Almost Reduction and Perturbation of Matrix Cocycles

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Abstract

In this note, we show that if all Lyapunov exponents of a matrix cocycle vanish, then it can be perturbed to become cohomologous to a cocycle taking values in the orthogonal group. This extends a result of Avila, Bochi and Damanik to general base dynamics and arbitrary dimension. We actually prove a fibered version of this result, and apply it to study the existence of dominated splittings into conformal subbundles for general matrix cocycles.

1 From zero Lyapunov exponents to rotation cocycles

1.1 Basic definitions

Let $F\colon \Omega \to \Omega$ be a homeomorphism of a compact metric space Ω. Let V be a finite-dimensional real vector bundle over Ω, whose fiber over ω is denoted by V_ω. Let $\mathcal{A}$ be a vector-bundle automorphism that fibers over F; this means that the restriction of $\mathcal{A}$ to each fiber V_ω is a linear automorphism $A(\omega)$ onto $V_{F\omega}$. In the case of trivial vector bundles, $\mathcal{A}$ is usually called a *linear cocycle*.

As a convention, automorphisms of V will be denoted by calligraphic letters, and the restrictions to the fibers will be denoted by the corresponding roman letters. Analogously, for any integer n, the restriction of the power $\mathcal{A}^n$ to the fiber V_ω is denoted by $A^n(\omega)$; thus $A^n(\omega) = A(F^{n-1}\omega) \circ \cdots \circ A(\omega)$ for $n > 0$.

A *Riemannian metric* on V is a continuous choice of inner product $\langle \cdot, \cdot \rangle_\omega$ on each fiber V_ω. It induces a *Riemannian norm* $\|v\|_\omega = \sqrt{\langle v, v \rangle_\omega}$. Given a linear map $L\colon V_\omega \to V_{\omega'}$, its *norm* $\|L\|$ and its *conorm* $\mathfrak{m}(L)$ are defined respectively as the supremum and the infimum of $\|Lv\|_{\omega'}$ over all unit vectors $v \in V_\omega$.

Let $\mathrm{Aut}(V, F)$ denote the space of all automorphisms of V that fiber over F, endowed with the topology induced by the distance $d(\mathcal{A}, \mathcal{B}) = \sup_\omega \|A(\omega) - B(\omega)\|$, for some choice of a Riemannian norm on V.

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1.2 Uniform subexponential growth and its consequences

Define

$$\lambda^+(\mathcal{A}) = \lim_{n\to+\infty} \frac{1}{n} \sup_{\omega\in\Omega} \log \|A^n(\omega)\| \quad \text{and} \quad \lambda^-(\mathcal{A}) = \lim_{n\to+\infty} \frac{1}{n} \inf_{\omega\in\Omega} \log \mathfrak{m}(A^n(\omega)) .$$

which exist by subadditivity and supraadditivity, respectively.

If μ is ergodic probability measure for $F\colon \Omega \to \Omega$, then there are constants $\lambda^+(\mathcal{A},\mu)$, $\lambda^-(\mathcal{A},\mu)$, called the *top* and *bottom Lyapunov exponents*, such that, for μ-almost every $\omega \in \Omega$,

$$\frac{1}{n} \log \|A^n(\omega)\| \to \lambda^+(\mathcal{A},\mu) \quad \text{and} \quad \frac{1}{n} \log \mathfrak{m}(A^n(\omega)) \to \lambda^-(\mathcal{A},\mu) \quad \text{as } n \to +\infty.$$

Moreover, the following "variational principle" holds[1]:

$$\lambda^+(\mathcal{A}) = \sup_\mu \lambda^+(\mathcal{A},\mu) \quad \text{and} \quad \lambda^-(\mathcal{A}) = \inf_\mu \lambda^-(\mathcal{A},\mu) . \tag{1.1}$$

where μ runs over all invariant ergodic probabilities for F.

Let us say that the automorphism $\mathcal{A}$ has *uniform subexponential growth* if $\lambda^+(\mathcal{A}) = \lambda^-(\mathcal{A}) = 0$. By (1.1), this is equivalent to the vanishing of all Lyapunov exponents with respect to all ergodic probability measures.

