathcal{G}}$ of $\mathcal{G}$ (to represent the set H of generators) such that $\tilde{\mathcal{G}}$ becomes a Cayley graph, i.e., the group of isomorphisms of the coloured graph is simply transitive on vertices. Notice that $h \in H$ implies $h^{-1} \in H$. Similarly, a *Cayley structure* for a polytope is a Cayley structure for its 1-skeleton. Most semi-regular polytopes admit Cayley structures: in dimension 3, only the dodecahedron and $\langle 3, 5, 3, 5 \rangle$ do not ([C1]).

Let $V_{\mathcal{G}}$ be the complex vector space of functions from the vertices of the Cayley graph $\mathcal{G}$ to $\mathbb{C}$. Define e_k by $e_k(g) = \delta_{kg}$. The *canonical left representation* of Γ on $V_{\mathcal{G}}$ is given by $L_g e_k = e_{gk}$ or, equivalently, $(L_h v)(g) = v(h^{-1}g)$. For a polytope with a Cayley structure, the canonical left action corresponds to an action by isometries. The *canonical right representation* is given by $R_g e_k = e_{kg^{-1}}$. The two canonical representations are equivalent, being intertwined by the linear involution $e_k \mapsto e_{k^{-1}}$, and commute: $L_{g_1} R_{g_2} = R_{g_2} L_{g_1}$. Geometrically, this action can be interpreted in terms of the Cayley graph $\mathcal{G}$ as follows: for a generator h, the value of $R_h v$ at a vertex g is the value of v at the target vertex of the h-edge starting from g (which is, of course, gh), i.e., $(R_h v)(g) = v(gh)$. Thus, unless Γ is abelian, R_h does not preserve adjacencies among vertices.

From representation theory for finite groups ([Se]), the space $V_{\mathcal{G}}$ splits as a direct sum,

$$V_{\mathcal{G}} = \bigoplus_{1\le i\le t, 1\le j\le d_i} W_{i,j},$$

where each $W_{i,j}$ is invariant under the canonical right action and the restriction of the action to $W_{i,j}$ is isomorphic to the i-th irreducible representation.

As in Babai ([B]), the adjacency matrix X of the oriented graph $\mathcal{G}$, defined by $x_{ij} = 1$ if there is an (oriented) edge from the i-th to the j-th vertex, and 0 otherwise, satisfies

$$X = \sum_{h\in H} R_h.$$

In particular, X commutes with the left canonical action. Invariant subspaces for the canonical right representation are therefore invariant under X and the restriction $X_{i,j}$ of X to each $W_{i,j}$ is the sum of the restrictions of the linear transformations corresponding to the elements of H. Thus, if we have matrices $M_{h,i}$ for the generators in H in any representation isomorphic to the i-th irreducible representation of Γ, $X_{i,j}$ is conjugate to

$$X_i = \sum_{h\in H} M_{h,i}.$$

Clearly, the spectrum $\sigma(X)$ of X is the union (as a multiset) of d_i copies of $\sigma(X_i)$, $i = 1, \dots, t$.

Although we use the right canonical action to decompose $V_{\mathcal{G}}$, the left canonical action is geometrically more helpful in the identification of a Cayley structure for a polytope. Like Babai's method, this technique applies to polytopes with different weights for edges of different colours.

Instead of computing X_i, Lovász and Babai have a formula for traces of powers of X_i ([B]):

$$\mathrm{tr}(X_i^\ell) = \lambda_{i,1}^\ell + \cdots + \lambda_{i,d_i}^\ell = \sum_{h_1,\dots,h_\ell\in H} \chi_i(h_1 \cdot \cdots \cdot h_\ell),$$

where $\lambda_{i,1}, \dots, \lambda_{i,d_i}$ are the eigenvalues of X_i, with dimensions d_i, and χ_i, $i = 1, \dots, t$, are the characters of the irreducible representations of Γ. The traces of X_i^ℓ for $\ell = 1, \dots, d_i$, determine $\sigma(X_i)$.

In our examples, we find the explicit computation of the right hand side of the above formula cumbersome since it involves counting special paths and we prefer to specify matrices for each irreducible representation.

5. Archimedean solids II

In this section we compute the spectra of the six remaining Archimedean solids using the methods of the previous section.

