Critical sets of proper Whitney functions in the plane

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Abstract: A characterization of critical sets and their images for suitable proper Whitney functions in the plane is given. More precisely, necessary and sufficient conditions are provided for the existence of proper Whitney extensions F of $f : C \to \mathbb{R}^2$ such that C is the critical set of F. This result relies strongly on theorems of Blank and Troyer.

Introduction

The purpose of this paper is to provide a characterization of critical sets and their images for suitable proper Whitney functions from the plane to the plane. Knowledge of the critical set of such a function F is essentially sufficient for the understanding of the global behaviour of F and is most useful for the numerical inversion of F (i.e., for the computation of all the solutions of the equation $F(x) = y$) by continuation methods. In [MST1], numerical inversion and a geometric description of F are provided for generic proper Whitney functions with bounded critical set. Interest in this last problem arose from the study of Rankine-Hugoniot equations for hyperbolic conservation laws as considered in [MT]. Given the motivation by the inversion problem, only functions from the plane to the plane are considered here. The results in this paper are described in a form suitable for eventual implementation of numerical inversion routines.

The numerical computation of the critical set of a function F often gives rise to the following situation: some critical curves have been found but it is possible, at least in principle, to infer the existence of additional unknown curves. A simple example of this phenomenon is when the signs of $\det(DF)$ near neighbouring curves are not compatible. One is then led to the following purely topological question:

- In which conditions a given set C of smooth curves and a smooth function $f : C \to \mathbb{R}^2$ can be the critical set of a Whitney function $F : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F|_C = f$?

In other words,

- When is it possible to extend $f : C \to \mathbb{R}^2$ to a Whitney function $F : \mathbb{R}^2 \to \mathbb{R}^2$ whose critical set is C?

In this paper, a complete answer to this question is provided for C being a finite union of smooth curves, $f(C)$ having a finite number of cusps and intersection points, all of them being generic, and the additional requirement that the extension F be proper.

This problem can be reduced to two virtually independent issues: the existence of a local extension of f to a tubular neighbourhood of C and the existence of extensions

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to regular regions (the connected components of the complement of a neighbourhood of C) of proper immersions defined on their boundaries. The first issue was considered by Francis and Troyer ([FT1]) and admits a simple answer. The second one is subtler: for disks, it corresponds to Blank's theorem ([Bl], [P] or, for our own slightly modified version, [MST2]). Troyer ([T], [MST2]) generalized Blank's result for disks with finitely many holes. Analogous criteria for unbounded regions are obtained in this paper.

Many authors, among them Bailey, Ezell, Francis, Marx and Verhey ([Ba], [E], [EM], [F], [FT2], [MV]), extended Blank's and Troyer's theorems to functions form surfaces to surfaces, often allowing functions to have cusps or branch points. Our interest in proper functions from the plane to the plane leads us to consider in detail a highly singular point at infinity. Techniques related to more general domains and images don't seem to be helpful for us and we thus stick to Troyer's original approach, which, as seen in [MST1], is also conveniently implementable.

In section 1, we define *nice* functions F and *adequate* functions f, for which we later obtain a complete answer to the two questions above. We also review the results of Blank, Francis and Troyer.

In section 2 Troyer's theorem is extended to unbounded regions in the plane with finitely many boundary curves, assuming finitely many intersection points in the image of the boundary. Immersions are assumed to be proper and the behaviour at infinity, which plays a central role, is described in combinatorial terms.

Finally, in section 3, we characterize critical sets. As with Blank's theorem, this result is essentially constructive. Formulae for the topological degree and number of pre-images of a regular point are also given. Such formulae are specially convenient for the numerical inversion problem.

1. Preliminaries

In this section, we introduce notation and review some results of Blank, Francis and Troyer ([Bl], [FT1], [P], [T]).

Let C be a finite union of disjoint smooth curves in $\mathbb{R}^2$ and let $f : C \to \mathbb{R}^2$ be a smooth proper function. Since C is to be the critical set of a Whitney function, curves in C are assumed to be images of smooth proper embeddings of S^1 or $\mathbb{R}$ in $\mathbb{R}^2$.

As mentioned in the introduction, to prove the existence of a proper Whitney extension F of f, we first extend f to a proper Whitney function $\tilde{f}$, defined in a thin closed tubular neighbourhood $\overline{U}$ of C whose critical set is C. We then extend $\tilde{f}|_{\partial U}$ to obtain a function $\tilde{F}$ which is a topological immersion outside C and is a Whitney function outside ∂U. To get the desired Whitney function F it is enough to regularize $\tilde{F}$ at ∂U, a standard procedure which we shall not discuss in this paper.

The existence of F is thus equivalent to:

(a) the existence of an extension $\tilde{f}$ of f as above,

(b) the existence of an immersion of a region in the plane with a prescribed behaviour at its boundary.

Item (a) was studied by Francis and Troyer ([FT1]); item (b) for bounded regions is Troyer's theorem ([T]). In this section we review these results and in section 2 we discuss item (b) for unbounded regions.

Recall that a Whitney function is a smooth function whose critical points are all *folds* or *cusps* ([W]). A critical curve for a Whitney function F admits a natural orientation, which will be called the *sense of folding*, leaving nearby points with positive $\det(DF)$ to the left of the oriented curve. For changes of coordinates preserving orientation, a cusp admits normal forms $(x, y) \mapsto (x, \pm y^3 - xy)$; the choice of sign indicates on which side of its critical curve the cusp acts. More precisely, if we take a smooth curve following parallel to the critical curve slightly to the left (resp. right), the image of the curve will form loops precisely at the cusps for which the sign in the above normal form is $+$ (resp. $-$). A cusp is thus said to be *effective to the left* (resp. *right*) if the sign is $+$ (resp. $-$).

Let γ be a critical curve of a Whitney function F, parametrized, according to sense of folding, by a smooth regular function g. Then $\alpha = F \circ g$ satisfies $\alpha''' \wedge \alpha'' > 0$ at all critical points of α. Francis and Troyer ([FT1]) proved that, if γ is an oriented proper curve embedded in $\mathbb{R}^2$ and $f : \gamma \to \mathbb{R}^2$ is a smooth function then the following conditions are equivalent:

- There is an open neighbourhood U of γ and a Whitney function $\tilde{f} : U \to \mathbb{R}^2$ extending f so that the critical set of $\tilde{f}$ is γ, with sense of folding corresponding to the orientation of γ.
- If $g : \mathbb{R} \to \gamma$ is a smooth orientation preserving parametrization and $\alpha = f \circ g$ then $\alpha''' \wedge \alpha'' > 0$ at all critical points of α.

Furthermore, if these conditions hold, $\tilde{f}$ can be chosen so that cusps are effective to whichever side we prescribe. We call $f : \gamma \to \mathbb{R}^2$ *suitable* if it satisfies these conditions.

In this paper, we consider only certain very well behaved Whitney functions, or better, excellent functions with additional hypotheses of finiteness.

Definition 1.1: *A smooth Whitney function $F : \mathbb{R}^2 \to \mathbb{R}^2$ is called* nice *if it satisfies the following properties:*

- *F is proper.*
- *The critical set C is a union of finitely many curves (necessarily disjoint).*
- *There are only finitely many cusps.*
- *There are finitely many intersection points in $F(C)$, i.e., points in $F(C)$ with more than one pre-image in C.*
- *All intersection points are double, transversal and are not cusps, that is, all intersection points have precisely two pre-images in C, both of them being fold points, so that tangent vectors to C at these two points are taken by DF to linearly independent vectors.*

Given a nice function F, with the curves in the critical set C of F oriented by sense of folding, the definition of a nice function imposes conditions on C and $f = F|_C$ motivating the next definition.

Definition 1.2: *Let C be a disjoint union of a finite number of embedded proper smooth oriented curves. A smooth proper function $f : C \to \mathbb{R}^2$ is called* adequate *if it satisfies the following conditions:*

- *Each connected component of $\mathbb{R}^2 - C$ admits an orientation such that the induced orientation on its boundary coincides with the given orientation in C.*
- *The restriction of f to each curve in C is suitable, with finitely many cusps.*
- *There exist only finitely many intersection points (i.e., points in $f(C)$ with more than one pre-image), all of them being double (i.e., with two pre-images) and transversal (tangent vectors to C at the two pre-images are taken by f' to linearly independent vectors).*

Except for the requirement that the number of intersections be finite (which is automatically satisfied if C is compact), adequate immersions correspond to what is known as *normal immersions*.

For a nice function F with critical set C oriented by sense of folding, $F|_C$ is adequate. Thus, the precise question we answer in this paper is: when is it possible to extend an adequate function $f : C \to \mathbb{R}^2$ to a nice function $F : \mathbb{R}^2 \to \mathbb{R}^2$ whose critical set is C?

