Classical invariants of Legendrian knots in the 3-dimensional torus

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Abstract

All knots in $\mathbb{R}^3$ possess Seifert surfaces, and so the classical Thurston-Bennequin invariant for Legendrian knots in a contact structure on $\mathbb{R}^3$ can be defined. The definitions extend easily to null-homologous knots in a 3-manifold M endowed with a contact structure ξ. We generalize the definition of Seifert surfaces and use them to define the Thurston-Bennequin invariant for all Legendrian knots, including those that are not null-homologous, in a contact structure on the 3-torus T^3. Finally, we show how to compute the Thurston-Bennequin and Maslov (or rotation) invariants in a tight oriented contact structure on T^3 using projections.

Keywords— Legendrian knots, Thurston-Bennequin invariant, Maslov invariant, contact structures, 3-torus T^3, Seifert surfaces

MSC 57R17, 57M27

1 Introduction and Statement of Results

Let ξ be an *oriented contact structure* on a smooth oriented 3-manifold M, i.e., a 2-plane field that locally is the kernel of a totally non-integrable 1-form ω, so that locally $\xi = \ker(\omega)$ with the induced orientation and $\omega \wedge d\omega$ is non-vanishing. A *knot* in a smooth 3-manifold M is a smooth embedding $\alpha : S^1 \to M$. A knot in the contact manifold (M, ξ) is *Legendrian* if α is everywhere tangent to ξ. Two Legendrian knots α_0 and α_1 are *Legendrian homotopic* if there is a smooth 1-parameter family of Legendrian knots α_t, $t \in [0, 1]$, that connects them.

The Thurston-Bennequin and Maslov numbers are well-known classical invariants of null-homologous oriented Legendrian knots in (M^3, ξ) that depend only on their Legendrian homotopy class and, for the Maslov invariant, on a fixed Legendrian vector field [1, 4]. Given a Seifert surface Σ for the Legendrian knot α in (M^3, ξ), the *Thurston-Bennequin invariant* $\mathrm{tb}(\alpha)$ is defined to be the number of times the contact plane ξ rotates relative to the tangent plane to Σ in one circuit of α. The *Maslov invariant* $\mu(\alpha)$ is the number of times the tangent vector α' rotates in ξ relative to a fixed Legendrian vector field Z in a single circuit of α. For the standard contact structure $\xi_{std} = \ker(dz - ydx)$ on $\mathbb{R}^3$, both invariants of a generic Legendrian knot can be calculated using the front and Lagrangian projections of $\mathbb{R}^3$ to $\mathbb{R}^2$ (see Sections 3.1 and 4.1).

On the 3-dimensional torus T^3, we define generalized Seifert surfaces for knots that are not null-homologous (Definition 2.1) and use them to extend the definition of the Thurston-Bennequin invariant to all Legendrian knots in an arbitrary contact structure ξ on T^3. Let $\alpha : S^1 = \mathbb{R}/\mathbb{Z} \to T^3$ be a knot in T^3 and let $\widetilde{\alpha} : \mathbb{R} \to \mathbb{R}^3$ be a lift of α to the universal cover $\widetilde{T^3} = \mathbb{R}^3$, where T^3 is identified with $\mathbb{R}^3/\mathbb{Z}^3$. If a component of $\widetilde{\alpha}$ is compact, then it is a knot on $\mathbb{R}^3$ and the usual definition of Seifert surfaces applies. Hence we usually assume the following General Hypothesis, and then prove the following Proposition.

General Hypothesis 1.1. *Each component of $\widetilde{\alpha}$ is assumed to be non-compact.*

Proposition 1.2. *Let α be a smooth knot on T^3 whose lift $\widetilde{\alpha}$ has non-compact components. Then*

1. *There is a (generalized) Seifert surface for the knot α;*

2. *If Σ_1 and Σ_2 are both Seifert surfaces for the knot α, then the relative rotation number $\rho(\Sigma_1, \Sigma_2)$, defined to be the number of times Σ_2 rotates relative to Σ_1 along one circuit of α, is zero.*

Using these Seifert surfaces, we extend the definition of the Thurston-Bennequin invariant to all Legendrian knots in (T^3, ξ).

Theorem 1. *Let α be a Legendrian knot for (T^3, ξ). Then Definition 3.1 gives a natural extension of the definition of the Thurston-Bennequin invariant for* $\mathrm{tb}(\alpha)$. *If α is contractible, then $\mathrm{tb}(\alpha) = \mathrm{tb}(\tilde{\alpha})$ where $\tilde{\alpha}$ and $\tilde{\xi}$ are the lifts of α and ξ to the universal cover $(\tilde{T}^3, \tilde{\xi})$ of (T^3, ξ).*

It is obvious that Definition 3.1 extends the usual definition in $\mathbb{R}^3$ in the sense that if α lifts to a knot with compact components $\widetilde{\alpha}$ in $\widetilde{T}^3$, then $\mathrm{tb}(\alpha) = \mathrm{tb}(\widetilde{\alpha})$. It also extends the definition of the Thurston-Bennequin invariant given by Kanda [7] for *linear Legendrian knots* on T^3, i.e., those that are isotopic to knots with constant slope, since Kanda uses an incompressible torus containing the knot to replace the Seifert surface, and in the universal cover, the torus lifts to a plane, half of which is isotopic to our generalized Seifert surface. The contact structures

$$\xi_n = \ker(\cos(2\pi nz)dx + \sin(2\pi nz)dy)$$

on $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, where n is a positive integer, are tight, and Kanda [7] has shown that for every tight contact structure ξ on T^3 there is a contactomorphism (i.e., a diffeomorphism that preserves the contact structure) from ξ to ξ_n, for some $n > 0$.

