QUADRATIC FORMS AND QUATERNION ALGEBRAS

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Scope. This course introduces the arithmetic of quadratic forms over number fields and their completions, culminating with the HasseMinkowski theorem. As a parallel and motivating thread, we develop the basic theory of quaternion algebras (ramification, Hilbert symbols, localglobal principles), emphasizing explicit calculations. The presentation follows the first half of Serre's *Cours d'Arithmétique* [1] and selected chapters of Vignéras *Arithmétique des algèbres de quaternions* [2].

Prerequisites. Algebraic structures; some exposure to elementary number theory (e.g. quadratic reciprocity), local fields (p-adic numbers), and Galois theory would be desirable but not strictly necessary.

Topics/outline. We propose the following tentative outline of topics.

- Quadratic forms over $\mathbf{Q}$ and $\mathbf{Z}$: diagonalization, isotropy, discriminant; equivalence, obstructions.
- Local fields and Hilbert symbols: brief p-adic primer; quadratic forms over $\mathbf{Q}_p$ and $\mathbf{R}$; the Hilbert symbol $(\cdot,\cdot)_v$ and its properties.
- Hasse-Minkowski theorem: statement and proof for quadratic forms over $\mathbf{Q}$; invariants (discriminant, Hasse invariant); localglobal principle and consequences.
- Genus, class, and representation problems: local conditions vs. global representation; examples and counterexamples; integral theory highlights from [1].
- Quaternion algebras: definition as central simple algebras; presentation $(a, b)_F$; reduced norm and trace; split versus division algebras (Wedderburn's theorem).
- Ramification and classification: local quaternion algebras, Hilbert symbol interpretation; finite set of ramified places; product formula and the discriminant of a quaternion algebra.
- Orders and arithmetic: maximal and Eichler orders (basic properties); units; norms; embeddings of quadratic fields.
- Connections between forms and quaternions: norm forms from quaternion algebras; isotropy and splitting; representing numbers by quadratic forms via localglobal criteria.
- Examples and computations: explicit Hilbert symbol calculations at p and ∞; deciding isotropy for ternary/quaternary forms; classifying quaternion algebras over $\mathbf{Q}$ by ramification; sample computations with orders.
- Additional topics: zeta functions, correspondence of Jacquet-Langlands.

Evaluation.

- Problem sets (every 2–3 weeks): 40%

- Mini-project (short exposition or computational notebook): 20%
- Final oral exam (with brief written report): 40%

References

[1] J.-P. Serre, *Cours d'arithmétique*, Presses Universitaires de France, Saint-Germain Paris (1970).
[2] M.-F. Vignéras, *Arithmétique des algèbres de quaternions*, Springer Lecture Notes in Math. **800** (1980).