# Star flows

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# Definition of Star flows

$M$ : $d$-dim, compact boundaryless Riemannian manifold

Star flow: $X \in \mathcal{X}^1(M)$ is a star flow if $\exists$ a $C^1$ neighborhood $\mathcal{U}$ of $X$ s.t. $\forall Y \in \mathcal{U}$, every critical element (periodic orbit or singularity) of $Y$ is hyperbolic.

# Background

Franks, 1971, $\Omega$-structural stability implies star condition.

Guckenheimer, 1976, $\exists$ 3-dim star flow, which is not Axiom A.

Liao, 1981: Nonsingular 3-dim star flow is Axiom A

Liao's selecting lemma and shadowing lemma

Mañé, 1982: 2-dim star diff. is Axiom A.

Sannami, 1983

Mañé's ergodic closing lemma.

Aoki, Hayashi, 1992: n-dim star diff. is Axiom A.

G-Wen, 2006: n-dim nonsingular star flows is Axiom A

# Examples

Singular star flow may not be Axiom A.

Counterexamples: Guckenheimer, Ding, Li-Wen

How to describe the hyperbolicity of singular star flow?

Given $X \in \mathcal{X}^1(M)$, denote:

$\phi_t : M \to M$

$\Phi_t = \mathrm{d}\phi_t : TM \to TM$

# Hyperbolicity for flow

Let $\Lambda$ be a compact invariant set of $X$. $\Lambda$ is called hyperbolic if there exists a continuous subbundle splitting

$$T_\Lambda M = E^s \oplus \langle X \rangle \oplus E^u$$

s.t. $E^s$ is uniformly contracting and $E^u$ is uniformly expanding.

If $\Lambda$ is a singularity or a periodic orbit, $\dim E^s$ is called the index of $\Lambda$, denoted by $\mathrm{Ind}(\Lambda)$.

By definition, singularities in $\Lambda$ are isolated in $\Lambda$.

# Breakthrough: Singular hyperbolicity

Assume that $\dim M = 3$.

Morales, Pacifico and Pujals, 1999.

Let $\Lambda$ be a compact invariant set of $X$. $\Lambda$ is **singular hyperbolic** if $\exists$ partially hyperbolic splitting:

$$T_{C(\sigma)}M = E^{ss} \oplus_{\prec} F, \quad \dim E^{ss} = 1,\ \dim F = 2,$$

and $E^{ss}$ is uniformly contracting and $F$ is **area-expanding**, i.e., $\exists\ C \geq 1, \lambda > 0$ such that for every $t \geq 0$,

$$|\det \Phi_t|_F| \geq C^{-1}\mathrm{e}^{\lambda t}.$$

**Theorem (Morales-Pacifico-Pujals, 2003).** Assume that $X \in \mathcal{X}^1(M)$, $\dim M = 3$. Every robustly transitive invariant set of $X$ is singular hyperbolic.

# Generalization

Let $\Lambda$ be a compact invariant set of $X$. $\Lambda$ is **singular hyperbolic** if ($X$ or $-X$) $\exists$ partially hyperbolic splitting:

$$T_{C(\sigma)}M = E^{ss} \oplus_{\prec} F,$$

$E^{ss}$ is uniformly contracting and $F$ is **sectional-expanding**, i.e., $\exists$ $C \geq 1, \lambda > 0$ such that for every $t \geq 0$, and every 2 dim subspace $L \subset F$,

$$|\det \Phi_t|_L| \geq C^{-1} \mathrm{e}^{\lambda t}.$$

Equivalently,

$$m(\wedge^2 \Phi_t|_F) \geq C^{-1} \mathrm{e}^{\lambda t}$$

# Generalization

Strongly homogeneous: $\Lambda$ is strongly homogeneous if $\exists$ integer $I$ and neighborhoods $\mathcal{U}$ of $X$ and $U$ of $\Lambda$ such that every periodic orbit of every $Y \in \mathcal{U}$ has index $I$.

**Theorem (Li-G-Wen, 2005).** If $\Lambda$ is

1. robustly transitive,
2. strongly homogeneous, and
3. indices of singularities in $\Lambda$ are greater than $I$,

then $\Lambda$ is partially hyperbolic.

**Theorem (Metzger-Morales, 2008)** Under the conditions of LGW, $\Lambda$ is singular hyperbolic.

**Theorem (Zhu-G-Wen, 2008)** The third condition in LGW is automatically satisfied and $\Lambda$ is singular hyperbolic.

