

# NON-HYPERBOLIC ITERATED FUNCTION SYSTEMS: ATTRACTORS AND STATIONARY MEASURES

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ABSTRACT. We consider iterated function systems  $\text{IFS}(T_1, \dots, T_k)$  consisting of continuous self maps of a compact metric space  $X$ . We introduce the subset  $S_t$  of *weakly hyperbolic sequences*  $\xi = \xi_0 \dots \xi_n \dots \in \Sigma_k^+$  having the property that  $\bigcap_n T_{\xi_0} \circ \dots \circ T_{\xi_n}(X)$  is a point  $\{\pi(\xi)\}$ . The *target set*  $\pi(S_t)$  plays a role similar to the semifractal introduced by Lasota-Myjak.

Assuming that  $S_t \neq \emptyset$  (the only hyperbolic-like condition we assume) we prove that the IFS has at most one strict attractor and we state a sufficient condition guaranteeing that the strict attractor is the closure of the target set. Our approach applies to a large class of genuinely non-hyperbolic IFSs (e.g. with maps with expanding fixed points) and provides a necessary and sufficient condition for the existence of a globally attracting fixed point of the Barnsley-Hutchinson operator. We provide sufficient conditions under which the disjunctive chaos game yields the target set (even when it is not a strict attractor).

We state a sufficient condition for the asymptotic stability of the Markov operator of a recurrent IFS. For IFSs defined on  $[0, 1]$  we give a simple condition for their asymptotic stability. In the particular case of IFSs with probabilities satisfying a “locally injectivity” condition, we prove that if the target set has at least two elements then the Markov operator is asymptotically stable and its stationary measure is supported in the closure of the target set.

## 1. INTRODUCTION

In this paper we study iterated function systems (IFSs) associated to continuous self-maps  $T_1, \dots, T_k$ ,  $k \geq 2$ , defined on a compact metric space  $(X, d)$  (denoted by  $\text{IFS}(T_1, \dots, T_k)$ ). In his fundamental paper [14], Hutchinson considered hyperbolic (uniformly contracting) IFSs and proved the existence and uniqueness of global attractors and stationary measures for such IFSs. The aim of this paper is to obtain similar results for genuinely non-hyperbolic IFSs having contracting and expanding regions as well as contracting and expanding fixed points.

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A key ingredient in this study is the so-called *Barnsley-Hutchinson operator* of an IFS  $\mathfrak{F} = \text{IFS}(T_1, \dots, T_k)$  that associates to each subset  $A$  of  $X$  the set

$$(1.1) \quad \mathcal{B}_{\mathfrak{F}}(A) \stackrel{\text{def}}{=} \bigcup_{i=1}^k T_i(A).$$

This operator acts continuously in the space of non-empty compact subsets of  $X$  endowed with the Hausdorff metric. In the hyperbolic setting (all maps  $T_i$  are uniform contractions) the operator  $\mathcal{B}_{\mathfrak{F}}$  has a unique global attractor: there exists a compact set  $A_{\mathfrak{F}}$ , called the *attractor of the IFS*, such that

$$\lim_{n \rightarrow \infty} \mathcal{B}_{\mathfrak{F}}^n(K) = A_{\mathfrak{F}} \quad \text{for every compact set } K \subset X, K \neq \emptyset,$$

see [14]. Edalat [13] extended this result to *weakly hyperbolic IFSs*, that is, IFSs satisfying the following “reverse” contracting condition

$$(1.2) \quad \text{diam}(T_{\xi_0} \circ \dots \circ T_{\xi_n}(X)) \rightarrow 0 \quad \text{for every } \xi = \xi_0 \xi_1 \xi_2 \dots \in \Sigma_k^+,$$

where  $\Sigma_k^+ \stackrel{\text{def}}{=} \{1, \dots, k\}^{\mathbb{N}}$ .

In this paper we will study a more general setting than the above one, considering genuinely non-hyperbolic IFSs. One of our goals is to describe the global and local “attractors” of  $\mathcal{B}_{\mathfrak{F}}$ . More precisely, we will consider so-called *strict* and *Conley attractors*. A compact set  $A \subset X$  is a *strict attractor* of the IFS  $\mathfrak{F}$  if there is an open neighbourhood  $U$  of  $A$  such that

$$\lim_{n \rightarrow \infty} \mathcal{B}_{\mathfrak{F}}^n(K) = A \quad \text{for every compact set } K \subset U.$$

The *basin of attraction* of  $A$  is the largest open neighbourhood of  $A$  for which the above property holds. A strict attractor whose basin of attraction is the whole space is a *global attractor*. A compact set  $S \subset X$  is a *Conley attractor* of the IFS  $\mathfrak{F}$  if there exists an open neighbourhood  $U$  of  $S$  such that

$$\lim_{n \rightarrow \infty} \mathcal{B}_{\mathfrak{F}}^n(\overline{U}) = S.$$

The continuity of the Barnsley-Hutchinson operator  $\mathcal{B}_{\mathfrak{F}}$  implies that Conley and strict attractors both are fixed points of  $\mathcal{B}_{\mathfrak{F}}$ . Note also that strict attractors are Conley attractors but the converse is not true in general. Finally, we say that the IFS  $\mathfrak{F}$  is *asymptotically stable* if there is a (unique) global attractor.

The above mentioned results in [14, 13] require some sort of global contraction (hyperbolicity) of the IFS. Having in mind the definition of weakly hyperbolicity in (1.2), we introduce the subset  $S_t \subset \Sigma_k^+$  of *weakly hyperbolic sequences* defined by

$$(1.3) \quad S_t \stackrel{\text{def}}{=} \left\{ \xi \in \Sigma_k^+ : \lim_{n \rightarrow \infty} \text{diam}(T_{\xi_0} \circ \dots \circ T_{\xi_n}(X)) = 0 \right\}.$$

Note that for a weakly hyperbolic IFS one has  $S_t = \Sigma_k^+$ . If  $S_t \neq \Sigma_k^+$  we will call the IFS *non-weakly hyperbolic*. We say that an IFS *has a weakly hyperbolic sequence* if  $S_t \neq \emptyset$ . When  $S_t \neq \emptyset$  then it contains a residual subset of  $\Sigma_k^{+1}$ . We replace the condition *every* sequence is weakly hyperbolic by the condition *there is at least one* weakly hyperbolic sequence. The goal of this paper is to recover results in the spirit of [14, 13] in such a setting.

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<sup>1</sup>This follows using genericity standard arguments, see for instance the construction in [11, Proposition 3.15]

We briefly sketch our main results and philosophy of our approach, postponing the precise statements. As a general principle, rephrasing Pugh-Shub principle [21], we show that “a little hyperbolicity goes a long way guaranteeing stability-like properties”. Here by a “little hyperbolicity” we understand either the almost-sure existence of weakly hyperbolic sequences or the existence of at least one, according to the case. First, assuming that the set  $S_t$  has “probability one”, we prove that the Markov operator is asymptotically stable (here we consider Markov measures associated to transition matrices and the particular case of Bernoulli probabilities). Second, we prove that if the Barnsley-Hutchinson operator has a unique fixed point then the IFS is asymptotically stable. Finally, in the case when  $X$  is an interval, to establish the stability of the Markov operator we show that it is enough to assume that there are no common fixed points for the maps of the IFS and that there exists at least one weakly hyperbolic sequence.

We now provide more details for our main results (for the precise definitions and statements see Section 2). Associated to the set  $S_t$  of weakly hyperbolic sequences we consider the *coding map*  $\pi: S_t \rightarrow X$  that projects  $S_t$  into the phase space  $X$ , see equation (2.1). The set  $A_t \stackrel{\text{def}}{=} \pi(S_t)$  is called the *target* set and contains relevant dynamical information of the IFS. Assuming that  $S_t \neq \emptyset$ , we prove the following results:

- The closure of the target set  $\overline{A_t}$  is a Conley attractor if and only if it is a strict attractor (Theorem 1).
- The set  $\overline{A_t}$  is the global maximal fixed point of the IFS if and only if the IFS is asymptotically stable. Moreover, the Barnsley-Hutchinson operator has a unique fixed point if and only if it is asymptotically stable (Theorem 2).

We will investigate more closely the relation between target sets and semifractals introduced in [19]. An IFS  $\mathfrak{F} = \text{IFS}(T_1, \dots, T_k)$  is said to be *regular* if there are numbers  $1 \leq i_1 < i_2 < \dots < i_\ell \leq k$  such that  $\mathfrak{F}' = \text{IFS}(T_{i_1}, \dots, T_{i_\ell})$  is asymptotically stable. The global attractor of  $\mathfrak{F}'$  is called a *nucleus* of  $\mathfrak{F}$  (an IFS may have several nuclei). By [19] for any regular IFS  $\mathfrak{F}$  there exists *the* minimum fixed point of  $\mathfrak{F}$ , called its *semifractal* and denoted by  $\text{Semi}(\mathfrak{F})$ . It is obtained from any nucleus of  $\mathfrak{F}$  and attracts every compact set inside it, where iterations are taken with respect the Barnsley-Hutchinson operator of  $\mathfrak{F}$ . On the other hand, when  $S_t \neq \emptyset$ , the set  $\overline{A_t}$  is a minimum fixed point that attracts every compact set inside it. This provides the following characterisation of semifractals:

- If an IFS  $\mathfrak{F}$  is regular and satisfies  $S_t \neq \emptyset$  then  $\text{Semi}(\mathfrak{F}) = \overline{A_t}$ .

For a non-regular IFS with  $S_t \neq \emptyset$  (see Example 6.1) the set  $\overline{A_t}$  plays the same role as a semifractal plays for a regular IFS. We refer to Remark 3.13 to support this assertion.

We will also study the consequence of our approach for the so-called chaos game. The *chaos game* is an algorithm for generating fractals using random iterations of an IFS, see [2]. It has probabilistic and disjunctive (deterministic) versions, see [2, 6, 8, 5]. Given an initial point  $x = x_0 \in X$ , one considers the orbit  $x_{n+1} = T_{\xi_n}(x_n)$ , where the sequence  $\xi \in \Sigma_k^+$  is chosen according to some probability (*probabilistic game*) or is a disjunctive sequence (*disjunctive game*). Recall that  $\xi \in \Sigma_k^+$  is *disjunctive* if its orbit (with respect to the usual left shift  $\sigma$  defined by  $\sigma(\xi)_n = \xi_{n+1}$ ) is dense in  $\Sigma_k^+$ . The *chaos game holds* when the sequence of *tails*  $(\{x_n : n \geq \ell\})_\ell$

in the Hausdorff distance converges to some attracting “fractal” (in such a case we also say that *chaos game yields the fractal*).

A natural question is how typically this game holds, where the term typical either refers to sequences in  $\Sigma_k^+$  or points in the phase space  $X$ . By [6], the probabilistic chaos game holds when the fractal is a strict attractor and the initial point is in its basin of attraction. By [8], the disjunctive chaos game holds for a special class of attractors<sup>2</sup> and every point in the pointwise basin of attraction.

In the context of the chaos game, [19] considers IFSs whose maps are Lipschitz with constants less than or equal to 1 and have at least one uniformly contracting map. It is proved that the probabilistic chaos game starting at any point of the phase space yields the semifractal (even if the semifractal is not an attractor). In our setting, we get a similar result for the disjunctive chaos game where the fractal is the closure of the target set.

A fixed point  $A$  of the Barnsley-Hutchinson operator is *stable* if for every open neighbourhood  $V$  of  $A$  there is an open neighbourhood  $V_0$  of  $A$  such that

$$(1.4) \quad \mathcal{B}^n(V_0) \subset V \quad \text{for every } n \geq 0.$$

For instance, the set  $\overline{A_t}$  is stable when it is a Conley attractor or when all the maps of the IFS are Lipschitz with constants less than or equal to 1 (the existence of a contracting map is not required). See Section 3.2 for an example where  $\overline{A_t}$  is stable but is not a Conley attractor.

- When  $\overline{A_t}$  is a stable fixed point the *disjunctive chaos game* holds for every point in the phase space (Theorem 3).

Finally we consider IFSs from the ergodic point of view, studying the existence and uniqueness of stationary measures. Recall that given an space of finite measures  $\mathfrak{M}(X)$  defined on a set  $X$ , an operator  $\mathfrak{T}: \mathfrak{M}(X) \rightarrow \mathfrak{M}(X)$  such that

- $\mathfrak{T}$  is linear and
- $\mathfrak{T}\nu(X) = \nu(X)$  for every  $\nu \in \mathfrak{M}(X)$

is called a *Markov operator*. A *stationary measure of  $\mathfrak{T}$*  is a fixed point of  $\mathfrak{T}$ . The operator  $\mathfrak{T}$  is *asymptotically stable* if it has a stationary measure  $\nu$  such that  $\lim \mathfrak{T}^n \mu = \nu$  for every  $\mu \in \mathfrak{M}(X)$ , in the weak\* topology. The ergodic study of IFSs deals with two main settings:

- *IFSs with probabilities* given by a Bernoulli probability  $\mathfrak{b}$  that assigns (positive) weights to each map;
- *Recurrent IFSs* associated to an irreducible transition matrix  $P$  inducing a Markov probability  $\mathbb{P}^+$ .

From the ergodic viewpoint one studies the iterations of points by an IFS (random orbits) as a Markov process and each type of IFS has associated a special type of Markov operator (associated to Bernoulli probabilities and associated to transition matrices). For a discussion see [3, 4].

When  $S_t \neq \emptyset$  and  $X = [0, 1]$  our ergodic results are summarised as follows:

- Every injective IFS with Bernoulli probability  $\mathfrak{b}$  whose target set  $A_t$  is not a *singleton* (i.e., has at least two points) is asymptotically stable and its unique stationary measure is  $\pi_* \mathfrak{b}$  and satisfies  $\text{supp}(\pi_* \mathfrak{b}) = \overline{A_t}$ . In this case,  $\overline{A_t}$  is uncountable and the stationary measure is continuous (Theorem 4).

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<sup>2</sup>Called *well-fibered* attractors, see also the strongly fibered case in [5].

We will see that condition  $\#(A_t) \geq 2$  ( $\#(A)$  means the cardinality of the set  $A$ ) implies that  $\mathfrak{b}(S_t) = 1$ . For IFSs with  $S_t \neq \emptyset$  we see that if the Markov operator associated to a Bernoulli probability  $\overline{\mathfrak{b}}$  is asymptotically stable then the support of its stationary measure is  $\overline{A_t}$ , even when  $\mathfrak{b}(S_t) = 0$ , see Proposition 5.4 (this proposition does not require  $X = [0, 1]$ ).

- An injective recurrent IFS with a *splitting Markov measure*<sup>3</sup>  $\mathbb{P}^+$  satisfies  $\mathbb{P}^+(S_t) = 1$  (Theorem 5). We also get sufficient conditions for the asymptotic stability of a recurrent IFS and characterise its unique stationary measure (Theorem 6).

This paper is organised as follows. In Section 2 we state the main definitions and the precise statements of our results. Section 3 is devoted to the study of different types of attractors of IFSs and to the proofs of Theorems 1, 2, and 3. In Section 4, we consider IFSs defined on the interval  $[0, 1]$ , study the measure of  $S_t$  for Markov measures, and prove Theorem 5. We also get results about probabilistic rigidity of  $S_t$  (Theorem 4.8) and characterise separable IFSs (Theorem 4.10). In Section 5 we prove Theorems 4 and 6 about stability of the Markov operator. Finally, in Section 6 we present some examples.

## 2. PRECISE STATEMENT OF RESULTS

**2.1. Topological properties of IFSs.** Consider the set  $S_t$  of weakly hyperbolic sequences in (1.3) and define the *coding map*<sup>4</sup>

$$(2.1) \quad \pi: S_t \rightarrow X \quad \text{by} \quad \pi(\xi) \stackrel{\text{def}}{=} \lim_{n \rightarrow \infty} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(p),$$

where  $p$  is any point of  $X$ . By definition of the set  $S_t$ , this limit always exists and is independent of  $p \in X$ . We introduce the *target set*  $A_t \stackrel{\text{def}}{=} \pi(S_t)$ . This name is justified by the following characterisation

$$(2.2) \quad A_t = \{x \in X : \text{there is } \xi \in \Sigma_k^+ \text{ with } \{x\} = \bigcap_n T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X)\},$$

see (3.2). The target set plays a key role in the study of strict attractors. We prove that if  $S_t \neq \emptyset$  then the IFS has at most one strict attractor. Moreover, if such a strict attractor exists then it is equal to  $\overline{A_t}$ , see Proposition 3.7.

