NOISE AND DISSIPATION IN RIGID BODY MOTION

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Abstract. Using the rigid body as an example, we illustrate some ideas from stochastic geometric mechanics, including coadjoint motion, stationary solutions of the Fokker-Planck equation, double bracket dissipation, the connection to statistical physics and random dynamical systems.

1. Introduction

The rigid body sets the paradigm for geometric mechanics. Any new ideas in this field must always be tested on the rigid body. Therefore, to illustrate the effects of stochasticity in geometric mechanics we may begin with the rigid body. The key idea underlying geometric mechanics is coadjoint motion on level sets of momentum maps corresponding to symmetries of the Hamiltonian or Lagrangian which generates the dynamics. Since the Lie-Poisson bracket preserves coadjoint orbits, we have incorporated dissipation in the double bracket form, which is compatible with coadjoint motion. The probability distribution on the coadjoint orbit for the dynamics of the stochastic rigid body without dissipation is described by the Lie-Poisson Fokker-Planck equation, whose asymptotic solution tends to a constant. However, when the double bracket dissipation is included, the nonlinear interaction between noise and dissipation leads to an equilibrium distribution given by a Gibbs measure of the standard form $\mathbb{P}_\infty(\Pi) \approx \exp(-E(\Pi)/k_BT)$, where $E(\Pi)$ is the energy, depending on body angular momentum Π, and k_B (Boltzmann) and T (temperature) are constants. The time-dependent approach to equilibrium with double bracket dissipation creates SRB (Sinai Ruelle Bowen) measures. Thus, introducing double bracket dissipation has allowed us to study random attractors on level sets of coadjoint orbits for stochastic rigid body dynamics. For more details about stochastic coadjoint motion, existence of SRB measures and extension to semidirect product theory for the heavy top, we refer to [ACH16].

2. Theory

2.1. **Preliminaries.** As mentioned in the Introduction, we choose to focus here on dynamical systems of rigid body types, which are written on the dual of a semi-simple Lie algebras, such as $\mathfrak{so}(3)$ for the classical rigid body. The semi-simplicity guarantees the existence of a non-degenerate pairing, called a Killing form, which allows the identification of the Lie algebra with its dual. We will denote this pairing by $\langle \xi, \eta \rangle := \mathrm{Tr}(\mathrm{ad}_\xi \mathrm{ad}_\eta) = \epsilon \mathrm{Tr}(\xi\eta)$, for $\xi, \eta \in \mathfrak{g}$ and a number ϵ which depends on the Lie algebra. This pairing is bi-invariant, i.e., $\kappa(\xi, \mathrm{ad}_\zeta \eta) = \kappa(\mathrm{ad}_\xi \zeta, \eta)$ and furthermore, if the Lie algebra is compact, $\epsilon < 0$ and (minus) the Killing form defines a norm. The Killing form will allow us to reformulate equations on the dual Lie algebra, in terms of ad-operations (Lie brackets) on the Lie algebra. (See [ACH16] for the corresponding equations written on the dual Lie algebra.) We will find that all our equations of motion will have a Lie bracket on the right hand side, which of course depends on the Lie algebra. This characterises coadjoint motion which are always restricted to some particular submanifolds called coadjoint orbits parameterised by the initial conditions. In the free rigid body example the coadjoint orbits are spheres in the three-dimensional space of angular momenta.

Here, we will only use the simplest type of noise, composed of n independent Wiener processes W_t^i indexed by $i = 1, 2, \dots, n$. See for example [CCR15, IW14] for more details about stochastic processes. For simplicity here, n will be the dimension of the Lie algebra, which are the dynamical variables of our system, but, in principle, n could be arbitrary. In the stochastic integrals we discuss here, we will use the multiplication symbol ($\circ$), which denotes a stochastic integral in the Stratonovich sense. While the Stratonovich integral admits the normal rules of calculus, the Itô integral requires the Itô calculus in the computations. Both representations are equivalent, provided a correction term is added, but the Stratonovich integral will be more convenient for us in dealing with variational calculus. Our strategy is to stay with the Stratonovich sense up until the end of our calculations, then transform to the Itô representation when passing to the Fokker-Planck equation.