# Chain recurrent class

$x \sim y \Leftrightarrow \forall \epsilon > 0$, $\exists\epsilon$-chains from $x$ to $y$ and from $y$ to $x$.

$$\mathrm{CR} = \{x : x \sim x\}$$

$\sim$: closed equivalence relation over CR.

**Chain (recurrent) class:** Equivalent class of CR under $\sim$

# Dream for star flows

**Conjecture I.** For every star v.f. $X$, CR$(X)$ is singular hyperbolic and consists of finitely many chain classes.

!!! Open even for $\dim M = 2$ !!!

(Connecting difficulty)

**Conjecture II.** For generic star v.f. $X$, CR$(X)$ is singular hyperbolic and consists of finitely many chain classes.

$\dim M = 2$, Peixoto.

$\dim M = 3$, Morales-Pacifico, 2003.

# 4 dim star flows

**Theorem (Shi-G-Wen).** For 4 dim generic star v.f. $X$, $\mathrm{CR}(X)$ is singular hyperbolic.

D. Yang and M. Li, a special case.

Arbieto-Morales, 2011, finiteness for some special cases.

# 2 steps

**Proposition A.** For 4 dim generic star v.f. $X$, if $C(\sigma)$ is nontrivial, then (for $X$ or $-X$)

1. for every singularity $\rho$ in $C(\sigma)$,

$$W^{ss}(\rho) \cap C(\sigma) = \{\rho\}.$$

2. for every singularity $\rho$ in $C(\sigma)$,

$$\mathrm{Ind}(\rho) = \mathrm{Ind}(\sigma), \qquad \mathrm{sv}(\rho) > 0.$$

**Proposition B.** For generic star v.f. $X$, if $C(\sigma)$ is nontrivial and the conclusions of Proposition A are satisfied, then $C(\sigma)$ is singular hyperbolic.

**Corollary C.** For generic star v.f. $X$, if $C(\sigma)$ is Lyapunov stable, then $C(\sigma)$ is singular hyperbolic.

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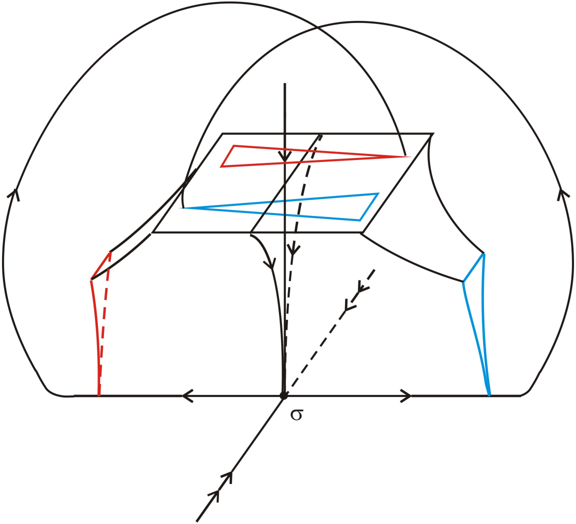
# Geometric Lorenz flow



# Basic properties of star flows

**Theorem.** **[Liao79]** $\forall$ star v.f. $\exists$ a neighborhood $\mathcal{U}$ and numbers $\eta > 0, C \geq 1, T \geq 1$ s.t. if $X \in \mathcal{U}$ then

1. For every periodic point $x$ of $X$ and $t \geq T$, one has

$$\frac{\|\psi_t|_{N^s(x)}\|}{m(\psi_t|_{N^u(x)})} \leq C\mathrm{e}^{-2\eta t};$$

2. For every periodic point $x$ of $X$, one has

$$\prod_{i=0}^{[\pi(x)/T]-1} \|\psi_T|_{N^s(\phi_{iT}(x))}\| \leq C\mathrm{e}^{-\eta\pi(x)},$$

$$\prod_{i=0}^{[\pi(x)/T]-1} m(\psi_T|_{N^u(\phi_{iT}(x))}) \geq C^{-1}\mathrm{e}^{\eta\pi(x)}.$$

# Basic properties of star flows: 2-form

Let $E$ be a finitely dimensional vector space. Denote $\wedge^2 E$ the second exterior power of $E$. Given a linear isomorphism: $A : E \to F$ between finitely dimensional vector space $E$ and $F$, denote $\wedge^2 A : \wedge^2 E \to \wedge^2 F$ the linear isomorphism induced by $A$. Now the item 2 of last theorem has the following consequence:

