Time of instability for near-integrable Hamiltonian systems

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IMPA, Rio de Janeiro, Brazil

EDAÍ, UFF, July 1

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1. Introduction

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2. A stability result

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2. A stability result
3. An instability result

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Gevrey regularity: H is $C^\infty(D)$, and for $\alpha \geq 1$ and $L > 0$, one has

$$|H|_{\alpha,L} = \sum_{k \in \mathbb{N}^{2n}} L^{|k|\alpha}(k!)^{-\alpha}|\partial^k H|_{C^0(D)} < \infty.$$

For $\alpha = 1$, H is real-analytic, and one can take $\sigma = L$.
For $\alpha > 1$, compactly-supported functions exist.

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Consider the interaction between planets: the system is near-integrable (with $\varepsilon \simeq 10^{-3}$), and studying the evolution of the action variables $I(t)$ is a way of studying the "deformation" of the ellipses.

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Estimates on the time of instability $\tau(\varepsilon)$?

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Specific problem

What is the maximal (minimal) time of stability $T(\varepsilon)$ (instability $\tau(\varepsilon)$)?

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From stability to instability I

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Assume the system is α-Gevrey, $\alpha \geq 1$, and let $0 \leq \delta \leq \frac{1}{2\alpha n(n-1)}$. If ε is small enough, then for all solutions

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Consequence: $T(\varepsilon) \gtrsim \exp\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha(n-1)}-\delta}$, hence $\tau(\varepsilon) \gtrsim \exp\left(\frac{1}{\varepsilon}\right)^{\frac{1}{2\alpha(n-1)}}$.

From stability to instability II

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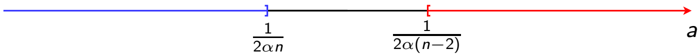

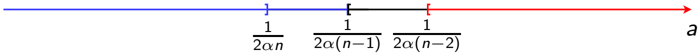

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A first step will be to construct an example of instability with an optimal time in the much simpler setting of *a priori* unstable systems.

From stability to instability III

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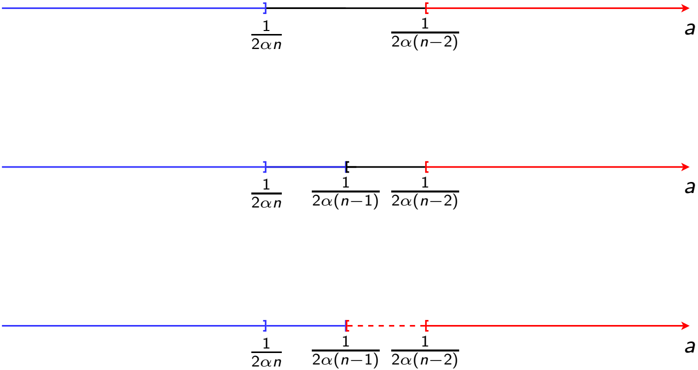

Plan

1. Introduction
2. A stability result
3. An instability result

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Prototype: product of a pendulum and a rotator

$$h(\theta_1, \theta_2, I_1, I_2) = \frac{1}{2}(I_1^2 + I_2^2) + \cos 2\pi\theta_1.$$

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Goal: explain the construction of an example of *a priori* unstable system with an optimal time of instability, that we hope to adapt to an *a priori* stable system (only α-Gevrey for $\alpha > 1$).

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Theorem 2

There exist positive constants C, μ_0 and a α-Gevrey function $f : \mathbb{T}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $\alpha > 1$, such that for $0 < \mu \leq \mu_0$, the system

$$H(\theta, I) = h(\theta, I) + \mu f(\theta, I), \quad (\theta, I) \in \mathbb{T}^3 \times \mathbb{R}^3,$$

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- The construction is possible in the analytic case
- The time is still optimal for non-analytic systems

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Consider the (Liouville) integrable system on $\mathbb{A}^2 = \mathbb{T}^2 \times \mathbb{R}^2$

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Proposition 1

There exist positive constants $\tilde{C}$, $\tilde{\mu}_0$ and a α-Gevrey function $\tilde{f} : \mathbb{A}^2 \to \mathbb{R}$, $\alpha > 1$, such that for $0 < \mu \leq \tilde{\mu}_0$, the diffeomorphism

$$\Phi^{\tilde{h}} \circ \Phi^{\mu\tilde{f}} : \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$

has an orbit $(\theta^k, I^k)_{k\in\mathbb{Z}}$ with

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Lemma

There exist a α-Gevrey function $\tilde{f} : \mathbb{A}^2 \to \mathbb{R}$, $\alpha > 1$, such that for the diffeomorphism $\mathcal{F}_\mu = \mathcal{F}_0 \circ \Phi^{\mu\tilde{f}} = \Phi^{\tilde{h}} \circ \Phi^{\mu\tilde{f}}$

- *the annulus $\mathcal{A}$ remains invariant and normally hyperbolic*
- *the stable and unstable manifolds of $\mathcal{A}$ intersect transversally along a homoclinic annulus $\mathcal{I}_\mu$.*
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Existence of a homoclinic annulus is a non-generic feature.

