Symmetrization of Convex Planar Curves

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Abstract. Given a closed convex planar curve, we call great chords the segment connecting two points with parallel tangents. We call great diagonals the support lines of the great chords and mid-parallels the lines through the mid-point of a great chord parallel to the corresponding tangents. Two curves are called parallel if the corresponding great diagonals are parallel.

In this paper, we define the parallel diagonal (PD) transform of a convex curve γ as a convex curve δ whose great diagonals coincide with the mid-parallels of γ and whose mid-parallels are parallel to the great diagonals of γ. Applying twice the PD transform, we obtain a transformation S that preserves parallelism of the curves. The main result of the paper says that the sequence of iterations $S^n(\gamma)$ converges uniformly to a symmetric curve parallel to γ.

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1. Introduction

Given a closed convex planar curve γ and a direction w, let chord(w) denote the segment connecting the points of γ whose tangents are parallel to w. We call *great diagonal* the support line $d(w)$ of chord(w) and *mid-parallel* the line $m(w)$ through the center of chord(w) parallel to w.

Given two closed convex planar curves γ_1 and γ_2, we say that γ_1 and γ_2 are *parallel* if $d_1(w)$ is parallel to $d_2(w)$, for any w, and *equidistant* if $d_1(w) = d_2(w)$ and $m_1(w) = m_2(w)$, for any w.

In this paper, we shall consider classes of parallel closed convex curves. We begin by showing that in any class of convex parallel curves, there exists one

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and only one, up to homothety, symmetric curve. A symmetric closed planar curve γ_0 admits a dual symmetric curve δ_0, defined up to homothety (see [2]). We extend this duality to classes of parallel curves as a transformation in the space of closed convex curves. Given a curve γ, we define its *parallel diagonal* (PD) transform δ by the condition that the great diagonals of δ coincide with the mid-parallels of γ and the mid-parallels of δ are parallel to the great diagonals of γ. The curve δ is unique up to equidistants. A similar transform for polygons was defined in [1].

If we apply twice the PD transform, we obtain a transformation S that preserves the parallelism between the great diagonals and the mid-parallels. Thus it preserves the parallelism of the curve. Denoting by S^n the n-iteration of the transformation S, the main result of the paper says that the sequence $S^n(\gamma)$ converges uniformly to a symmetric curve parallel to γ.

In order to compare the rate of symmetry of parallel curves, we define a real non-negative number $\rho(\gamma)$ which is zero if and only if γ is symmetric. We call this number the *asymmetry number* of γ. Two equidistants have the same asymmetry number although homothetic curves may have different ones. We prove that

$$\rho(S(\gamma)) \leq b \cdot \rho(\gamma),$$

where b is a real number, $0 < b < 1$, that depends only on the class of parallel curves defined by γ. This estimate is the main tool used to prove the convergence of $S^n(\gamma)$.

The paper is organized as follows: In section 2 we provide the basic definitions and results concerning parallel curves and equidistants. In section 3 we define the parallel diagonal transform and provide its geometrical interpretation. In section 4 we define the asymmetry number and prove the main result of the paper, namely, the uniform convergence of the sequence $S^n(\gamma)$ to a symmetric curve.

2. Parallel curves and equidistants

Given a convex planar curve γ, parameterize it by the oriented angle θ between the tangent line and some fixed direction. Denote by $d(\theta)$ the great diagonal passing through $\gamma(\theta)$ and $\gamma(\theta+\pi)$ and by $m(\theta)$ the mid-parallel line, which passes through the midpoint $M(\theta) = \frac{1}{2}\left(\gamma(\theta) + \gamma(\theta+\pi)\right)$ and is parallel to $\gamma'(\theta)$.

For two vectors $w_1, w_2 \in \mathbb{R}^2$, we denote by $[w_1, w_2]$ the determinant of the 2×2 matrix whose columns are w_1 and w_2. The convexity of γ implies that $[\gamma'(\theta), \gamma''(\theta)] \geq 0$. For the sake of simplicity, we shall assume along the paper that

$$[\gamma'(\theta), \gamma''(\theta)] > 0, \tag{2.1}$$

although this hypothesis is not always strictly necessary.

