ANALYTIC NUMBER THEORY

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Abstract. This will be a motivated course in analytic number theory for senior undergraduate and graduate students. We will cover the fundamental theory following Murty [3], and develop a more Fourier analytic approach to the main theorems. Here, the canonical topics of Dirichlet's theorem on primes in arithmetic progressions, Riemann's proof of the analytic continuation of the Riemann-zeta function, and the Prime Number Theorem will be covered first. This will include a more sustained discussion of Gauss sums and Fourier analysis on abelian groups. We will then proceed to the classical theory of quadratic fields and class numbers following [2, Chapter 22], leading to the construction of the L-function of a character of the class group of an imaginary quadratic field. Finally, a different sort of proof of the analytic continuation of this L-function would be presented – rather than using theta series and Poisson summation in the style of Riemann, we will present a proof using Fourier series expansions of Eisenstein series. If time permits, we will explore advanced topics such as the Kronecker limit formula and Weil's converse theorem (for instance).

Prerequisites. Analysis, Algebra (covering group theory).

Topics. Each topic should take between two to four lectures to complete. There will be regular assignments, as well as a final examination.

(1). Arithmetic functions. Review arithmetic functions, introduce the Möbius function and the inversion formula, formal Dirichlet series, orders and average orders of arithmetic functions following [3, Chapter 1].

(2). Dirichlet characters and Gauss sums. Introduce Dirichlet characters and Gauss sums, together with calculations and exercises to give a taste of the Fourier analytic flavour, following [2, §3.1-3.4].

(3). Primes in arithmetic progressions. Introduce Dirichlet L-series and their basic analytic properties, together with related summation techniques, then prove Dirichlet's theorem on primes in arithmetic progressions following [3, Chapter 2].

(4). Functional equations. Introduce the Riemann zeta function; derive its analytic continuation via Poisson summation as in [3, Chapter 5]. Treat the more general case of Dirichlet L-functions after that.

(5). Prime Number Theorem. State and prove the Prime Number Theorem following [3, Chapter 3]: Introduce Chebyshev's theorem with related exercises, then the Ikehara-Wiener Tauberian theorem, and then the corresponding proof.

(6). Imaginary quadratic fields. Introduce imaginary quadratic fields from the classical perspective of binary quadratic forms following [2, Chapter 22]. Introduce the ideal class group of an imaginary quadratic field in this way following [2, §22.2], without assuming background knowledge of number fields. Introduce characters of the ideal class group, and their corresponding L-functions in the style of [2, § 22.3], deriving the analytic continuation via identification with values of some Eisenstein series. Here, the derivation of the Fourier series expansion of the Eisenstein should also be given, in the style of [1, Chapter 1, Theorem 1.6.1].

(7). Advanced topics. If time permits, we cover more on Eisenstein series and Hecke operators, leading to a proof of Weil's converse theorem following [1, Chapter I]. We also develop the approach of [2, Ch. 22] to prove Kronecker's limit formula. Other topics may also be covered.

References

[1] D. Bump, *Automorphic Forms and Representations*, Cambridge Stud. Adv. Math. 55, Cambridge University Press (1998).
[2] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, Amer. Math. Soc. Colloq. Publ. 53 (2004).
[3] M.R. Murty, *Problems in Analytic Number Theory*, Grad. Texts in Math., Springer **206** (2001).