A knot in T^3 is *generic* relative to a projection $p : T^3 \to T^2$ if its curvature vanishes only at isolated points and the only singularities are transverse double points and, for the case of p_{xy} and ξ_n, cusps. Note that every Legendrian knot can be made generic by an arbitrarily small Legendrian homotopy. The following Theorem shows how to calculate both invariants of generic Legendrian knots for ξ_n using the projections $p_{xy}, p_{xz} : T^3 \to T^2$, defined by setting $p_{xy}(x, y, z) = (x, y)$ and $p_{xz}(x, y, z) = (x, z)$, where x, y, z are the coordinates modulo 1 in T^3 and in T^2.

Theorem 2. *Let α be a generic oriented Legendrian knot on (T^3, ξ_n).*

1. *For the projection p_{xy} of α,* $\mathrm{tb}(\alpha) = \mathrm{N} - \mathrm{P} + \mathrm{C}/2$, *where P and N are the numbers of positive and negative crossings and C is the number of cusps for $p_{xy} \circ \alpha$ in one circuit of α;*

2. *For the projection p_{xz} of α, $\mathrm{tb}(\alpha) = \mathrm{P} - \mathrm{N}$, where P and N are the numbers of positive and negative crossings for $p_{xz} \circ \alpha$ in one circuit of α, and there are no cusps;*

3. *For the projection p_{xy} of α, the Maslov invariant relative to the Legendrian vector field $Z = \partial/\partial z$ is $\mu(\alpha) = 1/2(C_+ - C_-)$, where C_+ and C_- are the numbers of positive and negative cusps of $p_{xy} \circ \alpha$ in one circuit of α;*

4. *For the projection p_{xz} of α, the Maslov invariant relative to $Z = \partial/\partial z$ is $\mu(\alpha) = 1/2 \sum_{t \in V} a(t)b(t)$, where $V = \{t \in S^1 \mid (x'(t), y'(t)) = (0,0)\}$ in one circuit of α, provided that $(x'(t), y'(t)) \neq (0,0)$ whenever $2nz(t) \in \mathbb{Z}$, where $a(t) = (-1)^{[2nz(t)]}$ and $b(t) = \pm 1$ according to whether $p_{xz} \circ \alpha'(t)$ is turning in the positive or negative direction in the xz-plane.*

Generalized Seifert surfaces for knots in T^3 will be defined and studied in §2. The Thurston-Bennequin invariant $\mathrm{tb}(\alpha)$ and the Maslov invariant $\mu(\alpha)$ for a generic oriented Legendrian knot in T^3 will be treated in §3 and §4, respectively. The proofs of the four assertions of Theorem 2 are given in the proofs of the Propositions 3.5, 3.3, 4.4, and 4.3, respectively, in the Subsections 3.1 and 4.1.

This paper is a continuation of the work of Fábio Souza in his masters thesis [9] at the Pontifícia Universidade Católica of Rio de Janeiro under the direction of Paul Schweitzer.

2 Generalized Seifert surfaces in T^3

In this section we consider smooth knots in T^3 and their Seifert surfaces, without reference to any contact structure, as a preparation for studying Legendrian knots and their Thurston-Bennequin and Maslov invariants in the next two sections.

Recall that a *Seifert surface* for an oriented knot (or link) α in $\mathbb{R}^3$ is a compact connected oriented surface Σ whose boundary is α with the induced orientation. Every knot and link α has Seifert surfaces, and there is a well-known method of constructing one using a regular projection of α in the plane ([2], pp. 16-18). Each crossing is replaced by a non-crossing that respects the orientation, the resulting circles are capped off by disjoint embedded disks, and then a twisted interval is inserted at each crossing, as in Figure 1. Finally the *Seifert circles*, the boundary of the resulting surface, are capped off by disjoint embedded disks.

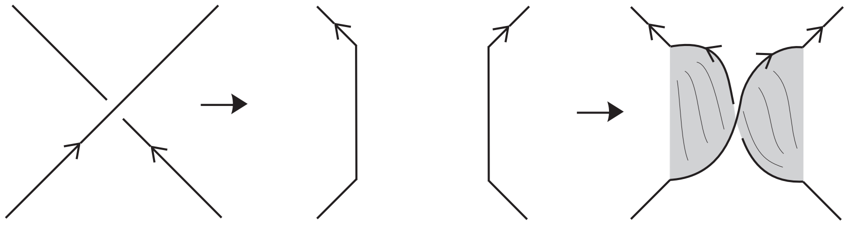

Figure 1: Inserting a twisted strip at a crossing.

Let $\alpha : S^1 = \mathbb{R}/\mathbb{Z} \to T^3$ be a knot in T^3 and let $\widetilde{\alpha} : \mathbb{R} \to \mathbb{R}^3$ be a lift of α to the universal cover $\widetilde{T^3} = \mathbb{R}^3$, where T^3 is identified with $\mathbb{R}^3/\mathbb{Z}^3$. Let $(p, q, r) \in \mathbb{Z}^3$ be a generator of the cyclic group of translations that preserve $\widetilde{\alpha}$. Note that (p, q, r) is determined up to multiplication by ± 1 and we choose the sign so that $\widetilde{\alpha}(t+1) = \widetilde{\alpha}(t) + (p, q, r)$. Any subset of $\mathbb{R}^3$ that is invariant under this group will be said to be (p, q, r)*-periodic*. The following definition adapts the classical concept of Seifert surfaces for knots in $\mathbb{R}^3$ to the present context.

Definition 2.1. Let α be a knot in T^3 whose lift $\widetilde{\alpha}$ has non-compact components and let (p, q, r) generate the cyclic group of translations that preserve $\widetilde{\alpha}$. A smooth surface $\Sigma \subset \mathbb{R}^3$ is a (generalized) *Seifert surface* for α if it satisfies the following conditions:

1. Σ is connected, orientable, properly embedded in $\mathbb{R}^3$, and (p, q, r)-periodic;
2. $\partial\Sigma = \widetilde{\alpha}$; and
3. There is an affine half-plane $P_+ \subset \mathbb{R}^3$ with boundary a straight line S such that Σ coincides with P_+ outside a δ-neighborhood N of S, for some sufficiently large δ.

