

SPELLING RULES FOR THE MONSTER/SEMPLÉ TOWER.

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ABSTRACT. We further study the incidence relations that arise from the various subtowers, known as Baby Monsters, which exist within the \mathbb{R}^3 -Monster Tower. This allows us to complete the *RVT* class spelling rules. We also present a method of calculating the various Baby Monsters that appear within the Monster Tower.

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1. INTRODUCTION

This paper further investigates the various critical hyperplane relations that occur within a rank-3 distribution associated to each level of the \mathbb{R}^3 -Monster Tower. Studying these hyperplane interactions yields the complete spelling rules for the *RVT* coding system. In [CH12] we studied how critical hyperplanes appeared within a rank 3 distribution that is associated to each of the points within the Monster Tower. We used these hyperplane relations in order to classify the orbits within the Monster Tower. Classifying these orbits enabled us to classify a certain geometric distribution known as a Goursat n -flag.

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A *Goursat n -flag of length k* , also called a *Goursat Multi-Flag* for $n \geq 2$, is a distribution, say D , of rank $(n + 1)$ sitting inside of a $(n + 1) + kn$ dimensional ambient manifold. The rank of the associated flag of distributions

$$D \subset D + [D, D] \subset D + [D, D] + [[D, D], [D, D]] \dots,$$

increases by n at each bracketing step.

R. Montgomery and Z. Zhitomirskii showed in [MZ01] that the problem of classifying up to local diffeomorphism of Goursat Flags is equivalent to the problem of classifying points within an iterated tower of manifolds that they called the \mathbb{R}^2 -*Monster Tower*, also known as the *Semple Tower* in algebraic geometry ([Sem54] and [Bér12]). Then in [MZ10] they focused on classifying the various points by partitioning them into equivalence classes known as the *RVT* coding system. This coding system served as an important tool which allowed them to completely classify the points within the \mathbb{R}^2 -Monster Tower. In [CM12] A. Castro and R. Montgomery took the first steps toward extending this *RVT* coding system for the \mathbb{R}^3 -Monster Tower. In the \mathbb{R}^3 case there are distinguished directions, labelled by the letter L , which come from the intersection of the vertical and tangency hyperplanes. While the critical hyperplane relations are completely understood in the \mathbb{R}^3 case over regular, vertical, tangency, and L points, it is not obvious what happens over the new critical directions that appear over the L directions. The word *over* here, and throughout the rest of the paper, refers to the rank 3 distribution which is “over” each point within the Monster Tower. In [CH12] we began studying the sources of these critical hyperplanes. Over L points there exist not just one, but two types of tangency hyperplanes and three new distinguished directions, see Figure 2a. These tangency hyperplanes above the L directions are labeled as T_1 and T_2 and the new L directions by L_j for $j = 1, 2, 3$. One natural question to ask is the following:

What are the critical hyperplane incidence relations over each of the directions T_i and L_j for $i = 1, 2$ and $j = 1, 2, 3$?

This is exactly the question we are solving in this paper, where we determine the complete *RVT* spelling rules. We want to examine this question because it will enable us to classify points at higher levels of the Monster Tower. In [CH12] we used a technique called the *isotropy method* that allowed us to classify not only the points within the fourth level of the \mathbb{R}^3 -Monster Tower, but to classify points at any other level of the tower so long as we know how to characterize the *RVT*-classes in Kumpera-Rubin coordinates. In [CH12] we only understood the critical hyperplane configurations over R, V, T , and L points. However, after level 4 of the Monster Tower it was not entirely clear what the spelling rules are for the various *RVT* classes. In particular, we pointed out how the existence of the L direction is the source of this confusion. This means that once we complete the spelling rules, we can then apply the isotropy method to determine how many orbits lie within any level of the tower. This is the main motivation for the work presented in this paper. We will derive the *RVT* spelling rules for the \mathbb{R}^3 - Monster Tower over each of the previously unknown planes and directions listed above. As a result, this will yield the complete *RVT* spelling rules. They are as follows:

- (1) Any RVT code string must begin with the letter R .
- (2) R : R and V .
- (3) V and $T (= T_1)$: R, V, T , and L .
- (4) $L (= L_1)$ and L_j for $j = 2, 3$: R, V, T_i for $i = 1, 2$, and L_j for $j = 1, 2, 3$.
- (5) T_2 : R, V, T_2 , and L_3 .

