GENERIC LINEAR COCYCLES OVER A MINIMAL BASE

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Abstract. We prove that a generic linear cocycle over a minimal base dynamics of finite dimension has the property that the Oseledets splitting with respect to any invariant probability coincides almost everywhere with the finest dominated splitting. Therefore the restriction of the generic cocycle to a subbundle of the finest dominated splitting is uniformly subexponentially quasiconformal. This extends a previous result for $\mathrm{SL}(2,\mathbb{R})$-cocycles due to Avila and the author.

1. Introduction

1.1. **Statement of the result.** Let X be a compact Hausdorff space, and let $\mathbb{E}$ be a real vector bundle with base space X. We will always assume that fibers $\mathbb{E}(x)$ have constant finite dimension.

Let $T\colon X \to X$ be an homeomorphism. A vector bundle automorphism covering T is a map $A\colon \mathbb{E} \to \mathbb{E}$ whose restriction to an arbitrary fiber $\mathbb{E}(x)$ is a linear isomorphism onto the fiber $\mathbb{E}(Tx)$; this isomorphism will be denoted by $A(x)$. Let $\mathrm{Aut}(\mathbb{E}, T)$ the set of these automorphisms. When the vector bundle is trivial, an automorphism is usually called a *linear cocycle*.

We endow $\mathbb{E}$ with a Riemannian metric, and $\mathrm{Aut}(\mathbb{E}, T)$ with the uniform topology, that is, the topology induced by the distance

$$d(A, B) := \sup_{x \in X} \|A(x) - B(x)\| ,$$

where $\|\cdot\|$ denotes the operator norm induced by the Riemannian metric.

Given $A \in \mathrm{Aut}(\mathbb{E}, T)$, an (ordered) splitting of the vector bundle

$$\mathbb{E} = \mathbb{E}_1 \oplus \mathbb{E}_2 \oplus \cdots \oplus \mathbb{E}_k$$

is called *dominated* (also *exponentially separated*) if it is A-invariant and there are constants $c > 0$ and $\tau > 1$ such that for all $x \in X$ and all unit vectors $v_1 \in \mathbb{E}_1(x)$, …, $v_k \in \mathbb{E}_k(x)$, we have

$$\frac{\|A^n(x) \cdot v_i\|}{\|A^n(x) \cdot v_{i+1}\|} > c\tau^n , \quad \forall n \geq 0.$$

(In fact, it is always possible to choose an *adapted* Riemannian metric so that $c = 1$; see [Go].)

There exists an unique such splitting into a maximal number k of bundles, which is called the *finest dominated splitting* of A. If $k = 1$, this is just a *trivial* splitting. The finest dominated splitting refines any other dominated splitting of A. (See e.g. [BDV] for these and other properties of dominated spittings.)

Given $A \in \mathrm{Aut}(\mathbb{E}, T)$, Oseledets theorem (see e.g. [Ar]) provides a set $R \subset X$ of full probability (i.e., such that $\mu(R) = 1$ for every T-invariant probability measure

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μ) such that each fiber $\mathbb{E}(x)$ over a point $x \in R$ splits into subspaces having the same Lyapunov exponents. This *Oseledets splitting* is A-invariant, measurable, but in general not continuous. For example, the dimensions of the subbundles may depend on the basepoint. Notice that the Oseledets splitting always refines the finest dominated splitting, since domination forces a gap between Lyapunov exponents.

It is shown in [BV] that for any ergodic measure μ, the generic automorphism A has the property that the Oseledets splitting coincides μ-almost everywhere with the finest dominated splitting above the support of the measure. In this paper we obtain this property simultaneously for all measures, under suitable assumptions:

We say the space X has *finite dimension* if it is homeomorphic to a subset of some euclidean space. For instance, subsets of manifolds (assumed as usual to be Hausdorff and second countable) have finite dimension. We say that the homeomorphism T is *minimal* if every orbit is dense.

Main Theorem 1.1. *Let $T\colon X \to X$ be a minimal homeomorphism of a compact space X of finite dimension, and let $\mathbb{E}$ be a vector bundle over X. Let $\mathcal{R}$ be the set of $A \in \mathrm{Aut}(\mathbb{E}, T)$ with the following property: for every T-invariant probability measure μ, the Oseledets splitting with respect to μ coincides μ-almost everywhere with the finest dominated splitting of A. Then $\mathcal{R}$ is a residual subset of* $\mathrm{Aut}(\mathbb{E}, T)$.

Thus if $A \in \mathcal{R}$ has a finest dominated splitting into k subbundles then at almost every point x with respect to each invariant probability measure, there are exactly k different Lyapunov exponents at x. Of course, these values are a.e. constant if the measure is ergodic; they may however depend on the measure.

Since a minimal homeomorphism may have uncountably many ergodic measures, Theorem 1.1 is not a consequence of the aforementioned result of [BV]. Actually, the theorem was proved first in the case of $\mathrm{SL}(2,\mathbb{R})$-cocycles in [AB].

It is evident that the minimality assumption is necessary for the validity of Theorem 1.1; it is easy to see that it cannot be replaced e.g. by transitivity. An example from [AB] shows that it is not sufficient to assume that T has a unique minimal set. As in [AB], we do not know whether the assumption that X has finite dimension is actually necessary.

1.2. **Uniform properties.** An immediate consequence of the Main Theorem 1.1 is that for the generic automorphism, the Oseledets splitting varies continuously. Another consequence is that the time needed to see a definite separation between expansion rates along different Oseledets subbundles is uniform. All these properties are much stronger than those provided by the Oseledets theorem itself. Let us discuss another uniform property that follows from Theorem 1.1, and that depends on information on all invariant measures.

If L is a linear automorphism between inner product vector spaces, define the *mininorm* of L as $\mathfrak{m}(L) := \|L^{-1}\|^{-1}$, and the *quasiconformal distortion* of L as

$$\kappa(L) := \log\left(\frac{\|L\|}{\mathfrak{m}(L)}\right) . \tag{1.1}$$

For an interpretation of this quantity in terms of angle distortion, see [BV, Lemma 2.7].

Let us say that an automorphism $A \in \mathrm{Aut}(\mathbb{E}, T)$ is *uniformly subexponentially quasiconformal* if for every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$\kappa\big(A^n(x)\big) \le c_\varepsilon + \varepsilon n \quad \text{for all } x \in X,\ n \ge 0.$$

Then, as an addendum to the Main Theorem, we have:

Proposition 1.2. *The elements of $\mathcal{R}$ are exactly the automorphisms $A \in \mathrm{Aut}(\mathbb{E}, T)$ whose restrictions $A|\mathbb{E}_i$ to the each bundle of the finest dominated splitting $\mathbb{E}_1 \oplus \cdots \oplus E_k$ are uniformly subexponentially quasiconformal.*

1.3. **Applications.** It is shown in [BN] that if $A \in \mathrm{Aut}(\mathbb{E}, T)$ is uniformly subexponentially quasiconformal then for every $\varepsilon > 0$, there is a Riemannian metric on $\mathbb{E}$ with respect to which the quasiconformal distortion is less than ε; moreover if ε is small then a perturbation of A is conformal with respect to this metric. Putting these results together with Main Theorem 1.1, it is possible to show the following:

Theorem 1.3 ([BN, Thrm. 2.3])**.** *Let $T \colon X \to X$ be a minimal homeomorphism of a compact space X of finite dimension, and let $\mathbb{E}$ be a vector bundle over X. Then there exists a dense subset $\mathcal{D} \subset \mathrm{Aut}(\mathbb{E}, T)$ with the following properties: For every $A \in \mathcal{D}$ there exists a Riemannian metric on the vector bundle $\mathbb{E}$ with respect to which the subbundles of the finest dominated splitting of A are orthogonal, and the restriction of A to each of these subbundles is conformal. Moreover, this metric is adapted in the sense of* [Go]*.*

This result should be useful to study the following question: *When can an automorphism $A \in \mathrm{Aut}(\mathbb{E}, T)$ be approximated by another with a nontrivial dominated splitting?*

1.4. **Comments on the proof and organization of the paper.** To prove Theorem 1.1 we used ideas and tools developed in [AB] to deal with the $\mathrm{SL}(2, \mathbb{R})$ case. The basic strategy for mixing different expansion rates on higher dimensions is similar to that from [BV], but using a characterization of domination from [BG] to find the suitable places to perturb. As in [BV], the desired residual set is obtained as the set of continuity points of some semicontinuous function.

Despite these overlaps, dealing simultaneously with several Lyapunov exponents with respect to all invariant measures presented substantial new difficulties. We introduce an especially convenient semicontinuous function Z to measure quasiconformal distortion. This function was in fact suggested by some ideas from [BB]. The proof that the mixing mechanism actually produces a discontinuity of Z is also more delicate: it is essential not to be too greedy, and instead attack only the points on X where the distortion is comparatively large. This is explained in § 3.2.