If the components of $\widetilde{\alpha}$ are compact, then a Seifert surface for one of the components can be translated by the action of $\mathbb{Z}^3$ to give a periodic Seifert surface.

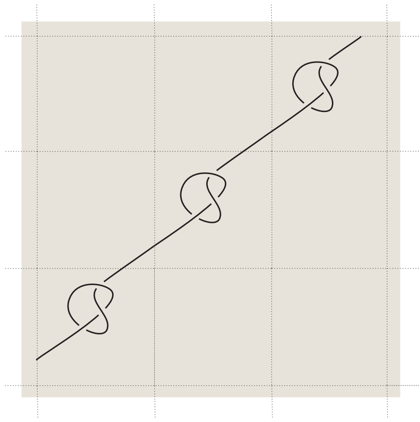

Figure 2: A (p, q, r)-periodic knot projected into an affine plane.

Clearly the half-plane P_+ and its boundary S are also (p, q, r)-periodic. It is convenient to choose P_+ to be disjoint from $\widetilde{\alpha}$ and such that $P_+ \subset \Sigma$. Given a Seifert surface Σ of α, we define the *associated vector field* $X = X(\alpha, \Sigma)$ along $\widetilde{\alpha}$ in $\mathbb{R}^3$ to be the (p, q, r)-periodic unit vector field along $\widetilde{\alpha}$ that is orthogonal to $\widetilde{\alpha}$, tangent to Σ, and directed towards the interior of Σ.

Given two Seifert surfaces Σ_i with associated vector fields X_i, $i = 1, 2$, the *relative rotation number* $\rho(\Sigma_1, \Sigma_2)$ of Σ_2 with respect to Σ_1 is defined to be the number of revolutions that X_2 makes with respect to X_1 in the positive direction in the normal plane field to $\widetilde{\alpha}$ along $\widetilde{\alpha}$ from a point on $\widetilde{\alpha}$ to its first (p, q, r)-translate in the positive direction, i.e., in one circuit of α. Note that this number is independent of the orientation of α, since changing the orientation of α also changes the orientation of the normal plane.

Proof of Proposition 1.2. First we construct a (generalized) Seifert surface for a knot α in T^3 with period (p, q, r). Choose a (p, q, r)-periodic plane P meeting the proper curve $\widetilde{\alpha}$ such that the orthogonal projection of $\widetilde{\alpha}$ onto P is regular (i.e., the only singularities are transverse double points). Fix an orientation of $\widetilde{\alpha}$. The (p, q, r)-periodicity of both $\widetilde{\alpha}$ and the affine plane P permits us to adapt the classical construction of the Seifert surface of a knot in $\mathbb{R}^3$ (see [2], pp. 16-18) in a (p, q, r)-periodic fashion. At each crossing of the image of $\widetilde{\alpha}$ in P, which we call the the knot diagram, replace the crossing by two arcs, respecting the orientation of $\widetilde{\alpha}$, and insert a twisted strip, as in Figure 1. Do this so that the resulting collection of "Seifert curves" is pairwise disjoint and (p, q, r)-periodic. The Seifert curves that are simple closed curves are capped off in a periodic fashion by mutually disjoint disks meeting P only in their boundaries. Then there will be a number of noncompact proper (p,q,r)-periodic Seifert curves left over. It is easy to check that this number will be odd, say $2k+1$, with $k+1$ of them oriented in the positive direction of $\widetilde{\alpha}$ and the other k in the opposite direction. (To see this, consider a plane perpendicular to the direction (p,q,r) that meets $\widetilde{\alpha}$ transversely and examine the sign of the intersections of $\widetilde{\alpha}$ with this plane.) These curves can be capped off in pairs with opposite orientations by disjoint oriented periodic infinite strips, starting with a pair whose projections are adjacent in the plane P. This process will leave one (p,q,r)-periodic Seifert curve which can be joined to a half plane P_+ contained in P by another infinite periodic strip so that the result is embedded. The whole construction is done so as to preserve (p,q,r)-periodicity. The result is a Seifert surface for α that coincides with P_+ outside a sufficiently large tubular neighborhood N of the line S.

Now suppose that Σ_1 and Σ_2 are two Seifert surfaces for α. Take a (p,q,r)-periodic line S in Σ_1 and a tubular neighborhood N of S sufficiently large so that the parts of Σ_1 and Σ_2 outside N are half-planes. Remove these half-planes and add an infinite periodic strip in $\partial\bar{N}$ to connect Σ_1 and Σ_2, if necessary. Thus we obtain a new proper (p,q,r)-periodic surface Σ which agrees with the union of Σ_1 and Σ_2 inside N. This surface Σ will be a proper immersed surface contained in $\bar{N}$. Note that Σ projects onto a compact oriented immersed surface on T^3. The following lemma will complete the proof, since by definition $\rho(\Sigma_1,\Sigma_2)=\rho(X_1,X_2)$. $\square$

Lemma 2.2. *Let Σ be a properly immersed (p,q,r)-periodic oriented surface in $\mathbb{R}^3$ whose boundary has two components, one being $\widetilde{\alpha}$ with the positive orientation and the other $\widetilde{\alpha}$ with the negative orientation, and which projects to a compact surface in $\widehat{T}^3=\mathbb{R}^3/\mathbb{Z}(p,q,r)$. Let X_1 and X_2 be the vector fields associated to the two boundary components of Σ. Then the rotation number $\rho(X_1,X_2)$ of X_2 relative to X_1 along $\widetilde{\alpha}$ is zero.*

Proof. The quotient mappings $\pi':\mathbb{R}^3\to\widehat{T}^3$ and $\pi:\widehat{T}^3\to T^3$ are projections of covering spaces. Note that $\widehat{T}^3$ is diffeomorphic to $\mathbb{R}^2\times S^1$. The curve $\widetilde{\alpha}$ projects under π' to a compact knot $\hat{\alpha}$ in $\widehat{T}^3$. Let V be a small closed tubular neighborhood of $\hat{\alpha}$ (so that V is diffeomorphic to $S^1\times D^2$, where D^2 is the closed unit disk in the plane) and set $M=\widehat{T}^3\setminus\operatorname{Int}V$. Let α_1 and α_2 be the loops on the torus $\partial V=\partial M$ obtained by isotoping α in the directions of the vector fields X_1 and X_2. We claim that their homology classes satisfy $[\alpha_1] = [\alpha_2] \in H_1(\partial V)$, which implies that the mutual rotation number $\rho(X_1, X_2)$ vanishes, as claimed.

