

THE SECTIONAL CURVATURE OF THE INFINITE DIMENSIONAL MANIFOLD OF HÖLDER EQUILIBRIUM PROBABILITIES

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ABSTRACT. Here we consider the discrete time dynamics described by a transformation $T : M \rightarrow M$, where T is either the action of shift $T = \sigma$ on the symbolic space $M = \{1, 2, \dots, d\}^{\mathbb{N}}$, or, T describes the action of a d to 1 expanding transformation $T : S^1 \rightarrow S^1$ of class $C^{1+\alpha}$ (for example $x \rightarrow T(x) = dx \pmod{1}$), where $M = S^1$ is the unit circle. It is known that the infinite-dimensional manifold \mathcal{N} of Hölder equilibrium probabilities is an analytical manifold and carries a natural Riemannian metric. Given a certain normalized Hölder potential A denote by $\mu_A \in \mathcal{N}$ the associated equilibrium probability. The set of tangent vectors X (functions $X : M \rightarrow \mathbb{R}$) to the manifold \mathcal{N} at the point μ_A (a subspace of the Hilbert space $L^2(\mu_A)$) coincides with the kernel of the Ruelle operator for the normalized potential A . The Riemannian norm $|X| = |X|_A$ of the vector X , which is tangent to \mathcal{N} at the point μ_A , is described via the asymptotic variance, that is, satisfies

$$|X|^2 = \langle X, X \rangle = \lim_{n \rightarrow \infty} \frac{1}{n} \int (\sum_{i=0}^{n-1} X \circ T^i)^2 d\mu_A.$$

Consider an orthonormal basis X_i , $i \in \mathbb{N}$, for the tangent space at μ_A . Given two unit tangent vectors X and Y the curvature $K(X, Y)$ satisfies

$$K(X, Y) = \frac{1}{4} \left[\sum_{i=1}^{\infty} \left(\int X Y X_i d\mu_A \right)^2 - \sum_{i=1}^{\infty} \int X^2 X_i d\mu_A \int Y^2 X_i d\mu_A \right].$$

When the equilibrium probabilities μ_A are invariant Markov probabilities on $\{0, 1\}^{\mathbb{N}}$, introducing a special orthonormal basis \hat{a}_y , indexed by finite words y , we shall get explicit expressions for $K(\hat{a}_x, \hat{a}_z)$, which will be a finite sum. These values can be positive or negative depending on A and the words x and z . Words x and z with large length can produce extremely negative curvature $K(\hat{a}_x, \hat{a}_z)$. If x and z do not share a common prefix, then $K(\hat{a}_x, \hat{a}_z) = 0$.

1. INTRODUCTION

We denote by $T : M \rightarrow M$ a transformation acting on the metric space M , which is either the shift σ acting on $M = \{1, 2, \dots, d\}^{\mathbb{N}}$, or, T is the action of a d to 1 expanding transformation $T : S^1 \rightarrow S^1$, of class $C^{1+\alpha}$, where $M = S^1$ is the unit circle.

For a fixed $\alpha > 0$ we denote by Hol the set of α -Hölder functions on M .

For a Hölder potential $A : M \rightarrow \mathbb{R}$ we define the Ruelle operator (sometimes called transfer operator) - which acts on Hölder functions $f : M \rightarrow \mathbb{R}$ - by

$$(1) \quad f \rightarrow \mathcal{L}_A f(x) = \sum_{T(y)=x} e^{A(y)} f(y)$$

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It is known (see for instance [22] or [2]) that \mathcal{L}_A has a positive, simple leading eigenvalue λ_A with a positive Hölder eigenfunction h_A . Moreover, the dual operator acting on measures \mathcal{L}_A^* has a unique eigenprobability ν_A which is associated to the same eigenvalue λ_A .

Given a Hölder potential A we say that the probability μ_A - defined on the Borel sigma-algebra of M - is the equilibrium probability for A , if μ_A maximizes the values

$$h(\mu) + \int A d\mu,$$

among Borel T -invariant probabilities μ and where $h(\mu)$ is the Kolmogorov-Sinai entropy of μ .

The theory of thermodynamics formalism shows that the probability μ_A is unique and is given by the expression $\mu_A = h_A \nu_A$.

In some particular cases, the equilibrium probability (also called Gibbs probability) μ_A is the one observed on the thermodynamical equilibrium in the Statistical Mechanics of the one-dimensional lattice \mathbb{N} (under an interaction described by the potential A). As an example (where the spin in each site of the lattice \mathbb{N} could be $+$ or $-$) one can take $M = \{+, -\}^{\mathbb{N}}$, $A : M \rightarrow \mathbb{R}$ and T is the shift.

Taking into account the above definitions, we say that a Hölder potential A is normalized if $\mathcal{L}_A 1 = 1$. In this case $\lambda_A = 1$ and $\mu_A = \nu_A$.

Two potentials A, B in Hol will be called cohomologous to each other (up to a constant), if there exists a continuous function $g : M \rightarrow \mathbb{R}$ and a constant c , such that,

$$(2) \quad A = B + g - g \circ T - c.$$

Note that the equilibrium probability for A , respectively B , is the same if A and B are coboundaries to each other. In each coboundary class (an equivalence relation) there exists a unique normalized potential A (see [22]). Therefore, the set of equilibrium probabilities for Hölder potentials \mathcal{N} can be indexed by Hölder potentials A which are normalized. We will use this point of view here: $A \leftrightarrow \mu_A$.

The infinite-dimensional manifold \mathcal{N} of Hölder equilibrium probabilities μ_A is an analytic manifold (see [24], [11], [22], [8]) and it was shown in [13] that it carries a natural Riemannian structure. In order to provide a context for our main result, let us review first some of the main properties of this infinite-dimensional manifold and some definitions described on [13].

The set of tangent vectors X (a function $X : M \rightarrow \mathbb{R}$) to \mathcal{N} at the point μ_A coincides with the kernel of \mathcal{L}_A . The Riemannian norm $|X| = |X|_{\mu_A}$ of the vector X , which is tangent to \mathcal{N} at the point μ_A , is described (see Theorem D in [13]) via the asymptotic variance, that is, satisfies

$$(3) \quad |X| = \sqrt{\langle X, X \rangle} = \sqrt{\lim_{n \rightarrow \infty} \frac{1}{n} \int \left(\sum_{j=0}^{n-1} X \circ T^j \right)^2 d\mu_A}$$

The associated bilinear form on the tangent space at the point μ_A can be described (see Theorem D in [13]) by

$$(4) \quad \langle X, Y \rangle = \int X Y d\mu_A.$$

This bilinear form is positive semi-definite and in order to make it definite one can consider equivalence classes (cohomologous up to a constant) as described by Definition 5.4 in [13]. In this way, we finally get a Riemannian structure on \mathcal{N} (as anticipated in some paragraphs above). Elements X on the tangent space at μ_A have the property $\int X d\mu_A = 0$. The tangent space to \mathcal{N} at μ_A is denoted by $T_A\mathcal{N}$.

Given a normalized potential A let $\{X_i\}$ be an orthonormal basis of $T_A\mathcal{N}$, $i \in \mathbb{N}$. Our main result is :

Theorem 1.1. *Let A be a normalized potential, and let $\{X_i\}$ be an orthonormal basis of $T_A\mathcal{N}$. Let $X = X_1, Y = X_2$, then the sectional curvature $K(X, Y)$ is given by*

$$(5) \quad K(X, Y) = \frac{1}{4} \left[\sum_{i=1}^{\infty} \left(\int X Y X_i d\mu_A \right)^2 - \sum_{i=1}^{\infty} \int X^2 X_i d\mu_A \int Y^2 X_i d\mu_A \right].$$

The expression of $K(X, Y)$ applies of course to any pair of vectors in the basis $\{X_i\}$, we can always change the enumeration of the vectors in the basis without changing the basis. The work consists of two distinct parts: the first part, from Sections 2 to 5, has a more geometric nature and deals with the calculation of the Levi-Civita connection and the curvature tensor. This estimate becomes quite complex because we are dealing with an infinitely dimensional Riemannian manifold. Our goal was to express the sectional curvature for sections on the tangent space at μ_A in terms of integrals of functions with respect to μ_A . An important tool which will be used here is item (iv) on Theorem 5.1 in [13]: for all normalized $A \in \mathcal{N}$, $X \in T_A\mathcal{N}$ and φ a continuous function it holds:

$$(6) \quad \frac{d}{dt} \int \varphi d\mu_{A+tX} \Big|_{t=0} = \int \varphi X d\mu_A.$$

In Section 4.3 we describe the expression of sectional curvature $K(X, Y)$ in terms of the calculus of thermodynamics formalism.

The nature of the second part of the paper, from Sections 6 to 9, is more dynamic and analytical. We focus on the sectional curvatures at equilibrium measures μ_A which are stationary Markov probabilities on $\{0, 1\}^{\mathbb{N}}$. We get explicit results in this case since we are able to exhibit a special orthonormal basis for the tangent space $\{\hat{a}_y\}$, indexed by finite words y on the alphabet $\{0, 1\}$ (see expression (21)). This second part of the article is perhaps the more technical and subtle part; after some computations we will get the explicit expression for sectional curvature $K(\hat{a}_x, \hat{a}_z)$ (see Theorem 7.8 and Propositions 7.10 and 7.13). These values can be positive or negative depending on A and the words x and z . Words x and z with large length can produce extremely negative curvature $K(\hat{a}_x, \hat{a}_z)$ (see end of Example 7.20). If x and z do not share a common prefix (in particular one can not be a subprefix of the other), then $K(\hat{a}_x, \hat{a}_z) = 0$ (see Example 7.11).

We also show that if μ_A corresponds to the measure of maximal entropy on $\{0, 1\}^{\mathbb{N}}$, most of the sectional curvatures $K(\hat{a}_x, \hat{a}_z)$ are equal to $-1/2$ (see Proposition 7.17). Proposition 7.19 shows, in this case, an example where the sectional curvature is $1/2$.

Considering the two dimensional manifold M of the Markov invariant probabilities (the set of equilibrium probabilities for potentials depending on two coordinates) we show that for any points in M the sectional curvature for the pair of tangent vectors to M is always zero (see Theorem 7.15).

One remarkable fact appearing in the proof of Theorem 1.1 is that the expression (5) of the sectional curvature $K(\hat{a}_x, \hat{a}_z)$ is actually a finite sum (see Theorem 7.8 and Remark 7.12).

Given a finite word $y = (y_1, y_2, \dots, y_n)$ we denote by $[y] = [y_1, y_2, \dots, y_n]$ the associated cylinder set in $\{0, 1\}^{\mathbb{N}}$. The function \hat{a}_y is constant in each of the cylinder sets $[a, y_1, y_2, \dots, y_n, b]$, where $a, b = 0, 1$. The support of \hat{a}_y is the union of these cylinder sets. In this way if the word y has large length, then the support of \hat{a}_y is contained on very small sets.

We point out that section 8 in [13], which considers a simplified model for Markov measures on the symbolic space $\{0, 1\}^{\mathbb{N}}$, there was an indication that the curvature should be non-negative. The results in that section would correspond to consider in our setting words x and z with very small lengths. It follows from the results described on the present paper, as we said before, that in the general case the sectional curvature $K(\hat{a}_x, \hat{a}_z)$ can be negative for certain words x and z with large length.

In [21], [4] and [23] the authors consider a similar kind of Riemannian structure. The bilinear form considered in [21] is the one we consider here divided by the entropy of μ_A . As mentioned in section 8 in [13] in that case the curvature can be positive and also negative in some parts.

The main motivation for the results obtained on [21] (and also [4]) is related to the study of a particular norm on the Teichmüller space.

The results presented in [13] and here are related to the topic of Information Geometry (see [1] for general results on the subject) and this is described in Section 5 in [17]. We point out that in the setting of Thermodynamic Formalism the asymptotic variance is the Fisher information (see Definition 4.3 and Proposition 4.4 in [14]). Results about Kullback-Leibler divergence on Thermodynamic Formalism appeared recently in [18].

General references for analyticity (and inverse function theorems and implicit function theorems) in Banach spaces are [8] and [25].

A reference for general results in infinite-dimensional Riemannian manifolds is [3].

In section 6 in [13] it is explained that the Riemannian metric considered here is not compatible with the 2-Wasserstein Riemannian structure on the space of probabilities.

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2. PRELIMINARIES OF RIEMANNIAN GEOMETRY

Let us introduce some basic notions of Riemannian geometry. Given an infinite-dimensional C^∞ manifold (M, g) equipped with a smooth Riemannian metric g , let TM be the tangent bundle and T_1M be the set of unit norm tangent vectors of (M, g) , the unit tangent bundle. Let $\chi(M)$ be the set of C^∞ vector fields of M .

In [3] several results for Riemannian metrics on infinite-dimensional manifolds are presented. We will not use any of the results of that paper.

The only infinite-dimensional manifold we will be interested in here is \mathcal{N} which is the set of Hölder equilibrium probabilities (which was initially defined in [13]). Tangent vectors, differentiability, analyticity, etc, should be always considered in the sense of the setting described in sections 2.3 and 5.1 in [13] (see also [6] and [11]). We will elaborate on this later.

So in our case, $M = \mathcal{N}$, and g is the L^2 metric, $g_A(X, Y) = \int XY d\mu_A$,

For practical purposes, we shall call *Energy* the function $E(v) = g(v, v)$, $v \in T\mathcal{N}$, although in mechanics the energy is rather defined by $\frac{1}{2}g(v, v)$.

Given a smooth function $f : \mathcal{N} \rightarrow \mathbb{R}$, the derivative of f with respect to a vector field $X \in \chi(\mathcal{N})$ will be denoted by $X(f)$. The Lie bracket of two vector fields $X, Y \in \chi(\mathcal{N})$ is the vector field whose action on the set of functions $f : \mathcal{N} \rightarrow \mathbb{R}$ is given by $[X, Y](f) = X(Y(f)) - Y(X(f))$.

The *Levi-Civita connection* of (\mathcal{N}, g) , $\nabla : \chi(\mathcal{N}) \times \chi(\mathcal{N}) \rightarrow \chi(\mathcal{N})$, with notation $\nabla(X, Y) = \nabla_X Y$, is the affine operator characterized by the following properties:

- (1) Compatibility with the metric g :

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

for every triple of vector fields X, Y, Z .

- (2) Absence of torsion:

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

- (3) For every smooth scalar function f and vector fields $X, Y \in \chi(\mathcal{N})$ we have

- $\nabla_{fX} Y = f\nabla_X Y$,
- Leibniz rule: $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

The expression of $\nabla_X Y$ can be obtained explicitly from the expression of the Riemannian metric, in dual form. Namely, given two vector fields $X, Y \in \chi(\mathcal{N})$, and $Z \in \chi(\mathcal{N})$ we have

$$\begin{aligned} g(\nabla_X Y, Z) &= \frac{1}{2}(Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)) \\ &\quad - g([X, Z], Y) - g([Y, Z], X) - g([X, Y], Z), \end{aligned}$$

2.1. Curvature tensor and sectional curvatures. We follow [9] for the definitions in the subsection. To simplify the notation, from now on we shall adopt the convention $g(X, Y) = \langle X, Y \rangle$. The curvature tensor

$$\mathcal{R} : \chi(\mathcal{N}) \times \chi(\mathcal{N}) \times \chi(\mathcal{N}) \rightarrow \chi(\mathcal{N})$$

is defined in terms of the Levi-Civita connection as follows

$$(7) \quad \mathcal{R}(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X, Y]} Z.$$

The sectional curvature of the plane generated by two vector fields X, Y at the point $A \in \mathcal{N}$, which are orthonormal at A , is given by

$$(8) \quad K(X, Y) = \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X + \nabla_{[X, Y]} X, Y \rangle = \langle \mathcal{R}(X, Y)X, Y \rangle.$$

Let A be a normalized Hölder potential. Let us consider a local smooth surface $S(t, s)$, for $|t|, |s| \leq \epsilon$ small, tangent to the plane $\{A + tX + sY\}$ generated by X, Y at the point $A = S(0, 0)$. Let \bar{X}, \bar{Y} be the coordinate vector fields of the surface, and suppose that $\bar{X}_A = X, \bar{Y}_A = Y$. In Subsection 4.2 we shall exhibit such local surfaces.

Lemma 2.1. *The expression of the sectional curvature of the plane generated by the two orthonormal vectors X, Y is*

$$(9) \quad K(X, Y) = -\frac{1}{2}(\bar{X}(\bar{X}(\|\bar{Y}\|^2)) + \bar{Y}(\bar{Y}(\|\bar{X}\|^2))) + \|\nabla_{\bar{Y}}\bar{X}\|^2 + \bar{Y}(\bar{X}\langle\bar{X}, \bar{Y}\rangle) - \langle\nabla_{\bar{X}}\bar{X}, \nabla_{\bar{Y}}\bar{Y}\rangle.$$

Proof. The fact that \bar{X} and \bar{Y} commute implies that $\nabla_{\bar{X}}\bar{Y} = \nabla_{\bar{Y}}\bar{X}$ and

$$\langle\mathcal{R}(\bar{X}, \bar{Y})\bar{X}, \bar{Y}\rangle = \langle\nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{X} - \nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{X}, \bar{Y}\rangle.$$

The first term of $\langle\mathcal{R}(\bar{X}, \bar{Y})\bar{X}, \bar{Y}\rangle$ gives

$$\begin{aligned} \langle\nabla_{\bar{Y}}\nabla_{\bar{X}}\bar{X}, \bar{Y}\rangle &= \bar{Y}\langle\nabla_{\bar{X}}\bar{X}, \bar{Y}\rangle - \langle\nabla_{\bar{X}}\bar{X}, \nabla_{\bar{Y}}\bar{Y}\rangle \\ &= \bar{Y}(\bar{X}\langle\bar{X}, \bar{Y}\rangle - \langle\bar{X}, \nabla_{\bar{X}}\bar{Y}\rangle) - \langle\nabla_{\bar{X}}\bar{X}, \nabla_{\bar{Y}}\bar{Y}\rangle \\ &= \bar{Y}(\bar{X}\langle\bar{X}, \bar{Y}\rangle - \langle\bar{X}, \nabla_{\bar{Y}}\bar{X}\rangle) - \langle\nabla_{\bar{X}}\bar{X}, \nabla_{\bar{Y}}\bar{Y}\rangle \\ &= \bar{Y}(\bar{X}\langle\bar{X}, \bar{Y}\rangle - \frac{1}{2}\bar{Y}(\|\bar{X}\|^2)) - \langle\nabla_{\bar{X}}\bar{X}, \nabla_{\bar{Y}}\bar{Y}\rangle \\ &= -\frac{1}{2}\bar{Y}(\bar{Y}(\|\bar{X}\|^2)) + \bar{Y}(\bar{X}\langle\bar{X}, \bar{Y}\rangle) - \langle\nabla_{\bar{X}}\bar{X}, \nabla_{\bar{Y}}\bar{Y}\rangle \end{aligned}$$

The second term of the formula gives

$$\begin{aligned} \langle\nabla_{\bar{X}}\nabla_{\bar{Y}}\bar{X}, \bar{Y}\rangle &= \bar{X}\langle\nabla_{\bar{Y}}\bar{X}, \bar{Y}\rangle - \langle\nabla_{\bar{Y}}\bar{X}, \nabla_{\bar{X}}\bar{Y}\rangle \\ &= \bar{X}\langle\nabla_{\bar{X}}\bar{Y}, \bar{Y}\rangle - \langle\nabla_{\bar{Y}}\bar{X}, \nabla_{\bar{Y}}\bar{X}\rangle \\ &= \frac{1}{2}\bar{X}(\bar{X}(\|\bar{Y}\|^2)) - \|\nabla_{\bar{Y}}\bar{X}\|^2 \end{aligned}$$

Subtracting the second term from the first one we obtain the lemma. \square

3. THE ANALYTIC STRUCTURE OF THE SET OF NORMALIZED POTENTIALS

Definition 3.1. Let $(X, |\cdot|)$ and $(Y, |\cdot|)$ Banach spaces and V an open subset of X . Given $k \in \mathbb{N}$, a function $F : V \rightarrow Y$ is called k -differentiable in x , if for each $j = 1, \dots, k$, there exists a j -linear bounded transformation

$$D^j F(x) : \underbrace{X \times X \times \dots \times X}_j \rightarrow Y,$$

such that,

$$D^{j-1}F(x+v_j)(v_1, \dots, v_{j-1}) - D^{j-1}F(x)(v_1, \dots, v_{j-1}) = D^j F(x)(v_1, \dots, v_j) + o_j(v_j),$$

where

$$o_j : X \rightarrow Y, \quad \text{satisfies, } \lim_{v \rightarrow 0} \frac{|o_j(v)|_Y}{|v|_X} = 0$$

By definition F has derivatives of all orders in V , if for any $x \in V$ and any $k \in \mathbb{N}$, the function F is k -differentiable in x .

Definition 3.2. Let X, Y be Banach spaces and V an open subset of X . A function $F : V \rightarrow X$ is called analytic on V , when F has derivatives of all orders in V , and for each $x \in V$ there exists an open neighborhood V_x of x in V , such that, for all $v \in V_x$, we have that

$$F(x+v) - F(x) = \sum_{j=1}^{\infty} \frac{1}{j!} D^j F(x)v^j,$$

where $D^j F(x)v^j = D^j F(x)(v, \dots, v)$ and $D_j F(x)$ is the j -th derivative of F in x .

Above we use the notation of section 3.2 in [11].

