Robust exponential decay of correlations for singular-flows

The Lorenz Attractor exhibits robust exponential decay of correlations

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1 Lorenz equations and geometric Lorenz flow

The Lorenz flow

The Lorenz equations

$$\dot{x} = 10(y-x), \quad \dot{y} = 28x - y - xz, \quad \dot{z} = xy - 8z/3$$

were presented by Lorenz in 1963 as a simplified model of convection on the Earth's atmosphere.

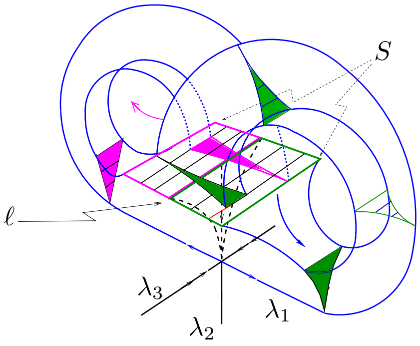
This was one of the first examples of systems exhibiting sensitive dependence on initial conditions, which became the "definition" of *chaotic dynamics*.

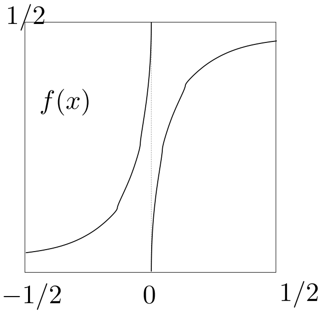
The Lorenz flow is best understood through the *geometric Lorenz flows* constructed to mimic the features of the Lorenz equations.

The geometric Lorenz flow

Recently Tucker with a computer assisted proof showed that the above equations and similar equations with nearby parameters define a geometric Lorenz flow: we may find a ***global cross-section*** given by a (embedded) square whose Poincaré first return map provides a ***roof function with logarithmic growth near the singularity line*** and having a ***uniformly contracting one-dimensional foliation***.

Tucker's work shows in particular the ***the attractor of the Lorenz equations is equivalent to a geometric Lorenz flow***.