**Corollary.** $\forall$ star v.f. $\exists$ a neighborhood $\mathcal{U}$ and numbers $\eta > 0, C \geq 1, T \geq 1$ s.t. if $X \in \mathcal{U}$ then for every periodic point $x$ of $X$, one has

$$\prod_{i=0}^{[\pi(x)/T]-1} \| \wedge^2 \Phi_T|_{E^{cs}(\phi_{iT}(x))}\| \leq C\mathrm{e}^{-\eta\pi(x)},$$

$$\prod_{i=0}^{[\pi(x)/T]-1} m(\wedge^2 \Phi_T|_{E^{cu}(\phi_{iT}(x))}) \geq C^{-1}\mathrm{e}^{\eta\pi(x)}.$$

For simplicity, we always assume that $C = T = 1$.

# Lorenz-like singularity

**Lemma.** Let $X$ be a star v.f. and $\sigma$ a singularity of $X$. If $C(\sigma)$ is nontrivial, then $\sigma$ is Lorenz-like, i.e., if the Lyapunov exponents of $\Phi_t(\sigma)$ is

$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d,$$

then

1. either $\lambda_{s-1} \neq \lambda_s$ or $\lambda_{s+1} \neq \lambda_{s+2}$,
2. if $\lambda_{s-1} = \lambda_s$, then $\lambda_s + \lambda_{s+1} < 0$.
3. if $\lambda_{s+1} = \lambda_{s+2}$, then $\lambda_s + \lambda_{s+1} > 0$.
4. if $\lambda_{s-1} \neq \lambda_s$ and $\lambda_{s+1} \neq \lambda_{s+2}$, then $\lambda_s + \lambda_{s+1} \neq 0$.

Idea: use the dominated splitting of extended linear Poincaré flow (natural compactification of linear Poincaré flow).

# Saddle value

**Corollary.** Let $\sigma$ be a singularity of a star v.f. $X$ such that $C(\sigma)$ is nontrivial. Then $\mathrm{sv}(\sigma) \neq 0$.

If the Lyapunov exponents of $\Phi_t(\sigma)$ is

$$\lambda_1 \leq \cdots \leq \lambda_s < 0 < \lambda_{s+1} \leq \cdots \leq \lambda_d,$$

the saddle value of $\sigma$ is

$$\mathrm{sv}(\sigma) = \lambda_s + \lambda_{s+1}.$$

# Periodic index of singularity

Now, define the periodic index of $\sigma$ by

$$\mathrm{Ind}_p(\sigma) = \begin{cases} s, & \text{if } \mathrm{sv}(\sigma) < 0, \\ s-1, & \text{if } \mathrm{sv}(\sigma) > 0. \end{cases}$$

For periodic orbit $P$, $\mathrm{Ind}_p(P) = \mathrm{Ind}(P)$.

**Lemma.** Let $\sigma$ be a singularity of a star v.f. $X$ and $\Gamma = \mathrm{Orb}(x)$ a homoclinic orbit of $\sigma$. Assume that $X_n \to X$ and $P_n = \mathrm{Orb}(p_n)$ a periodic orbit of $X_n$ such that $P_n \to \Gamma \cup \{\sigma\}$. Then

$$\lim_{n\to\infty} \mathrm{Ind}(P_n) = \mathrm{Ind}_p(\sigma),$$

i.e., for $n$ large enough $\mathrm{Ind}(P_n) = \mathrm{Ind}_p(\sigma)$.

Idea: Assume that $\mathrm{sv}(\sigma) > 0$ and

1. $\mathrm{Ind}(P_n) < s - 1$. Consider the subspace
$$F_n = E^u(P_n) \oplus \langle X_n(P_n) \rangle.$$
Let $F_n \to F$. Then
$$\dim(F \cap E^s(\sigma)) \geq \dim F + \dim E^s(\sigma) - d \geq d + 2 - s + s - d = 2.$$
Contradicting to the sectional expanding of $F$ and sectional contracting of $E^s$.
2. $\mathrm{Ind}(P_n) > s - 1$. Consider the subspace
$$E_n = E^s(P_n) \oplus \langle X_n(P_n) \rangle.$$
Let $E_n \to E$. Then $\dim(E \cap (E^u(\sigma) \oplus E^c(\sigma)))$
$$\geq \dim E + \dim(E^u(\sigma) \oplus E^c(\sigma)) \geq s + 1 + d - s + 1 = 2.$$
Contradicting to the sectional expanding of $E^u(\sigma) \oplus E^c(\sigma)$ and the sectional contracting of $E$.

