A FIRST INTEGRAL FOR C^∞, K-BASIC FINSLER SURFACES AND APPLICATIONS TO RIGIDITY

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Abstract. We show that a compact C^∞, k-basic Finsler surface without conjugate points and genus greater than one is Riemannian. This result is a C^∞ version of the fact, proved by G. Paternain, that analytic, compact, k-basic Finsler surfaces with genus greater than one are Riemannian. The proof in the C^∞ case relies mainly in two facts: first of all the existence of a first integral for the geodesic flow of any k-basic Finsler surface, one of the main contributions of this note; and secondly the triviality of every first integral assuming the absence of conjugate points.

Introduction

The characterization of Finsler manifolds which are Riemannian is one of the main problems of what is called Finsler rigidity theory. Roughly speaking, a C^∞ Finsler metric of a manifold M is given by a field of compact convex sets in the tangent space of M, each of which is contained in a tangent plane and whose interior contains the origin of the corresponding tangent plane (for a rigorous definition of Finsler manifold see Section 1). The set of unit vectors of the Finsler metric in a tangent plane is precisely the convex set in the plane. When the field of convex sets is a field of ellipsoids, then the metric is Riemannian. Finsler manifolds can be regarded as geometric models of high energy levels of convex, superlinear Hamiltonians, the so-called Tonelli Hamiltonians : it is known [6] that the Hamiltonian flow in a super critical energy level of a Tonelli Hamiltonian is, up to reparametrization, the geodesic flow of a Finsler manifold.

Finsler geometry extends the shape operators of Riemannian geometry. The so called flag curvatures are generalizations of the Riemannian sectional curvatures: they are given by the Jacobi tensor of the second variation formula associated to a certain Tonelli Hamiltonian. But Finsler geometry introduces many other geometric functions and tensors which can be viewed as curvatures as well, like the Cartan and the Landsberg curvatures (see for instance [2] for the definitions). In the case of surfaces, the flag curvature, the Cartan scalar and the Landsberg scalar determine completely the local geometry of the Finsler manifold (see Subsection 1.3 for details). The term rigidity is related to the study of Riemannian metrics in the context of Finsler geometry in a quite natural way. We know that whenever the Cartan scalar or the Landsberg scalar is constant, the Finsler surface is Riemannian. Yet, there are many examples of Finsler spheres and tori with constant flag curvature which are not Riemannian (see for instance the survey by Bao-Robles [3] and Katok-Ziller spheres (P. Foulon, [9]). The most famous rigidity result linked to

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the flag curvature is Akbar-Zadeh's Theorem [1]: if the flag curvatures are equal to a negative constant and the Finsler manifold is compact, then the Finsler metric is Riemannian.

Despite all ressemblances, Finsler geometry might be drastically different from Riemannian geometry in many fundamental aspects. There is a natural Finsler length of smooth curves that is not reversible, the Finsler length of a parametrized curve might be different from the length of the same curve endowed with a orientation reversing parametrization. The Gauss-Bonet Theorem holds for the so-called Landsberg surfaces (where the Landsberg scalar is zero), but there is no Gauss-Bonet Theorem for Finsler surfaces in general. The lack of reversibiilty of the Finsler metric implies as well that the Jacobi tensor is not reversible, in particular the flag curvature of a surface, which is actually defined in the tangent space of the unit tangent bundle, might change if we change the unit vector in the tangent plane of a given point. This is not the case of the Gaussian curvature of a surface which only depends on the base point of the tangent plane. The solutions of the Finsler Jacobi equation for negative time might not be just the opposite of the solutions of the Jacobi equation in positive time, like in Riemannian geometry. In particular, if a parametrized curve in a manifold M $\gamma : (a, b) \longrightarrow M$ is a Finsler geodesic (namely, it minimizes locally the length of curves joining $\gamma(t)$ and $\gamma(s)$ for every $a < t < s < b$) there is no reason why the same curve with an orientation reversing parametrization should be a geodesic as well.