The curve $\gamma(\theta)$ is symmetric with respect to the origin if $\gamma(\theta+\pi) = -\gamma(\theta)$, $0 \le \theta \le \pi$. Unless otherwise stated, along the paper the word symmetry means symmetry with respect to the origin.

2.1. Parallel curves and symmetry

Definition 2.1. Two curves γ_1 and γ_2 are said to be *parallel* if $d_1(\theta)$ is parallel to $d_2(\theta)$ for any $0 \le \theta \le \pi$.

Lemma 2.2. *Given a convex curve γ, there exists a symmetric convex curve u which is parallel to γ. If u_1 is symmetric and parallel to u, then $u_1 = cu$, for some constant c.*

Proof. Take

$$u(\theta) = \frac{1}{2}\left(\gamma(\theta) - \gamma(\theta+\pi)\right).$$

Then u is symmetric and

$$4[u', u''](\theta) = [\gamma', \gamma''](\theta) + [\gamma', \gamma''](\theta+\pi) - [\gamma'(\theta), \gamma''(\theta+\pi)] - [\gamma'(\theta+\pi), \gamma''(\theta)].$$

But $\gamma'(\theta+\pi) = -k(\theta)\gamma'(\theta)$, for some $k(\theta) > 0$ and so

$$4[u', u''](\theta) = (k^2 + 2k + 1)[\gamma', \gamma''](\theta)$$

is positive by (2.1), which implies that u is convex.

If u_1 is symmetric and $d_1(\theta)$ parallel to $d(\theta)$, we write $u_1(\theta) = \lambda(\theta)u(\theta)$, with $\lambda(\theta+\pi) = \lambda(\theta)$. Since

$$u_1'(\theta) = \lambda'(\theta)u(\theta) + \lambda(\theta)u'(\theta)$$

must be parallel to $u'(\theta)$, we conclude that $\lambda' = 0$ and thus $u_1 = cu$, for some constant c. □

We shall denote by u_0 the unique symmetric curve parallel to γ and normalized by

$$\frac{1}{2}\int_0^{2\pi} [u_0, u_0'](\theta)d\theta = 2. \tag{2.2}$$

2.2. Equidistants and the area evolute

Definition 2.3. Two curves γ_1 and γ_2 are called *equidistant* if $d_1(\theta)$ and $m_1(\theta)$ coincide with $d_2(\theta)$ and $m_2(\theta)$, for any $0 \le \theta \le \pi$.

For an equivalent definition of equidistants of a given curve, see ([4]).

Lemma 2.4. *Any equidistant of γ can be written in the form*

$$\gamma_c(\theta) = M(\theta) + cu_0(\theta), \tag{2.3}$$

for some constant c. Reciprocally, any curve given by (2.3) is an equidistant of γ.

Proof. Any curve with great diagonal coinciding with $d(\theta)$ can be written as $\gamma(\theta) = M(\theta) + \lambda(\theta)u_0(\theta)$. Since

$$\gamma'(\theta) = M'(\theta) + \lambda'(\theta)u_0(\theta) + \lambda(\theta)u_0'(\theta)$$

must me parallel to $u_0'(\theta)$, we conclude that $\lambda' = 0$ and thus equation (2.3) holds. The reciprocal result can be easily verified. □

The equidistants may have cusps for small values of c. In fact, writing

$$M'(\theta) = \alpha(\theta)u_0'(\theta), \tag{2.4}$$

we obtain

$$\gamma_c'(\theta) = (\alpha + c)u_0'(\theta),$$

and so $\gamma_c'(\theta)$ may have zeros. The equidistant γ_c has a cusp at θ if $\alpha + c$ is changing sign at this point.

Observe that

$$\gamma_c'' = (\alpha + c)'u_0' + (\alpha + c)u_0''$$

and so

$$[\gamma_c', \gamma_c''] = (\alpha + c)^2[u_0', u_0''].$$

We conclude that γ_c is convex outside cusps.