2.2. **Structure preserving stochastic deformations.** A natural framework for understanding dynamical systems with symmetry is Lie group reduction [MR99]. This approach leads to reduced equations in terms of an equivariant momentum map taking values on the dual Lie algebra of the Lie symmetry group, and evolving by coadjoint motion. However, here we will use a different approach, which yields equivalent dynamical equations for the same momentum map, while avoiding the use of Lie groups. Thus, stochasticity on Lie groups will not be discussed here. (See [ACC14], for discussions of Lie group reduction by symmetry for stochastic variational principles.) The equivalent formulation we discuss is the so-called Clebsch principle, which constrains the variations to respect certain auxiliary evolution equations for a set of configuration variables q constrained by Lagrange multipliers p. These are the Clebsch variables (q, p), which in the present case we take to be $(q, p) \in \mathfrak{g} \times \mathfrak{g}$, after using the Killing form to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ for the p variables. This method is often used for control systems, where our dynamical variable $\xi \in \mathfrak{g}$ below would play the role of a control parameter.

To be concise, we directly combine the Clebsch variational principle with noise. For this, we first introduce the so-called stochastic potentials $\Phi_i : \mathfrak{g} \to \mathbb{R}$ which are prescribed functions of the momentum map $\mu := \frac{\partial l(\xi)}{\partial \xi} = \mathrm{ad}_q p$, for the reduced Lagrangian $l(\xi)$ whose Hamilton's principle governs the deterministic system for ξ. Then, we write the following constrained stochastic variational principle,

$$S(\xi, q, p) = \int l(\xi) dt + \int \langle p, dq + \mathrm{ad}_\xi q \, dt \rangle + \int \sum_{i=1}^{n} \Phi_i(\mu) \circ dW_t^i. \qquad (2.1)$$

Following the detailed calculations in [Hol15], the free variations of this action functional yield the stochastic Euler-Poincaré equation,

$$d\frac{\partial l(\xi)}{\partial \xi} + \mathrm{ad}_\xi \frac{\partial l(\xi)}{\partial \xi} dt - \sum_i \mathrm{ad}_{\frac{\partial \Phi_i(\mu)}{\partial \mu}} \frac{\partial l(\xi)}{\partial \xi} \circ dW_t^i = 0\,. \tag{2.2}$$

After having defined the Stratonovich stochastic process (2.2), one may compute its corresponding Itô form. For convenience, we denote the field $\sigma_i := -\frac{\partial \Phi_i(\mu)}{\partial \mu} \in \mathfrak{g}$. A direct calculation gives the Itô correction $-\frac{1}{2}\sum_i \mathrm{ad}_{\sigma_i}\left(\mathrm{ad}_{\sigma_i}\frac{\partial l(\xi)}{\partial \xi}\right) dt$ which must be added to (2.2) in order to interpret the stochastic integral as an Itô integral. Let us simplify matters further by rewriting the Itô version of (2.2) in term of $\mu := \frac{\partial l(\xi)}{\partial \xi}$ as

$$d\mu + \mathrm{ad}_\xi \mu \; dt + \sum_i \mathrm{ad}_{\sigma_i}\mu \; dW_t^i - \frac{1}{2}\sum_i \mathrm{ad}_{\sigma_i}\left(\mathrm{ad}_{\sigma_i}\mu\right) dt = 0\,. \tag{2.3}$$

A straightforward Lie-Poisson formulation of this equation results:

$$\begin{aligned} df(\mu) &= \left\langle \mu, \left[\frac{\partial f}{\partial \mu}, \frac{\partial h}{\partial \mu}\right]\right\rangle dt + \sum_i \left\langle \mu, \left[\frac{\partial f}{\partial \mu}, \frac{\partial \Phi_i}{\partial \mu}\right]\right\rangle \circ dW_t^i \\ &=: \{f,h\}dt + \sum_i \{f,\Phi_i\} \circ dW_i\,, \\ &= \{f,h\}dt + \sum_i \{f,\Phi_i\} dW_i + \frac{1}{2}\sum_i \{\{f,\Phi_i\},\Phi_i\}\, dt\,, \end{aligned} \tag{2.4}$$

where the Lie-Poisson bracket $\{\cdot,\cdot\}$ is defined as in the deterministic case, so the motion takes place on coadjoint orbits. We refer to [ACH16, CHR16, GBH16] for more details.