Our first result is:

Theorem 1.1. *Assume that $\mathcal{A} \in \mathrm{Aut}(V,F)$ has uniform subexponential growth. Then:*

(a) For any $\varepsilon > 0$, there exists a Riemannian norm $|||\cdot|||$ on V such that

$$e^{-\varepsilon} |||v|||_\omega < |||A(\omega)v|||_{F\omega} < e^{\varepsilon} |||v|||_\omega , \quad \textit{for all } \omega \in \Omega,\ v \in V_\omega. \tag{1.2}$$

(b) There exists a perturbation of $\mathcal{A}$ that preserves some Riemannian norm on V.

As we will see, part (a) follows from a standard construction in Pesin theory, and part (b) follows form part (a). However, the latter implication is *not* straightforward, because if ε is small then the Riemannian norm constructed in part (a) may be very distorted with respect to a fixed reference Riemannian norm on V.

For a reformulation of the theorem in terms of conjugacy to isometric automorphisms, see § 1.4.

Despite making stringent assumptions about the automorphism $\mathcal{A}$, Theorem 1.1 can be used to obtain very strong properties for a dense subset D of $\mathrm{Aut}(V,F)$. More precisely, under the assumption that F is minimal, we show that every automorphism in the subset D has a dominated splitting into subbundles which are conformal with respect to a suitable Riemannian metric; see Section 2.

[1] This follows from [Sc, Thrm. 1] or [SS, Thrm. 1.7]. Although these references assume Ω to be compact metrizable, the proofs also work for compact Hausdorff Ω. (See also the proof of Proposition 1 in [AB].) A particular case was considered in [Fu].

In the paper [BN], we prove results about cocycles of isometries of spaces of nonpositive curvature that generalize Theorem 1.1. Actually, we first obtained Theorem 1.1 as a corollary of the geometrical results of [BN]. Later, we realized that the constructions could be modified or adapted to produce an elementary proof of Theorem 1.1, which we present, together with its applications, in this note.

1.3 Proof of Theorem 1.1

We need a few preliminaries.

Recall that V is a finite-dimensional vector bundle over the compact space Ω. We choose and fix a Riemannian metric $\langle\cdot,\cdot\rangle$ on V. Let $\mathcal{B}$ be an automorphism of V over a homeomorphism $G\colon \Omega \to \Omega$. The *transpose* of $\mathcal{B}$ is the automorphism $\mathcal{B}^*$ over G^{-1} defined by

$$\langle B(\omega)u, v\rangle_\omega = \langle u, B^*(G\omega)v\rangle_{G\omega}, \quad \text{for all } u \in V_\omega,\ v \in V_{G\omega}\,.$$

If $\mathcal{B}^* = \mathcal{B}$ (and thus G is the identity), then $\mathcal{B}$ is called *symmetric*. An automorphism $\mathcal{P}$ is called *positive* if it is symmetric and $\langle P(\omega)v, v\rangle_\omega > 0$ for all nonzero $v \in V_\omega$. We write $\mathcal{B} < \mathcal{C}$ if $\mathcal{B}$ and $\mathcal{C}$ are symmetric and $\mathcal{C} - \mathcal{B}$ is positive.

The following observations will be useful:

- If $\mathcal{A}$ is any automorphism and $\mathcal{B}$ is symmetric, then $\mathcal{A}^*\mathcal{B}\mathcal{A}$ is symmetric; moreover, if $\mathcal{B} < \mathcal{C}$, then $\mathcal{A}^*\mathcal{B}\mathcal{A} < \mathcal{A}^*\mathcal{C}\mathcal{A}$.
- Each positive automorphism $\mathcal{P}$ has a unique positive square root $\mathcal{P}^{1/2}$; moreover, $\mathcal{P}^{1/2}$ commutes with $\mathcal{P}$, and the map $\mathcal{P} \mapsto \mathcal{P}^{1/2}$ is continuous.
- The square root map is monotonic: if $\mathcal{P}$, $\mathcal{Q}$ are positive and $\mathcal{P} < \mathcal{Q}$, then $\mathcal{P}^{1/2} < \mathcal{Q}^{1/2}$. (See [Bh, p. 9] for a proof.)

Proof of Theorem 1.1. Let $\mathcal{A}$ be an automorphism of V over the homeomorphism F having uniform subexponential growth. Fix a small $\varepsilon > 0$.