\mathcal{N} can be expressed locally in coordinates via analytic charts (see [13]).

3.1. Some more estimates from Thermodynamic Formalism. Given a potential $B \in \text{Hol}$ we consider the associated Ruelle operator \mathcal{L}_B and the corresponding main eigenvalue λ_B and eigenfunction h_B .

The function

$$(10) \quad \Pi(B) = B + \log(h_B) - \log(h_B(T)) - \log(\lambda_B)$$

describes the projection of the space of potentials B on Hol onto the analytic manifold of normalized potentials \mathcal{N} .

We identify below $T_A\mathcal{N}$ with the affine subspace $\{A + X : X \in T_A\mathcal{N}\}$.

The function Π is analytic (see [13]) and therefore has first and second derivatives. Given the potential B , then the map $D_B\Pi : T_B\mathcal{N} \rightarrow T_{\Pi(B)}\mathcal{N}$ given by

$$D_B\Pi(X) = \frac{\partial}{\partial t}(\Pi(B + tX))_{t=0}$$

should be considered as a linear map from Hol to itself (with the Hölder norm on Hol). Moreover, the second derivative $D_B^2\Pi$ should be interpreted as a bilinear form from $\text{Hol} \times \text{Hol}$ to Hol , and is given by

$$D_B^2\Pi(X, Y) = \frac{\partial^2}{\partial t \partial s}(\Pi(B + tX + sY))_{t=s=0}.$$

We denote by $\|A\|_\alpha$ the α -Hölder norm of an α -Hölder function A .

When B is normalized the eigenvalue is 1 and the eigenfunction is equal to 1. We would like to study the geometry of the projection Π restricted to the tangent space $T_A\mathcal{N}$ into the manifold \mathcal{N} (namely, to get bounds for its first and second derivatives with respect to the potential viewed as a variable) for a given normalized potential A .

The space $T_A\mathcal{N}$ is a linear subspace of functions and the derivative map $D\Pi$ is analytic when restricted to it.

We denote by $E_0 = E_0^A$ the set of Hölder functions g , such that, $\int g d\mu_A = 0$, where μ_A is the equilibrium probability for the normalized potential A . Note that E_0^A is contained in $T_A(\mathcal{N})$.

The claims of the next Lemma are based mainly on results of [13] (see also [11], [6]). In Proposition 10.1 in the Appendix of our article, we outline the proof of Item (4).

Lemma 3.3. *Let $\Lambda : \text{Hol} \rightarrow \mathbb{R}$, $H : \text{Hol} \rightarrow \text{Hol}$ be given, respectively, by $\Lambda(B) = \lambda_B$, $H(B) = h_B$. Then we have*

- (1) *The maps Λ , H , and $A \rightarrow \mu_A$ are analytic.*
- (2) *For a normalized B we get that $D_B \log(\Lambda)(\psi) = \int \psi d\mu_B$,*
- (3) *$D_B^2 \log(\Lambda)(\eta, \psi) = \int \eta \psi d\mu_B$, where ψ, η are at $T_B\mathcal{N}$.*
- (4) *For any Hölder potential A we have*

$$D_A H(X) = h_A \int ([(I - \mathcal{L}_{T,A}|_{E_0^A})^{-1} (1 - h_A)] \cdot X) d\mu_A.$$

If A is normalized, we have $D_A H = 0$,

- (5) *If A is a normalized potential, then for every function $X \in T_A\mathcal{N}$ we have*
 - $\int X d\mu_A = 0$.

- $D_A \Pi(X) = X$.

In order to simplify the notation, from now on, unless is necessary for the understanding, we will denote $(I - \mathcal{L}_{T,A}|_{E_0^A})^{-1}$ by $(I - \mathcal{L}_{T,A})^{-1}$.

Remark 2: The proof of item (4) appears in an old version in arXiv in the year 2012 - see Proposition 4.6 in arXiv:1205.5361v1 - of the published paper [6]. The authors of [6] authorized us to present it in the Appendix. This claim and its proof did not appear in the final published version of [6].

Items (2) and (3) are taken from Theorem D in [13]. Item $\int X d\mu_A = 0$ in (5) follows from Theorem A and Corollary B in [13] the other item in (5) is trivial.

The analyticity of Λ and H of the item (1) are well-known facts (see chapter 4 in [22] or Corollary B in [13]) which was also proved in [6].

The law that takes a Hölder potential B to its normalization A is differentiable according to section 2.2 in [13].

Note that the derivative linear operator $X \rightarrow D_A H(X)$ is zero when A is normalized.

Remark 3: Note that item (2) implies by item (5) that $D_B \log(\Lambda)(\psi) = \int \psi d\mu_B = 0$, when B is normalized and $\psi \in T_{\mu_B}(\mathcal{N})$.

Remark 4: Item (1) above means that for a fixed Hölder function f the map $A \rightarrow \int f d\mu_A$ is differentiable on A (see theorem B in [6])

Questions related to second derivatives on Thermodynamic Formalism are considered in [19] and [23].

4. EVALUATING THE SECTIONAL CURVATURES OF THE RIEMANNIAN METRIC

The goal of the section is to calculate the sectional curvature $K(X, Y)$ of the plane generated by two orthogonal vector fields tangent to $A \in \mathcal{N}$ applying the calculus of Thermodynamics formalism. We start with a technical result that is a consequence of formula 6. This lemma will be extensively used in the article.

4.1. Leibniz rule of differentiation.

Lemma 4.1. *Let $A \in \mathcal{N}$ and let $\gamma : (-\epsilon, \epsilon) \rightarrow \mathcal{N}$ be a smooth curve such that $\gamma(0) = A$. Let $X(t) = \gamma'(t)$, and let Y be a smooth vector field tangent to \mathcal{N} defined in an open neighborhood of A . Denote by $Y(t) = Y(\gamma(t))$. Then the derivative of $\int Y(t) d\mu_{\gamma(t)}$ with respect to the parameter t is*

$$\frac{d}{dt} \int Y(t) d\mu_{\gamma(t)} = \int \frac{dY(t)}{dt} d\mu_{\gamma(t)} + \int Y(t) X(t) d\mu_{\gamma(t)}$$

for every $t \in (-\epsilon, \epsilon)$.

Proof. The idea of the proof is very simple and based on the fact that the function $Q : \chi(\mathcal{N}) \times m_T \rightarrow \mathbb{R}$ given by

$$Q(X, \mu) = \int X d\mu$$

is a bilinear form, where $\chi(\mathcal{N})$ is the set of C^1 vector fields tangent to \mathcal{N} and m_T is the set of invariant measures of the map T . So the derivative of a function of the type $Q(X(t), \mu(t))$ satisfies a sort of Leibniz rule. Let us check.

Let us calculate the derivative at $t = 0$, for every other $t \in (-\epsilon, \epsilon)$ the calculation is analogous. We have

$$\begin{aligned}
 \frac{d}{dt} \int Y(t) d\mu_{\gamma(t)} \Big|_{t=0} &= \lim_{t \rightarrow 0} \frac{1}{t} \left(\int Y(t) d\mu_{\gamma(t)} - \int Y(0) d\mu_A \right) \\
 &= \int \lim_{t \rightarrow 0} \frac{1}{t} (Y(t) - Y(0)) d\mu_{\gamma(t)} \\
 &+ \lim_{t \rightarrow 0} \frac{1}{t} \left(\int Y(0) d\mu_{\gamma(t)} - \int Y(0) d\mu_A \right) \\
 &= \int \frac{dY(t)}{dt} d\mu_A + \lim_{t \rightarrow 0} \frac{1}{t} \left(\int Y(0) d\mu_{A+tX(0)} - \int Y(0) d\mu_A \right)
 \end{aligned}$$

where in the last step we use the fact that the derivative with respect to t only depends on the vector $X(0)$ and not on the curve through A tangent to $X(0)$. By equation (6) the second term in the above equality is just $\frac{d}{dt} \int Y(0) d\mu_{A+tX(0)} \Big|_{t=0}$, which equals $\int X(0)Y(0) d\mu_A$. This finishes the proof of the lemma. \square

From now on, we shall adopt the notations $\frac{\partial Y}{\partial t} = Y' = Y_t$, the second one applies when there is only one parameter involved in the calculations, the third one will be used otherwise.

4.2. Auxiliary local surfaces in \mathcal{N} . Next, given a normalized potential A and X, Y orthonormal vector fields in the tangent space of A , we proceed to construct a local surface $S(t, s)$, $|t|, |s| < \epsilon$ small, such that $S(0, 0) = A$, and the tangent space of $S(t, s)$ at A is the plane generated by X, Y . Let us consider the plane

$$P(t, s) = A + tX + sY$$

where $t, s, \in \mathbb{R}$, that is a subset of $T_A\mathcal{N}$, and let Π be the projection into \mathcal{N} defined in equation 10. The vector fields $X_{P(t,s)} = \frac{\partial}{\partial t} P(t, s) = X$, $Y_{P(t,s)} = \frac{\partial}{\partial s} P(t, s) = Y$ are tangent to the plane P of course.

Let $S(t, s) = \Pi(P(t, s))$. By Lemma 3.3 item (5), the restriction of the map Π to the plane $P(t, s)$ is a local diffeomorphism onto its image, so there exists $\epsilon > 0$ small such that $S(t, s)$ is an analytic embedding of the rectangle $\{|t| < \epsilon\} \times \{|s| < \epsilon\}$.

The coordinate vector fields of $S(t, s)$ are $\bar{X}_{S(t,s)} = \frac{\partial}{\partial t} (\Pi(P(t, s))) = D_{P(t,s)} \Pi(X)$, $\bar{Y}_{S(t,s)} = \frac{\partial}{\partial s} (\Pi(P(t, s))) = D_{P(t,s)} \Pi(Y)$, so \bar{X}, \bar{Y} are extensions of X, Y .

Moreover, we have the following

Lemma 4.2. *The derivatives with respect to t, s of the coordinate vector fields \bar{X}, \bar{Y} at the point A (a normalized potential) are*

- (1) $\frac{\partial}{\partial t} \bar{X} = \frac{\partial}{\partial s} \bar{Y} = -1$
- (2) $\frac{\partial}{\partial s} \bar{X} = \frac{\partial}{\partial t} \bar{Y} = 0$.

Proof. This claim will be a consequence of Lemma 3.3. Indeed, since the local surface $S(t, s)$ is contained in the manifold of normalized potentials, the eigenvalues $\lambda_{S(t,s)}$ of the Ruelle operator associated to the functions $S(t, s)$, and the eigenfunctions $h_{S(t,s)}$ are equal to 1 for every t, s . Thus, the differentials of the functions $\Lambda(B) = \lambda_B$, $\log(\Lambda)(B)$, $H(B) = h_B$ and $\log(H)(B)$ at the point $B = A$ are equal to zero.

By definition, we have

$$(11) \quad \frac{\partial}{\partial t}(\bar{X}_{S(t,0)})_{t=0} = \frac{\partial}{\partial t}(D_{P(t,0)}\Pi(X_{P(t,0)}))_{t=0}$$

Let I be the identity map. The expression of the projection Π (equation (10)) is

$$\Pi(B) = I(B) + \log(h_B) - \log(h_B(T)) - \log(\lambda_B).$$

Lemma 3.3 grants that all the functions involved in the expression of Π are differentiable, so we get at the point $t = 0$,

$$(12) \quad \begin{aligned} \frac{\partial}{\partial t}(D_{P(t,0)}\Pi(X_{P(t,0)}))_{t=0} &= \frac{\partial}{\partial t}(D_{P(t,0)}I(X_{P(t,0)}))_{t=0} \\ &+ \frac{\partial}{\partial t}((D_{P(t,0)}\log(H))(X_{P(t,0)}))_{t=0} \\ &- \frac{\partial}{\partial t}((D_{P(t,0)}\log(H \circ T))(X_{P(t,0)}))_{t=0} \\ &- \frac{\partial}{\partial t}((D_{P(t,0)}\log(\Lambda))(X_{P(t,0)}))_{t=0} \end{aligned}$$

The first term gives at $t = 0$,

$$\frac{\partial}{\partial t}(D_{P(t,0)}I(X_{P(t,0)}))_{t=0} = \frac{\partial}{\partial t}(X)_{t=0} = 0$$

since X does not depend on t .

By Lemma 3.3 item (3), the fourth term of this equality gives at $t = 0$,

$$-\frac{\partial^2}{\partial t^2}\log(\Lambda)_{A+tX} = -\int X^2 d\mu_A = -1$$

To evaluate the second and third terms let us first show that

Claim 1: $\frac{\partial}{\partial t}(D_{P(t,0)}H(X_{P(t,0)})) = 0$ at $t = 0$.

We shall prove Claim 1 in two steps. Let $\beta(t) = \Pi(P(t,0))$, that is an analytic curve in the manifold \mathcal{N} with $\beta(0) = A$, $\beta'(0) = X$, the same values at $t = 0$ for the curve $P(t,0) = A + tX$ and its first derivative.

Claim: 2 $\frac{\partial}{\partial t}(D_{P(t,0)}H(X_{P(t,0)})) = \frac{\partial}{\partial t}(D_{\beta(t)}H(\beta'(t)))$ at $t = 0$.

Indeed, the function $F(B) = D_B(H)$ is analytic as a map from Hol to Hol . Then we have,

$$\begin{aligned} \|F(P(t,0)) - F(\beta(t))\| &= \|D_{P(t,0)}H(X) - D_{\beta(t)}H(\beta(t))\| \\ &\leq \|D_{P(t,0)}H(X) - D_{\beta(t)}H(X)\| \\ &+ \|D_{\beta(t)}H(X) - D_{\beta(t)}H(\beta(t))\| \\ &\leq \|D_{P(t,0)}H - D_{\beta(t)}H\| \|X\| \\ &+ \|D_{\beta(t)}H\| \|X - \beta'(t)\| \end{aligned}$$

Since $P(t,0) - \beta(t) = O(t^2)$ because these curves are tangent at $t = 0$, Taylor expansion yields that the term $\|D_{P(t,0)}H - D_{\beta(t)}H\| \|X\|$ is of order $O(t^2)$.

Moreover, the term $\|X - \beta'(t)\|$ is of order $O(t)$ since the curve $\beta(t)$ is tangent to X at $t = 0$, and the term $\|D_{\beta(t)}H\|$ vanishes because $\beta(t)$ is a curve of normalized potentials (item (4) of Lemma 3.3). Hence, the whole above expression is of order $O(t^2)$ and Claim 2 holds.

So let us proceed to evaluate $\frac{\partial}{\partial t}(D_{\beta(t)}H(\beta'(t)))$ at $t = 0$. Recall that $\beta(t)$ is a curve of normalized potentials, so $h_{\beta(t)} = 1$ for every t . Therefore,

$$D_{\beta(t)}H(\beta'(t)) = \frac{\partial}{\partial t}(H(\beta(t))) = \frac{\partial}{\partial t}1 = 0,$$

This yields the proof of Claim 1.

To finish the proof of the first part of item (1) in the Lemma, it remains to show that $\frac{\partial}{\partial t}((D_{P(t,0)}\log(H))(X_{P(t,0)}))_{t=0} = 0$. This result can be deduced by either applying the chain rule to $\log(H)$ or following the same steps of the proof of Claim 1, carried out replacing H by $\log(H)$.

Replacing t by s and X by Y , and $P(t, 0) = A + tX$ by $P(0, s) = A + sY$ in the above argument we get

$$\frac{\partial}{\partial s}\bar{Y} = - \int Y^2 d\mu_A = -1.$$

Since X, Y are unit vector fields, we get item (1) in Lemma 4.2.

Item (2) follows the same type of reasoning. By definition we have,

$$\frac{\partial}{\partial s}(\bar{X}_{S(0,s)})_{s=0} = \frac{\partial}{\partial s}(D_{P(t,s)}\Pi(X_{P(t,s)}))_{t=0,s=0}.$$

This expression, according to equation (13) is

$$\begin{aligned} \frac{\partial}{\partial s}(D_{P(t,s)}\Pi(X_{P(t,s)}))_{t=s=0} &= \frac{\partial}{\partial s}(D_{P(t,s)}I(X_{P(t,s)}))_{t=s=0} \\ &+ \frac{\partial}{\partial s}((D_{P(t,s)}\log(H))(X_{P(t,s)}))_{t=s=0} \\ &- \frac{\partial}{\partial s}((D_{P(t,s)}\log(H \circ T))(X_{P(t,s)}))_{t=s=0} \\ &- \frac{\partial}{\partial s}((D_{P(t,s)}\log(\Lambda))(X_{P(t,s)}))_{t=s=0} \end{aligned}$$

The first term gives at $t = 0$,

$$\frac{\partial}{\partial s}(D_{P(t,0)}I(X_{P(t,0)}))_{t=0} = \frac{\partial}{\partial s}(X)_{t=0} = 0$$

since X does not depend on t, s . The fourth term is, by Lemma 3.3 item (3),

$$- \frac{\partial}{\partial s}((D_{P(t,s)}\log(\Lambda))(X_{P(t,s)}))_{t=s=0} = -D_A^2 \log(\Lambda)(Y, X) = - \int XY d\mu_A = 0$$

To evaluate the second and third terms we proceed in the same way as in the proof of item (1), we shall show that they vanish (this follows from the chain rule). Similar to Claims 1 and 2 in the proof of item (1) we have

Claim 3 $\frac{\partial}{\partial s}(D_{P(t,s)}H(X_{P(t,s)})) = 0$ at $t = s = 0$.

We shall prove Claim 3 in two steps. Let $S_{s_0}(t) = S(t, s_0) = \Pi(P(t, s_0))$, that is an analytic curve, for each s_0 , in the manifold \mathcal{N} .

Claim: 4 $\frac{\partial}{\partial s}(D_{P(t,s)}H(X_{P(t,s)})) = \frac{\partial}{\partial s}(D_{S_{s(t)}}H(\bar{X}_{S_{s(t)}}))$ at $t = s = 0$, recalling that $\bar{X} = D\Pi(X)$.

The proof of Claim 4 is an extension of the proof of Claim 2. We shall show that the difference $D_{P(t,s)}H(X_{P(t,s)}) - D_{S_s(t)}H(\bar{X}_{S_s(t)})$ is of the order of a quadratic term in t, s in an open neighborhood of $(0, 0)$.

Let us start by observing that Taylor approximation of order 2 of the function $S(t, s)$ in terms of t, s is

$$\begin{aligned} S(t, s) &= \Pi(P(t, s)) = A + (D_A\Pi) \circ (D_{(0,0)}P)(t, s) + O(2) \\ &= A + tX + sY + O(2) = P(t, s) + O(2) \end{aligned}$$

since $D_A\Pi = I$, where $O(2)$ is a term of order 2 in t, s . In particular, notice that $D_{(0,0)}S = D_{(0,0)}P$.

To estimate the difference $D_{P(t,s)}H(X_{P(t,s)}) - D_{S_s(t)}H(\bar{X}_{S_s(t)})$ in an open neighborhood of $(0, 0)$, let us first extend the vector field X as a constant vector field in an open neighborhood B of A in the Banach space of Hölder functions. We shall denote by X this extension, to simplify notation. We have

$$\begin{aligned} \| D_{P(t,s)}H(X_{P(t,s)}) - D_{S_s(t)}H(\bar{X}_{S_s(t)}) \| &\leq \| D_{P(t,s)}H(X_{P(t,s)}) - D_{S_s(t)}H(X_{P(t,s)}) \| \\ &\quad + \| D_{S_s(t)}H(X_{P(t,s)}) - D_{S_s(t)}H(\bar{X}_{S_s(t)}) \| \\ &\leq \| D_{P(t,s)}H - D_{S_s(t)}H \| \| X_{P(t,s)} \| \\ &\quad + \| D_{S_s(t)}H \| \| X_{P(t,s)} - \bar{X}_{S_s(t)} \| . \end{aligned}$$

Since $S_s(t)$ is a curve of normalized potentials we have that $D_{S_s(t)}H = 0$, moreover, since $B \rightarrow D_B H$ is an analytic function of B , there exists a constant $L > 0$ such that for every t, s sufficiently small we have

$$\| D_{P(t,s)}H - D_{S_s(t)}H \| \leq L \| P(t, s) - S_s(t) \| = LO(2)$$

which implies Claim 4. Claim 3 follows from the fact that

$$\| D_{P(t,s)}H(X_{P(t,s)}) \| = \| D_{P(t,s)}H(X_{P(t,s)}) - D_{S_s(t)}H(\bar{X}_{S_s(t)}) \| .$$

As in the proof of item (1), to obtain the values of the derivative of $D \log(H)$ we either apply the chain rule and Claim 3 or we follow step by step the proof of Claims 3 and 4 replacing H by $\log(H)$.

The Claims also hold for the functions $H \circ T$ and $\log(H) \circ T$. Indeed, since $H(A) = h_A$, $(H \circ T)(A) = h_{A \circ T}$, and if A is a normalized potential, the function $A \circ T$ is a normalized potential as well. So the calculations in the proof of the Claims for normalized potentials apply to $A \circ T$. This yields item (2). \square

4.3. The expression of $K(X, Y)$ in terms of the calculus of thermodynamics formalism. Let us first state some notations. Let \bar{X}_t be the derivative of the vector field \bar{X} with respect to the parameter t and \bar{X}_s be the derivative of the vector field \bar{X} with respect to the parameter s . The same convention applies to \bar{Y}_t, \bar{Y}_s . The notations $\bar{X}(Y) = \frac{\partial}{\partial t} \bar{Y} = \bar{Y}_t$ will always represent derivatives with respect to the vector field \bar{X} , while $\bar{X}\bar{Y}$ or $\bar{X} \times \bar{Y}$ will represent the product of the functions \bar{X} and \bar{Y} . Through the section this double character of the vectors tangent to the manifold \mathcal{N} which are also functions will show up in all statements and proofs.