The paper is organized as follows:

In § 2 we explain several preliminaries, and reduce the proof of Main Theorem to a result (Lemma 2.9) on the existence of discontinuities of a certain function (related to Z).

In § 3 we prove Main Lemma 3.1, which produces the suitable perturbations along a segment of orbit.

In § 4 we explain how to patch those local perturbations to prove Lemma 2.9 and therefore conclude.

2. Initial considerations

In this section, X is a compact Hausdorff space X, the map $T \colon X \to X$ is at least continuous, and $\mathbb{E}$ is a vector bundle over X of dimension d.

We denote the set of all T-invariant probability measures by $\mathcal{M}(T)$. A Borel set $B \subset X$ is said to have *zero probability* (resp. *full probability*) with respect to a continuous map $T \colon X \to X$ if $\mu(B)$ is 0 (resp. 1) for every T-invariant probability measure μ.

2.1. **Semi-uniform subadditive ergodic theorem.** Proposition 1.2 is an equivalence between a uniform property on $\mathcal{M}(T)$ and a uniform property on X. The following Theorem 2.1 is often useful to obtain equivalences of this kind.

Recall that a sequence of $f_n \colon X \to \mathbb{R}$ is called *subadditive* if $f_{n+m} \le f_n + f_m \circ T^n$.

Theorem 2.1 (Semi-uniform subadditive ergodic theorem; [Sc, Thrm. 1], [SS, Thrm. 1.7])**.** *Let $T \colon X \to X$ be a continuous map of a compact Hausdorff space X. Given a subadditive sequence of continuous functions $f_n \colon X \to \mathbb{R}$, we have*

$$\sup_{\mu \in \mathcal{M}(T)} \lim_{n\to\infty} \frac{1}{n} \int_X f_n \, d\mu = \lim_{n\to\infty} \frac{1}{n} \sup_{x\in X} f_n(x) .$$

Notice that, by Fekete's lemma both limits above can be replaced by inf's. Also recall that for every $\mu \in \mathcal{M}(T)$, by Kingman's subadditive ergodic theorem the sequence $f_n(x)/n$ actually converges to a value in $[-\infty, +\infty)$ for every point x on a full probability subset.

2.2. **Maximal asymptotic distortion.** Recall the definition (1.1) of the quasiconformal distortion κ. Notice that κ is *subadditive*, i.e., if $L_i \colon E_i \to E_{i+1}$ ($i = 1$, 2) are isomorphisms between inner product spaces, then $\kappa(L_2 L_1) \le \kappa(L_2) + \kappa(L_1)$.

Given an automorphism $A \in \mathrm{Aut}(\mathbb{E}, T)$, define

$$K(A) := \inf_{n\ge1} \frac{1}{n} \sup_{x\in X} \kappa(A^n(x)) . \tag{2.1}$$

(By Fekete's lemma, the inf can be replaced by a limit.) Being an infimum of continuous functions, $K \colon \mathrm{Aut}(A, \mathbb{E}) \to [0, \infty)$ is upper semicontinuous.

Notice that A is uniformly subexponentially quasiconformal (as defined in the Introduction) if and only if $K(A) = 0$.

If L is an isomorphism between inner product vector spaces of dimension d, its singular values (i.e., the eigenvalues of $(L^*L)^{1/2}$) will be written as $\mathfrak{s}_1(L) \ge \cdots \ge \mathfrak{s}_d(L)$; so $\mathfrak{s}_1(L) = \|L\|$ and $\mathfrak{s}_d(L) = \mathfrak{m}(L)$.

Given $A \in \mathrm{Aut}(\mathbb{E}, T)$, the following *Lyapunov exponents* exist for every x in a full probability subset of X:

$$\chi_i(A, x) := \lim_{n\to+\infty} \frac{1}{n} \log \mathfrak{s}_i(A^n(x)) , \quad (i = 1, \dots, d).$$

Let us denote their averages with respect to some $\mu \in \mathcal{M}(T)$ as:

$$\chi_i(A, \mu) := \int_X \chi_i(A, x) \, d\mu(x).$$

It follows from Theorem 2.1 that:

$$K(A) = \sup_{\mu\in\mathcal{M}(T)} \left[\chi_1(A, \mu) - \chi_d(A, \mu) \right] . \tag{2.2}$$

In particular, A is uniformly subexponentially quasiconformal if and only if for every point x in a full probability subset, all Lyapunov exponents of A at x are equal.

2.3. **Distortion inside the bundles of a dominated splitting.** Let us review the basic robustness property of dominated splittings:

Proposition 2.2. *Suppose that the automorphism $A \in \mathrm{Aut}(\mathbb{E}, T)$ has a dominated splitting $\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k$. Then every automorphism $\tilde{A}$ sufficiently close to A has a dominated splitting $\tilde{\mathbb{E}}_1 \oplus \cdots \oplus \tilde{\mathbb{E}}_k$ such that, for each $i = 1, \dots, k$, the fibers of $\tilde{\mathbb{E}}_i$ have the same dimension and are uniformly close to the fibers of $\mathbb{E}_i$.*

We call $\tilde{\mathbb{E}}_1 \oplus \cdots \oplus \tilde{\mathbb{E}}_k$ the *continuation* of the originally given dominated splitting for A. We remark that the continuation of a finest dominated splitting is not necessarily finest.

For any $A \in \mathrm{Aut}(\mathbb{E}, T)$, define

$$K_{\mathrm{fine}}(A) := \max_i K(A|\mathbb{E}_i)\,,$$

where $\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k$ is the finest dominated splitting of A.

Notice that if $A \in \mathrm{Aut}(\mathbb{E}, T)$ and $\mathbb{F} \subset \mathbb{E}$ is an A-invariant subbundle then $K(A) \geq K(A|\mathbb{F})$. In particular, we have:

Proposition 2.3. $K_{\mathrm{fine}}(A) \leq K(A)$.

We use this to show the following:

Proposition 2.4. *The map* $K_{\mathrm{fine}} \colon \mathrm{Aut}(\mathbb{E}, T) \to [0, \infty)$ *is upper semicontinuous.*

Proof. Let $A \in \mathrm{Aut}(\mathbb{E}, T)$ have finest dominated splitting $\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k$, and let $\varepsilon > 0$ Let $\tilde{A}$ be a perturbation of A, and let $\tilde{\mathbb{E}}_1 \oplus \cdots \oplus \tilde{\mathbb{E}}_k$ be the continuation of the splitting, as given by Proposition 2.2. Each restriction $\tilde{A}|\tilde{\mathbb{E}}_i$ is conjugated to a perturbation of $A|\mathbb{E}_i$. Since K is upper-semicontinuous and invariant under conjugation, we have $K(\tilde{A}|\tilde{\mathbb{E}}_i) \leq K(A|\tilde{\mathbb{E}}_i) + \varepsilon$. Since the finest dominated splitting of $\tilde{A}$ refines $\tilde{\mathbb{E}}_1 \oplus \cdots \oplus \tilde{\mathbb{E}}_k$, it follows from Proposition 2.4 that $K_{\mathrm{fine}}(\tilde{A}) \leq K_{\mathrm{fine}}(A) + \varepsilon$. ☐

Notice that the set $\mathcal{R}$ from the statement of the Main Theorem 1.1 (or from Proposition 1.2, which is now obvious) is precisely $\{A \in \mathrm{Aut}(\mathbb{E}, T);\ K_{\mathrm{fine}}(A) = 0\}$, which by the proposition above is a G_δ set. The hard part of the proof of the Main Theorem is to show that $\mathcal{R}$ is dense.

Actually, we will see later that $\mathcal{R}$ is the set of points of continuity of K_{fine}, and therefore it is a residual set. However, it is not convenient to work with K_{fine} directly. We will introduce alternative ways of measuring quasiconformal distortion that will turn out to be more appropriate.

2.4. **Another measure of quasiconformal distortion.** Let E and F be inner product spaces of dimension d and let $L \colon E \to F$ be an isomorphism. Recall that $\mathfrak{s}_1(L) \geq \cdots \geq \mathfrak{s}_d(L)$ denote the singular values of L. Let $\lambda_i(L) := \log \mathfrak{s}_i(L)$. Define also

$$\sigma_0(L) := 0 \quad \text{and} \quad \sigma_i(L) := \lambda_1(L) + \cdots + \lambda_i(L) \text{ for } i = 1, \ldots, d.$$

In particular, $\sigma_1(L) = \log \|L\|$ and $\sigma_d(L) = \log |\det L|$.