2.3. **The Fokker-Planck equation and invariant distributions.** We derive here a geometric version of the classical Fokker-Planck equation using our SDE (2.2). Recall that the Fokker-Planck equation describes the time evolution of the probability distribution $\mathbb{P}$ for the process driven by (2.2). See, for example, [IW14] for the standard reference for stochastic processes. Here, we will consider $\mathbb{P}$ as a normalised function on $\mathfrak{g}$ with values in $\mathbb{R}$. First, the generator of the process (2.2) can be readily found from its Lie-Poisson form (2.4)

$$Lf(\mu) = \left\langle \mathrm{ad}_\xi \mu, \frac{\partial f}{\partial \mu}\right\rangle - \sum_i \left\langle \mathrm{ad}_{\sigma_i}\mu, \frac{\partial}{\partial \mu}\left\langle \mathrm{ad}_{\sigma_i}\mu, \frac{\partial f}{\partial \mu}\right\rangle\right\rangle, \tag{2.5}$$

where $f : \mathfrak{g} \to \mathbb{R}$ is an arbitrary function of μ. Then, provided that the Φ_i's are linear functions of the momentum μ, the diffusion terms of the infinitesimal generator L will be *self-adjoint* with respect to the L^2 pairing $\langle\, f\,,\,\mathbb{P}\,\rangle_{L^2} := \int_{\mathfrak{g}} f(\mu)\mathbb{P}(\mu)d\mu$.

The Fokker-Planck equation, $\frac{d\mathbb{P}}{dt} = -L\mathbb{P}(\mu)$, describes the dynamics of the probability distribution $\mathbb{P}$ associated to the stochastic process for μ, in the standard advection diffusion form. The underlying geometry of the Fokker-Planck equation may be highlighted by rewriting it in terms of the Lie-Poisson bracket structure, as

$$\frac{d}{dt}\mathbb{P} = -L\mathbb{P}(\mu) = -\{h,\mathbb{P}\} + \sum_i \{\Phi_i, \{\Phi_i, \mathbb{P}\}\}\,, \tag{2.6}$$

where $h(\mu)$ is the Hamiltonian associated to $l(\xi)$ by the Legendre transform. In (2.6), we have recovered the Lie-Poisson formulation (2.4) of the Euler-Poincaré equation together with a dissipative term arising from the noise of the original SDE in a double Lie-Poisson bracket form; see again [ACH16, CHR16, GBH16] for more details.

This formulation gives the following theorem for invariant distributions of (2.5).

Theorem 2.1. *The invariant distribution $\mathbb{P}_\infty$ of the Fokker-Planck equation (2.5), i.e, $L\mathbb{P}_\infty = 0$ is* uniform *on the coadjoint orbits on which the SDE (2.2) evolves.*

The proof of this theorem is based on hypo-coercive property of the Fokker-Planck operator in (2.6); see [ACH16]. Notice that the compactness of the coadjoint orbits is necessary for having a non-vanishing invariant measure.

2.4. **Double bracket dissipation.** We can now add dissipation in our systems for which the solutions of the stochastic process will still lie on the deterministic coadjoint orbit. For this purpose, we will use *double bracket dissipation*, which was studied in detail in [BKMR96] and was generalised recently in [GBH13, GBH14].