Theorem 4.3. *Let A be a normalized potential, let $X, Y \in T_A \mathcal{N}$ be a pair of orthonormal vector fields, and let $S : (-\epsilon, \epsilon) \times (-\delta, \delta) \rightarrow \mathcal{N}$ be the local surface defined in the previous subsection with $S(0, 0) = A$, \bar{X} , whose coordinate vector*

fields are \bar{X}, \bar{Y} , with $\bar{X}(A) = X, \bar{Y}(A) = Y$. Then the sectional curvature $K(X, Y)$ at A of the plane generated by X, Y is given by the expression

$$K(X, Y) = \|\nabla_{\bar{Y}} \bar{X}\|^2 - \langle \nabla_{\bar{X}} \bar{X}, \nabla_{\bar{Y}} \bar{Y} \rangle$$

We shall subdivide the proof into several steps.

Lemma 4.4. *We have that $\bar{X}_s = \bar{Y}_t$ in the local surface S .*

This is a straightforward consequence of the fact that the vector fields \bar{X}, \bar{Y} commute.

Next, let us evaluate the terms of the sectional curvature in Lemma 2.1,

$$\begin{aligned} K(X, Y) &= -\frac{1}{2}(\bar{X}(\bar{X}(\|\bar{Y}\|^2)) + \bar{Y}(\bar{Y}(\|\bar{X}\|^2))) + \|\nabla_{\bar{Y}} \bar{X}\|^2 \\ &\quad + \bar{Y}(\bar{X}\langle \bar{X}, \bar{Y} \rangle) - \langle \nabla_{\bar{X}} \bar{X}, \nabla_{\bar{Y}} \bar{Y} \rangle. \end{aligned}$$

Lemma 4.5. *At every point $p \in S(t, s)$ we have*

- (1) $\bar{X}(\bar{X}(\|\bar{Y}\|^2)) = 2 \int \bar{Y} \bar{Y}_{tt} d\mu_p - \int \bar{Y}^2 d\mu_p + \int \bar{X}^2 \bar{Y}^2 d\mu_p.$
- (2) $\bar{Y}(\bar{Y}(\|\bar{X}\|^2)) = 2 \int \bar{X} \bar{X}_{ss} d\mu_p - \int \bar{X}^2 d\mu_p + \int \bar{X}^2 \bar{Y}^2 d\mu_p.$

In particular, if $p = A$ we have

- (1) $\bar{X}(\bar{X}(\|\bar{Y}\|^2)) = 2 \int \bar{Y} \bar{Y}_{tt} d\mu_A - 1 + \int \bar{X}^2 \bar{Y}^2 d\mu_A.$
- (2) $\bar{Y}(\bar{Y}(\|\bar{X}\|^2)) = 2 \int \bar{X} \bar{X}_{ss} d\mu_A - 1 + \int \bar{X}^2 \bar{Y}^2 d\mu_A.$

Proof. The expression follows from the application of the Leibniz rule to differentiate $\|\bar{Y}\|^2 = \int \bar{Y}^2 d\mu_p$ (we shall omit for convenience the p in the notation of the measure $d\mu_p$):

$$\begin{aligned} \bar{X}(\bar{X} \int \bar{Y}^2 d\mu) &= \bar{X}(2 \int \bar{Y} \bar{Y}_t d\mu + \int \bar{X} \bar{Y}^2 d\mu) \\ &= 2 \int (\bar{Y}_t)^2 d\mu + 2 \int \bar{Y} \bar{Y}_{tt} d\mu + 2 \int \bar{Y} \bar{X} \bar{Y}_t d\mu \\ &\quad + \int \bar{X}_t \bar{Y}^2 d\mu + 2 \int \bar{X} \bar{Y} \bar{Y}_t d\mu + \int \bar{X}^2 \bar{Y}^2 d\mu \\ &= 2 \int (\bar{Y}_t)^2 d\mu + 2 \int \bar{Y} \bar{Y}_{tt} d\mu + 4 \int \bar{X} \bar{Y} \bar{Y}_t d\mu \\ &\quad + \int \bar{X}_t \bar{Y}^2 d\mu + \int \bar{X}^2 \bar{Y}^2 d\mu. \end{aligned}$$

Since by Lemma 4.2 we have that $\bar{X}_s = \bar{Y}_t = 0, \bar{X}_t = \bar{Y}_s = -1$, we get item (1) just by replacing this values in the integral expressions above.

Interchanging \bar{X} and \bar{Y} , t and s , in the above formula, we get item (2). At the point $p = A$ we have that $\int \bar{X}^2 d\mu_A = \int \bar{Y}^2 d\mu_A = 1$, so replacing these values in the formula we finish the proof of the lemma. \square

Lemma 4.6. *The expression of $\bar{Y}(\bar{X}\langle \bar{X}, \bar{Y} \rangle) = \bar{Y}(\bar{X} \int \bar{X} \bar{Y} d\mu_p)$ is*

$$\bar{Y}(\bar{X} \int \bar{X} \bar{Y} d\mu_p) = \int \bar{Y} \bar{X}_{ts} d\mu_p + 1 - \int \bar{Y}^2 d\mu_p + \int \bar{X} \bar{Y}_{ts} d\mu_p - \int \bar{X}^2 d\mu_p + \int \bar{X}^2 \bar{Y}^2 d\mu_p$$

at every point $p \in S(t, s)$. In particular, at $p = A$ we have

$$\bar{Y}(\bar{X} \int \bar{X}\bar{Y} d\mu_A) = \int \bar{Y}\bar{X}_{ts} d\mu_A + \int \bar{X}\bar{Y}_{ts} d\mu_A - 1 + \int \bar{X}^2\bar{Y}^2 d\mu_A$$

Proof. We apply the Leibniz rule,

$$\begin{aligned} \bar{Y}(\bar{X} \int \bar{X}\bar{Y} d\mu) &= \bar{Y}(\int \bar{X}_t\bar{Y} d\mu + \int \bar{X}\bar{Y}_t d\mu + \int \bar{X}^2\bar{Y} d\mu) \\ &= \int \bar{X}_{ts}\bar{Y} d\mu + \int \bar{X}_t\bar{Y}_s d\mu + \int \bar{X}_t\bar{Y}^2 d\mu \\ &\quad + \int \bar{X}_s\bar{Y}_t d\mu + \int \bar{X}\bar{Y}_{ts} d\mu + \int \bar{X}\bar{Y}_t\bar{Y} d\mu \\ &\quad + \int \bar{Y}_s\bar{X}^2 d\mu + 2 \int \bar{Y}\bar{X}\bar{X}_s d\mu + \int \bar{X}^2\bar{Y}^2 d\mu \end{aligned}$$

Since by Lemma 4.4 we have that $\bar{X}_s = \bar{Y}_t$ we get the following formula just adding the terms in the above formula:

$$\begin{aligned} \bar{Y}(\bar{X} \int \bar{X}\bar{Y} d\mu) &= \int \bar{Y}\bar{X}_{ts} d\mu + \int \bar{X}_t\bar{Y}_s d\mu + \int \bar{X}_t\bar{Y}^2 d\mu + \int (\bar{X}_s)^2 d\mu \\ &\quad + \int \bar{X}\bar{Y}_{ts} d\mu + 3 \int \bar{X}\bar{X}_s\bar{Y} d\mu + \int \bar{Y}_s\bar{X}^2 d\mu + \int \bar{X}^2\bar{Y}^2 d\mu. \end{aligned}$$

By Lemma 4.2 $\bar{X}_s = \bar{Y}_t = 0$, $\bar{X}_t = \bar{Y}_s = -1$, and replacing these values in the integral expression above we obtain the formula in the statement. Moreover, if $p = A$ we know that $\int \bar{X}^2 d\mu_A = \int^2 d\mu_A = 1$, as well as $\int \bar{Y}^2 d\mu_A = \int Y^2 d\mu_A = 1$, thus concluding the proof of the Lemma. \square

Corollary 4.7. *The term $-\frac{1}{2}(\bar{X}(\bar{X}(\|\bar{Y}\|^2)) + \bar{Y}(\bar{Y}(\|\bar{X}\|^2))) + \bar{Y}(\bar{X}\langle\bar{X}, \bar{Y}\rangle)$ in the expression of $K(X, Y)$ at the point A vanishes.*

Proof. To shorten notation, we shall omit the dependence of A in the expressions. According to Lemma 4.4, we have that

- (1) $\int \bar{X}\bar{X}_{ss} d\mu = \int \bar{X}\bar{Y}_{ts} d\mu.$
- (2) $\int \bar{Y}\bar{X}_{st} d\mu = \int \bar{Y}\bar{Y}_{tt} d\mu.$

Replacing the above equalities in the expressions of Lemmas 4.5, 4.6, and adding the resulting formulae we get Corollary 4.7. \square

Theorem 4.3 follows at once from Corollary 4.7.

5. CRISTOFFEL COEFFICIENTS AT THE EXPRESSION OF $K(X, Y)$

We denote by $\{X_i\}$, $i \in \mathbb{N}$, a complete orthonormal base of the vector space $T_A\mathcal{N} \subset L^2(\mu)$ (for the Gibbs probability μ associated to the normalized potential A).

The main goal of the section is to obtain the expression for the sectional curvature in Theorem 1.1.

Namely, let $A \in \mathcal{N}$ be a point in the manifold of normalized potentials, let $X, Y \in T_A\mathcal{N}$ be two orthonormal tangent vectors. Then the expression of the

curvature of the plane generated by X, Y is

$$(13) \quad K(X, Y) = \frac{1}{4} \left[\sum_{i=1}^{\infty} \left(\int X Y X_i d\mu \right)^2 - \sum_{i=1}^{\infty} \int X^2 X_i d\mu \int Y^2 X_i d\mu \right].$$

In Proposition 5.2 we will show that the above sum is well defined.

The proof is a direct calculation of the terms $\| \nabla_{\bar{Y}} \bar{X} \|^2, \langle \nabla_{\bar{X}} \bar{X}, \nabla_{\bar{Y}} \bar{Y} \rangle$ that appear in the expression of the curvature in Theorem 4.3. We shall subdivide the calculation in several lemmas.

We follow the notations of the previous section. Let $S(t, s)$ be the local surface given in Section 4 tangent to the plane generated by the vectors X, Y , satisfying $S(0, 0) = A$, let \bar{X}, \bar{Y} be the local extensions of the vectors X, Y obtained by projecting by the map Π the plane generated by X, Y at $T_A \mathcal{N}$ into the tangent space of \mathcal{N} .

Let us define local extensions \bar{X}_i of the vector fields X_i in an analogous way we defined the extensions of X, Y : let S_k be the plane generated by X_1, X_2, \dots, X_k and let us project by Π the tangent space of S_k into $T\mathcal{N}$ by the differential of the projection into \mathcal{N} .

The terms $\| \nabla_{\bar{Y}} \bar{X} \|^2, \langle \nabla_{\bar{X}} \bar{X}, \nabla_{\bar{Y}} \bar{Y} \rangle$ involve the Cristoffel symbols of the vector fields \bar{X}, \bar{Y} , at the point A we have:

$$\nabla_{\bar{X}_k} \bar{X}_l = \sum_{i=1}^{\infty} \Gamma_{kl}^i X_i$$

where $\Gamma_{kl}^i = \langle \nabla_{\bar{X}_k} \bar{X}_l, \bar{X}_i \rangle$ is the Cristoffel coefficient. We follow [9] for the definitions and basic properties of Cristoffel coefficients.

The coefficient Γ_{ij}^k can be calculated in terms of the coefficients of the first fundamental form of the metric at A , the inner products $g_{ij} = \langle X_i, X_j \rangle$ by the following formula:

$$\Gamma_{kl}^i = \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m})$$

where g^{im} is the coefficient of the inverse of the first fundamental form of index im , $g_{mk,l}$ is the derivative with respect to \bar{X}_l of the coefficient g_{mk} , and the above notation is Einstein's convention for the sum on the index m .

The expression "inverse of the first fundamental form" requires some explanation since we are dealing with an infinite-dimensional Riemannian manifold. One natural rigorous approach is to evaluate the series $\sum_{i=1}^{\infty} \Gamma_{kl}^i X_i$ as the limit of its partial sums $\sum_{i=1}^n \Gamma_{kl}^i X_i$, that includes the Cristoffel coefficients in the subspace of $T_A \mathcal{N}$ generated by $\{X_1, X_2, \dots, X_n\}$. The first fundamental form restricted to this subspace is a $n \times n$ matrix that, under our assumptions, is the identity. Its inverse is of course the identity. This allows us to define all the terms in the partial sum, then we take the limit as $n \rightarrow \infty$ to get the series. We shall prove that the series converges absolutely, so the above procedure provides the expression of $\nabla_{\bar{X}_k} \bar{X}_l$ as an infinite series.

In particular, since the basis $\{X_1, X_2, \dots, X_n, \dots\}$ is orthonormal, the indices in the sum of the expression of $\nabla_{\bar{X}_k} \bar{X}_l$ according to Einstein's convention just reduce to ii, kk, ll , depending on the case, and $g_{kl} = g^{kl} = \delta_{kl}$. So at the point A we get the formula

$$\Gamma_{kl}^i = \frac{1}{2} (g_{ik,l} + g_{il,k} - g_{kl,i}).$$

Lemma 5.1. *The term $g_{ik,l}$ at A , for any permutation of the indices, is*

$$g_{ik,l} = \int X_i X_k X_l d\mu.$$

Then,

$$\nabla_{\bar{X}_k} \bar{X}_l = \frac{1}{2} \sum_{i=1}^{\infty} \left(\int X_i X_k X_l d\mu_A \right) X_i.$$

Proof. We have that $g_{ik,l} = \bar{X}_l \langle \bar{X}_i, \bar{X}_k \rangle = \bar{X}_l \int \bar{X}_i \bar{X}_k d\mu$. By the Leibniz rule we have

$$\bar{X}_l \int \bar{X}_i \bar{X}_k d\mu = \int \frac{\partial}{\partial \bar{X}_l} (\bar{X}_i) \bar{X}_k d\mu + \int \bar{X}_i \frac{\partial}{\partial \bar{X}_l} (\bar{X}_k) d\mu + \int \bar{X}_i \bar{X}_k \bar{X}_l d\mu$$

where $\frac{\partial}{\partial \bar{X}_l} (\bar{X}_i)$ is the derivative of the vector field \bar{X}_i in the direction of \bar{X}_l .

Notice that Lemma 4.2 extends to the submanifolds S_k for every $k \in \mathbb{N}$. So we have

- (1) $\frac{\partial}{\partial \bar{X}_l} (\bar{X}_i) = 0$ if $l \neq i$,
- (2) $\frac{\partial}{\partial \bar{X}_l} (\bar{X}_i) = -1$ if $l = i$.

In both cases, since $\int \bar{X}_i d\mu = 0$ for every i , we get $g_{ik,l} = \int X_i X_k X_l d\mu$ as claimed.

The expression for $\nabla_{\bar{X}_k} \bar{X}_l$ is straightforward from this formula. □

Corollary 5.2. *Let us assume that $X = X_1$ and $Y = X_2$ are the first two vectors of the orthonormal base $\{X_i\}$. For the normalized potential $A = S(0, 0)$ we get the following expressions*

$$\begin{aligned} \nabla_{\bar{X}_1} \bar{X}_1 &= \frac{1}{2} \sum_{i=1}^{\infty} \left(\int X_1^2 X_i d\mu_A \right) X_i \\ \nabla_{\bar{X}_2} \bar{X}_2 &= \frac{1}{2} \sum_{i=1}^{\infty} \left(\int X_2^2 X_i d\mu_A \right) X_i \\ \nabla_{\bar{X}_1} \bar{X}_2 &= \frac{1}{2} \sum_{i=1}^{\infty} \left(\int X_1 X_2 X_i d\mu_A \right) X_i. \end{aligned}$$

Moreover, for any pair $X, Y \in T_A \mathcal{N}$ the sums

$$\sum_{i=1}^{\infty} \left(\int X Y X_i d\mu \right)^2 \quad \text{and} \quad \sum_{i=1}^{\infty} \int X^2 X_i d\mu \int Y^2 X_i d\mu$$

are both finite.

Proof. We consider an extension of the family X_r , $r \in \mathbb{N}$, to all $L^2(\mu)$ and we get a complete orthonormal base of the vector space $L^2(\mu)$, given by X_r, Y_s , $r, s \in \mathbb{N}$. The first three expressions in the statement are straightforward from Lemma 5.1.

Given two elements $X, Y \in T_A \mathcal{N}$ consider $f = X Y = \sum_r a_r^f X_r + \sum_s b_s^f Y_s \in L^2(\mu)$, then,

$$\left(\int X Y X_i d\mu \right)^2 = |a_i^f|^2.$$

It follows that $\sum_{i=1}^{\infty} \left(\int X Y X_i d\mu \right)^2 = \sum_{i=1}^{\infty} |a_i^f|^2 \leq \|f\|^2$ is finite.

Denote $g = X^2 = \sum_r a_r^g X_r + \sum_s b_s^g Y_s$ and $h = Y^2 = \sum_r a_r^h X_r + \sum_s b_s^h Y_s$. Therefore,

$$\int g h d\mu = \sum_{i=1}^{\infty} a_i^g a_i^h + \sum_{j=1}^{\infty} b_j^g b_j^h.$$

Form this follows that $\sum_{i=1}^{\infty} a_i^g a_i^h$ converges. Note that $\int X^2 X_i d\mu = a_i^g$ and $\int Y^2 X_i d\mu = a_i^h$. Then,

$$\sum_{i=1}^{\infty} \int X^2 X_i d\mu \int Y^2 X_i d\mu = \sum_{i=1}^{\infty} a_i^g a_i^h$$

converges. □

Theorem 1.1 follows from direct calculation applying Corollary 5.2 to the expression of $K(X, Y)$.

6. A WORKED EXAMPLE IN THE MARKOV CASE: AN ORTHONORMAL BASIS FOR THE KERNEL OF THE RUELLE OPERATOR

In this section, we will exhibit an orthonormal basis for the kernel of the Ruelle operator in the case of Markov stationary probabilities on $\Omega = \{0, 1\}^{\mathbb{N}}$.

Given a finite word $x = (x_1, x_2, \dots, x_k) \in \{0, 1\}^k$, $k \in \mathbb{N}$, we denote by $[x]$ the associated cylinder set in $\Omega = \{0, 1\}^{\mathbb{N}}$.

Consider an invariant Markov probability μ obtained from a **row** stochastic matrix $(P_{i,j})_{i,j=0,1}$ and an initial left invariant vector of probability $\pi = (\pi_0, \pi_1) \in \mathbb{R}^2$.

Given $r \in (0, 1)$ and $s \in (0, 1)$ we denote

$$(14) \quad P = \begin{pmatrix} P_{0,0} & P_{0,1} \\ P_{1,0} & P_{1,1} \end{pmatrix} = \begin{pmatrix} r & 1-r \\ 1-s & s \end{pmatrix}.$$

In this way $(r, s) \in (0, 1) \times (0, 1)$ parametrize all **row** stochastic matrices.

The explicit expression is

$$(15) \quad \mu[x_1, x_2, \dots, x_n] = \pi_{x_1} P_{x_1, x_2} P_{x_2, x_3} \dots P_{x_{n-1}, x_n}.$$

Definition 6.1. Denote by $J : \{0, 1\}^{\mathbb{N}} \rightarrow \mathbb{R}$ the Jacobian associated to P . This function J is such that is constant equal

$$J_{i,j} = \frac{\pi_i P_{i,j}}{\pi_j}$$

on the cylinder $[i, j]$, $i, j = 0, 1$.

According to our previous notation $\mu_A = \mu_{\log J}$ (which in this section will be called just μ).

Definition 6.2. The Ruelle operator for $\log J$ acts on continuous functions φ and is given by: for each $\varphi : \Omega \rightarrow \mathbb{R}$, we get that

$$(16) \quad \mathcal{L}_{\log J}(\varphi)(x_1, x_2, x_3, \dots) = \frac{\pi_0 P_{0,x_1}}{\pi_{x_1}} \varphi(0, x_1, x_2, \dots) + \frac{\pi_1 P_{1,x_1}}{\pi_{x_1}} \varphi(1, x_1, x_2, \dots).$$

It is known that $\mathcal{L}_{\log J}^*(\mu) = \mu$. (see [22])

We also consider the action of $\mathcal{L}_{\log J}$ on $L^2(\mu)$ and we are interested in the kernel of this operator when acting on Holder functions.

Given a finite word $x = (x_1, x_2, \dots, x_n)$, depending of the context $[x]$ will either denote the word or the corresponding cylinder set in $\{0, 1\}^{\mathbb{N}}$. The empty word is also considered a finite word.

