Billiards and an ‘improved’ LET

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We prove a criterion for the local ergodicity of non-uniformly hyperbolic symplectic maps with singularities. Our result is an extension of a theorem by Liverani and Wojtkowski.

This is an improvement of one of the seminal results by Ya. G Sinai. Philosophy:

“Expansion prevails over fractioning”

We consider maps that model billiards and other physical systems:
invertible maps with discontinuities, unbounded derivatives, preserve a symplectic form and are non-uniformly hyperbolic.

The last condition means that the Lyapunov exp. of $\mathcal{F}$ are non-zero almost everywhere, but the angles of the invariant subspaces can go to zero.

Our main result establishes sufficient conditions for a point in the domain of $\mathcal{F}$ to have a neighborhood contained (mod zero) in one ergodic component of $\mathcal{F}$.

In fact, we prove a stronger result, namely, that the mentioned neighborhood is contained up to a set of zero measure in one Bernoulli component of $\mathcal{F}$.

$\mathcal{F}_{|A}$ is Bernoulli on invariant set A if it is (meas.theor) isomorphic (equivalent) to a Bernoulli shift (on a compact metric space C; $B(C) = \{\theta : \mathbb{Z} \to C\}$).

$$Bernoulli \Rightarrow K - System \Rightarrow Mixing \Rightarrow Ergodic$$

We recall that if E is an ergodic component of positive measure and with non-zero Lyapunov exponents, then (Pesin, Katok-Strelcyn) there exist $m > 0$ disjoint measurable sets $B_1, \ldots, B_m = B_0$ of $\mathcal{M}$ such that

1. $E = \bigcup_{i=0}^{m-1} B_i$,
2. $\mathcal{F} B_i = B_{i+1}$ for each $i = 0, \ldots, m-1$,
3. $\mathcal{F}^m|_{B_i}$ is a K-automorphism for each i.

The map $\mathcal{F}^m|_{B_i}$ is in fact Bernoulli [Chernov - Haskell], [Ornstein - Weiss]. The sets $B_1, \ldots, B_m$ are uniquely defined up to a set of zero measure, and are called *Bernoulli components* of $\mathcal{F}$.

Local ergodic theorems (LET's for short) play an essential role in the proof of the ergodicity of non-uniformly hyperbolic systems.

From LET, one can derive criterions for the ergodicity of $\mathcal{F}$ based on the topology of the set X:

Suppose that the set X where our LET applies has full measure, and that the map $\mathcal{F}$ is topological transitive. Then, we can conclude quite easily that $\mathcal{F}$ is ergodic.

This paper was originated from an attempt to use the LET of LWto prove that the billiards introduced by Donnay and Bunimovich are ergodic.

Derivative D_xT of the billiard map T, on the billiard table Γ, at $x = (q_0, v_0) = (r_0, \varphi_0)$.

$$\frac{-1}{\cos\varphi_1}\begin{bmatrix} \tau K_0 + \cos\varphi_0 & \tau \\ \tau K_0 K_1 + K_0\cos\varphi_1 + K_1\cos\varphi_0 & \tau K_1 + \cos\varphi_1 \end{bmatrix},$$

$x_1 = (r_1, \varphi_1) = T(r_0, \varphi_0) = T(x)$;

$K_i = K(x_i), i = 0, 1$, curvatures of $\partial\Gamma$ at q_i;
τ, distance between q_0 and q_1.

Ideas behind the proof of LETs can be traced back to several seminal works:

- E. Hopf (1937) on the ergodicity of the geodesic flow on a surface of negative curvature

- Anosov (1967) on uniformly hyperbolic systems

- Sinai (1970) on the ergodicity of dispersing billiards which outlines a general method for proving ergodicity of non-uniformly hyperbolic systems with discontinuities and unbounded derivatives.

The proof of the LET of Liverani and Wojtkowski (1995), on which the proof of our LET builds, is an improved and 'simplified' version of Sinai's method.

Other improvements of Sinai's method were accomplished by Sinai and Bunimovich (1973), Sinai and Chernov (1987), Krámli, Simányi and Szász (1991), Chernov (1993) and Markarian (1993).

Our proof also relies on the Katok-Strelcyn theory (LNM 1222, 1986), which extends Pesin's theory (1977) to maps with singularities and techniques to prove non-vanishing Lyapunov exponents.