As we have seen, it is always possible to extend an adequate f to a Whitney function $\tilde{f}$ defined on a neighbourhood U of C such that the critical set of $\tilde{f}$ is C; we may even choose to which side cusps are effective. Notice that U can be taken to be a *tubular* neighbourhood of C, with one connected component per curve and boundary made of smooth curves; furthermore, given $\tilde{f}$, U can be made smaller if necessary so that $\tilde{f}|_{\partial U}$ is an adequate immersion.

We now recall Troyer's result ([T]) on the existence of extensions of boundary immersions for bounded regions in the plane. Blank's theorem ([Bl], [P]) is a special case of Troyer's where the region is a disk; in fact, Troyer's theorem is proved by induction on the number of holes so that Blank's result is the first inductive step. Our statement of Troyer's theorem takes into account regions with arbitrary orientations.

Let A be a disk with k holes in the plane with orientation $\sigma_A = \pm 1$. Let $f : \partial A \to \mathbb{R}^2$ be an adequate immersion where ∂A is oriented consistently with A. Troyer's theorem provides a criterion for the existence of an immersion $F : A \to \mathbb{R}^2$ with $\operatorname{sgn}(\det(DF)) = \sigma_A$ extending f.

Let $g : \mathbb{S}^1 \to \mathbb{R}^2$ be a locally injective continuous function and $\delta > 0$ be such that g is injective in any interval of size δ. The *turning* of g, $\tau(g)$, is defined as the degree of the function $\theta \mapsto (g(\theta + \delta) - g(\theta))/|g(\theta + \delta) - g(\theta)|$ from $\mathbb{S}^1$ to $\mathbb{S}^1$. Notice that $\tau(g)$ does not depend on the choice of δ. In particular, if g is regular, $\tau(g)$ is the degree of $\theta \mapsto g'(\theta)/|g'(\theta)|$, the usual *tangent winding number* (TWN). If γ is an oriented simple closed curve and $f : \gamma \to \mathbb{R}^2$ is continuous and locally injective, $\tau(f)$ is defined as $\tau(f \circ g)$ where g is any orientation preserving smooth parametrization of γ by $\mathbb{S}^1$. For a disjoint union C of oriented simple closed curves and $f : C \to \mathbb{R}^2$ a locally injective function, $\tau(f)$ is the sum of the turnings of the restrictions of f to the connected components of C. The following proposition, whose proof is omitted for being standard, shall be useful in our applications. As usual, $\chi(A)$ is the Euler characteristic of A.

Proposition 1.3: *Let A be an oriented disk with holes and let $\phi : A \to \mathbb{R}^2$ be an immersion such that* $\mathrm{sgn}(\det(D\phi)) = \sigma_A$. *Then* $\tau(\phi|_{\partial A}) = \chi(A)$.

For a closed subset X of $\mathbb{R}^2$, we call the connected components of $\mathbb{R}^2 - X$ the *tiles* for X. A *ray* for f is a proper embedding $r : [0, +\infty) \to \mathbb{R}^2$ which is transversal to $f(C)$ and never goes through an intersection point in $f(C)$. Images of rays, oriented by the given parametrization, are also called rays. A *system of rays* for f is a finite family of disjoint rays with the following properties:

- the origin of each ray is in some bounded tile for $f(C)$,
- each bounded tile for $f(C)$ contains the origin of a unique ray.

The orientations of $f(C)$ and the rays induce orientations, or signs, for their intersections: when the curve crosses the ray from right to left, we call the intersection *positive*, otherwise *negative*. Each intersection also has a *height* associated to it: it is the number of other intersections on the same ray which are closer to its origin. Thus, the first intersection of a ray with a curve always has height zero. The *Blank word* for a curve γ in C is obtained following $f(\gamma)$ once, respecting orientation, and writing down, at each intersection, a letter corresponding to the ray, the sign of the intersection (as an exponent) and its height (as an lower index). Blank words are defined only up to cyclic permutation: any intersection can be taken as the beginning of the word. In the example of Figure 1.1, the Blank words are

$$Bw_0 = a_0^+ b_0^+ c_1^+ d_1^+ e_1^+ f_1^+ b_1^+ c_2^+ d_2^+ e_2^+ f_2^+$$
$$Bw_1 = c_0^- d_0^+ e_0^- f_0^- .$$

A *concatenation* of two Blank words from a pair of intersections in the same ray $\mathbf{z}^-$ and $\mathbf{z}^+$, one in each word, is obtained by cyclically permuting the two words so as to leave $\mathbf{z}^-$ at the right extreme and $\mathbf{z}^+$ at the left extreme of their respective words, juxtaposing both words and eliminating the pair $\mathbf{z}^-\mathbf{z}^+$ produced at the juxtaposition. A concatenation is *positive* when the height of $\mathbf{z}^-$ is smaller than that of $\mathbf{z}^+$. By concatenating a family of Blank words, we obtain a Blank word for the family.

A Blank word admits a *simplification* if there exists a pair of letters $\mathbf{z}^+$, $\mathbf{z}^-$, such that, after a cyclic permutation if necessary, there are no letters with negative exponent between $\mathbf{z}^+$ and $\mathbf{z}^-$. In this case, the simplification is obtained by deleting from the word $\mathbf{z}^+$, $\mathbf{z}^-$ and the letters between them; we say that $\mathbf{z}^-$ was *cancelled* with $\mathbf{z}^+$. A simplification is *positive* if the height of $\mathbf{z}^-$ is smaller than that of $\mathbf{z}^+$. A Blank word *groups* (or *admits a grouping*) if we can sequentially simplify it until we get to a word with no negative exponents. A grouping is *positive* when all simplifications are positive. Finally, a family of Blank words *groups positively* if there exist positive concatenations giving rise to a single word which in turn groups positively.

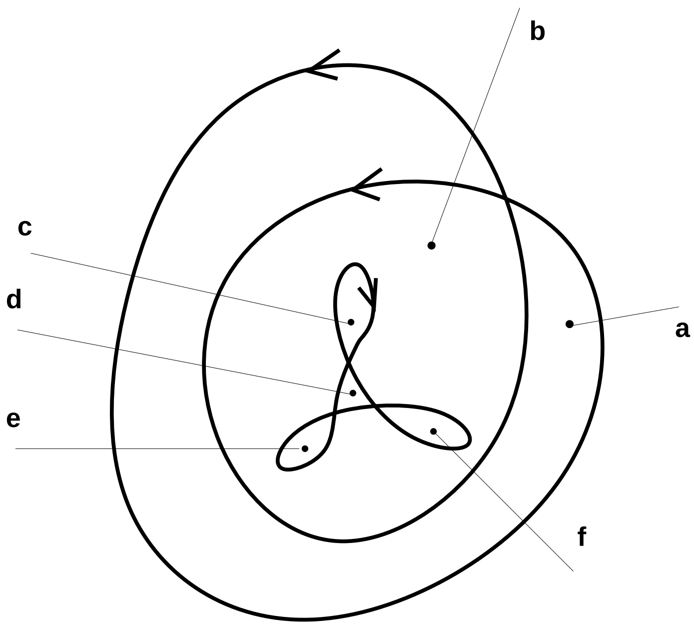

Figure 1.1

The Blank words Bw_0 and Bw_1 group positively, with concatenation $(f_0^- f_1^+)$ and simplifications indicated by brackets as follows:

$$\overbrace{c_0^- d_0^+ \overbrace{e_0^- (f_0^- \; f_1^+) b_1^+ c_2^+ d_2^+ e_2^+} f_2^+ a_0^+ b_0^+ c_1^+} d_1^+ e_1^+ .$$

Notice that, in a grouping, brackets are not allowed to intersect.

Theorem 1.4: (Troyer) *Let A be an oriented disk with k holes and $f : \partial A \to \mathbb{R}^2$ be an adequate immersion. Given a system of rays, consider the associated Blank words. Then there exists an immersion $F : A \to \mathbb{R}^2$ extending f with* $\operatorname{sgn}(\det(DF)) = \sigma_A$ *if and only if*

(a) $\tau(f) = \chi(A)$,

(b) *the Blank words group positively.*

2. Boundary immersions for unbounded regions

In this section we extend Troyer's theorem to unbounded regions. Let A be an open connected oriented subset of the plane with boundary given by a finite number of smooth curves, which are either smooth embeddings of S^1 or smooth proper embeddings of R. We divide such regions A into three types: A is of type I if bounded, of type II if unbounded with bounded boundary, and of type III if unbounded with unbounded boundary. As before, σ_A indicates the orientation of A, ∂A is compatibly oriented and $f : \partial A \to \mathbb{R}^2$ is assumed to be an adequate immersion. The bounded boundary components of A will be called $\gamma_i, i = 1, \ldots, k$ and the unbounded ones, $\beta_j, j = 1, \ldots, \ell$. We are concerned with the possible existence of an immersion F from A to $\mathbb{R}^2$ extending f with $\mathrm{sgn}(\det(DF)) = \sigma_A$. Troyer's result, Theorem 1.4, settles this for regions of type I; in this section, we present similar theorems for regions of type II and III.