To see this claim, note that there is a compact oriented surface Σ' immersed in M obtained from the projection of Σ into $\widehat{T}^3$ by a small isotopy so that its boundary $\partial\Sigma'$ is the union of α_1 and α_2 with opposite orientations. Consequently $i''_*([\alpha_1]-[\alpha_2]) = 0 \in H_1 M$, where $i'' : \partial V \to M$ is the inclusion. Now let ℓ and m be the oriented longitude and meridian of ∂V, so that there are integers n_1 and n_2 such that the homology classes of α_1 and α_2 on ∂V satisfy $[\alpha_r] = [\ell] + n_r[m], r = 1, 2$. Since m is contractible on the solid torus V, $i'_*([\alpha_1] - [\alpha_2]) = 0 \in H_1 V$, where $i' : \partial V \to V$ is the inclusion. In the Mayer-Vietoris exact sequence

$$\cdots \to H_2\widehat{T}^3 \xrightarrow{\partial_2} H_1\partial V \underset{\approx}{\xrightarrow{i_*}} H_1 V \oplus H_1 M \xrightarrow{j_*} H_1\widehat{T}^3 \to \cdots$$

$i_* = i'_* + i''_*$ so $i_*([\alpha_1] - [\alpha_2]) = 0$. Since $H_2\widehat{T}^3 \approx H_2(\mathbb{R}^3/\mathbb{Z}(p, q, r)) = 0$, i_* is injective, so $[\alpha_1] = [\alpha_2]$, as claimed. $\square$

3 The Thurston-Bennequin invariant

As mentioned in the Introduction, Kanda has extended the definition of the Thurston-Bennequin invariant tb to linear Legendrian knots in (T^3, ξ), but not to non-linear Legendrian knots. In this section we shall extend the definition to fill this gap and show how to compute the invariant using projections.

The Thurston-Bennequin invariant for null-homologous knots. First we recall the definition of $\mathrm{tb}(\alpha)$ for an oriented null-homologous Legendrian knot α relative to a contact structure ξ on an oriented 3-manifold M^3. Since α is null-homologous it has a Seifert surface Σ, which by definition is an oriented compact connected surface embedded in M^3 with oriented boundary α. Let X and Y be unit vector fields orthogonal to α (with respect to a metric on M), with X tangent to Σ and Y tangent to ξ. Then $\mathrm{tb}(\alpha)$ is defined to be the algebraic number of rotations of Y relative to X in the normal plane field $\alpha^\perp$, which is oriented by the orientations of M^3 and α, as we make one circuit of α in the positive direction. If we let α^+ be a knot obtained by pushing α a short distance in the direction Y, then it is easy to see that $\mathrm{tb}(\alpha)$ is the intersection number of α^+ with Σ. This is just the linking number of α^+ with α because Σ is a compact oriented surface with boundary α. An argument analogous to the proof of Lemma 2.2, taking Σ to be the disjoint union of two Seifert surfaces Σ_1 and Σ_2 for α with opposite orientations, shows that $\mathrm{tb}(\alpha)$ is independent of the choice of the Seifert surface.

The Thurston-Bennequin invariant in T^3. Now consider T^3 with an oriented contact structure ξ and let α be a Legendrian knot in T^3 with a (generalized) Seifert surface Σ for α using the lift $\widetilde{\alpha}$ to the universal cover $\mathbb{R}^3$. We can define the rotation number of the lifted contact structure $\widetilde{\xi}$ with respect to Σ to be the number of rotations of one of the two unit orthogonal vector fields Y along $\widetilde{\alpha}$ that is tangent to $\widetilde{\xi}$ with respect to a unit orthogonal vector field X along $\widetilde{\alpha}$ that is tangent to Σ, in one circuit of α.

Definition 3.1. The *Thurston-Bennequin invariant* $\mathrm{tb}(\alpha)$ for a Legendrian knot α in T^3 is the rotation number of the contact structure $\widetilde{\xi}$ with respect to a (generalized) Seifert surface for α in one circuit of α.

We note that this is an extension of the above definition of $\mathrm{tb}(\alpha)$ for a null-homologous Legendrian knot α in an oriented 3-manifold M^3 endowed with a contact structure ξ. As in the null-homologous case, the following holds.

Lemma 3.2. *The Thurston-Bennequin invariant for a Legendrian knot α in T^3 is independent of the choice of the Seifert surface and the orientation of α.*

Proof. According to Proposition 1.2, the rotation number of one Seifert surface for the knot α with respect to another one is zero. Hence the rotation numbers of the two Seifert surfaces with respect to the contact structure coincide. Given an orientation of α, we choose the orientation of the plane field orthogonal to α such that the orientations of α and the plane field determine the standard orientation of T^3, so reversing the orientation of α reverses the orientation of the plane orthogonal field as well and the rotation number does not change. $\square$

We also note that $\mathrm{tb}(\alpha)$ does not change if we replace the vector field Y tangent to $\widetilde{\xi}$ by $-Y$ or by a vector field $Y^{\pitchfork}$ transverse to $\widetilde{\xi}$, since the mutual rotation number of these vector fields with respect to Y is zero. Similarly, we could use $-X$ or a vector field $X^{\pitchfork}$ transverse to Σ in place of X without changing the value of $\mathrm{tb}(\alpha)$.

As for knots in $\mathbb{R}^3$, $\mathrm{tb}(\alpha)$ will be the intersection number of $\widetilde{\alpha}^+$, the lifted knot $\widetilde{\alpha}$ pushed a short distance in a direction transverse to the lifted contact structure $\widetilde{\xi}$, with the Seifert surface Σ, in one circuit of α. In this case, however, if α is not null-homologous, then $\mathrm{tb}(\alpha)$ is not a linking number, since Σ will not be compact.