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Let $A \subseteq \mathbb{N}$ be an alphabet, $\Sigma_A = A^{\mathbb{Z}}$ and $\sigma : \Sigma_A \to \Sigma_A$ the (left) shift: $\bar{n} = (n_k)_{k \in \mathbb{Z}} \in \Sigma_A$, then $\sigma(\bar{n}) = (n'_k)_{k \in \mathbb{Z}}$ with $n'_k = n_{k+1}$.

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Skew-product on M over σ: $F : \Sigma_A \times M \to \Sigma_A \times M$

$$F(\bar{n}, x) = (\sigma(\bar{n}), F_{\bar{n}}(x)), \quad \bar{n} \in \Sigma_A,\ x \in M.$$

Denote by $[[F_{\bar{n}}]]_{\bar{n}\in\Sigma_A}$ the skew-product, where $F_{\bar{n}} : M \to M$.

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Let $\mathcal{O}_\mu \supseteq \mathcal{I}_\mu$ be a nbhd of the homoclinic annulus, and define the transverse map $\mathcal{F}_\mu^T : \mathcal{O}_\mu \to \mathcal{O}_\mu$ by

$$\mathcal{F}_\mu^T(x) = \mathcal{F}_\mu^{n(x)}(x), \quad n(x) = \inf\{n \geq 1,\ \mathcal{F}_\mu^n(x) \in \mathcal{O}_\mu\} < \infty.$$

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IFS on M: skew-product on M over σ such that $F_{\bar{n}} = f_{n_0}$, *i.e.* the maps depend only on the first element n_0 of the symbol $\bar{n} = (n_k)_{k\in\mathbb{Z}} \in \Sigma_A$.

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Proof of proposition 3: use a partition of $\mathbb{A}$ into "resonant" zones and results on near-ergodization time of rotation on the circle

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Unstable orbit for the skew-product $[[F_{\bar{n}}]]_{\bar{n}\in\Sigma_A}$ with a control on the symbol $\bar{n}$ + Proposition 2 $\Longrightarrow$ unstable orbit for the transverse map $\mathcal{F}_\mu^T$ with a control on the return times $n(x)$

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Proposition 3: unstable orbit $(\sigma^k(\bar{n}), \theta^k, I^k)_{k\in\mathbb{Z}} \in \Sigma_A \times \mathbb{A}$ for the IFS $[[f_n]]_{n\in A}$ with a control on the symbol $\bar{n}$

Proposition 3 + shadowing arguments ("hyperbolic" fixed point theorem) $\Longrightarrow$ unstable orbit for the skew-product $[[F_{\bar{n}}]]_{\bar{n}\in\Sigma_A}$ with the same symbol $\bar{n}$

Unstable orbit for the skew-product $[[F_{\bar{n}}]]_{\bar{n}\in\Sigma_A}$ with a control on the symbol $\bar{n}$ + Proposition 2 $\Longrightarrow$ unstable orbit for the transverse map $\mathcal{F}_\mu^T$ with a control on the return times $n(x)$

Unstable orbit for the transverse map $\mathcal{F}_\mu^T$ with a control on the return times $n(x) \Longrightarrow$ Proposition 1: unstable orbit for $\mathcal{F}_\mu$ with an estimate on the time of instability

Proof of Theorem 2: conclusion

Proposition 3: unstable orbit $(\sigma^k(\bar{n}), \theta^k, I^k)_{k\in\mathbb{Z}} \in \Sigma_A \times \mathbb{A}$ for the IFS $[[f_n]]_{n\in A}$ with a control on the symbol $\bar{n}$

Proposition 3 + shadowing arguments ("hyperbolic" fixed point theorem) $\Longrightarrow$ unstable orbit for the skew-product $[[F_{\bar{n}}]]_{\bar{n}\in\Sigma_A}$ with the same symbol $\bar{n}$

Unstable orbit for the skew-product $[[F_{\bar{n}}]]_{\bar{n}\in\Sigma_A}$ with a control on the symbol $\bar{n}$ + Proposition 2 $\Longrightarrow$ unstable orbit for the transverse map $\mathcal{F}_\mu^T$ with a control on the return times $n(x)$

Unstable orbit for the transverse map $\mathcal{F}_\mu^T$ with a control on the return times $n(x)$ $\Longrightarrow$ Proposition 1: unstable orbit for $\mathcal{F}_\mu$ with an estimate on the time of instability

Proposition 1 + suspension arguments $\Longrightarrow$ Theorem 2.

Obrigado