Consider the graph of the function $i \in \{0, 1, \ldots, d\} \mapsto \sigma_i(L) \in \mathbb{R}$. By affine interpolation we obtain a graph over the interval $[0, d]$, which we call the *σ-graph* of L. The fact that the sequence $\lambda_i(L)$ is non-increasing means that this graph is concave. In particular, the σ-graph of L is above the line joining $(0, 0)$ and $(d, \sigma_d(L))$. Let us define $\zeta(L)$ as the area between this line and the σ-graph (see Fig. 1). This amounts to:

$$\zeta(L) = \sigma_1(L) + \sigma_2(L) + \cdots + \sigma_{d-1}(L) - \left(\frac{d-1}{2}\right) \sigma_d(L). \tag{2.3}$$

Of course, $\zeta(L) \geq 0$, and equality holds if and only if all singular values of L are equal, i.e., L is conformal. (Actually, it is not difficult to show that for every fixed dimension d, each quantity κ and ζ is bounded by an uniform multiple of the other.)

Like κ, the functions we have just defined enjoy the property of subadditivity:

Proposition 2.5. *The functions* σ_1, …, σ_{d-1} *and* ζ *are subadditive and the function* σ_d *is additive.*

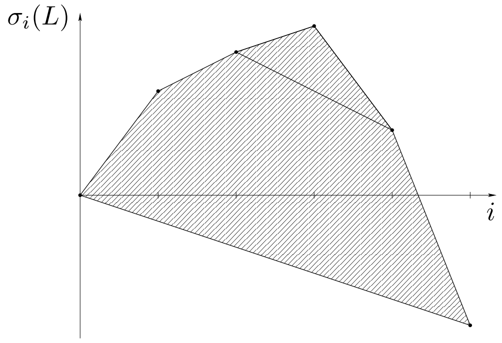

FIGURE 1. The upper curve is the σ-graph of some L. The shaded area is $\zeta(L)$. The area of the marked triangle is $\gamma_3(L)$.

Proof. We recall some facts about exterior powers (see e.g. [Ar, § 3.2.3]). Let $\wedge^i E$ denote the i-th exterior power of E. The inner product in E induces an inner product on $\wedge^i E$; actually if $\{e_1, \dots, e_d\}$ is an orthonormal basis of E then $\{e_{j_1} \wedge \cdots \wedge e_{j_i};\ 1 \le j_1 < \cdots < j_i \le d\}$ is an orthonormal basis of $\wedge^i E$. The isomorphism $L\colon E \to F$ induces an isomorphism $\wedge^i L\colon \wedge^i E \to \wedge^i F$, and its norm is:

$$\|\wedge^i L\| = \exp \sigma_i(L)\,.$$

Since operator norms are submultiplicative, it follows that $\sigma_i(\cdot)$ is subadditive. Moreover, since $\wedge^d E$ is 1-dimensional, $\sigma_d(\cdot)$ is additive. It follows from the definition (2.3) that $\zeta(\cdot)$ is subadditive. □

Let us introduce other quantities that will be used later, namely the following "half-gaps" between the λ's:

$$\gamma_i(L) := \frac{\lambda_i(L) - \lambda_{i+1}(L)}{2} = \frac{-\sigma_{i-1}(L) + 2\sigma_i(L) - \sigma_{i+1}(L)}{2}\,, \quad (i = 1, \dots, d-1).$$

Geometrically, these numbers are the areas of the triangles determined by three consecutive vertices in the σ-graph: see Fig. 1. In particular, $\gamma_i(L) \le \zeta(L)$ for each i. On the other hand, the maximal half-gap is comparable to $\zeta(L)$, as the following lemma shows:

Lemma 2.6. *If L is an isomorphism between inner product spaces of dimension $d \ge 2$ then*

$$\max_{i \in \{1, \dots, d-1\}} \gamma_i(L) \ge b_d\, \zeta(L)\,,$$

where $b_d \in (0, 1]$ is a constant depending only on d.

Proof. A calculation shows that $\zeta(L) = \sum_{i=1}^{d-1} i(d-i)\gamma_i(L)$. Therefore the lemma holds with

$$b_d := \left(\sum_{i=1}^{d-1} i(d-i)\right)^{-1} = \frac{6}{d(d^2-1)}\,. \qquad \square$$

Of course, Lemma 2.6 is just a property about concave graphs. Despite its simplicity, this property will play a significant role here, as is does (to a lesser extent) in [BB].

2.5. **Maximal quantities.** Given $A \in \mathrm{Aut}(\mathbb{E}, T)$, we define

$$Z(A) := \inf_{n \ge 1} \frac{1}{n} \sup_{x \in X} \zeta(A^n(x))\,. \tag{2.4}$$

Then the function $Z\colon \mathrm{Aut}(\mathbb{E}, T) \to [0, \infty)$ is upper semicontinuous.

The analog of formula (2.2) for Z is:

$$Z(A) = \sup_{\mu \in \mathcal{M}(T)} \zeta\Big(\operatorname{diag}\big(\chi_1(A,\mu), \chi_2(A,\mu), \dots, \chi_d(A,\mu)\big)\Big) . \tag{2.5}$$

For any $A \in \operatorname{Aut}(\mathbb{E}, T)$, define

$$K_{\mathrm{fine}}(A) := \max_i K(A|\mathbb{E}_i) ,$$

where $\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k$ is the finest dominated splitting of A.

Proposition 2.7. *$Z_{\mathrm{fine}}(A) \le Z(A)$ for every $A \in \operatorname{Aut}(\mathbb{E}, T)$.*

Proof. Let $A \in \operatorname{Aut}(\mathbb{E}, T)$, and let $E_1 \oplus \cdots \oplus E_k$ be the finest dominated splitting of A. Take i such that $K(A|\mathbb{E}_i) = K_{\mathrm{fine}}(A)$. Let $m := \dim(\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_{i-1})$ and $\ell := \dim \mathbb{E}_i$. Applying (2.5) to the automorphism $A|\mathbb{E}_i$, we have

$$Z(A|\mathbb{E}_i) = \sup_{\mu \in \mathcal{M}(T)} \zeta\Big(\operatorname{diag}\big(\chi_{m+1}(A,\mu), \dots, \chi_{m+\ell}(A,\mu)\big)\Big) .$$

It follows from the interpretation of ζ as an area that

$$\zeta\Big(\operatorname{diag}\big(\chi_{m+1}(A,\mu), \dots, \chi_{m+\ell}(A,\mu)\big)\Big) \le \zeta\Big(\operatorname{diag}\big(\chi_1(A,\mu), \dots, \chi_d(A,\mu)\big)\Big) ,$$

for every $\mu \in \mathcal{M}(T)$. Therefore $Z(A|\mathbb{E}_i) \le Z(A)$, as we wanted to show. □

Using Proposition 2.7 instead Proposition 2.3, the same argument that proved Proposition 2.4 yields:

Proposition 2.8. *The map $Z_{\mathrm{fine}} \colon \operatorname{Aut}(\mathbb{E}, T) \to [0, \infty)$ is upper semicontinuous.*

Of course, Z (resp. Z_{fine}) vanishes if and only if K (resp. K_{fine}) vanishes. Actually the main conclusions of §§ 2.2 and 2.3 could have been obtained using the functions Z and Z_{fine} instead; but we have preferred the proofs that seemed more natural.

2.6. **Setting up the proof.** In the next sections, we will prove the following:

Lemma 2.9. *Let T be a minimal homeomorphism of a space of finite dimension. Then for every $\varepsilon > 0$ there exists $\tilde{A} \in \operatorname{Aut}(\mathbb{E}, T)$ such that $\|\tilde{A}(x) - A(x)\| < \varepsilon$ for each $x \in X$ and*

$$Z_{\mathrm{fine}}(\tilde{A}) < a_d Z_{\mathrm{fine}}(A) + \varepsilon ,$$

where $a_d \in (0, 1)$ is a constant depending only on the dimension d.

An immediate consequence of Lemma 2.9 is that A is a point of continuity of the function $Z_{\mathrm{fine}}(\cdot)$ if and only if $Z_{\mathrm{fine}}(A) = 0$. Since the points of continuity of a semicontinuous function on a Baire space form a residual set, the Main Theorem 1.1 follows.

Therefore we are reduced to proving Lemma 2.9. Actually, if suffices to prove it in the particular case that A has no nontrivial dominated splitting:

Proof of the general case assuming the particular case. Assume that Lemma 2.9 is already proved for automorphisms of bundles of any dimension without nontrivial dominated splittings, thus providing a sequence (a_d). Replacing each a_d with $\max(a_1, \dots, a_d)$, we can assume that this sequence is nondecreasing.