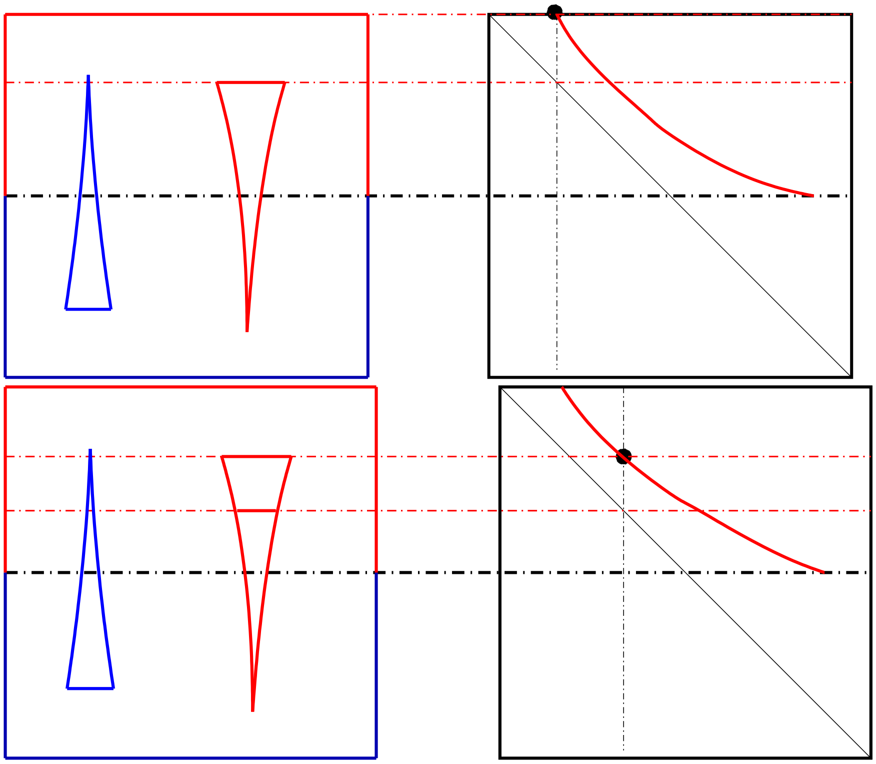


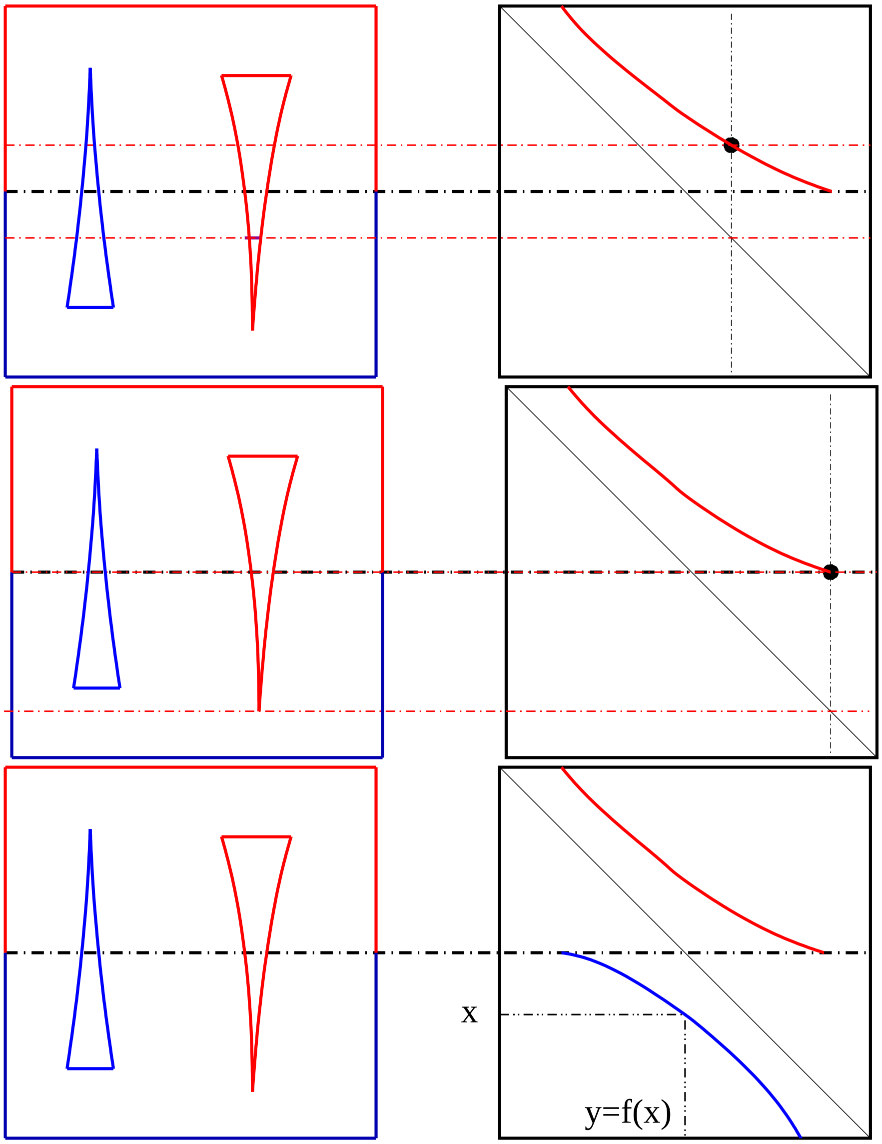
Lorenz attractor = geometric Lorenz attractor

The one-dimensional map $f : [-1/2, 1/2] \setminus \{0\} \to [-1/2, 1/2]$, obtained has the quotient over the leaves of the stable foliation, is piecewise expanding with $Df > \sqrt{2}$, exhibits ***exponentially slow recurrence to the singular set***, and Df has logarithmic growth near $\mathcal{S} = \{0\}$.

The one-dimensional reduction

To obtain the one-dimensional map f we proceed as follows





1.1 Suspension semiflows

Given a C^3 local diffeomorphism $f : M \setminus \mathcal{S} \to M$ outside a volume zero non-flat singular set, let $X^t : M_r \to M_r$ be a semiflow with roof function $r : M \setminus \mathcal{S} \to \mathbb{R}$ over the base transformation f, as follows.

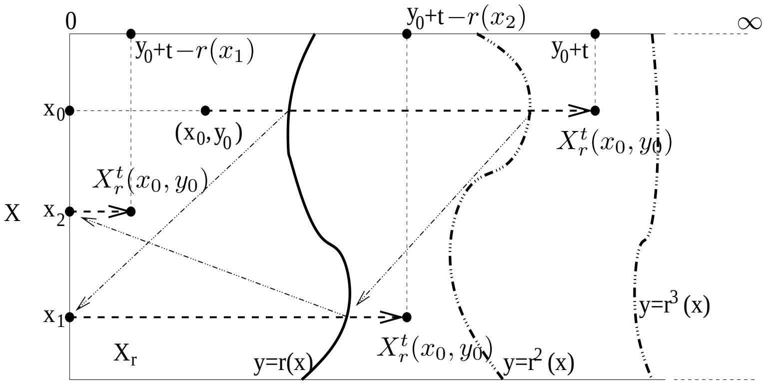
Set $M_r = \{(x, y) \in M \times [0, +\infty) : 0 \le y < r(x)\}$ and X^0 the identity on M_r.

For $x = x_0 \in M$ denote by x_n the nth iterate $f^n(x_0)$ for $n \ge 0$.

Then $S_n\varphi(x_0) = \sum_{j=0}^{n-1} \varphi(x_j)$ for $n \ge 1$ and for any given real function φ.
Then for each pair $(x_0, s_0) \in X_r$ and $t > 0$ there exists a unique $n \ge 1$ such that $S_n r(x_0) \le s_0 + t < S_{n+1} r(x_0)$ and we define

$$X^t(x_0, s_0) = \big(x_n, s_0 + t - S_n r(x_0)\big).$$

The semiflow dynamics can be pictured as follows



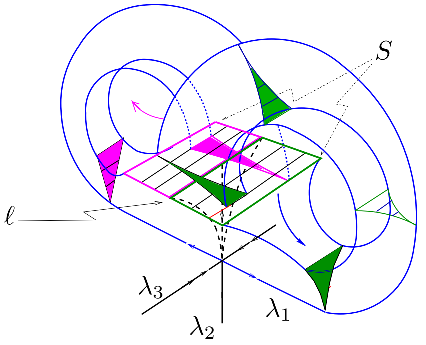
Lorenz flow and skew-product

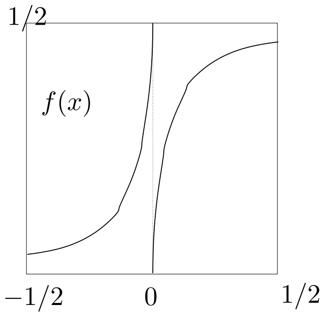
If $\pi : S \to I$ is the quotient map over the stable leaves, where $I = [-1/2, 1/2]$; $P : S \setminus \ell \to S$ is the Poincare return map from $S \setminus \ell$ to S, and $r : S \setminus \ell \to [r_0, +\infty)$ is the Poincare return time map, then we can define a pair of suspension semiflows:

- X_r^t with base map P and roof function r;
- $X_{r\circ\pi}^t$ with base map f and roof function $r \circ \pi$.

We have $f \circ \pi = \pi \circ P$ and the two flows are essentially conjugated.