We start by recalling that the family of Holder functions

$$(17) \quad e_{[x]} = \frac{1}{\sqrt{\mu([x])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \mathbf{1}_{[x_0]} - \frac{1}{\sqrt{\mu([x])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \mathbf{1}_{[x_1]},$$

where $x = (x_1, x_2, \dots, x_n)$ is a finite word on the symbols $\{0, 1\}$, is an orthonormal (Haar) basis for $\mathcal{L}^2(\mu)$ (see [15] for a general expression and [10] for the specific expression we are using here). In order to be more precise, we should also add $e_{[\emptyset]}^0 = \frac{1}{\sqrt{\mu(\{0\})}} \mathbf{1}_{[0]}$ and $e_{[\emptyset]}^1 = \frac{1}{\sqrt{\mu(\{1\})}} \mathbf{1}_{[1]}$ for the (Haar) basis claim.

Definition 6.3. Given a finite word $x = (x_1, x_2, \dots, x_n)$, we denote

$$(18) \quad a_x = \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_n]},$$

The main result of the section is the following:

Theorem 6.4. *The family a_x , indexed by all words $x = (x_1, x_2, \dots, x_n)$, is an orthogonal set on the kernel of the Ruelle operator $\mathcal{L}_{\log J}$.*

First note that as the family $e_{[x]}$, where x is a finite word, is orthonormal, then, a_x , where x is a finite word with size bigger or equal to 1, is an orthogonal family.

Indeed, it follows from the fact that the family $e_{[x]}$ defined by (17) is orthogonal, and the bilinearity of the inner product, that

$$\begin{aligned} \langle a_x, a_z \rangle &= \\ \langle \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} P_{0,x_1}} e_{[0,x]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} P_{1,x_1}} e_{[1,x]}, \frac{\sqrt{\pi_{z_1}}}{\sqrt{\pi_0} P_{0,z_1}} e_{[0,z]} - \frac{\sqrt{\pi_{z_1}}}{\sqrt{\pi_1} P_{1,z_1}} e_{[1,z]} \rangle &= 0, \end{aligned}$$

for all $x = (x_1, x_2, \dots, x_n) \neq z = (z_1, z_2, \dots, z_k)$.

We shall subdivide the proof of Theorem 6.4 into several steps. First of all, we have that:

Proposition 6.5. *Given $x = [x_1, x_2, \dots, x_n]$ with a size larger or equal to 1,*

$$(19) \quad \mathcal{L}_{\log J}(e_{[x_1, x_2, \dots, x_n]}) = \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_{x_2}}} \sqrt{P_{x_1, x_2}} e_{[x_2, x_3, \dots, x_n]}.$$

From this follows that all elements in the orthogonal family

$$(20) \quad a_x = \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_n]},$$

indexed by words $x = (x_1, x_2, \dots, x_n)$, are in the kernel of the Ruelle operator $\mathcal{L}_{\log J}$.

Proof. We consider finite words x with size larger or equal to 1.

Indeed, given the word $x = (x_1, x_2, \dots, x_n)$, let $L = \mathcal{L}_{\log J}(e_{[x_1, x_2, \dots, x_n]})$, then we get

$$\begin{aligned} L &= \frac{\pi_{x_1}}{\pi_{x_2}} P_{x_1, x_2} \left[\frac{1}{\sqrt{\mu([x])}} \sqrt{\frac{P_{x_n, 1}}{P_{x_n, 0}}} \mathbf{1}_{[x_2, \dots, x_n, 0]} \right. \\ &\quad \left. - \frac{1}{\sqrt{\mu([x])}} \sqrt{\frac{P_{x_n, 0}}{P_{x_n, 1}}} \mathbf{1}_{[x_2, \dots, x_n, 1]} \right] \\ &= \frac{\pi_{x_1}}{\pi_{x_2}} \sqrt{P_{x_1, x_2}} \left[\frac{\sqrt{P_{x_1, x_2}}}{\sqrt{\mu([x])}} \sqrt{\frac{P_{x_n, 1}}{P_{x_n, 0}}} \mathbf{1}_{[x_2, \dots, x_n, 0]} \right. \\ &\quad \left. - \frac{\sqrt{P_{x_1, x_2}}}{\sqrt{\mu([x])}} \sqrt{\frac{P_{x_n, 0}}{P_{x_n, 1}}} \mathbf{1}_{[x_2, \dots, x_n, 1]} \right] \end{aligned}$$

This is equal to

$$\begin{aligned} &\frac{\pi_{x_1}}{\pi_{x_2}} \frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_{x_2}}} \sqrt{P_{x_1, x_2}} \frac{\sqrt{P_{x_1, x_2}}}{\sqrt{\pi_{x_1} P_{x_1, x_2} P_{x_2, x_3} \dots P_{x_{n-1}, x_n}}} \sqrt{\frac{P_{x_n, 1}}{P_{x_n, 0}}} \mathbf{1}_{[x_2, \dots, x_n, 0]} \\ &- \frac{\pi_{x_1}}{\pi_{x_2}} \frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_{x_2}}} \sqrt{P_{x_1, x_2}} \frac{\sqrt{P_{x_1, x_2}}}{\sqrt{\pi_{x_1} P_{x_1, x_2} P_{x_2, x_3} \dots P_{x_{n-1}, x_n}}} \sqrt{\frac{P_{x_n, 0}}{P_{x_n, 1}}} \mathbf{1}_{[x_2, \dots, x_n, 1]} \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\frac{\pi_{x_1}}{\pi_{x_2}} \frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_{x_1}}} \sqrt{P_{x_1, x_2}} \frac{1}{\sqrt{\pi_{x_2} P_{x_2, x_3} \dots P_{x_{n-1}, x_n}}} \sqrt{\frac{P_{x_n, 1}}{P_{x_n, 0}}} \mathbf{1}_{[x_2, \dots, x_n, 0]} \\ &- \frac{\pi_{x_1}}{\pi_{x_2}} \frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_{x_1}}} \sqrt{P_{x_1, x_2}} \frac{1}{\sqrt{\pi_{x_2} P_{x_2, x_3} \dots P_{x_{n-1}, x_n}}} \sqrt{\frac{P_{x_n, 0}}{P_{x_n, 1}}} \mathbf{1}_{[x_2, \dots, x_n, 1]} \end{aligned}$$

that yields

$$\begin{aligned} L &= \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_{x_2}}} \sqrt{P_{x_1, x_2}} \frac{1}{\sqrt{\mu[x_2, x_3, \dots, x_n]}} \left[\sqrt{\frac{P_{x_n, 1}}{P_{x_n, 0}}} \mathbf{1}_{[x_2, \dots, x_n, 0]} - \sqrt{\frac{P_{x_n, 0}}{P_{x_n, 1}}} \mathbf{1}_{[x_2, \dots, x_n, 1]} \right] \\ &= \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_{x_2}}} \sqrt{P_{x_1, x_2}} e_{x_2, x_3, \dots, x_n}. \end{aligned}$$

Then,

$$\begin{aligned} &\mathcal{L}_{\log J} \left(\frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_{x_1}} \sqrt{P_{x_1, x_2}}} e_{[x_1, x_2, \dots, x_n]} \right) \\ &= \frac{1}{\sqrt{\mu[x_2, x_3, \dots, x_n]}} \left[\sqrt{\frac{P_{x_n, 1}}{P_{x_n, 0}}} \mathbf{1}_{[x_2, \dots, x_n, 0]} - \sqrt{\frac{P_{x_n, 0}}{P_{x_n, 1}}} \mathbf{1}_{[x_2, \dots, x_n, 1]} \right] = e_{[x_2, x_3, \dots, x_n]} \end{aligned}$$

and therefore,

$$\mathcal{L}_{\log J} \left(\frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_0} \sqrt{P_{0, x_2}}} e_{[0, x_2, \dots, x_n]} \right) = \mathcal{L}_{\log J} \left(\frac{\sqrt{\pi_{x_2}}}{\sqrt{\pi_1} \sqrt{P_{1, x_2}}} e_{[1, x_2, \dots, x_n]} \right).$$

For each finite word (x_1, x_2, \dots, x_n) denote

$$\begin{aligned}
a_x &= \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_n]} \\
&= \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} \frac{1}{\sqrt{\mu([0x])}} \left[\sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \mathbf{1}_{[0x0]} - \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \mathbf{1}_{[0x1]} \right] \\
(21) \quad &- \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} \frac{1}{\sqrt{\mu([1x])}} \left[\sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \mathbf{1}_{[1x0]} - \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \mathbf{1}_{[1x1]} \right].
\end{aligned}$$

From the above reasoning, it follows that the family a_x is in the kernel of the Ruelle operator. \square

For words, x of size greater or equal to 1 the function a_x is constant in cylinder sets of size equal to the length of x plus 2.

As an example, we get that

$$\begin{aligned}
a_0 &= \frac{\sqrt{\pi_0}}{\sqrt{\pi_0} \sqrt{P_{0,0}}} \frac{1}{\sqrt{\mu([00])}} \left[\sqrt{\frac{P_{0,1}}{P_{0,0}}} \mathbf{1}_{[000]} - \sqrt{\frac{P_{0,0}}{P_{0,1}}} \mathbf{1}_{[001]} \right] \\
(22) \quad &- \frac{\sqrt{\pi_0}}{\sqrt{\pi_1} \sqrt{P_{1,0}}} \frac{1}{\sqrt{\mu([10])}} \left[\sqrt{\frac{P_{0,1}}{P_{0,0}}} \mathbf{1}_{[100]} - \sqrt{\frac{P_{0,0}}{P_{0,1}}} \mathbf{1}_{[101]} \right]
\end{aligned}$$

is constant on cylinders of size 3.

Note that if x and z are different words, then, $1x$, $0x$, $0z$ and $1z$ are four different words.

Note that

$$(23) \quad e_{[x]}^2 = \frac{1}{\mu([x])} \frac{P_{x_n,1}}{P_{x_n,0}} \mathbf{1}_{[x0]} + \frac{1}{\mu([x])} \frac{P_{x_n,0}}{P_{x_n,1}} \mathbf{1}_{[x1]}.$$

Therefore,

$$\begin{aligned}
a_x^2 = a_x a_x &= \frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} e_{[0,x_1,x_2,\dots,x_n]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} e_{[1,x_1,x_2,\dots,x_n]}^2 \\
&= \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \frac{1}{\mu([0x])} \frac{P_{x_n,1}}{P_{x_n,0}} \mathbf{1}_{[0x0]} + \frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \frac{1}{\mu([0x])} \frac{P_{x_n,0}}{P_{x_n,1}} \mathbf{1}_{[0x1]} \right] \\
(24) \quad &+ \left[\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \frac{1}{\mu([1x])} \frac{P_{x_n,1}}{P_{x_n,0}} \mathbf{1}_{[1x0]} + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \frac{1}{\mu([1x])} \frac{P_{x_n,0}}{P_{x_n,1}} \mathbf{1}_{[1x1]} \right].
\end{aligned}$$

From the above it follows that

$$(25) \quad |a_x| = \sqrt{\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}}.$$

Using the notation in the variables r, s for the matrix P , when $x_1 = 0$ we get

$$(26) \quad |a_x| = \sqrt{\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}} = (\sqrt{r(1-r)})^{-1}$$

and when $x_1 = 1$ we get

$$(27) \quad |a_x| = \sqrt{\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}} = (\sqrt{s(1-s)})^{-1}.$$

Note also that the terms $|a_x|$ are uniformly bounded away from zero (the minimum value is 2). They depend just on the first letter of the word $[x]$.

Definition 6.6. We denote by

$$(28) \quad \hat{a}_x = \frac{1}{|a_x|} a_x,$$

the normalization of a_x .

Definition 6.7. We denote by $\tilde{\mathcal{F}}$ the orthonormal set of normalized functions \hat{a}_x , where $x = (x_1, x_2, \dots, x_k)$ is a finite word with size equal or larger than 1.

We will have to add two more functions in order to get a basis (a completely orthogonal set in the Hilbert space) for the kernel of the Ruelle operator $\mathcal{L}_{\log J}$.

We claim that the orthogonal pair (constant in cylinders of size 2)

$$(29) \quad \begin{aligned} V_1 &= \pi_1 P_{1,0} \mathbf{1}_{[00]} - \pi_0 P_{0,0} \mathbf{1}_{[10]} \\ V_2 &= \pi_0 P_{0,1} \mathbf{1}_{[11]} - \pi_1 P_{1,1} \mathbf{1}_{[01]} \end{aligned}$$

is in the kernel of the Ruelle operator (see Proposition 6.10). The functions V_1 and V_2 are orthogonal to all $a_x \in \mathcal{F}$ and they depend on the first two coordinates x_1, x_2 of x .

The vectors $\hat{V}_1 = \frac{1}{|V_1|}$ and $\hat{V}_2 = \frac{1}{|V_2|}$ are normalized and orthogonal to all \hat{a}_x . This claim will be proved in Proposition 6.10.

One can show that

$$(30) \quad |V_1| = \sqrt{\pi_1^2 P_{1,0}^2 \pi_0 P_{0,0} + \pi_0^2 P_{0,0}^2 \pi_1 P_{1,0}} = \sqrt{\frac{(1-r)r(s-1)^3}{(-2+r+s)^3}}$$

and

$$(31) \quad |V_2| = \sqrt{\pi_0^2 P_{0,1}^2 \pi_1 P_{1,1} + \pi_1^2 P_{1,1}^2 \pi_0 P_{0,1}} = \sqrt{\frac{(1-s)s(r-1)^3}{(-2+r+s)^3}}.$$

Definition 6.8. As a matter of notation we denote $\hat{a}_{[\emptyset]}^0 = \hat{V}_1$ and $\hat{a}_{[\emptyset]}^1 = \hat{V}_2$.

These two functions are constant in cylinders of size 2

Definition 6.9. We add $\hat{a}_{[\emptyset]}^0$ and $\hat{a}_{[\emptyset]}^1$ to the family $\tilde{\mathcal{F}}$ in order to get the family \mathcal{F} .

Now the elements in \mathcal{F} range in all possible words of size larger or equal to zero. A generic element in \mathcal{F} is denoted by \hat{a}_x , and by this we mean that \hat{a}_x can eventually represent $\hat{a}_{[\emptyset]}^0$ or $\hat{a}_{[\emptyset]}^1$.

Proposition 6.10. *The orthogonal pair*

$$(32) \quad \begin{aligned} V_1 &= \pi_1 P_{1,0} \mathbf{1}_{[00]} - \pi_0 P_{0,0} \mathbf{1}_{[10]} \\ V_2 &= \pi_0 P_{0,1} \mathbf{1}_{[11]} - \pi_1 P_{1,1} \mathbf{1}_{[01]} \end{aligned}$$

is such that, each one of them is orthogonal to the other elements \hat{a}_x , where x ranges in all finite words with size bigger or equal to 1. V_1 and V_2 are on the kernel of the Ruelle operator $\mathcal{L}_{\log J}$.

Proof. Note first that $\mathbf{1}_{[00]}$ is orthogonal to all a_x , where $x = (x_1, x_2, \dots, x_n)$ is a word with size equal or greater than 1. This claim follows from (21). Indeed, if $x_1 = 0$, we get that

$$\begin{aligned} & \langle \mathbf{1}_{[00]}, \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \mathbf{1}_{[0x0]} - \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \mathbf{1}_{[0x1]} \rangle = \\ & \sqrt{P_{x_n,1}} \pi_0 P_{0,x_1} P_{x_1,x_2} \dots P_{x_{n-1},x_n} \sqrt{P_{x_n,0}} - \\ & \sqrt{P_{x_n,0}} \pi_0 P_{0,x_1} P_{x_1,x_2} \dots P_{x_{n-1},x_n} \sqrt{P_{x_n,1}} = 0. \end{aligned}$$

If $x_1 = 1$ the claim follows at once.

Using the same reasoning one can show that $\mathbf{1}_{[01]}, \mathbf{1}_{[10]}, \mathbf{1}_{[11]}$ are orthogonal to all a_x , where length of x is bigger than zero. It follows that linear combinations of this functions are also orthogonal to all a_x . It follows that V_1 and V_2 are orthogonal to all a_x , where the length of x is bigger than zero.

We will show that V_1 is in the kernel of the Ruelle operator (for V_2 the proof is similar). Given $y = (y_1, y_2, \dots, y_n, \dots) \in \Omega$, suppose first that $y_1 = 0$, then, we get

$$\begin{aligned} \mathcal{L}_{\log J}(V_1) &= \mathcal{L}_{\log J}(\pi_1 P_{1,0} \mathbf{1}_{[00]} - \pi_0 P_{0,0} \mathbf{1}_{[10]})(y) = \\ & \pi_1 P_{1,0} (J_{0,y_1} \mathbf{1}_{[00]}(0, y_1, y_2, \dots) + J_{1,y_1} \mathbf{1}_{[00]}(1, y_1, y_2, \dots)) - \\ & \pi_0 P_{0,0} (J_{0,y_1} \mathbf{1}_{[10]}(0, y_1, y_2, \dots) + J_{1,y_1} \mathbf{1}_{[10]}(1, y_1, y_2, \dots)) = \\ & \pi_1 P_{1,0} J_{0,0} - \pi_0 P_{0,0} J_{1,0} = \pi_1 P_{1,0} \frac{\pi_0 P_{0,0}}{\pi_0} - \pi_0 P_{0,0} \frac{\pi_1 P_{1,0}}{\pi_0} = 0. \end{aligned}$$

In the case $y_1 = 1$, we get

$$\begin{aligned} \mathcal{L}_{\log J}(V_1) &= \pi_1 P_{1,0} (J_{0,y_1} \mathbf{1}_{[00]}(0, y_1, y_2, \dots) + J_{1,y_1} \mathbf{1}_{[00]}(1, y_1, y_2, \dots)) - \\ & \pi_0 P_{0,0} (J_{0,y_1} \mathbf{1}_{[10]}(0, y_1, y_2, \dots) + J_{1,y_1} \mathbf{1}_{[10]}(1, y_1, y_2, \dots)) = 0. \end{aligned}$$

□

Proposition 6.11. *The family of elements $\hat{a}_x \in \mathcal{F}$ is an orthonormal basis for the kernel of the Ruelle operator $\mathcal{L}_{\log J}$.*

Proof. From Proposition 6.5 we know that given $x = [x_1, x_2, \dots, x_n]$

$$(33) \quad \mathcal{L}_{\log J}(e_{[x_1, x_2, \dots, x_n]}) = \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_{x_2}}} \sqrt{P_{x_1, x_2}} e_{[x_2, x_3, \dots, x_n]}.$$

Suppose φ is in the kernel of the Ruelle operator. We will show that φ can be expressed as an infinite linear combination of the normalized functions $\hat{a}_x \in \mathcal{F}$.

We can express φ as

$$\varphi = \sum_{\text{words } y} c_y e_{[y]}.$$

When applying $\mathcal{L}_{\log J}$ on φ we separate the infinite sum in subsums of the form

$$c_{0, \alpha_2, \dots, \alpha_n} e_{[0, \alpha_2, \dots, \alpha_n]} + c_{1, \alpha_2, \dots, \alpha_n} e_{[1, \alpha_2, \dots, \alpha_n]}.$$

Assuming that φ is on the Kernel of $\mathcal{L}_{\log J}$, we get from (33) that

$$\begin{aligned} 0 &= \mathcal{L}_{\log J} \left(\sum_n \sum_{\alpha_2, \dots, \alpha_n} [c_{0, \alpha_2, \dots, \alpha_n} e_{[0, \alpha_2, \dots, \alpha_n]} + c_{1, \alpha_2, \dots, \alpha_n} e_{[1, \alpha_2, \dots, \alpha_n]}] \right) \\ &= \sum_n \sum_{\alpha_2, \dots, \alpha_n} \left[\frac{\sqrt{\pi_0}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{0, \alpha_2}} c_{0, \alpha_2, \dots, \alpha_n} e_{[\alpha_2, \dots, \alpha_n]} + \frac{\sqrt{\pi_1}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{1, \alpha_2}} c_{1, \alpha_2, \dots, \alpha_n} e_{[\alpha_2, \dots, \alpha_n]} \right] \\ &= \sum_n \sum_{\alpha_2, \dots, \alpha_n} \left[\frac{\sqrt{\pi_0}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{0, \alpha_2}} c_{0, \alpha_2, \dots, \alpha_n} + \frac{\sqrt{\pi_1}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{1, \alpha_2}} c_{1, \alpha_2, \dots, \alpha_n} \right] e_{[\alpha_2, \dots, \alpha_n]}. \end{aligned}$$

Then, for fixed n and $(\alpha_2, \alpha_3, \dots, \alpha_n)$

$$\frac{\sqrt{\pi_0}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{0, \alpha_2}} c_{0, \alpha_2, \dots, \alpha_n} = - \frac{\sqrt{\pi_1}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{1, \alpha_2}} c_{1, \alpha_2, \dots, \alpha_n},$$

which means

$$c_{0, \alpha_2, \dots, \alpha_n} = - \frac{\sqrt{\pi_1}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{1, \alpha_2}} \frac{\sqrt{\pi_{\alpha_2}}}{\sqrt{\pi_0}} \frac{1}{\sqrt{P_{0, \alpha_2}}} c_{1, \alpha_2, \dots, \alpha_n}.$$

Then, the sum

$$c_{0, \alpha_2, \dots, \alpha_n} e_{[0, \alpha_2, \dots, \alpha_n]} + c_{1, \alpha_2, \dots, \alpha_n} e_{[1, \alpha_2, \dots, \alpha_n]}$$

is equal to

$$- c_{1, \alpha_2, \dots, \alpha_n} \left[\frac{\sqrt{\pi_1}}{\sqrt{\pi_{\alpha_2}}} \sqrt{P_{1, \alpha_2}} \frac{\sqrt{\pi_{\alpha_2}}}{\sqrt{\pi_0}} \frac{1}{\sqrt{P_{0, \alpha_2}}} e_{[0, \alpha_2, \dots, \alpha_n]} - e_{[1, \alpha_2, \dots, \alpha_n]} \right].$$

Multiplying the above expression by $\frac{\sqrt{\pi_{\alpha_2}}}{\sqrt{\pi_1}} \frac{1}{\sqrt{P_{1, \alpha_2}}}$ we get

$$\frac{\sqrt{\pi_{\alpha_2}}}{\sqrt{\pi_1}} \frac{1}{\sqrt{P_{1, \alpha_2}}} [c_{0, \alpha_2, \dots, \alpha_n} e_{[0, \alpha_2, \dots, \alpha_n]} + c_{1, \alpha_2, \dots, \alpha_n} e_{[1, \alpha_2, \dots, \alpha_n]}]$$

which is equal to

$$\begin{aligned} &- c_{1, \alpha_2, \dots, \alpha_n} \left[\frac{\sqrt{\pi_{\alpha_2}}}{\sqrt{\pi_0}} \frac{1}{\sqrt{P_{0, \alpha_2}}} e_{[0, \alpha_2, \dots, \alpha_n]} - \frac{\sqrt{\pi_{\alpha_2}}}{\sqrt{\pi_1}} \frac{1}{\sqrt{P_{1, \alpha_2}}} e_{[1, \alpha_2, \dots, \alpha_n]} \right] \\ &= - c_{1, \alpha_2, \dots, \alpha_n} a_{[\alpha_2, \dots, \alpha_n]}. \end{aligned}$$

Then, $(c_{0, \alpha_2, \dots, \alpha_n} e_{[0, \alpha_2, \dots, \alpha_n]} + c_{1, \alpha_2, \dots, \alpha_n} e_{[1, \alpha_2, \dots, \alpha_n]})$ is a multiple of the function $\hat{a}_{[\alpha_2, \dots, \alpha_n]}$. Since the above reasoning was done for a generic choice of $(\alpha_2, \alpha_3, \dots, \alpha_n)$, we conclude that for each n the sum $\sum_{\text{words } y \text{ of length } n} c_y e_{[y]}$ can be expressed as a linear combination of elements \hat{a}_x , using words of length $n-1$, $n > 1$.