Let $A \in \operatorname{Aut}(\mathbb{E}, T)$, and let $\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k$ be the finest dominated splitting of A. Let $\varepsilon > 0$, and take a positive $\varepsilon' \ll \varepsilon$. Each restriction $A|\mathbb{E}_i$ is an automorphism with no dominated splitting and therefore, by the particular case, there exists an ε'-perturbation $B_i \in \operatorname{Aut}(\mathbb{E}_i, T)$ such that $Z(B_i) < a_d Z(A|\mathbb{E}_i) + \varepsilon'$. Let $\tilde{A} \in \operatorname{Aut}(\mathbb{E}, T)$ be such that $\tilde{A}|\mathbb{E}^i = B_i$; then $\tilde{A}$ is ε-close to A. The finest dominated splitting of $\tilde{A}$ refines $\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_k$, and thus by Proposition 2.7,

$$Z_{\mathrm{fine}}(\tilde{A}) \le \max_i Z(\tilde{A}|\mathbb{E}_i) \le \max_i \big(a_d Z(\mathbb{E}_i) + \varepsilon\big) = a_d Z_{\mathrm{fine}}(A) + \varepsilon . \qquad \square$$

Remark 2.10. The validity of Lemma 2.9 is equivalent to the validity of an analog statement for K_{fine}. The reason why Z_{fine} is more convenient to work with is that we know how to prove (the particular case of) Lemma 2.9 with a single perturbation, while producing a discontinuity of K_{fine} would probably require a more complicated procedure. ◁

Remark 2.11. Other upper semicontinuous functions on $\mathrm{Aut}(\mathbb{E}, T)$ that suggest themselves are:

$$\Sigma_i(A) := \inf_{n \geq 1} \frac{1}{n} \sup_{x \in X} \sigma_i(A^n(x)), \quad i = 1, \dots, d.$$

At first sight, these may seem the "right" functions to consider, especially since the proof from [BV] consists in finding a discontinuity of an analogue function (where the sup is replaced by an integral). However, it is not clear how to actually use these functions to prove the Main Theorem 1.1. ◁

3. Reducing non-conformality along segments of orbit

This section is devoted to the proof of the following result, which plays a role similar to Lemma 2 in [AB]:

Main Lemma 3.1. *Suppose that T is minimal and without periodic orbits, $A \in \mathrm{Aut}(\mathbb{E}, T)$ has no nontrivial dominated splitting, and $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ with the following properties: For every $x \in X$ and every $n \geq N$, there exist a sequence of linear maps*

$$\mathbb{E}(x) \xrightarrow{L_0} \mathbb{E}(Tx) \xrightarrow{L_1} \mathbb{E}(T^2x) \xrightarrow{L_2} \cdots \xrightarrow{L_{n-1}} \mathbb{E}(T^n x)$$

with $\|L_j - A(T^j(x))\| < \varepsilon$ for each j and such that

$$\frac{1}{n} \zeta\big(L_{n-1} \dots L_0\big) < a_d \, Z(A) + \varepsilon \, .$$

where $a_d \in (0, 1)$ is a constant depending only on the dimension d.

3.1. Preliminary lemmas.

If $\mathbb{E}_1 \oplus \cdots \oplus E_k$ is a nontrivial dominated splitting for some $A \in \mathrm{Aut}(\mathbb{E}, T)$, then its *indices* are the numbers:

$$\dim(\mathbb{E}_1), \ \dim(\mathbb{E}_1 \oplus \mathbb{E}_2), \ \dots, \ \dim(\mathbb{E}_1 \oplus \cdots \oplus \mathbb{E}_{k-1}).$$

We will need the following implicit characterization of these indices:

Theorem 3.2 ([BG, Thrm. A])**.** *An automorphism $A \in \mathrm{Aut}(\mathbb{E}, T)$ has a dominated splitting of index i if and only if there exist $c > 0$, $\tau > 1$ such that*

$$\frac{\mathfrak{s}_i(A^n(x))}{\mathfrak{s}_{i+1}(A^n(x))} > c\tau^n \quad \textit{for all } x \in X \textit{ and } n \geq 0.$$

In other words, the indices of domination correspond to exponentially large gaps between the singular values.

Absence of domination permits us to significantly change the orbits of vectors by performing small perturbations. One operation of this kind is described by the following lemma:

Lemma 3.3. *Assume that $A \in \mathrm{Aut}(\mathbb{E}, T)$ has no dominated splitting of index i. Then for every $\varepsilon > 0$ there exist $m \in \mathbb{N}$ and a nonempty open set $W \subset X$ with the following properties: For every $x \in W$ and every pair of subspaces $E \subset \mathbb{E}(x)$, $F \subset \mathbb{E}(T^m x)$ with respective dimensions i and $d - i$, there exist a sequence of linear maps*

$$\mathbb{E}(x) \xrightarrow{L_0} \mathbb{E}(Tx) \xrightarrow{L_1} \mathbb{E}(T^2x) \xrightarrow{L_2} \cdots \xrightarrow{L_{m-1}} \mathbb{E}(T^m x)$$

with $\|L_j - A(T^j x)\| < \varepsilon$ *for each* j *and such that*

$$L_{m-1} \cdots L_0(E) \cap F \neq \{0\}.$$

For the proof, we will need the following standard result, which can be shown by the same arguments as in the proof of [BV, Prop. 7.1].

Lemma 3.4. *For any* $C > 0$ *and any* $\alpha > 0$*, there exists* $m \in \mathbb{N}$ *with the following properties. If* $L_0, L_1, \ldots, L_{k-1} \in \mathrm{GL}(d, \mathbb{R})$ *satisfy* $\|L_k^{\pm 1}\| \le C$*, and* $v, w \in \mathbb{R}^d$ *are non-zero vectors such that*

$$\frac{\|L_{k-1} \cdots L_0 w\| \,/\, \|w\|}{\|L_{k-1} \cdots L_0 v\| \,/\, \|v\|} > \frac{1}{2}\,,$$

then there exist non-zero vectors $u_0, u_1, \ldots, u_k \in \mathbb{R}^d$ *such that* $u_0 = v$*,* $u_k = L_{k-1} \cdots L_0(w)$*, and*

$$\measuredangle\big(u_{j+1}, L_j(u_j)\big) < \alpha \quad \text{for each } j = 0, \ldots, k-1.$$

Proof of Lemma 3.3. Suppose $A \in \mathrm{Aut}(\mathbb{E}, T)$ has no dominated splitting of index i. Let $\varepsilon > 0$ be given. Let $C > 1$ be such that $\|A(x)^{\pm 1}\| \le C$ for all $x \in X$. Fix a positive $\alpha \ll \varepsilon$, and let $k = k(C, \alpha) \in \mathbb{N}$ be given by Lemma 3.4. Define open sets

$$W(m) := \left\{ x \in M;\ \frac{\mathfrak{s}_{i+1}(A^m(x))}{\mathfrak{s}_i(A^m(x))} > C^{2k} (1/2)^{m/k-1} \right\}.$$

Notice that if $W(m) = \varnothing$ for every sufficiently large m then by Theorem 3.2 there is a dominated splitting of index i, contradicting the hypothesis. Therefore we can fix $m > k$ such that $W = W(m) \neq \varnothing$.

Now fix a point $x \in W$ and spaces $E \subset \mathbb{E}(x)$, $F \subset \mathbb{E}(T^m x)$ with respective dimensions i and $d - i$. For simplicity, write $P = A^m(x)$.

Claim 3.5. There exist unit vectors $v \in E$ and $w \in P^{-1}(F)$ such that $\|Pv\| \le \mathfrak{s}_i(P)$ and $\|Pw\| \ge \mathfrak{s}_{i+1}(P)$.

Proof of the claim. Let $\{e_1, \ldots, e_d\}$ be a basis of $\mathbb{E}(x)$ formed by eigenvectors of $(P^* P)^{1/2}$ corresponding to the eigenvalues $\mathfrak{s}_1(P) \ge \cdots \ge \mathfrak{s}_d(P)$. Let $\tilde{E}$ be the space spanned by $e_i, \ldots, e_d$. Since $\dim E = i$, the intersection $E \cap \tilde{E}$ contains a unit vector v. Then $\|Pv\| \le \mathfrak{s}_i(P)$, proving the first part of the claim. The proof of the second part is analogous. ☐

Claim 3.6. There exists ℓ with $0 \le \ell < m - k$ such that

$$\frac{\|A^{k+\ell}(x) \cdot w\| \,/\, \|A^\ell(x) \cdot w\|}{\|A^{k+\ell}(x) \cdot v\| \,/\, \|A^\ell(x) \cdot v\|} > \frac{1}{2}\,.$$

Proof of the claim. Assume the contrary. It follows that:

$$\frac{\mathfrak{s}_{i+1}(P)}{\mathfrak{s}_i(P)} \le \frac{\|Pw\|}{\|Pv\|} \le \left(\frac{1}{2}\right)^{\lfloor m/k \rfloor} C^{2k}\,,$$

which contradicts the fact that $x \in W$. ☐

Next we apply Lemma 3.4 to the vectors $\tilde{v} = A^\ell(x) \cdot v$, $\tilde{w} = A^\ell(x) \cdot w$ and the linear maps $\tilde{L}_0 = A(T^\ell x), \ldots, \tilde{L}_{k-1} = A(T^{\ell+k-1} x)$. We obtain non-zero vectors $u_0, \ldots, u_k$ such that $u_0 = v$, $u_k = A^{\ell+k}(x) \cdot w$, and $\measuredangle(u_{j+1}, A(T^{\ell+j} x) \cdot u_j) < \alpha$ for each $j = 0, \ldots, k-1$.