The LET of LW assumes that the cone field $\mathcal{C}$ is defined everywhere on the interior of the domain of $\mathcal{F}$; we assume $\mathcal{C}$ defined only on an open subset of the domain.

Our proof follows closely that of LW, but several additions to their proof are required. The main one is an improved version of the Tail Bound (Sinai's Th.).

Lebesgue measure of "bad" rectangles (proportion of long unstable manifolds) is small.

Because of the difference in the assumption on the cone field $\mathcal{C}$, hypotheses of our LET are slightly different than the corresponding hypotheses of [LW].

Ergodic, mixing, ... plane billiards

Dispersing (Sinai), Semidispersing, 70.

Stadiums (Bunimovich), 73-79.

“Wojtkowski’s billiards” (Szász, Markarian), 92 - 93.

Dispersing with zero angle (Rehacek), 95.

Truncated elliptical billards (Del Magno), 01.

Semidispersing billiards with infinite cusp (Lenci) 02.

Elliptical stadiums (Del Magno - Markarian), 03.

$d^2R/ds^2 < 0$,
Lenci-Bussolari: *Hyperbolic billiards with nearly flat focusing boundaries.*

1 Basic definitions

A compact subset $\mathcal{A}$ of a C^2 manifold $\mathcal{M}$ of dimension $k \geq 2$ is called *regular* if it is a union of finitely many compact subsets $\mathcal{A}_1, \ldots, \mathcal{A}_n$ of $(k-1)$-dimensional C^2 submanifolds of $\mathcal{M}$ such that

1. each $\mathcal{A}_i$ is equal to the closure of its interior,
2. $\mathcal{A}_i \cap \mathcal{A}_j \subset \partial \mathcal{A}_i$ for $i \neq j$,
3. the boundary of each $\mathcal{A}_i$ is a union of finitely many compact subsets of $(k-2)$-dim. subman.

$\mathcal{A}_1, \ldots, \mathcal{A}_n$ are called *(regular) components of* $\mathcal{A}$.

Let $(\mathcal{M}, \omega)$ be a C^2 symplectic compact 2d-manifold, possibly with boundary and corners. $\partial\mathcal{M}$ is regular. The symplectic form ω is allowed to degenerate, but only on $\partial\mathcal{M}$.

$$\omega(y, x) = -\omega(x, y)$$
$$\omega(\lambda_1 x_1 + \lambda_2 x_2, y) = \lambda_1 \omega(x_1, y) + \lambda_2 \omega(x_2, y),$$
$$\text{non-degenerate: } \omega(x, y) = 0, \forall y \text{ implies that } x = 0.$$

Riemannian metric g in $\mathcal{M}$. Corresponding norm and distance are denoted by $\|\cdot\|$ and d, resp. Measures on $\mathcal{M}$ generated by g and ω are $\mathcal{L}$ and μ, respectively.

Under previous hypotheses $\mu \leq a\mathcal{L}$ for some constant $a > 0$.

Plane billiards $g : ds^2 + d\theta^2, \;\; \omega_x = \cos\theta ds \wedge d\theta.$

Definition. $(\mathcal{M}, \omega, g, \mathcal{F})$ is a *symplectomorphism with singular set* $\mathcal{R}$ if there exist two regular subsets $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$ of $\mathcal{M}$ and a C^2 diffeomorphism

$$\mathcal{F} : \mathcal{M} \setminus (\partial\mathcal{M} \cup \mathcal{S}_1^+) \to \mathcal{M} \setminus (\partial\mathcal{M} \cup \mathcal{S}_1^-) \text{ such that}$$

1. $\mathcal{S}_1^\pm \cap \partial\mathcal{M} \subset \partial\mathcal{S}_1^\pm$;
2. $\mathcal{F}$ preserves ω;
3. $\log^+ \|D_x\mathcal{F}\|$, $\log^+ \|D_x\mathcal{F}^{-1}\| \in L^1(\mu)$ (Oseledets),

$$\|D_x^2\mathcal{F}\| \leq \frac{A}{d(x, \mathcal{R})^b},$$

where $\mathcal{R} = \mathcal{R}_1^- \cup \mathcal{R}_1^+$ and $\mathcal{R}_1^\pm = \partial\mathcal{M} \cup \mathcal{S}_1^\pm$.