**Theorem 1.** *Consider an IFS defined on a compact metric space such that  $S_t \neq \emptyset$ . Then  $\overline{A_t}$  is a Conley attractor if and only if it is a strict attractor.*

In [7] Barnsley and Vince consider IFSs consisting either of affine maps or of Möbius maps and introduce sufficient conditions that guarantee the existence of a unique strict attractor. The proof involves some type of local hyperbolicity in a neighbourhood of a Conley attractor, see [1, 25]. We point out that Theorem 1 only requires the existence of *at least one* weakly hyperbolic sequence.

Given an IFS  $\mathfrak{F}$  and its Barnsley-Hutchinson operator  $\mathcal{B}_{\mathfrak{F}}$ , a subset  $Y \subset X$  is  $\mathcal{B}_{\mathfrak{F}}$ -invariant if  $\mathcal{B}_{\mathfrak{F}}(Y) \subset Y$ . The closure of any  $\mathcal{B}_{\mathfrak{F}}$ -invariant set contains some fixed point of  $\mathcal{B}_{\mathfrak{F}}$  (see the discussion below). Therefore, since  $X$  is  $\mathcal{B}_{\mathfrak{F}}$ -invariant,

<sup>3</sup>This is an ergodic version of the condition “the set  $A_t$  is not a singleton” and means that there is  $i$  such that the restriction of  $\pi$  to  $[i] \cap \text{supp}(\mathbb{P}^+)$  is not constant.

<sup>4</sup>This is the standard terminology for the map  $\pi$  when  $S_t = \Sigma_k^+$ .

the operator  $\mathcal{B}_{\mathfrak{F}}$  always has at least one fixed point. Indeed, we have a more precise description of the fixed points of  $\mathcal{B}_{\mathfrak{F}}$ . Following [13], given  $Y \subset X$  define the set

$$(2.3) \quad Y^* \stackrel{\text{def}}{=} \bigcap_{n \geq 0} \mathcal{B}_{\mathfrak{F}}^n(Y).$$

If  $Y$  is  $\mathcal{B}_{\mathfrak{F}}$ -invariant then the set  $(\overline{Y})^*$  is the *global maximal fixed point of the restriction of  $\mathcal{B}_{\mathfrak{F}}$  (or of the IFS) to the subsets of  $\overline{Y}$* , see Proposition 3.1. The next theorem generalizes [13] in two ways: it applies also to IFSs which are not weakly hyperbolic and it provides a necessary and sufficient condition for the existence of a global attractor.

**Theorem 2.** *Consider an IFS  $\mathfrak{F}$  defined on a compact metric space  $X$  such that  $S_t \neq \emptyset$ . Then the following three assertions are equivalent:*

- (1)  $\overline{A_t} = X^*$ ,
- (2) the Barnsley-Hutchinson operator  $\mathcal{B}_{\mathfrak{F}}$  has a unique fixed point,
- (3)  $X^*$  is a global attractor of the IFS  $\mathfrak{F}$ .

We observe that the statement in Theorem 2 is sharp. Indeed, there are examples of non-weakly hyperbolic IFSs where  $A_t \subsetneq \overline{A_t} = X^*$ , see Section 6.

Let us observe that for weakly hyperbolic IFSs it holds  $A_t = X^*$ , see Lemma 3.3 and also [13]. We observe that there are IFSs that are non-weakly hyperbolic such that  $A_t = \overline{A_t} = X^*$ , see Section 6.

**Theorem 3** (Disjunctive chaos game). *Consider an IFS  $(T_1, \dots, T_k)$  defined on a compact metric space  $X$  such that  $\overline{A_t}$  is a stable fixed point of the Barnsley-Hutchinson operator. Then for every  $x \in X$  and every disjunctive sequence  $\xi \in \Sigma_k^+$  we have*

$$\overline{A_t} = \bigcap_{\ell \geq 0} \overline{\{x_{n,\xi} : n \geq \ell\}}, \quad \text{where } x_{n,\xi} \stackrel{\text{def}}{=} T_{\xi_n} \circ \dots \circ T_{\xi_0}(x).$$

In particular

$$\lim_{\ell \rightarrow \infty} \overline{\{x_{n,\xi} : n \geq \ell\}} = \overline{A_t},$$

where the limit is considered in the Hausdorff distance.

## 2.2. Ergodic properties of IFSs.

2.2.1. *IFSs with probabilities.* Consider an IFS  $(T_1, \dots, T_k)$  defined on a compact metric space  $X$  and strictly positive numbers  $p_1, \dots, p_k$  (called *weights*) such that  $\sum_{i=1}^k p_i = 1$ . We denote by  $\mathfrak{b} = \mathfrak{b}(p_1, \dots, p_k)$  the (non-trivial) Bernoulli probability measure with weights  $p_1, \dots, p_k$  defined on  $\Sigma_k^+$ . We denote by IFS  $(T_1, \dots, T_k; \mathfrak{b})$  the IFS with the corresponding Bernoulli probability and say that it is an *IFS with probabilities*.

Let  $\mathcal{M}_1(X)$  be the space of Borel probability measures defined on  $X$  equipped with the weak\*-topology. The *Markov operator* associated to the IFS  $(T_1, \dots, T_k; \mathfrak{b})$  is defined by

$$(2.4) \quad \mathfrak{T}_{\mathfrak{b}} : \mathcal{M}_1(X) \rightarrow \mathcal{M}_1(X), \quad \mathfrak{T}_{\mathfrak{b}}\mu \stackrel{\text{def}}{=} \sum_{i=1}^k p_i T_{i*}\mu,$$

where  $T_{i*}\mu(A) = \mu(T_i^{-1}(A))$  for every Borel set  $A$ . Note that the Markov operator  $\mathfrak{T}_{\mathfrak{b}}$  is continuous. Hence, if  $\mathfrak{T}_{\mathfrak{b}}$  is asymptotically stable then its attracting measure  $\mu$  is stationary, that is, satisfies  $\mathfrak{T}_{\mathfrak{b}}\mu = \mu$ .

An IFS with probabilities  $\text{IFS}(T_1, \dots, T_k; \mathbf{b})$  is called *asymptotically stable* if its Markov operator  $\mathfrak{T}_{\mathbf{b}}$  is asymptotically stable. It is a folklore result that if  $\mathbf{b}(S_t) = 1$  then the IFS is asymptotically stable and  $\pi_* \mathbf{b}$  is the unique stationary measure, see for instance [22, 18]. In Proposition 5.1 we prove this fact and we see that  $\text{supp}(\pi_* \mathbf{b}) = \overline{A_t}$ . Note that, since that  $\sigma^{-1}(S_t) \subset S_t$ , the ergodicity of the Bernoulli measure (with positive weights)  $\mathbf{b}$  with respect to the shift implies that either  $\mathbf{b}(S_t) = 1$  or  $\mathbf{b}(S_t) = 0$ .

A combination of Theorem 2 and Proposition 5.1 allows us to recover properties of hyperbolic IFSs in non-hyperbolic settings provided that the sets  $A_t$  and  $S_t$  are “big enough” (from the topological and probabilistic points of view, respectively): there are a unique global attractor and the IFS with probabilities is asymptotically stable.

Proposition 5.1 assumes that  $\mathbf{b}(S_t) = 1$  (which is often difficult to verify). When  $X = [0, 1]$  we improve this proposition replacing the condition  $\mathbf{b}(S_t) = 1$  by the topological condition  $\#(A_t) \geq 2$  that we call *separability* and it is quite straightforward to verify.

**Theorem 4.** *Consider an IFS  $(T_1, \dots, T_k)$  defined on  $[0, 1]$  such that*

- *the target set  $A_t$  has at least two elements and*
- *there is a non-trivial closed interval  $J \subset [0, 1]$  such that  $T_i(J) \subset J$  and  $T_i|_J$  is injective for every  $j \in \{1, \dots, k\}$ .*

*Then for every (non-trivial) Bernoulli probability  $\mathbf{b}$  the IFS  $(T_1, \dots, T_k; \mathbf{b})$  is asymptotically stable.*

*Moreover,  $\pi_* \mathbf{b}$  is the (unique) stationary measure of IFS  $(T_1, \dots, T_k; \mathbf{b})$ , satisfies  $\text{supp}(\pi_* \mathbf{b}) = \overline{A_t}$ , and is continuous. As a consequence, the set  $A_t$  has no isolated points.*

In the previous theorem, the purely topological condition  $\#(A_t) \geq 2$  depending only on  $\text{IFS}(T_1, \dots, T_k)$  implies the asymptotic stability of the Markov operator  $\mathfrak{T}_{\mathbf{b}}$  of  $\text{IFS}(T_1, \dots, T_k; \mathbf{b})$  for any (non-trivial) Bernoulli probability  $\mathbf{b}$ . Moreover, we also obtain properties of the stationary measure. The support of this stationary measure is independent of the Bernoulli probability. In Proposition 5.4 we state a result about the support of stationary measures that holds for general compact metric spaces: if  $S_t \neq \emptyset$  and the Markov operator associated to  $\mathbf{b}$  is asymptotically stable then the support of its stationary measure always is  $\overline{A_t}$ , even when  $\mathbf{b}(S_t) = 0$ .

The asymptotic stability of an IFS with probabilities has been obtained in several contexts such as, for example, *contracting on average* [3], weakly hyperbolic [13], and *non-overlapping*<sup>5</sup> [24]. Observe that the contexts of [3, 13] have a hyperbolic flavour. Let us also observe that [23] states the asymptotic stability of admissible IFSs consisting of circle homeomorphisms (these homeomorphisms preserve the orientation and some homeomorphism of the IFS is transitive). Note that in this case the set  $S_t$  is empty. Let us compare these results with Theorem 4. First, the condition to be contracting on average depends on the selected Bernoulli probability (an IFS may be contracting in average with respect to some probabilities but not with respect to *all* Bernoulli probabilities). In contrast, weak hyperbolicity, separability, non-overlapping, and admissibility conditions are topological conditions that

<sup>5</sup>An IFS is called non-overlapping if the maps  $T_i$  are injective and the sets  $T_i(I)$  have disjoint interiors. We will see that separability is a weak form of non-overlapping, see Theorem 4.10. We observe that [13] and [3] do not involve injective-like conditions of the IFS.

do not involve probabilities. These conditions guarantee the asymptotic stability of the Markov operator  $\mathfrak{T}_{\mathfrak{b}}$  of the IFS with respect to *any* Bernoulli probability  $\mathfrak{b}$ .

Finally, note that checking the properties of weak hyperbolicity and contracting on average may be rather complicated, while the separability condition is comparably much simpler, thus Theorem 4 can also be useful in these contexts.

**2.2.2. Recurrent IFSs.** A generalization of IFSs with probabilities are the so-called *recurrent IFSs* introduced in [4], where the weights  $p_i$  are replaced by a transition matrix.

To be more precise, recall that a  $k \times k$  matrix  $P = (p_{ij})$  is a *transition matrix* if  $p_{ij} \geq 0$  for all  $i, j$  and for every  $i$  it holds  $\sum_{j=1}^k p_{ij} = 1$ . An *stationary probability vector* associated to  $P$  is a vector  $\bar{p} = (p_1, \dots, p_k)$  whose elements are non-negative real numbers and sum up to 1 and satisfies  $\bar{p}P = \bar{p}$ . The transition matrix  $P$  is called *irreducible* if for every  $\ell, r \in \{1, \dots, k\}$  there is  $n = n(\ell, r)$  such that  $P^n = (p_{ij}^n)$  satisfies  $p_{\ell, r}^n > 0$ . An irreducible transition matrix has a unique stationary probability vector  $\bar{p} = (p_i)$ , see [15, page 100]. We consider the *cylinders*

$$[a_0 \dots a_\ell] \stackrel{\text{def}}{=} \{\omega \in \Sigma_k^+ : \omega_0 = a_0, \dots, \omega_\ell = a_\ell\} \subset \Sigma_k^+$$

which is a semi-algebra that generates the Borel  $\sigma$ -algebra of  $\Sigma_k^+$ . We denote by  $\mathbb{P}^+$  the *Markov measure* associated to  $(P, \bar{p})$  defined on  $\Sigma_k^+$ , this measure is defined on the cylinders  $[a_0 \dots a_\ell]$  by

$$\mathbb{P}^+([a_0 \dots a_\ell]) \stackrel{\text{def}}{=} p_{a_0} p_{a_0 a_1} \dots p_{a_{\ell-1} a_\ell}.$$

Given an IFS  $(T_1, \dots, T_k)$  and an irreducible transition matrix  $P = (p_{ij})$ , we call IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  a *recurrent IFS*. We now introduce the Markov operator in this context. Consider the set  $\hat{X} \stackrel{\text{def}}{=} X \times \{1, \dots, k\}$  with the product topology and the corresponding Borel sets. Given a subset  $\hat{B} \subset \hat{X}$ , its  *$i$ -section* is defined by

$$\hat{B}_i \stackrel{\text{def}}{=} \{x \in X : (x, i) \in \hat{B}\}.$$

The  *$i$ -section of a probability measure*  $\hat{\mu}$  on  $\hat{X}$  is defined on the set  $X$  by

$$\mu_i(B) \stackrel{\text{def}}{=} \hat{\mu}(B \times \{i\}), \quad \text{where } B \text{ is any Borel subset of } X.$$

Observe that  $\mu_i$  is a finite measure on  $X$  but, in general, it is not a probability measure. Since the measure  $\hat{\mu}$  is completely defined by its sections we write  $\hat{\mu} = (\mu_1, \dots, \mu_k)$  and note that

$$\hat{\mu}(\hat{B}) = \sum_{j=1}^k \mu_j(\hat{B}_j) \quad \text{for every Borel subset } \hat{B} \text{ of } \hat{X}.$$

The (*generalised*) *Markov operator* of recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  is defined by

$$(2.5) \quad \mathfrak{S}_{\mathbb{P}^+} : \mathcal{M}_1(\hat{X}) \rightarrow \mathcal{M}_1(\hat{X}), \quad \hat{\mu} \mapsto \mathfrak{S}_{\mathbb{P}^+}(\hat{\mu}),$$

where

$$\mathfrak{S}_{\mathbb{P}^+}(\hat{\mu})(\hat{B}) \stackrel{\text{def}}{=} \sum_{i,j} p_{ij} T_j * \mu_i(\hat{B}_j).$$

A recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  is called *asymptotically stable* if the Markov operator  $\mathfrak{S}_{\mathbb{P}^+}$  is asymptotically stable.



Given a Markov measure  $\mathbb{P}^+$  there is associated its *inverse Markov measure*  $\mathbb{P}^-$  defined on  $\Sigma_k^+$  by

$$(2.6) \quad \mathbb{P}^-([a_0 a_1 \dots a_n]) \stackrel{\text{def}}{=} \mathbb{P}^+([a_n \dots a_1 a_0]), \quad \text{for a cylinder } [a_0 a_1 \dots a_n].$$

The measure  $\mathbb{P}^-$  is also Markov (see Section 2.3.2).

There is the following *generalised coding map* from  $S_t$  to  $\widehat{X}$  defined by

$$(2.7) \quad \varpi: S_t \rightarrow \widehat{X}, \quad \varpi(\xi) \stackrel{\text{def}}{=} (\pi(\xi), \xi_0).$$

In Theorem 5.5 we see that if a recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  is such that  $\mathbb{P}^-(S_t) = 1$  and  $\mathbb{P}^-$  is mixing then it is asymptotically stable and the stationary measure of  $\mathfrak{S}_{\mathbb{P}^+}$  is  $\varpi_* \mathbb{P}^-$ , that is,

$$\mathfrak{S}_{\mathbb{P}^+}^n(\widehat{\mu}) \xrightarrow[*]{} \varpi_* \mathbb{P}^- \quad \text{for every } \widehat{\mu} \in \mathcal{M}_1(\widehat{X}).$$

This is a version of Proposition 5.1 for recurrent IFSs.

As in the case of IFSs with probabilities, when  $X = [0, 1]$  we can improve Theorem 5.5. In this proposition it is assumed that  $\mathbb{P}^-(S_t) = 1$  (verifying this assumption is in general difficult). When  $X = [0, 1]$  we can replace this condition by a “splitting condition” that is quite straightforward to verify.

Consider a recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  defined on a compact metric space  $X$ . A cylinder  $[j_1 \dots j_s]$  is called *admissible* if  $\mathbb{P}^+([j_1 \dots j_s]) > 0$ .

