Partially hyperbolic diffeomorphisms on the 3-torus

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Motivation

Theorem (Mañe-Franks)

Let S be a closed surface. The following are equivalent:

- *f is C^1-robustly transitive*
- *f is Anosov*
- *f is conjugated to a linear Anosov automorphism*

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This gives relationship between:

- Robust dynamical properties
- Invariant geometric structures
- Topological properties

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In higher dimensions: Bonatti-Diaz-Pujals prove existence of Invariant geometric structures for robustly transitive diffeomorphisms.

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In general, we do not have results in the line of:

- Invariant geometric structures $\Rightarrow$ Robust dynamical properties

In dimension 2 yes: Pujals and Sambarino's work.

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Conjecture (Pujals)

$f : M^3 \to M^3$ a transitive Strong Partially Hyperbolic (SPH) diffeomorphism ($TM = E^s \oplus E^c \oplus E^u$)

- *f is leaf conjugate to a linear Anosov in $\mathbb{T}^3$*
- *f is leaf conjugate to a skew product (on $\mathbb{T}^3$ or nilmanifold).*
- *f is leaf conjugate to a time one map of an Anosov flow*

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Some Progress:

- Bonatti and Wilkinson when there exists a compact center leaf.
- Hammerlindl solves the case of 3-torus or nilmanifolds without transitivity hypothesis but under an absolute version of strong partial hyperbolicity (not the one given by Diaz-Pujals-Ures).

Definitions

Definition

We say that $f : M^3 \to M^3$ is *partially hyperbolic* (PH) (in the pointwise sense) if $TM = E^{cs} \oplus E^u$ a Df-invariant splitting such that $\exists N > 0$ and for every $x \in M^3$:

$$\|Df^N|_{E^{cs}(x)}\| < \frac{1}{2}\|Df^N|_{E^u(x)}\| > 1$$

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We say that $f : M^3 \to M^3$ is *strongly partially hyperbolic* (SPH) (in the *pointwise* sense) iff $TM = E^s \oplus E^c \oplus E^u$ a Df-invariant splitting such that $\exists N > 0$ and for every $x \in M^3$:

$$1 > 2\|Df^N|_{E^s(x)}\| < \|Df^N|_{E^c(x)}\| < \frac{1}{2}\|Df^N|_{E^u(x)}\| > 1$$

Absolute definitions

There are also absolute definitions of partial hyperbolicity which are sometimes used: The definition covers many examples but it is in some sense artificial (and not the one given by Diaz-Pujals-Ures' result). The norm of the differential of f in E^u must dominate the expansion on E^{cs} for any pair of points in M.
This can be compared with pinching conditions used by Brin and Manning for classifying Anosov systems.

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A strong partially hyperbolic diffeomorphism f is *dynamically coherent* if $\exists$ f-invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^s \oplus E^c$ and $E^c \oplus E^u$ respectively.

In particular $\exists$ f-invariant foliation $\mathcal{F}^c$ tangent to E^c.

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In particular $\exists$ f-invariant foliation $\mathcal{F}^c$ tangent to E^c.
There always exists $\mathcal{F}^s$ and $\mathcal{F}^u$ f-invariant foliations tangent to E^s and E^u called strong manifolds.

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Proposition

Being almost dynamically coherent is an open and closed property among partially hyperbolic diffeomorphisms. In particular, it contains entire connected components of the space of partially hyperbolic diffeomorphisms.

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Moreover:

Proposition (Brin-Burago-Ivanov)

An foliation transverse to E^u cannot have Reeb components. In particular, if M is not "large" this gives that the action in homology is partially hyperbolic.

Previous Results

Theorem (Brin-Burago-Ivanov)

Under a stronger (absolute) version of SPH, if $f : \mathbb{T}^3 \to \mathbb{T}^3$ *is SPH then it is dynamically coherent.*

This was used by Hammerlindl to get *leaf conjugacy*.

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Theorem (Rodriguez Hertz-Rodriguez Hertz-Ures)

There exists a (non transitive) SPH diffeomorphism in $\mathbb{T}^3$ which is NOT dynamically coherent.

Statement of results

Theorem

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ a PH diffeomorphism isotopic to Anosov and almost dynamically coherent. Then f is dynamically coherent.

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Theorem

Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ a SPH diffeomorphism.

- *Either there exists a repelling torus T tangent to $E^s \oplus E^u$ or,*
- *There exists an f-invariant foliation $\mathcal{F}^{cs}$ tangent to $E^s \oplus E^c$.*

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Corollary

If $f : \mathbb{T}^3 \to \mathbb{T}^3$ is SPH and $\Omega(f) = \mathbb{T}^3$ then f is dynamically coherent.

Results of Brin-Burago-Ivanov

- (BBI) $f_* : \mathbb{R}^3 \to \mathbb{R}^3$ is SPH (either f_* is hyperbolic or f_* "is" Anosov$\times Id_{S^1}$.

Results of Brin-Burago-Ivanov

- (BBI) $f_* : \mathbb{R}^3 \to \mathbb{R}^3$ is SPH (either f_* is hyperbolic or f_* "is" Anosov$\times Id_{S^1}$.
- (BI) There exists $\mathcal{F}^1$ and $\mathcal{F}^2$ Reebless foliations transverse to E^u and E^s respectively.

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    - The plane projects into a torus: We find a repelling torus.
    - The plane close to the center stable leaf is the center unstable plane: Estimate growth of diameter and apply Novikov's theorem.

Obrigado!