As a first example, let $P = \langle 4, 6, 6 \rangle$. The cube and P have the same group of symmetries (of order 48); the subgroup Γ of orientation preserving isometries acts simply transitively on the vertices of P, thus giving P a Cayley structure. As is well known, Γ is isomorphic to S_4, the permutations of $\{1,2,3,4\}$ (number pairs of antipodal hexagons $1,2,3,4$ as in Figure 5.1). With this notation for Γ, $H = \{(12), (1234), (1432)\}$: the first generator corresponds in Figure 5.1 to the edge joining e to a, the second to the edge going southwest from e to d and the third to the remaining edge starting from e. Generators for the five irreducible representations of S_4 are given in Table 5.2. The corresponding $X_i = M_{(12),i} + M_{(1234),i} + M_{(1432),i}$, with their spectra, are listed in Table 5.3.

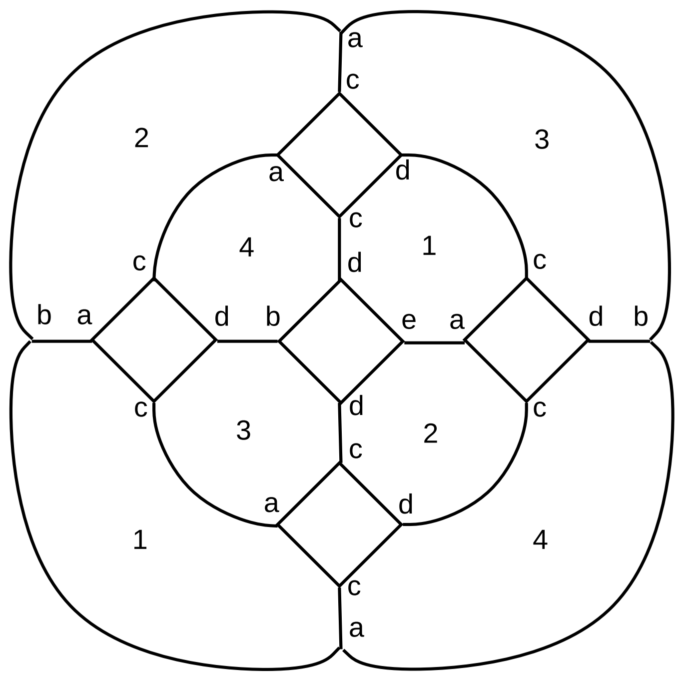

Figure 5.1

Consider now $\langle 3, 3, 3, 3, 4 \rangle$, which admits a Cayley structure for its full symmetry group $\Gamma = S_4$. Comparing Figures 5.1 and 5.4, we see two additional generators (234) and (243), so $H = \{(12), (1234), (1432), (234), (243)\}$. From Table 5.2 we have $M_{(12),i}$ and $M_{(1234),i}$ and thus we have also $M_{(234),i} = M_{(12),i} M_{(1234),i}$ and $M_{(243),i} = M^{-1}_{(234),i}$. The restrictions X_i and their spectra follow immediately.

To handle $\langle 3, 4, 5, 4 \rangle$, consider the planes containing the triangular faces: they form the faces of a circumscribed icosahedron. The group Γ of orientation preserving isometries of these two polyhedra is therefore the same. It is well known that this group can be

$$M_{(12),1} = 1 \qquad M_{(1234),1} = 1$$

$$M_{(12),2} = -1 \qquad M_{(1234),2} = -1$$

$$M_{(12),3} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad M_{(1234),3} = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$M_{(12),4} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{(1234),4} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$M_{(12),5} = -M_{(12),4} \qquad M_{(1234),5} = -M_{(1234),4}$$

Table 5.2

$$X_1 = 3 \qquad \sigma(X_1) = \{3\}$$

$$X_2 = -3 \qquad \sigma(X_2) = \{-3\}$$

$$X_3 = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix} \qquad \sigma(X_3) = \{\pm\sqrt{3}\}$$

$$X_4 = \begin{pmatrix} -1 & 2 & -1 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix} \qquad \sigma(X_4) = \{1, -1 \pm \sqrt{2}\}$$

$$X_5 = -X_4 \qquad \sigma(X_5) = \{-1, 1 \pm \sqrt{2}\}$$

Table 5.3

identified with A_5, the even permutations of $\{1,2,3,4,5\}$. The polytope $\langle 3,4,5,4\rangle$ admits a Cayley structure since A_5 acts simply transitively on its 60 vertices. With an appropriate identification of A_5 with the group of isometries of the icosahedron, $H = \{a, a^{-1}, b, b^{-1}\}$, for $a = (12345)$ and $b = (253)$. Generators for the five irreducible representations of A_5 are given in Table 5.5 where, again, $\tau = (1+\sqrt{5})/2$ and $\overline{\tau} = (1-\sqrt{5})/2$. The matrices X_i and their spectra are easily obtained.