To prove part (a), we will use an standard construction in Pesin theory called *Lyapunov norms* (see e.g. [KH, p. 667]). Define

$$|\!|\!| v |\!|\!|_\omega^2 := \sum_{n\in\mathbb{Z}} e^{-2\varepsilon|n|}\, \|A^n(\omega)v\|_{F^n\omega}^2\,. \tag{1.3}$$

Since the cocycle has uniform subexponential growth, the series converges uniformly on compact subsets of V, and hence defines a (continuous) Riemannian norm. Property (1.2) is straightforward to check. This proves part (a),

To prove part (b), let $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ be the inner product that induces the norm (1.3). Then there are positive automorphisms $\mathcal{R}$, $\mathcal{Q}$ such that

$$\langle\!\langle u, v\rangle\!\rangle_\omega = \langle R(\omega)u, v\rangle_\omega\,, \quad \langle\!\langle A(\omega)u, A(\omega)v\rangle\!\rangle_{F\omega} = \langle Q(\omega)u, v\rangle_\omega\,, \quad \forall u, v \in V_\omega\,.$$

The almost-invariance property (1.2) can now be expressed as:

$$e^{-2\varepsilon}\mathcal{Q} < \mathcal{R} < e^{2\varepsilon}\mathcal{Q}\,. \tag{1.4}$$

We want to find an automorphism $\tilde{\mathcal{A}}$ over F that is close to $\mathcal{A}$ and leaves the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ invariant. As it is straightforward to check, invariance means that the automorphism $\mathcal{P} = \mathcal{A}^{-1}\tilde{\mathcal{A}}$ (over the identity) satisfies:

$$\mathcal{P}^* \mathcal{Q} \mathcal{P} = \mathcal{R} . \tag{1.5}$$

Equivalently,

$$(\mathcal{Q}^{1/2} \mathcal{P} \mathcal{Q}^{1/2})^* (\mathcal{Q}^{1/2} \mathcal{P} \mathcal{Q}^{1/2}) = \mathcal{Q}^{1/2} \mathcal{R} \mathcal{Q}^{1/2} .$$

Let us try to find a *positive* solution $\mathcal{P}$. Then the relation above becomes $(\mathcal{Q}^{1/2} \mathcal{P} \mathcal{Q}^{1/2})^2 = \mathcal{Q}^{1/2} \mathcal{R} \mathcal{Q}^{1/2}$, and using the uniqueness of positive square positive roots, we obtain

$$\mathcal{P} = \mathcal{Q}^{-1/2} (\mathcal{Q}^{1/2} \mathcal{R} \mathcal{Q}^{1/2})^{1/2} \mathcal{Q}^{-1/2} .$$

One checks directly that this formula solves the invariance equation (1.5), and thus gives the unique positive solution.

To estimate $\mathcal{P}$, we follow an observation due to [PT]. By the second inequality in (1.4) and the monotonicity of the square root, we obtain $\mathcal{P} < e^{\varepsilon} \mathcal{I}$ where $\mathcal{I}$ is the identity automorphism. This means that $\|P(\omega)\| < e^{\varepsilon}$ for every ω. Analogously, the first inequality in (1.4) implies that $\mathfrak{m}(P(\omega)) > e^{-\varepsilon}$. This shows that $\mathcal{P}$ is close to the identity, and therefore the automorphism $\tilde{\mathcal{A}} := \mathcal{A}\mathcal{P}$ is close to $\mathcal{A}$. As we have seen, $\tilde{\mathcal{A}}$ preserves the new Riemannian metric, thus completing the proof of the theorem. □

Remark 1.2. Equation (1.5) obviously has infinitely many solutions $\mathcal{P}$, not all of them close to the identity. As we have seen, restricting to positive automorphisms we have a unique solution, which is close to the identity and varies continuously with the data.

In [BN], we obtain a generalization of Theorem 1.1 to cocycles of isometries of symmetric spaces of non-positive curvature. If specialized to the present situation, the construction presented in [BN] is the same given here for part (b), thus "explaining" the efficiency of positive matrices. ◁

Remark 1.3. Notice that the Riemannian norm and the perturbed automorphism constructed in the proof of Theorem 1.1 depend continuously on the parameter ε and also on the automorphism $\mathcal{A}$ itself. These properties are relevant for the applications obtained in [ABD2]. ◁

1.4 Conjugacy

Let us put Theorem 1.1 under a different perspective.

Two automorphisms $\mathcal{A}, \mathcal{B} \in \mathrm{Aut}(V, F)$ are said to be *conjugate* if there exists $\mathcal{U} \in \mathrm{Aut}(V, \mathrm{id})$ such that $\mathcal{A} = \mathcal{U}\mathcal{B}\mathcal{U}^{-1}$. (In the case of a trivial vector bundle, we say that the two linear cocycles are *cohomologous*.)