In our proofs, we often make use of auxiliary closed curves. We define an *enveloping curve in the domain* to be a smooth simple oriented closed curve δ with the following properties:

- δ is oriented counterclockwise iff $\sigma_A > 0$.
- δ encloses the bounded components of ∂A.
- δ is transversal to ∂A.
- For each unbounded component β_j of ∂A (if any), β_j meets δ at exactly two points.

If A is of type II the two last conditions in the definition of δ are vacuosly satisfied. Similarly, an *enveloping curve in the image* is a smooth simple oriented closed curve ζ with the following properties:

- ζ is oriented counterclockwise.
- ζ encloses the image of the bounded components of ∂A.
- ζ surrounds the intersection points of the image.
- ζ is transversal to $f(\partial A)$.
- For each unbounded component β_j of ∂A (if any), $f(\beta_j)$ meets ζ at exactly two points.

As before, if A is of type II the three last conditions are vacuosly satisfied.

Let A be a region of type III and δ be an enveloping curve in the domain. The unbounded connected components of $A - \delta$ naturally correspond to the *ends* of A (see [S]). The set $\beta_j - \delta$, on the other hand, has two unbounded connected components: following the orientation induced by β_j, one component, α_j, goes from infinity to δ, while the other, ω_j, goes from δ to infinity. The symbols α_j and ω_j shall be interpreted as names for the two ends of β_j. Tracing δ according to its orientation and keeping track of the ends of the β_j's, we build a cyclic word with letters α_j and ω_j, the *word for the domain*. This is a precise formulation of the rather geometric concept of *order of arrival at infinity* of the components of the boundary of A. It can easily be verified that this word is independent of the choice of the enveloping curve δ. We assume the curves β_j to be labeled so that the word for the domain is $\alpha_1\omega_1\alpha_2\ldots\omega_\ell$. In particular, the boundary of an unbounded component of $A - \delta$ is composed of ω_j, an arc of δ and α_{j+1} (where $\beta_{\ell+1} = \beta_1$). Similarly, using an enveloping curve ζ in the image and letters $f(\alpha_j)$ and $f(\omega_j)$, we define the order of arrival at infinity of the curves $f(\beta_j)$, or the *word for the image* $\mathcal{W}$.

The cyclic word $\mathcal{W}$ induces a permutation π of $\{1, 2, \ldots, 2\ell - 1\}$. Turn $\mathcal{W}$ into a linear word leaving $f(\omega_\ell)$ at the last position. The numbers $\pi(2j - 1)$ and $\pi(2j)$ give the positions of $f(\alpha_j)$ and $f(\omega_j)$ in the linear word, respectively. We now define ρ_A to be the *number of runs* of π ([K]), i.e., the number of maximal intervals in $\{1, 2, \ldots, 2\ell - 1\}$ where π is increasing. Equivalently, ρ_A is the number of times required to go through $\mathcal{W}$ in order to pass sequentially by the letters $f(\alpha_1), f(\omega_1), \ldots, f(\omega_\ell), f(\alpha_1)$. A more relevant interpretation is the following: let δ and ζ be enveloping curves in the domain and image, respectively. Consider orientation preserving immersions from δ to ζ taking α_j to $f(\alpha_j)$ and ω_j to $f(\omega_j)$, i.e., taking the points in $\beta_j \cap \delta$ to the corresponding points in $f(\beta_j) \cap \zeta$. The minimum degree (or turning) of such immersions is ρ_A.

In Figure 2.1, the word for the image is $f(\alpha_1)f(\alpha_2)f(\alpha_3)f(\omega_2)f(\omega_1)f(\omega_3)$, the permutation π is given by $\pi(1) = 1$, $\pi(2) = 5$, $\pi(3) = 2$, $\pi(4) = 4$ and $\pi(5) = 3$ and therefore $\rho_A = 3$.

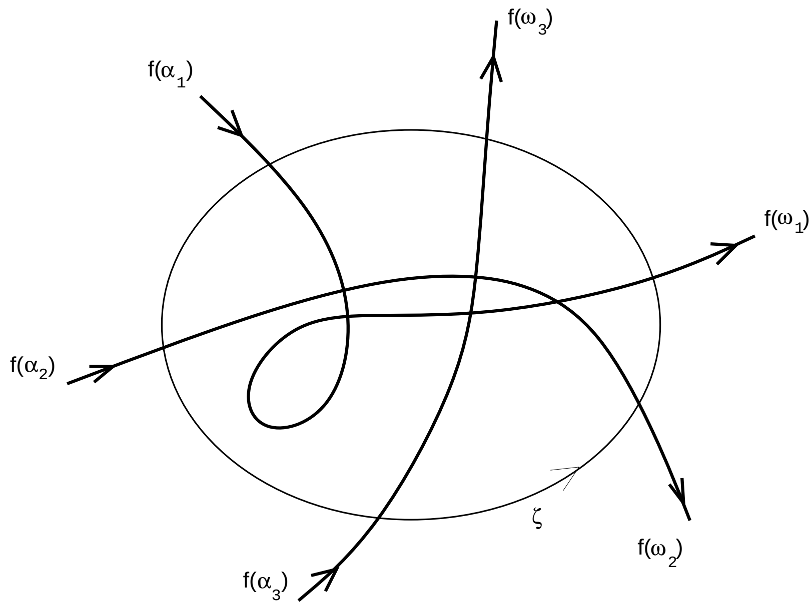

Figure 2.1

The turning of an oriented closed curve was defined in the previous section: we now define the *turning* of an unbounded oriented curve with a finite number of self-intersections. Take two points p and q on the curve, one on each unbounded connected component of the curve minus its self-intersections. Connect p and q by a simple arc which does not intersect the portion pq of the curve between p and q and consider the oriented closed curve formed by the arc and pq, respecting in pq the original orientation of the curve. There are two possible values, differing by 2, for the turning of the closed curve thus constructed, depending on the choice of the simple arc. The turning of the unbounded curve is by definition the average of these two values. In particular, a curve with no self-intersections has turning number 0 and the turning of the curve in Figure 2.2 is -2. We then define $\tau(\beta_j)$ as the turning of $f(\beta_j)$, where $f(\beta_j)$ has the orientation induced by f and the orientation of β_j. Similarly, $\tau(\gamma_i) = \tau(f|_{\gamma_i})$.

Lemma 2.1: *Let A be a region of type* II *or* III, $F : A \to \mathbb{R}^2$ *a smooth proper immersion such that $F|_{\partial A}$ is an adequate immersion. Suppose that A is oriented so that $\sigma_A =$* $\mathrm{sgn}(\det(DF))$. *Let ζ be an enveloping curve in the image of $f = F|_{\partial A}$. Then*

(i) *If A is of type* II, *the pre-image of ζ by F is a simple closed curve.*

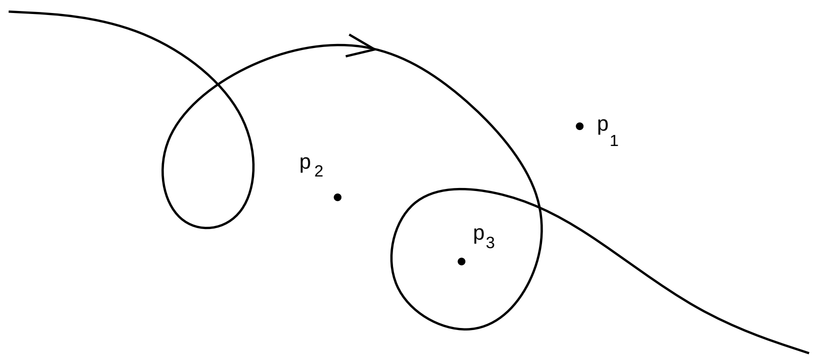

Figure 2.2

(ii) *If A is of type* III, *then each connected component of the pre-image of ζ by F is a simple arc lying in A joining end ω_j to α_{j+1}.*

Proof: Since F is a proper immersion, the connected components of the pre-image of a simple, closed, regular curve are either simple closed curves or simple arcs whose endpoints belong to ∂A. If this closed curve is the enveloping curve ζ, any arc must go from some ω_j to $\alpha_{j'}$.

Let A be of type II. From the previous paragraph, the connected components of the pre-image of ζ are simple closed curves. It suffices to show that there is only one such component. We first show this for an auxiliary curve $\tilde{\zeta}$, defined as follows. By properness of F, there is an enveloping curve δ bounding a disk D_δ which contains the pre-image of $F(\partial A)$. Notice that, since A is of type II, δ has to be contained in A. Now, let $\tilde{\zeta}$ be an enveloping curve in the image surrounding $F(D_\delta)$ and ζ. The connected components of the pre-image of $\tilde{\zeta}$ are necessarily simple closed curves surrounding the disk D_δ, by construction. The existence of two such components would give rise to a critical point of F in the annulus between them, contradicting the fact that F is an immersion. Thus the pre-image of $\tilde{\zeta}$ is connected; the connectivity of the pre-image of ζ follows from the fact that, if R is the annulus bounded by ζ and $\tilde{\zeta}$, the restriction of F to $F^{-1}(R)$ is a covering map.