To conclude the proof, we need to define the linear maps $L_0, \ldots, L_{m-1}$. Since α is small, for each $j = 0, \ldots, k-1$ we can find an ε-perturbation $L_{\ell+j}$ of $A(T^{\ell+j} x)$ such that $L_j(u_j)$ and u_{j+1} are collinear. We define the remaining maps as:

$$L_j = A(T^j x) \quad \text{if } 0 \le j \le \ell \text{ or } \ell + k \le j \le m\,.$$

Then $L_{m-1} \ldots L_0(v)$ is collinear to $A^m(w)$. This proves Lemma 3.3. ☐

The next lemma indicates how the perturbations that Lemma 3.3 provides can be used to manipulate singular values. For simplicity of notation, we state the lemma in terms of matrices instead of bundle maps.

Lemma 3.7. *Let P, $Q \in \mathrm{GL}(d,\mathbb{R})$ and $i \in \{1,\dots,d-1\}$. Then there are subspaces E, $F \subset \mathbb{R}^d$ with respective dimensions i, $d-i$ and with the following property: If $R \in \mathrm{GL}(d,\mathbb{R})$ satisfies $R(E) \cap F \neq \{0\}$ then*

$$\sigma_i(QRP) \le \sigma_i(P) + \sigma_i(Q) - 2\min\big\{\gamma_i(P), \gamma_i(Q)\big\} + c_d \max\{1, \log\|R\|\}\,,$$

where $c_d > 0$ depends only on d.

A similar estimate appears in the proof of [BV, Prop 4.2].

Proof. Let P, Q, and i be given. Fix an orthonormal basis $\{e_1,\dots,e_d\}$ of eigenvectors of $(PP^*)^{1/2}$ corresponding to the eigenvalues $\mathfrak{s}_1(P)$, …, $\mathfrak{s}_d(P)$, and let E be the subspace spanned e_1, …, e_i. Analogously, fix an orthonormal basis $\{f_1,\dots,f_d\}$ of eigenvectors of $(Q^*Q)^{1/2}$ corresponding to the eigenvalues $\mathfrak{s}_1(Q)$, …, $\mathfrak{s}_d(Q)$, and let F be the subspace spanned by f_{i+1}, …, f_d.

Now take $R \in \mathrm{GL}(d,\mathbb{R})$ such that $R(E) \cap F \neq \{0\}$.

Define also $\bar{e}_j := \mathfrak{s}_j(P)\, P^{-1}(e_j)$ and $\bar{f}_j := (\mathfrak{s}_j(Q))^{-1}\, Q(e_j)$, for $j = 1,\dots,d$. Then $\{\bar{e}_1,\dots,\bar{e}_d\}$ and $\{\bar{f}_1,\dots,\bar{f}_d\}$ are orthonormal bases formed by eigenvectors of $(P^*P)^{1/2}$ and $(QQ^*)^{1/2}$, respectively.

As in the proof of Proposition 2.5, we will use exterior powers. Consider the following subsets of $\wedge^i\mathbb{R}^d$:

$$\begin{aligned}
\mathcal{B}_0 &= \big\{\bar{e}_{j_1} \wedge \cdots \wedge \bar{e}_{j_i};\ 1 \le j_1 < \cdots < j_i \le d\big\},\\
\mathcal{B}_1 &= \big\{e_{j_1} \wedge \cdots \wedge e_{j_i};\ 1 \le j_1 < \cdots < j_i \le d\big\},\\
\mathcal{B}_2 &= \big\{f_{j_1} \wedge \cdots \wedge e_{j_i};\ 1 \le j_1 < \cdots < j_i \le d\big\},\\
\mathcal{B}_3 &= \big\{\bar{f}_{j_1} \wedge \cdots \wedge \bar{f}_{j_i};\ 1 \le j_1 < \cdots < j_i \le d\big\},
\end{aligned}$$

each of them endowed with the lexicographical order. These are all orthonormal bases of $\wedge^i\mathbb{R}^d$. We represent the maps $\wedge^i P$, $\wedge^i R$, $\wedge^i Q$ as $\binom{d}{i} \times \binom{d}{i}$ matrices $\mathbf{P}$, $\mathbf{R}$, $\mathbf{Q}$ with respect to these bases

$$(\wedge^i\mathbb{R}^d, \mathcal{B}_0) \xrightarrow{\wedge^i P} (\wedge^i\mathbb{R}^d, \mathcal{B}_1) \xrightarrow{\wedge^i R} (\wedge^i\mathbb{R}^d, \mathcal{B}_2) \xrightarrow{\wedge^i Q} (\wedge^i\mathbb{R}^d, \mathcal{B}_3)\,.$$

Then the matrices $\mathbf{P}$ and $\mathbf{Q}$ are diagonal with positive diagonal entries. The biggest and the second biggest entries of $\mathbf{P}$ are respectively

$$\mathbf{P}_{11} = \mathfrak{s}_1(P)\dots\mathfrak{s}_i(P) \quad \text{and} \quad \mathbf{P}_{22} = \mathfrak{s}_1(P)\dots\mathfrak{s}_{i-1}(P)\mathfrak{s}_{i+1}(P)\,.$$

Analogously, the biggest and the second biggest entries of $\mathbf{Q}$ are respectively

$$\mathbf{Q}_{11} = \mathfrak{s}_1(Q)\dots\mathfrak{s}_i(Q) \quad \text{and} \quad \mathbf{Q}_{22} = \mathfrak{s}_1(Q)\dots\mathfrak{s}_{i-1}(Q)\mathfrak{s}_{i+1}(Q)\,.$$

Claim 3.8. $\mathbf{R}_{11} = 0$.

Proof of the claim. By assumption, there exist a non-zero vectors $w \in E \cap R^{-1}(F)$. Choose $\ell \in \{1,\dots,i\}$ such that $\{e_1,\dots,e_{\ell-1}, w, e_{\ell+1},\dots,e_i\}$ is a basis for E. Therefore the first element of the basis $\mathcal{B}_1$ is a multiple of $\xi := e_1 \wedge \cdots \wedge e_{\ell-1} \wedge w \wedge e_{\ell+1} \wedge \cdots \wedge e_i$. We have

$$(\wedge^i R)(\xi) = R(e_1) \wedge \cdots \wedge R(e_{\ell-1}) \wedge R(w) \wedge R(e_{\ell+1}) \wedge \cdots \wedge R(e_i)\,.$$

Write each $R(e_j)$ as a linear combination of vectors f_1, …, f_d, write $R(w)$ (which is in F) as a linear combination of vectors f_{i+1}, …, f_d, and substitute in the expression above. We obtain a linear combination of vectors $f_{j_1} \wedge \cdots \wedge f_{j_i}$ where $f_1 \wedge \cdots \wedge f_i$ does not appear. This means that the first coordinate of $(\wedge^i R)(\xi)$ with respect to the basis $\mathcal{B}_2$ is zero. Therefore $\mathbf{R}_{11} = 0$. □

Now let $\mathbf{M} = \mathbf{QRP}$, i.e., the matrix that represents $\wedge^i(QRP)$ with respect to the bases $\mathcal{B}_0$ and $\mathcal{B}_3$. Then the norm of $\mathbf{M}$ is $\exp \sigma_i(QRP)$. This norm is comparable to $\max_{\alpha,\beta} |\mathbf{M}_{\alpha\beta}|$. We estimate each entry as follows:

$$|\mathbf{M}_{\alpha\beta}| = \mathbf{Q}_{\alpha\alpha}\, |\mathbf{R}_{\alpha\beta}|\, \mathbf{P}_{\beta\beta} \le |\mathbf{R}_{\alpha\beta}| \, \max\{\mathbf{Q}_{11}\mathbf{P}_{22}, \mathbf{Q}_{22}\mathbf{P}_{11}\}\,.$$

On one hand, $\max_{\alpha,\beta} |\mathbf{R}_{\alpha\beta}|$ is comparable to $\|\mathbf{R}\| = e^{\sigma_i(R)} \le \|R\|^i$. On the other hand,

$$\log(\mathbf{Q}_{11}\mathbf{P}_{22}) = \sigma_i(P) + \sigma_i(Q) - 2\gamma_i(P), \quad \log(\mathbf{Q}_{22}\mathbf{P}_{11}) = \sigma_i(P) + \sigma_i(Q) - 2\gamma_i(Q),$$

and so the lemma follows. □

3.2. **Proof of the Main Lemma 3.1.** First, let us give an outline of the proof. If the segment of orbit $\{x, Tx, \dots, T^{n-1}x\}$ is long, then by minimality it will regularly visit the sets from Lemma 3.3 where the lack of domination is manifest. We will choose a single one of those visits, and then perform a perturbation of the kind given by Lemma 3.3 on a relatively short subsegment, in order to obtain by Lemma 3.7 a drop in one σ_i value of the long product. We have to assure ourselves that this drop is a significant one.