Lorenz flow and skew-product





Note that $\Phi : S \times [0, +\infty) \to \mathbb{R}^3, (x,t) \mapsto X^t(x)$ is a natural smooth conjugation between the first semiflow X_r^t and the original geometric Lorenz flow X^t.

This will be usefull to reduce the study of the speed of decay of correlations to the second and simpler semiflow.

1.2 Invariant/physical measure/decay of correlations

Invariant measure for the semiflow

Let μ be an invariant probability measure with respect to f on M (or P on S) and assume that the roof function r (or $r \circ \pi$) is μ-integrable and Leb-integrable.

Denote by $\nu = \mu \ltimes \mathrm{Leb}^1$ the natural X^t-invariant extension of μ to M_r: for any subset $A \subset M_r$

$$\nu(A) = \frac{1}{\mu(r)} \int d\mu(x) \int_0^{r(x)} ds\, \chi_A(x,s).$$

Then ν is invariant under the semiflow. Note that for $\psi : M \to \mathbb{R}$ we have $\nu(\psi) = \mu(\varphi)/\mu(r)$ with

$$\varphi(x) := \int_0^{r(x)} \psi(x,t)\,dt.$$

Physical probability measure

An invariant probability measure ν for a flow X_t on a compact manifold is ***physical*** if the points z satisfying for every continuous function ψ

$$\lim_{t\to+\infty} \frac{1}{t}\int_0^t \psi\big(X_s(z)\big)\,ds = \int \psi\,d\nu,$$

form a subset with positive volume on the ambient space.

These time averages are in principle physically observable if the flow models a real world phenomenon admitting some measurable features.

Mixing/Decay of correlations

Let us study the properties of this measure. Besides ergodicity there are various degrees of mixing.

Given a flow X and an invariant ergodic probability measure μ, we say that the system (X,μ) is ***mixing*** if for any two measurable sets A, B

$$\mu\big(A \cap X^{-t}B\big) \xrightarrow[t\to\infty]{} \mu(A)\cdot\mu(B)$$

or equivalently

$$\int \varphi\cdot\big(\psi\circ X^t\big)\,d\mu \xrightarrow[t\to\infty]{} \int \varphi\,d\mu \int \psi\,d\mu$$

for any pair $\varphi, \psi : M \to \mathbb{R}$ of continuous functions.

Correlation decay

Considering φ and $\psi\circ X^t : M \to \mathbb{R}$ as random variables over the probability space (M,μ), this definition just says that **"the random variables φ and $\psi\circ X^t$ are asymptotically independent"** since

$$\mathbb{E}\big(\varphi\cdot(\psi\circ X^t)\big) \xrightarrow[t\to+\infty]{} \mathbb{E}(\varphi)\cdot\mathbb{E}(\psi).$$

The ***correlation function***

$$\begin{aligned} C_t(\varphi,\psi) &= \big|\mathbb{E}\big(\varphi\cdot(\psi\circ X^t)\big) - \mathbb{E}(\varphi)\cdot\mathbb{E}(\psi)\big| \\ &= \Big|\int \varphi\cdot\big(\psi\circ X^t\big)\,d\mu - \int\varphi\,d\mu\int\psi\,d\mu\Big| \end{aligned}$$

satisfies $C_t(\varphi,\psi) \xrightarrow[t\to\infty]{} 0$ in this case. The ***rate of approach to zero of the correlation function*** **is called** ***the rate of decay of correlations*** for the observables φ and ψ of the system (X,μ).

Exponential decay of correlations

The study of decay of correlations for hyperbolic systems goes back to the work of Sinai (1972) and Ruelle (1976), mostly for diffeomorphisms.

It was proved that the *physical* (SRB) measures for Axiom A ***diffeomorphisms*** are mixing and have ***exponential decay of correlations***: there exists $\alpha \in (0,1)$ such that given φ and ψ there exists $C = C(\varphi,\psi) > 0$ satisfying

$$C_n(\varphi,\psi) \le C\cdot e^{-\alpha n} \quad \text{for all} \quad n \ge 1,$$

for a suitable class of continuous functions $M \to \mathbb{R}$, in this case the Hölder continuous functions.

Exponential decay for flows

Seems to be much more complex!

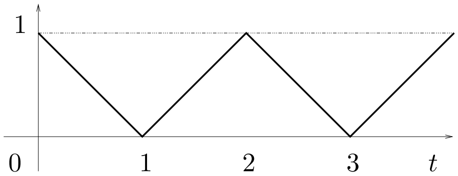
Ergodicity and mixing for geodesic flows on manifolds of negative curvature are known since the early half of the XXth century by Hopf and Hedlund (1939), Anosov and Sinai (1960,67).

The proof of exponential decay of correlations for geodesic flows on manifolds of constant negative curvature was first obtained in two dimensions by Collet-Epstein-Gallavotti (1984) and then in three dimensions by Pollicott (1982) through group theoretical arguments.

Non-mixing/slow decay of correlations

Let $f : M \to M$ be a diffeomorphism with an invariant probability measure μ and consider the suspension flow X_f over f with **constant roof function** $r \equiv 1$**.** Then $\nu = \mu \times \mathrm{Leb}$ on $M \times [0,1)$ is a X_f-invariant probability measure on X_r ***which is NOT mixing***, whatever f may be.

Indeed, consider $A = \pi\big(M \times [0,1/2)\big)$ and $B = M_r \setminus A$. Then $t \mapsto \nu\big(A \cap X^{-t}B\big)$ for $t > 0$ has the following graph which shows that **this flow is not even topologically mixing.**



Ergodic base implies ergodic suspension

However, if (X,f,μ) is ergodic, then ν is X_f-ergodic also!

Indeed, given $A \subset X_r$ such that $(X_f^t)^{-1}(A) = A$ for all $t > 0$ (an X_f-invariant set), then A is saturated, i.e., $p \in A$ if, and only if, $\mathcal{O}_{X_f}(p) \subset A$.