For the stochastic process (2.2), the dissipative stochastic Euler-Poincaré equation written in Hamiltonian form with double bracket dissipation is

$$d\mu + \mathrm{ad}_{\frac{\partial h}{\partial \mu}}\mu\, dt + \theta \left[\frac{\partial C}{\partial \mu}, \left[\frac{\partial C}{\partial \mu}, \frac{\partial h}{\partial \mu}\right]\right] dt + \sum_i \mathrm{ad}_{\sigma_i}\mu \circ dW_t^i = 0\,, \tag{2.7}$$

where $\theta > 0$ parametrises the rate of energy dissipation and C is a chosen Casimir of the coadjoint orbit, i.e. an constant function on the space of solutions of the original equation.

As before, we compute the Fokker-Planck equation for the Euler-Poincaré stochastic process (2.7) which is now modified by the double bracket dissipative term

$$\frac{d}{dt}\mathbb{P}(\mu) + \{h, \mathbb{P}\} + \theta \left\langle \left[\frac{\partial \mathbb{P}}{\partial \mu}, \frac{\partial C}{\partial \mu}\right], \left[\frac{\partial h}{\partial \mu}, \frac{\partial C}{\partial \mu}\right] \right\rangle - \frac{1}{2}\sum_i \{\Phi_i, \{\Phi_i, \mathbb{P}\}\} = 0. \tag{2.8}$$

The invariant distribution of this Fokker-Planck equation is no longer a constant on the coadjoint orbits. Instead, it now depends on the energy, as summarized in the following theorem.

Theorem 2.2. *Let the noise amplitude be of the form $\sigma_i = \sigma e_i$ for an arbitrary $\sigma \in \mathbb{R}$, where the e_i's span the underlying vector space of the Lie algebra $\mathfrak{g}$. The invariant distribution of the Fokker-Planck equation (2.8) associated to (2.7) with Casimir $C = \kappa(\mu, \mu)$ is given on coadjoint orbits by*

$$\mathbb{P}_\infty(\mu) = Z^{-1} e^{-\frac{2\theta}{\sigma^2} h(\mu)}, \tag{2.9}$$

where Z is the normalisation constant that enforces $\int \mathbb{P}_\infty(\mu) d\mu = 1$.

Measures of the form (2.9) are called *Maxwellians*, or *Gibbs measures*, for canonical ensembles in statistical physics. In statistical physics, the constant damping-to-forcing ratio $2\theta/\sigma^2$ would be associated with the inverse temperature $\beta = 1/(k_B T)$, where k_B is the Boltzmann constant and T is the Kelvin temperature. In this same context, the normalisation constant $Z(\beta)$ is called the partition function, [Chi09].

2.5. **Random attractors.** When having both noise and dissipation in a dynamical system, the study of objects such as random attractors is an interesting topic. We briefly expose here the main steps to take in order to understand these object, that we will explicitly expose in the example section, using numerical simulations.

We refer the interested reader to [CF94, CDF97, Arn95, BDV06, KR11] for extensive accounts of the topic of random attractors in the random dynamical systems theory. Briefly, the invariant distribution $\mathbb{P}_\infty(\mu)$ of the Fokker-Planck equation represents the average solution of our dynamical system over all possible realisations of the noise, asymptotically in time, $t \to \infty$. That is, averaging and taking the limit in time gives the probability measure $\mathbb{P}_\infty$, which is usually smooth, covers the entire phase space and is independent of time and initial conditions. One can also do the opposite, instead of averaging over the ensemble of realisations of the noise, one may average only over the initial conditions, and let the system evolve toward large times. The distribution resulting from this procedure depends on time, and does not smoothly cover the whole phase space. Nevertheless, it can be shown that this distribution, called the random attractor, admits a nice probability measure, the Sinai-Ruelle-Bowen measure (SRB); see [You02]. We will denote the SRB measure by $\mathbb{P}_\omega(\mu)$ for a given realisation of the noise ω. Under certain conditions, a strong link exists between $\mathbb{P}_\infty$ and the SRB measure $\mathbb{P}_\omega$, given roughly by

$$\int_\Omega \mathbb{P}_\omega(\mu) d\omega = \mathbb{P}_\infty(\mu), \tag{2.10}$$

for the probability space Ω; see [CF98]. The stochastic process (2.7) does indeed admit random attractors, which may be singular sets. See [KCG15, SH98] and references therein for more details about this type of approach. The proof of this result is based on the observation that the energy is monotonically decaying in time, and thus provides a Lyapunov function, which in turn implies the existence of attractive random sets.

