Seminar (reading group)
"Semigroups for linear evolution equations"
DMAT PUC-Rio, 2024.2
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The central question for the seminar can be formulated as follows:
Find all maps T that satisfy the functional equation

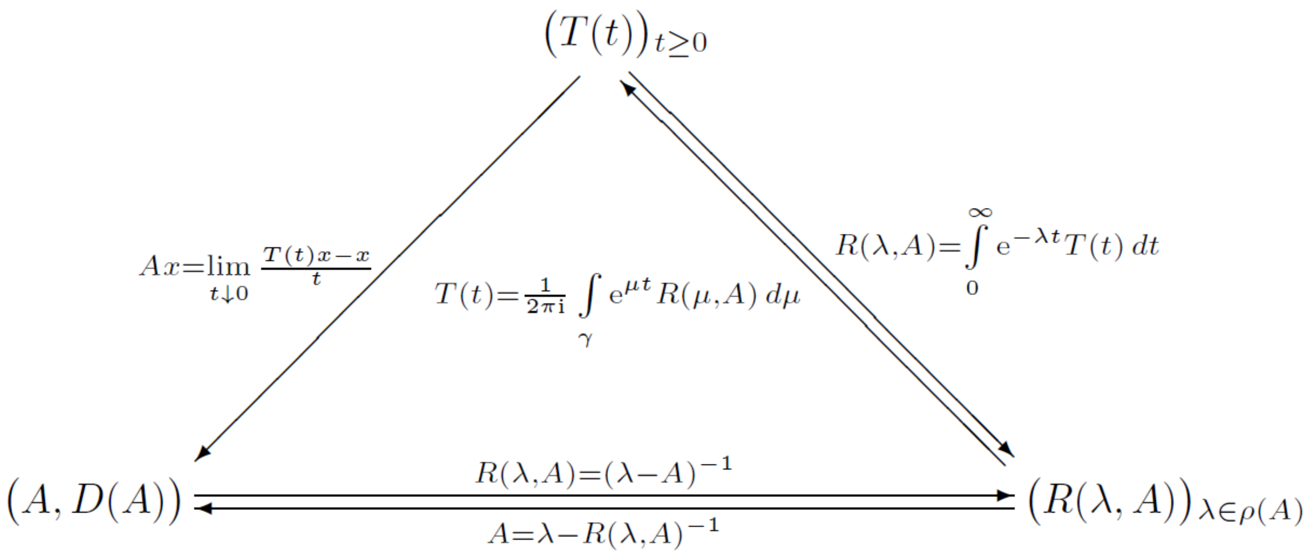
$$T(t+s) = T(t)T(s), \quad t, s \geq 0; \qquad T(0) = Id. \tag{1}$$

If we consider the continuous maps $T : \mathbb{R}_+ \to \mathbb{C}$, then the answer is clear $T(t) = e^{at}$ for some $a \in \mathbb{C}$.
If we consider finite-dimensional case, say matrix-valued function $T : \mathbb{R}_+ \to M_n(\mathbb{C})$, then again the answer is given by exponential $T(t) = e^{At}$ for some matrix A.

What happens in the infinite-dimensional case?
Let X be a Banach space and and $T : \mathbb{R}_+ \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is a Banach algebra of all bounded linear operators on X. We will see that one can represent $T(t) = e^{At}$ for some $A \in \mathcal{L}(X)$ (A is called an infinitesimal generator of a semigroup T). Moreover, for any element $x \in X$ the function $x(t) := T(t)x$ satisfies the abstract Cauchy problem (in other words, T provides us with the solution for the following evolution equation):

$$\frac{d}{dt}x(t) = Ax(t), \qquad x(0) = x.$$

One of the goals will be to understand the diagram below that describes the connection between semigroup $T(t), t \geq 0$, linear operator A and resolvent $R(\lambda, A)$. The spectral properties of the operator A play crucial role in determining the properties of the semigroup $T(t)$ (and as a consequnce of a solution of the evolution equation). Key theorems are: Hille-Yosida theorem, Lumers-Philips theorem.



We will follow the book [1]. One of possible ideas for your contribution for the seminar is as follows: look at Section 6 "Semigroups Everywhere" of [1], choose your favorite differential equation and explore how semigroup theory helps to prove existence, uniqueness and long-time behavior of solutions.

Another useful book is [2]. Some familiarity with functional analysis and spectral theory of operators is highly encouraged, but not necessary.

References

[1] K.-J. Egnel, R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer, 1999.

[2] A. Pazy, *Semigroups of linear operators and applications to partial differential equations (Vol. 44)*, Springer Science & Business Media, 2012.