From this follows that each element in the kernel of $\mathcal{L}_{\log J}$ can be expressed as an infinite linear combination of the functions \hat{a}_x . \square

Theorem 6.4 follows from the combination of Propositions 6.5 and 6.11

The above shows that the set \mathcal{F} is a complete orthonormal set for the kernel of the Ruelle operator acting on $\mathcal{L}^2(\mu)$.

Remark 6.12. A function of the form $w = r_1 \mathbf{1}_{[0]} + r_2 \mathbf{1}_{[1]}$ is in the kernel of $\mathcal{L}_{\log J}$ only in the case where $P_{01} = (1-r) = s = P_{11}$. In this case

$$(34) \quad w = (1-r) \mathbf{1}_{[0]} - (1-s) \mathbf{1}_{[1]}$$

is such that $\mathcal{L}_{\log J}(w) = 0$.

We do not have to take into account in our reasoning this function because

$$w = \frac{1}{r}V_1 + \frac{1}{r-1}V_2.$$

7. A WORKED EXAMPLE IN THE MARKOV CASE: PRELIMINARY CALCULATIONS OF THE TERMS IN $K(X, Y)$

In this section, we shall devote ourselves to the calculation of the sectional curvatures in the case of Markov stationary probabilities on $\Omega = \{0, 1\}^{\mathbb{N}}$.

We denote by $M \subset \mathcal{G}$, the set of Markov invariant probabilities. We will consider in this section the sectional curvature for points in M for general orthogonal pairs of tangent vectors to \mathcal{G} .

We can also consider M as a two-dimensional manifold carrying the Riemannian structure induced by \mathcal{G} . From this point of view, there exists just one orthonormal pair to be considered. One of our main results (see Theorem 7.15) claims that for the two dimensional manifold M , for any point in M , the sectional curvature for the pair of tangent vectors to M is always zero.

We will consider in our reasoning the empty word as a regular word. \hat{a}_\emptyset^0 and \hat{a}_\emptyset^1 are two elements in \mathcal{F} associated to the empty word.

Definition 7.1. We say that z is a subprefix of x , if x and z satisfy

$$[x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [z] = [x_1, x_2, \dots, x_k],$$

where $n \geq k$.

Note that, even when z is not a subprefix of x and x is not a subprefix of z , they can share some common subprefix. Note also that if x and z do not share a common prefix, then z is not a subprefix of x and x is not a subprefix of z .

If $[x] = [z]$, then, x is a subprefix of z .

Definition 7.2. We say that z is a strict subprefix of x , if x and z satisfy

$$[x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [z] = [x_1, x_2, \dots, x_k],$$

where $n > k$.

Two different words with the same length can not be subprefix of each other. If the length of z is strictly larger than the length of x , then, z can not be a subprefix of x .

Definition 7.3. Given the finite words x, z we denote by $D[x, z]$ the set of all finite words y such that are subprefix of x and z .

If for example $x = (0, 0, 0)$ and $z = (0, 0, 0, 1)$, then

$$D[x, z] = \{\hat{a}_\emptyset^0, (0), (0, 0), (0, 0, 0)\}.$$

In the case $x = (0, 1, 0, 0, 1)$ and $z = (0, 1, 1)$ we get that $D[x, z] = \{\hat{a}_\emptyset^0(0), (0, 1), \}$.

Another example: $D[a_{0,0}, \hat{a}_\emptyset^0] = \{\hat{a}_\emptyset^0\}$ and $D[a_{0,0}, \hat{a}_\emptyset^1] = \emptyset$.

Note that in the case $z = (z_1, z_2, \dots, z_k)$ is a subprefix of $x = (x_1, x_2, \dots, x_n)$, $n > k$, then, $z_1 = x_1$. Then, it follows from (25) that $|a_x| = |a_z|$.

Proposition 7.4. Assume that x is not a subprefix of z and z is not a subprefix of x . Then,

$$a_x a_z = 0.$$

Proof. Note that a_z is a linear combination of $\mathbf{1}_{[0z0]}$, $\mathbf{1}_{[0z1]}$, $\mathbf{1}_{[1z0]}$ and $\mathbf{1}_{[1z1]}$. As a_x is a linear combination of $\mathbf{1}_{[0x0]}$, $\mathbf{1}_{[0x1]}$, $\mathbf{1}_{[1x0]}$ and $\mathbf{1}_{[1x1]}$ the result follows. \square

Note that the hypothesis of the last proposition is equivalent to saying that the cylinders $[x]$ and $[z]$ are disjoint.

Proposition 7.5. *Given a word x assume that x is not a subprefix of y and y is not a subprefix of x . Then,*

$$\hat{a}_x^2 \hat{a}_y = 0.$$

Proof. Note that from (24) we get

$$\begin{aligned} a_x a_x = & \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \frac{1}{\mu([0x])} \frac{P_{x_n,1}}{P_{x_n,0}} \mathbf{1}_{[0x0]} + \frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \frac{1}{\mu([0x])} \frac{P_{x_n,0}}{P_{x_n,1}} \mathbf{1}_{[0x1]} \right] \\ & + \left[\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \frac{1}{\mu([1x])} \frac{P_{x_n,1}}{P_{x_n,0}} \mathbf{1}_{[1x0]} + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \frac{1}{\mu([1x])} \frac{P_{x_n,0}}{P_{x_n,1}} \mathbf{1}_{[1x1]} \right]. \end{aligned}$$

As \hat{a}_y is a linear combination of $\mathbf{1}_{[0y0]}$, $\mathbf{1}_{[0y1]}$, $\mathbf{1}_{[1y0]}$ and $\mathbf{1}_{[1y1]}$ the result follows. \square

Note that if x and y have the same length, but they are different, then $\int \hat{a}_x^2 \hat{a}_y d\mu = 0$.

From Proposition 7.4 it follows:

Corollary 7.6. *Assume that x is not a subprefix of z and z is not a subprefix of x . Then, we get that the products (part of the first sum contribution in (38)) satisfy*

$$(35) \quad \int \hat{a}_x \hat{a}_z \hat{a}_y d\mu = 0,$$

for all word y .

Remember that \mathcal{F} (defined in last section) is the set of all functions of the form

$$(36) \quad \hat{a}_x = \frac{1}{\sqrt{\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}}} a_x,$$

where $x = (x_1, x_2, \dots, x_k)$ is a general finite word, plus the functions $\hat{a}_{[\emptyset]}^0$ and $\hat{a}_{[\emptyset]}^1$.

The main result of last section claims that:

Theorem 7.7. *The family of functions \mathcal{F} determine an orthonormal basis for the Kernel of the Ruelle operator*

We want to estimate for $X = \hat{a}_x$, $Y = \hat{a}_z \in \mathcal{F}$ and the orthogonal basis $X_i = \hat{a}_y \in \mathcal{F}$ the explicit expression of the curvature which was described in Theorem 1.1

$$(37) \quad K(X, Y) = \frac{1}{4} \left[\sum_{i=1}^{\infty} \left(\int X Y X_i d\mu \right)^2 - \sum_{i=1}^{\infty} \int X^2 X_i d\mu \int Y^2 X_i d\mu \right].$$

We will not present the explicit expression of the sectional curvature $K(X, Y)$ for any pair of vectors X, Y in the kernel, but just for the case where the functions X, Y are part of the family $\hat{a}_x \in \mathcal{F}$.

An important issue is: $0 = \langle \hat{a}_z^2, \hat{a}_y \rangle = \int \hat{a}_z^2 \hat{a}_y d\mu$, when length of y is strictly larger than the length of z (see Sections 8 and 9).

Our main result in this section concerns the Markov case:

Theorem 7.8. *For a fixed pair $\hat{a}_x, \hat{a}_z \in \mathcal{F}$ (with z different from x) the value*

$$(38) \quad K(\hat{a}_z, \hat{a}_x) = \frac{1}{4} \left[\sum_{\text{word } y} \left(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu \right)^2 - \sum_{\text{word } y} \int \hat{a}_z^2 \hat{a}_y d\mu \int \hat{a}_x^2 \hat{a}_y d\mu \right].$$

In the case the length of x is strictly larger than the length of z we get that (38) can be expressed in a more simplified form as:

$$(39) \quad \frac{1}{4} \left[\left(\int \hat{a}_x^2 \hat{a}_z d\mu \right)^2 - \sum_{y \in D[x,z]} \int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu \right].$$

The sum (38) is finite. The value $\int \hat{a}_x^2 \hat{a}_z d\mu$ is zero if z is not a subprefix of x . If z is a strict subprefix of x and y is a strict sub prefix of z , then the term

$$(40) \quad - \int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$$

is non positive. Moreover, by (69) we get $\int \hat{a}_z^2 \hat{a}_x d\mu = 0$.

The proof of this result will take several sections and subsections. Proposition 7.13 will summarize several explicit computations justifying (38) and (40).

We will also provide an explicit expression for the curvature (39) in terms of the words x, z and the probability μ (which is indexed by (r, s) of expression (14)). This will follow from explicit expressions for $(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu)^2$, $\int \hat{a}_z^2 \hat{a}_y d\mu$ and $\int \hat{a}_x^2 \hat{a}_y d\mu$, for all finite words x, z, y , that will be presented the Propositions 7.10 and 7.13 (which will be proved in sections 8 and 9).

It will also follow that when x and z do not share a common subprefix y , then the curvature $K(\hat{a}_z, \hat{a}_x)$ is equal to 0 (see Proposition 7.11).

There are examples (for instance, the case $x = (0, 1, 0)$ and $z = (0, 1, 0, 0)$) where the curvature $K(\hat{a}_z, \hat{a}_x)$ is positive for some values of the parameters (r, s) and negative for others (see Example 7.20). We can show from the explicit expressions we obtain that for fixed values of the parameters (r, s) the curvature $K(\hat{a}_z, \hat{a}_x)$ can be very negative if both words x, z have large lengths and share common prefix with large length (see Remark 7.18). In Example 7.21 we show that $K(\hat{a}_{(0)}, \hat{a}_{(0,0)}) = -0.205714\dots$, when $r = 0.1, s = 0.3$. In Proposition 7.19 we show the curvature $K(\hat{a}_{[\emptyset]}^0, \hat{a}_0)$ can be positive for some pairs $r, s \in (0, 1)$. It follows from the expressions of Proposition 7.13 that all sectional curvatures $K(\hat{a}_z, \hat{a}_x)$ are equal to $-1/2$, when $r = 1/2 = s$, the size of z is bigger than 1 and z is a strict subprefix of x . See also Proposition 7.19, when $r = 1/2 = s$, for the computation of $K(\hat{a}_{[\emptyset]}^0, \hat{a}_0) = 1/2$.

Remark 7.9. Expression (67) in Subsection 8.3 shows that in the case the length of x is larger than the length of z , then $(\int \hat{a}_z^2 \hat{a}_x d\mu)^2 = 0$.

Proposition 7.10. *Assume that the length of x is larger than the length of z . The first sum on expression (38) is given by*

$$(41) \quad \sum_{\text{word } y} \left(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu \right)^2 = \left(\int \hat{a}_x^2 \hat{a}_z d\mu \right)^2 + \left(\int \hat{a}_z^2 \hat{a}_x d\mu \right)^2 = \left(\int \hat{a}_x^2 \hat{a}_z d\mu \right)^2.$$

For a proof of this claim see expression (72) in section 9. This term in the sum (38) is the part that contributes to the curvature to be more positive. The second term in the sum (38) will contribute to the curvature becoming more negative (see proposition 7.13).

Note that (41) does not depend on y . Note also that from expression (15) one can get explicitly the values (41) as a function of (r, s) .

In Proposition 7.4 we show that if x is not a subprefix of z and z is not a subprefix of x , we get that $\int \hat{a}_x^2 \hat{a}_z d\mu = 0$. In this case the contribution of (41) for the curvature will be null.

Proposition 7.11. *When z and x do not share common prefix the curvature*

$$K(\hat{a}_z, \hat{a}_x) = 0.$$

Proof. When z and x do not share a common prefix, it follows that x is not a subprefix of z and z is not a subprefix of x .

We will show that in this case $K(\hat{a}_z, \hat{a}_x) = 0$. Indeed, from Proposition 7.4 we get that $(\int \hat{a}_x^2 \hat{a}_z d\mu)^2 + (\int \hat{a}_z^2 \hat{a}_x d\mu)^2 = 0$. Fix the words z, x and consider a variable word y . In order to estimate the second sum in expression (39) we have to consider all different possible words y such that are subprefix of x and z . But there is no such kind of y .

Therefore, $K(\hat{a}_z, \hat{a}_x) = 0$.

See also Proposition 7.19, when $r = 1/2 = s$, for the computation of other sectional curvatures. \square

Remark 7.12. It follows from Remark 7.9 that $\sum_{\text{word } y} \int \hat{a}_z^2 \hat{a}_y d\mu \int \hat{a}_x^2 \hat{a}_y d\mu$ is a sum of a finite numbers of terms. This is so because in the estimation of $\int \hat{a}_z^2 \hat{a}_y d\mu \int \hat{a}_x^2 \hat{a}_y d\mu$ we do not have to take into account words y with length strictly larger than the minimum of the lengths of x and z . It also follows from Proposition 7.11 that if x is not a subprefix of y and y is not a subprefix of x , we get that $\int \hat{a}_x^2 \hat{a}_y d\mu = 0$.

Note that the above makes clear that in expression (38), the second sum has nonzero terms only when $y \in D[x, z]$. This justifies the simplified expression (39).

With all this in mind, in order to have explicit expressions, the next proposition deals just with the words y with lengths smaller than the length of a given word x .

Proposition 7.13. *Assume that the length of x is larger or equal to the length of y . Then we have:*

a) $\int \hat{a}_x^2 \hat{a}_y d\mu = 0$, if y is not a subprefix of x . This also includes the case where $x \neq y$ and length of x is equal to the length of y .

b.0) Assume that $[x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [y] = [x_1, x_2, \dots, x_k]$, where $n > k$, and $x_{k+1} = 0$. Note that from (25) we get that $|a_x| = |a_y|$. Then,

$$\int \hat{a}_x^2 \hat{a}_y d\mu = \frac{1}{|a_y|^3} \frac{\sqrt{P_{x_k,1}}}{\sqrt{P_{x_k,0}}} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y])}} - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y])}} \right\} =$$

$$(42) \quad \frac{1}{|a_x|^3} \sqrt{P_{x_k,1}} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y0])}} - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y0])}} \right\}.$$

b.1) Assume, that $[x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [y] = [x_1, x_2, \dots, x_k]$, where $n > k$, and $x_{k+1} = 1$. Then,

$$(43) \quad \int \hat{a}_x^2 \hat{a}_y d\mu = \frac{1}{|a_y|^3} \sqrt{P_{x_k,0}} \left\{ - \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y1])}} + \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y1])}} \right\} = \frac{1}{|a_x|^3} \sqrt{P_{x_k,0}} \left\{ - \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y1])}} + \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y1])}} \right\}$$

b.2) Assume, $[x] = [x_1, x_2, \dots, x_n] = [y]$.
Then,

$$(44) \quad \int \hat{a}_x^2 \hat{a}_y d\mu = \int \hat{a}_y^3 d\mu = \frac{1}{|a_x|^3} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\frac{P_{x_n,1}^{3/2}}{\sqrt{\mu([0y0])}} - \frac{P_{x_n,0}^{3/2}}{\sqrt{\mu([0y1])}} \right] - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\frac{P_{x_n,1}^{3/2}}{\sqrt{\mu([1y0])}} - \frac{P_{x_n,0}^{3/2}}{\sqrt{\mu([1y1])}} \right] \right\}.$$

b3) If $x_1 = 0$, then

$$(45) \quad \int \hat{a}_x^2 \hat{a}_{[\emptyset]}^0 d\mu = \frac{1}{|a_x|^2 |V_1|} \left(\frac{\pi_1 P_{1,0}}{P_{0,0}} + \frac{\pi_0^2 P_{0,0}}{\pi_1 P_{1,0}} \right) = \frac{(s-1)(1-2r+2r^2)}{\sqrt{\frac{(1-r)r(s-1)^3}{(-2+r+s)^3}} (-2+r+s)} > 0,$$

and

$$\int \hat{a}_x^2 \hat{a}_{[\emptyset]}^1 d\mu = 0.$$

When $r = 1/2 = s$ we get that for any word x (with size bigger or equal to 1), such that, $x_1 = 0$

$$(46) \quad \int \hat{a}_x^2 \hat{a}_{[\emptyset]}^0 d\mu = \sqrt{2}.$$

For the proof of this proposition see sections 8.1 and 8.2.

Remark 7.14.

- We point out that (42) and (43) do not depend on $x_{k+2}, \dots, x_{n-1}, x_n$.
- If $y \in D[x, z] - \{z\}$, then, the product $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$ is non negative for any choice of (p, q) (the product will not depend on x and z). This follows from the expressions in b.0) and b.1). This shows (40).
- The term $\int \hat{a}_x^2 \hat{a}_z d\mu \int \hat{a}_z^2 \hat{a}_z d\mu$ may be sometimes negative.

We previously denoted by M the two dimensional manifold of Markov invariant probabilities (the set of equilibrium probabilities for potentials depending on two coordinates and parametrized by r, s)

Given the Markov invariant probability μ associated to the parameters r, s , the set of vectors which are tangent to M at this point is the set of functions that depend on two coordinates (x_1, x_2) . The ones that are on the kernel of $\mathcal{L}_{\log J}$ are $\hat{a}_{[\emptyset]}^0$ and $\hat{a}_{[\emptyset]}^1$.

Theorem 7.15. *Given the two-dimensional manifold of Markov invariant probabilities M , for any point in M the sectional curvature for the pair of tangent vectors to M is always zero.*

Proof. Remember that $V_1 = \pi_1 P_{1,0} \mathbf{1}_{[00]} - \pi_0 P_{0,0} \mathbf{1}_{[10]}$ and $V_2 = \pi_0 P_{0,1} \mathbf{1}_{[11]} - \pi_1 P_{1,1} \mathbf{1}_{[01]}$ determine an orthogonal basis for the tangent space to M at μ .

We claim that the curvature $K(\hat{a}_{[\emptyset]}^0, \hat{a}_{[\emptyset]}^1) = 0$.

Indeed, take $X_i = \hat{a}_y$, for some finite word $z = (x_1, x_2, \dots, x_k)$. If we assume that $x_1 = 1$, then,

$$\begin{aligned} V_1^2 a_y &= \left[\pi_1^2 P_{1,0}^2 \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \frac{1}{\mu([0z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right)^{1/2} \mathbf{1}_{[0z0]} \mathbf{1}_{[00]} \right. \\ &\quad - \pi_1^2 P_{1,0}^2 \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \frac{1}{\mu([0z])} \frac{P_{x_k,0}}{P_{x_k,1}} \right)^{1/2} \mathbf{1}_{[0z1]} \mathbf{1}_{[00]} \left. \right] \\ &\quad - \left[\pi_0^2 P_{0,0}^2 \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \frac{1}{\mu([1z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right)^{1/2} \mathbf{1}_{[1z0]} \mathbf{1}_{[10]} \right. \\ &\quad \left. - \pi_0^2 P_{0,0}^2 \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \frac{1}{\mu([1z])} \frac{P_{x_k,0}}{P_{x_k,1}} \right)^{1/2} \mathbf{1}_{[1z1]} \mathbf{1}_{[00]} \right] = 0. \end{aligned}$$

From the above it follows that:

$$\int \hat{V}_1^2 \hat{a}_y d\mu - \int \hat{V}_2^2 \hat{a}_y d\mu = 0.$$

If we assume that $x_1 = 0$, then, in a similar way $\int \hat{a}_z V_2^2 d\mu = 0$, and therefore, $\int \hat{a}_z \hat{V}_1^2 d\mu - \int \hat{a}_z \hat{V}_2^2 d\mu = 0$.