The three solids $\langle 5,6,6\rangle$, $\langle 3,10,10\rangle$ and $\langle 3,3,3,3,5\rangle$ admit Cayley structures for the same $\Gamma = A_5$ with H in each case being $\{a, a^{-1}, ab\}$, $\{b, b^{-1}, ab\}$ and $\{a, a^{-1}, b, b^{-1}, ab\}$. The rest of the procedure is analogous.

The polyhedron $\langle 4,6,10\rangle$, unlike the previous four examples, has 120 vertices and admits a Cayley structure for its *full* isometry group, which is isomorphic to $A_5 \oplus \mathbb{Z}/(2)$, where the generator of $\mathbb{Z}/(2)$ is the (orientation reversing) antipodal map. The characters and representations for this larger group are trivially obtained from those of A_5 ([Se]) and the ten X_i matrices are thus easily obtained. We prefer, however, to consider X^2, the square of the adjacency matrix. Since $\langle 4,6,10\rangle$ is bipartite, X^2 has two obvious invariant subspaces corresponding to the white and black vertices. The white vertices

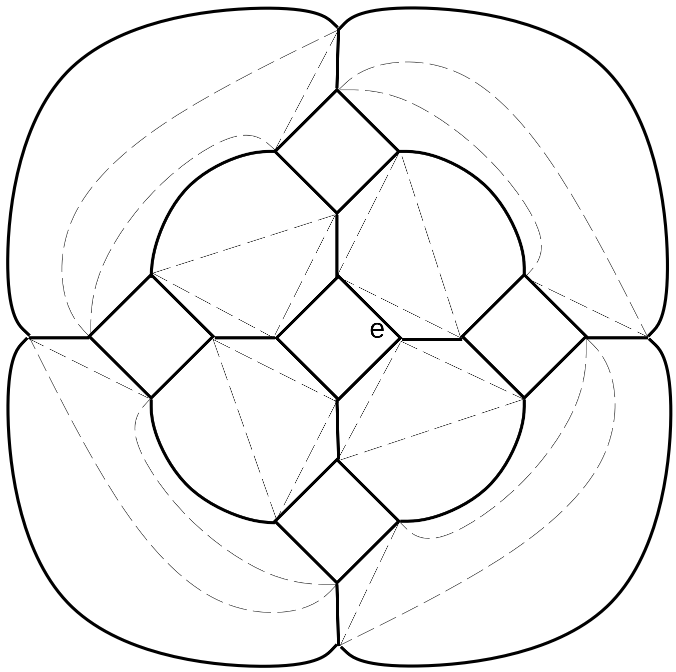

Figure 5.4

$$M_{a,1} = 1 \qquad\qquad M_{b,1} = 1$$

$$M_{a,2} = \frac{1}{2}\begin{pmatrix} 1 & \tau-1 & -\tau \\ \tau-1 & \tau & 1 \\ \tau & -1 & \tau-1 \end{pmatrix} \quad M_{b,2} = \frac{1}{2}\begin{pmatrix} -1 & \tau-1 & -\tau \\ -\tau+1 & \tau & 1 \\ \tau & 1 & -\tau+1 \end{pmatrix}$$

$$M_{a,3} = \frac{1}{2}\begin{pmatrix} 1 & \overline{\tau}-1 & -\overline{\tau} \\ \overline{\tau}-1 & \overline{\tau} & 1 \\ \overline{\tau} & -1 & \overline{\tau}-1 \end{pmatrix} \quad M_{b,3} = \frac{1}{2}\begin{pmatrix} -1 & \overline{\tau}-1 & -\overline{\tau} \\ -\overline{\tau}+1 & \overline{\tau} & 1 \\ \overline{\tau} & 1 & -\overline{\tau}+1 \end{pmatrix}$$

$$M_{a,4} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \qquad M_{b,4} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$M_{a,5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad M_{b,5} = \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Table 5.5

are naturally identified with those of $\langle 3,3,3,3,5\rangle$ and admit a simply transitive action of A_5. Let Y be the restriction of X^2 to the white vertices: in the notation of section 4,

$Y = 3I + R_a + R_{a^{-1}} + R_b + R_{b^{-1}} + 2R_{ab}$. Thus, the computation of $\sigma(Y)$ reduces as usual to the computations of spectra of matrices in each irreducible representation of A_5. Finally, $\sigma(X)$ contains the two square roots of each element of $\sigma(Y)$.

6. Three discrete subgroups of S^3

Among the six regular polytopes in four dimensions, three of them, $\{3,3,4\}$, $\{3,4,3\}$ and $\{3,3,5\}$, have a most remarkable property ([C], [C2]): when suitably inscribed in the unit quaternionic sphere S^3, their vertices form finite groups. In this section, we briefly describe these three groups, to be called Q_8, Q_{24} and Q_{120}.