The goal of this article is to study the following rigidity problem in Finsler geometry : suppose that the flag curvatures of a compact, C^∞ Finsler surface of genus greater than one at each point of the surface just depend on the point. Then the surface is Riemannian. This problem has a negative answer for the sphere and the torus (see [8] and [9]. We would like to remark that for higher dimensional manifolds the above assumption on the flag curvatures implies that the curvature is constant, by Schur's Lemma. Even in this case the problem has negative answer in the sphere and the torus.

This problem is related to Akbar- Zadeh's work [1]: if the flag curvatures of a compact, C^∞ Finsler manifold equal a negative constant, then the manifold is Riemannian. The proof extends without much changes to Finsler surfaces where the flag curvature at each point of the manifold depends only on the base point and the geodesic flow is Anosov. Akbar-Zade was perhaps the first to explore the relationship between hyperbolic dynamics and Finsler rigidity, and subsequence results on the subject have shown a fruitful interaction between dynamics and Finsler global geometry.

Finsler manifolds where the flag curvature just depend on the base point in the manifold are called **k-basic** Finsler manifolds. G. Paternain [13] improves Akbar- Zadeh's result in the case of analytic compact surfaces of genus greater than one, he shows that either the k-basic property or the Landsberg character of the surface imply that the Finsler surface is Riemannian. The two main ingredients of the proof of this fact are the analyticity of the metric and the existence of a hyperbolic invariant set for the geodesic flow. If we assume that the surface is just C^∞ the proof for the analytic case does not extend, the question turns out to be much more subtle. Some partial positive answers are in [7], [8], [4]) assuming that the surface has no conjugate points. This latter assumption proved to be meaninful in the study of this rigidity issue because of the particular dynamical properties of the geodesic flow of such surfaces. In [8] it is proved that every first integral of the geodesic flow is constant, and since many Finsler special surfaces have first integrals involving curvatures (like Landsberg surfaces where the Cartan scalar is a first integral) then we might expect that the triviality of first integrals related to Finsler shape operators leads to some kind of rigidity. In particular, C^∞ compact Landsberg surfaces without conjugate points and genus greater than one are Riemannian. In [4] it is shown that the geodesic flow of compact Finsler surfaces without conjugate points is transitive, this fact combined with a closed relationship between the Cartan scalar and the Jacobi equation yields that k-basic compact surfaces without conjugate points and higher genus are Riemannian provided that the Green bundles are continuous. The main result of this article finally settles the question for compact surfaces without conjugate points.

Theorem A: Let (M, F) be a compact, connected, C^∞, k-basic Finsler surface without conjugate points and genus greater than one. Then (M, F) is Riemannian.

The proof is based on the combination of two facts: first integrals for geodesic flows in compact surfaces without conjugate points are constant [8], and the existence of a special first integral for k-basic Finsler surfaces (see Section 2) which involves certain derivatives of the Cartan and the Landsberg scalars. Let T_1M be the unit tangent bundle of (M, F), let X be the unit vector field tangent to the geodesic flow, V a unit vector field tangent to the vertical fibers, and let X, V, H be a Cartan frame of unit vector fields. For a smooth function $g : T_1M \longrightarrow \mathbb{R}$ and a vector field Y of T_1M let $Y(g)$ be the derivative of g with respect to Y.

Theorem B : Let (M, F) be a compact, connected, C^∞, k-basic Finsler surface. Then the function

$$f = V(J) + H(I)$$

is a first integral for the geodesic flow, where I is the Cartan scalar and J is the Landsberg scalar.

The existence of a first integral for the geodesic flow of any k-basic Finsler surface is interesting by its own sake in Finsler theory. The first integral exists regardless of the no conjugate points assumption, and may have more applications to the study of the global geometry of k-basic surfaces.

1. Preliminaries

We recall briefly some fundamental notions of Finsler geometry, we follow [2] as main reference.