Let A be of type III. Construct in the domain an enveloping curve δ surrounding $F^{-1}(D_\zeta)$. Let $\tilde{\zeta}$ be an enveloping curve in the image surrounding $F(D_\delta)$. We first prove (ii) for the auxiliary curve $\tilde{\zeta}$. Remember that since δ is an enveloping curve in the domain, each connected component of $A - D_\delta$ is an end bounded by ω_j and α_{j+1}. Thus, from the fact that $\tilde{\zeta}$ surrounds $F(D_\delta)$ it is clear that any connected component of $F^{-1}(\tilde{\zeta})$ is contained in one of these ends. Arcs must therefore join ω_j to α_{j+1} and it remains to prove that there are no closed curves in the pre-image of $\tilde{\zeta}$. Indeed, the image of the disk bounded by such a closed curve would be $D_{\tilde{\zeta}}$, which contains D_ζ, contradicting the fact that $F^{-1}(D_\zeta)$ is contained in D_δ. In order to transfer the result from $\tilde{\zeta}$ to ζ, consider a smooth non-zero vector field in the annulus contained between ζ and $\tilde{\zeta}$ transversal to these two curves, coming in through ζ and going out through $\tilde{\zeta}$ and tangent to $F(\beta_j)$. The pullback of this vector field by F defines a smooth non-zero vector field on the pre-image of the said annulus. Each connected component of this pre-image must therefore be a disk with boundary consisting of an arc in ω_j, a connected component of $F^{-1}(\tilde{\zeta})$, an arc in α_{j+1} and a connected component of $F^{-1}(\zeta)$, in this order. Item (ii) is therefore proved for ζ. ∎

From the above proof, the inverse image of the disk D_ζ bounded by ζ is a disk with finitely many holes. The outer boundary γ_∞ of this pre-image is simply the pre-image of ζ if A is of type II. If A is of type III, γ_∞ is formed by arcs which are alternatedly connected components of $F^{-1}(\zeta)$ and segments of β_j; these segments come in the order indicated by the indices. In either case, the inner boundaries are the γ_i. We orient γ_∞ so that it is positively oriented iff $\sigma_A > 0$; define $\mathcal{T}(\gamma_\infty) = \tau(F|_{\gamma_\infty})$.

Let F and A be as in the previous lemma. If A is of type II, let d be the number of pre-images of an arbitrary point on an enveloping curve in the image; clearly, this number is independent of the point or the curve. We define the *degree at infinity* of F by $\deg(F) = \operatorname{sgn}(\det(DF))d$ (notice that degrees are thus taken with respect to the usual orientation of $\mathbb{R}^2$, not that of A). Any continuous function extending F to the plane has topological degree equal to the degree of F at infinity.

If A is of type III, we assign a non-negative integer d_j (somewhat similar to d) to each end E_j of A, where E_j is enclosed between ω_j and α_{j+1}. Consider an arbitrary enveloping curve ζ in the image and let p be the only intersection of ζ with $F(\omega_j)$. The number d_j is defined to be the number of pre-images of p in the interior of the connected component of the pre-image of ζ corresponding to the end E_j. Clearly, d_j thus defined does not depend on the choice of ζ since there are d_j pre-images of $F(\omega_j)$ (i.e., connected components of $F^{-1}(F(\omega_j)))$ in the interior of E_j. We call $\operatorname{sgn}(\det(DF))d_j$ the *partial degree of F at E_j*.

For example, the behaviour at infinity of a nice proper polynomial function from $\mathbb{R}^2$ to $\mathbb{R}^2$ can be determined by methods such as Newton polygons. This yields the number of unbounded critical curves, the extrema of such curves, their order of arrival at infinity and the partial degrees at each end.

Lemma 2.2: *Let A be an oriented region of type* II *or* III *and $F : A \to \mathbb{R}^2$ be an immersion such that* $\operatorname{sgn}(\det(DF)) = \sigma_A$ *and $F|_{\partial A}$ is an adequate immersion. Let ζ be an enveloping curve in the image and consider γ_∞ as above. Then*

$$\mathcal{T}(\gamma_\infty) = \begin{cases} d & \text{if } A \text{ is of type II,} \\ \sum_j \mathcal{T}(\beta_j) + \sum_j d_j + \rho_A & \text{if } A \text{ is of type III.} \end{cases}$$

Also,

$$\mathcal{T}(\gamma_\infty) + \sum_i \mathcal{T}(\gamma_i) = \chi(A).$$

Proof: The first identity is easy for A of type II (be careful with orientations). For A of type III, if all d_j and $\mathcal{T}(\beta_j)$ are zero the curve $F(\gamma_\infty)$ can be deformed without changing turning numbers to the orientation preserving immersion from γ_∞ to ζ with minimum degree taking α_j and ω_j to $F(\alpha_j)$ and $F(\omega_j)$, respectively, hence the formula in this special case. Clearly, changing a d_j amounts to introducing extra turns to the above immersion.

Finally, the additive property of turnings takes care of the $\mathcal{T}(\beta_j)$, thus proving the first identity for A of type III. The last identity follows from Proposition 1.3. ■

In order to obtain generalizations of Theorem 1.4 to regions of type II or III we make use of Blank words. We first generalize the notion of systems of rays for an adequate immersion f defined on the boundary of an unbounded oriented region A.

For regions of type II, a system of rays for f is defined exactly as in Section 1. For regions of type III, a system of rays for f is again a finite set of embeddings of the closed positive half-line, with the same properties of disjointness and transversality to the image of the boundary curves as described in Section 1 plus the additional condition that a ray may only intersect $f(\partial A)$ finitely many times. Also, there must be one ray with origin at each of the (finitely many) *bounded* tiles for the image of the curves. In each case, Blank words for each bounded curve in ∂A are constructed exactly as in Troyer's theorem but we now need an extra ingredient: the *word at infinity*.

Consider a system of rays for f as described above. Let ζ be an enveloping curve in the image surrounding all intersections of rays with $f(\partial A)$ crossing each ray transversally and exactly once. Tracing ζ counterclockwise and keeping track of intersections with rays and images of boundary curves, we build a cyclic word $\mathcal{W}^\star$ with the letters used for the rays, $f(\alpha_j)$ and $f(\omega_j)$, describing the order of arrival at infinity of rays and images of unbounded boundary curves (if any). This word is independent of the choice of ζ (with the properties above). Notice that, for A of type III, by ommiting the letters for rays we obtain the word for the image $\mathcal{W}$.

We first consider regions of type II. Let d be a positive integer (the absolute value of the degree at infinity of the desired extension of f). The *word at infinity* is made up of d juxtaposed copies of the word of $\mathcal{W}^\star$ giving all letters a positive sign and height index equal to ∞ (indicating that such heights are always greater than those of intersections with $f(\partial A)$).

For A of type III, given non-negative integers d_j (the absolute value of the partial degrees at E_j of the desired extension), we first construct auxiliary strings $\mathcal{S}_j$ and $\mathcal{R}_j$. The string $\mathcal{S}_j$ is obtained by following $f(\beta_j)$, keeping track of oriented intersections with the rays as before. The string $\mathcal{R}_j$ is constructed by following $\mathcal{W}^\star$ starting at ω_j and reaching α_{j+1} (or α_1 if $j = \ell$) after making d_j full turns around $\mathcal{W}^\star$, ignoring α's and ω's. Finally, the word at infinity is obtained by concatenating $\mathcal{S}_1, \mathcal{R}_1, \mathcal{S}_2, \mathcal{R}_2, \ldots, \mathcal{S}_\ell, \mathcal{R}_\ell$, in this order. Letters in the strings $\mathcal{S}_j$ receive height indices as before while letters in the strings $\mathcal{R}_j$ have height indices equal to ∞. An example is provided immediately after the statement of Theorem 2.4.

Thus, for unbounded regions, we have, given an adequate function and a system of rays, a Blank word for each bounded boundary curve and the word at infinity. The notions of adjunction and grouping are exactly as in Section 1 taking into account both kinds of words.