Following [C], we display appropriate choices of unit quaternions for the vertices of these polytopes. We identify (a, b, c, d) to $a + bi + cj + dk$. The elements of Q_8 are the vertices of $\{3,3,4\}$ with coordinates ± 1, $\pm i$, $\pm j$ and $\pm k$. In order to obtain the vertices of $\{3,4,3\}$, add to the above list all 16 points of the form $(\pm 1 \pm i \pm j \pm k)/2$, producing Q_{24}. Finally, consider the 96 points obtained by even permutations of the coordinates of $(\pm\tau, \pm 1, \pm\overline{\tau}, 0)/2$: adding these to the 24 points already defined, we obtain the vertices of $\{3,3,5\}$, i.e., the elements of Q_{120}.

The four groups S^3, Q_8, Q_{24} and Q_{120} have the same centre $\{\pm 1\}$. As is well known ([MT]), the quotient $S^3/\{\pm 1\}$ is isomorphic to $SO(3)$. The quotient $Q_8/\{\pm 1\} = \mathbb{Z}/(2) \times \mathbb{Z}/(2)$, as a group of isometries in $\mathbb{R}^3$, is generated by 180° rotations around the three axis. Similarly, $Q_{24}/\{\pm 1\} = A_4$ and $Q_{120}/\{\pm 1\} = A_5$ are the groups of orientation preserving isometries of a tetrahedron and an icosahedron, respectively.

Conjugacy classes in S^3 are determined by the first coordinate, i.e., the real part; in the groups Q_n, therefore, points with different real parts are never conjugate. The conjugacy classes of Q_8 are $\{1\}$, $\{\pm i\}$, $\{\pm j\}$, $\{\pm k\}$, $\{-1\}$. Representatives for the conjugacy classes of Q_{24} are 1, $(1+i+j+k)/2$, $(1+i+j-k)/2$, i, $(-1+i+j+k)/2$, $(-1+i+j-k)/2$ and -1; these classes have 1, 8, 8, 6, 8, 8 and 1 elements, respectively. For Q_{120}, the real part determines the conjugacy class: there are thus 9 conjugacy classes denoted by 1, b, c, d, e, $-d$, $-c$, $-b$, -1 (decreasing real parts) with 1, 12, 20, 12, 30, 12, 20, 12, 1 elements, respectively; the order of an element in each conjugacy class is 1, 10, 6, 5, 4, 10, 3, 5, 2. The conjugacy class b consists of the 12 neighbours of the vertex 1 and these 12 points form the vertices of an icosahedron.

The irreducible representations of S^3 are well known ([Su]): we call them $\mathcal{R}_n$, where $\mathcal{R}_n$ has dimension $n+1$. The representation $\mathcal{R}_0$ is trivial and $\mathcal{R}_1$ corresponds to the identification $S^3 = SU(2)$. A basis for the space where $\mathcal{R}_n$ acts is given by monomials of degree n in two variables e_1 and e_2, identified with the basis of $\mathbb{C}^2$; for a homogeneous polynomial ϕ of degree n, define $\mathcal{R}_n(g)(\phi(e1, e2)) = \phi(\mathcal{R}_1(g)(e_1), \mathcal{R}_1(g)(e_2))$. Restrictions of $\mathcal{R}_n$ to Q_n are still representations, but are usually not irreducible. Call an irreducible representation of S^3 or Q_n *even* (resp., *odd*) if -1 is taken to I (resp., $-I$); $\mathcal{R}_n$ is even if and only if n is even. The number of even (resp. odd) irreducible representations of Q_8, Q_{24} and Q_{120} are 4, 4, 5 (resp. 1, 3, 4); character tables for Q_8 and Q_{24} are easy to obtain and that of Q_{120} is in [CCNPW].

We show how to obtain a matrix form for $\mathcal{M}_i(G)$, the i-th irreducible representation of G (even representations come first; within each parity, representations are ordered by dimension). By restrictions, $\mathcal{R}_1$ yield $\mathcal{M}_5(Q_8)$, $\mathcal{M}_5(Q_{24})$ and $\mathcal{M}_6(Q_{120})$. Also, $\mathcal{R}_2$ yields $\mathcal{M}_4(Q_{24})$ and $\mathcal{M}_2(Q_{120})$; $\mathcal{R}_3$, $\mathcal{R}_4$ and $\mathcal{R}_5$ yield $\mathcal{M}_8(Q_{120})$, $\mathcal{M}_5(Q_{120})$ and $\mathcal{M}_9(Q_{120})$. The other representations can be obtained by algebraic conjugation and tensor products: $\mathcal{M}_3(Q_{120})$ and $\mathcal{M}_7(Q_{120})$ are the conjugates of $\mathcal{M}_2(Q_{120})$ and $\mathcal{M}_6(Q_{120})$, respectively. Also, $\mathcal{M}_6(Q_{24}) = \mathcal{M}_2(Q_{24}) \otimes \mathcal{M}_5(Q_{24})$, $\mathcal{M}_7(Q_{24}) = \mathcal{M}_3(Q_{24}) \otimes \mathcal{M}_5(Q_{24})$ and $\mathcal{M}_4(Q_{120}) = \mathcal{M}_6(Q_{120}) \otimes \mathcal{M}_7(Q_{120})$.