Let M be a n-dimensional, C^∞ manifold, let T_pM be the tangent space at $p \in M$, and let TM be its tangent bundle. In canonical coordinates, an element of T_xM can be expressed as a pair (x, y), where y is a vector tangent to x. Let $TM_0 = \{(x, y) \in TM;\ y \neq 0\}$ be the complement of the zero section. A C^k ($k \geq 2$) *Finsler structure* on M is a function $F : TM \to [0, +\infty)$ with the following properties:
(i) F is C^k on TM_0;
(ii) F is positively homogeneous of degree one in y, where $(x, y) \in TM$, that is,

$$F(x, \lambda y) = \lambda F(x, y)\ \forall\ \lambda > 0$$

(iii) The Hessian matrix of $F^2 = F \cdot F$

$$g_{ij} = \frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j} F^2$$

is positive definite on TM_0.

A C^k *Finsler manifold* (or just a Finsler manifold) is a pair (M, F) consisting of a C^∞ manifold M and a C^k Finsler structure F on M.

Item (iii) implies that the function $L(p, v) = \frac{1}{2}F(x, y)^2$ defines a Tonelli Lagrangian in the tangent space, so Finsler metrics have geodesics and their local theory is just the local theory of existence and uniqueness of solutions of the Euler-Lagrange equation. We shall assume throughout the paper that geodesics have unit speed.

For a non-vanishing vector $y \in T_xM$, we shall denote by $\gamma_{(x,y)}(t)$ the geodesic with initial conditions $\gamma_{(x,y)}(0) = x$ and $\gamma'_{(x,y)}(0) = y$. The *exponential map* at x, $exp_x : T_xM \to M$ is defined as usual: $exp_x(y) := \gamma_{(x,y)}(1)$.

The Finsler manifold (M, F) induces naturally a Finsler structure in the universal covering $\tilde{M}$ of M, just by pulling back the Finsler structure F to the tangent space of $\tilde{M}$ by the covering map. Let us denote by $(\tilde{M}, \tilde{F})$ this Finsler manifold.

1.1. **Chern-Rund connection (or Chern connection) and Jacobi fields.** One of the main tools of Finsler geometry used to study geodesics is the so-called Chern-Rund connection that we describe briefly in this subsection. We follow see [2], [14].

There exists a Riemannian metric in the tangent bundle, called in the literature the **fundamental tensor** of the Finsler metric, that is given by the Hessian of the square of the Finsler metric in canonical coordinates divided by $\frac{1}{2}$. This tensor is a Riemannian metric in the cotangent bundle by the convexity assumptions on the Finsler metric. The fundamental tensor has many remarkable properties, the most important is that the orbits of the geodesic flow of the Finsler metric are geodesics of the fundamental tensor. This gives a sort of covariant differentiation for the geodesics of the Finsler metric, the Chern-Rund connection, that we describe next.

A piecewise C^1 variation of a smooth curve $\sigma(t)$ in M is a continuous map

$$\sigma(t, u) : \triangle = \{(t, u); 0 \le t \le r, -\varepsilon < u < \varepsilon\} \longrightarrow M$$

which is C^1 on each $[t_{i-1}, t_i] \times (-\varepsilon, \varepsilon)$, and such that $\sigma(t, 0) = \sigma(t)$ for every $t \in [0, r]$. When the variation is piecewise C^2 and the curves $\sigma_u(t) = \sigma(t, u)$, $t \in [0, r]$, are geodesics, then the derivative of the variation is a *Jacobi field* of the Finsler metric.

Lemma 1.1. *Let (M, F) be a C^4 Finsler manifold. For $(x, v) \in TM$, let $g_{ij}(x, v)$ be as defined in item (3) of the definition of Finsler metric. Let σ be a smooth curve and $\sigma(t, u)$ be a variation as before. Then, in the tangent space $T_{\sigma(t,u)}M$ the inner product*

$$g_T := g_{ij(\sigma(t,u),T(t,u))} dx^i \otimes dx^j$$

where $T = T(t, u) := \sigma_* \frac{\partial}{\partial t} = \frac{\partial \sigma}{\partial t}$, satisfies the following properties:

(1) $g_T(T, T) = F^2(T)$.
(2) *σ is a Finslerian geodesic if and only if*

$$\frac{d}{dt} g_T(V, W) = g_T(D_T V, W) + g_T(V, D_T W)$$

where V and W are two arbitrary vector fields along σ. The operator $D_T = \frac{d}{dt}$ is called covariant differentiation with reference vector T.