3.1 Computation of tb using projections

In this subsection we compute the Thurston-Bennequin invariant $\mathrm{tb}(\alpha)$ of an oriented Legendrian knot α in T^3 relative to Kanda's tight contact structure ξ_n, $n > 0$, using the projections $p_{xy}, p_{xz} : T^3 \to T^2$ defined in the Introduction and a (generalized) Seifert surface Σ for α, as defined in Section 2.

First, we recall how to do this for a generic oriented Legendrian knot α in $\mathbb{R}^3$ with the standard contact structure ξ_{std} utilizing its Lagrangian and front projections. The *front (*resp., *Lagrangian) projection* of a Legendrian knot α in $(\mathbb{R}^3, \xi_{std})$ is the map $\bar{\alpha} = pr_F \circ \alpha$ (resp., $\bar{\alpha} = pr_L \circ \alpha$) where the map $pr_F : \mathbb{R}^3 \to \mathbb{R}^2$ (resp., $pr_L : \mathbb{R}^3 \to \mathbb{R}^2$) is defined by $pr_F(x, y, z) = (x, z)$ (resp., $pr_L(x, y, z) = (x, y)$).

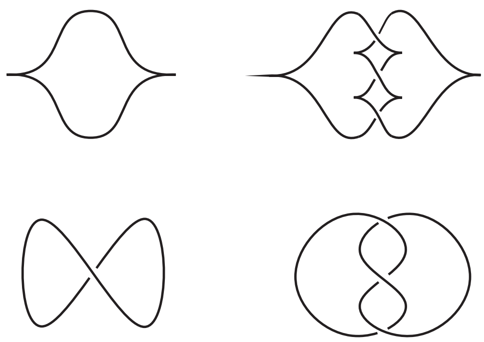

Figure 3: Front and Lagrangian projections of a Legendrian unknot and trefoil for ξ_{std} on $\mathbb{R}^3$.

Computation of tb using projections of $\mathbb{R}^3$. Let $\bar{\alpha} = pr_F \circ \alpha$ be the front projection of α. The vector field $Y = \partial/\partial z$ is transverse to $\xi_{std} = \ker(dz - ydx)$ along α, and we let α^+ be a knot obtained by shifting α slightly in the direction Y. Then, as observed above, $\mathrm{tb}(\alpha)$ is the intersection number of α^+ with the Seifert surface Σ, and this is the definition of the linking number of α^+ with α. This linking number is known to be half of the algebraic number of crossings of $\bar{\alpha}$ and $\bar{\alpha}^+$, where a crossing is positive if it is right handed and negative if it is left handed (see Figure 4). One can check this directly by observing the intersections of α^+ and Σ, if α^+ is chosen to be slightly above the (x, y)-plane, and the part of Σ near to $\bar{\alpha}$ is chosen to be in this plane. Each crossing of $\bar{\alpha}$ will yield one intersection point and contribute $(+1)$ if the crossing is positive and (-1) if it is negative.

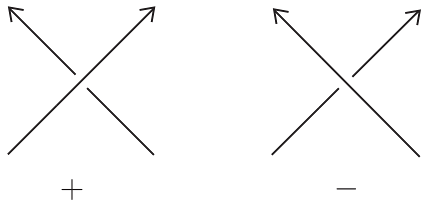

Figure 4: Positive and negative crossings.

The crossings and cusps of $\bar{\alpha}$ and $\bar{\alpha}^+$ in the front projection are shown in Figure 5, with $\bar{\alpha}$ in black and $\bar{\alpha}^+$ in gray. Each cusp of $\bar{\alpha}$ pointing to the left contributes 0 to the intersection of α and Σ since α^+ does not meet Σ near the cusp, and a cusp pointing to the right contributes (-1), so two adjacent cusps contribute (-1). Hence if P and N are the number of positive and negative crossings of the front projection $\bar{\alpha}$, respectively, and C is the number of cusps, we conclude that

$$\mathrm{tb}(\alpha) = P - N - C/2. \tag{1}$$

Figure 5: The pieces of the curves $\bar{\alpha}$ and $\bar{\alpha}^+$.

In the Lagrangian projection $pr_L(x, y, z) = (x, y)$) the knots α and α^+ project to similar diagrams, since α^+ is obtained by moving α a small distance in the y-direction. We can see that $\mathrm{tb}(\alpha)$, the linking number of α and α^+, is the algebraic number of the positive and negative crossings of the Lagrangian projection of α, $\mathrm{tb}(\alpha) = P - N$.

Computation of tb for Legendrian knots in (T^3, ξ_n)**.** Now consider an oriented Legendrian knot α in (T^3, ξ_n) for a fixed $n > 0$. We shall show how to compute $\mathrm{tb}(\alpha)$ using the lifted projections $\hat{p}_{xy}, \hat{p}_{xz} : T^2 \times \mathbb{R} \to T^2$ in a similar way to the case of $(\mathbb{R}^3, \xi_{std})$ treated above. We cannot use a linking number here, since the lifted knot $\hat{\alpha}$ does not bound a compact surface, but we can use the intersection number of a perturbed lifted knot $\hat{\alpha}^+$ with the image $\hat{\Sigma}$ in $T^2 \times \mathbb{R}$ of the (generalized) Seifert surface of Σ of α in $\mathbb{R}^3$ in one circuit of α.

Proposition 3.3. *For a generic Legendrian knot α in (T^3, ξ_n) and the projection p_{xy},*

$$\mathrm{tb}(\alpha) = \mathrm{P} - \mathrm{N} + \mathrm{C}/2$$

where P is the number of positive crossings, N is the number of negative crossings, and C is the number of cusps of $p_{xy} \circ \alpha$, which must be even, in one circuit of α.