Similar strategies are used in [AB] and [BV]. In [BV], the short perturbative subsegment is chosen basically halfway along the segment; that this is a suitable position for perturbation is a consequence of Oseledets theorem. In the minimal $\mathrm{SL}(2,\mathbb{R})$ situation considered in [AB], the middle position is not necessarily the most convenient one, but nevertheless it is easy to see that there exists a suitable position that produces a big drop.

The considerations here are more delicate. We actually apply Lemmas 3.3 and 3.7 to the index i_0 which maximizes the half-gap $\gamma_{i_0}(A^n(x))$ and so is likely to produce a bigger drop in the ζ-area (see Fig. 1). Suppose we break $A^n(x) = QP$ into left and right unperturbed subsegments (disregarding the short middle term). Similarly to [AB], we choose the breaking point so that $\gamma_{i_0}(P) \simeq \gamma_{i_0}(Q)$. Then we need to estimate the drop in ζ. By subadditivity, $\sigma_i(A^n(x)) \le \sigma_i(P) + \sigma_i(Q)$ for each i. On the other hand, since the lengths k and $n-k$ of P and Q are big, the values $k^{-1}\zeta(P)$ and $(n-k)^{-1}\zeta(Q)$ are essentially bounded by $Z(A)$. We can assume that for the point x under consideration, the value $n^{-1}\zeta(A^n(x))$ is already sufficiently close to $Z(A)$, because otherwise no perturbation is needed. It follows that $\zeta(A^n(x)) \simeq \zeta(P) + \zeta(Q)$ and therefore $\sigma_i(A^n(x)) \simeq \sigma_i(P) + \sigma_i(Q)$ for each i. This allows us to recover an "Oseledets-like" situation and carry on the estimates easily. The actual argument is more subtle, because in order to prove the Main Lemma we need to consider points x such that $n^{-1}\zeta(A^n(x))$ is close, but not extremely close, to $Z(A)$. We proceed with the formal proof.

Proof of the Main Lemma. Let $b = b_d$ be given by Lemma 2.6, and define

$$a = a_d := \frac{1}{1 + b/2}\,. \tag{3.1}$$

Let $A \in \mathrm{Aut}(\mathbb{E}, T)$ be without nontrivial dominated splitting, and let $\varepsilon > 0$. Take a positive number $\delta \ll \varepsilon$; how small it needs to be will become apparent along the proof.

For each $i = 1, \dots, d-1$, we apply Lemma 3.3 and thus obtain an integer m_i and a nonempty open set $W_i \subset X$ with the following property: along segments of orbits of length m_i starting from W_i, we can ε-perturb the linear maps in order to make any given i-dimensional space intersect any given $(d-i)$-dimensional space.

Since T is minimal, there exists $m' \in \mathbb{N}$ such that

$$\bigcup_{j=0}^{m'} T^j(W_i) = X \quad \text{for each } i = 1, \dots, d-1. \tag{3.2}$$

Let also $m'' \in \mathbb{N}$ be such that

$$j \ge m'' \;\Rightarrow\; \zeta(A^j(y)) < \big(Z(A) + \delta\big) j, \ \forall y \in X\,. \tag{3.3}$$

Take

$$N \ge \delta^{-1} \max\{m_1, \dots, m_{d-1}, m', m''\}. \tag{3.4}$$

Fix any point $x \in X$ and any $n \ge N$. We can assume that

$$\frac{1}{n}\zeta(A^n(x)) \ge a\, Z(A)\,, \tag{3.5}$$

because otherwise the unperturbed maps $L_j = A(T^j(x))$ satisfy the conclusion of the Main Lemma.

Let $i_0 \in \{1, \dots, d-1\}$ be such that $\gamma_{i_0}(A^n(x)) = \max_i \gamma_i(A^n(x))$. Thus, by Lemma 2.6,

$$\gamma_{i_0}(A^n(x)) \ge b\, \zeta(A^n(x))\,. \tag{3.6}$$

Let us write $m_0 = m_{i_0}$, for simplicity. Given an integer $k \in [0, n - m_0]$, we factorize $A^n(x)$ as $Q_k R_k P_k$, where

$$P_k := A^k(x), \quad R_k := A^{m_0}(T^k x), \quad Q_k := A^{n-k-m_0}(T^{k+m_0} x)\,;$$

In what follows, we will use big O notation; the comparison constants are allowed to depend only on A (and d).

Claim 3.9. We can find $k \in [m'', n - m_0 - m'']$ such that $T^k x \in W_{i_0}$ and

$$\big|\gamma_{i_0}(P_k) - \gamma_{i_0}(Q_k)\big| \le O(\delta n)\,. \tag{3.7}$$

Proof of the claim. Notice the following facts:

- $\big|\gamma_{i_0}(A^{j+1}(x)) - \gamma_{i_0}(A^j(x))\big| \le O(1)$ for every j.
- So, letting $\Delta_j := \gamma_{i_0}(A^j(x)) - \gamma_{i_0}(A^{n-j}(T^j x))$, we have $|\Delta_{j+1} - \Delta_j| \le O(1)$.
- Since $\Delta_0 = -\Delta_n$, there exists $j_0 \in [0, n]$ such that $|\Delta_{j_0}| \le O(1)$.
- So there exists $j_1 \in [m'', n - m_0 - m'']$ such that $|\Delta_{j_1}| \le O(m'' + m_0)$.
- So, by (3.2), there exists $k \in [m'', n - m_0 - m'']$ such that $T^k x \in W_{i_0}$ and $|\Delta_k| \le O(m'' + m_0 + m')$.

Since the right hand side of (3.7) is $\le |\Delta_k| + O(m_0)$, the claim follows from (3.4). ☐

Let k be fixed from now on, and write $P = P_k$, $R = R_k$, $Q = Q_k$.

Let $E \subset \mathbb{E}(T^k x)$ and $F \subset \mathbb{E}(T^{k+m_0} x)$ be the subspaces with respective dimensions i_0 and $d - i_0$ obtained by applying Lemma 3.7 to the maps P and Q. Since $T^k x \in W_{i_0}$, we can apply Lemma 3.3 and find linear maps $\tilde{L}_j \colon \mathbb{E}(T^{k+j} x) \to \mathbb{E}(T^{k+j+1} x)$ (where $j = 0, \dots, m_0 - 1$) each ε-close to the respective $A(T^{k+j} x)$, whose product $\tilde{R} := \tilde{L}_{m_0-1} \cdots \tilde{L}_0$ satisfies $\tilde{R}(E) \cap F \neq \{0\}$. The maps L_j ($j = 0, \dots, n-1$) that we are looking for are $L_j = \tilde{L}_{j-k}$ if $k \le j < k + m_0$, and $L^j = A(T^j x)$ otherwise. So their product is $L_{n-1} \cdots L_0 = Q\tilde{R}P$. Notice that $\|\tilde{R}\| \le O(m_0) \le O(\delta n)$. Therefore Lemma 3.7 gives:

$$\sigma_{i_0}(Q\tilde{R}P) \le \sigma_{i_0}(P) + \sigma_{i_0}(Q) - 2\min\big\{\gamma_{i_0}(P), \gamma_{i_0}(Q)\big\} + O(\delta n). \tag{3.8}$$

To conclude the proof, we need to estimate $\zeta(Q\tilde{R}P)$. Begin by noticing that, as a consequence of (3.3),

$$\zeta(P) + \zeta(Q) \le Z(A)\, n + O(\delta n). \tag{3.9}$$

Also, since $\sigma_i(R) \le O(m_0) \le O(\delta n)$, subadditivity and additivity give:

$$\sigma_i(P) + \sigma_i(Q) \begin{cases} \ge \sigma_i(A^n(x)) - O(\delta n) & \text{for each } i = 1, \dots, d-1, \\ \le \sigma_d(A^n(x)) + O(\delta n) & \text{for each } i = d. \end{cases} \tag{3.10}$$

Claim 3.10. $\zeta(P) + \zeta(Q) - \zeta(A^n(x)) \ge -\gamma_{i_0}(P) - \gamma_{i_0}(Q) + \gamma_{i_0}(A^n(x)) - O(\delta n)$.