Fixed a Riemannian metric on V, we say that a automorphism $\mathcal{A}$ is *isometric* if it preserves this metric. (In the case of a trivial vector bundle, the cocycle will take values in the orthogonal group, i.e., it will be a *rotation cocycle*.)

Then we have:

Theorem 1.4. *Fix a Riemannian metric on the vector bundle V. Assume that $\mathcal{A} \in \mathrm{Aut}(V, F)$ has uniform subexponential growth. Then:*

(a) There exists an automorphism conjugate to $\mathcal{A}$ that is close to an isometric automorphism.

(b) There exists an automorphism close to $\mathcal{A}$ that is conjugate to an isometric automorphism.

For $\mathrm{SL}(2, \mathbb{R})$-cocycles and under extra assumptions on the dynamics F, the result above was shown by Avila, Bochi and Damanik as a step in the proofs of their results about spectra of Schrödinger operators, see [ABD1, ABD2].[2]

In the case of cocycles (i.e., trivial vector bundles), it is natural to look for conditions under which we can improve the conclusion of Theorem 1.4.(b) and find a perturbed cocycle cohomologous to a constant rotation, or even to the identity. The case of $\mathrm{SL}(2, \mathbb{R})$-cocycles is studied in [ABD2].

Proof of Theorem 1.4. Let $\varepsilon > 0$ be small. We follow the notation of the proof of Theorem 1.1.

Notice that $\|R(\omega)^{-1/2} v\|_\omega = \|v\|_\omega$ for every $v \in V_\omega$. Thus $\mathcal{B} := \mathcal{R}^{-1/2} \mathcal{A} \mathcal{R}^{1/2}$ satisfies

$$e^{-\varepsilon} \|v\|_\omega < \|B(\omega) v\|_{F\omega} < e^{\varepsilon} \|v\|_\omega .$$

This implies that $\mathcal{B}$ is close to an isometric isomorphism, thus proving part (a).

To prove part (b), it suffices to notice that $\mathcal{R}^{-1/2} \tilde{\mathcal{A}} \mathcal{R}^{1/2}$ is an isometric isomorphism. □

There is another property which is closely related to what we have seen so far. Let us say that $\mathcal{A} \in \mathrm{Aut}(V, F)$ is *product-bounded* if

$$0 < \inf_{\substack{\omega \in \Omega \\ n \in \mathbb{Z}}} \mathfrak{m}(A^n(\omega)) \leqslant \sup_{\substack{\omega \in \Omega \\ n \in \mathbb{Z}}} \|A^n(\omega)\| < \infty .$$

If an automorphism $\mathcal{A}$ is conjugate to an isometric automorphism then $\mathcal{A}$ is product-bounded, as it is easy to check. Although product-bounded cocycles are not always conjugate to isometric automorphisms[3], this happens whenever F is minimal, according to a result shown by Coronel, Navas and Ponce in [CNP].[4]

2 Existence of conformal subbundles

2.1 Results

The following is an immediate consequence of Theorem 1.1:

[2] However, they haven't explicitly stated the result: see the proof of Theorem 1 in [ABD1] and Proposition 6.3 in [ABD2].

[3] See e.g. [KH, Exercise 2.9.2], [Qu].

[4] In the non-minimal case, one can still ensure the existence of a bounded and measurable conjugacy.

Corollary 2.1. *Let $\mathcal{A} \in \mathrm{Aut}(V, F)$ be such that $\lambda^+(\mathcal{A}) = \lambda^-(\mathcal{A}) =: \lambda$. Then there exists a perturbation $\tilde{\mathcal{A}}$ of $\mathcal{A}$ and a Riemannian norm $|\!|\!|\cdot|\!|\!|$ on V such that*

$$|\!|\!|\tilde{A}(\omega)v|\!|\!|_{F\omega} = e^{\lambda} |\!|\!|v|\!|\!|_{\omega}, \quad \textit{for all } \omega \in \Omega,\ v \in V_\omega.$$

In other words, if all Lyapunov exponents of an automorphism are equal, then we can perturb it to become conformal with respect to a new Riemannian metric.[5]

Using the corollary and a theorem from [BV], we will obtain the following result:

Theorem 2.2. *Assume that $F\colon \Omega \to \Omega$ is a uniquely ergodic homeomorphism with an invariant measure of full support. Then for every automorphism $\mathcal{A}$ in a dense subset of $\mathrm{Aut}(V, F)$, there exist:*