Note that $V_1 V_2 = 0$.

Then, we get that for any word y we have that $\int \hat{V}_1 \hat{V}_2 \hat{a}_y d\mu = 0$. In the same way $\int \hat{V}_1^2 \hat{V}_2 d\mu = 0$ and $\int \hat{V}_2^2 \hat{V}_1 d\mu = 0$. Therefore,

$$(47) \quad \frac{1}{4} \left[\left(\int (\hat{a}_{[\emptyset]}^0)^2 \hat{a}_{[\emptyset]}^1 d\mu \right)^2 - \sum_y \left(\int (\hat{a}_{[\emptyset]}^0)^2 \hat{a}_y d\mu \right) \left(\int (\hat{a}_{[\emptyset]}^1)^2 \hat{a}_y d\mu \right) \right] = 0.$$

□

Remark 7.16. : Recall that the expression of the Gauss sectional curvature $K_M(X, Y)$ of an isometric immersion (M, g_M) , submanifold of the Riemannian manifold (N, g) , at the plane generated by two orthogonal vector fields X, Y tangent to M , is given by

$$K_M(X, Y) = K(X, Y) + \langle \nabla_X^\perp X, \nabla_Y^\perp Y \rangle - \|\nabla_X^\perp Y\|^2$$

according to Gauss formula (see for instance [9]). Here, the operator $\nabla_{\bar{X}}^\perp Y$ is the component of the covariant derivative $\nabla_X Y$ of the Riemannian manifold (N, g) that is normal to (M, g_M) .

Notice that the sectional curvature

$$K(X, Y) = \|\nabla_{\bar{Y}} \bar{X}\|^2 - \langle \nabla_{\bar{X}} \bar{X}, \nabla_{\bar{Y}} \bar{Y} \rangle$$

includes all the terms of the normal component of the covariant derivative of X, Y . By Theorem 7.15, all the components of the covariant derivative of a certain pair of orthogonal vector fields tangent to the surface of Markov probabilities vanish. In particular, all the terms of the normal covariant derivative of X, Y vanish. Therefore, Theorem 7.15 yields that the Gaussian curvature of the surface of Markov probabilities vanishes, its intrinsic curvature as an isometric immersion of the manifold of normalized potentials is zero. This is a remarkable fact, which implies for instance that the surface would be totally geodesic in the manifold of normalized potentials provided that geodesics exist. We won't consider the problem of the existence of geodesics in this article, we shall study this problem in further papers.

Proposition 7.17. *When $r = 0.5$ and $s = 0.5$, we get that*

$$(48) \quad K(\hat{a}_y, \hat{a}_x) = -1/2,$$

for words x, y with size bigger or equal to 1

Proof. It follows from the above proposition that due to symmetry, when $r = 0.5$ and $s = 0.5$, we get $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu = 0$, for words with size bigger or equal to 1. Moreover, $\int \hat{a}_x^2 w d\mu = 0$, for any a_x . In this case, if $x_1 = 0$ and x is a subprefix of y , we get that for words with size bigger or equal to 1 (see (46)),

$$(49) \quad K(\hat{a}_y, \hat{a}_x) = -1/4 \int \hat{a}_x^2 \hat{a}_{[\emptyset]}^0 d\mu \int \hat{a}_y^2 \hat{a}_{[\emptyset]}^0 d\mu = -1/2.$$

□

Remark 7.18. From the explicit expressions we obtain (for fixed values of the parameters (r, s)) the curvature $K(\hat{a}_z, \hat{a}_x)$ can be very negative if both words x, z have large lengths and have common prefix y with large length. Indeed, for fixed \hat{a}_z, \hat{a}_x , as $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$ is non negative for any common word y , in the calculus of the curvature $K(\hat{a}_z, \hat{a}_x)$, we get a sum of several expressions $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$. Note that $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$ does depend on y (but not on x and z). Note also that for fixed x the expression (42) can be very large if the length of y is very large (and, so $\mu([ayb])$, $a, b = 0, 1$, is very small).

Proposition 7.19. *The curvature $K(\hat{a}_{[\emptyset]}^0, \hat{a}_0) = 1/2$, when $r = 1/2 = s$.*

Proof. Note that (39) can be expressed as

$$K(\hat{a}_{[\emptyset]}^0, \hat{a}_0) = \frac{1}{4} \left[\left(\int \hat{a}_0^2 \hat{a}_{[\emptyset]}^0 d\mu \right)^2 - \int \hat{a}_0^2 \hat{a}_{[\emptyset]}^0 d\mu \int (\hat{a}_{[\emptyset]}^0)^2 \hat{a}_{[\emptyset]}^0 d\mu \right].$$

For any r, s , it is known from (45) that

$$(50) \quad \int \hat{a}_0^2 \hat{a}_{[\emptyset]}^0 d\mu = \frac{1}{|a_0|^2 |V_1|} \left(\frac{\pi_1 P_{1,0}}{P_{0,0}} + \frac{\pi_0^2 P_{0,0}}{\pi_1 P_{1,0}} \right) > 0.$$

Note that

$$V_1^2 V_1 =$$

$$(\pi_1^2 P_{1,0}^2 \mathbf{1}_{[00]} + \pi_0^2 P_{0,0}^2 \mathbf{1}_{[10]}) \times (\pi_1 P_{1,0} \mathbf{1}_{[00]} - \pi_0 P_{0,0} \mathbf{1}_{[10]}) = \\ \pi_1^3 P_{1,0}^3 \mathbf{1}_{[00]} - \pi_0^3 P_{0,0}^3 \mathbf{1}_{[10]}.$$

Then,

$$\int V_1^2 V_1 = \pi_1^3 P_{1,0}^3 \mu([00]) - \pi_0^3 P_{0,0}^3 \mu([10]),$$

which is equal to $\frac{1}{2} \frac{6}{2} \frac{1}{2}^2 - \frac{1}{2} \frac{6}{2} \frac{1}{2}^2 = 0$, in the case $r = 1/2 = s$.

Therefore,

$$K(\hat{a}_{[\emptyset]}^0, \hat{a}_0) = \frac{1}{4} \left(\int \hat{a}_0^2 \hat{a}_{[\emptyset]}^0 d\mu \right)^2 = \frac{1}{4} \sqrt{2}^2 = 1/2 > 0.$$

□

In other examples, we used the software Mathematica for getting explicit computations.

Example 7.20. Consider the case where $z = (0, 1, 0)$ and $x = (0, 1, 0, 0)$.

$\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu = 0$, unless \hat{a}_y is such that

$$y \in D[(0, 1, 0), (0, 1, 0, 0)] = \{(0), (0, 1), (0, 1, 0)\},$$

or $\hat{a}_y \in \{\hat{a}_{[\emptyset]}^0, \hat{a}_{[\emptyset]}^1\}$.

Using Mathematica and the formulas of Proposition 7.13, we made computations when $r = 0.1$ and $s = 0.3$. In this case $\pi_0 = 0.4375$ and $\pi_1 = 0.5625$ and from (28) we get $|a_{(0,1,0)}| = |a_{(0,1,0,0)}| = 3.33..$ and $|V_1| = 0.086..$ Finally, $\frac{1}{|a_x|^2 |a_z|} = \frac{1}{|a_z|^3} = \frac{1}{|a_x|^2 |a_y|} = 0.027..$

We will show that $K(\hat{a}_{(0,1,0)}, \hat{a}_{(0,1,0,0)}) = 35.9142..$

We get the following values:

$$\text{using (44)} \quad \int \hat{a}_z^2 \hat{a}_z d\mu = \frac{1}{|a_z|^3} \int a_{(0,1,0)}^2 a_{(0,1,0)} d\mu = 107,51..,$$

$$\text{using (42)} \quad \int \hat{a}_x^2 \hat{a}_z d\mu = \frac{1}{|a_x|^2 |a_z|} \int a_{(0,1,0,0)}^2 a_{(0,1,0)} d\mu = 120.949..,$$

$$\left(\int \hat{a}_x^2 \hat{a}_z d\mu \right)^2 = \left(\int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0,1,0)} d\mu \right)^2 = (16.93..)^2 = 14628.7..,$$

$$\text{using (42)} \quad \int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0,1)} d\mu = 38.2473..,$$

$$\text{using (42)} \quad \int \hat{a}_{(0,1,0)}^2 \hat{a}_{(0,1)} d\mu = 38.2473..$$

$$\text{using (43)} \quad \int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0)} d\mu = -1.34387..$$

$$\text{using (43)} \quad \int \hat{a}_{(0,1,0)}^2 \hat{a}_{(0)} d\mu = -1.34387..,$$

and finally, using (45)

$$\int \hat{a}_{(0,1,0)}^2 \hat{a}_{[\emptyset]}^0 d\mu = \int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{[\emptyset]}^0 d\mu = \frac{1}{|a_{(0,1,0)}|^2 |V_1|} \int a_{(0,1,0)}^2 V_1 d\mu = 4.13241..$$

Using (45) and (30) (note that $x_1 = 0$), we get that the expression (39) can be written in this case as

$$\begin{aligned}
K(\hat{a}_{(0,1,0)}, \hat{a}_{(0,1,0,0)}) &= \frac{1}{4} \left(\int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0,1,0)} d\mu \right)^2 \\
&- \frac{1}{4} \left[\int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0,1,0)} d\mu \int \hat{a}_{(0,1,0)}^2 \hat{a}_{(0,1,0)} d\mu \right] \\
&- \frac{1}{4} \left[\int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0,1)} d\mu \int \hat{a}_{(0,1,0)}^2 \hat{a}_{(0,1)} d\mu \right] \\
&- \frac{1}{4} \left[\int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{(0)} d\mu \int \hat{a}_{(0,1,0)}^2 \hat{a}_{(0)} d\mu \right] \\
&- \frac{1}{4} \int \hat{a}_{(0,1,0,0)}^2 \hat{a}_{[\emptyset]}^0 d\mu \int \hat{a}_{(0,1,0)}^2 \hat{a}_{[\emptyset]}^0 d\mu = 35.9142...
\end{aligned}$$

Taking $r = 0.8, s = 0.5$ we get $K(\hat{a}_{(0,1,0)}, \hat{a}_{(0,1,0,0)}) = -3.17713...$ When $r = 1/2 = s$ we get $K(\hat{a}_{(0,1,0)}, \hat{a}_{(0,1,0,0)}) = -1/2$.

◇

Example 7.21. Consider the case where $z = (0)$ and $x = (0, 0)$. Then, $D[(0), (0, 0)] = \{\hat{a}_0, \hat{a}_{[\emptyset]}^0\}$. Therefore,

$$\begin{aligned}
K(\hat{a}_{(0)}, \hat{a}_{(0,0)}) &= \frac{1}{4} \left(\int \hat{a}_{(0,0)}^2 \hat{a}_{(0)} d\mu \right)^2 \\
&- \frac{1}{4} \left[\int \hat{a}_{(0,0)}^2 \hat{a}_{(0)} d\mu \int \hat{a}_{(0)}^2 \hat{a}_{(0)} d\mu \right] \\
&- \frac{1}{4} \int \hat{a}_{(0,0)}^2 \hat{a}_{[\emptyset]}^0 d\mu \int \hat{a}_{(0)}^2 \hat{a}_{[\emptyset]}^0 d\mu.
\end{aligned}$$

In this case, using Mathematica, one can show that $K(\hat{a}_{(0)}, \hat{a}_{(0,0)}) \leq 0$, for all values $r, s \in (0, 1)$. For $r = 0.1, s = 0.3$, we will show that $K(\hat{a}_{(0)}, \hat{a}_{(0,0)}) = -0.205714...$

When, $r = 0.1, s = 0.3$, we get

$$|a_0| = 3.333...,$$

$$|V_1| = 0.086...,$$

$$\int \hat{a}_{(0,0)}^2 \hat{a}_{(0)} d\mu = \frac{1}{|a_{(0)}|^3}$$

$$\int \hat{a}_{(0)}^2 \hat{a}_{(0)} d\mu = \frac{1}{|a_{(0)}|^3}$$

$$\int \hat{a}_{(0,0)}^2 \hat{a}_{[\emptyset]}^0 d\mu = \frac{1}{|a_{(0)}|^2 |V_1|} \int a_{(0)}^2 V_1 d\mu = \frac{1}{|a_{(0)}|^2 |V_1|} 3.96.$$

Finally, when $r = 0.1, s = 0.3$ we get $K(\hat{a}_{(0)}, \hat{a}_{(0,0)}) = -0.205714...$

◇

8. COMPUTATIONS FOR THE INTEGRAL $\int X^2Y$

Our purpose in this section is to evaluate the integral

$$(51) \quad \sum_{\text{word } y} \int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu,$$

for any given pair of words x, z . This corresponds to the second term in the sum given by expression (38).

We assume that x is different from z .

From proposition 7.5 if x is not a subprefix of y and y is not a subprefix of x , and $x \neq y$, then:

$$\hat{a}_x^2 \hat{a}_y = 0.$$

In the same way, if z is not a subprefix of y and y is not a subprefix of z , and $z \neq y$, then:

$$\hat{a}_z^2 \hat{a}_y = 0.$$

If y has the same length as x but $y \neq x$, then $\hat{a}_z^2 \hat{a}_y = 0$.

In this way, several words y do not contribute to the above sum.

8.1. The value of $\langle \hat{a}_x^2, \hat{a}_y \rangle$ when length of x is larger or equal than the length of y . We want to compute $\langle \hat{a}_x^2, \hat{a}_y \rangle = \int \hat{a}_x^2 \hat{a}_y d\mu$ in the case where the length of x is larger or equal to the length of z .

Our computation is in fact for $\langle a_x^2, a_y \rangle$ and after that, of course, to get $\langle \hat{a}_x^2, \hat{a}_y \rangle$ it will be necessary to divide by $|a_x|^2 |a_y|$.

We assume that $[x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [y] = [x_1, x_2, \dots, x_k]$, where $n \geq k$ (otherwise we get zero).

Note that these assumptions include the integral $\int \hat{a}_x^3 d\mu$, that is, the case $x = y$ (see III) below).

I) Case $n > k$ - We will assume first that $x_{k+1} = 0$ in the word $[x]$.

Given the words $z = (v_1, \dots, v_t)$ and $v = (v_1, v_2, \dots, v_t, v_{t+1}, \dots, v_m)$, assume $v_{t+1} = 0$, then, from (17) and (23)

$$(52) \quad \begin{aligned} e_{[v]}^2 e_{[z]} &= \left[\frac{1}{\mu([v])} \frac{P_{v_m,1}}{P_{v_m,0}} \mathbf{1}_{[v_1, \dots, v_t, 0, v_{t+2}, \dots, v_m, 0]} + \frac{1}{\mu([v])} \frac{P_{v_m,0}}{P_{v_m,1}} \mathbf{1}_{[v_1, \dots, v_t, 0, v_{t+2}, \dots, v_m, 1]} \right] \\ &\quad \times \left[\frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,1}}{P_{v_t,0}}} \mathbf{1}_{[v_1, \dots, v_t, 0]} - \frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,0}}{P_{v_t,1}}} \mathbf{1}_{[v_1, \dots, v_t, 1]} \right] \\ &= \left(\frac{1}{\mu([v])} \frac{P_{v_m,1}}{P_{v_m,0}} \right) \left(\frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,1}}{P_{v_t,0}}} \mathbf{1}_{[v_1, \dots, 0, v_{t+2}, \dots, v_m, 0]} \right. \\ &\quad \left. + \left(\frac{1}{\mu([v])} \frac{P_{v_m,0}}{P_{v_m,1}} \right) \left(\frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,1}}{P_{v_t,0}}} \mathbf{1}_{[v_1, \dots, 0, v_{t+2}, \dots, v_m, 1]} \right) \right. \end{aligned}$$

Note that in the above reasoning when going from the second to the third line the term multiplying $\mathbf{1}_{[v_1, \dots, v_t, 1]}$ disappear because we assume that $v_{t+1} = 0$.

We are going to apply the above when $z = [0y]$, $z = [1y]$, $v = [0x]$, $v = [1x]$, $m = n$ and $t + 1 = k$.

Then, from (20), (24), (55) and using the fact that

$$\begin{aligned} e_{[0, x_1, x_2, \dots, x_k, 0, x_{k+2}, \dots, x_n]}^2 e_{[1, x_1, x_2, \dots, x_k]} &= 0, \\ e_{[1, x_1, x_2, \dots, x_k, 0, x_{k+2}, \dots, x_n]}^2 e_{[0, x_1, x_2, \dots, x_k]} &= 0, \end{aligned}$$

we get

$$\begin{aligned}
a_x^2 a_y &= \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} e_{[0,x_1,x_2,\dots,x_k,0,x_{k+2},\dots,x_n]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} e_{[1,x_1,x_2,\dots,x_k,0,x_{k+2},\dots,x_n]}^2 \right] \\
&\quad \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_k]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_k]} \right] \\
&= \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([0x])} \frac{P_{x_n,1}}{P_{x_n,0}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) \mathbf{1}_{[0x0]} \right. \\
&\quad \left. + \left(\frac{1}{\mu([0x])} \frac{P_{x_n,0}}{P_{x_n,1}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) \mathbf{1}_{[0x1]} \right] \\
&\quad - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([1x])} \frac{P_{x_n,1}}{P_{x_n,0}} \right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) \mathbf{1}_{[1x0]} \right. \\
&\quad \left. + \left(\frac{1}{\mu([1x])} \frac{P_{x_n,0}}{P_{x_n,1}} \right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) \mathbf{1}_{[1x1]} \right].
\end{aligned}$$

Finally, as the matrix P is row stochastic

$$\begin{aligned}
\int a_x^2 a_y d\mu &= \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\left(\frac{P_{x_n,1}}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) + \left(\frac{P_{x_n,0}}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) \right] \\
&\quad - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\left(\frac{P_{x_n,1}}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) + \left(\frac{P_{x_n,0}}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k,1}}{P_{x_k,0}}} \right) \right] \\
&= (P_{x_n,1} + P_{x_n,0}) \sqrt{P_{x_k,1}} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\frac{1}{\sqrt{\mu([0y0])}} \right] \right. \\
&\quad \left. - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\frac{1}{\sqrt{\mu([1y0])}} \right] \right\} = \\
(53) \quad &\sqrt{P_{x_k,1}} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y0])}} - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y0])}} \right\}.
\end{aligned}$$

II) Case $n > k$ - If we assume $x_{k+1} = 1$ in the word $[x]$, then we get in a similar way as before

$$\begin{aligned}
&\int a_x^2 a_y d\mu = \\
(54) \quad &\sqrt{P_{x_k,0}} \left\{ - \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y1])}} + \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y1])}} \right\}
\end{aligned}$$

Indeed, given the words $z = (v_1, \dots, v_t)$ and $v = (v_1, v_2, \dots, v_t, v_{t+1}, \dots, v_m)$, assume $v_{t+1} = 1$, then, from (17) and (23)

$$\begin{aligned}
e_{[v]}^2 e_{[z]} &= \left[\frac{1}{\mu([v])} \frac{P_{v_m,1}}{P_{v_m,0}} \mathbf{1}_{[v_1,\dots,v_t,1,v_{t+2},\dots,v_m,0]} + \frac{1}{\mu([v])} \frac{P_{v_m,0}}{P_{v_m,1}} \mathbf{1}_{[v_1,\dots,v_t,1,v_{t+2},\dots,v_m,1]} \right] \\
&\quad \times \left[\frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,1}}{P_{v_t,0}}} \mathbf{1}_{[v_1,\dots,v_t,0]} - \frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,0}}{P_{v_t,1}}} \mathbf{1}_{[v_1,\dots,v_t,1]} \right] \\
&= - \left[\left(\frac{1}{\mu([v])} \frac{P_{v_m,1}}{P_{v_m,0}} \right) \left(\frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,0}}{P_{v_t,1}}} \right) \mathbf{1}_{[v_1,\dots,1,v_{t+2},\dots,v_m,0]} \right]
\end{aligned}$$

$$(55) \quad + \left(\frac{1}{\mu([v])} \frac{P_{v_m,0}}{P_{v_m,1}} \right) \left(\frac{1}{\sqrt{\mu([z])}} \sqrt{\frac{P_{v_t,0}}{P_{v_t,1}}} \right) \mathbf{1}_{[v_1, \dots, 1, v_{t+2}, \dots, v_m, 1]}.$$

We are going to apply the above when $z = [0y]$, $z = [1y]$, $v = [0x]$, $v = [1x]$, $m = n$ and $t = k + 1$.