Remark 3.11. Since Claim 3.10 is an important estimate in the proof, it is worthwhile to interpret it geometrically. Consider the concave graphs of $\sigma_i(A^n(x))$ and $\sigma_i(P) + \sigma_i(Q)$. By (3.10), modulo a small error, the first graph is below the second one and their endpoints meet. The quantities $\gamma_{i_0}(A^n(x))$ and $\gamma_{i_0}(P) + \gamma_{i_0}(Q)$ are the areas of triangles touching the corresponding graphs, as in Fig. 1. Now, if the first quantity is substantially bigger than the second quantity, then concavity forces the existence of a large hole between the two graphs, and therefore the ζ-area of the second graph is substantially bigger than the ζ-area of the first one. ◁

Proof of the claim. Since the functions γ_{i_0} and ζ are invariant under composition with homothecies, we can assume for simplicity that $\sigma_d = 0$, i.e., $|\det| = 1$, for all the linear maps involved. Notice that for any L with $|\det L| = 1$, we have

$$\zeta(L) + \gamma_{i_0}(L) = \sum_{i=1}^{d-1} u_i \, \sigma_i(L), \quad \text{where } u_i := \begin{cases} 1 & \text{if } |i - i_0| > 1, \\ 1/2 & \text{if } |i - i_0| = 1, \\ 2 & \text{if } i = i_0. \end{cases}$$

In particular,

$$\zeta(P) + \gamma_{i_0}(P) + \zeta(Q) + \gamma_{i_0}(Q) - \zeta(A^n(x)) - \gamma_{i_0}(A^n(x)) = \sum_{i=1}^{d-1} u_i \underbrace{\left[\sigma_i(P) + \sigma_i(Q) - \sigma_i(A^n(x))\right]}_{\ge -\delta n \text{ (by (3.10))}} \ge -d\delta n,$$

which completes the proof of the claim. □

Next, we estimate

$$\begin{aligned} \gamma_{i_0}(P) + \gamma_{i_0}(Q) &\ge \gamma_{i_0}(A^n(x)) + \zeta(A^n(x)) - \zeta(P) - \zeta(Q) - O(\delta n) && \text{(by Claim 3.10)} \\ &\ge (b+1)\zeta(A^n(x)) - \zeta(P) - \zeta(Q) - O(\delta n) && \text{(by (3.6))} \\ &\ge (b+1)aZ(A)\, n - Z(A)\, n - O(\delta n) && \text{(by (3.9))} \\ &= (ab + a - 1)Z(A)\, n - O(\delta n)\,. \end{aligned}$$

Therefore, using (3.7)

$$\begin{aligned} 2\min\{\gamma_{i_0}(P), \gamma_{i_0}(Q)\} &= \gamma_{i_0}(P) + \gamma_{i_0}(Q) - |\gamma_{i_0}(P) - \gamma_{i_0}(Q)| \\ &\ge (ab + a - 1)Z(A)\, n - O(\delta n) \end{aligned}$$

Substituting this into (3.8) we obtain

$$\sigma_{i_0}(Q\tilde{R}P) \le \sigma_{i_0}(P) + \sigma_{i_0}(Q) - (ab + a - 1)Z(A)\, n + O(\delta n).$$

So it follows from (3.10) that

$$\zeta(Q\tilde{R}P) \le \zeta(P) + \zeta(Q) - (ab + a - 1)Z(A)\, n + O(\delta n).$$

Using (3.9) we obtain

$$\zeta(Q\tilde{R}P) \le \underbrace{(2 - ab - a)}_{= a \text{ (by (3.1))}} Z(A)\, n + \underbrace{O(\delta n)}_{< \varepsilon n}\,.$$

This concludes the proof of the Main Lemma. □

4. Patching the perturbations

Here we will use the Main Lemma 3.1 to prove Lemma 2.9 and therefore the Main Theorem. The arguments are essentially the same as in [AB].

To begin, we recall some results from [AB] on zero probability sets.

Theorem 4.1 ([AB, Lemma 3]). *Let X be a compact space of finite dimension, and let $T\colon X \to X$ be a homeomorphism without periodic orbits. Then there exists a basis of the topology of X consisting of sets U such that ∂U has zero probability.*

This is the only place where we use the assumption that X has finite dimension. (Actually, the proof of the theorem consists in finding sets U such that no point in X visits ∂U more than $\dim X$ times.)

The next result follows from a simple Krylov–Bogoliubov argument:

Lemma 4.2 ([AB, Lemma 7]). *Let $T\colon X \to X$ be a continuous mapping of a compact space X. If $K \subset X$ is a compact set with zero probability then for every $\varepsilon > 0$, there exists an open set $V \supset K$ and $n_* \in \mathbb{N}$ such that*

$$x \in X,\ n \ge n_* \quad \Rightarrow \quad \#\{x, Tx, \dots, T^{n-1}x\} \cap V < \varepsilon n.$$

We also need the following result that decomposes the space into two Rokhlin towers:

Lemma 4.3 ([AB, Lemma 6]). *Let X be a non-discrete compact space, and let $T\colon X \to X$ be a minimal homeomorphism. Then for any $N \in \mathbb{N}$, there exists an open set $B \subset X$ such that:*

- *the return time from B to itself under iterations of T assumes only the values N and $N+1$;*
- *∂B has zero probability.*

Since we are working with non-necessarily trivial vector bundles $\mathbb{E}$, we need to introduce local coordinates.

Let us fix a finite open cover $\{\hat{D}_m\}$ of X by trivializing domains, together with bundle charts $\xi_m \colon \hat{D}_m \times \mathbb{R}^d \to \mathbb{E}$. For each $x \in \hat{D}_m$, the map $H_m(x) := \xi_m(x, \cdot)$ is an isomorphism from $\mathbb{R}^d$ to $\mathbb{E}(x)$. We can assume that there is a finer cover $\{D_m\}$ of X with $\overline{D_m} \subset \hat{D}_m$ for each m.

It is convenient to fix a constant $C > 0$ such that:

$$\left\|(H_m(x))^{\pm 1}\right\| \le C \quad \text{and} \quad \zeta(H_m(x)) \le C, \quad \forall m,\ \forall x \in D_m\,. \tag{4.1}$$

Any $B \in \mathrm{Aut}(\mathbb{E}, X)$ can be represented in local coordinates by a family of (uniformly continuous) maps $B^{(m,m')} \colon X_m \cap T^{-1}(X_{m'}) \to \mathrm{GL}(d,\mathbb{R})$ defined by:

$$B^{(m,m')}(x) := \big(H_{m'}(Tx)\big)^{-1} \circ B(x) \circ H_m(x)\,, \quad x \in X_m \cap T^{-1}(X_{m'})\,. \tag{4.2}$$

Let us call this the (m,m')-*local representation* of $B(x)$.

Now we have all the tools we need to conclude the proof.

Proof of the Lemma 2.9. As explained in § 2.6, it is sufficient to consider the particular case where the automorphism $A \in \mathrm{Aut}(\mathbb{E}, T)$ has no nontrivial dominated splitting. If the space X is discrete then it consists of a single periodic orbit, and it follows that $Z(A) = 0$. So we can assume that X is non-discrete, i.e., T has no periodic orbits.