Remark 1.1. The colon indicates the letters which can come after the given letter(s). For example, after the letter R we can add either the letter R or V . The letters T will just be used for T_1 and L for L_1 from this point on. Additionally, T_i will just refer to one of the two tangency hyperplanes and L_j as one of the three distinguished directions.

Another motivation for studying these incidence relations is understanding how the RVT coding system relates to the EKR coding system studied by P. Mormul ([Mor04]). The initial relationship between these two coding systems that characterize Goursat Multi-Flags was first studied in an appendix to [CH12]. One other benefit of establishing this connection is that it helps explain the relationship between the RVT code and the articulated arm system. In [PS12] F. Pelletier and M. Slayman investigate the relationship between the RVT codes and the EKR system for points within the fourth level of the \mathbb{R}^3 -Monster Tower along with the restrictions placed upon the positions of the bars within the articulated arm system [LR11]. We discuss this line of research in more detail within the conclusion section of the paper.

We also want to point out that all of our work done under the name of the Monster Tower can be done in the same setting as the Semple Tower with a base of \mathbb{A}^3 , affine 3-space, and with the terminology of Baby Monsters replaced by Subtowers.

In Sections 2 and 3 we present the main definitions needed in order understand the statements of our main results. We also present a few illustrative examples that will help the reader understand the proofs of the paper. Section 4 consists of the proofs of our main results. Then in Section 5 we provide a summary of our findings and further research directions.

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2. PRELIMINARIES AND MAIN RESULTS

Before presenting the results we want to give a brief summary of the RVT coding system. A more detailed discussion is presented in Subsection 2.3.

The RVT coding system partitions the points at each level of the Monster Tower. An RVT code is a word in the letters R, V, T_i for $i = 1, 2$, and L_j for $j = 1, 2, 3$ subject to certain spelling rules. The partial spelling rules were given in [CM12] and [CH12] and state that the initial letter in any RVT code must be the letter R , the letters T_i and L_j cannot immediately follow the letter R , the first occurrence of the letter L_1 can only come after

the letters V and T_1 , and the first appearance of the letters T_2 and L_j for $j = 2, 3$ cannot appear until the letter L_1 has been introduced. This gives an incomplete list of spelling rules. For example, we do not know what letters can come after T_2 or L_j for $j = 2, 3$.

Example 2.1. The codes $RRRR$, RVT_1 , RVL , $RRVVT_1$, and RVL_1T_2 are allowable RVT codes under our partial spelling rules. The codes RRT_1 , $RVRL_2$, and RVT_2 are not allowed, though.

We complete these spelling rules for the \mathbb{R}^3 -Monster Tower with the following theorems. In Theorem 2.2 the “:” denotes which letters can be placed after a given letter. For example, given the letter R one can put either the letters R or V after it.

Theorem 2.2. The complete spelling rules for any RVT code are as follows:

- (1) Any RVT code string must begin with the letter R .
- (2) R : R and V .
- (3) V and $T (= T_1)$: R, V, T , and L .
- (4) $L (= L_1)$ and L_j for $j = 1, 2, 3$: R, V, T_i for $i = 1, 2$, and L_j for $j = 1, 2, 3$.

In order to prove this we use Theorem 2.3 below as the base case of an induction argument showing that the spelling rules above hold for the first occurrence of the distinguished directions L_j for $j = 2, 3$ and T_2 .

Theorem 2.3. Let ω be an RVT code of length k where the k -th letter (the last letter in the code) is the first occurrence in the code of either the letters L_j for $j = 2, 3$ or T_2 . Then the incidence relations between the various critical hyperplanes in $\Delta_k(p_k)$ for $p_k \in \omega$ are determined and summarized in Table 2.