Thus we may find $\hat{A} \subset X$ such that $A \cap \pi(X \times \{0\}) = \pi(\hat{A})$ is X_f^1-invariant by construction (because $r \equiv 1$), $\hat{A}$ is f-invariant and $\nu(A) = \mu(\hat{A}) \cdot \mathrm{Leb}([0,1))$.

Hence $\mu(\hat{A}) \cdot \mu(X \setminus \hat{A}) = 0$ by the ergodicity of (f,μ) which implies that $\nu(A) \cdot \nu(X_r \setminus A) = 0$.

Slow decay of correlations for Axiom A flows

In addition to examples of non-mixing suspension flows:

- **Ruelle (1983) and Pollicott (1984) exhibited suspensions semiflows with piecewise constant ceiling functions over uniformly expanding base dynamics, with arbitrarily slow decay rates of correlations.**

The example from Ruelle is simple to describe: take the full shift on 2 symbols $\sigma : \Sigma_2 \to \Sigma_2$ and the roof function $r(\xi) = \lambda_0$ if $\xi_0 = 0$ and $r(\xi) = \lambda_1$ if $\xi_0 = 1$; where $\lambda_0, \lambda_1 > 0$ and λ_0/λ_1 is not rational. Take any equilibrium state μ for σ with respect to a Hölder continuous potential $\phi : \Sigma_2 \to \mathbb{R}$ and consider the induced probability $\nu = \mu \times \mathrm{Leb}$ on $\{(\xi, s), 0 \le s < r(\xi)\}$. The suspension semiflow over σ with roof function r does not have exponential decay of correlations for ν.

Mixing versus suspension with constant roof

Anosov (1967) showed that geodesic flows for negatively curved compact Riemannian manifolds are mixing and obtained the ***Anosov alternative***: given a transitive volume preserving Anosov flow, either it is mixing (with respect to the volume measure), or it is a suspension of an Anosov diffeomorphism by a constant roof function.

Bowen (1976) showed that, if a mixing Anosov flow is the suspension of an Anosov diffeomorphism, then it is ***stably mixing***, that is, the mixing property remains true for all nearby flows (which are Anosov also by the structural stability of Axiom A flows).

Generic topologically mixing

Bowen (1976) also showed that the class of C^r Axiom A flows, $r \geq 1$, admits a residual subset $\mathcal{R}$ such that for every $X \in \mathcal{R}$ the spectral decomposition of $\Omega(X)$ is formed by pairwise disjoint pieces $\Omega_1 \cup \cdots \cup \Omega_k$ each of which is ***topologically mixing***.

That is, given any pair of open sets U, V in Ω_i, there exists $T_0 = T_0(U, V) > 0$ such that $U \cap X^t(V) \neq \emptyset$ for all $t > T_0$, for a residual subset of C^r Axiom A flows.

Exponential decay of correlations for Axiom A flows

Breakthrough obtained by Dolgopyat (1998): **smooth (C^r with $r \geq 7$) geodesic flows on manifolds of negative curvature, under a non-integrability condition on both the stable and unstable distributions exhibit exponential decay of correlations.** Liverani (2004) obtained exponential decay of correlations for C^4 contact Anosov flows.

Field-Melbourne-Törok (2007) obtained ***stability of rapid mixing*** **among Axiom A flows**:

$$C_t(\varphi, \psi) \lesssim t^{-k}, \quad k \in \mathbb{N}, \quad t \to \infty$$

for a C^2-open and C^r-dense set of flows among the family of C^r Axiom A flows, for each $r \geq 2$.

Question Are there stable exponentially mixing Axiom A flows?

Exponential decay for suspension flows...

Dolgopyat (1998): Generic suspension flows over subshifts of finite type have exponential decay of correlations.

Using Dolgopyat ideas applied to a **suspension over a uniformly expanding base**, a conjecture of Ruelle was proved by Pollicott (1999): on a cohomological condition on the roof function, the decay of correlations for this type of suspension flows is exponential for observables not supported on the base.

This was extended to non-uniformly expanding base by Baladi-Vallée (2005), clarifying the assumptions on the base and on the ceiling function which suffice to obtain **exponential decay of correlations** for **suspension of one-dimensional expanding maps with countable Markov partitions** (modelled by countable subshifts).

Exponential decay for suspension flows

All these ideas were used, in a more abstract setting, by Avila-Gouezel-Yoccoz (2006) to obtain exponential decay of correlations for the Teichmüller flow on flat surfaces, after extending the assumptions of Baladi-Vallée to **suspension of multidimensional piecewise expanding maps**.

Relating the geometric Lorenz flow to a suspension semiflow, Luzzatto-Melbourne-Paccaut (2005) proved that the **physical measure for the geometric Lorenz flow is mixing.**

The speed of mixing for the Lorenz flow has been an open problem.

2 Decay correlations for geometric Lorenz flow

2.1 Good hyperbolic skew-product semiflow

Suspension flow over piecewise expanding map

Theorem 1 (Avila-Gouezel-Yoccoz, 2006)**.** *Let Y_t be a good hyperbolic skew-product semi-flow on a space $\widehat{\Delta}_r$, preserving the probability measure $\bar{\eta}$. There exist constants $C > 0$ and $\delta > 0$ such that, for each pair of functions $\varphi, \psi \in C^1(\widehat{\Delta}_r)$, for all $t \geq 0$,*

$$\left| \int \varphi \cdot \psi \circ Y_t \, d\bar{\eta} - \left(\int \varphi \, d\bar{\eta} \right) \left(\int \psi \, d\bar{\eta} \right) \right| \leq C \|\varphi\|_1 \|\psi\|_1 e^{-\delta t}.$$

We show that an open class of geometric Lorenz flows can be conjugated to semiflows in the above setting, concluding robust exponential decay of correlations for a wide class of singular flows.

Geometric Lorenz flow is a good hyperbolic skew-product semiflow

This is our main result.