As before, α is generic if the only singularities are transverse double points and isolated cusps. To determine which crossings are positive and which are negative we use the lift $\hat{\alpha}$ of α to $T^2 \times \mathbb{R}$; then following $\hat{\alpha}$ from the double point on one arc to the same double point on the other arc, the change in the vertical coordinate z determines which arc is above the other, and hence whether the crossing is positive or negative (see Figure 4).

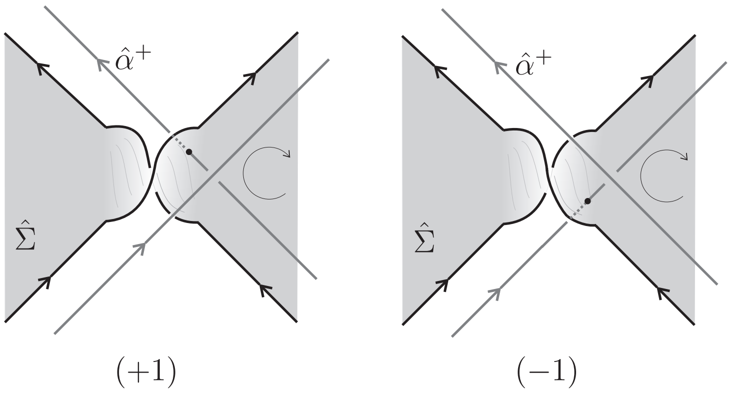

Figure 6: The curve $\hat{\alpha}^+$, in gray, intersects $\hat{\Sigma}$ positively and negatively close to positive and negative crossings, respectively.

Proof. Lift the contact structure ξ_n to the contact structure $\hat{\xi}_n$ on $T^2 \times \mathbb{R}$. The perpendicular vector field $\hat{Y} = (\cos 2\pi nz, \sin 2\pi nz, 0)$ determines the orientation of $\hat{\xi}_n$. Let $\hat{\alpha}^+$ be a copy of $\hat{\alpha}$ obtained by shifting $\hat{\alpha}$ slightly in the positive direction of $\hat{Y}$. By Definition 3.1 the Thurston-Bennequin invariant of α is equal to the signed intersection number of $\hat{\alpha}^+$ with a Seifert surface $\hat{\Sigma}$ of α in one circuit of $\hat{\alpha}$. It is convenient to choose the Seifert surface Σ to descend vertically near the Seifert curves, except near the cusp points on T^2, where the Seifert surface must move out horizontally for a small distance before descending. Then it is easy to check that the contribution of a crossing will be $(+1)$ for a positive crossing and (-1) for a negative crossing, since the upper strand of $\hat{\alpha}^+$ near the crossing will not meet $\hat{\Sigma}$, and the lower strand will pierce $\hat{\Sigma}$ just once, with the appropriate orientation, as shown in Figure 6.

The contribution of a cusp point of $\hat{p}_{xy} \circ \hat{\alpha}$ is illustrated in Figure 7, which shows $\hat{\alpha}$ and $\hat{\alpha}^+$ on $T^2 \times \mathbb{R}^3$. The arrows show the direction in which the vertical coordinate z increases. If the vector field Y points to the left of $\hat{\alpha}$ as $\hat{\alpha}$ approaches the cusp point, as in Figure 7 (a), then the perturbed knot $\hat{\alpha}^+$ will be above $\hat{\Sigma}$, so there is no intersection and the contribution will be 0. If, on the other hand, Y points toward the right as $\hat{\alpha}$ approaches the cusp point, then there will be a single intersection point p where $\hat{\alpha}^+$ pierces $\hat{\Sigma}$, and the contribution will be $+1$, as the orientations in Figure 7 (b) show.

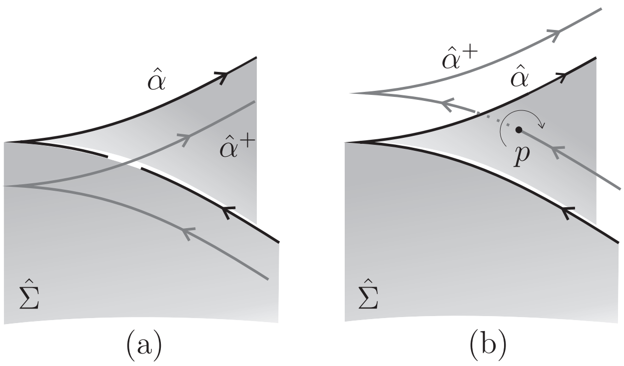

Figure 7: The curve $\hat{\alpha}^+$ (gray) does not pierce $\hat{\Sigma}$ and in the other case $\hat{\alpha}^+$ does.

The following lemma will complete the proof of the Proposition. □

Lemma 3.4. *The contributions of the cusps alternate between* $+1$ *and* 0*, so the total contribution of the cusps is* $C/2$*, where* C *is the number of cusps.*

Proof. If the direction of increasing z is the same from a cusp with value $+1$ to the next cusp, then Y will point to the left as the next cusp is approached and its contribution will be 0. On the other hand, if the direction of increasing z reverses, then again the contribution will be 0, since the direction of increasing z will be reversed, so Y will point to the right leaving the next cusp in the direction of increasing z. In a similar manner, if a cusp has contribution 0, the next cusp will contribute $+1$. $\square$

The projection onto the xz-plane. Now we shall compute $\mathrm{tb}(\alpha)$ using the front projection p_{xz} of a generic Legendrian knot α. Set $\alpha(t) = (x(t), y(t), z(t))$ and note that by a small Legendrian perturbation of α we can suppose that

$$2nz(t) \notin \mathbb{Z} \quad \text{whenever} \quad (x'(t), y'(t)) = (0, 0). \tag{2}$$

In other words, for these values of t the vertical component of $\alpha'(t)$ is not zero. For other values of t, the plane ξ_n of the contact structure projects onto the tangent plane of T^2 under p_{xz}, and so the image $p_{xz} \circ \alpha$ is a smooth non-singular curve.