**Definition 2.1** (Splitting Markov measure). *Consider an IFS  $\mathfrak{F} = \text{IFS}(T_1, \dots, T_k)$  defined on  $[0, 1]$  and a non-trivial closed interval  $J$  of  $[0, 1]$ . A Markov measure  $\mathbb{P}^+$  defined on  $\Sigma_k^+$  splits the IFS  $\mathfrak{F}$  in  $J$  if*

- $T_i(J) \subset J$  and  $T_i|_J$  is injective for every  $i \in \{1, \dots, k\}$ ,
- there are admissible cylinders  $[i_1 \dots i_\ell]$  and  $[j_1 \dots j_s]$  of  $\mathbb{P}^+$  with  $i_1 = j_1$  such that

$$T_{j_1} \circ \dots \circ T_{j_s}(I) \cap T_{i_1} \circ \dots \circ T_{i_\ell}(I) = \emptyset$$

and

$$T_{j_1} \circ \dots \circ T_{j_s}(I) \cup T_{i_1} \circ \dots \circ T_{i_\ell}(I) \subset J.$$

When  $J = I$  we say that  $\mathbb{P}^+$  splits  $\mathfrak{F}$ .

Let  $T$  be a measure-preserving transformation on a probability space  $(X, \mathbb{B}, \mu)$ . Recall that  $(T, \mu)$  is *ergodic* if for every measurable set  $A$  with  $T^{-1}(A) = A$  it holds  $\mu(A) = 0$  or  $\mu(A) = 1$ . Recall that  $(T, \mu)$  is *mixing* if

$$\lim_{n \rightarrow \infty} \mu(T^{-n}(A) \cap B) = \mu(A) \mu(B) \quad \text{for every } A, B \in \mathbb{B}.$$

A Borel measure  $\mu$  on  $\Sigma_k^+$  is *mixing* if the system  $(\sigma, \mu)$  is mixing.

Next theorem states consequences of the splitting property of a Markov measure and is the main tool to get the asymptotic stability of the Markov operator.

**Theorem 5.** *Consider an IFS  $(T_1, \dots, T_k)$  defined on the interval  $[0, 1]$ . If  $\mathbb{P}^+$  is mixing Markov measure that splits the IFS in some non-trivial closed interval  $J$  then  $\mathbb{P}^+(S_t) = 1$ .*

Next theorem gives sufficient conditions for the asymptotic stability of the Markov operator.

**Theorem 6.** *Consider a recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  defined on the interval  $[0, 1]$ . Suppose that the inverse Markov measure  $\mathbb{P}^-$  is mixing and splits the IFS in some non-trivial closed interval  $J$ . Then  $\text{IFS}(T_1, \dots, T_k; \mathbb{P}^+)$  is asymptotically stable and  $\varpi_* \mathbb{P}^-$  is the stationary measure of the Markov operator  $\mathfrak{S}_{\mathbb{P}^+}$ .*

Note that  $\mathbb{P}^+$  is mixing if and only if  $\mathbb{P}^-$  is mixing. However, a splitting property for  $\mathbb{P}^+$  does not imply a splitting property for  $\mathbb{P}^-$  (and vice-versa).

**2.3. Preliminaries and notation.** We now establish some basic definitions and notations.

**2.3.1. Distances.** Throughout this paper  $(X, d)$  is a compact metric space and  $\mathcal{P}(X)$  denotes the power set of  $X$ . Given a point  $x \in X$  and a set  $A \subset X$ , distance between  $x$  and  $A$  is defined by

$$d(x, A) \stackrel{\text{def}}{=} \inf\{d(x, a) : a \in A\}.$$

The *Hausdorff distance* between two sets  $A, B \subset X$  is defined by

$$d_H(A, B) \stackrel{\text{def}}{=} \max\{h_s(A, B), h_s(B, A)\}, \quad \text{where} \quad h_s(A, B) \stackrel{\text{def}}{=} \sup_{a \in A} d(a, B).$$

Note that, in general,  $d_H$  is only a pseudo-metric defined on  $\mathcal{P}(X)$ . Let  $\mathcal{H}(X) \subset \mathcal{P}(X)$  be the set of all non-empty compact subsets of  $X$ . Then  $(\mathcal{H}(X), d_H)$  is a compact metric space, see [2].

**2.3.2. Inverse Markov measures.** Consider a transition matrix  $P = (p_{ij})$  and a stationary probability vector  $\bar{p} = (p_1, \dots, p_k)$  of  $P$ . If all entries of  $\bar{p}$  are (strictly) positive then the *inverse transition matrix* associated to  $(P, \bar{p})$  is the matrix  $Q_{(P, \bar{p})} = (q_{ij})$  where

$$q_{ij} \stackrel{\text{def}}{=} \frac{p_j}{p_i} p_{ji}.$$

Note that  $Q = Q_{(P, \bar{p})}$  is a transition matrix and  $\bar{p}$  is a stationary probability vector of  $Q_{(P, \bar{p})}$ . We observe that if  $P$  is primitive if and only if  $Q$  is primitive.

Denote by  $\mathbb{P}^-$  the Markov measure associated to  $(Q, \bar{p})$ . For every cylinder  $[a_0 \dots a_\ell]$  it holds

$$\mathbb{P}^-([a_0 \dots a_\ell]) = \mathbb{P}^+([a_\ell \dots a_0]),$$

where  $\mathbb{P}^+$  is the Markov measure associated to  $(P, \bar{p})$ .

Let us observe that a Markov measure  $\mathbb{P}^+$  is mixing if and only if the transition matrix is *primitive*<sup>6</sup> (i.e. there is  $n \geq 1$  such that all the entries of  $P^n$  are strictly positive), see for instance [10, page 79]. As a consequence,  $\mathbb{P}^-$  is mixing if and only if  $P$  is primitive.

### 3. ATTRACTORS OF ITERATED FUNCTION SYSTEMS

This section is devoted to the study of fixed points and the attractors of the Barnsley-Hutchinson operator of an IFS (see Sections 3.1 and 3.2). Our goal is to prove Theorems 1, 2, and 3 (see Sections 3.3, 3.4, and 3.6, respectively). We also get some topological properties of the target set  $A_t$  in Section 3.5.

In what follows we consider  $\mathfrak{F} = \text{IFS}(T_1, \dots, T_k)$  and denote by  $\mathcal{B}_{\mathfrak{F}} = \mathcal{B}$  its Barnsley-Hutchinson operator, recall (1.1).

<sup>6</sup>Also called *aperiodic*.

**3.1. Fixed points for the Barnsley-Hutchinson operator.** We will show that every compact invariant set  $A$  of  $X$  (i.e.,  $\mathcal{B}(A) \subset A$ ) contains some fixed point of  $\mathcal{B}$ . Since  $X$  is invariant for  $\mathcal{B}$  this implies that  $\mathcal{B}$  always has at least one fixed point. To each set  $A$  we associate the set  $A^* \stackrel{\text{def}}{=} \bigcap_{n \geq 0} \mathcal{B}^n(A)$ , recall (2.3).

Recall that  $\mathcal{H}(X)$  denotes the set consisting of all non-empty compact subsets of  $X$ . We consider in  $\mathcal{H}(X)$  the Hausdorff distance  $d_H$ .

**Proposition 3.1** (Existence of fixed points of  $\mathcal{B}$ ). *Consider  $A \in \mathcal{H}(X)$  such that  $\mathcal{B}(A) \subset A$ . Then  $A^*$  is a fixed point of  $\mathcal{B}$ . In particular,  $X^*$  is a fixed point of  $\mathcal{B}$ .*

*Proof.* The proposition follows from the next lemma and the continuity of  $\mathcal{B}$ .

**Lemma 3.2.** *Let  $(A_n)$  be a sequence of nested compact sets,  $A_{n+1} \subset A_n$ , and  $A = \bigcap_{n \geq 0} A_n$ . Then  $d_H(A_n, A) \rightarrow 0$ .*

*Proof.* The proof is by contradiction. If the lemma is false there are  $\epsilon > 0$  and a subsequence  $(n_\ell)$ ,  $n_\ell \rightarrow \infty$ , such that  $d_H(A_{n_\ell}, A) \geq \epsilon$  for all  $\ell$ . Since  $A \subset A_n$ , for each  $\ell$  there is a point  $p_\ell \in A_{n_\ell}$  such that  $d(p_\ell, A) \geq \epsilon$ . By compactness, taking a subsequence if necessary, we can assume that  $p_\ell \rightarrow p$ . As  $(A_n)$  is nested it follows that  $p \in A$ , contradicting that  $d(p_\ell, A) \geq \epsilon$  for all  $\ell$ .  $\square$

To prove the proposition it is enough to apply the lemma to nested sequence  $A_n = \mathcal{B}^n(A)$ .  $\square$

Now let us look more closely to the fixed point  $X^*$  of  $\mathcal{B}$ . For that to each  $\xi \in \Sigma_k^+$  we consider its *fibre* defined by

$$(3.1) \quad I_\xi \stackrel{\text{def}}{=} \bigcap_{n \geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X), \quad \text{if } \xi = \xi_0 \xi_1 \dots$$

We will see in Lemma 3.3 that the set  $X^*$  is the union of the fibres  $I_\xi$ .

Note that every fibre is a non-empty set: just note that  $(T_{\xi_0} \circ \cdots \circ T_{\xi_n}(X))_{n \in \mathbb{N}}$  is a sequence of nested compact sets. Moreover, when  $X$  is an interval, the fibres also are intervals (may be trivial ones). With this definition, the set  $S_t$  of weakly hyperbolic sequences, recall (1.3), is given by

$$S_t = \{\xi \in \Sigma_k^+ : I_\xi \text{ is a singleton}\}.$$

From the definition of the target set  $A_t$  in (2.2) it immediately follows that

$$(3.2) \quad A_t = \bigcup_{\xi \in S_t} I_\xi.$$

Recall that by definition for every set  $A$  we have

$$(3.3) \quad A^* = \bigcap_{n \geq 0} \mathcal{B}^n(A) = \bigcap_{n \geq 0} \bigcup_{\xi \in \Sigma_k^+} T_{\xi_0} \circ \cdots \circ T_{\xi_{n-1}}(A).$$

Next lemma just says that the operations “ $\cup$ ” and “ $\cap$ ” above commute.

**Lemma 3.3.** *Let  $A \in \mathcal{H}(X)$  such that  $\mathcal{B}(A) \subset A$ . Then*

$$A^* = \bigcap_{n \geq 0} \mathcal{B}^n(A) = \bigcup_{\xi \in \Sigma_k^+} \bigcap_{n \geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(A).$$

*In particular,*

$$X^* = \bigcup_{\xi \in \Sigma_k^+} I_\xi.$$

*Proof.* Condition  $\mathcal{B}(A) \subset A$  implies that  $\mathcal{B}^n(A)$  is a decreasing nested family of compact subsets and  $T_i(A) \subset A$  for all  $i = 1, \dots, k$ . From equation (3.3) it follows immediately that

$$\bigcup_{\xi \in \Sigma_k^+} \bigcap_{n \geq 0} T_{\xi_0} \circ \dots \circ T_{\xi_n}(A) \subset \bigcap_{n \geq 0} \bigcup_{\xi \in \Sigma_k^+} T_{\xi_0} \circ \dots \circ T_{\xi_n}(A) = A^*.$$

which implies the inclusion “ $\supset$ ”.

To prove the inclusion “ $\subset$ ” we use the following classical combinatorial lemma, see for instance [16, page 302].

**Lemma 3.4** (König’s lemma). *Let  $G$  be a connected graph with infinitely many vertices such that every vertex has finite degree. Then  $G$  contains an infinite path with no repeated vertices.*

Take now any point  $p \in \bigcap_{n \geq 0} \mathcal{B}^n(A)$ . Then for each  $n \geq 1$  there is a finite sequence  $\beta_0^n \dots \beta_{n-1}^n$  such that  $p \in T_{\beta_0^n} \circ \dots \circ T_{\beta_{n-1}^n}(A)$ . As  $\mathcal{B}(A) \subset A$  this implies that  $p \in T_{\beta_0^n} \circ \dots \circ T_{\beta_\ell^n}(A)$  for all  $\ell \leq n-1$ .

We now apply Lemma 3.4 to the graph  $G$  whose vertices are the sets

$$\bigcup_{n \geq 0} \left\{ T_{\beta_0^n} \circ \dots \circ T_{\beta_{n-1}^n}(A), T_{\beta_0^n} \circ \dots \circ T_{\beta_{n-2}^n}(A), \dots, T_{\beta_0^n}(A), A \right\}.$$

Note that, in principle, a vertex can be obtained using different compositions.

The edges of the graph are defined as follows: the vertex  $T_{\beta_0^n} \circ \dots \circ T_{\beta_\ell^n}(A)$  has edges joining to  $T_{\beta_0^n} \circ \dots \circ T_{\beta_{\ell-1}^n}(A)$  and  $T_{\beta_0^n} \circ \dots \circ T_{\beta_{\ell+1}^n}(A)$  (provided  $\ell-1 \geq 0$  and  $\ell+1 \leq n$ ).

Observe that the way we define the graph  $G$  allows that a pair of adjacent vertices may have infinite links a, thus in such a case the graph has not finite degree. To bypass this difficulty, we consider the *underlying simple* graph  $G_0$  of  $G$  obtained by deleting from every pair of adjacent vertices all but one edge joining them. For the underlying simple graph  $G_0$  the “top vertex”  $A$  has at most  $k$  edges (joining to the sets  $T_1(A), \dots, T_k(A)$ ) and the other vertices has at most  $k+1$  edges. In this way, the graph  $G_0$  has finite degree at most  $k+1$ .

Lemma 3.4 now gives a simple path with infinite length. This simple path provides a sequence  $\xi = \xi_0 \dots \xi_n \dots$  such that  $p \in T_{\xi_0} \circ \dots \circ T_{\xi_n}(A)$  for every  $n$ . This implies the inclusion “ $\subset$ ”.  $\square$

**3.2. Conley and strict attractors.** In this section we introduce the notion of a minimum fixed point of an IFS and prove that if  $S_t \neq \emptyset$  then the closure of the target set is a minimum fixed point of  $\mathcal{B}$ . We also characterise strict attractors for IFSs with  $S_t \neq \emptyset$ .

**3.2.1. Minimal fixed points and minimum fixed point.** Note that the set  $X^*$  is the *maximum fixed point* (ordered by inclusion) of the map  $\mathcal{B}$ , meaning that if  $K$  is another fixed point of  $\mathcal{B}$  then  $K \subset X^*$ . A natural question is about the existence of a *minimum fixed point*  $Y$  of  $\mathcal{B}$ , meaning that if  $K$  is any fixed point of  $\mathcal{B}$  then  $Y \subset K$ . By definition, maximum and minimum fixed points are unique. We see that, in general, may no exist a minimum fixed point. Observe that an application of Zorn’s lemma immediately provides a *minimal fixed point* for  $\mathcal{B}$ , that is, a fixed point that does not contain properly another fixed point. Note that, by definition, a minimum fixed point is minimal, but the converse is not true in general.

To get a simple example of an IFS without a minimum fixed point just consider the IFS  $(T_1, T_2)$  defined on the interval  $[0, 1]$  with  $T_1(x) = x$  and  $T_2(x) = 1 - x$ . For each  $x \in [0, 1]$ , the set  $\{x, 1 - x\}$  is a fixed point of  $\mathcal{B}$ . Clearly, the set  $\{x, 1 - x\}$  is minimal. It is also obvious, that there is not a fixed point contained in all fixed points. Thus the minimal fixed points cannot be minimum fixed points.

In the previous example we have  $S_t = \emptyset$ . Next proposition shows that the condition  $S_t \neq \emptyset$  guarantees the existence of a minimum fixed point. For the next result recall the characterisation of the set  $A_t$  in (3.2).

**Proposition 3.5.** *Suppose that  $S_t \neq \emptyset$ . Then  $\overline{A_t}$  is the minimum fixed point of  $\mathcal{B}$ .*

*Proof.* We need to see that  $\mathcal{B}(\overline{A_t}) = \overline{A_t}$  and  $\overline{A_t} \subset K$  for every compact set  $K$  with  $\mathcal{B}(K) = K$ .

To prove the second assertion, fix any compact set  $K$  that is fixed point of  $\mathcal{B}$  and take any point  $p \in A_t$ . By the characterisation of  $A_t$  in (3.2) there is a sequence  $\xi$  such that

$$(3.4) \quad \{p\} = \bigcap_{n \geq 0} T_{\xi_0} \circ \dots \circ T_{\xi_n}(X) \supset \bigcap_{n \geq 0} T_{\xi_0} \circ \dots \circ T_{\xi_n}(K).$$

Since the last intersection is non-empty and contained in  $K$  it follows  $p \in K$ . This implies that  $A_t$  (and hence  $\overline{A_t}$ ) is contained in  $K$ .