In this situation, following for example [CSG11], and provided that the largest Lyapunov exponent of the random system is positive (meaning there is chaos in the system), the existence of the non-singular SRB measure can be derived. The key step in establishing the existence of an SRB measure is to determine a condition for the positivity of the top Lyapunov exponent, as a function of the system parameters, and especially the noise and dissipation amplitudes (σ^2, θ) [ACH16].

3. Euler-Poincaré example: the stochastic free rigid body

This section treats the classic example of the Euler-Poincaré dynamical equation; namely, the equation for free rigid body motion with three dimensions, described by the Lie group $SO(3)$. For a complete treatment from the viewpoint of reduction we refer to [MR99]. Although noise in the rigid body has already been considered in a number of previous works (see for example [Chi12, Chi09] and references therein), the system that we will obtain from this theory is quite different, as it fully take into account the geometry of the rigid body motion, that is the preservation of the norm of the angular momentum.

3.1. **The stochastic rigid body.** Before applying the previous theory, we should mention that we will use the isomorphism $\mathfrak{so}(3) \cong \mathbb{R}^3$ which translates the commutator in the Lie algebra to the cross product of three-dimensional vectors, via $[A, B] \to \boldsymbol{A} \times \boldsymbol{B}$, where vectors in $\mathbb{R}^3$ are denoted with bold font. This allows us to use the scalar product of vectors as our pairing, via the formula $\boldsymbol{A} \cdot \boldsymbol{B} = -\frac{1}{2}\kappa(A, B)$. We skip the details and directly use the reduced Lagrangian of the free rigid body

$$l(\boldsymbol{\Omega}) = \frac{1}{2}\boldsymbol{\Omega} \cdot \mathbb{I}\boldsymbol{\Omega} := \frac{1}{2}\boldsymbol{\Omega} \cdot \boldsymbol{\Pi}\,, \tag{3.1}$$

where $\boldsymbol{\Omega}$ is the angular velocity, $\mathbb{I} = \mathrm{diag}(I_1, I_2, I_3)$ is a prescribed moment of inertia and $\boldsymbol{\Pi}$ is the angular momentum. Notice that the Legendre transform gives the reduced Hamiltonian $h(\boldsymbol{\Pi}) = \frac{1}{2}\boldsymbol{\Pi} \cdot \mathbb{I}^{-1}\boldsymbol{\Pi}$. We take the stochastic potential to be linear in the momentum variable $\boldsymbol{\Pi}$

$$\Phi_i(\boldsymbol{\Pi}) = \sum_{i=0}^{3} \boldsymbol{\sigma}_i \cdot \boldsymbol{\Pi}\,, \tag{3.2}$$

where the constant vectors $\boldsymbol{\sigma}_i$ span $\mathbb{R}^3$. The stochastic process for $\boldsymbol{\Pi}$ is then computed from (2.2) to be

$$d\boldsymbol{\Pi} + \boldsymbol{\Pi} \times \boldsymbol{\Omega}\, dt + \sum_i \boldsymbol{\Pi} \times \boldsymbol{\sigma}_i \circ dW_t^i = 0. \tag{3.3}$$

One can check via either the Itô or Stratonovich stochastic process that the Casimir level set $\|\boldsymbol{\Pi}\|^2 = \mathrm{c}^2$, which defines the momentum sphere of radius c, or coadjoint orbit, is preserved even with the noise. Notice that the energy $h(\boldsymbol{\Pi})$ is not a conserved quantity, but it stays bounded within the maximum and minimum energies of the deterministic system, as the dynamics takes place on the momentum sphere, [ACH16]. As seen in the theory, the invariant solution of the associated Fokker-Planck equation (see below in (3.6) with $\theta = 0$) is a constant on the coadjoint surface, or momentum sphere. Thus, this system is similar to the heat equation on a compact domain, but with the non-trivial geometry of the Casimir level set.