The argument used for the projection p_{xy} shows the following result. By analogy with the previous analysis, we let the generalized Seifert surface move off the knot $\hat{\alpha}$ in the direction of the y-axis, instead of the z-axis. To determine whether a crossing in positive or negative, lift the knot to $S^1 \times \mathbb{R} \times S^1$, where the order of the arcs passing through a double point is well defined.

Proposition 3.5. *For a generic Legendrian knot α in (T^3, ξ_n) that satisfies (2),*

$$\mathrm{tb}(\alpha) = \mathrm{P} - \mathrm{N}$$

where P is the number of positive crossings and N is the number of negative crossings of $p_{xz} \circ \alpha$ in one circuit of α.

It is possible to calculate $\mathrm{tb}(\alpha)$ for a generic Legendrian knot α that does not satisfy (2) using its projection in the xz-plane, but the formula is more complicated, so we omit it.

These calculations prove the first two assertions of Theorem 2, which relate to the Thurston-Bennequin invariant.

4 The Maslov Number

Recall that a null homologous oriented Legendrian knot α in a 3-manifold M with an oriented contact structure ξ has an invariant $\mu(\alpha)$, called the Maslov number, which depends on the choice of a non-vanishing section Z of ξ, as follows.

Definition 4.1. The *Maslov number* of the oriented Legendrian knot α, $\mu(\alpha)$, is the algebraic number of rotations of the tangent vector α' with respect to Z in the plane field ξ in a single circuit of α.

The Maslov number of a null-homologous knot α does not depend on the choice of the section Z.

Proposition 4.2. *If α is a null homologous oriented Legendrian knot, the Maslov number $\mu(\alpha)$ does not depend on the section Z. Furthermore, two Legendrian knots that are isotopic through Legendrian knots have the same Maslov number with respect to the same section Z.*

Proof. The second affirmation in obvious, since the Maslov number is an integer that varies continuously as the Legendrian knot varies.

Now let Z' be another global section of ξ and let $f : M \to S^1$ be the function which gives the angle from Z to Z'. Since α is null homologous in M, the image $f_*[\alpha]$ of its homology class must vanish in $H_1(S^1)$, so the mutual rotation number of Z' relative to Z is 0. Hence the Maslov numbers are the same. □

Consequently $\mu(\alpha)$ for a null-homologous knot α depends only on the orientations of α and ξ. Reversing one of these orientations changes the sign of $\mu(\alpha)$. As in the case of the definition of the Thurston-Bennequin invariant, the Maslov number can also be defined for non-null homologous oriented Legendrian knots, but then it does depend on the choice of the section Z of ξ. This dependence holds, in particular, when $M = T^3$ (see [6]).

4.1 Computation of the Maslov invariant using projections

Computation of μ for Legendrian knots in $(\mathbb{R}^3, \xi_{std})$. Let α be an oriented generic Legendrian knot in the standard contact structure $\xi_{std} = \ker(dz - ydx)$ on $\mathbb{R}^3$. In order to calculate the Maslov number $\mu(\alpha)$ we fix the Legendrian vector field $Y = \partial/\partial y$. Then $\mu(\alpha)$ is the algebraic number of times the field of tangent vectors α' rotates in ξ_{std} relative to Y, so $\mu(\alpha)$ can be obtained by counting how many times α' and $\pm Y$ point in the same direction. The sign is determined by whether α' passes $\pm Y$ counterclockwise (+1) or clockwise (-1), and then we must divide by two, since in one rotation α' passes both Y and $-Y$.

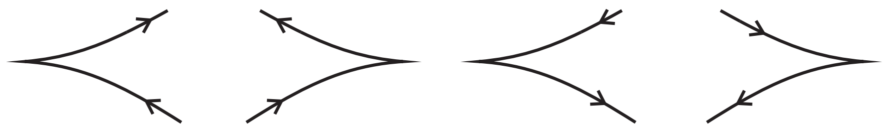

Figure 8: Up cusps and down cusps.

If $\bar{\alpha}$ denotes the front projection of α, the field of tangent vectors to $\bar{\alpha}$, $\bar{\alpha}'$, points in the direction of $\pm Y = \pm\partial/\partial y$ at the cusps, which are horizontal in the xz-plane. Let us analyze the upwards left-pointing cusp, the first of the four cusps in Figure 8. The value of y is just the slope of $\bar{\alpha}'$, so y is negative before the cusp and becomes positive, and thus at the cusp $y(t)$ is increasing so $y'(t)$ is positive and α' passes $+Y$ at the cusp point. Before the cusp, x is decreasing, so $x'(t)$ passes from negative to positive at the cusp. Thus the vector $\bar{\alpha}'(t)$ turns in the negative direction, and the contribution is (-1). By a similar analysis of the other three cases, we see that a cusp going upwards (the first two cusps in the figure) contributes (-1), while a cusp going downwards (the third and fourth cusps in the figure) contributes $(+1)$. Therefore we have shown that the Maslov number of α in the front projection is

$$\mu(\alpha) = 1/2(C_d - C_u),$$

where C_u is the number of up cusps and C_d is the number of down cusps in the front projection of α. Since α is null-homologous, $\mu(\alpha)$ does not depend on the choice of the vector field Y, as we observed above.

In the Lagrangian projection $pr_L(x, y, z) = (x, y))$, the vector field Y projects to $\partial/\partial y$, thus the Maslov number of α is simply the winding number of the field of tangent vectors of the Lagrangian projection $pr_L \circ \alpha$ of α in ξ_{std},

$$\mu(\alpha) = \text{winding}(pr_L(\alpha)).$$

Computation of μ for Legendrian knots in (T^3, ξ_n). Let α be an oriented generic Legendrian knot in T^3 for the tight contact structure ξ_n. We shall calculate the Maslov invariant $\mu(\alpha)$ relative to the vertical vector field $Z = \partial/\partial z \in \xi_n$.