Theorem 2 (V.A.-P.Varandas)**.** *Given any compact* 3*-manifold* M*, we can find an open subset* $\mathcal{U}$ *of* $\mathcal{X}^3(M)$ *such that each* $X \in \mathcal{U}$ *exhibits a geometric Lorenz flow which is smoothly semi-conjugated to a good hyperbolic skew-product semi-flow.*

Now we explain what ***good hyperbolic skew-product semiflow*** means, and how we relate a geometric Lorenz flow to these semiflows.

Uniformly expanding Markov map

We assume that $\cup_{\ell\in L}\Delta^{(\ell)}$ is an at most countable partition (Lebesgue modulo zero) of an open domain Δ of some manifold by open subsets and let $F : \cup_{\ell\in L}\Delta^{(\ell)} \to \Delta$ be a C^r **uniformly expanding Markov map**, $r \geq 2$, that is

1. $F : \Delta^\ell \to \Delta$ is a C^r diffeomorphism for every ℓ;
2. there are $C > 0$ and $0 < \lambda < 1$ such that
   (a) for every inverse branch h_n of F^n, with $n \geq 1$, $d(h_n(x), h_n(y)) \leq C\lambda^n d(x, y)$; and
   (b) if JF is the Jacobian of F with respect to the Lebesgue measure, then $\log JF$ is a C^1 function and $\|D((\log JF) \circ h)\|_0 \leq C$ for every inverse branch h of F.

We denote by $\mathcal{H}_n$ **the family of inverse branches of** F^n**.** It is well known that F **admits an invariant probability measure** ν **which is absolutely continuous with respect to Lebesgue.**

Good roof function with exponential tail

We say that the roof function $r : \Delta \to \mathbb{R}^+$ has ***exponential tail*** if there exists $\sigma_0 > 0$ such that $\int e^{\sigma_0 r} d\nu < \infty$.

We say that the roof function r is ***good*** if

1. r is bounded from below by some positive constant r_0;
2. there exists $C > 0$ such that $\sup_{h\in\mathcal{H}} \|D(r \circ h)\|_0 \leq C < \infty$;
3. it is not possible to write $r = v + u \circ F - u$ on Δ, where $v : \Delta \to \mathcal{R}$ is constant on each Δ^ℓ and $u : \Delta \to \mathbb{R}$ is a C^1-function.

The last cohomological condition corresponds to **uniform non-integrability, or aperiodicity,** as defined by Baladi-Vallée adapted from the work of Dolgopyat.

Hyperbolic skew-product structure

Let $F : \bigcup_l \Delta^{(l)} \to \Delta$ be a uniformly expanding Markov map preserving an abs. cont. probability ν. An ***hyperbolic skew-product*** **over** F is a map $\widehat{F}$ from a dense open subset of an open domain $\widehat{\Delta}$, to $\widehat{\Delta}$, satisfying

1. there exists a continuous map $\pi : \widehat{\Delta} \to \Delta$ such that $F \circ \pi = \pi \circ \widehat{F}$ whenever both members of the equality are defined;
2. there is $\kappa > 1$ such that, for all $w_1, w_2 \in \widehat{\Delta}$ in the same leaf, i.e. $\pi(w_1) = \pi(w_2)$, we have $d(\widehat{F}w_1, \widehat{F}w_2) \leq \kappa^{-1} d(w_1, w_2)$.
3. there is a $\widehat{F}$-invariant probability measure η on $\widehat{\Delta}$, giving full mass to $\widehat{\Delta}$;
4. there exists a smooth disintegration of η along the stable leaves $\pi^{-1}(w), w \in \Delta$, as follows.

Existence of smooth disintegration

(4) cont. there exists a family of probability measures $\{\eta_x\}_{x\in\Delta}$ on $\widehat{\Delta}$ which is a **disintegration** of η over ν:

- (a) $x \mapsto \eta_x$ is measurable;
- (b) η_x is supported on $\pi^{-1}(x)$, and
- (c) for each measurable subset A of $\widehat{\Delta}$ we have $\eta(A) = \int \eta_x(A)\, d\nu(x)$.

Moreover, **this disintegration is smooth**: we can find a constant $C > 0$ such that, for any open subset $V \subset \bigcup \Delta^{(l)}$ and for each $u \in C^1(\pi^{-1}(V))$, the function $\bar{u} : V \to \mathbb{R}, x \mapsto \bar{u}(x) := \int u(y)\, d\eta_x(y)$ belongs to $C^1(V)$ and satisfies

$$\sup_{x\in V} \|D\bar{u}(x)\| \le C \sup_{y\in\pi^{-1}(V)} \|Du(y)\|.$$

Good hyperbolic skew-product semiflow

Given $r : \cup_{\ell\in L}\Delta^{(\ell)} \to [r_0, +\infty)$ for some $r_0 > 0$ we define

$$\widehat{\Delta}_r = \{(w,t) : w \in \widehat{\Delta},\ 0 \le t \le r(\pi(w))\}/\sim,$$

where $\sim$ is the equivalence relation $(w, r(\pi(w))) \sim (F(w), 0)$.

Consider the suspension semiflow $Y_t(w,s) = (w, s+t)$.

If Y_t is a semiflow over a hyperbolic skew-product with a good roof function which, moreover, has exponential tail, then we say that Y_t is a ***good hyperbolic skew-product semi-flow***.

Note that, if η is an $\widehat{F}$-invariant probability measure so that $\int r\, d\eta < \infty$, then $(Y_t)_t$ preserves the probability measure $\bar{\eta} = (\eta \otimes \mathrm{Leb})/\int r\, d\eta$.