In our examples, we frequently consider isometry groups in $\mathbb{R}^4$. As is well known ([MT]), $SO(4) = (S^3 \times S^3)/(-1,-1)$ by unit quaternion bilateral multiplication: $(q,r) \cdot v = qvr^{-1}$. Thus, the finite groups $(Q_n \times Q_m)/(-1,-1)$ act on $\mathbb{R}^4$ by isometries. The irreducible representations of such groups are obtained by tensoring representations of each factor having the same parity.

7. P_{96}

In this section, we compute the spectrum of P_{96} (Coxeter's $s\{3,4,3\}$), one of the three semi-regular polytopes of dimension 4. Each of its 96 vertices is surrounded by three icosahedra and five tetrahedra, arranged according to the vertex figure shown in Figure 8.9A of [C]; each vertex has nine neighbours. Our main task is to obtain a group Γ of isometries of P_{96} acting simply transitively on vertices.

We remind the reader of a construction for P_{96}, detailed in [C], section 8.4. Edges of $\{3,4,3\}$ can be oriented so that, given any vertex p, there are four edges pointing outwards from p, no two of which belong to the same (triangular) 2-cell. Now, divide each oriented edge in two segments a and b (in this order) satisfying $b/a = \tau$; the points thus obtained are the vertices of a P_{96}. Alternatively, the 120 vertices of $\{3,3,5\}$ are the disjoint union of the 24 vertices of a $\{3,4,3\}$ and the 96 vertices of a P_{96}; from the coordinates for $\{3,3,5\}$ in the previous section, we thus obtain, as in [C] (section 8.7) coordinates for P_{96}.

The finite group $(Q_{24} \times Q_{24})/(-1,-1)$ acts on $\{3,4,3\}$ by isometries preserving edge orientation, therefore acting also on P_{96}. This group is too large to act simply transitively on the vertices of P_{96}: the subgroup $\Gamma = (Q_{24} \times Q_8)/(-1,-1)$ has the right order. Clearly, Γ acts transitively on the vertices of $\{3,4,3\}$ by elements of the form $(q,1)$. Also, Γ acts transitively on oriented edges: by the transitivity on vertices, it is enough to show that it acts transitively on the four edges starting from 1, which is done by elements of the form (r,r). Adding up, Γ acts simply transitively on the vertices of P_{96} and this polytope therefore admits a Cayley structure. The set H of generators has 9 elements; we explain how to obtain it. We begin by identifying vertices of P_{96} with oriented edges of $\{3,4,3\}$ as in the construction above. Let p_0 be the vertex $(1,(1+i+j+k)/2)$: by simple geometric considerations, its nine neighbours are $(1,(1+i-j-k)/2)$, $(1,(1-i+j-k)/2)$, $(1,(1-i-j+k)/2)$, $((1+i+j-k)/2,1)$, $((1+i-j+k)/2,1)$, $((1-i+j+k)/2,1)$, $((1+i+j+k)/2,(1+i+j-k)/2)$, $((1+i+j+k)/2,(1+i-j+k)/2)$ and $((1+i+j+k)/2,(1-i+j+k)/2)$. The (unique) elements of Γ taking p_0 to its nine neighbours are (i,i), (j,j), (k,k), $((1+i-j+k)/2,k)$, $((1-i+j+k)/2,j)$, $((1+i+j-k)/2,i)$,

$((-1+i+j-k)/2, i)$, $((-1+i-j+k)/2, k)$ and $((-1-i+j+k)/2, j)$. Thus, if we choose p_0 to be the identity for the Cayley structure, the nine elements of Γ above are the nine elements of H. From the previous section, we have explicit matrices for the 19 irreducible representations of Γ; their dimensions are at most 4. The spectrum of the adjacency matrix X can now be computed as in section 4.

8. P_{720}

Following Coxeter ([C], sections 8.1 and 8.9), we take for vertices of $P_{720} = \left\{ {3 \atop 3,5} \right\}$ the midpoints of the edges of the regular polytope $\{3, 3, 5\}$. Each vertex of P_{720} is surrounded by two icosahedra and five octahedra; its vertex figure is a pentagonal prism.