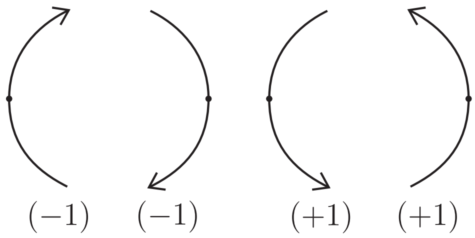

Figure 9: Values of $b(t)$ in the projection p_{xz}.

The projection p_{xz}. First we use the projection $p_{xz} : T^3 \to T^2$. By a small Legendrian perturbation, if necessary, we guarantee that if $2nz(t) \in \mathbb{Z}$ then the tangent vector $\alpha'(t)$ is not vertical, i.e., $(x'(t), y'(t)) \neq (0, 0)$. The only contributions to the Maslov number $\mu(\alpha)$ occur for points where $\alpha'(t)$ is vertical, and then since $2nz(t) \notin \mathbb{Z}$ the projection p_{xz} takes ξ_n onto the tangent plane to T^2. Near to where $\alpha'(t)$ is vertical the tangent vector, which must be non-zero, will be turning in either the positive direction with respect to the orientation of the xz-plane and pass the vertical line in the positive direction, and then we set $b(t) = +1$ (as in the last two cases of Figure 9), or in the negative direction (as in the first two cases) where we set $b(t) = -1$. Let $a(t) = (-1)^{[2nz(t)]}$, where the brackets indicate the largest integer function, so that $a(t)$ is positive where the projection p_{xz} of ξ_n onto the tangent xz-plane preserves the orientation and negative where the orientation is reversed, except when $2nz(t) \in \mathbb{Z}$, but we have guaranteed that then α' will not be vertical. Thus the contribution of a point where α' is vertical is half the product of $a(t)$ and $b(t)$. We have shown the following.

Proposition 4.3. *The Maslov invariant $\mu(\alpha)$ of a generic oriented knot α in T^3 with respect to the projection $p_{xz} : T^3 \to T^2$ is*

$$\mu(\alpha) = 1/2 \sum_{t \in V} a(t)b(t)$$

where $V = \{t \in S^1 \mid (x'(t), y'(t)) = (0, 0)\}$ in one circuit of α, provided that $(x'(t), y'(t)) \neq (0, 0)$ whenever $2nz(t) \in \mathbb{Z}$.

The projection p_{xy}. For the projection $p_{xy} : T^3 \to T^2$, we must count how many times the tangent field of $\bar{\alpha} = p_{xy} \circ \alpha$ and $Z = \partial/\partial z$ point in the same direction, and this will happen where $\bar{\alpha}$ has a cusp since the projection $\bar{\alpha}$ will have velocity $\bar{\alpha}'(t_0) = 0$ at such a point. Observe that the horizontal normal vector $Y = (\cos 2\pi nz, \sin 2\pi nz, 0)$, which determines the orientation of the perpendicular contact plane ξ_n, projects to a vector $\bar{Y} = p_{xy*}(Y) = (\cos 2\pi nz, \sin 2\pi nz)$ perpendicular to the line tangent to the cusp in the xy-plane. The slope of this line, determined by the value of $z(t_0)$ at the cusp, may have any value.

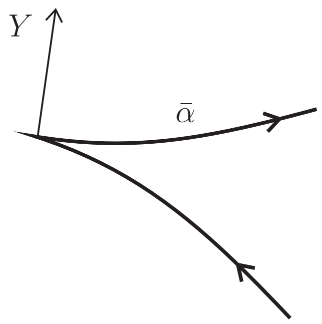

Figure 10: A cusp of $p_{xz} \circ \alpha$.

Consider the orientation of α and the direction of Y in Figure 10. Since the tangent vector $\bar{\alpha}'(t)$ is turning in the positive direction in the xy-plane, $z'(t_0) > 0$ at the cusp. Before the cusp $\bar{\alpha}'(t)$ is directed toward the cusp, and afterwards, it is directed away from the cusp. Hence it is clear that $\alpha'(t)$ passes the vertical vector Z in the positive direction in the contact plane ξ_n, so in this case the contribution of the cusp is $+1$, and we call the cusp *positive*. In this case the projection of $\bar{\alpha}'(t)$ onto the line through Y has the same direction at Y, both before and after the cusp point. The result is the same if the diagram in Figure 10 is rotated in the xy-plane.

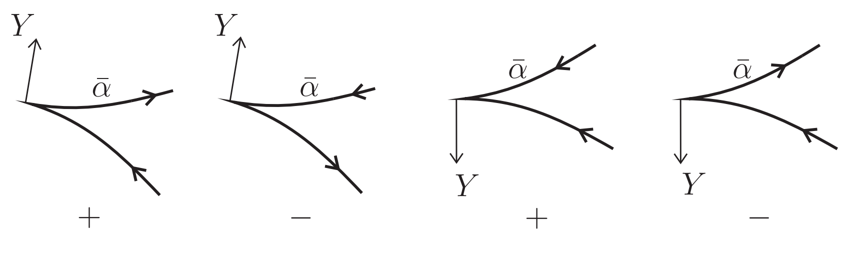

Figure 11: Positive and negative cusps for the projection p_{xz}.

Now it is clear that if the orientation of α or the direction of Y is reversed, the sign of the contribution of the cusp changes. It follows that in all four cases of the orientation of α and the perpendicular direction of Y, the contribution of the cusp is $+1$ and the cusp is *positive* if the projection of $\bar{\alpha}'(t)$ onto the line through Y both before and after the cusp has the same direction as Y, and the cusp is *negative*, with contribution -1, if the direction is opposite to Y, as shown in Figure 11.
Thus we have shown the following.

Proposition 4.4. *The Maslov number of a generic oriented knot α in (T^3, ξ_n) with respect to the projection p_{xy} is*

$$\mu(\alpha) = 1/2(C_+ - C_-)$$

where C_+ is the number of positive cusps and C_- is the number of negative cusps of $p_{xy} \circ \alpha$ in one circuit of α.

This completes the calculation of the Maslov invariant using the projections p_{xz} and p_{xy} as in Theorem 2, so its proof is complete.

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