2.2 Steps of the proof

Steps of the proof

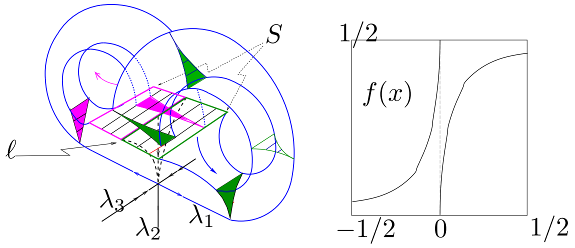
- obtain a robust C^3-smooth strong-stable foliation for the geometric Lorenz flow together with robust transitivity for the associated attractor;
- obtain a uniformly expanding Markov map F as an induced map of the one-dimensional Lorenz transformation f;
- show that there is a related induced map $\hat{F}$ from the Poincare return map P to S and a natural choice r of roof function over $\hat{F}$ such that the original flow is conjugated to the semiflow over $\hat{F}$ with roof r;
- prove that r has exponential tail, satisfies the aperiodicity condition and that the disintegration property holds for the right choice of measure on the semiflow over $\hat{F}$;
- check that each of the above steps are robust for C^3 close flows.

3 Some details of the proof

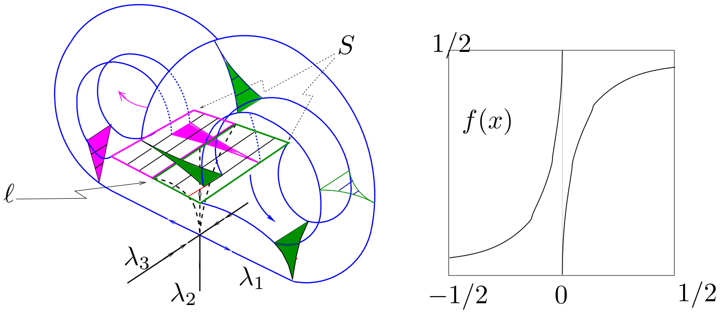
3.1 Smooth stable foliation, smooth density

Persistent strong-stable foliation

Theorem 3 (Consequence of domination)**.** *Let X be the vector field obtained in the construction of the geometric Lorenz model and $\mathcal{F}_X$ the invariant contracting foliation of the cross-section S. Then any vector field Y which is sufficiently C^1-close to X admits an invariant contracting continuous foliation $\mathcal{F}_Y$ on the cross-section S with C^1 leaves.*



Robust transitivity



The fact that the geometric Lorenz attractors are robustly transitive in the C^1 topology is a well-known consequence of the persistence of the strong-stable foliation on the cross-section S. This is a “classical” result.

Smooth strong-stable foliation

It can be shown that holonomies along the leaves are Hölder-C^1 in general. But if we have a ***strong dissipative* condition** on the equilibrium $\sigma = (0,0,0)$, that is, if $\beta := -\lambda_2/\lambda_1$ and $\alpha := -\lambda_3/\lambda_1$ satisfy $\beta > \alpha + k$ for some $k \in \mathbb{Z}^+$, then $\mathcal{F}_Y$ is a C^k smooth foliation (it can be trivialized by C^k charts on M), and so the holonomies along its leaves are C^k maps.

Theorem 4 (Consequence of strong dissipation/domination)**.** *For strongly dissipative Lorenz attractors, with $\beta > \alpha + k$, the one-dimensional quotient map f is C^k smooth away from the singularity. Moreover, this smoothness property is also persistent for all nearby C^k flows, since the condition $\beta > \alpha + k$ is open in the C^1 topology.*

3.2 Induced map with exponential tail of induced times

Slow recurrence to the singular set

In general, for C^1+Hölder piecewise expanding maps, we know that the invariant density is of generalized bounded variation (Keller, 1985) and so is **bounded from above**.

In particular, $\log|x| \cdot (d\nu_0/d\lambda)(x)$ is λ-integrable. By the Ergodic Theorem, for every $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\sum_{i=0}^{n-1} -\log|f^i(x)|_\delta = \int_{-\delta}^{\delta} -\log|x|\,d\nu_0(x) < \varepsilon$$

for λ-almost every $x \in I$; where $|y|_\delta = |y|$ if $|y| < \delta$ and 1 otherwise.

This last property is known as **slow recurrence** to the singular set (recall that the singularity for f sits at 0).

Uniformly expanding Markov structure

We use the theory built around non-uniformly expanding maps by several people in the last decades. We state the main results whose concatenation provides the conclusion we need.

First piecewise expansion together with slow recurrence guarantees the existence of hyperbolic times.

Lemma 5 (Existence hyperbolic times)**.** *There exists $\theta > 0$ and $\delta \in (0,1)$, depending only on f and the expanding rate $\sqrt{2}$ such that, for Lebesgue almost every $x \in I$, we can find $n_0 \geq 1$ satisfying: for each $n > n_0$ there are (σ, δ)-hyperbolic times $1 \leq n_1 < \cdots < n_l \leq n$ for x with $l \geq \theta n$.*

Properties of hyperbolic times

Lemma 6 (Hyperbolic times and hyperbolic pre-balls)**.** *Given $0 < \sigma < 1/\sqrt{2}$ and $\delta > 0$, there exist $\delta_1, D_1, \kappa > 0$, depending only on σ, δ and on the map f, such that for any $x \in M$ and $n \geq 1$ a (σ,δ)-hyperbolic time for x, there exists a neighborhood $V_n(x)$ of x with the following properties:*