3.2. **Adding dissipation.** The double bracket dissipation for the rigid body involves the only Casimir $\|\boldsymbol{\Pi}\|^2$ and gives the dissipative stochastic process in (2.7),

$$d\boldsymbol{\Pi} + \boldsymbol{\Pi} \times \boldsymbol{\Omega}\, dt + \theta\, \boldsymbol{\Pi} \times (\boldsymbol{\Pi} \times \boldsymbol{\Omega})\, dt + \sum_i \boldsymbol{\Pi} \times \boldsymbol{\sigma}_i \circ dW_t^i = 0. \tag{3.4}$$

In the absence of noise, the energy decay of the deterministic dissipative rigid body is given by,

$$\frac{dh}{dt} = -\theta\|\boldsymbol{\Pi} \times \boldsymbol{\Omega}\|^2\,. \tag{3.5}$$

Consequently, the dissipation will bring the system to one of the minimal energy positions, where Π and Ω are aligned, which corresponds to (relative) equilibria.

In the presence of the noise, the associated the Fokker-Planck equation may be found as

$$\frac{d}{dt}\mathbb{P} + (\boldsymbol{\Pi} \times \boldsymbol{\Omega}) \cdot (\nabla\mathbb{P} - \theta\, \boldsymbol{\Pi} \times \nabla\mathbb{P}) + \frac{1}{2}\sum_i (\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \nabla[(\boldsymbol{\Pi} \times \boldsymbol{\sigma}_i) \cdot \nabla\mathbb{P}] = 0. \tag{3.6}$$

Unlike the case with $\theta = 0$, this equation will not have a constant invariant solution. As an illustration, we derive the invariant distribution $\mathbb{P}_\infty$ of Theorem 2.2 for this simple case. First, we rewrite the Fokker-Planck equation (3.6) as

$$\frac{d}{dt}\mathbb{P} + (\mathbf{\Pi}\times\mathbf{\Omega})\cdot\nabla\mathbb{P} + \nabla\cdot\left(\theta\,\mathbf{\Pi}\times(\mathbf{\Pi}\times\mathbf{\Omega})\mathbb{P} - \frac{1}{2}\sigma^2\,\mathbf{\Pi}\times(\mathbf{\Pi}\times\nabla\mathbb{P})\right) = 0\,, \quad (3.7)$$

where we have used $\nabla\cdot(\mathbf{\Pi}\times(\mathbf{\Pi}\times\mathbf{\Omega})) = 0$. The last term in (3.6) simplifies as

$$\sum_i(\mathbf{\Pi}\times\boldsymbol{\sigma}_i)[(\mathbf{\Pi}\times\boldsymbol{\sigma}_i)\cdot\nabla\mathbb{P}] = \sum_i(\mathbf{\Pi}\times\boldsymbol{\sigma}_i)[(\nabla\mathbb{P}\times\mathbf{\Pi})\cdot\boldsymbol{\sigma}_i] = \mathbf{\Pi}\times(\nabla\mathbb{P}\times\mathbf{\Pi}),$$

since the sum over i is simply the decomposition of the vector $(\nabla\mathbb{P}\times\mathbf{\Pi})$ into its $\boldsymbol{\sigma}_i$ basis components. Hence, the invariant solution is the Gibbs measure, given by

$$\mathbb{P}_\infty(\mathbf{\Pi}) = Z^{-1}e^{-\frac{2\theta}{\sigma^2}h(\mathbf{\Pi})}\,. \quad (3.8)$$

We recover the constant solution when $\theta = 0$. Notice that when $\sigma = 0$, the $\mathbb{P}_\infty$ has singular support, which comprises two Dirac delta functions at the lowest energy equilibrium points, as expected from the double bracket dissipation.