Fix $\varepsilon > 0$; we can assume that:

$$\varepsilon < \inf_{x \in X} \mathfrak{m}(A(x))\,. \tag{4.3}$$

As a consequence, if a linear map $L\colon \mathbb{E}(x) \to \mathbb{E}(Tx)$ is such that $\|L - A(x)\| < \varepsilon$ then it is invertible; moreover $\zeta(L)$ is bounded by some $C_0 = C_0(A, \varepsilon)$. Let $\varepsilon' > 0$ be small enough so that:

$$(1 + C_0)\varepsilon' < \varepsilon/3\,, \tag{4.4}$$

$$C^2(C^2 + 1)\varepsilon' < \varepsilon\,, \tag{4.5}$$

where C as in (4.1). Let $N = N(A, \varepsilon') \in \mathbb{N}$ be given by the Main Lemma 3.1. We can assume that N is large enough so that

$$\frac{2C}{N} < \frac{\varepsilon}{3}\,. \tag{4.6}$$

Recall that $\{D_m\}$ is a cover of X by trivializing domains. By uniform continuity of the local representations (4.2), there exists $\rho > 0$ such that

$$x, y \in D_m \cap T^{-1}(D_{m'}),\ d(x, y) < \rho \Rightarrow \ \left\|A^{(m,m')}(x) - A^{(m,m')}(y)\right\| < \varepsilon'\,. \tag{4.7}$$

Choose an open cover $\{W_i\}_{i=1,\dots,k}$ of X with the following properties:

- it refines the cover $\left\{D_{m_0} \cap T^{-1}(D_{m_1}) \cap \dots \cap T^{-N-1}(D_{m_{N+1}})\right\}_{m_0,\dots,m_{N+1}}$;
- $\operatorname{diam} T^j(W_i) < \rho$ for each $i = 1, \dots, k$ and $j = 0, 1, \dots, N + 1$;
- the sets ∂W_i have zero probability.

(To guarantee the last requirement we use Theorem 4.1.) For each $i = 1, \dots, k$ and each $j = 0, 1, \dots, N + 1$, we fix an index $m(i, j)$ such that $T^j(W_i) \subset D_{m(i,j)}$.

Let B be the set given by Lemma 4.3. Let B_ℓ be the set of points in B whose first return to B occurs in time ℓ. Then $B_N = B \cap T^{-N}(B)$ and $B_{N+1} = B \smallsetminus B_N$, and in particular ∂B_ℓ has zero probability. Let

$$B_{\ell,i} := B_\ell \cap W_i \smallsetminus (W_1 \cup W_2 \cup \dots \cup W_{i-1})\,, \text{ for each } (\ell, i) \in \{N, N + 1\} \times \{1, \dots, k\}.$$

Let I be the set of pairs (ℓ, i) such that $B_{\ell,i} \neq \varnothing$. Let also J be the set of (ℓ, i, j) such that $(\ell, i) \in I$ and $0 \le j \le \ell - 1$. For each $\alpha = (\ell, i, j) \in J$, let $X_\alpha := T^j(B_{\ell,i})$. Notice that $\{X_\alpha\}_{\alpha \in J}$ is a finite partition of X. Moreover, each ∂X_α has zero probability, and so by Lemma 4.2 there exists an open set $V \supset \bigcup_{\alpha \in J} \partial X_\alpha$ and $n_* \in \mathbb{N}$ such that

$$x \in X,\ n \ge n_* \quad \Rightarrow \quad \#\{x, Tx, \dots, T^{n-1}x\} \cap V < \frac{\varepsilon' n}{N + 1}\,. \tag{4.8}$$

For each $(\ell, i) \in I$, choose a point $y_{\ell,i} \in B_{\ell,i}$. For each $j = 0, 1, \dots, \ell$, let $y_{\ell,i,j} := T^j(y_{\ell,i})$. Applying the Main Lemma 3.1, we find $L_{\ell,i,0}$, …, $L_{\ell,i,\ell-1}$ so that

$$\|L_{\ell,i,j} - A(y_{\ell,i,j})\| < \varepsilon' \quad \forall j = 0, \dots, \ell - 1, \quad \text{and} \tag{4.9}$$

$$\zeta\big(L_{\ell,i,\ell-1} \cdots L_{\ell,i,0}\big) < \big(aZ(A) + \varepsilon'\big)\ell, \tag{4.10}$$

where $a = a_d \in (0, 1)$ is a constant.

For each $\alpha = (\ell, i, j) \in J$, let $\left\{\tilde{A}_\alpha(x)\colon \mathbb{E}(x) \to \mathbb{E}(Tx)\right\}_{x \in X_\alpha}$ be the family of linear maps uniquely characterized by the following properties:

- $\tilde{A}_\alpha(y_\alpha) = L_\alpha$;
- letting $m = m(i, j)$, $m' = m(i, j + 1)$, the local representation $\tilde{A}_\alpha^{(m,m')}(x)$ does not depend on $x \in X_\alpha$.

It follows from (4.9) and (4.1) that

$$\left\|\tilde{A}_\alpha^{(m,m')}(y_\alpha) - A_\alpha^{(m,m')}(y_\alpha)\right\| < C^2\varepsilon'.$$

So, by (4.7),

$$\left\|\tilde{A}_\alpha^{(m,m')}(y_\alpha) - A_\alpha^{(m,m')}(x)\right\| < (C^2 + 1)\varepsilon'\,, \quad \text{for all } x \in X_\alpha.$$

It follows that

$$\|\tilde{A}_\alpha(x) - A(x)\| < \underbrace{C^2(C^2+1)\varepsilon'}_{<\varepsilon \text{ (by (4.5))}}, \quad \text{for all } x \in X_\alpha. \tag{4.11}$$

For every $x \in B_{\ell,i}$, the products $\tilde{A}_{\ell,i,\ell-1}(T^{\ell-1}x)\cdots\tilde{A}_{\ell,i,0}(x)$ and $L_{\ell,i,\ell-1}\cdots L_{\ell,i,0}$ have the same $(m(i,0),m(i,\ell))$-local representation. It follows from (4.10) and (4.1) that

$$x \in B_{\ell,i} \;\Rightarrow\; \zeta\Big(\tilde{A}_{\ell,i,\ell-1}(T^{\ell-1}x)\cdots\tilde{A}_{\ell,i,0}(x)\Big) < \big(aZ(A)+\varepsilon'\big)\ell + 2C. \tag{4.12}$$

Now consider the open cover $\{V\} \cup \{\operatorname{int} X_\alpha\}_{\alpha\in J}$ of X. Since X is compact Hausdorff, we can find a continuous partition of unity $\{\psi\}\cup\{\varphi_\alpha\}_{\alpha\in J}$ subordinate to this cover. For each $x \in X$, define a linear map $\tilde{A}(x)\colon \mathbb{E}(x)\to\mathbb{E}(Tx)$ by

$$\tilde{A}(x) := \psi(x)A(x) + \sum_{\alpha\in J}\varphi_\alpha(x)\tilde{A}_\alpha(x)\,.$$

By (4.11), we have $\|\tilde{A}(x)-A(x)\| < \varepsilon$, and it follows from (4.3) that $\tilde{A}(x)$ is invertible. Thus $\tilde{A}\in\operatorname{Aut}(\mathbb{E},T)$. Also, $\zeta(\tilde{A}(x)) \le C_0$ for every x.

Take n large enough so that

$$n \ge n_* \quad\text{and}\quad 2C_0N < (\varepsilon/3)n\,. \tag{4.13}$$

We will give a uniform upper bound for $\zeta(\tilde{A}^n(x))$. Fix $x\in X$ and write

$$n = p + \ell_1 + \ell_2 + \cdots + \ell_r + q$$

in such a way that the points

$$x_1 = T^p(x),\ x_2 = T^{p+\ell_1}(x),\ \ldots,\ x_{r+1} = T^{p+\ell_1+\cdots+\ell_r}(x)$$

are exactly the points in the segment of orbit x, $T(x)$, …, $T^{n-1}(x)$ that belong to B. Then p, $q\in[0,N]$ and ℓ_1, …, $\ell_r\in[N,N+1]$.

The points x_j such that $j\ne r+1$ and $\{x_j, Tx_j, \ldots, T^{\ell_j-1}x_j\}\cap V=\varnothing$ will be called *good*. By subadditivity,

$$\zeta\big(\tilde{A}^n(x)\big) \le \sum_{x_j \text{ is good}} \zeta\big(\tilde{A}^{\ell_j}(x_j)\big) + C_0\left(n - \sum_{x_j \text{ is good}} \ell_j\right).$$

Notice the following estimates:

- If x_j is good then $\zeta\big(\tilde{A}^{\ell_j}(x_j)\big)$ is less than the right hand side of (4.12) with $\ell=\ell_j$;
- There are at most $r\le N^{-1}n$ good points;
- By (4.8), the number between large brackets is at most $2N+\varepsilon' n$; equality may hold only in the case that each segment $\{x_j, Tx_j, \ldots, T^{\ell_j-1}x_j\}$ (for $j=1,\ldots,r$) contains at most one point of V.

Then we obtain:

$$\zeta\big(\tilde{A}^n(x)\big) \le (aZ(A)+\varepsilon')n + 2CN^{-1}n + C_0(2N+\varepsilon' n)\,.$$

Using (4.4), (4.6), and (4.13), we conclude that $\zeta\big(\tilde{A}^n(x)\big) < (aZ(A)+\varepsilon)n$. So $Z(\tilde{A}) < aZ(A)+\varepsilon$, as we wanted to prove. ☐

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