1. *f^n maps $V_n(x)$ diffeomorphically onto the ball $B(f^n(x), \delta_1)$;*
2. *we have $|f^{n-k}(y) - f^{n-k}(z)| \leq \sigma^k |f^n(y) - f^n(z)|$ for all $y, z \in V_n(x)$ $1 \leq k < n$, so that $V_n(x) \subset B(x, 2\delta_1\sigma^n)$;*
3. *f^n has distortion bounded by a factor D_1 on $V_n(x)$: for every y, z in the hyperbolic pre-ball $V_n(x)$*

$$\frac{1}{D_1} \leq \frac{|\det Df^n(y)|}{|\det Df^n(z)|} \leq D_1 \, .$$

Existence expanding Markov structure

Theorem 7 (Alves-Luzzatto-Pinheiro (2004), Gouezel (2006))**.** *There exists a neighborhood $\Delta := (-a, a)$, for some $0 < a < 1/2$, of the singular point 0; a countable Lebesgue modulo zero partition $\mathcal{Q}$ of Δ into sub-intervals; a function $R : \Delta \to \mathbb{Z}^+$ defined almost everywhere, constant on elements of the partition $\mathcal{Q}$; and constants $b, c, \gamma > 0, \kappa > 1, N \in \mathbb{Z}^+$ such that, for all $\omega \in \mathcal{Q}$ and $R = R(\omega)$, the map $F := f^R : \omega \to \Delta$ is a C^3 diffeomorphism satisfying: for each $\omega \in \mathcal{Q}$ there exists $0 < k \leq N$ such that*

1. *$n := R(\omega) - k$ is a (σ, δ_1)-hyperbolic time for each $x \in \omega$;*
2. *$\omega \subset V_n(x)$ and $f^j(\omega) \subset I \setminus \Delta$ for all $n \leq j < R(\omega)$.*

The Renyi condition

By a routine calculation using the properties of hyperbolic times we get

$$\frac{|D^2F|}{|DF|^2}(x) \leq B \approx \sum_{i \geq 0} \sigma^{(1-b\alpha)i} < \infty,$$

a uniform bound for every $x \in \omega$, $\omega \in \mathcal{Q}$ with $R = R(\omega)$.

The usefulness of this comes from the next bound: for $x, y \in \omega$

$$\left| \frac{1}{DF}(x) - \frac{1}{DF}(y) \right| \leq |x-y| \frac{|D^2F|}{|DF|^2}(z) \leq B|x-y|$$

where z is some point in ω between x and y by the Mean Value Theorem (“**bounded distortion**”).

Consequences of the Renyi condition

We can deduce the following useful consequences

- (useful bound) for $n > 1$

  $$\left|\frac{D^2F^n(x)}{(DF^n(x))^2}\right| \leq B \cdot n \cdot \sigma^{n-1}$$

  which is an infinitesimal when $n \to +\infty$.

- **Smooth absolutely continuous invariant measure**

  In general, for C^1+Hölder piecewise expanding maps, we know that the invariant density is of generalized bounded variation (Keller, 1985) and so is **bounded from above**.

  But we need a smooth density. However the Renyi condition is sufficient to get a C^1 density, as shown by Baladi-Vallée.

Exponential tail for the inducing time function

It is known that the Lorenz map has ***exponentially slow recurrence*** **to the singular set**, that is, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{1}{n} \log \mathrm{Leb}\left(\left\{x \in I : \frac{1}{n}\sum_{i=0}^{n-1} -\log|f^i(x))|_\delta > \varepsilon\right\}\right) < 0.$$

This ensures (following the work of Gouezel) that the tail of the return time function is exponential.

Theorem 8. *In the same setting of the previous Theorem (on the existence of Markov structure), the "induced time function" R has exponential tail: $\lambda(\{R > n\}) < ce^{-\gamma n}$ for some $c, \gamma > 0$.*

Proof of existence of Markov map is complete

At this point we have all the properties showing that **the one-dimensional Lorenz transformation is a uniformly expanding Markov map**:

1. $F : \Delta^\ell \to \Delta$ is a C^3 diffeomorphism for every ℓ since Δ^ℓ is a hyperbolic pre-ball whose image is Δ;
2. we can find $C > 0$ and $0 < \lambda < 1$ satisfying:
   - since every inverse branch h_n of F^n corresponds to some hyperbolic time, with $n \geq 1$, we have $d(h_n(x), h_n(y)) \leq C\lambda^n d(x, y)$; and
   - $\log JF = \log|DF|$ is C^1 and $\|D((\log|DF|) \circ h)\|_0 \leq C$ for every inverse branch h of F by the very definition of hyperbolic times (another routine calculation).

Now we need to show that the roof function is good and obtain the hyperbolic skew-product structure.

3.3 Good roof function, smooth disintegration

The roof function

The roof function r over F is defined from the Poincaré return time function as

$$r(x) = S_R\varrho(x) := \sum_{j=0}^{R(x)-1} \varrho(f^j(x)), \quad x \in \bigcup_{\omega\in\Omega} \omega,$$

where

$$\varrho(x) := \inf\left\{\tau(z) : z \in \pi^{-1}(\{x\})\right\} = \tau(x, 0, 1), \quad x \in I \setminus \{0\},$$

since the Poincaré return time to S, denoted by τ, does not depend of the point we choose on some strong-stable leaf in S^*.

Logarithmic growth near the singular set

We say that $r : M \setminus \mathcal{S} \to \mathbb{R}$ has ***logarithmic growth near*** $\mathcal{S}$ if there exists a constant $L => 0$ such that

$$r \leq L \cdot |\log d(x, \mathcal{S})|$$

for all $x \in M \setminus \mathcal{S}$ with $d(x, \mathcal{S}) < \delta$ for all small enough $\delta > 0$.

We also have that r is bounded below by some $r_0 > 0$.

These are direct consequences of the presence of a hyperbolic singularity of saddle-type attached to the attractor in the geometric model.