To see that  $\mathcal{B}(\overline{A_t}) = \overline{A_t}$  note that the continuity of the maps  $T_i$  implies that for  $p$  as in (3.4) and every  $i = 1, \dots, k$  it holds

$$\{T_i(p)\} = \bigcap_{n \geq 0} T_i \circ T_{\xi_0} \circ \dots \circ T_{\xi_n}(X).$$

This implies that  $\mathcal{B}(A_t) \subset A_t$ . Hence, by continuity of the maps  $T_i$ ,  $\mathcal{B}(\overline{A_t}) \subset \overline{A_t}$ . By definition this implies that  $(\overline{A_t})^* \subset \overline{A_t}$ . By Proposition 3.1,  $(\overline{A_t})^*$  is a fixed point of  $\mathcal{B}$ . The minimality property proved before implies that  $\overline{A_t} \subset (\overline{A_t})^*$ . This ends the proof of the proposition.  $\square$

Note that in the proof of the proposition we obtained the following.

**Scholium 3.6.** *Given an IFS  $(T_1, \dots, T_k)$  with  $S_t \neq \emptyset$  it holds  $T_i(A_t) \subset A_t$ .*

3.2.2. *Characterisation of strict attractors.* The proposition below claims that an IFS with a weakly hyperbolic sequence has at most one strict attractor and describes such an attractor.

**Proposition 3.7.** *Consider an IFS defined on a compact metric space such that  $S_t \neq \emptyset$ . Then there exists at most one strict attractor. If such a strict attractor exists then it is equal to  $\overline{A_t}$ .*

*Proof.* If there are no strict attractor we are done. Otherwise assume that there is a strict attractor  $K$ . Since  $K$  is a fixed point of  $\mathcal{B}$ , Proposition 3.5 implies that  $\overline{A_t} \subset K$ . Since by definition of a strict attractor the set  $K$  attracts every compact set in a neighbourhood of it, the minimum fixed point  $\overline{A_t}$  is attracted by  $K$ . Therefore  $\overline{A_t} = K$ , proving the proposition.  $\square$

The following example shows that there are IFSs with  $S_t \neq \emptyset$  without strict attractors. In this example  $\overline{A_t}$  is stable (recall (1.4)).

**Example 3.8.** Consider the maps  $T_1, T_2: [0, 2] \rightarrow [0, 2]$  depicted in Figure 1 and defined by

- $T_1(x) = \frac{1}{3}x$  and
- $T_2: [0, 2] \rightarrow [0, 2]$  is the piecewise-linear map defined by  $T_2(x) = \frac{1}{3}x + \frac{2}{3}$  for  $x \in [0, 1]$  and  $T_2(x) = x$  for  $x \in [1, 2]$ ,

Let  $\mathcal{C}$  be the standard ternary Cantor set in the interval  $[0, 1]$ . We claim that  $S_t \neq \emptyset$ ,  $\overline{A}_t = \mathcal{C}$ , and  $\mathcal{C}$  is not a strict attractor. Indeed,  $\mathcal{C}$  is not a Conley attractor. We now prove these assertions.

First, as  $T_1$  is a contraction  $\bar{1} \in S_t$ , where  $\bar{1}$  is the sequence whose terms are all equal to 1.

For the second assertion, consider the auxiliary IFS  $(f_1, f_2)$  where  $f_1 = T_1|_{[0,1]}$  and  $f_2 = T_2|_{[0,1]}$ . Note that  $\mathcal{C}$  is the attractor of IFS  $(f_1, f_2)$  (see, for instance, Example 1 in [14, Section 3.3]). In particular, the set  $\mathcal{C}$  is the unique fixed point of the Barnsley-Hutchinson operator  $\mathcal{B}$  of IFS  $(T_1, T_2)$  contained in  $[0, 1]$ . Since  $[0, 1]$  is  $\mathcal{B}$ -invariant, by Propositions 3.1 and 3.5 we have  $\overline{A}_t \subset [0, 1]^* \subset [0, 1]$  and therefore  $\overline{A}_t = \mathcal{C}$ .

To see that  $\mathcal{C}$  is not a strict attractor, just note that every open neighbourhood of  $\mathcal{C}$  necessarily contains an interval of the form  $[1, \delta)$ . Since  $T_2(x) = x$  for all  $x \in [1, \delta)$  the assertion follows.

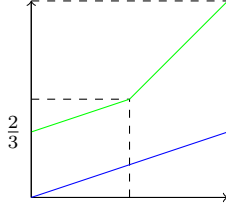


FIGURE 1. The set  $\overline{A}_t$  is not a Conley attractor

**3.3. Proof of Theorem 1.** Since every strict attractor is a Conley attractor, to prove the theorem it is enough to see that given an IFS  $(T_1, \dots, T_k)$  such that  $\overline{A}_t$  is a non-empty Conley attractor then  $\overline{A}_t$  is a strict attractor. We need the following preparatory lemma:

**Lemma 3.9.** *Consider sequences  $(A_n)$  of compact sets in  $\mathcal{H}(X)$  and  $(p_n)$  of points in  $X$  with  $A_n \rightarrow A$  and  $p_n \rightarrow p$  in the Hausdorff distance  $d_H$ . Then*

$$d(p, A) = \lim_{n \rightarrow \infty} d(p_n, A_n).$$

*Proof.* We use the following “triangular” inequality: given a point  $q$  and two compact sets  $A$  and  $B$  it holds

$$d(q, A) \leq d(q, B) + d_H(A, B).$$

Consider the sequences  $(A_n)$  and  $(p_n)$  in the lemma. Applying twice the “triangular” inequality above we get

$$d(p, A) \leq d(p, p_n) + d(p_n, A) \leq d(p, p_n) + d(p_n, A_n) + d_H(A_n, A).$$

By hypothesis,  $d(p, p_n) \rightarrow 0$  and  $d_H(A_n, A) \rightarrow 0$ . We conclude that

$$(3.5) \quad d(p, A) \leq \liminf_n d(p_n, A_n).$$

Applying again twice the “triangular” inequality, we get

$$d(p_n, A_n) \leq d(p_n, p) + d(p, A_n) \leq d(p_n, p) + d(p, A) + d_H(A, A_n).$$

This implies that

$$(3.6) \quad \limsup_n d(p_n, A_n) \leq d(p, A).$$

Equations (3.5) and (3.6) imply the lemma.  $\square$

We are now ready to prove the theorem. Since  $\overline{A_t}$  is a Conley attractor it has an open neighbourhood  $U$  such that  $\mathcal{B}^n(\overline{U}) \rightarrow \overline{A_t}$ . To prove that  $\overline{A_t}$  is a strict attractor we need to check that for every compact set  $K \in \mathcal{H}(\overline{U})$  it holds  $\mathcal{B}^n(K) \rightarrow \overline{A_t}$ . For that it is enough to see that for any  $\epsilon > 0$  there is  $n_0 \in \mathbb{N}$  such that for every  $n \geq n_0$  it holds

$$(3.7) \quad d_H(\overline{A_t}, \mathcal{B}^n(K)) = \max\{h_s(\overline{A_t}, \mathcal{B}^n(K)), h_s(\mathcal{B}^n(K), \overline{A_t})\} \leq \epsilon.$$

By hypothesis,  $\mathcal{B}^n(\overline{U}) \rightarrow \overline{A_t}$ . Thus there is  $n_0$  such that for every  $n \geq n_0$  we have

$$h_s(\mathcal{B}^n(\overline{U}), \overline{A_t}) \leq \epsilon.$$

Therefore, for every  $n \geq n_0$ ,

$$h_s(\mathcal{B}^n(K), \overline{A_t}) \leq h_s(\mathcal{B}^n(\overline{U}), \overline{A_t}) \leq \epsilon.$$

Hence to prove (3.7) it remains to see that  $h_s(\overline{A_t}, \mathcal{B}^n(K)) \leq \epsilon$  for every  $n$  sufficiently large. This is proved in the next lemma.

**Lemma 3.10.** *For every  $K \in \mathcal{H}(X)$  it holds  $\lim_{n \rightarrow \infty} h_s(\overline{A_t}, \mathcal{B}^n(K)) = 0$ .*

*Proof.* The proof is by contradiction. Assume that there are a compact set  $K \in \mathcal{H}(X)$  and a sequence  $(n_\ell)$  such that  $h_s(\overline{A_t}, \mathcal{B}^{n_\ell}(K)) > \epsilon$  for every  $\ell$ . Note that for each  $\ell$  there is  $p_{n_\ell} \in \overline{A_t}$  with  $d(p_{n_\ell}, \mathcal{B}^{n_\ell}(K)) > \epsilon$ . By compactness we can assume that  $p_{n_\ell} \rightarrow p^* \in \overline{A_t}$  and that  $\mathcal{B}^{n_\ell}(K) \rightarrow \widehat{K}$ . By Lemma 3.9,

$$(3.8) \quad d(p^*, \widehat{K}) \geq \epsilon.$$

We now derive a contradiction from this inequality. By construction, there is  $\ell_0$  such that

$$(3.9) \quad h_s(\mathcal{B}^{n_\ell}(K), \widehat{K}) < \frac{\epsilon}{2}, \quad \text{for all } \ell \geq \ell_0.$$

Take  $q \in B_{\frac{\epsilon}{2}}(p^*) \cap A_t$  and note that there is a sequence  $\omega = \omega_0 \omega_1 \dots \in S_t$  such that

$$\bigcap_{n \geq 0} T_{\omega_0} \circ \dots \circ T_{\omega_n}(X) = \{q\}.$$

Therefore there is  $m_0$  such that

$$T_{\omega_0} \circ \dots \circ T_{\omega_{m-1}}(K) \subset T_{\omega_0} \circ \dots \circ T_{\omega_{m-1}}(X) \subset B_{\frac{\epsilon}{2}}(p^*) \quad \text{for every } m \geq m_0.$$

Since  $T_{\omega_0} \circ \dots \circ T_{\omega_{m-1}}(K) \subset \mathcal{B}^m(K)$ , for every  $\ell$  big enough we have  $\mathcal{B}^{n_\ell}(K) \cap B_{\frac{\epsilon}{2}}(p^*) \neq \emptyset$ .

Note that for every  $\ell$  sufficiently large  $\mathcal{B}^{n_\ell}(K) \cap B_{\frac{\epsilon}{2}}(p^*) \neq \emptyset$  and equation (3.9) holds. Hence for every  $z \in \mathcal{B}^{n_\ell}(K) \cap B_{\frac{\epsilon}{2}}(p^*)$  we have  $d(z, \widehat{K}) < \frac{\epsilon}{2}$  and  $d(z, p^*) < \frac{\epsilon}{2}$ . Hence  $d(p^*, \widehat{K}) < \epsilon$  contradicting (3.8). This ends the proof of the lemma.  $\square$

The proof of the theorem is now complete.  $\square$

**Scholium 3.11.** *If  $U$  is a neighbourhood of  $\overline{A_t}$  such that  $\mathcal{B}^n(\overline{U}) \rightarrow \overline{A_t}$  then every compact subset of  $\overline{U}$  also satisfies  $\mathcal{B}^n(K) \rightarrow \overline{A_t}$ .*

We have the following corollary that allows us to establish a connection between the set  $\overline{A_t}$  and semifractals.

**Corollary 3.12.** *Consider an IFS such that  $S_t \neq \emptyset$ . Then*

$$\lim_{n \rightarrow \infty} \mathcal{B}^n(K) = \overline{A_t}, \quad \text{for every compact set } K \subset \overline{A_t}.$$

*Proof.* The statement is an immediate consequence of Lemma 3.10 and the invariance of  $\overline{A_t}$ .  $\square$

**Remark 3.13.** Combining Propositions 3.5 and Corollary 3.12 one gets the following: if  $S_t \neq \emptyset$  then set  $\overline{A_t}$  is a minimum fixed point that attracts every compact set inside it.

**3.4. Proof of Theorem 2.** Suppose that the set  $A_t$  is non-empty. We need to prove the equivalence of the following three assertions:

- (1)  $\overline{A_t} = X^*$ ;
- (2) the Barnsley-Hutchinson operator  $\mathcal{B}$  has a unique fixed point;
- (3)  $X^*$  is a global attractor (a strict attractor whose basin is the whole space).

The equivalence  $1 \Leftrightarrow 2$  follows immediately from Proposition 3.5 (“the minimum fixed point  $\overline{A_t}$  is equal to the maximum fixed point  $X^*$ ”).

The implication  $3 \Rightarrow 2$  follows noting that if  $K$  is a fixed point of  $\mathcal{B}$  and since  $X^*$  is a global attractor then  $K = \mathcal{B}^n(K) \rightarrow X^*$  and thus  $K = X^*$ .

To prove  $1 \Rightarrow 3$  note that, by Lemma 3.2,  $X^* = \lim_{n \rightarrow \infty} \mathcal{B}^n(X)$  and thus  $X^*$  is a Conley attractor. Then if  $\overline{A_t} = X^*$  we have that  $\overline{A_t}$  is a Conley attractor, by Theorem 1 and Scholium 3.11 this set is a strict attractor whose basin is the whole space.  $\square$

**3.5. Structure of set  $A_t$ .** The main result of this section is Proposition 3.14 about the topological structure of the target set  $A_t$ . This result will be used in Section 4.3.

We begin by observing that, in general, the set  $A_t$  is not necessarily closed. The IFS in Example 3.8 illustrates this case. In this example  $\overline{A_t}$  is the ternary Cantor set  $\mathcal{C}$  in  $[0, 1]$ , thus  $1 \in \overline{A_t}$ . We claim that  $1 \notin A_t$  and thus  $A_t$  is not closed. Recall the definitions of  $\text{IFS}(T_1, T_2)$  and  $\text{IFS}(f_1, f_2)$  in this example and consider their natural associated projections  $\pi_T$  and  $\pi_f$ , see (2.1). Arguing by contradiction, if  $1 \in A_t$  then there is a sequence  $\xi \in S_t$  with  $\pi_T(\xi) = 1$ . In this case we also have  $\pi_f(\xi) = 1$ . It is easy to check that  $\xi = \bar{2}$  and that  $\bar{2} \notin S_t$ , where  $\bar{2} = (\xi_i = 2)$ . This gives a contradiction.

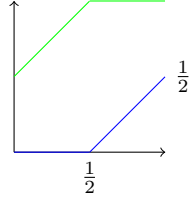
**Proposition 3.14.** *Consider  $\text{IFS}(T_1, \dots, T_k)$  defined on a compact set  $X$  such that  $A_t \neq \emptyset$ .*

- (1) *Assume that the IFS is injective in  $A_t$ . Then either  $A_t$  is a singleton or  $A_t$  has no isolated points (thus it is infinite).*
- (2) *Assume that the maps  $T_i$  are open. Then either  $A_t$  has empty interior or  $\text{int}(A_t) \subset A_t \subset \text{int}(A_t)$ .*

We observe that in the proof of the first item of the proposition we only use the injectivity of the maps  $T_i$  on  $A_t$ .

Let us also observe that if the maps  $T_i$  are not injective then the set  $A_t$  can be finite with more than one element. The maps depicted in Figure 2 give an example of this case, where  $A_t = \{0, \frac{1}{2}, 1\}$ .



FIGURE 2.  $\#(A_t) = 3$ 

**Remark 3.15.** Every injective IFS  $(T_1, \dots, T_k)$  defined on  $[0, 1]$  satisfies the hypotheses in the second part of Proposition 3.14.

*Proof of Proposition 3.14.* We prove the first item in the proposition. If  $A_t$  is a singleton we are done. Otherwise  $\#(A_t) \geq 2$ . To see that every  $p \in A_t$  is not isolated we check that for every neighbourhood  $V$  of  $p$  the set  $A_t \cap V$  contains at least two points. By definition of  $A_t$ , there is a finite sequence  $\xi_0 \dots \xi_n$  such that

$$T_{\xi_0} \circ \dots \circ T_{\xi_n}(X) \subset V.$$

In particular,

$$T_{\xi_0} \circ \dots \circ T_{\xi_n}(A_t) \subset V.$$

Since  $T_{\xi_0} \circ \dots \circ T_{\xi_n}(A_t) \subset A_t$  (recall Scholium 3.6) and  $T_{\xi_0} \circ \dots \circ T_{\xi_n}$  is one-to-one in  $A_t$ , we have that  $V$  contains at least two points of  $A_t$ , proving the first part of the proposition.