Idea: Assume that $\mathrm{sv}(\sigma) > 0$ and

1. $\mathrm{Ind}(P_n) < s - 1$. Consider the subspace

   $$F_n = E^u(P_n) \oplus \langle X_n(P_n) \rangle.$$

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   Let $E_n \to E$. Then $\dim(E \cap (E^u(\sigma) \oplus E^c(\sigma)))$

   $$\geq \dim E + \dim(E^u(\sigma) \oplus E^c(\sigma)) \geq s + 1 + d - s + 1 = 2.$$

   Contradicting to the sectional expanding of $E^u(\sigma) \oplus E^c(\sigma)$ and the sectional contracting of $E$.

Precisely, $\exists p_n \in P_n \to y \in \Gamma$ s.t.

$$\frac{1}{k}\sum_{i=0}^{k-1} \log m(\wedge^2 \Phi_{1,n}|_{\Phi_{i,n}(F_n(p_n))}) \geq \eta, \;\; k = 1, 2, \cdots$$

$$\frac{1}{k}\sum_{i=0}^{k-1} \log m(\wedge^2 \Phi_1|_{\Phi_i(F(y))}) \geq \eta, \;\; k = 1, 2, \cdots$$

One can find $n_j \to \infty$ such that

$$\frac{1}{k}\sum_{i=0}^{k-1} \log m(\wedge^2 \Phi_1|_{\Phi_{i+n_j}(F(y))}) \geq \eta/2, \;\; k = 1, 2, \cdots$$

Since $\phi_{n_j}(y) \to \sigma$, $\exists$ subspace $F \subset T_\sigma M$ with $\dim F = \dim F_n(p_n)$ and

$$\frac{1}{k}\sum_{i=0}^{k-1} \log m(\wedge^2 \Phi_1|_{\Phi_i(F)}) \geq \eta/2, \;\; k = 1, 2, \cdots$$

# Heterodimensional cycle

**Lemma 1.** For generic star v.f. $X$, for every critical element $c$ in $C(\sigma)$, $\mathrm{Ind}_p(c) = \mathrm{Ind}_p(\sigma)$.

For periodic orbit $P$, $\mathrm{Ind}_p(P) = \mathrm{Ind}(P)$.

Idea: singularity $\rho$ can be considered as a periodic orbit with index $\mathrm{Ind}_p(\rho)$.

**Proposition A'.** For generic star v.f. $X$, if $C(\sigma)$ is nontrivial, then (for $X$ or $-X$)

1. for every singularity $\rho$ in $C(\sigma)$, if $\mathrm{sv}(\rho) > 0$,
$$W^{ss}(\rho) \cap C(\sigma) = \{\rho\}.$$
2. for every singularity $\rho$ in $C(\sigma)$,
$$\mathrm{Ind}_p(\rho) = \mathrm{Ind}_p(\sigma).$$

Idea (for item 1): If the conclusion is false, one can get a strong homoclinic orbit $\Gamma$ w.r.t $\rho$. [LGW] has proven that if the periodic orbits approximating $\Gamma$ has index $\mathrm{Ind}_p(\rho)$, we will get a contradiction to the dominated splitting of index $\mathrm{Ind}_p(\rho)$.

# Lyapunov stability and Proof of Proposition A

We have to use Lyapunov stability of $C(\sigma)$ to prove that $\forall$ singularity $\rho$ in $C(\sigma)$, $\mathrm{Ind}(\rho) = \mathrm{Ind}(\sigma)$.

**Lemma [G-Yang].** For $C^1$ generic vector field $X$ and a hyperbolic critical element $P$ of $X$, if $W^u(P, X) \subset C(P, X)$, then $\exists$ $C^1$ neighborhood $\mathcal{U}$ of $X$, such that $\forall Y \in \mathcal{U}$, we have $W^u(P_Y, Y) \subset C(P_Y, Y)$. Moreover, if $Y$ is star v.f., then $C(P_Y)$ is Lyapunov stable.