Then, from (20), (24), (55) and using the fact that

$$\begin{aligned} e_{[1, x_1, x_2, \dots, x_k, 1, x_{k+2}, \dots, x_n]}^2 e_{[0, x_1, x_2, \dots, x_k]} &= 0, \\ e_{[0, x_1, x_2, \dots, x_k, 1, x_{k+2}, \dots, x_n]}^2 e_{[1, x_1, x_2, \dots, x_k]} &= 0, \end{aligned}$$

we get

$$\begin{aligned} a_x^2 a_y &= \left[\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} e_{[0, x_1, x_2, \dots, x_k, 1, x_{k+2}, \dots, x_n]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} e_{[1, x_1, x_2, \dots, x_k, 1, x_{k+2}, \dots, x_n]}^2 \right] \\ &\quad \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0, x_1}}} e_{[0, x_1, x_2, \dots, x_k]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1, x_1}}} e_{[1, x_1, x_2, \dots, x_k]} \right] \\ &= - \left(\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([0x])} \frac{P_{x_n, 1}}{P_{x_n, 0}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) \mathbf{1}_{[0x0]} \right. \\ &\quad \left. + \left(\frac{1}{\mu([0x])} \frac{P_{x_n, 0}}{P_{x_n, 1}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) \mathbf{1}_{[0x1]} \right] \\ &\quad + \left(\frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([1x])} \frac{P_{x_n, 1}}{P_{x_n, 0}} \right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) \mathbf{1}_{[1x0]} \right. \\ &\quad \left. + \left(\frac{1}{\mu([1x])} \frac{P_{x_n, 0}}{P_{x_n, 1}} \right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) \mathbf{1}_{[1x1]} \right]. \end{aligned}$$

Finally, as the matrix P is row stochastic

$$\begin{aligned} \int a_x^2 a_y d\mu &= - \left(\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \right)^{3/2} \left[\left(\frac{P_{x_n, 1}}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) + \left(\frac{P_{x_n, 0}}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) \right] \\ &\quad + \left(\frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \right)^{3/2} \left[\left(\frac{P_{x_n, 1}}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) + \left(\frac{P_{x_n, 0}}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_k, 0}}{P_{x_k, 1}}} \right) \right] = \\ &\quad \sqrt{P_{x_k, 0}} \left\{ - \left(\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \right)^{3/2} \left[\frac{P_{x_n, 1} + P_{x_n, 0}}{\sqrt{\mu([0y1])}} \right] + \left(\frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \right)^{3/2} \left[\frac{P_{x_n, 1} + P_{x_n, 0}}{\sqrt{\mu([1y1])}} \right] \right\} = \\ (56) \quad &\quad \sqrt{P_{x_k, 0}} \left\{ - \left(\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0y1])}} + \left(\frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1y1])}} \right\} \end{aligned}$$

III) Case $n = k$ - We assume $[x] = [x_1, x_2, \dots, x_n] = [y]$, otherwise $\int \hat{a}_x^2 \hat{a}_y d\mu = 0$. Then, one can show that

$$(57) \quad \int a_x^2 a_y d\mu = \int a_x^3 d\mu = \left(\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \right)^{3/2} \left[\frac{P_{x_n, 1}^{3/2}}{\sqrt{\mu([0x0])}} - \frac{P_{x_n, 0}^{3/2}}{\sqrt{\mu([0x1])}} \right] - \left(\frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \right)^{3/2} \left[\frac{P_{x_n, 1}^{3/2}}{\sqrt{\mu([1x0])}} - \frac{P_{x_n, 0}^{3/2}}{\sqrt{\mu([1x1])}} \right].$$

Indeed, note first that from (23), $v = [v_1, v_2, \dots, v_m]$

$$e_{[v]}^2 e_{[v]} = \left[\frac{1}{\mu([v])} \frac{P_{v_m, 1}}{P_{v_m, 0}} \mathbf{1}_{[v_1, \dots, v_m, 0]} + \frac{1}{\mu([v])} \frac{P_{v_m, 0}}{P_{v_m, 1}} \mathbf{1}_{[v_1, \dots, v_m, 1]} \right]$$

$$\begin{aligned}
& \times \left[\frac{1}{\sqrt{\mu([v])}} \sqrt{\frac{P_{v_m,1}}{P_{v_m,0}}} \mathbf{1}_{[v_1, \dots, v_m, 0]} - \frac{1}{\sqrt{\mu([v])}} \sqrt{\frac{P_{v_m,0}}{P_{v_m,1}}} \mathbf{1}_{[v_1, \dots, v_m, 1]} \right] \\
& = \left(\frac{1}{\mu([v])} \frac{P_{v_m,1}}{P_{v_m,0}} \right) \left(\frac{1}{\sqrt{\mu([v])}} \sqrt{\frac{P_{v_m,1}}{P_{v_m,0}}} \right) \mathbf{1}_{[v_1, \dots, v_m, 0]} \\
(58) \quad & - \left(\frac{1}{\mu([v])} \frac{P_{v_m,0}}{P_{v_m,1}} \right) \left(\frac{1}{\sqrt{\mu([v])}} \sqrt{\frac{P_{v_m,0}}{P_{v_m,1}}} \right) \mathbf{1}_{[v_1, \dots, v_m, 1]}.
\end{aligned}$$

Then, from (20), (24), (55)

$$\begin{aligned}
a_x^2 a_x & = \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} e_{[0,x_1,x_2,\dots,x_n]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} e_{[1,x_1,x_2,\dots,x_n]}^2 \right] \\
& \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_n]} \right] \\
& = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([0x])} \frac{P_{x_n,1}}{P_{x_n,0}} \right) \left(\frac{1}{\sqrt{\mu([0x])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[0x0]} \right. \\
& \quad \left. - \left(\frac{1}{\mu([0x])} \frac{P_{x_n,0}}{P_{x_n,1}} \right) \left(\frac{1}{\sqrt{\mu([0x])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \mathbf{1}_{[0x1]} \right] \\
& - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([1x])} \frac{P_{x_n,1}}{P_{x_n,0}} \right) \left(\frac{1}{\sqrt{\mu([1x])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[1x0]} \right. \\
& \quad \left. - \left(\frac{1}{\mu([1x])} \frac{P_{x_n,0}}{P_{x_n,1}} \right) \left(\frac{1}{\sqrt{\mu([1x])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \mathbf{1}_{[1x1]} \right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int a_x^2 a_x d\mu & = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\left(\frac{P_{x_n,1}}{\sqrt{\mu([0x])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) - \left(\frac{P_{x_n,0}}{\sqrt{\mu([0x])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \right] \\
& - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\left(\frac{P_{x_n,1}}{\sqrt{\mu([1x])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) - \left(\frac{P_{x_n,0}}{\sqrt{\mu([1x])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \right] = \\
(59) \quad & \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\frac{P_{x_n,1}^{3/2}}{\sqrt{\mu([0x0])}} - \frac{P_{x_n,0}^{3/2}}{\sqrt{\mu([0x1])}} \right] - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\frac{P_{x_n,1}^{3/2}}{\sqrt{\mu([1x0])}} - \frac{P_{x_n,0}^{3/2}}{\sqrt{\mu([1x1])}} \right] = \\
& \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\frac{P_{x_n,1}^{3/2}}{\sqrt{\mu([0x0])}} - \frac{P_{x_n,0}^{3/2}}{\sqrt{\mu([0x1])}} \right] - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\frac{P_{x_n,1}^{3/2}}{\sqrt{\mu([1x0])}} - \frac{P_{x_n,0}^{3/2}}{\sqrt{\mu([1x1])}} \right].
\end{aligned}$$

The above reasoning shows III).

Given the word $[x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n]$ we get n words y , such that, the cylinder $[x] \subset [y] = [x_1, x_2, \dots, x_k]$, where $n \geq k$.

Given x and z , with length larger than y , then $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$ will be nonzero only for the subprefixes y which are common to both x and z (see Proposition 7.5). If there are no common subprefixes for x and z , then the contribution $\int \hat{a}_x^2 \hat{a}_y d\mu \int \hat{a}_z^2 \hat{a}_y d\mu$, for words y of length strictly smaller than the length of x and z , in the sum (39) is null.

8.2. **The values of $\langle \hat{a}_x^2, \hat{a}_{[\emptyset]}^0 \rangle$ and $\langle \hat{a}_x^2, \hat{a}_{[\emptyset]}^1 \rangle$ when x is a finite word.** Denote $[x] = [x_1, x_2, \dots, x_n]$. We assume that $n \geq 2$.

In fact, we will compute $\langle a_x^2, V_1 \rangle$ and $\langle a_x^2, V_2 \rangle$. In order to compute $\langle \hat{a}_x^2, \hat{a}_{[\emptyset]}^0 \rangle$ and $\langle \hat{a}_x^2, \hat{a}_{[\emptyset]}^1 \rangle$ it will be necessary to normalize.

I) Case $\langle a_x^2, V_1 \rangle$

We will consider first the case $x_1 = 0$.

Denote $y = (y_1, y_2, \dots, y_k)$. If we assume $y_1 = 0, y_2 = 0$, then, from (23)

$$(60) \quad \begin{aligned} e_{[y]}^2 V_1 &= \left[\frac{1}{\mu([y])} \frac{P_{y_k,1}}{P_{y_k,0}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 0]} + \frac{1}{\mu([y])} \frac{P_{y_k,0}}{P_{y_k,1}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 1]} \right] \\ &\quad \times [\pi_1 P_{1,0} \mathbf{1}_{[0,0]} - \pi_0 P_{0,0} \mathbf{1}_{[1,0]}] = \\ &= \frac{1}{\mu([y])} \frac{P_{y_k,1}}{P_{y_k,0}} \pi_1 P_{1,0} \mathbf{1}_{[y_1, y_2, \dots, y_k, 0]} + \frac{1}{\mu([y])} \frac{P_{y_k,0}}{P_{y_k,1}} \pi_1 P_{1,0} \mathbf{1}_{[y_1, y_2, \dots, y_k, 1]}. \end{aligned}$$

If we assume $y_1 = 1, y_2 = 0$, then, from (23)

$$(61) \quad \begin{aligned} e_{[y]}^2 V_1 &= \left[\frac{1}{\mu([y])} \frac{P_{y_k,1}}{P_{y_k,0}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 0]} + \frac{1}{\mu([y])} \frac{P_{y_k,0}}{P_{y_k,1}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 1]} \right] \\ &\quad \times [\pi_1 P_{1,0} \mathbf{1}_{[0,0]} - \pi_0 P_{0,0} \mathbf{1}_{[1,0]}] = \\ &= - \left[\frac{1}{\mu([y])} \frac{P_{y_k,1}}{P_{y_k,0}} \pi_0 P_{0,0} \mathbf{1}_{[y_1, y_2, \dots, y_k, 0]} + \frac{1}{\mu([y])} \frac{P_{y_k,0}}{P_{y_k,1}} \pi_0 P_{0,0} \mathbf{1}_{[y_1, y_2, \dots, y_k, 1]} \right]. \end{aligned}$$

As we assume that $x_1 = 0$, then, from (20), (24), (55) we get

$$\begin{aligned} a_x^2 V_1 &= \left[\frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} e_{[0, x_1, x_2, \dots, x_k, 0, x_{k+2}, \dots, x_n]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} e_{[1, x_1, x_2, \dots, x_k, 0, x_{k+2}, \dots, x_n]}^2 \right] \\ &\quad \times [\pi_1 P_{1,0} \mathbf{1}_{[0,0]} - \pi_0 P_{0,0} \mathbf{1}_{[1,0]}] = \\ &= \frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \left[\frac{1}{\mu([0x])} \frac{P_{x_n,1}}{P_{x_n,0}} \pi_1 P_{1,0} \mathbf{1}_{[0, x_1, x_2, \dots, x_n, 0]} + \frac{1}{\mu([0x])} \frac{P_{x_n,0}}{P_{x_n,1}} \pi_1 P_{1,0} \mathbf{1}_{[0, x_1, x_2, \dots, x_n, 1]} \right] + \\ &= \frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \left[\frac{1}{\mu([1x])} \frac{P_{x_n,1}}{P_{x_n,0}} \pi_0 P_{0,0} \mathbf{1}_{[1, x_1, x_2, \dots, x_n, 0]} + \frac{1}{\mu([1x])} \frac{P_{x_n,0}}{P_{x_n,1}} \pi_0 P_{0,0} \mathbf{1}_{[1, x_1, x_2, \dots, x_n, 1]} \right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \int a_x^2 V_1 d\mu &= \\ &= \frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \left[\frac{1}{\mu([0x])} \frac{P_{x_n,1}}{P_{x_n,0}} \pi_1 P_{1,0} \mu[0, x, 0] + \frac{1}{\mu([0x])} \frac{P_{x_n,0}}{P_{x_n,1}} \pi_1 P_{1,0} \mu[0, x, 1] \right] + \\ &= \frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \left[\frac{1}{\mu([1x])} \frac{P_{x_n,1}}{P_{x_n,0}} \pi_0 P_{0,0} \mu[1, x, 0] + \frac{1}{\mu([1x])} \frac{P_{x_n,0}}{P_{x_n,1}} \pi_0 P_{0,0} \mu[1, x, 1] \right] = \\ &= \frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} [P_{x_n,1} \pi_1 P_{1,0} + P_{x_n,0} \pi_1 P_{1,0}] + \frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} [P_{x_n,1} \pi_0 P_{0,0} + P_{x_n,0} \pi_0 P_{0,0}] = \\ &= \frac{\pi_{x_1}}{\pi_0 P_{0, x_1}} \pi_1 P_{1,0} + \frac{\pi_{x_1}}{\pi_1 P_{1, x_1}} \pi_0 P_{0,0}. \end{aligned}$$

As we assumed that $x_1 = 0$ we get

$$(62) \quad \int \hat{a}_x^2 \hat{a}_{[\emptyset]}^0 d\mu = \frac{1}{|a_x|^2 |V_1|} \left(\frac{\pi_1 P_{1,0}}{P_{0,0}} + \frac{\pi_0^2 P_{0,0}}{\pi_1 P_{1,0}} \right).$$

II) $\langle a_x^2, V_2 \rangle$

Now we will compute $\int a_x^2 V_2 d\mu$.

Denote $y = (y_1, y_2, \dots, y_k)$. If we assume $y_1 = 0, y_2 = 0$, then, from (23)

$$e_{[y]}^2 V_2 = \left[\frac{1}{\mu([y])} \frac{P_{y_k,1}}{P_{y_k,0}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 0]} + \frac{1}{\mu([y])} \frac{P_{y_k,0}}{P_{y_k,1}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 1]} \right] \\ \times [\pi_0 P_{0,1} \mathbf{1}_{[1,1]} - \pi_1 P_{1,1} \mathbf{1}_{[0,1]}] = 0.$$

If we assume $y_1 = 1, y_2 = 0$, then, from (23)

$$e_{[y]}^2 V_2 = \left[\frac{1}{\mu([y])} \frac{P_{y_k,1}}{P_{y_k,0}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 0]} + \frac{1}{\mu([y])} \frac{P_{y_k,0}}{P_{y_k,1}} \mathbf{1}_{[y_1, y_2, \dots, y_k, 1]} \right] \\ \times [\pi_0 P_{0,1} \mathbf{1}_{[1,1]} - \pi_1 P_{1,1} \mathbf{1}_{[0,1]}] = 0.$$

As we assumed that $x_1 = 0$, then, from (20), (24), (55) we get

$$a_x^2 V_2 = \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} e_{[0, x_1, x_2, \dots, x_n]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} e_{[1, x_1, x_2, \dots, x_n]}^2 \right] \\ \times [\pi_0 P_{0,1} \mathbf{1}_{[1,1]} - \pi_1 P_{1,1} \mathbf{1}_{[0,1]}] = 0.$$

Therefore, if $x_1 = 0$ we get

$$(63) \quad \int \hat{a}_x^2 \hat{a}_{[\emptyset]}^1 d\mu = 0.$$

The case $x_1 = 1$ is left for the reader.

8.3. The value of $\langle \hat{a}_z^2, \hat{a}_y \rangle$ when length of y is strictly larger than the length of z . Now we want to estimate $\langle \hat{a}_z^2, \hat{a}_y \rangle = \int \hat{a}_z^2 \hat{a}_y d\mu$ in the case that the length of y is strictly larger than the length of z . We will show that $\int \hat{a}_z^2 \hat{a}_y d\mu = 0$.

We assume that $[y] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [z] = [x_1, x_2, \dots, x_k]$, where $n > k$ (otherwise we get that $\int \hat{a}_z^2 \hat{a}_y d\mu$ is zero from Proposition 7.5).

In fact, we will show that $\int a_z^2 a_y d\mu = 0$.

I) If we assume $x_{k+1} = 0$ in the word $[y]$, then, from (23)

$$e_z^2 e_y = \left[\frac{1}{\mu([z])} \frac{P_{x_k,1}}{P_{x_k,0}} \mathbf{1}_{[x_1, \dots, x_k, 0]} + \frac{1}{\mu([z])} \frac{P_{x_k,0}}{P_{x_k,1}} \mathbf{1}_{[x_1, \dots, x_k, 1]} \right] \\ \times \left[\frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \mathbf{1}_{[x_1, \dots, x_k, 0, x_{k+2}, \dots, x_n, 0]} - \frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \mathbf{1}_{[x_1, \dots, x_k, 0, x_{k+2}, \dots, x_n, 1]} \right] \\ = \left(\frac{1}{\mu([z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right) \left(\frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[x_1, \dots, x_k, 0, x_{k+2}, \dots, x_n, 0]} \\ (64) \quad - \left(\frac{1}{\mu([z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right) \left(\frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \mathbf{1}_{[x_1, \dots, x_k, 0, x_{k+2}, \dots, x_n, 1]}.$$

Note that above, from the second to the third line, we use the fact that

$$\mathbf{1}_{[x_1, \dots, x_k, 1]} \mathbf{1}_{[x_1, \dots, x_k, 0, x_{k+2}, \dots, x_n, 0]} = 0$$

and

$$\mathbf{1}_{[x_1, \dots, x_k, 1]} \mathbf{1}_{[x_1, \dots, x_k, 0, x_{k+2}, \dots, x_n, 1]} = 0.$$

Then, from (20), (64) and (24)

$$a_z^2 a_y = \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} e_{[0, x_1, x_2, \dots, x_k]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} e_{[1, x_1, x_2, \dots, x_k]}^2 \right]$$

$$\begin{aligned}
 & \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0 P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_k,0,x_{k+2},\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1 P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_k,0,x_{k+2},\dots,x_n]} \right] \\
 & = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([0z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[0y0]} \right. \\
 & \quad \left. - \left(\frac{1}{\mu([0z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \mathbf{1}_{[0y1]} \right] \\
 & \quad - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([1z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[1y0]} \right. \\
 & \quad \left. - \left(\frac{1}{\mu([1z])} \frac{P_{x_k,1}}{P_{x_k,0}} \right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \mathbf{1}_{[1y1]} \right].
 \end{aligned}$$

Finally,

$$\begin{aligned}
 & \int a_z^2 a_y d\mu \\
 & = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\frac{P_{x_k,1}}{\mu([0z0])} \sqrt{\mu([0y0])} \sqrt{P_{x_n,1}} - \frac{P_{x_k,1}}{\mu([0z0])} \sqrt{\mu([0y1])} \sqrt{P_{x_n,0}} \right] \\
 & + \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[-\frac{P_{x_k,1}}{\mu([1z0])} \sqrt{\mu([1y0])} \sqrt{P_{x_n,1}} + \frac{P_{x_k,1}}{\mu([1z0])} \sqrt{\mu([1y1])} \sqrt{P_{x_n,0}} \right] \\
 & = P_{x_k,1} \sqrt{\mu([y])} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\frac{\sqrt{P_{x_n,1} P_{0,x_1} P_{x_n,0}}}{\mu([0z0])} - \frac{\sqrt{P_{x_n,0} P_{0,x_1} P_{x_n,1}}}{\mu([0z0])} \right] \right. \\
 (65) \quad & \left. + P_{x_k,1} \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \left[-\frac{\sqrt{P_{x_n,1} P_{1,x_1} P_{x_n,0}}}{\mu([1z0])} + \frac{\sqrt{P_{x_n,0} P_{1,x_1} P_{x_n,1}}}{\mu([1z0])} \right] \right\} = 0.
 \end{aligned}$$

II) If we assume $x_{k+1} = 1$ in the word $[y]$, then, from (23)

$$\begin{aligned}
 & e_z^2 e_y = \left[\frac{1}{\mu([z])} \frac{P_{x_k,1}}{P_{x_k,0}} \mathbf{1}_{[x_1,\dots,x_k,0]} + \frac{1}{\mu([z])} \frac{P_{x_k,0}}{P_{x_k,1}} \mathbf{1}_{[x_1,\dots,x_k,1]} \right] \\
 & \times \left[\frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \mathbf{1}_{[x_1,\dots,x_k,1,x_{k+2},\dots,x_n,0]} - \frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \mathbf{1}_{[x_1,\dots,x_k,1,x_{k+2},\dots,x_n,1]} \right] \\
 & = \left(\frac{1}{\mu([z])} \frac{P_{x_k,0}}{P_{x_k,1}} \right) \left(\frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[x_1,\dots,x_k,1,x_{k+2},\dots,x_n,0]} \\
 (66) \quad & - \left(\frac{1}{\mu([z])} \frac{P_{x_k,0}}{P_{x_k,1}} \right) \left(\frac{1}{\sqrt{\mu([y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}} \right) \mathbf{1}_{[x_1,\dots,x_k,1,x_{k+2},\dots,x_n,1]}
 \end{aligned}$$

Then, from (20), (66) and (24)

$$\begin{aligned}
 & a_z^2 a_y = \left[\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} e_{[0,x_1,x_2,\dots,x_k]}^2 + \frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} e_{[1,x_1,x_2,\dots,x_k]}^2 \right] \\
 & \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0 P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_k,0,x_{k+2},\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1 P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_k,0,x_{k+2},\dots,x_n]} \right] \\
 & = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \left[\left(\frac{1}{\mu([0z])} \frac{P_{x_k,0}}{P_{x_k,1}} \right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}} \right) \mathbf{1}_{[0y0]} \right.
 \end{aligned}$$

$$\begin{aligned}
& -\left(\frac{1}{\mu([0z]} \frac{P_{x_k,0}}{P_{x_k,1}}\right) \left(\frac{1}{\sqrt{\mu([0y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}}\right) \mathbf{1}_{[0y1]}] \\
& -\left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}\right)^{3/2} \left[\left(\frac{1}{\mu([1z])} \frac{P_{x_k,0}}{P_{x_k,1}}\right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_n,1}}{P_{x_n,0}}}\right) \mathbf{1}_{[1y0]} \right. \\
& \left. -\left(\frac{1}{\mu([1z]} \frac{P_{x_k,0}}{P_{x_k,1}}\right) \left(\frac{1}{\sqrt{\mu([1y])}} \sqrt{\frac{P_{x_n,0}}{P_{x_n,1}}}\right) \mathbf{1}_{[1y1]} \right].
\end{aligned}$$

Finally,

$$\begin{aligned}
& \int a_z^2 a_y d\mu \\
& = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}}\right)^{3/2} \left[\frac{P_{x_k,0}}{\mu([0z1])} \sqrt{\mu([0y0])} \sqrt{P_{x_n,1}} - \frac{P_{x_k,0}}{\mu([0z1])} \sqrt{\mu([0y1])} \sqrt{P_{x_n,0}} \right] \\
& + \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}\right)^{3/2} \left[-\frac{P_{x_k,1}}{\mu([1z0])} \sqrt{\mu([1y0])} \sqrt{P_{x_n,1}} + \frac{P_{x_k,1}}{\mu([1z0])} \sqrt{\mu([1y1])} \sqrt{P_{x_n,0}} \right] \\
& = P_{x_k,0} \sqrt{\mu([y])} \left\{ \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}}\right)^{3/2} \left[\frac{\sqrt{P_{x_n,1} P_{0,x_1} P_{x_n,0}}}{\mu([0z1])} - \frac{\sqrt{P_{x_n,0} P_{0,x_1} P_{x_n,1}}}{\mu([0z1])} \right] \right. \\
(67) \quad & \left. + P_{x_k,1} \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}\right)^{3/2} \left[-\frac{\sqrt{P_{x_n,1} P_{1,x_1} P_{x_n,0}}}{\mu([1z1])} + \frac{\sqrt{P_{x_n,0} P_{1,x_1} P_{x_n,1}}}{\mu([1z1])} \right] \right\} = 0.
\end{aligned}$$

9. COMPUTATIONS FOR THE INTEGRAL $\int XYZ$

Our purpose on this section is: given x and z we want to compute for all y

$$(68) \quad \sum_{\text{word } y} \left(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu \right)^2,$$

which corresponds to the first term in the sum given by expression (38).