We now prove the second item of the proposition. If  $A_t$  has empty interior we are done. Thus we can assume that  $\text{int}(A_t) \neq \emptyset$ . Since  $\text{int}(A_t) \subset A_t$  it only remains to see that  $A_t \subset \overline{\text{int}(A_t)}$ . Take a point  $x \in A_t$  and any open neighbourhood  $V$  of  $x$ . By definition of  $A_t$  there is a finite sequence  $\xi_0 \dots \xi_n$  such that

$$T_{\xi_0} \circ \dots \circ T_{\xi_n}(X) \subset V.$$

By  $T_i(A_t) \subset A_t$  it follows

$$T_{\xi_0} \circ \dots \circ T_{\xi_n}(\text{int}(A_t)) \subset V \cap A_t.$$

Since  $\text{int}(A_t)$  is an open set and the maps  $T_i$  are open, then  $T_{\xi_0} \circ \dots \circ T_{\xi_n}(\text{int}(A_t))$  is a non-empty and open subset of  $V \cap A_t$ , thus  $V \cap \text{int}(A_t) \neq \emptyset$ . Since this holds for every neighbourhood of  $x$  we get that  $x \in \overline{\text{int}(A_t)}$ . The proof of the proposition is now complete.  $\square$

**3.6. Proof of Theorem 3.** Suppose that  $\overline{A_t}$  is stable. We need to prove that given any disjunctive sequence  $\xi$  and any point  $x$  it holds

$$\overline{A_t} = \bigcap_{\ell \geq 0} \overline{\{x_{n,\xi} : n \geq \ell\}}, \quad \text{where } x_{n,\xi} \stackrel{\text{def}}{=} T_{\xi_n} \circ \dots \circ T_{\xi_0}(x).$$

To simplify notation write

$$Y_\ell \stackrel{\text{def}}{=} \{x_{n,\xi} : n \geq \ell\}.$$

For the inclusion “ $\subset$ ” take any point  $p \in A_t$  and fix  $\ell \geq 0$ . We need to see that for every neighbourhood  $V$  of  $p$  it holds

$$(3.10) \quad V \cap Y_\ell \neq \emptyset.$$

By definition of  $A_t$  there is a finite sequence  $c_0 \dots c_r$  such that

$$(3.11) \quad T_{c_r} \circ \dots \circ T_{c_0}(X) \subset V.$$

We can assume that  $r \geq \ell$ . Since  $\xi$  has dense orbit there is  $m_1$  such that

$$\xi_{m_1} = c_0, \xi_{m_1+1} = c_1, \dots, \xi_{m_1+r} = c_r.$$

Therefore, from (3.11) it follows

$$x_{m_1+r,\xi} = T_{\xi_{m_1+r}} \circ \dots \circ T_{\xi_{m_1}} \circ T_{\xi_{m_1-1}} \circ \dots \circ T_{\xi_0}(x) \in V.$$

Since  $m_1 + r \geq \ell$  we have that  $V \cap Y_\ell \neq \emptyset$ , proving (3.10).

We now prove the inclusion “ $\subset$ ”. Take any neighbourhood  $V$  of  $\overline{A_t}$ . Since  $\overline{A_t}$  is stable it has a neighbourhood  $V_0 \subset V$  such that  $\mathcal{B}^n(V_0) \subset V$  for every  $n \geq 0$ . Since  $\xi$  is a disjunctive sequence and  $A_t \subset V_0$  there is  $n_0 \in \mathbb{N}$  such that  $x_{n_0,\xi} \in V_0$ . Hence  $Y_{n_0} \subset V$  and thus  $\overline{Y_{n_0}} \subset \overline{V}$ . As the sequence of sets  $(\overline{Y_\ell})$  is nested, we have that  $\bigcap_{\ell \geq 0} \overline{Y_\ell} \subset \overline{V}$ . Since this holds for every neighbourhood  $V$  of  $\overline{A_t}$  we conclude that

$$\bigcap_{\ell \geq 0} \overline{Y_\ell} \subset \overline{A_t}.$$

Finally, as  $\overline{Y_\ell}$  is a nested sequence of compact sets, from Lemma 3.2 and the definition of a Hausdorff limit, it follows  $Y_\ell \rightarrow \overline{A_t}$ , where the convergence is in the Hausdorff distance.  $\square$

#### 4. MEASURE AND RIGIDITY OF $S_t$ FOR IFSs ON $[0, 1]$

In this section, for IFSs defined on  $[0, 1]$ , we study the measure of  $S_t$  for Markov measures and prove Theorem 5, see Section 4.1. In Section 4.2 we prove a result about probabilistic rigidity of the set  $S_t$ : under quite general conditions, if  $S_t$  intersects the support of a Markov measure it has full probability. Finally, in Section 4.3 we characterise separable IFSs.

**4.1. Proof of Theorem 5.** Given an IFS  $(T_1, \dots, T_k)$  defined on  $I = [0, 1]$  we need to see that every mixing Markov measure that splits the IFS in some non-trivial closed interval  $J$  satisfies  $\mathbb{P}^+(S_t) = 1$ .

Recall the definition of the fibre  $I_\xi$  of a sequence in  $\Sigma_k^+$  in (3.1). Given  $x \in [0, 1]$  we consider the set of sequences whose fibres contain  $x$  defined by

$$(4.1) \quad \Sigma_x \stackrel{\text{def}}{=} \{\xi \in \Sigma_k^+ : x \in I_\xi\}.$$

**Lemma 4.1.** *Suppose that  $\mathbb{P}^+(\Sigma_x) = 0$  for all  $x \in [0, 1]$ . Then  $\mathbb{P}^+(S_t) = 1$ .*

*Proof.* Note that if  $\xi \notin S_t$  then its fibre  $I_\xi$  is a non-trivial interval and hence contains a rational point. This implies that

$$(S_t)^c = \Sigma_k^+ \setminus S_t \subset \bigcup_{x \in \mathbb{Q} \cap [0,1]} \Sigma_x.$$

This union is countable and each set  $\Sigma_x$  satisfies  $\mathbb{P}^+(\Sigma_x) = 0$ , thus  $\mathbb{P}^+(S_t) = 1$ .  $\square$

In view of Lemma 4.1, to see that  $\mathbb{P}^+(S_t) = 1$  it is sufficient to show the following:

**Theorem 4.2.** *Consider an IFS defined on  $I = [0, 1]$  and a mixing Markov measure  $\mathbb{P}^+$  that splits the IFS in some non-trivial interval  $J$ . Then  $\mathbb{P}^+(\Sigma_x) = 0$  for all  $x \in [0, 1]$ .*

*Proof.* By the splitting hypothesis there is a pair of admissible cylinders  $[i_1 \dots i_\ell]$  and  $[j_1 \dots j_s]$  with  $i_1 = j_1$  such that

$$(4.2) \quad \begin{aligned} T_{j_1} \circ \dots \circ T_{j_s}(I) \cap T_{i_1} \circ \dots \circ T_{i_\ell}(I) &= \emptyset, \quad \text{and} \\ T_{j_1} \circ \dots \circ T_{j_s}(I) \cup T_{i_1} \circ \dots \circ T_{i_\ell}(I) &\subset J. \end{aligned}$$

Next claim restates the splitting condition:

**Claim 4.3.** *There are admissible cylinders  $[\xi_0 \dots \xi_{N-1}]$  and  $[\omega_0 \dots \omega_{N-1}]$  such that  $\xi_0 = \omega_0$ ,  $\xi_{N-1} = \omega_{N-1}$ ,*

$$\begin{aligned} T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cap T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I) &= \emptyset \quad \text{and} \\ T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cup T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I) &\subset J. \end{aligned}$$

*Proof.* Consider  $j_1, \dots, j_s$  and  $i_1, \dots, i_\ell$  as in (4.2). Since  $\mathbb{P}^+$  is mixing there is  $n_0$  such that for every  $n \geq n_0$  there are admissible cylinders of the form  $[i_\ell c_1 \dots c_{n-1} 0]$  and  $[j_s d_1 \dots d_{n-1} 0]$ . Take now  $n_1, n_2 \geq n_0$  and admissible cylinders  $[i_\ell c_1 \dots c_{n_1} 0]$  and  $[j_s d_1 \dots d_{n_2} 0]$  such that  $n_1 + \ell = n_2 + s$ . Let  $N = \ell + n_1 + 1$ . Then the cylinders

$$[\xi_0 \dots \xi_{N-1}] = [i_1 \dots i_\ell c_1 \dots c_{n_1} 0] \quad \text{and} \quad [\omega_0 \dots \omega_{N-1}] = [j_1 \dots j_s d_1 \dots d_{n_2} 0]$$

are admissible and satisfy the intersection and union properties in the claim. To see why this is so note that  $T_{c_1} \circ \dots \circ T_{c_{n_1}} \circ T_0(I) \subset I$  and  $T_{d_1} \circ \dots \circ T_{d_{n_2}} \circ T_0(I) \subset I$ .  $\square$

We now fix  $x \in I$  and prove that  $\mathbb{P}^+(\Sigma_x) = 0$ . For that fix  $N$ , the admissible cylinders  $[\xi_0 \dots \xi_{N-1}]$  and  $[\omega_0 \dots \omega_{N-1}]$  in the claim, and for  $j \geq 1$  define the sets

$$(4.3) \quad \Sigma_x^j \stackrel{\text{def}}{=} \{[a_0 \dots a_{jN-1}] \subset \Sigma_k^+ : x \in T_{a_0} \circ \dots \circ T_{a_{jN-1}}(I)\} \quad \text{and} \quad S_x^j \stackrel{\text{def}}{=} \bigcup_{C \in \Sigma_x^j} C.$$

Note that by definition  $S_x^{j+1} \subset S_x^j$  and that for each  $j \geq 1$  it holds  $\Sigma_x \subset S_x^j$ . Hence

$$\Sigma_x \subset \bigcap_{j \geq 1} S_x^j.$$

Therefore

$$\mathbb{P}^+(\Sigma_x) \leq \mathbb{P}^+\left(\bigcap_{j \geq 1} S_x^j\right) = \lim_{j \rightarrow \infty} \mathbb{P}^+(S_x^j).$$

Hence the assertion  $\mathbb{P}^+(\Sigma_x) = 0$  in the theorem follows from the next proposition:

**Proposition 4.4.**  $\lim_{j \rightarrow \infty} \mathbb{P}^+(S_x^j) = 0$ .

*Proof.* Suppose, for instance, that the cylinders in the claim satisfy

$$(4.4) \quad 0 < \mathbb{P}^+([\xi_0 \dots \xi_{N-1}]) \leq \mathbb{P}^+([\omega_0 \dots \omega_{N-1}]).$$

The first inequality follows from the admissibility of  $[\xi_0 \dots \xi_{N-1}]$ .

Define for  $j \geq 1$  the family of cylinders

$$E^j \stackrel{\text{def}}{=} \{[a_0 \dots a_{jN-1}] \subset \Sigma_k^+ : \sigma^{iN}([a_0 \dots a_{jN-1}]) \cap [\xi_0 \dots \xi_{N-1}] = \emptyset, \quad i = 0, \dots, j-1\}$$

and their union

$$Q^j \stackrel{\text{def}}{=} \bigcup_{C \in E^j} C.$$

Note that by definition  $Q^{j+1} \subset Q^j$ . Let

$$Q^\infty \stackrel{\text{def}}{=} \bigcap_{j \geq 1} Q^j = \{\omega \in \Sigma^+ : \sigma^{iN}(\omega) \cap [\xi_0 \dots \xi_{N-1}] = \emptyset \text{ for all } i \geq 0\}.$$

Recall that the mixing property of  $(\sigma, \mathbb{P}^+)$  implies the ergodicity of  $(\sigma^N, \mathbb{P}^+)$ . Thus the Birkhoff's ergodic theorem implies that  $\mathbb{P}^+(Q^\infty) = 0$ . Therefore condition  $Q^{j+1} \subset Q^j$  implies that

$$\lim_{j \rightarrow \infty} \mathbb{P}^+(Q^j) = 0.$$

In view of this property, the proposition follows from the next lemma.

**Lemma 4.5.**  $\mathbb{P}^+(S_x^j) \leq \mathbb{P}^+(Q^j)$  for all  $j \geq 1$ .

*Proof.* For each  $j \geq 1$  consider the auxiliary substitution function  $F_j : \Sigma_x^j \rightarrow E^j$  defined as follows. For each cylinder  $[\alpha_0 \dots \alpha_{jN-1}] \in \Sigma_x^j$  we consider its sub-cylinders  $[\alpha_0 \dots \alpha_{N-1}]$ ,  $[\alpha_N \dots \alpha_{2N-1}]$ ,  $\dots$ ,  $[\alpha_{(j-1)N} \dots \alpha_{jN-1}]$  and use the following concatenation notation

$$[\alpha_0 \dots \alpha_{jN-1}] = [\alpha_0 \dots \alpha_{N-1}] * [\alpha_N \dots \alpha_{2N-1}] * \dots * [\alpha_{(j-1)N} \dots \alpha_{jN-1}].$$

In a compact way, we write

$$C = C_0 * C_1 * \dots * C_{j-1}$$

where the cylinder  $C$  has size  $jN$  and each cylinder  $C_i$  has size  $N$ . With this notation we define  $F_j$  by

$$F_j(C) \stackrel{\text{def}}{=} F_j(C_0 * C_1 * \dots * C_{j-1}) = C'_0 * C'_1 * \dots * C'_{j-1},$$

where  $C'_i = C_i$  if  $C_i \neq [\xi_0 \dots \xi_{N-1}]$  and  $C'_i = [\omega_0 \dots \omega_{N-1}]$  otherwise.

**Claim 4.6.** For every  $j \geq 1$  it holds  $\mathbb{P}^+(C) \leq \mathbb{P}^+(F_j(C))$  for every  $C \in \Sigma_x^j$ .

*Proof.* Recalling that  $\omega_0 = \xi_0$  and  $\omega_{N-1} = \xi_{N-1}$ , from equation (4.4) we immediately get the following: For every  $m, s \geq 0$  and every pair of cylinders  $[a_0 \dots a_s]$  and  $[b_0 \dots b_m]$  it holds

- (1)  $\mathbb{P}^+([a_0 \dots a_s \xi_0 \dots \xi_{N-1} b_0 \dots b_m]) \leq \mathbb{P}^+([a_0 \dots a_s \omega_0 \dots \omega_{N-1} b_0 \dots b_m])$ ,
- (2)  $\mathbb{P}^+([\xi_0 \dots \xi_{N-1} b_0 \dots b_m]) \leq \mathbb{P}^+([\omega_0 \dots \omega_{N-1} b_0 \dots b_m])$ , and
- (3)  $\mathbb{P}^+([a_0 \dots a_s \xi_0 \dots \xi_{N-1}]) \leq \mathbb{P}^+([a_0 \dots a_s \omega_0 \dots \omega_{N-1}])$ .

The inequality  $\mathbb{P}^+(C) \leq \mathbb{P}^+(F_j(C))$  now follows from the definition of  $F_j$ .  $\square$

**Claim 4.7.** The map  $F_j$  is injective for every  $j \geq 1$ .

*Proof.* Fix  $j \geq 1$ . Given cylinders  $C, \tilde{C} \in \Sigma_x^j$ , using the notation above write  $C = C_0 * C_1 * \dots * C_{j-1}$  and  $\tilde{C} = \tilde{C}_0 * \tilde{C}_1 * \dots * \tilde{C}_{j-1}$ . Then

$$F_j(C) = C'_0 * C'_1 * \dots * C'_{j-1} \quad \text{and} \quad F_j(\tilde{C}) = \tilde{C}'_0 * \tilde{C}'_1 * \dots * \tilde{C}'_{j-1}.$$

Suppose that  $F_j(C) = F_j(\tilde{C})$ . Then  $C'_i = \tilde{C}'_i$  for all  $i = 0, \dots, j-1$ . If  $C \neq \tilde{C}$  there is a first  $i$  such that  $C_i \neq \tilde{C}_i$ . Then either  $C_i = [\xi_0 \dots \xi_{N-1}]$  and  $\tilde{C}_i = [\omega_0 \dots \omega_{N-1}]$  or vice-versa. Let us assume that the first case occurs.