Remark 2.4 (Theorem 2.3). We want to provide an example of what we mean by “first occurrence” in the statement of Theorem 2.3. The RVT code $RVLT_2$ of length 4 has the first occurrence of the letter T_2 as the last letter in the code. Another example is the code $RVVLL_3$ of length 5, where the first occurrence of the letter L_3 is the last letter of the code.

Also, the critical hyperplanes will tell us which letters can come after the first occurrence of the letters L_j for $j = 2, 3$ and T_2 . By knowing which hyperplanes appear above these letters we are able to assert that the letters R, V, L_j for $j = 1, 2, 3$, and T_i for $i = 1, 2$ can come after the first occurrence of L_2 and L_3 , and that the letters R, V, L_3 , and T_2 can be put after the first occurrence of the letter T_2 . This connection between the critical hyperplanes and RVT code will be discussed in more detail in Subsection 2.3.

A *geometric distribution* hereafter denotes a linear subbundle of the tangent bundle with fibers of constant dimension.

2.1. Prolongation. Let the pair (Z, Δ) denote a manifold Z of dimension d equipped with a distribution Δ of rank r . We denote by $\mathbb{P}(\Delta)$ the *projectivization* of Δ , meaning each vector space fiber $\Delta(p)$ for $p \in Z$ is projectivized resulting in fibers of dimension $(r - 1)$. As a manifold,

$$\mathbb{P}(\Delta) \equiv Z^1,$$

TABLE 1. Outline of Common Notation

Notation	Description	Subsection
$\mathcal{P}^k(n)$	k -th level of the \mathbb{R}^{n+1} Monster/Semple Tower	2.2
Δ_k	Rank n distribution associated to $\mathcal{P}^k(n)$	2.2
$\delta_i^j(p)$	Baby Monster through p in $\Delta_k(p)$, $k = i + j$	2.3 and 2.5
ω	The <i>RVT</i> code of a point; string of <i>R</i> 's, <i>V</i> 's, <i>T</i> 's, and <i>L</i> 's	2.2, 2.4, and 2.3

has dimension $d + (r - 1)$.

Example 2.5. Take $Z = \mathbb{R}^3$, $\Delta = TR^3$ viewed as a rank 3 distribution. Then Z^1 is simply the trivial bundle $\mathbb{R}^3 \times \mathbb{P}^2$, where the factor on the right denotes the projective plane.

Various geometric objects in Z can be canonically prolonged (lifted) to the new manifold Z^1 . In what follows prolongations of curves and transformations are essential.

The manifold Z^1 also comes equipped with a distribution Δ_1 called the *Cartan prolongation of Δ* ([BH93]) which is defined as follows. Let $\pi : Z^1 \rightarrow Z$ be the projection map $(p, \ell) \mapsto p$. Then

$$\Delta_1(p, \ell) = d\pi_{(p, \ell)}^{-1}(\ell),$$

i.e. $\Delta_1(p, \ell)$ is the subspace of $T_{(p, \ell)}Z^1$ consisting of all tangents to curves in Z that pass through p with a velocity vector contained in ℓ . It is easy to check using linear algebra that Δ_1 is also a distribution of rank r .

We note that the word prolongation will always be synonymous with Cartan prolongation.

2.2. The Monster Tower and RVT Coding. We start with \mathbb{R}^{n+1} as our base manifold Z and take $\Delta_0 = T\mathbb{R}^{n+1}$. Prolonging Δ_0 we get $\mathcal{P}^1(n) = \mathbb{P}(\Delta_0)$ equipped with the distribution Δ_1 of rank $n + 1$. By iterating this process we end up with the manifold $\mathcal{P}^k(n)$ which is endowed with the rank $n + 1$ distribution $\Delta_k = (\Delta_{k-1})^1$ and fibered over $\mathcal{P}^{k-1}(n)$. In this paper we will be studying the case $n = 2$.

Definition 2.1. The \mathbb{R}^{n+1} -*Monster Tower* is a sequence of manifolds with distributions, $(\mathcal{P}^k, \Delta_k)$, together with fibrations

$$\dots \rightarrow \mathcal{P}^k(n) \rightarrow \mathcal{P}^{k-1}(n) \rightarrow \dots \rightarrow \mathcal{P}^1(n) \rightarrow \mathcal{P}^0(n) = \mathbb{R}^{n+1}$$

and we write $\pi_{k,i} : \mathcal{P}^k(n) \rightarrow \mathcal{P}^i(n)$ for the respective bundle projections.