Conditions in [KS], sufficient to apply Pesin´s theory, follow from 3 in the previous definition, and from the regularity of $\mathcal{R}_1^-$ and $\mathcal{R}_1^+$: $\mu(\mathcal{R}_1^{\pm}(\epsilon)) \leq C\epsilon^a$ for every $\epsilon > 0$ sufficiently small. Given a subset $\mathcal{A} \subset \mathcal{M}$ and $\epsilon > 0$, let $\mathcal{A}(\epsilon) = \{y \in \mathcal{M} : d(y, \mathcal{A}) < \epsilon\}$ is ϵ-neighborhood of $\mathcal{A}$.

The sets $\mathcal{M} \setminus \mathcal{R}_1^+$ and $\mathcal{M} \setminus \mathcal{R}_1^-$ have the same finite number of connected components.

We do not require as in [LW] that the restriction of $\mathcal{F}(\mathcal{F}^{-1})$ to each connected component of $\mathcal{M} \setminus \mathcal{R}_1^+(\mathcal{M} \setminus \mathcal{R}_1^-)$ has a homeomorphic extension up to its boundary, because this condition is superfluous.

Definition. For every $n > 1$, define recursively

$$\mathcal{R}_n^+ = \mathcal{R}_{n-1}^+ \cup \mathcal{F}^{-1}\mathcal{R}_{n-1}^+,$$
$$\mathcal{R}_n^- = \mathcal{R}_{n-1}^- \cup \mathcal{F}\mathcal{R}_{n-1}^-.$$

$\mathcal{F}^n : \mathcal{M} \setminus \mathcal{R}_n^+ \to \mathcal{M} \setminus \mathcal{R}_n^-$ is a C^2 diffeomorphism.

Definition. Let $\mathcal{L}_+ = \mathcal{L}_{\mathcal{S}_1^+}$ and $\mathcal{L}_- = \mathcal{L}_{\mathcal{S}_1^-}$ be the volume measures induced by g on $S_1^{\pm}$.

1.1 Quadratic forms and invariant cone fields

A is a **Lagrangian subspace** of a 2d-symplectic space if has dimension d and $\omega(X,Y)=0$ for every $X,Y\in A$.

Let U be an open subset of $\mathcal{M}$, and consider two families $A=\{A_x\}_{x\in U}$ and $B=\{B_x\}_{x\in U}$ of transverse Lagrangian subspaces $A_x, B_x \subset T_x\mathcal{M}$ for $x\in U$

We define a quadratic form $\mathcal{Q}=\{\mathcal{Q}_x\}_{x\in U}$ on U associated to the transverse Lagrangian families A and B by $\mathcal{Q}_x(u) = \omega_x(u_1,u_2) \qquad$ for every $u \in T_x\mathcal{M}$ and $x\in U$, where $u_1\in A_x$ and $u_2\in B_x$ are uniquely defined by $u=u_1+u_2$.

$\mathcal{Q}$ is *continuous* if the mappings $x \mapsto A_x$ and $x \mapsto B_x$ are continuous;

The cone field $\mathcal{C} = \{\mathcal{C}(x)\}_{x \in U}$ on U associated to A and B, is the family of closed cones given by

$$\mathcal{C}(x) = \mathcal{Q}_x^{-1}\left([0, +\infty)\right) \subset T_x\mathcal{M} \qquad \text{for every } x \in U,$$

$\mathcal{C}$ is *invariant (with respect to $\mathcal{F}$)* if $D_x\mathcal{F}^k\mathcal{C}(x) \subset \mathcal{C}(\mathcal{F}^k x)$ for every $x \in U$ and $k > 0$ such that $\mathcal{F}^k x \in U$.

$\mathcal{C}$ is *eventually strictly invariant* if it is invariant, and for a.e. $x \in U$, there exists an integer $k(x) > 0$ such that $\mathcal{F}^{k(x)}x \in U$ and $D_x\mathcal{F}^{k(x)}\mathcal{C}(x) \subset \operatorname{int}\mathcal{C}(\mathcal{F}^{k(x)}x)$.

Related with the 'monotonicity' of Q.

Notion of complementary cone field $\mathcal{C}'$.

We formalize the notion of least expansion of the iterates of $D\mathcal{F}$ with respect to the quadratic form $\mathcal{Q}$, and with respect to the norm $\|\cdot\|$.

