ALGEBRAIC NUMBER THEORY

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Scope

The aim is to provide a foundational course on algebraic number theory, starting with standard topics such as the factorization of prime ideals, the ideal class group, Dirichlet's unit theorem, the analytic class number formula, density theorems, and a sketch of Hecke's proof of the analytic continuation of the Dedekind zeta function. If time permits, then we will also given an overview of some advanced topics such as local fields and basic class field theory. This course should be of interest to wide range of students across pure mathematics.

Prerequisites. Linear algebra, some knowledge of groups, rings, and fields. Ideally (but not necessarily), some knowledge of Galois theory would be desirable.

Outline of topics

Each topic should take several lectures, with additional topics covered depending on time remaining.

(1). Euclidean rings. Preliminaries on integral domains, principal ideal domains, unique factorization domains, Euclidean domains, and Gauss' lemma in the style of [2, Ch. 2]. Introduce Gaussian integers and Eisenstein integers.

(2). Algebraic numbers. Basic definitions and examples, Liouville's theorem, algebraic number fields, theorem of the primitive element in the style of [2, Ch. 3].

(3). Integral bases. Introduce norm and trace, existence of integral bases, ideals in $\mathcal{O}_K$, Stickelberger's criterion, and many examples for quadratic fields in the style of [2, Ch. 4].

(4). Dedekind domains. Introduce integral closure (cf. [1, §I.2]), noetherian rings, fractional ideals and unique factorization, the Chinese Remainder Theorem, the ramification degree, Dedekind's theorem, characterization of the different, and factorization of primes in $\mathcal{O}_K$ in the style of [2, Chapter 5].

(5). The ideal class group. Introduce equivalence relations on ideals, and show that the corresponding class group is finite ([2, Ch. 6]). Discuss Minkwoski's method and exponents of the ideal class group if time permits.

(6). Galois theoretic perspective. Introduce the action of Galois, decomposition groups, and inertia groups in the style of [1, Ch. I, §3 − 5].

(7). Dirichlet's unit theorem. State and prove Dirichlet's unit theorem in the style of [2, Ch. 8]. Discuss implication. Describe how this can be interpreted in terms of group actions and ergodic theory in the case of totally real number fields.

(8). Cyclotomic fields. To crystalize what is done so far, illustrate with the example of cyclotomic fields following parts of [4, Chs. 2-3]. State and prove the conductor discriminant formula.

(9). L-series and density functions. Derive the analytic class number formula ([2, Ch. 11]), discuss the distribution of prime ideals, and present the Cebotarev Density Theorem. Introduce Artin L-functions and Brauer's induction theorem.

(10). Additional topics. After sketching Hecke's proof of the analytic continuation and functional equation of the Dedekind zeta function following [1, Ch. XIII], the following topics could be covered:

Overview of class field theory. Introduce class fields, and the classical statement of the Artin-Takagi theorem without proof for motivation. Deduce the Kronecker-Weber theorem. Introduce ray class fields, and formal properties of the Artin map.

Valuations. Introduce norms, Ostrowski's theorem, the product formula, and Hensel's lemma in the style of [3, Ch. II]. Carry on by introducing completions, local fields, unramified and tamely ramfied extensions, and Galois theory of valuations.

Abstract framework. Introduce infinite Galois theory, projective and inductive limits, abstract Galois theory, abstract valuation theory, the reciprocity map, the general reciprocity law, and the Herbrand quotient following [3, Ch. IV].

Local class field theory. State and prove the local reciprocity law and the existence theorem following [3, Ch. V]. Introduce the conductor, the maximal abelian extension (and proof of Kronecker-Weber), the norm residue symbol, the Hilbert symbol, formal groups, generalized cyclotomic theory, and higher ramification groups following [3, Ch. V].

Global class field theory. Introduce ideles, idele class groups, and behaviour of ideles in field extensions. Introduce the Herbrand quotient again, the global class field theory axiom – with the Hasse norm principle and Hasse-Minkowski, the global reciprocity law, and global class fields following [3, Ch. VI].

References

[1] S. Lang, *Algebraic Number Theory* (Second Edition), Grad. Texts in Math., Springer **110** (1994).
[2] M.R. Murty and J. Esmonde, *Problems in Algebraic Number Theory*, Grad. Texts in Math., Springer **190** (2004).
[3] J. Neukirch, *Algebraische Zahlentheorie*, Springer-Verlag Berlin (1992).
[4] L. Washington, *Cyclotomic Fields* (Second Edition), Grad. Tests in Math., Springer **83** (1982).