If we take $c = 0$, the equidistant is called *area evolute* (AE). The AE is thus the envelope of the mid-parallels. Its cusps corresponds to points where α is changing sign. It is well-known that the number of cusps of the AE is odd and bigger than or equal to three ([3]).

2.3. Duality of symmetric curves

The construction of this section was proposed in [2] in the context of dual billiards. Let $u(\theta)$ be a symmetric convex planar curve and denote

$$v = \frac{u'}{[u, u']}$$

Then v is symmetric and has great diagonals coinciding with the mid-parallels of u. Moreover,

$$v' = \frac{u''}{[u, u']} - \frac{[u, u'']u'}{[u, u']^2},$$

and so $[u, v'] = 0$. We conclude that the mid-parallels parallels of v coincide with the great diagonals of u. Observe that

$$[v, v'] = \frac{[u', u'']}{[u, u']^2},$$

and thus the convexity of u implies that v is star-shaped. Since u is parallel to v' and $[u, v] = 1$, we conclude that

$$u = -\frac{v'}{[v, v']}.$$

Now the same argument as above shows that, since u is star-shaped, v is convex.

We have thus proved the following proposition:

Proposition 2.5. *Let u be a symmetric smooth closed convex curve. There exists a smooth closed convex curve v with great diagonals coinciding with the mid-parallels of u and mid-parallels coinciding with the great diagonals of u. If v_1 is another curve satisfying these conditions, then $v_1 = cv$, for some constant c.*

If we apply proposition 2.5 to the convex symmetric curve u_0 defined in section 2.1, we obtain a one-parameter family of dual symmetric curves. We shall denote by v_0 the symmetric convex curve of the dual family satisfying the condition

$$\frac{1}{2}\int_0^{2\pi}[v_0, v_0'](\theta)d\theta = 2 \tag{2.5}$$

(see figure 1).

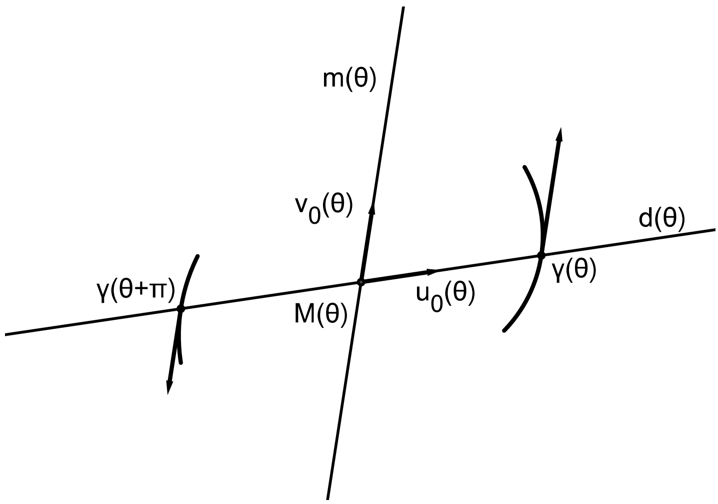

FIGURE 1. Illustration of the curve γ close to θ and $\theta+\pi$, the great diagonal $d(\theta)$, the mid-parallel $m(\theta)$, the mid-point $M(\theta)$ and the normalized vectors $u_0(\theta)$ and $v_0(\theta)$.

2.4. Center Symmetry Set

The envelope of $d(\theta)$ is called the *center symmetry set* (CSS) of γ ([4], [5]).

Lemma 2.6. *The CSS can be parameterized by*

$$x(\theta) = M(\theta) - \alpha(\theta)u_0(\theta). \tag{2.6}$$

Proof. The great diagonals correspond to the zero set of
$$F(x,\theta)=[x-M(\theta),u_0(\theta)].$$
The envelope of the great diagonals is the set of x such that $F=F_\theta=0$. But
$$F_\theta=\alpha[u_0,u_0']+[x-M,u_0'].$$
Thus $x-M=\lambda u_0$ with $\lambda+\alpha=0$. □

Proposition 2.7. *The CSS is the locus of cusps of the equidistants. The cusps of the CSS correspond to points where α' is changing sign.*

Proof. The first assertion follows from (2.6) and the discussion after lemma 2.4. To prove the second one observe that
$$x'(\theta)=\alpha'(\theta)u_0(\theta).$$
□

It is well-known that the CSS has an odd number of cusps, at least three ([4],[5]).