Now, assume that $\dim M = 4$ and $\mathrm{sv}(\sigma) > 0$. If $\mathrm{Ind}(\rho) \neq \mathrm{Ind}(\sigma)$,

1. $\mathrm{Ind}(\sigma) = 1$, impossible
2. $\mathrm{Ind}(\sigma) = 2$ and $\mathrm{Ind}(\rho) = 3$. $C(\sigma)$ is Lyapunov stable.
3. $\mathrm{Ind}(\sigma) = 3$ and $\mathrm{Ind}(\rho) = 2$. $C(\sigma)$ is Lyapunov stable.

Reason: $E^u(\sigma)$ or $E^s(\sigma)$ is 1 dimensional!

**Lemma.** For generic star v.f. $X$, if $C(\sigma)$ is Lyapunov stable, then for any singularity $\rho \in C(\sigma)$, $\mathrm{Ind}(\rho) = \mathrm{Ind}(\sigma)$.

Idea: Otherwise, use connecting lemma to get a heterodimensional cycle between $\rho$ and $\sigma$. [ZGW] proved that if the non-transverse connection is not isolated in $C(\sigma)$, we will get a contradiction (with domination). But [GY]'s lemma guarantees the non-isolating of the non-transverse connection.

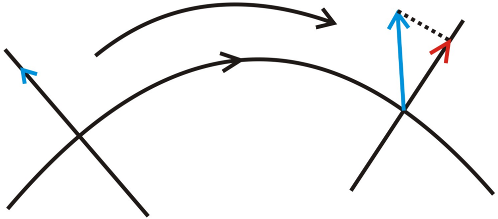
# Scaled linear Poincaré flow

$X : C^1$ vector field over $M$

$\phi_t : M \to M$: flow generated by $X$

$\Phi_t = \mathrm{d}\phi_t : TM \to TM$: tangent flow

Linear Poincaré flow $\psi_t$:



Scaled linear Poincaré flow:

$$\psi_t^* = \frac{\psi_t}{\|\mathrm{d}\phi_t|_X\|} = \frac{|X(x)|}{|X(\phi_t(x))|}\psi_t$$

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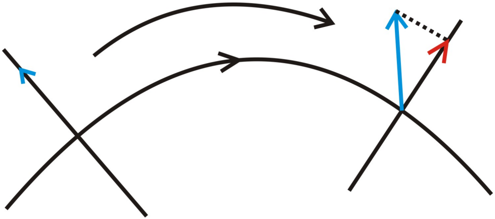

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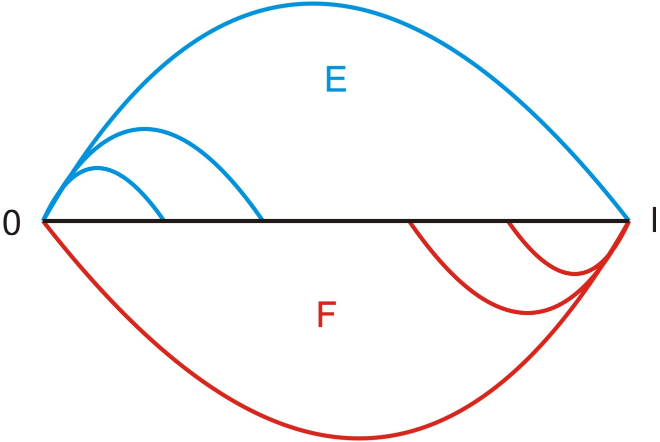
# Quasi hyperbolic orbit arc

An orbit arc $\phi_{[0,l]}(x)$ is $\lambda$ **quasi hyperbolic** (w.r.t a dominated splitting $E \oplus F$ and $\psi_t^*$), if

$$\prod_{i=0}^{k-1} \|\psi_1^*|_{E(\phi_i(x))}\| \leq \lambda^k, k = 1, 2, \cdots, l,$$

$$\prod_{i=k}^{l-1} m(\psi_1^*|_{F(\phi_i(x))}) \geq \lambda^{k-l}, k = 0, 1, \cdots, l-1.$$


E
0
l
F

# Shadowing lemma of Liao

**Theorem. (Liao 1985)** Let $\Lambda$ be a compact invariant set of $X$. And assume that $\psi_t$ has a dominated splitting over $\Lambda - \mathrm{Sing}(X)$