Theorem 2.3: *Let A be an oriented region of type* II *and $f : \partial A \to \mathbb{R}^2$ be an adequate immersion. Given $d > 0$ and a system of rays, consider the associated Blank words. Then,* *there exists a proper immersion* $F : A \to \mathbb{R}^2$ *extending* f, *with* $\operatorname{sgn}(\det(DF)) = \sigma_A$ *and degree at infinity equal to* $\sigma_A d$ *if and only if*

(a) $d + \sum_{1 \le i \le k} \mathcal{T}(\gamma_i) = \chi(A)$,

(b) *the Blank words together with the word at infinity group positively.*

Theorem 2.4: *Let* A *be an oriented region of type* III *and* $f : \partial A \to \mathbb{R}^2$ *be an adequate immersion. Given non-negative integers* d_j, $j = 1, \ldots, \ell$, *and a system of rays, consider the associated Blank words. Then, there exists a proper immersion* $F : A \to \mathbb{R}^2$ *extending* f *with* $\operatorname{sgn}(\det(DF)) = \sigma_A$ *and partial degree at* E_j *equal to* $\sigma_A d_j$ *if and only if*

(a) $\rho_A + \sum_j d_j + \sum_j \mathcal{T}(\beta_j) + \sum_i \mathcal{T}(\gamma_i) = \chi(A)$,

(b) *the Blank words together with the word at infinity group positively.*

Before proving the theorems, we describe an example. Consider the region A of type III in Figure 2.3(a) with image of the boundary under an adequate immersion f shown in 2.3(b). We assign to the two ends of A partial degrees $d_1 = 0$ and $d_2 = 2$; orientation and ends of curves are indicated in the figures. We then have $\mathcal{T}(\gamma_1) = -5$, $\mathcal{T}(\beta_1) = +1$ and $\mathcal{T}(\beta_2) = 0$. The word at the image $\mathcal{W}$ is $f(\omega_1)f(\alpha_2)f(\alpha_1)f(\omega_2)$ and the corresponding permutation is $\pi(1) = 3$, $\pi(2) = 1$ and $\pi(3) = 2$, whence $\rho_A = 2$. Condition (a) in Theorem 2.4 holds, since $\chi(A) = 0$. For the rays in the picture, the Blank word for $f(\gamma_1)$ is $b_0^+ c_0^- d_0^- e_0^- f_0^- g_0^- h_0^-$ and the substrings which concatenate to yield the word at infinity are

$$\begin{aligned}
\mathcal{S}_1 &= a_0^+ b_1^+ c_1^+ d_1^+ e_1^+ f_1^+ g_1^+ h_1^+,\\
\mathcal{R}_1 &= \text{(empty word)},\\
\mathcal{S}_2 &= a_1^+ b_2^+ c_2^+ d_2^+ e_2^+ f_2^+ g_2^+ h_2^+,\\
\mathcal{R}_2 &= a_\infty^+ b_\infty^+ c_\infty^+ d_\infty^+ e_\infty^+ f_\infty^+ g_\infty^+ h_\infty^+ a_\infty^+ b_\infty^+ c_\infty^+ d_\infty^+ e_\infty^+ f_\infty^+ g_\infty^+ h_\infty^+ a_\infty^+ b_\infty^+ c_\infty^+ d_\infty^+ e_\infty^+ f_\infty^+ g_\infty^+ h_\infty^+.
\end{aligned}$$

The reader is invited to check that these words indeed satisfy condition (b) in Theorem 2.4. An immersion F extending f therefore exists and the proofs of the previous theorems show how to construct it; we shall see a related example in Section 3.

Proof of Theorem 2.3: We apply Theorem 1.4 to a bounded region $A_0 \subseteq A$, with the same inner boundaries as A and outer boundary given by a simple closed curve γ_0, which we now construct.

Consider a closed regular curve $\tilde{\gamma}_0$ in the image, parametrized by $\tilde{g}_0 : [0,1] \to \mathbb{R}^2$ with $\tilde{g}_0(0) = \tilde{g}_0(1)$ satisfying the following properties (see Figure 2.4, where $d = 3$ and $\partial A = C$).

- The curve $\tilde{\gamma}_0$ turns d times conterclockwise around a fixed enveloping curve in the image.
- The curve $\tilde{\gamma}_0$ intersects each ray transversally exactly d times.
- The curve $\tilde{\gamma}_0$ has exactly $d - 1$ self-intersections, all transversal.

We initially prove that if an immersion exists, then items (a) and (b) hold. By continuation, there is a $g_0 : [0,1] \to \mathbb{R}^2$ with $\tilde{g}_0 = F \circ g_0$ for any choice of $g_0(0)$, a pre-image of $\tilde{g}_0(0)$. We want to show that g_0 is the parametrization of a simple closed curve.

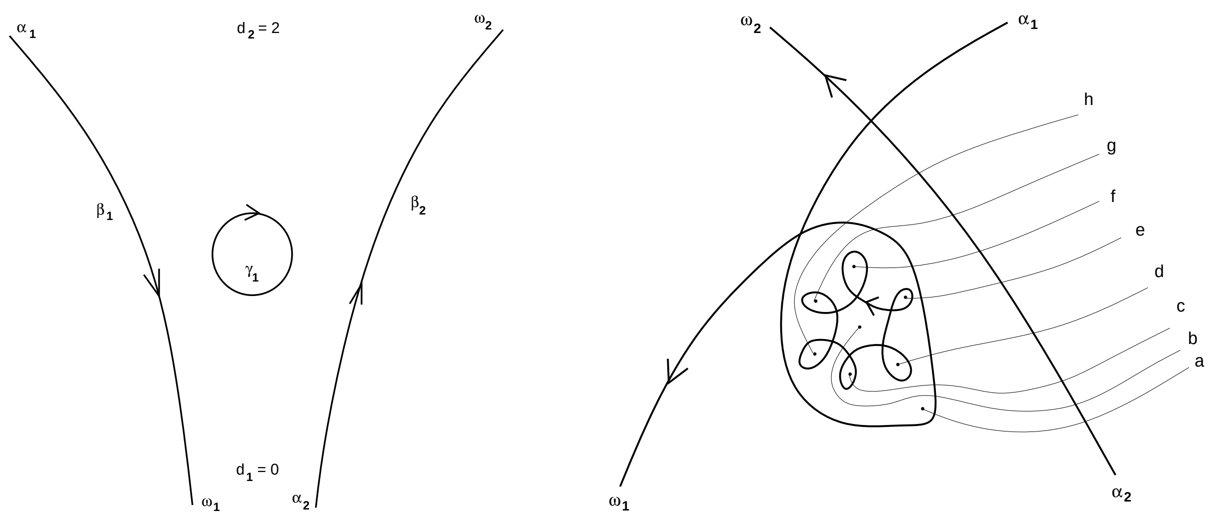

Figure 2.3

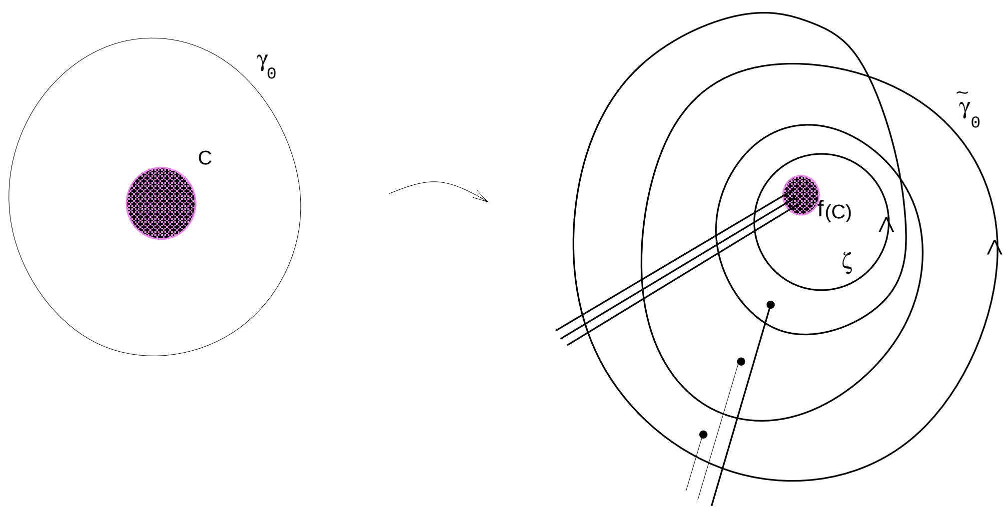

Figure 2.4

Indeed, let us first prove that there are no proper loops. A loop has to surround ∂A since its image surrounds an enveloping curve and therefore the turning of the restriction of F to the loop has to be equal to the turning of $F|_{\gamma_\infty}$, which is d. The image of a proper loop must of course have smaller turning than d, a contradiction. If $g_0(0) \neq g_0(1)$, continue the inversion process until g_0 comes back to $g_0(0)$, which it must eventually do since F is proper. The turning of $F \circ g_0$ is greater than d since $F \circ g_0$ traces $\tilde{\gamma}_0$ more than once. Let γ_0 be the image of g_0.

In order to obtain a system of rays for the new problem, it suffices to add to the old system a few rays starting from the tiles created by $\tilde{\gamma}_0$ with positive intersections only. The Blank word for γ_0 in the new problem is therefore, up to irrelevant new letters, equal to the word at infinity. Conditions (a) and (b) of our proposition follow from the corresponding conditions in Theorem 1.4.