3.3. **Random attractors.** We will end this short note by commenting on the random attractor of the rigid body. The proof of its existence was sketched earlier in the paper, and can be found in more detail in [ACH16].

The existence of random attractive sets is a direct consequence of the dissipation, whereas showing that this set is non-singular requires a certain amount of care. Indeed, chaotic motion only occurs, provided the noise amplitude is sufficiently large, compared to the dissipation. Chaotic motions are characterised by Lyapunov exponents, which describe the sensitivity of a dynamical system to the initial conditions. In particular, chaos can only occur, if the largest Lyapunov exponent is positive. This may be achieved for every rigid body, by choosing appropriate dissipation and noise coefficients, [ACH16].

Having shown that non-singular random attractors exist for the rigid body, one may hope to characterise their properties. At present, we have not been able to answer to this interesting question. However, some clues to the answers may be seen in numerical simulations. For example, we display in Figure 1 a realisation of a random attractor of the rigid body.[1] What is shown is the rigid body probability density, in log scale, calculated from a Monte-Carlo simulation. This approximates the SRB measure, which is supported on the random attractor, for a certain noise and dissipation. This measure is time dependent, and its motion exhibits stretching and folding. This is a known feature of some strange attractors with positive and negative Lyapunov exponents, such as the Smale horseshoe. Indeed, the positive exponent produces stretching of this set, and the negative one produces compression, which together with the original rigid body dynamics, creates the folding process. This folding mechanism in principle could create a structure similar to a horseshoe map, although we have not been able to prove this in the presence of stochasticity. An outstanding open problem would be to prove that the SRB measure for the rigid body undergoing stochastic coadjoint motion is indeed a fractal.

[1]See `http://wwwf.imperial.ac.uk/~aa10213/` for a video of this random attractor.

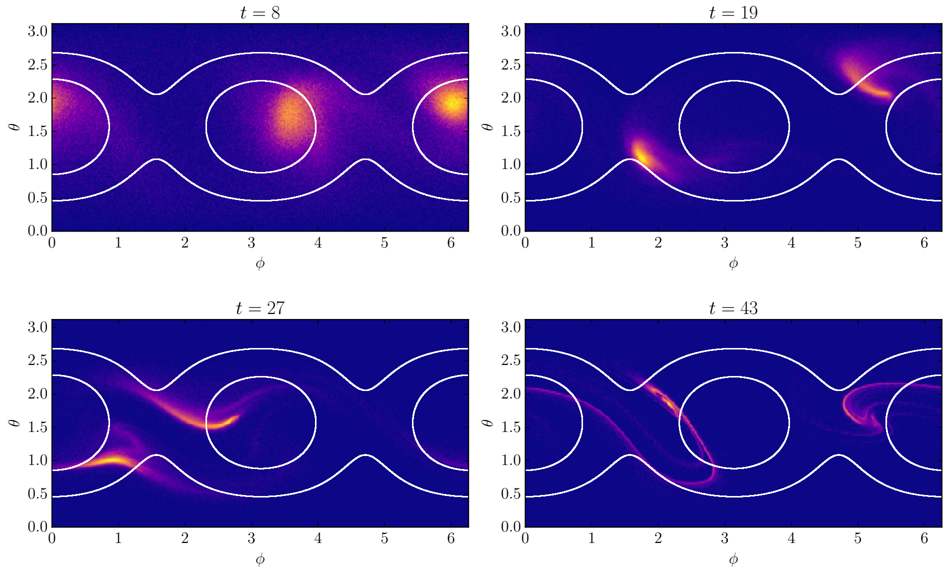

FIGURE 1. We display four snapshots of the same rigid body random attractor with $\mathbb{I} = \mathrm{diag}(1, 2, 3)$, $\theta = 0.5$ and $\sigma = 0.5$. The simulation started from a uniform distribution of rigid bodies on the momentum sphere at $t = 0$. The color is in log scale and we simulated 400 000 rigid body initial conditions with a split step numerical scheme.

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