If  $i = 0$  then the definition of  $\Sigma_x^j$  implies that

$$x \in T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cap T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I),$$

contradicting Claim 4.3. Thus we can assume that  $i > 0$  and define the cylinder

$$[\eta_0 \dots \eta_{(i-1)N-1}] \stackrel{\text{def}}{=} C_0 * C_1 * \dots * C_{i-1} = \tilde{C}_0 * \tilde{C}_1 * \dots * \tilde{C}_{i-1}.$$

Write  $(i-1)N - 1 = r$ . By the definition of  $\Sigma_x^j$  in (4.3) we have

$$(4.5) \quad x \in T_{\eta_0} \circ \dots \circ T_{\eta_r} \circ T_{\xi_0} \circ \dots \circ T_{\xi_{N-1}}(I) \cap T_{\eta_0} \circ \dots \circ T_{\eta_r} \circ T_{\omega_0} \circ \dots \circ T_{\omega_{N-1}}(I).$$

Since for every  $i$  we have that  $T_i(J) \subset J$  and  $T_i|_J$  is injective, the intersection and union inclusion properties in Claim 4.3 implies that

$$T_{\eta_0} \circ \cdots \circ T_{\eta_r} \circ T_{\xi_0} \circ \cdots \circ T_{\xi_{N-1}}(I) \cap T_{\eta_0} \circ \cdots \circ T_{\eta_r} \circ T_{\omega_0} \circ \cdots \circ T_{\omega_{N-1}}(I) = \emptyset,$$

contradicting (4.5). Thus  $C = \tilde{C}$  and proof of the claim is complete.  $\square$

To prove that  $\mathbb{P}^+(S_x^j) \leq \mathbb{P}^+(Q^j)$  note that

$$\mathbb{P}^+(S_x^j) \stackrel{(a)}{=} \sum_{C \in \Sigma_x^j} \mathbb{P}^+(C) \stackrel{(b)}{\leq} \sum_{C \in \Sigma_x^j} \mathbb{P}^+(F_j(C)) \stackrel{(c)}{=} \mathbb{P}^+\left(\bigcup_{C \in \Sigma_x^j} F_j(C)\right) \stackrel{(d)}{\leq} \mathbb{P}^+(Q^j),$$

where (a) follows from the disjointness of the cylinders  $C \in \Sigma_x^j$ , (b) from Claim 4.6, (c) from the injectivity of  $F_j$  (Claim 4.7), and (d) from  $F_j(C) \in E_j \subset Q^j$ . The proof of the lemma is now complete.  $\square$

This completes the proof of the proposition.  $\square$

The proof of Theorem 4.2 (i.e.,  $\mathbb{P}^+(\Sigma_x) = 0$ ) is now complete.  $\square$

The proof of Theorem 5 is now complete.  $\square$

**4.2. Probabilistic rigidity of  $S_t$ .** In this section we see that under quite general conditions the hypothesis  $S_t \cap \text{supp}(\mathbb{P}^+) \neq \emptyset$  implies that  $\mathbb{P}^+(S_t) = 1$ . Recall the definition of the projection  $\pi$  in (2.1).

**Theorem 4.8.** *Consider an injective IFS  $(T_1, \dots, T_k)$  defined on  $I = [0, 1]$ . Let  $\mathbb{P}^+$  be a mixing Markov measure defined on  $\Sigma_k^+$  with transition matrix  $P = (p_{ij})$ .*

- *If there is  $i \in \{1, \dots, k\}$  such that  $\pi$  is not constant in  $[i] \cap \text{supp}(\mathbb{P}^+)$  then  $\mathbb{P}^+(S_t) = 1$ . In particular,*

$$\#\pi(S_t \cap \text{supp}(\mathbb{P}^+)) \geq k + 1 \implies \mathbb{P}^+(S_t) = 1.$$

- *If the maps  $T_i$  have no common fixed points and for every  $i$  and  $j$ , with  $i \neq j$ , there is  $m \in \{1, \dots, k\}$  with  $p_{mi} p_{mj} > 0$ . Then*

$$S_t \cap \text{supp}(\mathbb{P}^+) \neq \emptyset \iff \mathbb{P}^+(S_t) = 1.$$

*Proof.* To prove the first item of the theorem note that by hypothesis there is  $i$  such that  $\pi$  is not constant in  $[i] \cap \text{supp}(\mathbb{P}^+)$ . Hence  $\xi, \omega \in [i] \cap \text{supp}(\mathbb{P}^+) \cap S_t$  such that  $\pi(\xi) \neq \pi(\omega)$ . Thus there are  $s$  and  $\ell$  such that

$$T_{\xi_0} \circ \cdots \circ T_{\xi_s}(I) \cap T_{\omega_0} \circ \cdots \circ T_{\omega_\ell}(I) = \emptyset.$$

As  $\xi, \omega \in [i] \cap \text{supp}(\mathbb{P}^+)$  the cylinders  $[\xi_0 \dots \xi_s]$  and  $[\omega_0 \dots \omega_\ell]$  are both admissible and satisfy  $\xi_0 = \omega_0 = i$ . This means that  $\mathbb{P}^+$  splits the IFS. Hence, by Theorem 5,  $\mathbb{P}^+(S_t) = 1$ .

For the second part of the first item, just note that if  $\#\pi(S_t \cap \text{supp}(\mathbb{P}^+)) \geq k + 1$  then from the pigeonhole principle there is  $i$  such that  $\pi$  is not constant in  $[i] \cap \text{supp}(\mathbb{P}^+)$ .

The implication ( $\Leftarrow$ ) in the second item of the theorem is immediate. For the implication ( $\Rightarrow$ ) we need the following lemma.

**Lemma 4.9.** *For every  $\xi \in S_t \cap \text{supp}(\mathbb{P}^+)$  there is  $\omega \in S_t \cap \text{supp}(\mathbb{P}^+)$  such that  $\pi(\xi) \neq \pi(\omega)$ .*

*Proof.* Fix  $\xi \in S_t$ . By definition of  $S_t$  we have that

$$\{\pi(\xi)\} = \bigcap_{n \geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(I).$$

As the maps  $T_i$  have no common fixed points there is  $i_0$  such that  $T_{i_0}(\pi(\xi)) \neq \pi(\xi)$ . The definition of an irreducible matrix implies that there is an admissible cylinder of the form  $[i_0 i_1 \dots i_m \xi_0]$ . Let

$$r \stackrel{\text{def}}{=} \max\{\ell \in \{0, \dots, m\} : T_{i_\ell}(\pi(\xi)) \neq \pi(\xi)\} \geq 0.$$

Consider the concatenation  $\omega = i_r \dots i_m * \xi$ . Note that, by definition,  $\pi(\zeta) = T_{\zeta_0}(\pi(\sigma(\zeta)))$  for every  $\zeta \in S_t$ . Hence

$$\pi(\omega) = T_{i_r} \circ \cdots \circ T_{i_m}(\pi(\xi)).$$

By definition of  $r$ ,

$$T_{i_m}(\pi(\xi)) = \cdots = T_{i_{r+1}}(\pi(\xi)) = \pi(\xi).$$

Therefore

$$\pi(\omega) = T_{i_r}(\pi(\xi)) \neq \pi(\xi).$$

It remains to see that  $\omega \in S_t \cap \text{supp}(\mathbb{P}^+)$ , for that just note that the cylinder  $[i_0 \dots i_m \xi_0]$  is admissible and  $\xi \in \text{supp}(\mathbb{P}^+)$ . This ends the proof of the lemma.  $\square$

Take sequences  $\xi$  and  $\omega$  as in Lemma 4.9. By definition of  $\pi$ ,

$$\{\pi(\xi)\} = \bigcap_{n \geq 0} T_{\xi_0} \circ \cdots \circ T_{\xi_n}(I) \quad \text{and} \quad \{\pi(\omega)\} = \bigcap_{n \geq 0} T_{\omega_0} \circ \cdots \circ T_{\omega_n}(I).$$

As  $\pi(\xi) \neq \pi(\omega)$  there are  $\ell$  and  $s$  such that

$$(4.6) \quad T_{\xi_0} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_{\omega_0} \circ \cdots \circ T_{\omega_s}(I) = \emptyset.$$

Note that the cylinders  $[\xi_0 \dots \xi_\ell]$  and  $[\omega_0 \dots \omega_s]$  are admissible. If  $\xi_0 = \omega_0$  we are done. Otherwise,  $\xi_0 \neq \omega_0$  and by hypothesis there is  $m$  such that  $p_{m\xi_0} > 0$  and  $p_{m\omega_0} > 0$ . This implies that the cylinders  $[m\xi_0 \dots \xi_\ell]$  and  $[m\omega_0 \dots \omega_s]$  are both admissible. Since the maps  $T_i$  are injective it follows from (4.6)

$$T_m \circ T_{\xi_0} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_m \circ T_{\omega_0} \circ \cdots \circ T_{\omega_s}(I) = \emptyset.$$

Therefore  $\mathbb{P}^+$  splits the IFS and by Theorem 5 we have  $\mathbb{P}^+(S_t) = 1$ . This ends the proof of the theorem.  $\square$

**4.3. Separability.** In this section we give some characterisations of a separable IFS. Note that item (2) in the next theorem means that the IFS is separable.

**Theorem 4.10.** *Consider an IFS  $(T_1, \dots, T_k)$  defined on  $I = [0, 1]$ . Suppose that there is some non-trivial closed interval  $J$  such that  $T_i(J) \subset J$  and  $T_i|_J$  is injective for every  $j \in \{1, \dots, k\}$ . Then the following assertions are equivalent:*

- (1) *The maps of the IFS have no common fixed points and  $S_t \neq \emptyset$ .*
- (2) *The target set  $A_t$  has at least two elements.*
- (3) *There are finite sequences  $\xi_1 \dots \xi_\ell$  and  $\omega_1 \dots \omega_s$  such that*

$$\begin{aligned} T_{\xi_1} \circ \cdots \circ T_{\xi_\ell}(I) \cap T_{\omega_1} \circ \cdots \circ T_{\omega_s}(I) &= \emptyset \quad \text{and} \\ T_{\xi_1} \circ \cdots \circ T_{\xi_\ell}(I) \cup T_{\omega_1} \circ \cdots \circ T_{\omega_s}(I) &\subset J. \end{aligned}$$

- (4) *The maps of the IFS have no common fixed point and  $\mathbb{P}^+(S_t) = 1$  for every mixing Markov measure  $\mathbb{P}^+$  whose support is the whole  $\Sigma_k^+$ .*

*Proof.* To prove the implication (1)  $\Rightarrow$  (2) note that since  $S_t \neq \emptyset$  there is  $p \in A_t$ . Since the maps of the IFS have no common fixed point there is  $i$  such that  $T_i(p) \neq p$ . The invariance of  $A_t$  implies that  $T_i(p) \in A_t$ . Thus  $\{p, T_i(p)\} \subset A_t$  and we are done.

To see that (2)  $\Rightarrow$  (3) we need the following claim:

**Claim 4.11.**  $\#(A_t \cap \text{int}(J)) \geq 2$ .

*Proof.* Since  $T_i(J) \subset J$  for every  $i$  we have that  $\mathcal{B}(J) \subset J$ . Hence Propositions 3.1 and 3.5 implies that  $A_t \subset J$ . The claim follows from Proposition 3.14.  $\square$

Take two different points  $p, q \in A_t \cap \text{int} J$  and consider disjoint neighbourhoods  $U$  and  $V$  of  $p$  and  $q$ , respectively, such that  $U \cup V \subset J$ . By the definition of  $A_t$  there are sequences  $\xi$  and  $\omega$  such that

$$\{p\} = \bigcap_{n \geq 0} T_{\xi_0} \circ \dots \circ T_{\xi_n}(I) \quad \text{and} \quad \{q\} = \bigcap_{n \geq 0} T_{\omega_0} \circ \dots \circ T_{\omega_n}(I).$$

Hence there are  $n_0$  and  $m_0$  such that  $T_{\xi_0} \circ \dots \circ T_{\xi_{n_0}}(I) \subset U$  and  $T_{\omega_0} \circ \dots \circ T_{\omega_{m_0}}(I) \subset V$ . Since  $U \cap V = \emptyset$  we get the implication (2)  $\Rightarrow$  (3).

To prove (3)  $\Rightarrow$  (4) consider the finite sequences  $\xi_1 \dots \xi_\ell \in \omega_1 \dots \omega_s$  in item (3). Clearly the condition in (3) prevents the existence of a common fixed point. On the other hand, since  $T_1(J) \subset J$  and  $T_1|_J$  is injective, we have that

$$\begin{aligned} T_1 \circ T_{\xi_1} \circ \dots \circ T_{\xi_\ell}(I) \cap T_1 \circ T_{\omega_1} \circ \dots \circ T_{\omega_s}(I) &= \emptyset \quad \text{and} \\ T_1 \circ T_{\xi_1} \circ \dots \circ T_{\xi_\ell}(I) \cup T_1 \circ T_{\omega_1} \circ \dots \circ T_{\omega_s}(I) &\subset J. \end{aligned}$$

Thus every mixing Markov measure with full support  $\mathbb{P}^+$  splits the IFS in  $J$ . Now Theorem 5 implies that  $\mathbb{P}^+(S_t) = 1$  and we are done.

The implication (4)  $\Rightarrow$  (1) is immediate.  $\square$

## 5. ASYMPTOTIC STABILITY ON MEASURES

In this section we prove Theorems 4 and 6 in Sections 5.1 and 5.2, respectively.

**5.1. Stationary measures for IFSs with probabilities in  $[0, 1]$ .** In this section we prove Theorem 4. For that we consider a separable IFS  $(T_1, \dots, T_k; \mathbf{b})$  defined on  $I = [0, 1]$ , its Markov operator  $\mathfrak{T} = \mathfrak{T}_{\mathbf{b}}$ , and its coding map  $\pi$  in (2.1), we see that for every probability measure  $\mu \in \mathcal{M}_1(I)$  it holds

$$\lim_{n \rightarrow \infty} \mathfrak{T}^n \mu = \pi_* \mathbf{b} \quad (\text{asymptotic stability}).$$

The main step of the proof of the theorem is the next proposition that states a sufficient condition for the asymptotic stability of an IFS with probabilities.

**Proposition 5.1.** *Consider an IFS  $(T_1, \dots, T_k; \mathbf{b})$  with probabilities defined on a compact metric space  $X$ . Suppose that  $\mathbf{b}(S_t) = 1$ . Then for every probability measure  $\mu \in \mathcal{M}_1(X)$  it holds*

$$\lim_{n \rightarrow \infty} \mathfrak{T}_{\mathbf{b}}^n \mu = \pi_* \mathbf{b}.$$

In particular,  $\mu_{\mathbf{b}} \stackrel{\text{def}}{=} \pi_* \mathbf{b}$  is the unique stationary measure of IFS  $(T_1, \dots, T_k; \mathbf{b})$ . Furthermore,  $\text{supp}(\mu_{\mathbf{b}}) = \overline{A_t}$ .

We postpone the proof of Proposition 5.1 and deduce the theorem from it.

5.1.1. *Proof of Theorem 4.* In view of Proposition 5.1 it is sufficient to prove that  $\mathfrak{b}(S_t) = 1$  and the measure  $\pi_*\mathfrak{b}$  is continuous. Since the IFS is separable and every Bernoulli measure (with strictly positive weights) is a mixing Markov measure, Theorem 4.10 implies that  $\mathfrak{b}(S_t) = 1$ . To see that  $\pi_*\mathfrak{b}$  is continuous we need to prove that  $\pi_*\mathfrak{b}(\{x\}) = 0$  for every  $x \in [0, 1]$ . Take  $x \in [0, 1]$  and recall the definition of the set  $\Sigma_x$  in (4.1). Since  $\pi^{-1}(x) \subset \Sigma_x$  we have that

$$\pi_*\mathfrak{b}(\{x\}) = \mathfrak{b}(\pi^{-1}(x)) \leq \mathfrak{b}(\Sigma_x) = 0,$$

where the last equality follows from Theorem 4.2. The proof of Theorem 4 is now complete.  $\square$

5.1.2. *Proof of Proposition 5.1.* We assume that  $\mathfrak{b} = \mathfrak{b}(p_1, \dots, p_k)$  and write  $\mathfrak{T} = \mathfrak{T}_{\mathfrak{b}}$ . We begin by proving two auxiliary lemmas:

**Lemma 5.2.** *For every stationary measure of  $\mathfrak{T}$  it holds  $\mathcal{B}(\text{supp}(\mu)) \subset \text{supp}(\mu)$ .*

*Proof.* It is sufficient to show that  $T_i(\text{supp}(\mu)) \subset \text{supp}(\mu)$  for every  $i$ . Given  $x \in \text{supp}(\mu)$  take a neighborhood  $V$  of  $T_i(x)$ . By the choice of  $x$ ,  $\mu(T_i^{-1}(V)) > 0$ . Since  $\mu$  is a stationary measure we have

$$\mu(V) = p_1\mu(T_1^{-1}(V)) + \dots + p_k\mu(T_k^{-1}(V)) \geq p_i\mu(T_i^{-1}(V)) > 0,$$

proving the lemma.  $\square$

**Lemma 5.3.** *Consider the IFS  $(T_1, \dots, T_k)$ . Then for every sequence  $(\mu_n)$  of probabilities of  $\mathcal{M}_1(X)$  and every  $\omega \in S_t$  it holds*

$$\lim_{n \rightarrow \infty} T_{\omega_0} * \dots * T_{\omega_n} * \mu_n = \delta_{\pi(\omega)}.$$

*Proof.* Consider a sequence of probabilities  $(\mu_n)$  and  $\omega \in S_t$ . Fix any  $g \in C^0(X)$ . Then given any  $\epsilon > 0$  there is  $\delta$  such that

$$|g(y) - g \circ \pi(\omega)| < \epsilon \quad \text{for all } y \in X \text{ with } d(y, \pi(\omega)) < \delta.$$

Since  $\omega \in S_t$  there is  $n_0$  such that  $d(T_{\omega_0} \circ \dots \circ T_{\omega_n}(x), \pi(\omega)) < \delta$  for every  $x \in X$  and every  $n \geq n_0$ . Therefore for  $n \geq n_0$  we have

$$\begin{aligned} \left| g \circ \pi(\omega) - \int g dT_{\omega_0} * \dots * T_{\omega_n} * \mu_n \right| &= \left| \int g \circ \pi(\omega) d\mu_n - \int g \circ T_{\omega_0} \circ \dots \circ T_{\omega_n}(x) d\mu_n \right| \\ &\leq \int |g \circ \pi(\omega) - g \circ T_{\omega_0} \circ \dots \circ T_{\omega_n}(x)| d\mu_n \leq \epsilon. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \int g dT_{\omega_0} * \dots * T_{\omega_n} * \mu_n = g \circ \pi(\omega)$$

Since this holds for every continuous map  $g$  the lemma follows.  $\square$

We will show that  $\lim_{n \rightarrow \infty} \mathfrak{T}^n \nu = \pi_*\mathfrak{b}$  for every  $\nu \in \mathcal{M}_1(X)$ . In particular, by the continuity of  $\mathfrak{T}$ ,  $\mathfrak{T}\pi_*\mathfrak{b} = \pi_*\mathfrak{b}$ .