Remember that from Corollary 7.13 if x is not a subprefix of z and z is not a subprefix of x , we get that for any y

$$\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu = 0.$$

Without loss of generality, we assume that z is a subprefix of x (see Proposition 7.4). The only possible nonzero value for (68) is $\int \hat{a}_x^2 \hat{a}_z d\mu$. This justifies the first term in the sum (39).

We assume first that:

$$[y] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n, x_{n+1}, \dots, x_j] \subset [x] = [x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] \subset [z] = [x_1, x_2, \dots, x_k], \text{ where } j > n \geq k.$$

We will show in all cases that $\int a_x a_y a_z d\mu = 0$. This includes the case

$$(69) \quad \int a_z^2 a_x d\mu = 0.$$

I) First we assume that $x_{k+1} = 0 = x_{n+1}$.

Then,

$$\begin{aligned}
& a_x a_y a_z = \\
& \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_k]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_k]} \right]
\end{aligned}$$

$$\begin{aligned}
 & \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_j]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_j]} \right] \\
 & \times \left[\frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_0} \sqrt{P_{0,x_1}}} e_{[0,x_1,x_2,\dots,x_n]} - \frac{\sqrt{\pi_{x_1}}}{\sqrt{\pi_1} \sqrt{P_{1,x_1}}} e_{[1,x_1,x_2,\dots,x_n]} \right] \\
 & = \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0x])\mu([0z])\mu([0y])}} \left[\sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,1}}{P_{x_n,0} P_{x_k,0} P_{x_j,0}}} \mathbf{1}_{[0y0]} \right. \\
 & \quad \left. - \sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,0}}{P_{x_n,0} P_{x_k,0} P_{x_j,1}}} \mathbf{1}_{[0y1]} \right] \\
 & - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1x])\mu([1z])\mu([1y])}} \left[\sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,1}}{P_{x_n,0} P_{x_k,0} P_{x_j,0}}} \mathbf{1}_{[1y0]} \right. \\
 & \quad \left. - \sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,0}}{P_{x_n,0} P_{x_k,0} P_{x_j,1}}} \mathbf{1}_{[1y1]} \right].
 \end{aligned}$$

Note that for all j

$$(70) \quad \sqrt{\frac{P_{x_j,1}}{P_{x_j,0}}} \mu([0y0]) = \sqrt{P_{x_j,1} P_{x_j,0}} \mu([0y]) = \sqrt{\frac{P_{x_j,0}}{P_{x_j,1}}} \mu([0y1]).$$

and

$$(71) \quad \sqrt{\frac{P_{x_j,1}}{P_{x_j,0}}} \mu([1y0]) = \sqrt{P_{x_j,1} P_{x_j,0}} \mu([1y]) = \sqrt{\frac{P_{x_j,0}}{P_{x_j,1}}} \mu([1y1])$$

Finally, from (70) and (71)

$$\begin{aligned}
 & \int a_x a_y a_z d\mu = \\
 & \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0x])\mu([0z])\mu([0y])}} \left[\sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,1}}{P_{x_n,0} P_{x_k,0} P_{x_j,0}}} \mu([0y0]) \right. \\
 & \quad \left. - \sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,0}}{P_{x_n,0} P_{x_k,0} P_{x_j,1}}} \mu([0y1]) \right] \\
 & - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1x])\mu([1z])\mu([1y])}} \left[\sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,1}}{P_{x_n,0} P_{x_k,0} P_{x_j,0}}} \mu([1y0]) \right. \\
 & \quad \left. - \sqrt{\frac{P_{x_n,1} P_{x_k,1} P_{x_j,0}}{P_{x_n,0} P_{x_k,0} P_{x_j,1}}} \mu([1y1]) \right] = \\
 & \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([0x])\mu([0z])\mu([0y])}} \left[\sqrt{\frac{P_{x_n,1} P_{x_k,1}}{P_{x_n,0} P_{x_k,0}}} \sqrt{\frac{P_{x_j,1}}{P_{x_j,0}}} \mu([0y0]) \right. \\
 & \quad \left. - \sqrt{\frac{P_{x_n,1} P_{x_k,1}}{P_{x_n,0} P_{x_k,0}}} \sqrt{\frac{P_{x_j,0}}{P_{x_j,1}}} \mu([0y1]) \right] \\
 & - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}} \right)^{3/2} \frac{1}{\sqrt{\mu([1x])\mu([1z])\mu([1y])}} \left[\sqrt{\frac{P_{x_n,1} P_{x_k,1}}{P_{x_n,0} P_{x_k,0}}} \sqrt{\frac{P_{x_j,1}}{P_{x_j,0}}} \mu([1y0]) \right. \\
 & \quad \left. - \sqrt{\frac{P_{x_n,1} P_{x_k,1}}{P_{x_n,0} P_{x_k,0}}} \sqrt{\frac{P_{x_j,0}}{P_{x_j,1}}} \mu([1y1]) \right]
 \end{aligned}$$

$$-\sqrt{\frac{P_{x_n,1}P_{x_k,1}}{P_{x_n,0}P_{x_k,0}}} \sqrt{\frac{P_{x_j,0}}{P_{x_j,1}}} \mu([1y1]) = 0 - 0 = 0.$$

II) Now we assume that $x_{k+1} = 1 = x_{n+1}$. In a similar way as before

$$\begin{aligned} & \int a_x a_y a_z d\mu \\ &= \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}}\right)^{3/2} \frac{1}{\sqrt{\mu([0x])\mu([0z])\mu([0y])}} \left[\sqrt{\frac{P_{x_n,0}P_{x_k,0}P_{x_j,1}}{P_{x_n,1}P_{x_k,1}P_{x_j,0}}} \mu([0y0]) \right. \\ & \quad \left. - \sqrt{\frac{P_{x_n,0}P_{x_k,0}P_{x_j,0}}{P_{x_n,1}P_{x_k,1}P_{x_j,1}}} \mu([0y1]) \right] \\ & - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}\right)^{3/2} \frac{1}{\sqrt{\mu([1x])\mu([1z])\mu([1y])}} \left[\sqrt{\frac{P_{x_n,0}P_{x_k,0}P_{x_j,1}}{P_{x_n,1}P_{x_k,1}P_{x_j,0}}} \mu([1y0]) \right. \\ & \quad \left. - \sqrt{\frac{P_{x_n,0}P_{x_k,0}P_{x_j,0}}{P_{x_n,1}P_{x_k,1}P_{x_j,1}}} \mu([1y1]) \right] \\ &= \left(\frac{\pi_{x_1}}{\pi_0 P_{0,x_1}}\right)^{3/2} \frac{\sqrt{\mu([0y0])} P_{x_j,1}}{\sqrt{\mu([0x])\mu([0z])}} \left[\sqrt{\frac{P_{x_n,0}P_{x_k,0}}{P_{x_n,1}P_{x_k,1}}} - \sqrt{\frac{P_{x_n,0}P_{x_k,0}}{P_{x_n,1}P_{x_k,1}}} \right] \\ & - \left(\frac{\pi_{x_1}}{\pi_1 P_{1,x_1}}\right)^{3/2} \frac{\sqrt{\mu([1y0])} P_{x_j,1}}{\sqrt{\mu([1x])\mu([1z])}} \left[\sqrt{\frac{P_{x_n,0}P_{x_k,0}}{P_{x_n,1}P_{x_k,1}}} - \sqrt{\frac{P_{x_n,0}P_{x_k,0}}{P_{x_n,1}P_{x_k,1}}} \right] = 0 - 0 = 0. \end{aligned}$$

II) Now if we assume that $x_{k+1} = 0$ and $x_{n+1} = 1$ or that $x_{k+1} = 1$ and $x_{n+1} = 0$, we get that in a similar way that

$$\int \hat{a}_x \hat{a}_y \hat{a}_z d\mu = 0.$$

After all these computations, for fixed \hat{a}_x and \hat{a}_z , we want to compute $K(\hat{a}_z, \hat{a}_x)$. In this direction we have to consider (68) which is the first sum in expression (38)

We wonder for which y we have that $(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu)^2 \neq 0$. We assumed without loss of generality that z is a subprefix of x . In this case, the length of x is strictly larger than the length of z .

Considering first the case where the length of y is larger than z and x , it follows from the above that

$$\sum_{\text{word } y \text{ with length larger than } x \text{ and } z} \left(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu \right)^2 = 0.$$

Now we consider the case where the length of y is strictly smaller than the length of z and x .

For the case where the length of y is strictly smaller than z and x we need to assume that y is a subprefix of z (otherwise $\hat{a}_y \hat{a}_z = 0$ and we get $(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu)^2 = 0$). If y is a strict subprefix of z and z is a strict subprefix of s we get from the above that $(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu)^2 = 0$.

Finally, we assume that the length of y is strictly smaller than x and strictly larger than z . In this case we have to assume that x is a subprefix of y and y is a subprefix of z (otherwise by Proposition 7.4 we have $\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu^2 = 0$). It follows from the above that also in this case $(\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu)^2 = 0$.

Therefore, in the estimation of expression (68) it follows from our reasoning that all elements in this sum are zero up to expressions $(\int \hat{a}_x^2 \hat{a}_z d\mu)^2$ and $(\int \hat{a}_z^2 \hat{a}_x d\mu)^2$, that is, the cases where $y = x$ or $y = z$. From Proposition 7.5 we have to assume that x is a subprefix of z or vice versa. The explicit expressions for these two cases were analyzed in sections 8.1 and 8.3.

If the length of x is larger than the length of z , then, from (67) we get $(\int \hat{a}_z^2 \hat{a}_x d\mu)^2 = 0$.

The final conclusion is that

$$(72) \quad \sum_{\text{word } y} (\int \hat{a}_x \hat{a}_z \hat{a}_y d\mu)^2 = (\int \hat{a}_x^2 \hat{a}_z d\mu)^2 + (\int \hat{a}_z^2 \hat{a}_x d\mu)^2 = (\int \hat{a}_x^2 \hat{a}_z d\mu)^2.$$

10. APPENDIX

We recall that $\mathcal{L}_{T,A} = \mathcal{L}_A$ denotes the Ruelle operator for the α -Holder potential $A : M \rightarrow \mathbb{R}$, λ_A its main eigenvalue, ν_A its eigenprobability and h_A the associated normalized eigenfunction for A . Remember that we say that the potential A is normalized if $\mathcal{L}_A(1) = 1$. If A is normalized the equilibrium probability $\mu_A = \nu_A$.

Remember that E_0^A denotes the set of Hölder functions g , such that, $\int g d\mu_A = 0$.

For a non normalized potential A the operator $\tilde{\mathcal{L}}_A = \lambda_A^{-1} \mathcal{L}_A$ denotes the renormalized Ruelle operator. Note that $\tilde{\mathcal{L}}(h_A) = h_A$. Moreover, if A_0 is normalized, then $\tilde{\mathcal{L}}_{A_0} = \mathcal{L}_{A_0}$. Note that $I - \tilde{\mathcal{L}}_A$ is not invertible, but is invertible when restricted to E_0^A .

The next result corresponds “exactly” to Proposition 4.6 page 14 in [7]. For completeness, we will outline the proof for the reader. We follow line by line the proof in [7] of this claim. This explicit expression does not appear in [6].

Proposition 10.1. *The map $A \mapsto H(A) = h_A \in C^0(M, \mathbb{R})$, where A is Holder, is differentiable. Furthermore, given a potential $A_0, X \in \text{Hol}$:*

$$D_{A_0} H(X) = h_{A_0} \int ([(I - \mathcal{L}_{A_0}|_{E_0^{A_0}})^{-1} (1 - h_{A_0})] \cdot X) d\mu_{A_0}.$$

Moreover, when A_0 is normalized we get $D_{A_0} H = 0$.

Proof. Let $A_0 \in \text{Hol}$ be fixed. It is known from Proposition 3.4 in [7] (or Proposition 3.8 in [6]) that $\lambda_A^{-n} \mathcal{L}_A^n(1)$ converges uniformly to h_A , with respect to A , in some sufficiently small neighborhood W of A_0 .

Consider the family of functionals $F_n : W \rightarrow \text{Hol}$ given by $F_n(A) = \lambda_A^{-n} \mathcal{L}_A^n(1)$. We claim that the derivatives of F_n are uniformly convergent. Indeed, taking into account Proposition 4.4 in [7] (or Proposition 4.1 of [6]) for the derivative of the spectral radius λ_A , one can write

$$(73) \quad \begin{aligned} D_A F_n \cdot X &= \lambda_A^{-n} D_A \mathcal{L}_A^n(1) \cdot X - n \lambda_A^{-(n+1)} \mathcal{L}_A^n 1 \cdot D_A \lambda_A \cdot X \\ &= \left[\sum_{i=1}^n \tilde{\mathcal{L}}_A^i (\tilde{\mathcal{L}}_A^{n-i}(1) \cdot X) - n \tilde{\mathcal{L}}_A^n(1) \cdot \int h_A \cdot X d\nu_A \right]. \end{aligned}$$

The above expression is precisely expression (4.2) in [7].

We will show that the latter expression is uniformly convergent (with respect to A and $\{X \in \text{Hol} : \|X\|_\alpha \leq 1\}$) to the series $h_A \cdot \sum_{i=0}^{+\infty} \int (\tilde{\mathcal{L}}_A^i(1) \cdot X - h_A \cdot X) d\nu_A$.

Note first that,

$$h_A \cdot \sum_{i=0}^{n-1} \int (\tilde{\mathcal{L}}_A^i(1) \cdot X - h_A \cdot X) d\nu_A = h_A \cdot \int \sum_{i=0}^{n-1} \tilde{\mathcal{L}}_A^i(1 - h_A) \cdot X d\nu_A.$$

This expression uniformly converges (by the spectral gap property of Theorem 3.7 in [6]) to $h_A \cdot \int [(I - \tilde{\mathcal{L}}_A|_{E_0})^{-1}(1 - h_A)] \cdot X d\nu_A$, as n tends to infinite. Note also that, using (73)

$$\begin{aligned} DF_n(A) \cdot X - h_A \cdot \sum_{i=0}^{n-1} \int [\tilde{\mathcal{L}}_A^i(1) \cdot X - h_A \cdot X] d\nu_A \\ = \sum_{i=1}^n \tilde{\mathcal{L}}_A^i [\tilde{\mathcal{L}}_A^{n-i}(1)X - \tilde{\mathcal{L}}_A^{n-i}(1) \int h_A X d\nu_A - h_A \int (\tilde{\mathcal{L}}_A^{n-i}(1)X - h_A X) d\nu_A], \end{aligned}$$

which is a sum of the form $\sum_{i=1}^n \tilde{\mathcal{L}}_A^i(\xi_{n,i})$, where

$$\xi_{n,i} = \tilde{\mathcal{L}}_A^{n-i}(1)X - \tilde{\mathcal{L}}_A^{n-i}(1) \int h_A X d\nu_A - h_A \int (\tilde{\mathcal{L}}_A^{n-i}(1)X - h_A X) d\nu_A.$$

Note that

$$(74) \quad \int \xi_{n,i} d\nu_A = - \int \tilde{\mathcal{L}}_A^{n-i}(1) d\nu_A \int h_A X d\nu_A + \int h_A X d\nu_A.$$

Moreover, as $\tilde{\mathcal{L}}_A^*(\nu_A) = \nu_A$, it follows that $\int \tilde{\mathcal{L}}_A^{n-i}(1) d\nu_A = 1$.

Then, from (74) we get that $\int \xi_{n,i} d\nu_A = 0$.

Theorem 3.7 in [6] claims that there exists $C > 0$ and $\tau \in (0, 1)$, such that,

$$\|\tilde{\mathcal{L}}_A^n(\varphi) - h_A \int \varphi d\nu_A\|_\alpha \leq C \tau^n \|\varphi\|_\alpha,$$

for all $n \geq 1$ and all α -Holder function φ , where h_A is the unique fix point for $\tilde{\mathcal{L}}_A$, such that $\int h_A d\nu_A = 1$.

Then, from this theorem we get that there exists $C > 0$ and $0 < \tau < 1$, such that,

$$\begin{aligned} \|DF_n(A) \cdot X - h_A \cdot \sum_{i=0}^{n-1} \int [\tilde{\mathcal{L}}_A^i(1) \cdot X - h_A \cdot X] d\nu_A\|_0 \\ \leq \sum_{i=1}^n C \tau^i \|\tilde{\mathcal{L}}_A^{n-i}(1)X - \tilde{\mathcal{L}}_A^{n-i}(1) \int h_A X d\nu_A - h_A \int (\tilde{\mathcal{L}}_A^{n-i}(1)X - h_A X) d\nu_A\|_0 \\ \leq \sum_{i=1}^n C \tau^i \|\tilde{\mathcal{L}}_A^{n-i}(X \circ f^{n-i}) - h_A \cdot \int \tilde{\mathcal{L}}_A^{n-i}(X \circ f^{n-i}) d\nu_A\| \\ + \sum_{i=1}^n C \tau^i \|\tilde{\mathcal{L}}_A^{n-i}(1) \cdot \int h_A \cdot X d\nu_A - h_A \cdot \int h_A \cdot X d\nu_A\|_0 \\ \leq \sum_{i=1}^n C^2 \tau^n \cdot \|X\|_0 \cdot (1 + \|h_A\|_0)^2, \end{aligned}$$

which is convergent to zero.

In the last inequality we used the facts that $\int h_A d\nu_A = 1$, $|\int h_A \cdot X d\nu_A| \leq \|X\|_0 \int h_A d\nu_A$ and the norm of $\tilde{\mathcal{L}}_A^{n-i}(1) - h_A$ decreases like τ^{n-i} . The last claim follows from Theorem 3.7 in [6] when $\varphi = 1$.

As in all of the above estimates, the convergence is uniform with respect to A and the α -Holder function X , where $\|X\|_\alpha \leq 1$, we get the proof of our claim.

Now we can finish the proof of the proposition by estimating

$$DF_n(A) \cdot X \rightarrow \int [(I - \tilde{\mathcal{L}}_A|_{E_\alpha^A})^{-1}(1 - h_A)] \cdot X \, d\nu_A \cdot h_A,$$

uniformly with respect to A and the Holder function X , where $\|X\|_\alpha = 1$. Finally, we get

$$D_{A_0}H(X) = h_{A_0} \cdot \int [(I - \mathcal{L}_{A_0}|_{E_\alpha^{A_0}})^{-1}(1 - h_{A_0})] \cdot X \, d\mu_{A_0}.$$

□

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