(3) *In particular, Finslerian geodesics satisfy*

$$D_T \left[\frac{T}{F(T)} \right] = 0.$$

The constant speed Finslerian geodesics $F(v) = c$ are the solutions of

$$D_T T = 0,$$

just like Riemannian geodesics.

(4) *Assume that σ has unit speed. Then a Jacobi field along σ satisfies*

$$D_T D_T J + R(J, T)T = 0,$$

where R is the Jacobi tensor of the Finsler metric (We shall denote as usual $J'' = D_T D_T J$, $J' = D_T J$). When $dim(M) = 2$,

$$R(y, u)u = K(y)[g_y(y, y)u - g_y(y, u)y], \;\; y, u \in T_x M \setminus \{0\}$$

where $K(y)$ is the Gaussian curvature, which coincides as well with the flag curvature.

(5) *Let σ have unit speed. Then, if $J(t)$ is a Jacobi field along σ, the component $J_\perp(t)$ of $J(t)$ that is perpendicular to $\sigma'(t)$ with respect to g_T satisfies the scalar Jacobi equation*

$$J''_\perp + K J_\perp = 0.$$

as in the Riemannian case. Moreover, if $g_T(T, J(t_0)) = g_T(T, J'(t_0)) = 0$ at some point t_0, then $g_T(T, J) = 0$ at every point.

Throughout the paper, all covariant differentiations will be carried out with reference vector T. Lemma 1.1 reduces many Finsler problems concerning Jacobi fields to Riemannian ones. We shall often call the inner product g_T the *adapted Riemannian metric*. The next result is proved in [**?**] and extends a well known result about bounded Jacobi fields of Riemannian metrics.

1.2. **Conjugate points.** We say that q is *conjugate* to p along a geodesic σ if there exists a nonzero Jacobi field J along σ which vanishes at p and q. We say that (M, F) *has no conjugate points* if no geodesic has conjugate points. The following result taken from [2] (Proposition 7.1.1) has a similar, well known counterpart in Riemannian geometry.

Proposition 1.2. *Let $\sigma(t) = exp_p(tv)$, $0 \leq t \leq r$, be a unit speed geodesic. Then the following statements are mutually equivalent:*

(1) *The point $q = \sigma(r)$ is not conjugate to $p = \sigma(0)$ along σ;*
(2) *Any Jacobi field defined along σ that vanishes at p and q must be identically zero;*
(3) *Given any $V \in T_p M$ and $W \in T_q M$, there exists a unique Jacobi field $J : [0, r] \longrightarrow TM$ defined along σ such that $J(0) = V$, $J(r) = W$;*
(4) *The derivative $(exp_p)_*$ of the exponential map exp_p is nonsingular at rv.*

1.3. **Cartan's structural equations.** Here we recall briefly Cartan's structural equations for Finsler metrics, for details we refer to [2]. Like in the Riemannian case, the tangent bundle of T_1M has a natural oriented frame of vectors $e_1 = H$, $e_2 = X$, $e_3 = V$, where $e_2 = X$ is the unit vector tangent to the geodesic flow and $e_3 = V$ is tangent to the vertical bundle.

The vectors e_1, e_2 are chosen in a way that they are orthonormal in each T_pM with respect to the Sasaki-like metric associated to the adapted Riemannian metric $g_T := g_{ij}dx^i \otimes dx^j$. The partial derivatives of a function $f : T_1M \longrightarrow \mathbb{R}$ with respect to the vectors fields e_i will be denoted by f_i.