3. The parallel diagonal transform

In this section we generalize the duality of symmetric convex curves described in section 2.3 as a transformation in the space of convex curves which are not necessarily symmetric. We call it the *parallel diagonal* (PD) transform.

3.1. Definition of the PD transform

We begin with the following proposition:

Proposition 3.1. *Given a closed convex curve γ, there exists a closed convex curve δ whose great diagonals coincides with the mid-parallels of γ and whose mid-parallels are parallels to the great diagonals of γ. If δ_1 is another curve satisfying these properties, then it is an equidistant of δ. The mid-parallels of δ are independent of the choice of the equidistant.*

Proof. Let
$$\delta(\theta)=M(\theta)+\lambda(\theta)v_0(\theta),$$
where λ is to be chosen in order that δ' is parallel to u_0. We have
$$\delta'=M'+\lambda'v_0+\lambda v_0'=(\alpha[u_0,u_0']+\lambda')v_0+\lambda v_0'.$$
Thus we must choose λ satisfying
$$\lambda'=-\alpha[u_0,u_0']. \tag{3.1}$$
But since $\alpha(\theta+\pi)=-\alpha(\theta)$, we have
$$\int_0^{2\pi}\alpha[u_0,u_0'](\theta)d\theta=0.$$

This implies in the existence of λ satisfying (3.1).

The mid-point of δ is given by

$$N(\theta) = \frac{1}{2}\left(\delta(\theta) + \delta(\theta+\pi)\right) = M(\theta) + \frac{1}{2}\left(\lambda(\theta) - \lambda(\theta+\pi)\right) v_0(\theta).$$

Let $\beta(\theta) = \frac{1}{2}\left(\lambda(\theta+\pi) - \lambda(\theta)\right)$ and observe that

$$\beta(\theta) = \frac{1}{2}\int_{\theta}^{\theta+\pi} \alpha[u_0, u_0'](s)ds \tag{3.2}$$

is independent of the choice of $\lambda(0)$. Thus

$$N(\theta) = M(\theta) - \beta(\theta)v_0(\theta) \tag{3.3}$$

does not depend on $\lambda(0)$. Since

$$\delta(\theta) = N(\theta) + \frac{1}{2}\left(\lambda(\theta) + \lambda(\theta+\pi)\right) v_0(\theta),$$

the same argument given in lemma 2.4 implies that

$$\delta(\theta) = N(\theta) + \lambda(0)v_0(\theta).$$

Thus any δ_1 satisfying the properties of the proposition must be an equidistant of δ.

It remains to show that we can choose δ convex. We have that

$$\delta' = \lambda v_0' = -\lambda[v_0, v_0']u_0.$$

and so

$$[\delta', \delta''] = \lambda^2[u_0, u_0'][v_0, v_0']^2,$$

which is strictly positive since we can chose $\lambda(0)$ such that $\lambda(\theta)$ becomes strictly positive. □

Definition 3.2. The PD transform of a convex closed curve γ is any convex closed curve δ of the 1-parameter family of equidistants given by proposition 3.1 (see figure 2).

Remark 3.3. Since the AE is the envelope of mid-parallels and the CSS the envelope of the great diagonals, the PD transform δ of γ has CSS coinciding with the AE of γ and AE parallel to the CSS of γ.

3.2. The inverse PD transform

Define

$$x(\theta) = M(\theta) + \lambda(\theta)u_0(\theta),$$

and impose the condition $x'(\theta)$ parallel to $u_0(\theta)$. We have

$$x' = (\alpha + \lambda)u_0' + \lambda' u_0,$$

and so $\lambda = -\alpha$. We conclude that

$$x(\theta) = M(\theta) - \alpha(\theta)u_0(\theta).$$

Comparing with equation (2.6), we observe that $x(\theta)$ is a point of the CSS.

