ADVANCED TOPICS IN ALGEBRAIC NUMBER THEORY

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Scope

Building on the first semester course of algebraic number theory (08.2023-12.2023), the course will develop the following topics. It is aimed primarily at graduate students wanting to access the literature or start research in this area, but will be taught in such a way that motivated undergraduate students can also participate. We propose to first describe the theory of local fields from scratch, moving through to class field theory, followed by an illustration of the theory of complex multiplication, and then a detailed account of Tate's thesis[1] "Fourier Analysis in Number Fields and Hecke's Zeta-Functions" (which gives a modern account of Hecke's proof of the analytic continuation and functional equations of "Hecke" L-functions). Each topic should take two or three lectures, with some degree of flexibility for later topics, and there are regular written assignments for assessment. The final exam is by graded oral presentation.

(1). Overview of class field theory. Review cyclotomic fields, introduce class fields, and introduce the classical statement of the Artin-Takagi theorem without proof for motivation. Deduce the Kronecker-Weber theorem. Introduce ray class fields, and formal properties of the Artin map.

(2). Valuations. Introduce norms, Ostrowski's theorem, the product formula, and Hensel's lemma in the style of [6, Ch. II]. Carry on by introducing completions, local fields, unramified and tamely ramfied extensions, and Galois theory of valuations.

(3). Abstract framework. Introduce infinite Galois theory, projective and inductive limits, abstract Galois theory, abstract valuation theory, the reciprocity map, the general reciprocity law, and the Herbrand quotient following [6, Ch. IV].

(4). Local class field theory. State and prove the local reciprocity law and the existence theorem following [6, Ch. V]. Introduce the conductor, the maximal abelian extension (and proof of Kronecker-Weber), the norm residue symbol, the Hilbert symbol, formal groups, generalized cyclotomic theory, and higher ramification groups following [6, Ch. V].

(5). Global class field theory. Introduce ideles, idele class groups, and behaviour of ideles in field extensions. Introduce the Herbrand quotient again, the global class field theory axiom – with the Hasse norm principle and Hasse-Minkowski, the global reciprocity law, and global class fields following [6, Ch. VI].

(6). Kroneckers Jugendtraum and complex multiplication. Go over the special case of imaginary quadratic fields following the beautiful article of Serre [8], filling in background on CM elliptic curves.

(7). Classical theory revisited. Go over the ideal theoretic formulation [6, Ch. VII, §7]. Introduce Hecke characters. Briefly, discuss Hecke's proof of the analytic continuation and functional equation of Hecke L-functions.

(8). Tate's thesis. Review Fourier analysis and the inversion theorem on abelian groups following [7, Chapter 1], taking for granted existence of the Haar measure. Go over local additive and multiplicative theory, restricted direct products, global additive theory (including Riemann-Roch) and multiplicative theory, and then the global functional equation following Tate [9] (or well-written accounts in [2, Ch. XIV] and [1, §3.1]).

[1] which was also discovered independently by Iwasawa

References

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