To prove the converse, let γ_0 be any enveloping curve in the domain. We define $f_0 : \gamma_0 \to \mathbb{R}^2$ as an orientation preserving regular parametrization of $\tilde{\gamma}_0$. Let A_0 be as above and apply to it Theorem 1.4. Clearly, heights for letters in the Blank word of γ_0 are greater than the height of any other intersection on the same ray with any other curve, so that the existence of a positive grouping is unaffected by the change of word. Therefore, conditions (a) and (b) of the proposition imply conditions (a) and (b) in Theorem 1.4. There exists then an F_0 extending simultaneously f and f_0 to A_0. The existence of an F_∞ extending f_0 to $A - A_0$ is trivial. By gluing F_0 and F_∞ we obtain an extension F of f to A which may be non smooth, but which can easily be rendered smooth. ∎

Proof of Theorem 2.4: The proof is similar to that of Proposition 2.3: we apply Theorem 1.4 to a bounded region $A_0 \subseteq A$ with the same inner boundaries as A and with outer boundary given by a curve γ_0 similar to γ_∞, i.e., composed of big chunks of β_j's together with arcs ξ_j (to be constructed) connecting $\omega_j \subseteq \beta_j$ to $\alpha_{j+1} \subseteq \beta_{j+1}$.

Let ζ be an arbitrary enveloping curve in the image. For each j take an oriented arc $\tilde{\xi}_j$ outside D_ζ with the following properties (see Figure 2.5):

- $\tilde{\xi}_j$ goes from $f(\omega_j)$ to $f(\alpha_{j+1})$,
- $\tilde{\xi}_j$ intersects any $f(\omega)$ transversally from right to left and any $f(\alpha)$ transversally from left to right,
- $\tilde{\xi}_j$ intersects any ray transversally from right to left,
- the intersection of the closed arc $\tilde{\xi}_j$ with $f(\omega_j)$ has exactly $d_j + 1$ points,
- the arcs $\tilde{\xi}_j$ are simple and disjoint.

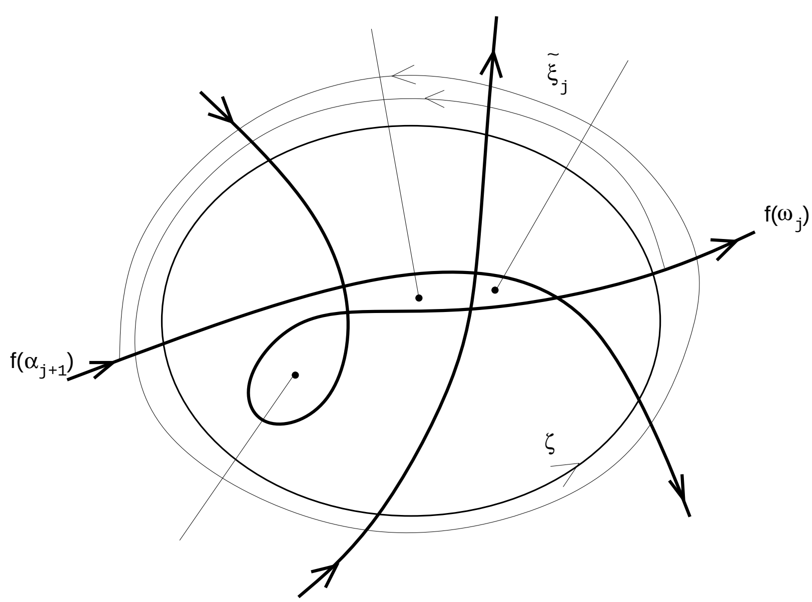

Figure 2.5

First we prove that the existence of F implies (a) and (b). For each j take by continuation a pre-image ξ_j of $\tilde{\xi}_j$, starting at the only pre-image of the beginning of $\tilde{\xi}_j$ in ω_j; the process can get started since F preserves orientation. This will of course produce a simple arc entirely contained in the end E_j. We have to prove that this arc ends at α_{j+1} and that its image is the entire arc $\tilde{\xi}_j$. The continuation process cannot fail before we reach the end of $\tilde{\xi}_j$ since we would then have less than d_j pre-images of $f(\omega_j)$ in the interior of E_j, contradicting the fact that d_j is the corresponding partial degree. On the other hand, the process must end by reaching α_{j+1}: otherwise, extend $\tilde{\xi}_j$ respecting sense of intersections until we reach ω_j and we have $d_j + 1$ pre-images of $f(\omega_j)$ in the interior of E_j, again a contradiction. We thus have the curve γ_0 constructed from these arcs $\tilde{\xi}_j$ and chunks of β_j's, oriented consistently with the β_j (and thus, automatically, consistently with the $\tilde{\xi}_j$). As usual, let $\mathcal{T}(\gamma_0) = \tau(F|_{\gamma_0})$. Clearly, since F is an immersion, $\mathcal{T}(\gamma_0) = \mathcal{T}(\gamma_\infty)$, where γ_∞ is obtained from ζ and F, and so Lemma 2.2 implies item (a). Again, in order to apply Theorem 1.4 to A_0 we need to add new rays but these can be taken with positive intersections only and are therefore irrelevant when considering grouping. The Blank word for γ_0 is, except for these new positive letters, the word at infinity; item (b) follows.

For the converse, let ξ_j be arbitrary simple arcs contained in E_j joining ω_j to α_{j+1}. We have thus defined the curve γ_0 and f is defined on those parts of it coming from β_j. In order to extend f to γ_0, define homeomorphisms from ξ_j onto $\tilde{\xi}_j$ respecting endpoints. In order to apply Theorem 1.4 to A_0 we first observe that the curve γ_0 and f on it were constructed in order to guarantee that

$$\mathcal{T}(\gamma_0) = \rho_A + \sum_j (\mathcal{T}(\beta_j) + d_j),$$

similarly to Lemma 2.2. Item (a) in Theorem 1.4 now follows from our item (a). Item (b) follows from our item (b) since again we need to introduce a few irrelevant rays and then the word for γ_0 is essentially the word at infinity. As in the previous theorem, the extension to each end is trivial; the overall smoothing is again done by classical methods.
■

3. The main theorems

In this section, we finally characterize critical sets of nice functions (Theorem 3.1) and provide formulae for degree and number of pre-images in terms of the critical set (Theorem 3.2).

Recall that for a closed subset X of $\mathbb{R}^2$ the connected components of $\mathbb{R}^2 - X$ are called *tiles* for X. Let C be the union of disjoint oriented curves γ_i, $1 \leq i \leq k$ and β_j, $1 \leq j \leq \ell$, where γ_i and β_j are images of proper embeddings of S^1 and $\mathbb{R}$ respectively. We assume that these curves are *consistently oriented*, i.e., that any tile S for C can be oriented so that the induced orientation in $\partial S \subseteq C$ is consistent with that of C: we then say that S is *consistently oriented* with C. Notice that if $\ell = 0$, we have bounded tiles (regions of type

I) and a single unbounded one (of type II). On the other hand, if $\ell > 0$, all unbounded tiles are regions of type III and there are at least two of them.

As we saw in Section 1, cusps of a Whitney function can be effective to the right or to the left of their critical curves. The function f alone does not determine to which side of a critical curve a cusp will be effective and it is thus natural to consider this information as another given of the problem. We therefore assign indices **l** or **r** to cusps indicating if they are to be effective to the left or right, respectively, for the desired Whitney function F. Thus, a cusp labeled **l** (resp., **r**) will be called *effective in* S if S is the tile for C immediately to the left (resp., right) of the cusp.

Again, we begin by constructing the Blank words. A *ray* for $f|_{\partial S}$ is now a proper embedding $r : [0, +\infty) \to \mathbb{R}^2$ with the previous properties and the extra requirements that, except possibly at the origin, the ray does not meet images of cusps. Similarly, a *system of rays* for $f|_{\partial S}$ (or, for simplicity, a system of rays for S) is a finite system of disjoint rays with the following properties:

- Given a bounded connected component of $\mathbb{R}^2 - f(\partial S)$, there exists exactly one ray such that its origin lies in this connected component.
- Given a effective cusp in S, there exists exactly one ray whose origin is the image of this cusp; furthermore, the ray leaves the cusp to the right of $f(\partial S)$.
- Rays start either in a bounded component of $\mathbb{R}^2 - f(\partial S)$ or at the image of an effective cusp in S.

Given a system of rays for S of type I, we now construct a Blank word for each γ_i. If S is of type II (resp. III), given also $d > 0$ (resp. $d_j \geq 0$ $(1 \leq j \leq \ell^S)$), we construct, besides the Blank words for each γ_i, an additional *word at infinity*. The words for each such γ_i are constructed by following $f(\gamma_i)$, keeping track of oriented intersections with the rays, as before. Letters corresponding to intersections at images of cusps receive a minus sign and a height index 0, by definition. The word at infinity is constructed as before, again assigning to letters corresponding to cusps a minus sign and a height index 0.

Given a tile S for C, let $\kappa(S)$ be the total number of effective cusps in S and $\tau(S)$ be the sum of the turning of f at each boundary curve of S.