Definition. Let $\mathcal{C}$ be an invariant cone field on U, and let $\mathcal{Q}$ be the quadratic form generating it. For every $x \in U$ and $k > 0$ such that $\mathcal{F}^k x \in U$, let

$$\sigma_{\mathcal{C}}(D_x\mathcal{F}^k) = \inf_{u \in \operatorname{int} \mathcal{C}(x)} \sqrt{\frac{\mathcal{Q}_{\mathcal{F}^k x}(D_x\mathcal{F}^k u)}{\mathcal{Q}_x(u)}},$$

$$\sigma^*_{\mathcal{C}}(D_x\mathcal{F}^k) = \inf_{u \in \operatorname{int} \mathcal{C}(x)} \frac{\sqrt{\mathcal{Q}_{\mathcal{F}^k x}(D_x\mathcal{F}^k u)}}{\|u\|}.$$

For $k < 0$, we define $\sigma_{\mathcal{C}}$ and $\sigma^*_{\mathcal{C}}$ by replacing the cone field $\mathcal{C}$ in the definitions above with its complementary cone field $\mathcal{C}'$.

From the invariance of $\mathcal{C}$, it follows immediately that $\sigma_{\mathcal{C}}(D_x\mathcal{F}^k) \geq 1$.

Furthermore, if $D_x\mathcal{F}^k\mathcal{C}(x) \subset \operatorname{int}\mathcal{C}(\mathcal{F}^k x)$, then $\sigma_{\mathcal{C}}(D_x\mathcal{F}^k) > 1$

Joint invariance for two cone fields. Roughly speaking, if the cone fields $\mathcal{C}_1$ and $\mathcal{C}_2$ are jointly invariant, then $\mathcal{C}_2$ can be thought as an extension of $\mathcal{C}_1$, and vice versa.

Definition. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two cone fields defined on the open sets U_1 and U_2, respectively. We say that $\mathcal{C}_1$ and $\mathcal{C}_2$ are *jointly invariant* if

- $D_x\mathcal{F}^k\mathcal{C}_1(x) \subset \mathcal{C}_2(\mathcal{F}^k x)$ for every $x \in U_1$ and $k > 0$ such that $\mathcal{F}^k x \in U_2$,
- $D_x\mathcal{F}^k\mathcal{C}_2(x) \subset \mathcal{C}_1(\mathcal{F}^k x)$ for every $x \in U_2$ and $k > 0$ such that $\mathcal{F}^k x \in U_1$.

We neither require that the sets U_1 and U_2 are disjoint nor that the cone fields $\mathcal{C}_1$ and $\mathcal{C}_2$ are invariant.

However, it is easy to see that $\mathcal{C}_1$ and $\mathcal{C}_2$ are invariant in the following sense: if $x \in U_1$ and $k_2 > k_1 > 0$ such that $\mathcal{F}^{k_1}x \in U_2$ and $\mathcal{F}^{k_2}x \in U_1$, then
$D_x\mathcal{F}^{k_2}\mathcal{C}_1(x) \subset \mathcal{C}_1(\mathcal{F}^{k_2}x)$.

The same is true for $\mathcal{C}_2$, once U_1 has been replaced by U_2.

2 Sufficient and essential points

Notions borrowed from Chernov (*Local ergodicity of hyperbolic systems with singularities*, 1993).

A point $x \in \mathcal{M} \setminus \partial\mathcal{M}$ is called *sufficient* if there exist

(i) an integer l such that $\mathcal{F}^l$ is a local diffeomorphism at x,

(ii) a neighborhood U of $\mathcal{F}^l x$ and an integer $N > 0$ such that $U \cap \mathcal{R}_N^- = \emptyset$,

(iii) an invariant continuous cone field $\mathcal{C}$ on $U \cup \mathcal{F}^{-N} U$ such that $\sigma_{\mathcal{C}}(D_y \mathcal{F}^N) > 3$ for every $y \in \mathcal{F}^{-N} U$.

To emphasize the role of l, N, U and $\mathcal{C}$ in this definition, we say that x is a sufficient point with quadruple $(l, N, U, \mathcal{C})$.