Proposition 1.3. *The structural equations of the Finsler metric (M, F) are written in terms of the 1-forms ω_i, $i = 1, 2, 3$, in the following way:*

$$\begin{aligned} d\omega_1 &= -I\omega_1 \wedge \omega_3 + \omega_2 \wedge \omega_3 \\ d\omega_2 &= -\omega_1 \wedge \omega_3 \\ d\omega_3 &= K\omega_1 \wedge \omega_2 - J\omega_1 \wedge \omega_3 \end{aligned}$$

The structural equations in terms of the dual basis $\{H, X, V\}$ are the following:

$$[V, X] = H$$

$$[H, V] = X + IH + JV$$

$$[X, H] = kV.$$

The scalar K is the *Gauss-Finsler curvature* (or Gaussian curvature) of the Finsler surface, J is called the *Landsberg scalar* and I the Cartan scalar. The three scalars I, J, K are all functions on T_1M.

It is possible to characterize Finsler metrics which are Riemannian in terms of the above functions. For our purposes, the following characterization will be the relevant one: I vanishes everywhere if and only if the Finsler structure is Riemannian.

The next result contains some properties of the Cartan tensor I that will be relevant for our purposes.

Proposition 1.4. *Let (M, F) be a C^∞ Finsler surface. Then*

(1) *$J = XI = I'$, where I' is the derivative with respect to the arc length parameter of geodesics.*
(2) *(Bianchi identity) $Vk + kI + XJ = 0$. In particular, if $Vk = 0$ then I satisfies*

$$X(XI) + kI = 0.$$

2. A first integral of the geodesic flow of k-basic Finsler surfaces without conjugate points

Let us consider the 1-form

$$\alpha = (XI)\omega_1 - I\omega_3 = I'\omega_1 - I\omega_3$$

defined in T_1M. One of the main properties of α, already observed in [10] is the following.

Lemma 2.1. *The form α is invariant by the geodesic flow.*

For the sake of completeness we give a rough idea of the proof of Lemma 2.1. By Proposition 1.4 the Cartan scalar is a solution of the Jacobi equation in its Riemannian form, that is the Jacobi equation of the Finsler metric in the frame of the Chern-Rund connection. In the Sasaki-like metric (see [2]) the geodesic flow preserves the bundle of subspaces N_θ, $\theta \in T_1M$, generated by V and H, which is actually perpendicular to the vector field X with respect to this metric. As in the Riemannian case, the vectors $(I(\theta), I'(\theta)) \in N_\theta$ in coordinates $H \oplus V$ are invariant by the geodesic flow, namely,

$$D_\theta \phi_t (I(\theta), I'(\theta)) = (I(\phi_t(\theta)), I'(\phi_t(\theta)))$$

for every $t \in \mathbb{R}$. Whenever $I \neq 0$, the field of planes generated by (I, I') and the geodesic vector field X is the kernel of the form α, and we have $(D\phi_t)_*(\alpha)_\theta = \alpha_{\phi_t(\theta)}$.

From now on the paper is devoted to the study of the form α, its algebraic properties will be crucial for the proof of the main Theorem.

Lemma 2.2. *The form $d\alpha$ is a multiple of the symplectic form $\omega_1 \wedge \omega_3$. Namely, we have*

$$d\alpha = (-V(X(I)) - H(I))\omega_1 \wedge \omega_3.$$

Proof. By the Cartan equations of the dual forms ω_i we have that

$$d\alpha = (-I'' - IK)\omega_1 \wedge \omega_2 + (-V(J) - IJ - H(I) + IJ)\omega_1 \wedge \omega_3 + (J - I')\omega_2 \wedge \omega_3$$

which implies

$$d\alpha = (-V(J) - H(I))\omega_1 \wedge \omega_3$$

since $I' = J$ and $I'' + KI = 0$. □

So Theorem B follows as a consequence of Lemma 2.2.