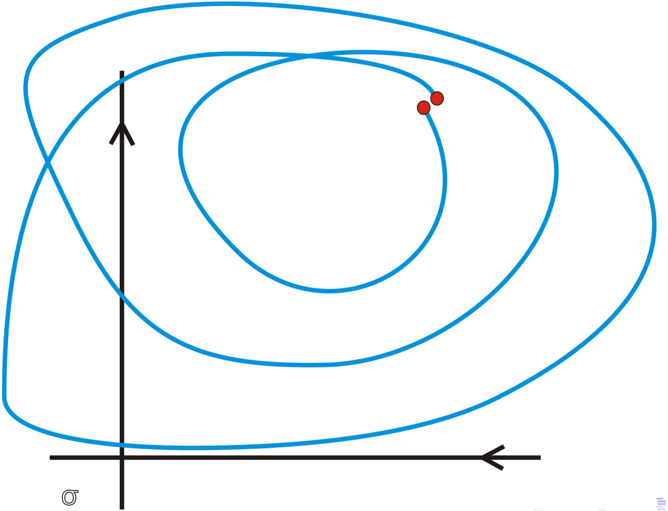
$$D_{\Lambda - \mathrm{Sing}(X)} = E \oplus F,$$

and $F \subset \Lambda$ is a nonsingular closed subset in $\Lambda$, then $\forall\ \epsilon > 0$, $\lambda \in (0,1)$, $\exists \delta > 0$, if $\phi_{[0,l]}(x)$ is a **$\lambda$ quasi hyperbolic string** (w.r.t $\psi^*$), and $x, \phi_l(x) \in F$, once $d(x, \phi_l(x)) \leq \delta$ there exists periodic point $p$ and homeomorphism $\theta : [0,l] \to [0,l']$ (where $l'$ is the period of $p$) such that

$$d(\phi_t(x), \phi_{\theta(t)}(p)) \leq \epsilon |X(\phi_t(x))|, \qquad \forall t \in [0,l].$$

# Shadowing lemma of Liao

If the two ends of a **quasi hyperbolic** string are **far from singularities**, then it can be shadowed by a periodic orbit.



# Remarks on the shadowing lemma

- Some part of the quasi hyperbolic string can be very close to singularities.
- If the ends of the quasi hyperbolic string are close to singularity, the conclusion may not hold.

# Proof of Proposition B

According to the assumption, the periodic indices of singularities in $C(\sigma)$ are the same, say $I$. And we may assume that saddle values are all positive. Now, the natural compactification $\widetilde{C(\sigma)}$ admits a continuous splitting

$$N^s \oplus E^s \oplus N^u,$$

where $N^s, N^u$ are $\psi_t$ invariant and dominated w.r.t $\psi_t$ and $E^c$ is invariant w.r.t $\Phi_t$.

**Lemma.** $(N^s, \psi_t) \prec (E^c, \Phi_t)$, i.e., $\psi_t^*$ is contracting.

Idea: Otherwise, $\exists$ ergodic measure $\mu$ s.t.

$$\int \log \|\psi_1^*|_{N^s(x)}\| \mathrm{d}\mu(x) \geq 0.$$

Since $\mu(\mathrm{Sing}(X)) = 0$, according to the Ergodic Closing Lemma, $\exists$ periodic orbits $P_n$ s.t. $\mu_n \to \mu$, where $\mu_n$ supports on $P_n$. $J \triangleq \mathrm{Ind}(P_n) < I$. Since

$$\int \log \|\psi_1|_{N^{s,J}(x)}\| \mathrm{d}\mu_n(x) \leq -\eta, \quad \int \log m(\psi_1|_{N^{u,J}(x)}) \mathrm{d}\mu_n(x) \geq \eta,$$

taking limits, we have that $\mu$ is a hyperbolic measure with index $J$. We can use Katok's argument to $\mu$ to obtain a quasi hyperbolic orbit arc with two ends very close, and then use Liao's shadowing lemma to get a periodic orbit of index $J$ inside $C(\sigma)$, Contradiction!

The sectional expanding of $E^{cu}$ can be proved similarly.

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The sectional expanding of $E^{cu}$ can be proved similarly.

**Proposition D.** Every ergodic measure of a star flow is hyperbolic. In fact, for every non-atomic ergodic measure $\mu$, $\exists T > 0, \eta > 0$ s.t.

$$\int \log \|\psi_T|_{N^s(x)}\| \mathrm{d}\mu(x) \leq -\eta,$$

$$\int \log \|\psi_{-T}|_{N^u(x)}\| \mathrm{d}\mu(x) \leq -\eta.$$

Obrigado!