Exponential tail for the roof function

The exponential tail for r: for some $\gamma > 0$ and every big $L > 1$

$$\mathrm{Leb}\{r > L\} < e^{-\gamma L}$$

is a consequence of

- the exponential tail of the inducing time, when the inducing time R is very big ($R > \xi L$);
- the logarithmic growth of τ (and ϱ) near the singularity, when the inducing time R is small ($R < N$);
- for intermediate inducing times we need a large deviations estimate for the piecewise expanding map f using ϱ as an observable (here $N \leq n \approx R \leq \xi L$ is relatively small)

$$\frac{1}{n} \log \mathrm{Leb}\left\{ x : \frac{1}{n} \sum_{i=0}^{n-1} \varrho \circ f^i(x) > L \right\} < 0, \quad n \geq N.$$

Large deviations for non-uniformly expanding maps

The large deviations estimate used here was obtained by one of the authors for

- C^2 local diffeomorphisms away from a non-flat critical/singular set;
- having hyperbolic times for Lebesgue almost every point;
- exhibiting exponentially slow recurrence to the critical/singular set (as the Lorenz transformation): for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\limsup_{n \to +\infty} \frac{1}{n} \log \mathrm{Leb}\left(\left\{ x \in I : \frac{1}{n} \sum_{i=0}^{n-1} -\log |f^i(x))|_\delta > \varepsilon \right\} \right) < 0.$$

Uniform bound on the derivative of the roof

Again this follows from the properties of hyperbolic times after some lengthy but routine calculations, where h is an inverse branch of $F = f^\ell$ and ℓ is a hyperbolic time

$$|D(r \circ h)(x)| = \frac{|Dr(h(x))|}{|DF(h(x))|} = \left| \sum_{i=0}^{\ell-1} \frac{(D\varrho \circ f^i) \cdot Df^i}{DF} \circ h(x) \right| \lesssim \sum_{i=0}^{\ell-1} \sigma^{(1-b)i}.$$

We use the logarithmic growth of ϱ near the singularity together with the control of the distance to the singularity provided by the hyperbolic times to bound the summation.

Aperiodicity/uniform non-integrability

We essentially use that $D\varrho(x) \approx |x|^{-1}$, while $Df(x) \approx |x|^{\alpha-1}$ so, for x near the singularity, these functions have different growth rates.

By contradiction: assume there is a C^1 **function** $u : \Delta \to \mathbb{R}$ and a **function** $v : \Delta \to \mathbb{R}$ **constant on each element** ω of $\mathcal{Q}$ satisfying $r = u \circ F - u + v$. Then we also have

$$r + r \circ F = v + v \circ F + u \circ F^2 - u$$

so that $v + v \circ F$ is constant on every element of $\mathcal{Q} \vee F^{-1}(\mathcal{Q})$ and $u \circ F^2 - u$ is bounded.

Vastly different growth rates

We also have the following relation between derivatives

$$Dr + Dr \circ F \cdot DF = Du \circ F^2 \cdot DF \circ F \cdot DF - Du.$$

Since $F : \omega \to \Delta$ is a diffeomorphism we take x_n in ω with $x_n \xrightarrow[n\to\infty]{} q \in \omega$ and $F(x_n) \xrightarrow[n\to\infty]{} 0^+$ (the singularity is at 0). Hence we get

$$Dr(x_n) \to Dr(q), \quad Du(x_n) \to Du(q), \quad DF(x_n) \to DF(q) > 0,$$

consider a rearrangement of the above identity

$$Dr(F(x_n)) - Du(F^2(x_n)) \cdot DF(F(x_n)) = \frac{Dr(x_n) + Du(z_n)}{DF(x_n)}$$

and show that **unbounded LHD = bounded RHS!!**.

Smooth conditional disintegration on the cross-section

Let η be the $\hat{F}$-invariant probability on $\widehat{\Delta}$ corresponding to F-invariant abs. cont. probability ν on I. Let $u : \widehat{\Delta} \to \mathbb{R}$ be C^1 and

$$\bar{u} : \widehat{\Delta} \to \mathbb{R}, \quad x \mapsto \bar{u}(x) := \int u(y)\, d\eta_x(y)$$

where $\{\eta_x\}_x$ is the disintegration of η with respect to the partition defined by the stable leaves on $S \supset \widehat{\Delta}$.

We prove a theorem showing that η_x can be obtained as the limit of a certain operator related to the Ruelle-Perron-Frobenius transfer operator; and this property ensures that $D\bar{u}$ exists and is continuous, and also provides an "explicit" expression for $D\bar{u}$.

Disintegration as limit of R-P-F operator

We prove the following relation

$$\bar{u}(x) = \int u\, d\bar{\omega}_x = \int \lim_{n\to+\infty} \frac{1}{\phi}\mathcal{P}^n(\phi \cdot u \circ \widehat{F}_t^n)(x)\; d\lambda(t)$$

where

- $\widehat{F}_t^n(z) := \widehat{F}^n(z,t)$ for $(z,t) \in \widehat{\Delta}$;
- $\phi := d\nu/d\lambda$ is the density of ν;
- $\mathcal{P}(\varphi)(x) := \sum_{h\in\mathcal{H}} \varphi(h(x))DF(h(x))^{-1}$ is the R-P-F operator.

Smooth disintegration

Using the last identity, we can prove that

$$D\bar{u}(x) = -\frac{D\phi(x)}{\phi(x)} \int \lim_{n\to+\infty} \frac{1}{\phi}\mathcal{P}^n(\phi \cdot u \circ \widehat{F}_t^n)\, d\lambda(t) = -\frac{D\phi(x)}{\phi(x)}\bar{u}(x)$$

and so $D\bar{u}$ exists, hence $\bar{u}$ is continuous, thus by the last identity $D\bar{u}$ is also continuous. This proves that the disintegration $(\eta_x)_x$ is smooth.

In particular: $\bar{u}(x) = \frac{1}{\phi(x)} + \int u\, d\eta - 1$.

This concludes the proof of the main result, since all steps can be performed for all C^3 nearby flows!

Some conjectures and extensions

- Similar results should hold with different roof functions; perhaps we can provide an explicit class of good roof functions.
- Similar results should hold for general singular-hyperbolic attractors in 3-manifolds.
- Analogous results should hold for sectional-hyperbolic attractors in higher-dimensions.
- Does the exponential rate of decay depend continuously or smoothly on the flow?

References to main related works

References

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Finally, we have reached...

THE END.

Thanks for your attention!