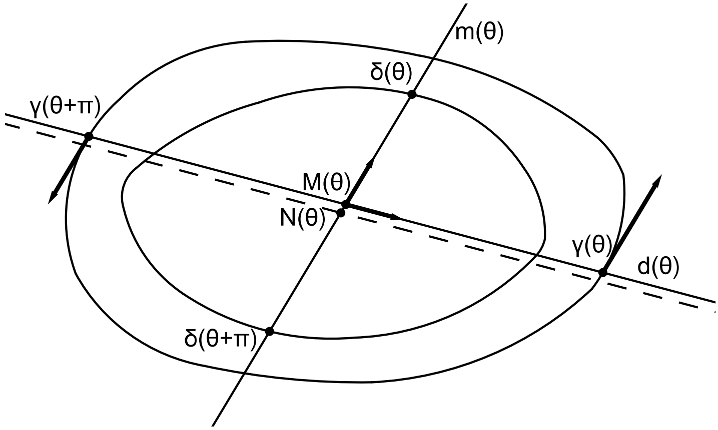

FIGURE 2. The curve γ and its PD transform δ. The great diagonal of δ coincides with $m(\theta)$ while its mid-parallel (traced) is parallel to $d(\theta)$ through $N(\theta) = \frac{1}{2}(\delta(\theta)+\delta(\theta+\pi))$.

Then, for any $c \in \mathbb{R}$, let

$$\eta_c(\theta) = x(\theta) + cv_0(\theta). \tag{3.4}$$

It is easy to show that the great diagonals of η_c are parallel to the mid-parallels of γ and the mid-parallels of η_c coincides with the great diagonals of γ. Moreover, any curve satisfying these 2 conditions must be of the form (3.4).

We now show that we can choose c such that η_c is convex. In fact,

$$\eta_c' = (c[v_0, v_0'] - \alpha'')u_0.$$

Take $|c|$ big enough such that $c[v_0, v_0'] - \alpha'' \neq 0$, for any $0 \leq \theta \leq 2\pi$. Thus

$$[\eta_c', \eta_c''] = (c[v_0, v_0'] - \alpha'')^2[u_0, u_0'] > 0,$$

and we conclude that η_c is convex. It is now clear that the inverse parallel diagonal transform of γ is any curve η_c defined by (3.4).

3.3. Geometric interpretation of the PD transform

The great diagonal $d(\theta)$ divide the region R into two regions. We denote by $A_1(\theta)$ the area of the region bounded by $\gamma(s)$, $\theta \leq s \leq \theta + \pi$ and $d(\theta)$. The area of the region bounded by $\gamma(s)$, $\theta + \pi \leq s \leq \theta + 2\pi$, and $d(\theta)$ will be denoted by $A_2(\theta)$.

Proposition 3.4. *Write $\gamma(\theta) = M(\theta) + cu_0(\theta)$. Then*

$$\frac{1}{2}\left(A_1(\theta) - A_2(\theta)\right) = c\beta(\theta). \tag{3.5}$$

Proof. We have that

$$2A_1(\theta) = \int_\theta^{\theta+\pi} [\gamma(s) - M(\theta), \gamma'(s)]ds$$

$$= \int_\theta^{\theta+\pi} [cu_0(s) + M(s) - M(\theta), cu_0'(s) + M'(s)]\, ds.$$

The area $A_2(\theta)$ is obtained by integrating the same integrand from $\theta + \pi$ to $\theta + 2\pi$. Thus

$$(A_1 - A_2)(\theta) = \int_\theta^{\theta+\pi} [M(s) - M(\theta), cu_0'(s)] - [M'(s), cu_0(s)]\, ds.$$

Since

$$\int_\theta^{\theta+\pi} [M(s) - M(\theta), cu_0'(s)]\, ds = -\int_\theta^{\theta+\pi} [M'(s), cu_0(s)]\, ds,$$

we conclude that

$$A_1(\theta) - A_2(\theta) = -2\int_\theta^{\theta+\pi} [M'(s), \gamma(s) - M(s)]ds.$$