Corollary 2.3. *The function*

$$f = V(J) + H(I) = V(X(I)) + H(I) = X(V(I)) + 2H(I)$$

is a first integral of the geodesic flow.

Proof. The Lie derivative $\mathfrak{L}_X d\alpha$ of $d\alpha$ with respect to the geodesic vector field vanishes since α is invariant by the geodesic flow. Therefore,

$$\begin{aligned} 0 = \mathfrak{L}_X d\alpha &= \mathfrak{L}_X(-f\omega_1 \wedge \omega_3) \\ &= -X(f)\omega_1 \wedge \omega_3 - f\mathfrak{L}_X(\omega_1 \wedge \omega_3) \end{aligned}$$

and since the symplectic form $\omega_1 \wedge \omega_3$ is invariant by the geodesic flow we get

$$0 = -X(f)\omega_1 \wedge \omega_3$$

which clearly implies that $X(f) = 0$ everywhere in T_1M. To finish the proof just observe that $X(I) = J$. □

Applying the main result in [8] we get

Lemma 2.4. *The function $f = V(X(I)) + H(I)$ is constant.*

3. The form α is closed

The goal of this section is to show

Proposition 3.1. *Let (M, F) be a compact k-basic Finsler surface without conjugate points and genus > 1. Then the form α defined in the previous section is closed.*

The proof of this Proposition follows from Lemma 2.4 and the following general property of the function $f = V(X(I)) + H(I)$ defined in the previous section.

Lemma 3.2. *Let (M, F) be a compact Finsler surface. Then*

$$\int f\omega_2 \wedge d\omega_2 = 0.$$

Proof. By Lemma 2.2 and Cartan's formulae we have that $d\alpha = f d\omega_2$. By Stokes Theorem we have that

$$\int d(\alpha \wedge \omega_2) = \int d\alpha \wedge \omega_2 - \int \alpha \wedge d\omega_2 = 0.$$

Now, observe that by the definition of $\alpha = I'\omega_1 - I\omega_3$,

$$\int \alpha \wedge d\omega_2 = \int I'\omega_1 \wedge d\omega_2 - \int I\omega_3 \wedge d\omega_2 = 0$$

since $d\omega_2 = -\omega_1 \wedge \omega_3$. Hence we get

$$\int d\alpha \wedge \omega_2 = \int f d\omega_2 \wedge \omega_2 = 0.$$

Since $\omega_2 \wedge d\omega_2$ is the volume form of the unit tangent bundle, we have

$$\int f d\omega_2 \wedge \omega_2 = 0$$

as claimed. □

Notice that the proof of Proposition 3.1 is straightforward from Lemma 3.2 and Lemma 2.4: since f is constant according to the latter lemma, then

$$0 = \int f\omega_2 \wedge d\omega_2 = f vol(T_1M),$$

where $vol(T_1M)$ is the volume of the unit tangent bundle with respect to the Liouville form $\omega_1 \wedge \omega_2 \wedge \omega_3$. Since $d\alpha = -f\omega_1 \wedge \omega_3$ we conclude that α is closed as claimed in Proposition 3.1.

4. Exactness of α and rigidity

In this section we get further properties of the form α.

Lemma 4.1. *Let (M, F) be a compact k-basic Finsler surface without conjugate points and genus > 1. Then the form α is exact.*

Proof. According to Proposition 3.1 the form $\alpha = I'\omega_1 - I\omega_3$ is closed. So let us consider a lift $\bar{\alpha}$ of α in $\widetilde{T_1M}$. The form $\bar{\alpha}$ is exact since $\widetilde{T_1M}$ is simply connected. So there exists a function $g : \widetilde{T_1M} \longrightarrow \mathbb{R}$ such that

$$\bar{\alpha} = dg.$$

Notice that the function g is a first integral, since its differential $\bar{\alpha}$ does not depend on the geodesic vector field.