- *a Riemannian norm $|\!|\!|\cdot|\!|\!|$ on V;*
- *a continuous $\mathcal{A}$-invariant splitting $V = V^1 \oplus \cdots \oplus V^k$ which is orthogonal with respect to the Riemannian norm;*
- *and constants $\lambda_1 > \cdots > \lambda_k$;*

such that

$$|\!|\!|A(\omega)v_i|\!|\!|_{F\omega} = e^{\lambda_i} |\!|\!|v_i|\!|\!|_{\omega}, \quad \textit{for all } \omega \in \Omega,\ i = 1, \dots, k,\ v_i \in V^i_\omega.$$

Weakening the assumption of unique ergodicity to minimality, we have the following result:

Theorem 2.3. *Assume that $F\colon \Omega \to \Omega$ is a minimal homeomorphism of a compact space of finite dimension*[6]*. Then for every $\mathcal{A}$ in a dense subset of $\mathrm{Aut}(V, F)$, there exist:*

- *a Riemannian norm $|\!|\!|\cdot|\!|\!|$ on V;*
- *a continuous $\mathcal{A}$-invariant splitting $V = V^1 \oplus \cdots \oplus V^k$ which is orthogonal with respect to the Riemannian norm;*
- *and continuous functions $\lambda_1 > \cdots > \lambda_k$ on Ω;*

such that

$$|\!|\!|A(\omega)v_i|\!|\!|_{F\omega} = e^{\lambda_i(\omega)} |\!|\!|v_i|\!|\!|_{\omega}, \quad \textit{for all } \omega \in \Omega,\ i = 1, \dots, k,\ v_i \in V^i_\omega.$$

As we will see, this has a similar proof as Theorem 2.2, basically replacing the result from [BV] by the result from [Bo].

We expect that Theorems 2.2 and 2.3 will be useful to answer the following question: *When can a linear cocycle over an uniquely ergodic or minimal base dynamics be approximated by a cocycle with a dominated (non-trivial) splitting?* Results on the 2-dimensional case were obtained in [ABD1, ABD2].

[5] See [KS] for a non-perturbative result with a similar conclusion.

[6] We say that Ω has *finite dimension* if it is homeomorphic to a subset of an euclidean space $\mathbb{R}^n$.

2.2 Proofs

Proof of Theorem 2.2. Assume that $F : \Omega \to \Omega$ has a unique invariant probability μ, and its support is Ω. Take any $\mathcal{A} \in \mathrm{Aut}(V, F)$; we will explain how to perturb it so that it has the desired properties. First, by [BV], one can perturb $\mathcal{A}$ so that the along μ-almost every orbit, the Oseledets splitting is trivial or dominated. Let

$$V_\omega^1 \oplus \cdots \oplus V_\omega^k = V_\omega\,, \quad \omega \in \Omega,$$

be the *finest dominated splitting* of the cocycle, that is, the unique everywhere defined global dominated splitting with a maximal number k of bundles (with $k = 1$ if there is no dominated splitting).[7]

We claim that for almost every point, there are exactly k different Lyapunov exponents. Indeed, on the one hand, there are at least k different exponents because there is a dominated splitting with k bundles. On the other hand, if there is a positive measure set of points with more than k different Lyapunov exponents, then one can select an orbit along which the Oseledets splitting is dominated. This orbit is dense on Ω (because the invariant measure has full support). Since dominated splittings extend to the closure (see [BDV]), one gets a global dominated splitting with more than k bundles; this is a contradiction.

For each $i = 1, \ldots, k$, let $\mathcal{A}_i$ be the restriction of $\mathcal{A}$ to the bundle V^i; this is a (continuous) vector bundle automorphism. By the claim above,

$$\lambda^+(\mathcal{A}_i) = \lambda^+(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i) =: \lambda_i\,.$$

Therefore, by Corollary 2.1, for each i there is a a perturbation $\tilde{\mathcal{A}}_i$ of $\mathcal{A}_i$ and a Riemannian norm $|\!|\!|\cdot|\!|\!|_i$ on V^i such that

$$|\!|\!|\tilde{A}_i(\omega) v_i|\!|\!|_{i, F\omega} = e^{\lambda} |\!|\!|v_i|\!|\!|_{i,\omega}\,, \quad \text{for all } \omega \in \Omega,\ v_i \in V_\omega^i.$$