Note that from the definition of the Markov operator in (2.4), for every  $\nu \in \mathcal{M}_1(X)$  and every continuous map  $f \in C^0(X)$  it holds

$$(5.1) \quad \int f d(\mathfrak{T}^n \nu) = \sum_{\xi_0, \dots, \xi_{n-1}} p_{\xi_0} p_{\xi_1} \dots p_{\xi_{n-1}} \int f dT_{\xi_0} * T_{\xi_1} * \dots * T_{\xi_{n-1}} * \nu.$$



Fixed  $\nu \in \mathcal{M}_1(X)$  consider the sequence of functions  $F_n : \Sigma_k^+ \rightarrow \mathbb{R}$  defined by

$$F_n(\xi) \stackrel{\text{def}}{=} \int f dT_{\xi_0} * T_{\xi_1} * \dots * T_{\xi_{n-1}} * \nu.$$

Since the map  $F_n$  is constant in the cylinders  $[\xi_0, \dots, \xi_{n-1}]$ , it is a measurable function. From this property, equation (5.1), and the definition of the Bernoulli measure  $\mathfrak{b}$  we have

$$\int f d(\mathfrak{T}^n \nu) = \int F_n d\mathfrak{b}.$$

By hypothesis  $\mathfrak{b}(S_t) = 1$ , thus applying Lemma 5.3 to the constant sequence  $\mu_n = \nu$  we have that

$$(5.2) \quad \lim_{n \rightarrow \infty} F_n(\xi) = f \circ \pi(\xi) \quad \text{for } \mathfrak{b}\text{-a.e. } \xi.$$

Since  $|F_n(\xi)| \leq \|f\|$ , from (5.2) using the dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \int f d(\mathfrak{T}^n \nu) = \lim_{n \rightarrow \infty} \int F_n d\mathfrak{b} = \int f \circ \pi d\mathfrak{b} = \int f d\pi_* \mathfrak{b}.$$

Since the previous equality holds for every continuous map  $f$  it follows that  $\pi_* \mathfrak{b}$  is an attracting measure.

It remains to see that  $\text{supp}(\pi_* \mathfrak{b}) = \overline{A_t}$ . For that note the following equalities

$$(5.3) \quad \pi_* \mathfrak{b}(A_t) = \mathfrak{b}(\pi^{-1}(A_t)) = \mathfrak{b}(S_t) = 1$$

that imply  $\text{supp}(\pi_* \mathfrak{b}) \subset \overline{A_t}$ .

To get  $\text{supp}(\pi_* \mathfrak{b}) \supset \overline{A_t}$  recall that, by Proposition 3.1, every  $\mathcal{B}$ -invariant compact set contains a fixed point of  $\mathcal{B}$ . By Lemma 5.2 we have  $\mathcal{B}(\text{supp}(\pi_* \mathfrak{b})) \subset \text{supp}(\pi_* \mathfrak{b})$ . Hence  $\text{supp}(\pi_* \mathfrak{b})$  contains a fixed point of  $\mathcal{B}$ . As  $\overline{A_t}$  is a minimum fixed point of  $\mathcal{B}$  (see Proposition 3.5) this implies that  $\overline{A_t} \subset \text{supp}(\pi_* \mathfrak{b})$ . Thus  $\text{supp}(\pi_* \mathfrak{b}) = \overline{A_t}$ , completing the proof of the proposition.  $\square$

The previous proposition provides a (unique) stationary measure whose support is  $\overline{A_t}$ . To prove that the support of this measure is the closure of the target we use the characterisation of the stationary measure in (5.3). Next proposition claims that the support of the stationary measure of an asymptotically stable Markov operator of an IFS with  $S_t \neq \emptyset$  always is  $\overline{A_t}$ , even when  $\mathfrak{b}(S_t) = 0$  (recall that either  $\mathfrak{b}(S_t) = 1$  or  $\mathfrak{b}(S_t) = 0$ ).

**Proposition 5.4.** *Consider an IFS  $(T_1, \dots, T_k; \mathfrak{b})$  with probabilities defined on a compact metric space whose Markov operator  $\mathfrak{T}_{\mathfrak{b}}$  is asymptotically stable and let  $\mu$  be its stationary measure. If  $S_t \neq \emptyset$  then  $\text{supp}(\mu) = \overline{A_t}$ .*

*Proof.* The inclusion  $\text{supp}(\mu) \supset \overline{A_t}$  follows from Lemma 5.2. To prove the inclusion “ $\subset$ ” take any point  $p \in \text{supp}(\mu)$  and an open neighbourhood  $V$  of  $p$ . We need to see that  $V \cap A_t \neq \emptyset$ . For this take any point  $x \in A_t$ . Since  $\mathfrak{T} = \mathfrak{T}_{\mathfrak{b}}$  is asymptotically stable Alexandrov’s theorem (see [9, page 60]) implies that

$$\liminf_n \mathfrak{T}^n \delta_x(V) \geq \mu(V) > 0.$$

Hence there is  $n_0$  such that  $\mathfrak{T}^{n_0} \delta_x(V) > 0$ . By definition of the Markov operator we have that

$$\mathfrak{T}^{n_0} \delta_x(V) = \sum_{\xi_0, \dots, \xi_{n_0-1}} p_{\xi_0} p_{\xi_1} \dots p_{\xi_{n_0-1}} T_{\xi_0} * T_{\xi_1} * \dots * T_{\xi_{n_0-1}} * \delta_x(V).$$

Therefore there is a finite sequence  $\xi_0 \dots \xi_{n_0-1}$  such that

$$\delta_x(T_{\xi_{n_0-1}}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(V)) > 0$$

and thus  $x \in T_{\xi_{n_0-1}}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(V)$ . The invariance of  $A_t$  now implies that  $V \cap A_t \neq \emptyset$ , proving the proposition.  $\square$

**5.2. Stationary measures for recurrent IFSs in  $[0, 1]$ .** In this section we prove Theorem 6. For that we consider a recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  defined on a compact metric space  $X$ , where  $\mathbb{P}^+$  is the Markov probability associated to  $(P = (p_{i,j}), \bar{p} = (p_i))$ . We also consider the set  $\hat{X} = X \times \{1, \dots, k\}$  and the (generalised) Markov operator  $\mathfrak{S} = \mathfrak{S}_{\mathbb{P}^+}$  (see (2.5)) and the generalised coding map  $\varpi: S_t \rightarrow \hat{X}$  given by  $\varpi(\xi) \stackrel{\text{def}}{=} (\pi(\xi), \xi_0)$  (see (2.7)) of the IFS. A final ingredient is the inverse Markov measure  $\mathbb{P}^-$  associated to  $\mathbb{P}^+$  defined in (2.6).

To prove Theorem 6 we need to see that every IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  such that the inverse Markov measure  $\mathbb{P}^-$  is mixing and splits the IFS in some non-trivial closed interval  $J$  satisfies

$$\lim_{n \rightarrow \infty} \mathfrak{S}^n(\hat{\mu}) = \varpi_* \mathbb{P}^- \quad \text{for every } \hat{\mu} \in \mathcal{M}_1([0, 1] \times \{1, \dots, k\}).$$

The main step of the proof of Theorem 6 is the following result whose proof is postponed.

**Theorem 5.5.** *Consider a recurrent IFS  $(T_1, \dots, T_k; \mathbb{P}^+)$  defined on a compact metric space  $X$  such that  $\mathbb{P}^-$  is mixing and  $\mathbb{P}^-(S_t) = 1$ . Then*

$$\lim_{n \rightarrow \infty} \mathfrak{S}^n(\hat{\mu}) = \varpi_* \mathbb{P}^- \quad \text{for every } \hat{\mu} \in \mathcal{M}_1(\hat{X}).$$

In particular,  $\varpi_* \mathbb{P}^-$  is the unique stationary measure of  $\mathfrak{S}$ .

*Proof of Theorem 6.* Since  $\mathbb{P}^-$  is mixing and splits the IFS in some non-trivial interval it follows from Theorem 5 that  $\mathbb{P}^-(S_t) = 1$ . Thus the theorem follows from Theorem 5.5.  $\square$

*Proof of Theorem 5.5.* Given a function  $\hat{f}: \hat{X} \rightarrow \mathbb{R}$  we define its  $i$ -section  $f_i: X \rightarrow \mathbb{R}$  by  $f_i(x) \stackrel{\text{def}}{=} \hat{f}(x, i)$  and write  $\hat{f} = \langle f_1, \dots, f_k \rangle$ . We need to see that for every measure  $\hat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\hat{X})$  and every continuous function  $\hat{f} = \langle f_1, \dots, f_k \rangle \in C^0(\hat{X})$  it holds

$$(5.4) \quad \lim_{n \rightarrow \infty} \int \hat{f} d\mathfrak{S}^n(\hat{\mu}) = \int \hat{f} d\varpi_* \mathbb{P}^-.$$

By definition, it follows that

$$\int \hat{f} d\hat{\mu} = \sum_{i=1}^k \int f_i d\mu_i, \quad \text{where } \hat{\mu} = (\mu_1, \dots, \mu_k),$$

and hence

$$(5.5) \quad \int \hat{f} d\mathfrak{S}^n(\hat{\mu}) = \sum_{j=1}^k \int f_j d(\mathfrak{S}^n(\hat{\mu}))_j, \quad \mathfrak{S}^n(\hat{\mu}) = ((\mathfrak{S}^n(\hat{\mu}))_1, \dots, (\mathfrak{S}^n(\hat{\mu}))_k).$$

To get the convergence of the integrals of the sum in (5.5) we need a preparatory lemma. First, denote by  $\|g\|$  the uniform norm of a continuous function  $g: X \rightarrow \mathbb{R}$ .

**Lemma 5.6.** Consider  $\hat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\hat{X})$  such that  $\mu_i(X) > 0$  for every  $i \in \{1, \dots, k\}$ . Then for every  $g \in C^0(X)$  it holds

$$\limsup_n \left| \int g d(\mathfrak{S}^n(\hat{\mu}))_j - \int_{[j]} (g \circ \pi) d\mathbb{P}^- \right| \leq k \|g\| \max_i |\mu_i(X) - p_i|,$$

where  $\bar{p} = (p_1, \dots, p_k)$  is the unique stationary vector of  $P$ .

*Proof.* Take  $\hat{\mu} \in \mathcal{M}_1(\hat{X})$  as in the statement of the lemma and for each  $i$  define the probability measure  $\bar{\mu}_i$

$$\bar{\mu}_i(B) \stackrel{\text{def}}{=} \frac{\mu_i(B)}{\mu_i(X)}, \quad \text{where } B \text{ is a Borel subset of } X.$$

A straightforward calculation and the previous definition imply that

$$\begin{aligned} (\mathfrak{S}^n(\hat{\mu}))_j &= \sum_{\xi_1, \dots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} p_{\xi_1 j} T_{j^*} T_{\xi_1^*} \cdots T_{\xi_{n-1}^*} \mu_{\xi_n} \\ &= \sum_{\xi_1, \dots, \xi_n} \mu_{\xi_n}(X) p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} p_{\xi_1 j} T_{j^*} T_{\xi_1^*} \cdots T_{\xi_{n-1}^*} \bar{\mu}_{\xi_n}. \end{aligned}$$

Thus given any  $g \in C^0(X)$  we have that

$$\int g d(\mathfrak{S}^n(\hat{\mu}))_j = \sum_{\xi_1, \dots, \xi_n} \mu_{\xi_n}(X) p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g dT_{j^*} T_{\xi_1^*} \cdots T_{\xi_{n-1}^*} \bar{\mu}_{\xi_n}.$$

Let

$$L_n \stackrel{\text{def}}{=} \left| \int g d(\mathfrak{S}^n(\hat{\mu}))_j - \int_{[j]} (g \circ \pi) d\mathbb{P}^- \right|$$

and write  $\mu_{\xi_n}(X) = (\mu_{\xi_n}(X) - p_{\xi_n}) + p_{\xi_n}$ . Then

$$\begin{aligned} L_n &\leq \left| \sum_{\xi_1, \dots, \xi_n} (\mu_{\xi_n}(X) - p_{\xi_n}) p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g dT_{j^*} T_{\xi_1^*} \cdots T_{\xi_{n-1}^*} \bar{\mu}_{\xi_n} \right| \\ &+ \left| \sum_{\xi_1, \dots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g dT_{j^*} T_{\xi_1^*} \cdots T_{\xi_{n-1}^*} \bar{\mu}_{\xi_n} - \int_{[j]} (g \circ \pi) d\mathbb{P}^- \right| \\ &\leq \max_i |\mu_i(X) - p_i| \|g\| \sum_{\xi_1, \dots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \\ &+ \left| \sum_{\xi_1, \dots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g dT_{j^*} T_{\xi_1^*} \cdots T_{\xi_{n-1}^*} \bar{\mu}_{\xi_n} - \int_{[j]} (g \circ \pi) d\mathbb{P}^- \right|. \end{aligned}$$

Note that  $\sum_{\xi_1, \dots, \xi_{n-1}} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j}$  is the entry  $(\xi_n, j)$  of the matrix  $P^n$ . Hence

$$\sum_{\xi_1, \dots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} = \sum_{\xi_n=1}^k \sum_{\xi_1, \dots, \xi_{n-1}} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \leq k.$$

Therefore

$$(5.6) \quad \max_i |\mu_i(X) - p_i| \|g\| \sum_{\xi_1, \dots, \xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \leq k \|g\| \max_i |\mu_i(X) - p_i|.$$

We now estimate the second parcel in the sum above.

**Claim 5.7.** *For every continuous function  $g$  it holds*

$$\lim_{n \rightarrow \infty} \sum_{\xi_1, \dots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g dT_{j*} T_{\xi_1*} \cdots T_{\xi_{n-1}*} \bar{\mu}_{\xi_n} = \int_{[j]} (g \circ \pi) d\mathbb{P}^-.$$

Observe that equation (5.6) and the claim imply the lemma.