The specific amount of expansion $\sigma_{\mathcal{C}}(D_y\mathcal{F}^N) > 3$ in the definition of a sufficient point is required only for maps with singularities. For smooth maps, the weaker condition $\sigma_{\mathcal{C}}(D_y\mathcal{F}^N) > 1$ suffices. The condition $\sigma_{\mathcal{C}}(D_y\mathcal{F}^N) > 3$ is used in one of the steps of the proof of Sinai's Theorem (Proposition **??**).

Every point of the neighborhood U is a sufficient point with quadruple $(0, N, U, \mathcal{C})$.

Proposition 2.1. Local stable (unstable) manifolds. *Let* $x \in \mathcal{M} \setminus \partial\mathcal{M}$ *be a sufficient point with quadruple* $(l, N, \mathcal{C}, U)$. *Then, there exist an invariant measurable set* $\Lambda \subset \bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U$ *with* $\mu(\bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U \setminus \Lambda) = 0$ *and* C^2 *submanifolds* $V^s = \{V^s_y\}_{y\in\Lambda}$ *and* $V^u = \{V^u_y\}_{y\in\Lambda}$ *such that for every* $y \in \Lambda$*:*

$V^s_y \cap V^u_y = \{y\}$; V^s_y *and* V^u_y *are d-dimensional balls,*

$T_y V^s_y \subset \mathcal{C}'(y)$ *and* $T_y V^u_y \subset \mathcal{C}(y)$ *if* $y \in U \cup \mathcal{F}^{-N} U$,

$\mathcal{F} V^s_y \subset V^s_{\mathcal{F}y}$ *and* $\mathcal{F}^{-1} V^u_y \subset V^u_{\mathcal{F}^{-1}y}$,

$d(\mathcal{F}^n y, \mathcal{F}^n z) \to 0$ *exponentially as* $n \to +\infty$ *for every* $z \in V^s_y$, *same is true as* $n \to -\infty$ *for every* $z \in V^u_y$.

Let x be a sufficient point of $\mathcal{M} \setminus \partial\mathcal{M}$, and let Λ be the set as in Proposition 2.1.

For every $y \in \Lambda$, W_y^u is the connected component of $\bigcup_{k\geq 0} \mathcal{F}^k V^u_{\mathcal{F}^{-k}y}$ containing y. Analogously, W_y^s.

W_y^s and W_y^u are immersed submanifolds of $\mathcal{M}$.

Proposition 2.1 remains valid if $\mathcal{C}$ and $\mathcal{D}$ are jointly invariant on an open set V. Then, for a.e. $y \in \Lambda$, we have

$$\begin{aligned} T_z W_y^u &\subset \mathcal{D}(z) \qquad \text{for } z \in W_y^u \cap V, \\ T_z W_y^s &\subset \mathcal{D}'(z) \qquad \text{for } z \in W_y^s \cap V. \end{aligned}$$

Essential points appear in the formulation of of our LET. Play the same role as the points with strictly unbounded derivatives in the Sinai-Chernov Ansatz.

A point $x \in \mathcal{M} \setminus \partial\mathcal{M}$ is called *u-essential* if for every $\alpha > 0$, there exist

a neighborhood U of x and an integer $n > 0$ such that $U \cap \mathcal{R}_n^+ = \emptyset$,

an invariant continuous cone field $\mathcal{C}$ on $U \cup \mathcal{F}^n U$ such that $\sigma_{\mathcal{C}}^*(D_y\mathcal{F}^n) > \alpha$ for every $y \in U$.

Analogously, *s-essential point* by replacing in the definition above $\mathcal{F}$ and $\mathcal{R}_n^+$ with $\mathcal{F}^{-1}$ and $\mathcal{R}_n^-$, respectively.

Theorem 2.2 (LET). *Let* $x \in \mathcal{M} \setminus \partial\mathcal{M}$ *be a sufficient point with quadruple* $(l, N, U, \mathcal{C})$. *Furthermore, let* Λ *be the subset of* $\bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U$ *as in Proposition 2.1, and suppose that Conditions L1-L4 below are satisfied.*

L1 (Regularity) *The sets* $\mathcal{R}_k^+$ *and* $\mathcal{R}_k^-$ *are regular for every* $k > 0$.