Now, let $\pi_1(T_1M)$ be the fundamental group of the unit tangent bundle of (M, F). The form α induces a group morphism $c : \pi_1(T_1M) \to \mathbb{R}$ given by

$$c(\gamma) := g(\gamma.z) - g(z)$$

for any $z \in \widetilde{T_1}1M$.

The exactness is equivalent to the triviality of the morphism c. Since g is flow invariant, then given an element $\gamma \in \pi_1(T_1M)$ represented by a closed orbit in T_1M of period T and a given z on this orbit we have $c(\gamma) = g(\varphi_T(z)) - g(z) = 0$. Now, the fundamental group of the unit tangent bundle of a closed surface is generated by the closed orbits of the flow and the vertical fibers. So in order to show that the morphism c vanishes in the whole fundamental group it is enough to show that the morphism c is trivial in the cyclic subgroup of $\pi_1(T_1M)$ generated by the vertical fibers. But it is well known that the integral of the Cartan scalar I in every vertical fiber is zero (see for instance [2]), therefore, for each $\theta \in T_1M$, and V_θ its vertical fiber, we have

$$\int_{V_\theta} \alpha = \int_{V_\theta} -I\omega_3 = 0,$$

and thus c is trivial. □

Proof of Theorem A

Let us consider the form $\alpha = I'\omega_1 - I\omega_3$. Since by Lemma 4.1 it is exact there exists a function $h : T_1M \longrightarrow \mathbb{R}$ such that $\alpha = dh$. The function h is a first integral for the geodesic flow since its differential does not depend on the geodesic vector field. By [8] the function h is constant and hence $J = I = 0$ everywhere in T_1M which implies that (M, F) is Riemannian.

5. Further properties of the form α: a default for k-basic metrics to be Riemannian

The 1-form $\alpha = I'\omega_1 - I\omega_3 = J\omega_1 - I\omega_3$ is an interesting object by its own in the theory of k-basic Finsler surfaces. First of all, in the set of vanishing points of α the Cartan scalar vanishes and therefore the Finsler metric is in some sense infinitesimally or locally Riemannian. In the interior of the support of α we have information about the existence of subspaces which are invariant by the geodesic flow (see [7], [10]): whenever α does not vanish its kernel is an invariant distribution by Lagrangian planes where the geodesic flow has central (and of course, nonhyperbolic) behavior. From Corollary 2.3, $d\alpha = fd\omega_2$ and f is a first integral for the geodesic flow. Moreover, the proof of the main Theorem is based in the following assertion,

Lemma 5.1. *Let (M, F) be a compact, k-basic Finsler surface. Suppose that every first integral for the geodesic flow is constant. Then the metric is Riemannian if and only if the form α is exact.*

Therefore, in the family of k-basic compact surfaces with transitive geodesic flows the non-exactness of α is the only obstruction for the metric to be Riemannian. If we do not assume anything on the dynamics, we may stratify some levels of obstructions to be Riemannian in terms of α and f. Namely, the supports of f, df and α certainly satisfy

$$supp(df) \subset supp(f) \subset supp(\alpha)$$

so the non-triviality of f and df can also be viewed as obstructions to be Riemannian and to the transitivity of the geodesic flow. This remark has interesting consequences even in the case of constant curvature Finsler surfaces. Since there exist many examples of flat, non-Riemannian tori, the function f is not zero, as well as in the case of constant curvature Finsler spheres (see [3], [9]). Moreover, for a flat Finsler torus we have by the Bianchi identity that $I'' + KI = 0 = I''$ so I being bounded has to be constant along geodesics. This means that $I' = 0 = J$, so the Finsler torus is Landsberg and by the definition of f,

$$f = 2VX(I) - XV(I) = -XV(I).$$

So the default to be Riemannian for a flat torus can be expressed in terms of a certain "torsion" of the vertical derivative of the Cartan scalar.

Let us explore in more detail the dynamics in the support of df, we shall see that the non-triviality of df imposes strong restrictions.