We now describe the group $G_{14400} \subset O(4)$ of all isometries of P_{720}. The group G_{7200} of orientation preserving isometries of $\{3, 3, 5\}$ has order 120×60, since it is transitive on the 120 vertices and the subgroup of such isometries keeping a given vertex fixed equals the group of orientation preserving isometries of the vertex figure, an icosahedron. Thus, $G_{7200} = (Q_{120} \times Q_{120})/(-1, -1)$. The group G_{14400} is generated by G_{7200} together with a reflection on a hyperplane preserving the vertices of $\{3, 3, 5\}$.

Unfortunately, the technique of the previous sections does not apply directly.

Proposition: *The polytope P_{720} admits no Cayley structure.*

Proof: Let G_{720} be an arbitrary subgroup of order 720 of G_{14400}: we prove that G_{720} does not act simply transitively on edges of $\{3, 3, 5\}$. Let G_{25} be a 5-Sylow subgroup of G_{720}. Clearly, G_{25} is contained in $G_{7200} = (Q_{120} \times Q_{120})/(-1, -1)$ and, by lifting, we obtain a 5-Sylow subgroup of $Q_{120} \times Q_{120}$ which is conjugate to $\{1, q, q^2, q^3, q^4\} \times \{1, q, q^2, q^3, q^4\}$, where q is some quaternion of order 5. Thus, G_{720} contains some element g conjugate to (q, q), whose action keeps some vertex $v \in \{3, 3, 5\}$ fixed (since (q, q) does). The element g permutes the 12 neighbours of v, splitting them into orbits of size 1 or 5; hence, there is a neighbour w of v, and hence an edge vw, which are kept fixed under g, as we wanted to show. ∎

Lovász describes a method ([L]) to reduce the problem of computing the spectrum of a graph with a transitive group Γ of isomorphisms to each irreducible decomposition of Γ, which could be applied to this example. As before, instead of counting paths, we prefer to work with the matrices for representations in a modified version of Lovász's technique. For simplicity, we start by applying the procedure to $\langle 3, 5, 3, 5 \rangle$, which also admits no Cayley structure, since its isometry group, $A_5 \oplus \mathbb{Z}/(2)$, has no subgroup of order 30 (the number of vertices).

As illustrated in Figure 8.1, the vertices of $\langle 3, 5, 3, 5 \rangle$ are the midpoints of edges of a dodecahedron. If instead of taking midpoints of edges we take two suitably spaced points per edge, we obtain $\langle 3, 10, 10 \rangle$, which admits a Cayley structure described in section 5. Recall that $a = (12345)$, $b = (253)$ and $H = \{b, b^{-1}, ab\}$. Edges between decagons

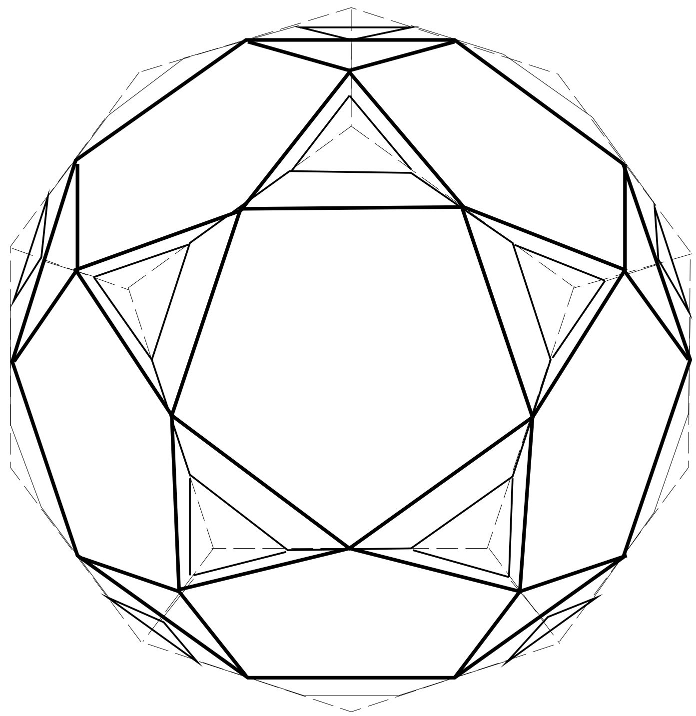

Figure 8.1

correspond to the generator $ab = (ab)^{-1}$. For each vertex of $\langle 3,5,3,5\rangle$ there are two elements of A_5 (counting the identity) which keep it fixed.