Theorem 3.1: *Let $f : C \to \mathbb{R}^2$ be an adequate function and let labels* **l** *and* **r** *be assigned to the cusps of f. Consider a system of rays for each tile S for C. Then there exists a proper Whitney function F extending f to $\mathbb{R}^2$ having critical set C, with sense of folding corresponding to the given orientation of C and such that a cusp is effective to the left (resp., right) if its label is* **l** *(resp.,* **r***) if and only if, for each tile S (consistently oriented with C), the following condition holds:*

- *For S of type* I*, the identity*
$$\tau(S) - \kappa(S) = \chi(S)$$
holds and the Blank words group positively.
- *For S of type* II*, we have that the number d, defined by*
$$\tau(S) - \kappa(S) + d = \chi(S)$$

is strictly positive and the associated Blank words group positively (in this case, $d = |\deg(F)|$).

- *For S of type* III*, with ends E_j, $1 \le j \le \ell^S$, there exist non-negative integers d_j, $1 \le j \le \ell^S$, with*
$$\rho_S + \mathcal{T}(S) - \kappa(S) + \sum_{1 \le j \le \ell^S} d_j = \chi(S),$$
such that the associated Blank words group positively (d_j is the absolute value of the partial degree of F at the end E_j).

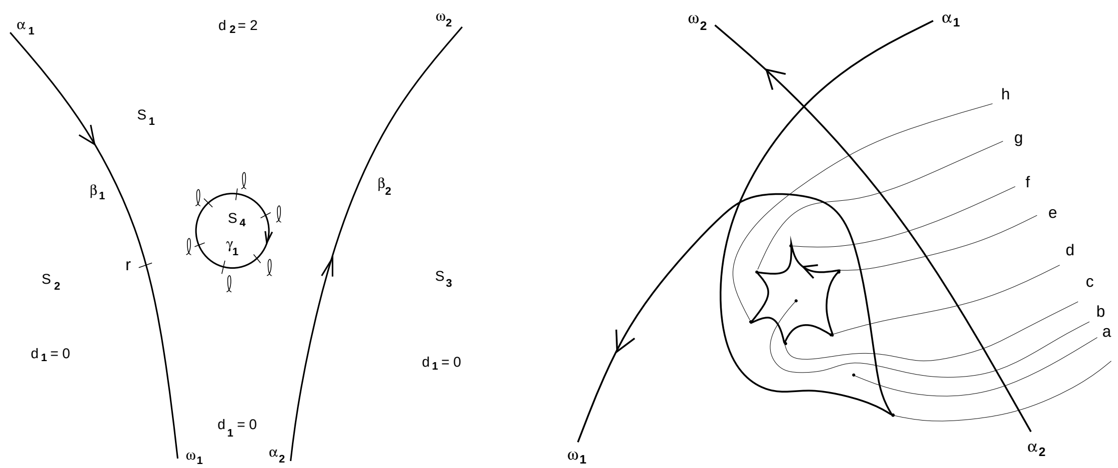

Figure 3.1

As an example, consider C and $f(C)$ as in Figure 3.1, (a) and (b), respectively. The tiles in the domain are S_1, S_2, S_3 and S_4, the first three being of type III and the last one of type I. The orientations of C and $f(C)$ (which are to be the sense of folding) are indicated, as are the labels for cusps and partial degrees. We now check the condition in Theorem 3.1 for tile S_1. It is easy to see that $\kappa(S_1) = 6$ and that $\mathcal{T}(\gamma_1) = 1$, $\mathcal{T}(\beta_1) = 1$ and $\mathcal{T}(\beta_2) = 0$, whence $\mathcal{T}(S_1) = 2$. Since the behaviour at infinity of this example for tile S_1 is identical to that of the example shown in Figure 3.3, we have $\rho_{S_1} = 2$ and the same word at infinity. The expression for $\chi(S_1) = 0$ therefore holds. Also, the Blank word for γ_1 is identical to that for the example in Figure 2.3, and again the words group positively. The reader will easily check that the remaining tiles also satisfy the appropriate conditions in the above theorem. Thus, there is a proper Whitney function F extending f with prescribed critical set C, partial degrees and senses of cusps.

In order to provide a geometric description of F, we show $F^{-1}(F(C))$ in Figure 3.2(a). Five of the six tiles for $F(C)$ are simply connected and thus diffeomorphic by F to the connected components of their pre-images, which are tiles for $F^{-1}(F(C))$. The pre-image of the tile for $F(C)$ surrounding $F(\gamma_1)$ has three connected components in which F is a diffeomorphism and a fourth where F is a five-fold covering map.

Proof of Theorem 3.1: Since f is adequate, it can be extended to a Whitney function $\tilde{f}$ defined on a thin tubular neighborhood U of C. In order to prove the existence of F, it

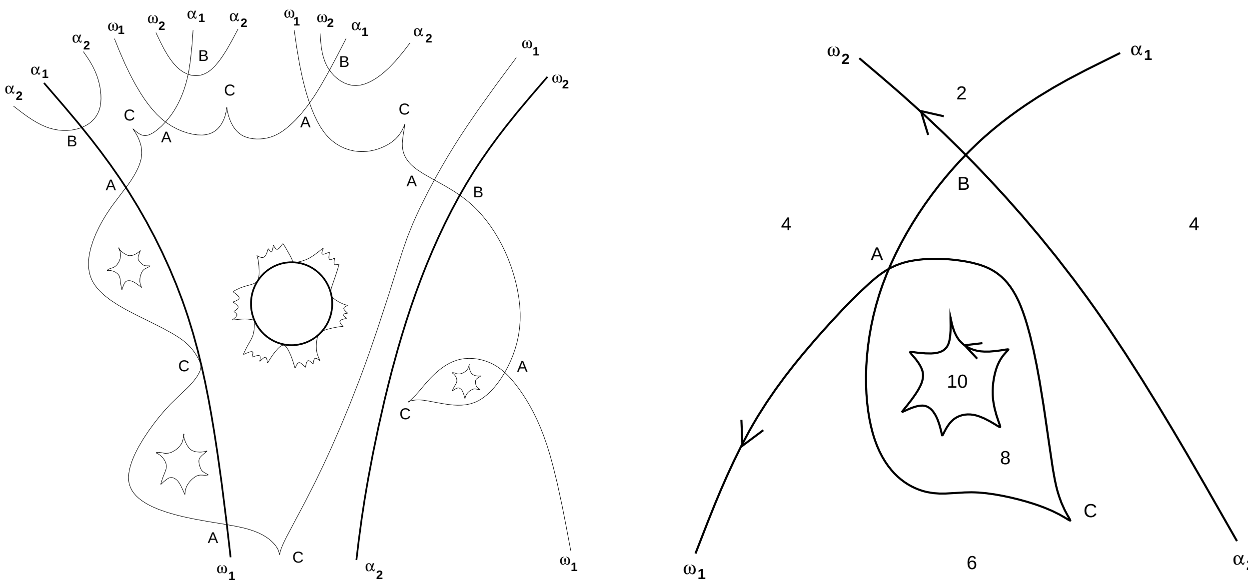

Figure 3.2

therefore suffices to extend to each tile S immersions which are already defined near the boundary.

Let S be a tile and let its boundary be composed of bounded curves γ_i^S, $1 \leq i \leq k^S$, and unbounded curves β_j^S, $1 \leq j \leq \ell^S$. We denote by $\kappa^S(\gamma_i^S)$ (resp., $\kappa^S(\beta_j^S)$) the number of cusps on γ_i^S (resp., β_j^S) which are effective in S. Thus,

$$\mathcal{T}(S) = \sum_{1 \leq i \leq k^S} \mathcal{T}(\gamma_i^S) + \sum_{1 \leq j \leq \ell^S} \mathcal{T}(\beta_j^S)$$

and

$$\kappa(S) = \sum_{1 \leq i \leq k^S} \kappa^S(\gamma_i^S) + \sum_{1 \leq j \leq \ell^S} \kappa^S(\beta_j^S).$$

Let $S^\star \subseteq S$ be a closed set whose boundary is composed of smooth curves $\gamma_i^\star$ and $\beta_j^\star$ which are sufficiently near the corresponding curves in ∂S and contained in U. Orient $\gamma_i^\star$ and $\beta_j^\star$ as the nearby γ_i^S and β_j^S, so that they form the oriented boundary of $S^\star$, which is oriented positively or negatively according to the constant sign of $\det(D\tilde{f})$ in $S \cap U$. For S of type I, II or III, we apply to $S^\star$ Theorem 1.4, Theorem 2.3 or Theorem 2.4, respectively. We now show the equivalence between the conditions in Theorem 3.1 and the hypotheses of these theorems.