Let X a codimension 1 subspace of $\mathcal{V}$. Then the characteristic line *$L_\omega(X)$ is the skew-orthogonal complement of X: $L_\omega(X)=\{u\in\mathcal{V}:\omega(u,v)=0$ for all $v\in X\}$.*

L2 (Alignment) *For every $k>0$, we have*

- *if Σ is a component of $\mathcal{R}_k^-$ and $y\in\Sigma\cap\mathcal{F}^{-N}U$, then*

$$L_\omega(T_y\Sigma)\subset\mathcal{C}(y),$$

- *if Σ is a component of $\mathcal{R}_k^+$ and $y\in\Sigma\cap U$, then*

$$L_\omega(T_y\Sigma)\subset\mathcal{C}'(y).$$

L3 (Sinai-Chernov Ansatz) *The set of all y, $u(s)$-essential points of $\mathcal{S}_1^-(\mathcal{S}_1^+)$ has full $\mathcal{L}_-(\mathcal{L}_+)$-measure, and $\mathcal{C}$ and each cone field of the family of the invariant cone fields associated to y are jointly invariant.*

L4 (Contraction) *There exist* $\beta > 0$ *and* $\xi > 0$ *such that*

- *if* $y \in \Lambda \cap U$, $z \in W_y^u$ *and* $\mathcal{F}^{-k} z \in \mathcal{S}_1^-(\xi)$ *with* $k > 0$, *then*
$$\left\| D_z \mathcal{F}^{-k}|_{T_z W_y^u} \right\| \leq \beta,$$
- *if* $y \in \Lambda \cap U$, $z \in W_y^s$ *and* $\mathcal{F}^{k} z \in \mathcal{S}_1^+(\xi)$ *with* $k > 0$, *then*
$$\left\| D_z \mathcal{F}^{k}|_{T_z W_y^s} \right\| \leq \beta.$$

Then, there exists a neighborhood $\mathcal{O}$ of x contained (mod 0) in a Bernoulli ergodic component of $\mathcal{F}$.

Since Conditions L1-L4 depend only on $N, U, \mathcal{C}$ and not on x, we see that if the LET applies to x, then it does to every point of U.

Corollary 2.3. *Under the same hypotheses of Theorem 2.2, we have*

1. *each Bernoulli component of* $\mathcal{F}$ *contained in* $\bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U$ *is open (mod 0);*
2. *each connected component of* $\bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U$ *is contained (mod 0) in a Bernoulli component of* $\mathcal{F}$*. In particular, if* $\bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U$ *is connected, then* $\bigcup_{k\in\mathbb{Z}} \mathcal{F}^k U$ *coincides (mod 0) with a Bernoulli component of* $\mathcal{F}$*.*

Differences between the hypotheses of our LET and those of the LET of Liverani and Wojtkowski.

Cone field: The invariant cone field $\mathcal{C}$ in our definition of a sufficient point is defined on $U \cup \mathcal{F}^{-N}U$ with U being an open subset of $\mathcal{M} \setminus \partial\mathcal{M}$. Instead, Liverani and Wojtkowski assume that $\mathcal{C}$ is defined on the entire set $\mathcal{M} \setminus \partial\mathcal{M}$. Our weaker condition on $\mathcal{C}$ suffices to prove the LET, BUT the other hypotheses of our LET turn out to be more involved than those of the LET of [LW].

L1: Condition L1 is identical to the condition called Regularity in [LW, Section 7]).

L2 and L4: Condition L2 corresponds to the condition called Proper Alignment (PA) in [LW]. They use L2 in the proof of their LET, and not the full PA.

A similar remark can be made for our L4 and the Noncontraction condition of [LW].

There is an important difference between L2 and the PA. PA requires the characteristic lines of the tangent spaces of the singular sets to be contained in the interior of the cones, whereas in L2 these characteristic lines have just to be in the cones. The strict inclusion assumed by the Proper Alignment is used in the original proof of the Tail Bound [LW], but not in our proof.

L3: The difference between Condition L3 and the Sinai-Chernov Ansatz of [LW] is due to the fact that we do not assume the cone field $\mathcal{C}$ to be defined on the singular sets $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$.

But we still need a condition similar to the ('just convenient and temporary technical') Si-Ch Ansatz.

To remedy this situation introduce essential points and joint invariant cone fields in the formulation of L3.

Recent Upgrading of the LET for planar semi-dispersing billiards without using the Ansatz [Chernov - Simány].

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