Define a linear injection $A_1 : V_{\langle 3,5,3,5\rangle} \to V_{\langle 3,10,10\rangle}$ such that the value of $A_1(v)$ at a vertex p of $\langle 3,10,10\rangle$ is the value of v at the midpoint of the only edge between decagons containing p (recall that V_P is the set of complex valued functions on the vertices of the polytope P). Conversely, define $A_2 : V_{\langle 3,10,10\rangle} \to V_{\langle 3,5,3,5\rangle}$ so that the value of $A_2(w)$ at a vertex q of $\langle 3,5,3,5\rangle$ is the sum of the values of w at the two ends of the edge of $\langle 3,10,10\rangle$ containing q. Thus, $A_2A_1 = 2I$ and $A_1A_2 = I + R_{ab}$, where R is the right multiplication action. We claim that

$$X_{3535} = A_2(R_b + R_{b^{-1}})A_1;$$

Figure 8.2 illustrates the equality at a basis vector of $V_{\langle 3,5,3,5\rangle}$. The matrix $Y = (R_b + R_{b^{-1}})A_1A_2 = (R_b + R_{b^{-1}})(I + R_{ab})$ has the same spectrum as X_{3535}, up to 30 extra zero eigenvalues. It is now clear that Y splits into the irreducible representations and its spectrum is computed in the usual manner.

We are ready to consider P_{720}. As with $\langle 3,5,3,5\rangle$, take two points in each edge of $\{3,3,5\}$ to obtain a (non-semi-regular) polytope P_{1440} with 1440 vertices. Call edges of P_{1440} contained in edges of $\{3,3,5\}$ *special*. The group G_{7200} of orientation preserving isometries of $\{3,3,5\}$ does not act simply transitively on the vertices of P_{1440}. Happily, its subgroup $G_{1440} = (Q_{120} \times Q_{24})/(-1,-1)$ does; in other words, P_{1440} admits a Cayley structure. Indeed, the first factor Q_{120} guarantees that G_{1440} acts transitively on the vertices of $\{3,3,5\}$. The subgroup of G_{1440} keeping the vertex 1 fixed consists of the 12 elements of the form (q,q), where $q \in Q_{24}$. These act simply transitively on the 12

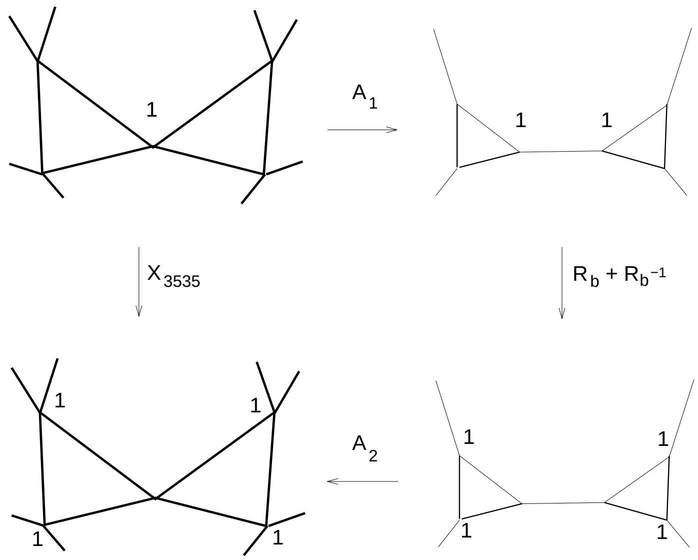

Figure 8.2

neighbours in $\{3,3,5\}$ of the vertex 1, as can be checked using the coordinate system in Section 6. The identity for the Cayley structure of P_{1440} is chosen to be the vertex between 1 and $(\tau + i - \overline{\tau} j)/2$ which is closer to 1. Using coordinates and quaternion multiplication, the reader may check that $H = \{g_0, g_1, g_2, g_3, g_4, g_s\}$, where $g_0 = (i,i)$, $g_1 = ((1+i+j+k)/2, (1+i+j+k)/2)$, $g_2 = ((1+i+j-k)/2, (1+i+j-k)/2)$, $g_3 = ((1-i-j+k)/2, (1-i-j+k)/2)$, $g_4 = ((1-i-j-k)/2, (1-i-j-k)/2)$ and $g_s = ((-\overline{\tau} i - j + \tau k)/2, k)$; special edges correspond to g_s. Notice that $g_0^{-1} = g_0$, $g_1^{-1} = g_4$, $g_2^{-1} = g_3$ and $g_s^{-1} = g_s$.