From the local behaviour of cusps, each effective cusp creates a little loop in $f(\gamma_i^\star)$ or $f(\beta_j^\star)$, always negatively oriented (see the paragraph immediately following the proof for an example) and if $\gamma_i^\star$ and $\beta_j^\star$ are sufficiently near γ_i^S and β_j^S these are the only new intersections. This implies

$$\tau(\tilde{f}(\gamma_i^\star)) = \mathcal{T}(\gamma_i^S) - \kappa^S(\gamma_i^S)$$

and

$$\tau(\tilde{f}(\beta_j^\star)) = \daleth(\beta_j^S) - \kappa^S(\beta_j^S).$$

Also, the connected components of $\mathbb{R}^2 - (\bigcup_i \tilde{f}(\gamma_i^\star) \cup \bigcup_i \tilde{f}(\gamma_i^\star))$ coincide with those of $\mathbb{R}^2 - (\bigcup_i f(\gamma_i^S) \cup \bigcup_i f(\gamma_i^S))$ except that one small disk has been created by the loop around the image of each effective cusp. As an example, Figure 2.3 shows $S_1^\star$ for the example in Figure 3.1. The Blank words (with cusps) as constructed for our theorem are therefore the Blank words for $S^\star$. Thus, the hypothesis of the appropriate theorem for the type of $S^\star$ hold and the desired immersion exists. If we take the immersions for all tiles and smooth out the resulting function we obtain the proper Whitney function F.

Conversely, if F exists, its restriction to $S^\star$ as above is an immersion and the same theorems show that the conditions in our theorem hold. ■

The next result gives formulae for the topological degree of F and for the number of pre-images of an arbitrary regular value. In the formulae, we use either information about the behaviour of the function near infinity (such as ρ_S or d_j) or finite information (such as $\daleth$ or κ).

For a closed oriented parametrized curve $\alpha : S^1 \to \mathbb{R}^2$ and a point p not on the image of α let $w(\alpha, p)$ be *winding number* of α around p (i.e., the topological degree of $\theta \mapsto \alpha(\theta) - p/|\alpha(\theta) - p|$). We extend this concept for parametrized curves of the form $\alpha : \mathbb{R} \to \mathbb{R}^2$, where α is a proper continuous function injective in the complement of a compact interval I. Consider a circle around the origin such that $\alpha(I)$ and p are in its interior. Taking the maximal arc of α which contains $\alpha(I)$ and is in the interior of the circle, we construct two oriented closed curves by using two complementary arcs in the circle. We define the winding number $w(\alpha, p)$ as the average of the two winding numbers of these auxiliary closed curves around p. Notice that $w(\alpha, p)$ independs on the choice of the circle and is always in $\mathbb{Z} + 1/2$. In Figure 2.2, we have $w(\alpha, p_1) = 1/2$, $w(\alpha, p_2) = -1/2$ and $w(\alpha, p_3) = -3/2$.

Let F be a nice function with bounded critical curves γ_i and unbounded critical curves β_j, always oriented by sense of folding. Consider a connected component T of $\mathbb{R}^2 - \bigcup \beta_j$. Let $\sigma(T)$ be the sign of $\det(DF)$ in the unbounded tile S contained in T, $\rho_T = \rho_S$ and d_T be the sum of the absolute values of the partial degrees of F on the ends of S. Let κ be the total number of cusps of F and $\daleth(C)$ be the sum of all $\daleth(\beta_j)$ and $\daleth(\gamma_i)$. For T as above, we define $\kappa(T)$ to be the number of cusps which are in the interior of T or are effective in the unbounded tile of T. Similarly, let $\daleth(T) = \sum_{\gamma_i \subseteq T} \daleth(\gamma_i)$; notice that only bounded critical curves are taken into account.

Theorem 3.2: *The topological degree of F is*

$$\begin{aligned}\deg(F) &= \sum_T \sigma(T)(\rho_T + d_T) \\ &= \sum_T \sigma(T)(\kappa(T) - 2\daleth(T) + 1)\end{aligned}$$

and the number of pre-images of an arbitrary regular value p is

$$\begin{aligned}\#F^{-1}(p) &= \sum_T (\rho_T + d_T) + 2\sum_j (w(F(\beta_j),p) - 1/2) + 2\sum_i w(F(\gamma_i),p) \\ &= 1 + \kappa - 2\tau(C) + 2\sum_j w(F(\beta_j),p) + 2\sum_i w(F(\gamma_i),p)\end{aligned}$$

where T ranges over all connected components of $\mathbb{R}^2 - \bigcup_{1\le j\le \ell}\beta_j$.

For instance, the degree of F given in Figures 3.1 and 3.2 equals 2 and the number of pre-images in each tile for $F(C)$ is indicated in Figure 3.2(b).

Proof of Theorem 3.2: We use the same notation for the boundary of a tile S as in the proof of Theorem 3.1.

Take a curve ζ in the image which is enveloping for every tile and consider its pre-image by F, a curve δ which is, up to orientation, an enveloping curve for every tile in the domain. For each unbounded tile S construct γ_∞^S as in Section 2. The turning of $F(\gamma_\infty^S)$, $\tau(\gamma_\infty^S)$, can be interpreted as the degree of a function ϕ^S from γ_∞^S to ζ. In order to construct ϕ^S, first smooth out the curves $F(\beta_j^S)$ near cusps without introducing new intersections and bend them near ζ so that they become tangent to ζ, with counterclockwise orientation in ζ corresponding to the sense of folding in $F(\beta_j^S)$. Clearly, $\tau(\gamma_\infty^S)$ coincides with the turning of this deformed curve. Now define $\phi^S(p) = F(p)$ for $p \in \delta$ and $\phi^S(p)$ as the direction of the normal vector to the auxiliary curve above near $F(p)$ for $p \in \beta_j^S$. By construction, the degree of ϕ^S is the turning of the deformed curve and so $\deg(\phi^S) = \tau(\gamma_\infty^S)$.

Clearly, the topological degree of F is the degree of $F|_\delta$ as a function from the oriented curve δ to the oriented curve ζ. Thus,

$$\deg(F) = \sum_{\text{unbounded } S} (\sigma(S)\deg(\phi^S)) = \sum_{\text{unbounded } S} (\sigma(S)\tau(\gamma_\infty^S)),$$

where each γ_∞^S is oriented according the sign of $\det(DF)$. The first formula for $\deg(F)$ then follows from Lemma 2.2 since the terms $\tau(\beta)$ cancel out.

Now, for an arbitrary unbounded tile S we have, from Lemma 2.2 and Theorem 3.1,

$$\tau(\gamma_\infty) = \chi(S) + \kappa(S) - \sum_{1\le i\le k^S} \tau(\gamma_i^S).$$

On the other hand, for a bounded tile S', again from Theorem 3.1,

$$0 = \chi(S') + \kappa(S') - \sum_{1\le i\le k^{S'}} \tau(\gamma_i^{S'}).$$

Adding all these equations over the tiles contained in a given T, we obtain

$$\tau(\gamma_\infty) = \chi(S) + \kappa(S) - \sum_{1\le i\le k^S} \tau(\gamma_i^S) + \sum_{S'\subseteq T}\left(\chi(S') + \kappa(S') - \sum_{1\le i\le k^{S'}} \tau(\gamma_i^{S'})\right).$$

Notice that the Euler characteristics add up to $\chi(T) = 1$. Also, for each bounded critical curve $\gamma \subseteq T$, $\daleth(\gamma)$ appears twice in the above expression. Finally, each cusp is counted exactly once since it is effective in one tile only. Thus, the above identity reduces to $\daleth(\gamma_\infty^S) = \kappa(T) - 2\daleth(T) + 1$, completing the proof of the degree formula.

Let p be a regular point. The number of pre-images of p in a bounded tile S is given by the sum of $w(F(\gamma_i), p)$ over all boundary components γ_i of S (each γ_i being oriented, as usual, by sense of folding), as the sign of $\det(DF)$ inside S is constant equal to σ_S. Similarly, the number of pre-images of p in an unbounded tile S is given by the sum of all $w(F(\gamma_i), p)$ (again over all bounded boundary components) with $w(F(\gamma_\infty^S), p)$. We have, however,

$$w(F(\gamma_\infty^S), p) = \rho_T + d_T + \sum_{1 \le j \le \ell^S} (w(F(\beta_j), p) - 1/2);$$

the proof of this identity is similar to that of Lemma 2.2 and is left to the reader. By adding all these terms, we get the first formula for the number of pre-images of a regular point.

For the other formula,

$$\#F^{-1}(p) = 1 + \sum_T (\rho_T + d_T - 1) + 2\sum_j w(F(\beta_j), p) + 2\sum_i w(F(\gamma_i), p)$$

and by Theorem 4.1,

$$\rho_S + d_S - 1 = \chi(S) - \daleth(S) + \kappa(S) - 1,$$

for the unbounded tile S contained in T and

$$0 = \chi(S') - \daleth(S') + \kappa(S')$$

for bounded tiles S' contained in T. Adding the identity for S and the identities for all S', we have

$$\rho_T + d_T - 1 = -\sum_j \daleth(\beta_j^S) - 2\sum_i \daleth(\gamma_i^S) + \kappa(T)$$

whence

$$\sum_T (\rho_T + d_T - 1) = -2\daleth(C) + \kappa,$$

yielding our theorem.

■

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