Hence

$$\frac{1}{2}(A_1 - A_2) = c\int_\theta^{\theta+\pi} \alpha[u_0, u_0'](s)ds = c\beta(\theta),$$

thus proving the proposition. □

4. Symmetrization

Let $\mathcal{C}$ denote the space of continuous 2π-periodic functions $f : [0, 2\pi] \to \mathbb{R}$ with the norm

$$||f|| = \sup_{\theta\in[0,2\pi]} |f(\theta)|.$$

4.1. A measure of symmetry

A curve γ is symmetric if and only if $A_1(\theta) = A_2(\theta)$, for any $0 \leq \theta \leq \pi$ (see [6]). Thus it is natural to consider the difference $A_1(\theta) - A_2(\theta)$ as a deviation from symmetry. By proposition 3.4, this difference is linear in c. We thus consider the rate of growth β of $A_1 - A_2$ with respect to c.

Definition 4.1. Define the *asymmetry number* $\rho(\gamma)$ of the curve γ as the norm of β in the space $\mathcal{C}$, i.e.,

$$\rho(\gamma) = ||\beta||.$$

Proposition 4.2. *The following properties of the asymmetry number hold:*

1. $\rho(\gamma) = 0$ *if and only if* γ *is symmetric.*
2. *If* γ_1 *and* γ_2 *are equidistants, then* $\rho(\gamma_1) = \rho(\gamma_2)$.
3. *If* $\gamma_2 = \lambda\gamma_1$, *then* $\rho(\gamma_2) = \lambda\rho(\gamma_1)$.

The proof of the above proposition is easy and left to the reader. Recall that the mid-point N of δ is given by equation (3.3). Differentiating we obtain

$$N'(\theta) = -\beta(\theta)v_0'(\theta). \tag{4.1}$$

Thus $-\beta$ plays the role of α for the curve δ.

4.2. The contractive property of S

Denote by S the iteration of the PD transform two times. We write $\gamma_1 = \gamma$ and $\gamma_2 = S(\gamma, c_2)$, where c_2 is a constant defined by equation (2.3). Let α_1, α_2 be defined by equation (2.4) and β_1, β_2 be defined by equation (3.2). It follows then from equations (4.1) and (3.2) applied to δ that

$$\alpha_2(\theta) = -\frac{1}{2}\int_\theta^{\theta+\pi} \beta_1[v_0, v_0'](s)ds. \tag{4.2}$$

Recall that we have chosen u_0 and v_0 satisfying equation (2.5) and hence

$$\frac{1}{2}\int_\theta^{\theta+\pi} [v_0, v_0'](s)ds = 1,$$

for any $\theta \in \mathbb{R}$. This implies that

$$||\alpha_2|| \leq ||\beta_1||. \tag{4.3}$$

Let $a = ||[v_0, v_0']||$. Then equation (4.2) implies that

$$||\alpha_2'|| \leq a||\beta_1||. \tag{4.4}$$

We shall now prove the contractive property of S, beginning with the following lemma:

Lemma 4.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a positive differentiable π-periodic function satisfying*

$$\int_\theta^{\theta+\pi} g(s)ds = 1,$$

for any $\theta \in \mathbb{R}$. Consider a differentiable function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(\theta + \pi) = -f(\theta)$, for any $\theta \in \mathbb{R}$. Assume that $|f(\theta)| \leq 1$ and $|f'(\theta)| \leq a$, for some positive number a. Then there exists $0 < b < 1$ depending on g and a but not on f or θ such that

$$\int_\theta^{\theta+\pi} f(s)g(s)ds \leq b.$$

Proof. Take any number $t \in [-1, 1]$ and suppose $f(\theta) = t$. Then $f(\theta+\pi) = -t$ and

$$\begin{cases} f(s) \leq a(s-\theta) + t, & \theta \leq s \leq \theta + \frac{1-t}{a}, \\ f(s) \leq -a(s-\theta) + a\pi - t, & \theta + \pi - \frac{1+t}{a} \leq s \leq \theta + \pi. \end{cases}$$