Again, define linear transformations $A_1 : V_{P_{720}} \to V_{P_{1440}}$ and $A_2 : V_{P_{1440}} \to V_{P_{720}}$: $A_1(v)$ at a vertex p of P_{1440} is the value of v at the midpoint of the special edge containing p and $A_2(w)$ at a vertex q of P_{720} is the sum of the values of w at the two ends of the special edge containing q. Thus, $A_2 A_1 = 2I$ and $A_1 A_2 = I + R_{g_s}$. Also,

$$X_{P_{720}} = A_2(R_{g_0} + R_{g_1} + R_{g_2} + R_{g_3} + R_{g_4})A_1.$$

In order to prove this equality, we relate the adjacencies of P_{720} and P_{1440}. A vertex p of P_{720} has 10 neighbours and is the midpoint of a special edge of P_{1440} with vertices q and q'. The vertex q has six neighbours: $qg_0, \ldots, qg_4$ and $q' = qg_s$. Similarly, the six neighbours of q' are $q'g_0, \ldots, q'g_4$ and $q = q'g_s$. Omitting the repetitions of q and q', the special edges containing the remaining 10 points have as midpoints the 10 neighbours $p_0, \ldots, p_9$ of p. The process $p \mapsto \{q, q'\} \mapsto \{qg_0, \ldots, qg_4, q'g_0, \ldots, q'g_4\} \mapsto \{p_0, \ldots, p_9\}$ corresponds the successive application on a basis vector of V_{720} of the transformations A_1, $R_{g_0} + \cdots + R_{g_4}$ and A_2, finishing the proof of the equality. The rest is routine by now: $Y = (R_{g_0} + \cdots + R_{g_4})(I + R_{g_s})$ has the same spectrum as $X_{P_{720}}$, up to 720 extra zero eigenvalues. Finally, split Y into the 32 irreducible representations of G_{1440} to compute its spectrum.

References

[B] Babai, L., *Spectra of Cayley graphs*, J. Comb. Theory, Ser. B, 27 (1979), 180-189.

[BB] Blind, G. and Blind, R., *The semiregular polytopes*, Comment. Math. Helvetici, 66 (1991), 150-154.

[BCN] Brouwer, A. E., Cohen, A. M. and Neumaier, A., *Distance-regular graphs*, Springer-Verlag, New York, 1989.

[C] Coxeter, H. S. M., *Regular polytopes*, Dover, New York, 1973.

[C1] Coxeter, H. S. M., *Regular and semi-regular polytopes, I*, Math. Zeitschrift, 46 (1940), 380-407.

[C2] Coxeter, H. S. M., *Regular and semi-regular polytopes, II*, Math. Zeitschrift, 188 (1985), 559-591.

[C3] Coxeter, H. S. M., *Regular and semi-regular polytopes, III*, Math. Zeitschrift, 200 (1988), 3-45.

[CCNPW] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A. and Wilson, R. A., *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[CDS] Cvetković, D. M., Doob, M. and Sachs, H., *Spectra of graphs*, Academic Press, New York, 1979.

[CS] Chung, F. R. K. and Sternberg, S., *Laplacian and vibrational spectra for homogeneous graphs*, J. Graph Theory, 16 (1992), 605-627.

[Ga] Gantmacher, F., *The theory of matrices*, Chelsea, New York, 1974.

[Go] Gosset, T., *On the regular and semi-regular figures in space of n dimensions*, Messenger of Math., 29 (1900), 43-48.

[K] Kepler, J., *Harmonice Mundi*, Opera Omnia, vol. 5, Frankfurt, 1864.

[L] Lovász, L., *Spectra of graphs with transitive groups*, Periodica Math. Hungarica, 6 (2) (1975), 191-195.

[MKS] Magnus, W., Karrass, A. and Solitar, D., *Combinatorial group theory*, Interscience, New York, 1966.

[MT] Mneimné, R. and Testard, F., *Introduction à la théorie des groupes de Lie classiques*, Hermann, Paris, 1986.

[PS] Petersdorf, M. and Sachs, H., *Spektrum und Automorphismengruppe eines Graphen*, Combinatorial Theory and its Applications (Colloq. Math. Soc. J. Bolyai, 4), Amsterdam–London, 1970, 891-907.

[ST] Saldanha, N. and Tomei, C., *Spectra of regular polytopes*, Discrete Comput. Geom., 7 (1992), 403-414.

[Sc] Schläfli, L., *Theorie der vielfachen Kontinuität*, Denkschriften der Schweizerischen naturforschenden Gesellschaft, 38 (1901), 1-237.

[Se] Serre, J. P., *Représentations linéaires des groupes finis*, Hermann, Paris, 1978.

[Su] Sugiura, M., *Unitary representations an harmonic analysis, an introduction*, second edition, North-Holland–Kodansha, Tokyo, 1990.

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