*Proof of Claim 5.7.* Consider the sequence of functions given by

$$G_n : \Sigma_k^+ \rightarrow \mathbb{R}, \quad G_n(\xi) \stackrel{\text{def}}{=} \int g dT_{\xi_0*} T_{\xi_1*} \cdots T_{\xi_{n-1}*} \bar{\mu}_{\xi_n}.$$

By definition, for every  $n$  the corresponding map  $G_n$  is constant in the cylinders  $[\xi_0, \dots, \xi_n]$  and thus it is measurable. By definition of  $\mathbb{P}^\pm$ , for every  $j$  we have that

$$p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_2 \xi_1} p_{\xi_1 j} = \mathbb{P}^+([\xi_n \xi_{n-1} \cdots \xi_1 j]) = \mathbb{P}^-([j \xi_1 \xi_2 \cdots \xi_n]).$$

Hence

$$\sum_{\xi_1, \dots, \xi_n} p_{\xi_n} p_{\xi_n \xi_{n-1}} \cdots p_{\xi_1 j} \int g dT_{j*} T_{\xi_1*} \cdots T_{\xi_{n-1}*} \bar{\mu}_{\xi_n} = \int_{[j]} G_n d\mathbb{P}^-.$$

It follows from the hypothesis  $\mathbb{P}^-(S_t) = 1$  and Lemma 5.3 that

$$(5.7) \quad \lim_{n \rightarrow \infty} G_n(\xi) = g \circ \pi(\xi) \quad \text{for } \mathbb{P}^- \text{-almost every } \xi.$$

Now note that  $|G_n(\xi)| \leq \|g\|$  for every  $\xi \in \Sigma_k^+$ . From (5.7), using the dominated convergence theorem, we get

$$\lim_{n \rightarrow \infty} \int_{[j]} G_n d\mathbb{P}^- = \int_{[j]} (g \circ \pi) d\mathbb{P}^-,$$

ending the proof of the claim.  $\square$

The proof of the lemma is now complete.  $\square$

To prove the theorem observe that since  $\mathbb{P}^-$  is mixing the transition matrix  $P$  associated to  $\mathbb{P}^+$  is primitive, recall Section 2.3.2. Take  $\hat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\hat{X})$ . Note that by definition of the Markov operator

$$((\mathfrak{S}\hat{\mu})_1(X), \dots, (\mathfrak{S}\hat{\mu})_k(X)) = \hat{p}P, \quad \text{where } \hat{p} = (\mu_1(X), \dots, \mu_k(X)).$$

Hence for every  $n \geq 1$

$$(5.8) \quad ((\mathfrak{S}^n \hat{\mu})_1(X), \dots, (\mathfrak{S}^n \hat{\mu})_k(X)) = \hat{p}P^n.$$

By the Perron-Frobenius theorem, see for instance [20, page 64], we have that  $P$  the stationary vector  $\bar{p} = (p_1, \dots, p_k)$  is positive<sup>7</sup> and

$$\lim_{n \rightarrow \infty} \hat{p}P^n = \bar{p} \quad \text{for every probability vector } \hat{p}.$$

Hence (5.8) gives  $n_0$  such that the vector  $((\mathfrak{S}^{n_1} \hat{\mu})_1(X), \dots, (\mathfrak{S}^{n_1} \hat{\mu})_k(X))$  is positive for every  $n_1 \geq n_0$ . Therefore we can apply Lemma 5.6 to the measure  $\mathfrak{S}^{n_1}(\hat{\mu})$  for every  $n_1 \geq n_0$ , obtaining for every  $g \in C^0(X)$  the inequality

$$\limsup_n \left| \int g d(\mathfrak{S}^{n+n_1}(\hat{\mu}))_j - \int_{[j]} (g \circ \pi) d\mathbb{P}^- \right| \leq k \|g\| \max_i |(\mathfrak{S}^{n_1} \hat{\mu})_i(X) - p_i|.$$

<sup>7</sup>A vector  $v = (v_1, \dots, v_k)$  is said positive if  $v_i > 0$  for all  $i$ .

It follows from the definition of limsup and the previous inequality that

$$\limsup_n \left| \int g d(\mathfrak{S}^n(\widehat{\mu}))_j - \int_{[j]} (g \circ \pi) d\mathbb{P}^- \right| \leq k \|g\| \max_i |(\mathfrak{S}^{n_1}\widehat{\mu})_i(X) - p_i|$$

for every  $n_1 \geq n_0$ . By (5.8) and the Perron-Frobenius theorem we get

$$\lim_{n_1 \rightarrow \infty} \max_i |(\mathfrak{S}^{n_1}\widehat{\mu})_i(X) - p_i| = 0.$$

Therefore

$$(5.9) \quad \lim_{n \rightarrow \infty} \int g d(\mathfrak{S}^n(\widehat{\mu}))_j = \int_{[j]} (g \circ \pi) d\mathbb{P}^- \quad \text{for every } g \in C^0(X).$$

To get equation (5.4), write  $\widehat{f} = \langle f_1, \dots, f_k \rangle$ , apply (5.9) to the maps  $f_i$ , and use (5.5) to get

$$\lim_{n \rightarrow \infty} \int \widehat{f} d\mathfrak{S}^n(\widehat{\nu}) \stackrel{(5.5)}{=} \sum_{j=1}^k \lim_{n \rightarrow \infty} \int f_j d(\mathfrak{S}^n(\widehat{\nu}))_j \stackrel{(5.9)}{=} \sum_{j=1}^k \int_{[j]} (f_j \circ \pi) d\mathbb{P}^-.$$

Now observing that  $f_j \circ \pi(\xi) = \widehat{f} \circ \varpi(\xi)$  for every  $\xi \in [j]$ , we conclude that

$$\lim_{n \rightarrow \infty} \int \widehat{f} d\mathfrak{S}^n(\widehat{\nu}) = \sum_{j=1}^k \int_{[j]} \widehat{f} \circ \varpi d\mathbb{P}^- = \int \widehat{f} d\varpi_*\mathbb{P}^-.$$

proving (5.4) and ending the proof of the theorem.  $\square$

In Proposition 5.8 we state a result that does not involve the mixing condition of the probability  $\mathbb{P}^-$ . For that we consider the subset  $\mathcal{M}_{\bar{p}}(\widehat{X})$  of  $\mathcal{M}_1(\widehat{X})$  defined by

$$\mathcal{M}_{\bar{p}}(\widehat{X}) \stackrel{\text{def}}{=} \{\widehat{\mu} = (\mu_1, \dots, \mu_k) : \mu_i(X) = p_i \text{ for every } i\},$$

where  $\bar{p} = (p_1, \dots, p_k)$  is the stationary vector of the irreducible transition matrix  $P$  associated to  $\mathbb{P}^+$ . The set  $\mathcal{M}_{\bar{p}}(\widehat{X})$  is invariant by  $\mathfrak{S}_{\mathbb{P}^+}$  and contains all stationary measures of  $\text{IFS}(T_1, \dots, T_k; \mathbb{P}^+)$ . For the first assertion observe that given any  $\widehat{\mu} \in \mathcal{M}_{\bar{p}}(\widehat{X})$  by the definition of  $\mathfrak{S}_{\mathbb{P}^+}$  we have

$$(\mathfrak{S}_{\mathbb{P}^+}\widehat{\mu})_j = \sum_{i=1}^k p_{ij} T_{j*}\mu_i \quad \text{for every } j.$$

Thus

$$(\mathfrak{S}_{\mathbb{P}^+}\widehat{\mu})_j(X) = \sum_{i=1}^k p_{ij} \mu_i(T_j^{-1}(X)) = \sum_{i=1}^k p_{ij} \mu_i(X) = \sum_{i=1}^k p_i p_{ij} = p_j$$

and hence  $\mathfrak{S}_{\mathbb{P}^+}(\widehat{\mu}) \in \mathcal{M}_{\bar{p}}(\widehat{X})$ .

For the second assertion note that a measure  $\widehat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_1(\widehat{X})$  is stationary if and only if

$$\mu_j = \sum_{i=1}^k p_{ij} T_{j*}\mu_i \quad \text{for every } j.$$

If  $\widehat{\mu} = (\mu_1, \dots, \mu_k)$  is stationary then  $(\mu_1(X), \dots, \mu_k(X))$  is the stationary probability vector for the transition matrix  $P$  of  $\mathbb{P}^+$ . Thus  $\mu_i(X) = p_i$  for every  $i$ .

A corollary of Lemma 5.6 is the following proposition.

**Proposition 5.8.** *Consider a recurrent IFS( $T_1, \dots, T_k; \mathbb{P}^+$ ) defined on a compact metric space  $X$  such that  $\mathbb{P}^-(S_t) = 1$ . Then*

$$\lim_{n \rightarrow \infty} \mathfrak{S}^n(\hat{\nu}) = \varpi_* \mathbb{P}^- \quad \text{for every } \hat{\nu} \in \mathcal{M}_{\bar{p}}(\hat{X}).$$

In particular,  $\varpi_* \mathbb{P}^-$  is the unique stationary measure of  $\mathfrak{S}$ .

*Proof.* Consider  $\hat{\mu} = (\mu_1, \dots, \mu_k) \in \mathcal{M}_{\bar{p}}(\hat{X})$  and note that  $\mu_i(X) = p_i$ . Lemma 5.6 implies that for every continuous function  $g$  it holds

$$(5.10) \quad \lim_{n \rightarrow \infty} \int g d(\mathfrak{S}^n(\hat{\mu}))_j = \int_{[j]} (g \circ \pi) d\mathbb{P}^-.$$

Consider a continuous map  $\hat{f} = \langle f_1, \dots, f_k \rangle$ . We apply (5.10) to the maps  $f_i$  and use equation (5.5) to get

$$\lim_{n \rightarrow \infty} \int \hat{f} d\mathfrak{S}^n(\hat{\nu}) \stackrel{(5.5)}{=} \sum_{j=1}^k \lim_{n \rightarrow \infty} \int f_j d(\mathfrak{S}^n(\hat{\nu}))_j \stackrel{(5.10)}{=} \sum_{j=1}^k \int_{[j]} (f_j \circ \pi) d\mathbb{P}^-.$$

Observing that  $f_j \circ \pi(\xi) = \hat{f} \circ \varpi(\xi)$  for every  $\xi \in [j]$ , we conclude that

$$\lim_{n \rightarrow \infty} \int \hat{f} d\mathfrak{S}^n(\hat{\nu}) = \sum_{j=1}^k \int_{[j]} \hat{f} \circ \varpi d\mathbb{P}^- = \int \hat{f} d\varpi_* \mathbb{P}^-,$$

proving the proposition.  $\square$

## 6. EXAMPLES

**Example 6.1** (A non-regular IFS with  $S_t \neq \emptyset$  and  $\#(A_t) \geq 2$ ). Consider an IFS defined on  $[0, 1]$  consisting of two injective continuous maps  $T_1$  and  $T_2$  as in Figure 3.

- The map  $T_1$  has exactly two fixed points  $0, 1$ , where  $0$  is a repeller and  $1$  is an attractor.
- The map  $T_2$  has (exactly three) fixed points  $p_1 < p_2 < p_3$ , where  $p_1$  and  $p_3$  are attractors and  $p_2$  is a repeller,  $T_2([0, 1]) = [\alpha, \beta] \subset (0, 1)$ , and  $T_1(p_1) < \beta$ .

Obviously,  $\text{IFS}(T_1)$  and  $\text{IFS}(T_2)$  are not asymptotically stable. To see that  $\text{IFS}(T_1, T_2)$  is not asymptotically stable just note that  $[0, 1]$  and  $[p_1, 1]$  are fixed points of the Barnsley-Hutchinson operator. For the last assertion we use that  $T_1(p_1) < \beta$ . This implies that  $\text{IFS}(T_1, T_2)$  is non-regular.

Finally, to see that  $S_t \neq \emptyset$  note that since  $1$  is an attracting fixed point of  $T_1$  and  $T_2([0, 1]) \subset (0, 1)$  we have that  $T_1^n \circ T_2([0, 1]) \cap T_2([0, 1]) = \emptyset$  for every  $n$  sufficiently large. Now Theorem 4.10 implies that  $S_t \neq \emptyset$ . To see that  $\#(A_t) \geq 2$  just note that given any  $x \in A_t$  then  $T_i(x) \in A_t$  and that  $T_1(x) \neq T_2(x)$ .

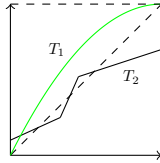


FIGURE 3. A non-regular IFS with a weakly hyperbolic sequence

**Example 6.2** ( $A_t \subsetneq \overline{A_t} = [0, 1]$ ). In this example we consider the underlying IFS of the porcupine-like horseshoes in [11]. We translate the construction in [12, page 12] to our context.

Consider an injective IFS( $T_1, T_2$ ) defined on  $[0, 1]$  such that  $T_1(x) = \lambda(1 - x)$ ,  $\lambda \in (0, 1)$ , and  $T_2$  is a continuous function with exactly two fixed points, the repelling fixed point 0 and the attracting fixed point 1, see Figure 4. We assume that  $T_2$  is a uniform contraction on  $[T_2^{-1}(\lambda), 1]$ . Then  $\overline{A_t} = [0, 1]$  and  $1 \notin A_t$ .

To prove the first assertion note that  $\lambda \in \overline{A_t}$ . For that take an open neighbourhood  $V \subset (0, 1)$  of  $\lambda$ . Note that  $T_1^{-1}(V)$  is a neighbourhood of 0. Consider the fixed point  $p = \frac{\lambda}{1+\lambda} \in (0, 1)$  of  $T_1$  and note that  $p \in A_t$ . Since  $T_2^n(p) \rightarrow 1$  as  $n \rightarrow \infty$  and  $T_1(1) = 0$ , there is  $\ell$  such that  $T_1 \circ T_2^\ell(p) \in T_1^{-1}(V)$ . Hence  $T_1^2 \circ T_2^\ell(p) \in V$ . By the invariance of  $A_t$  we have that  $A_t \cap V \neq \emptyset$ . Since this holds for every neighbourhood  $V$  of  $\lambda$  we get  $\lambda \in \overline{A_t}$ .

We now prove that  $A_t$  is dense in  $[0, 1]$ . Take any open interval  $J \subset (0, 1)$ . We need to see that  $J \cap A_t \neq \emptyset$ . If  $\lambda \in J$  we are done. Otherwise  $\lambda \notin J$  and either  $J \subset (\lambda, 1] = I_2$  or  $J \subset [0, \lambda) = I_1$ . We now construct a finite sequence  $\xi_0 \dots \xi_m$  such that

$$\lambda \in T_{\xi_m}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J).$$

For that let  $\xi_0 = i$  if  $J \subset I_i$  and define recursively  $\xi_{\ell+1} = i$  if  $T_{\xi_\ell}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J) \subset I_i$ . Note that if  $T_{\xi_\ell}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J) \cap I_i \neq \emptyset$  and  $T_{\xi_\ell}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J) \cap I_i \not\subset I_i$  some  $i = 1, 2$ , we are done. Since  $T_2^{-1}$  is a uniform expansion on  $(\lambda, 1]$  and  $T_1^{-1}$  is a uniform expansion on  $[0, \lambda]$  the recursion stops after a finitely many steps: there is  $m$  such that  $\lambda \in T_{\xi_m}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J)$ . Since  $\lambda \in \overline{A_t} \cap T_{\xi_m}^{-1} \circ \dots \circ T_{\xi_0}^{-1}(J)$ , the invariance of  $A_t$  implies that  $J \cap A_t \neq \emptyset$ .

The fact that  $1 \notin A_t$  follows observing that  $\bar{2} \notin S_t$  and that every finite sequence  $\xi_0 \dots \xi_n$  such that  $\xi_i = 1$  for some  $i$  satisfies  $1 \notin T_{\xi_0} \circ \dots \circ T_{\xi_n}([0, 1])$ .

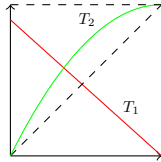


FIGURE 4. The underlying IFS of a porcupine-like horseshoe

**Example 6.3** (A non-weakly hyperbolic IFS in  $[0, 1]$  with  $A_t = [0, 1]$ ). We consider the underlying IFS of the bony attractors in [17].

Consider the IFS( $T_1, T_2$ ) defined on  $[0, 1]$  as follows,  $T_1$  is the piecewise-linear map with “vertices”  $(0, 0)$ ,  $(0.6, 0.2)$ , and  $(1, 0.8)$  and  $T_2$  is the piecewise-linear map with “vertices”  $(0, 0.15)$ ,  $(0.4, 0.8)$ , and  $(1, 1)$ , see Figure 5. We claim that the IFS( $T_1, T_2$ ) is not weakly hyperbolic and  $A_t = [0, 1]$ .

To prove the first assertion note that  $T_1 \circ T_2$  has a repelling fixed point, see [17]. Therefore the periodic sequence  $\bar{12}$  does not belong to  $S_t$ , hence the IFS is not weakly hyperbolic.

To see the second assertion, note that the compositions  $T_1^3$ ,  $T_1^2 \circ T_2$ ,  $T_2^2 \circ T_1$  and  $T_2^5$  are uniform contractions and that the union of their images is  $[0, 1]$ , see [17]. In other words, the IFS( $T_1^3, T_1^2 \circ T_2, T_2^2 \circ T_1, T_2^5$ ) is hyperbolic and  $[0, 1]$  is the unique

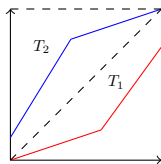


FIGURE 5. The underlying IFS of a bony attractor

fixed point of its Barnsley-Hutchinson operator. Consider the finite set of words

$$W = \{111, 112, 221, 22222\}$$

and let  $E_W$  be the subset of  $\Sigma_k^+$  consisting of sequences  $\xi$  that are a concatenation of words of  $W$ <sup>8</sup>. Let  $S_t$  be the set of weakly hyperbolic sequences corresponding to the IFS  $(T_1, T_2)$  and  $\pi$  the associated coding map. By construction we have that  $E_W \subset S_t$  and  $\pi(E_W) = [0, 1]$ . Since  $A_t = \pi(S_t)$  we have that  $A_t = [0, 1]$ .

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<sup>8</sup>There is an increasing sequence  $(i_\ell)_{\ell \in \mathbb{N}}$  with  $\xi_0 = 0$  such that  $\xi_{i_\ell} \dots \xi_{i_{\ell+1}-1} \in W$  for every  $\ell \in \mathbb{N}$ .



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