Lemma 5.2. *Let (M, F) be a compact k-basic Finsler surface. If df does not vanish then the interior of the support of df is foliated by lagrangian tori which are invariant by the geodesic flow. Moreover, if the genus is greater than one, then each of these tori has critical points for the restriction of the canonical projection, and each tori is either compressible or isotopic to a torus in T_1M foliated by vertical fibers.*

Proof. Since f is a C^∞ function, by Sard's theorem the set of singular values of f, namely, the image by f of the set of singular points. has measure zero. The level set of a regular value is a smooth compact surface that is invariant by the geodesic flow. So this surface has to be a torus or a Klein bottle. Since the unit tangent bundle is oriented, this level surface is a torus. Moreover, since each smooth surface that is invariant by the geodesic flow in T_1M is Lagrangian, each regular level set of f is Lagrangian.

Now suppose that the surface M has genus greater than one. Recalling Waldhaussen's Theorem, an incompressible torus in a Seifert fibered bundle has to be isotopic either to a covering of the basis of the bundle or to a union of vertical fibers. The former case is impossible because a torus cannot cover a surface with higher genus. In particular, the restriction of the canonical projection to one regular level set of f must have critical points. This finishes the proof of the lemma. □

So in the case of surfaces of higher genus, the non-Riemannian nature of a k-basic Finsler metric not only implies central behavior of the geodesic flow and lack of transitivity but also the existence of a foliation by invariant tori, each of which has critical points for the restriction of the canonical projection (usually called caustics), like in the case of integrable systems with elliptic periodic orbits. Moreover, all this rigid structure is persitent under C^5 perturbations in the family of k-basic Finsler metrics, because the first integral f is a continuous function of the second derivatives of the Cartan and Landsberg scalars. All these remarks strongly suggest that k-basic Finsler, non-Riemannian metrics in compact surfaces of higher genus do not exist, like in the analytic case.

As a final comment, we would like to point out that the non-triviality of f is equivalent to the existence of an invariant volume form in the support of f that is different from the Liouville form.

Lemma 5.3. *Let (M,F) be a compact k-basic Finsler surface. If (M,F) is not Riemannian, then the 3-form*

$$\Omega = df \wedge \omega_2 \wedge \alpha$$

is an invariant volume form in the set of non-vanishing points of f. The following formula holds:

$$\int \Omega = \int f^2 \omega_2 \wedge d\omega_2.$$

Proof. The form Ω is obviously invariant by the geodesic vector field X: first of all $df(X) = 0$ for the first integral f, secondly the two form $\omega_2 \wedge \alpha$ is invariant by the geodesic flow, and finally the Lie derivative of Ω with respect to X is the sum of the Lie derivatives of df and $\omega_2 \wedge \alpha$ by the algebraic properties of the Lie derivative. Hence, the Lie derivative of Ω with respect to X is zero.

Now, let us notice that

$$0 = \int d(f\omega_2 \wedge \alpha) = \int df \wedge \omega_2 \wedge \alpha + \int f d\omega_2 \wedge \alpha - \int f\omega_2 \wedge d\alpha,$$

and since $d\omega_2 = -\omega_1 \wedge \omega_3$ and α is a linear combination of ω_1 and ω_3, we have that $d\omega_2 \wedge \alpha = 0$ and therefore,

$$\int df \wedge \omega_2 \wedge \alpha = \int f\omega_2 \wedge d\alpha = \int f^2 \omega_2 \wedge d\omega_2.$$

The form $\omega_2 \wedge \omega_2$ is the Liouville volume form, so the form $df \wedge \omega_2 \wedge \alpha$ is a volume form in the region where f does not vanish. □

As a straightforward consequence we have

Corollary 5.4. *Let (M,F) be a compact k-basic Finsler surface. Then the set where the forms df and α are linearly independent coincides with the support of f almost everywhere with respect to the Liouville measure. In particular, the form α is exact if and only if $\int \Omega = \int df \wedge \omega_2 \wedge \alpha = 0$.*

So the integral of Ω can be viewed as an integral quantifier of the failure of α to be exact.

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