Let $|\!|\!|\cdot|\!|\!|$ be the Riemannian norm that makes the subbundles orthogonal and that coincides with $|\!|\!|\cdot|\!|\!|_i$ on V^i. Let $\tilde{\mathcal{A}}$ be the automorphism of V whose restriction to the subbundles V^i are the automorphisms $\tilde{\mathcal{A}}_i$. This automorphism has the desired properties, thus completing the proof. □

For the proof of Theorem 2.3, we need the following result:

Theorem 2.4. *Assume that $F : \Omega \to \Omega$ is a minimal homeomorphism of a compact space of finite dimension. Then every $\mathcal{A}$ in a residual subset of* $\mathrm{Aut}(V, F)$ *has the following property: the Oseledets splitting with respect to any invariant probability coincides almost everywhere with the finest dominated splitting of $\mathcal{A}$.*

This result is proved in full generality in [Bo]. (The case of $\mathrm{SL}(2, \mathbb{R})$-cocycles was previously considered in [AB].) Notice that, as we have seen in the proof of Theorem 2.2 above, under the additional assumption of unique ergodicity, Theorem 2.4 follows from [BV].

Proof of Theorem 2.3. Assume that $F : \Omega \to \Omega$ is minimal. Take any $\mathcal{A} \in \mathrm{Aut}(V, F)$; we will explain how to perturb it so that it has the desired properties. First, perturb $\mathcal{A}$ so that it has the property from Theorem 2.4. This means that if

$$V_\omega^1 \oplus \cdots \oplus V_\omega^k = V_\omega\,, \quad (\omega \in \Omega)$$

[7] See [BDV] for details on finest dominated splittings.

is the finest dominated splitting of the cocycle and $\mathcal{A}_i$ is the restriction of $\mathcal{A}$ to the bundle V^i then

$$\lambda^+(\mathcal{A}_i, \mu) = \lambda^-(\mathcal{A}_i, \mu) \quad \text{for every ergodic probability } \mu \text{ for } F. \tag{2.1}$$

By [Go], we can choose a Riemannian metric on V that is *adapted* to the dominated splitting, which means that

$$\inf_{\omega \in \Omega} \frac{\mathfrak{m}(A_i(\omega))}{\|A_{i+1}(\omega)\|} > 1, \quad \text{for every } i = 1, 2, \dots, k-1.$$

Let d_i be the fiber dimension of V^i, and let

$$\lambda_i(\omega) := \frac{1}{d_i} \log |\det A_i(\omega)| \, ;$$

here determinants are computed with respect to the adapted metric, and in particular $\lambda_1 > \lambda_2 > \cdots > \lambda_d$ pointwise.[8]

Let $\mathcal{B}_i = e^{-\lambda_i} \mathcal{A}_i$, so $|\det B_i(\omega)| = 1$ for every ω. It follows from (2.1) that

$$\lambda^+(\mathcal{B}_i, \mu) = \lambda^-(\mathcal{B}_i, \mu) = 0 \quad \text{for every ergodic probability } \mu \text{ for } F.$$

By the "variational principle" (1.1), this implies $\lambda^+(\mathcal{B}_i, \mu) = \lambda^-(\mathcal{B}_i, \mu) = 0$. Therefore, by Theorem 1.1, for each i there is a Riemannian norm $|\!|\!|\cdot|\!|\!|_i$ on V^i that is preserved by a perturbation $\tilde{\mathcal{B}}_i$ of $\mathcal{B}_i$.

Let $|\!|\!|\cdot|\!|\!|$ be the Riemannian norm that makes the subbundles orthogonal and that coincides with $|\!|\!|\cdot|\!|\!|_i$ on V^i. Let $\tilde{\mathcal{A}}$ be the automorphism of V whose restrictions to the subbundles V^i are the automorphisms $e^{\lambda_i} \tilde{\mathcal{B}}_i$. This automorphism has the desired properties, thus completing the proof. □

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[8] One can avoid using adapted metrics in the proof of Theorem 2.3 by using the following fact: if the functions $\lambda_1, \dots, \lambda_k$ satisfy $\int \lambda_1 \, d\mu > \cdots > \int \lambda_k \, d\mu$ for every ergodic probability μ, then there are functions $\hat{\lambda}_i$ cohomologous to the λ_i's such that $\hat{\lambda}_1 > \cdots > \hat{\lambda}_d$ pointwise.

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