Thus

$$1 - \int_\theta^{\theta+\pi} f(s)g(s)ds \geq \int_\theta^{\theta+\frac{1-t}{a}} (1 - a(s-\theta) - t)g(s)ds$$

$$+\int_{\theta+\pi-\frac{1+t}{a}}^{\theta+\pi}(1+a(s-\theta)-a\pi+t)g(s)ds.$$

The second member of the inequality is continuous in $(t,\theta)\in[-1,1]\times[0,\pi]$ and strictly positive. Thus has a minimum positive value independent of f, which proves the lemma. □

Proposition 4.4. *There exists a real number b, $0<b<1$, depending only on the class of parallel curves defined by γ, such that*

$$||\beta_2||\leq b\cdot||\beta_1||,$$

or equivalently,

$$\rho(S(\gamma))\leq b\cdot\rho(\gamma).$$

Proof. In lemma 4.3, take $g=\frac{1}{2}[v_0,v_0']$ and obtain b. Observe that b depends only on u_0 and v_0. By equations (4.3) and (4.4), $f=\frac{\alpha_2}{||\beta_1||}$ satisfies the conditions of the above lemma. Thus, using equation (3.2) we obtain

$$\beta_2(\theta)=\frac{1}{2}\int_\theta^{\theta+\pi}\alpha_2[v_0,v_0']ds\leq b||\beta_1||,$$

which implies that $||\beta_2||\leq b||\beta_1||$. □

4.3. Convergence of the iterations of S

Define a 2 sequences γ_i and δ_i of closed convex curves recursively by $\gamma_1=\gamma$, $\delta_1=\delta$ and

$$\gamma_{i+1}=S(\delta_i),\quad \delta_{i+1}=S(\delta_i).$$

We recall that the transformation S is defined only up to equidistants, so at each step we must choose a constant c defined by equation (2.3). We denote by c_n the constant associated with γ_n. It follows from proposition 4.4 that the sequences of asymmetry numbers $\rho(\gamma_n)$ and $\rho(\delta_n)$ are decreasing and converging to 0.

Let $\mathcal{C}_2$ denote the space of 2π-periodic continuous functions $w:\mathbb{R}\to\mathbb{R}^2$ with the norm

$$||w||_\infty=\sup_{\theta\in[0,2\pi]}||w(\theta)||. \qquad (4.5)$$

Lemma 4.5. *The sequence $M_n(\theta)$ is converging to a constant M_0 in $\mathcal{C}_2$.*

Proof. Applying proposition 4.4 to γ and δ we obtain b_1 and b_2. Let $b=\max(b_1,b_2)$. It follows that $||\alpha_n||\leq Kb^n$ and $||\beta_n||\leq Kb^n$, for some constant K. Thus equation (3.3) implies that

$$||M_{n+1}-M_n||=||M_{n+1}-N_n||+||N_n-M_n||\leq K_1b^n,$$

for some constant K_1. So

$$||M_{n+p}-M_n||\leq K_1\frac{b^n}{1-b},$$

for any $p > 0$, which implies that $M_n(\theta)$ is a Cauchy sequence in $\mathcal{C}_2$. Since $\mathcal{C}_2$ is a complete space, we conclude that $M_n(\theta)$ is converging to some limit $M_0(\theta)$. But $||M_n'||$ is converging to zero, so $M_0(\theta)$ is a constant function. □

Theorem 4.6. *Assume that we have chosen the constants c_n converging to c. Let γ_0 be the unique convex symmetric curve parallel to γ centered at M_0 with constant c defined by equation* (2.3). *Then γ_n is converging to γ_0 in $\mathcal{C}_2$.*

Proof. Since $\gamma_n(\theta) = M_n(\theta) + c_n u_0(\theta)$, lemma 4.5 implies that γ_n is converging to $M_0 + cu_0(\theta)$ in $\mathcal{C}_2$, which proves the theorem. □

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