Remark 2.6 (Notation from Table 2). We assume that each of the above *RVT* codes is of total length k . At the same time, each of the intermediate and undetermined *RVT* blocks, denoted by ω' , are assumed to be of total length r .

It is worth noting that the L_2 and L_3 directions will have the same critical hyperplane configuration as L points, but the level at which the Baby Monsters are born from which will produce these critical hyperplanes will be different than in the L case. In short, the source of the critical hyperplanes over the L_j 's depends on which letters come before these distinguished directions.

TABLE 2. Critical Hyperplanes Incidence Relations.

<i>RVT</i> Code	T_1	T_2
$R\omega L_1$	$\delta_{k-1}^1(p_k)$	$\delta_{k-2}^2(p_k)$
$R\omega L_2$:		
$R\omega'VT^mLL_2$ for $m \geq 0$	$\delta_{m+2+r}^2(p_{m+r+4})$	$\delta_{1+r}^{\overline{m+3}}(p_{m+r+4})$
$R\omega'LLL_2$	$\delta_{k-2}^2(p_k)$	$\delta_{k-3}^3(p_k)$
$R\omega L_3$:		
$R\omega'VT^mLL_3$ for $m \geq 0$	$\delta_{k-1}^1(p_k)$	$\delta_{r+1}^{\overline{m+3}}(p_{r+m+4})$
$R\omega'LLL_3$	$\delta_{k-1}^1(p_k)$	$\delta_{k-3}^3(p_k)$
$R\omega'VT^mLT_2L_3$ for $m \geq 0$	$\delta_{k-1}^1(p_k)$	$\delta_{r+1}^{m+4}(p_{m+r+5})$
$R\omega'LLT_2L_3$	$\delta_{k-1}^1(p_k)$	$\delta_{k-4}^4(p_k)$
$R\omega'VT^mLL_2T_2L_3$ for $m \geq 0$	$\delta_{k-1}^1(p_k)$	$\delta_{m+r+2}^4(p_{m+r+6})$
$R\omega'VT^mL^sL_2T_2L_3$ for $m \geq 0, s \geq 2$	$\delta_{k-1}^1(p_k)$	$\delta_{m+r+2}^4(p_{m+r+6})$
$R\omega T_1$:		
$R\omega'LT_1$	$\delta_{k-2}^2(p_k)$	None
$R\omega'L_1T_1$	$\delta_{k-2}^2(p_k)$	None
$R\omega'L_2T_1$	$\delta_{k-3}^3(p_{k+3})$	None
$R\omega'L_3T_1$	$\delta_{k-2}^2(p_k)$	None
$R\omega T_2$:		
$R\omega'VT^mLT_2$ for $m \geq 0$	None	$\delta_{1+r}^{\overline{m+3}}(p_{m+r+4})$
$R\omega'LLT_2$	None	$\delta_{k-3}^3(p_k)$
$R\omega'VT^mLL_2T_2$ for $m \geq 0$	None	$\delta_{r+1}^{m+4}(p_{m+r+5})$
$R\omega'VT^mL^sL_2T_2$ for $m \geq 0, s \geq 2$	None	$\delta_{r+m+s}^4(p_{m+s+r+4})$
$R\omega'VT^mLL_3T_2$ for $m \geq 0$	None	$\delta_{r+1}^{m+4}(p_{m+r+5})$
$R\omega'VT^mL^sL_3T_2$ for $m \geq 0, s \geq 2$	None	$\delta_{m+s+r}^3(p_{m+s+r+3})$

When we prove Theorem 2.2 we can restrict our attention to looking at the cases of L_j for $j = 2, 3$ and T_2 , since from Definition 2.7 (below) we can say $T = T_1$ and $L = L_1$, which from previous work, we know what the spelling rules are for these letters. Despite this fact, we will still list where the critical hyperplane come from for these cases in Table 2.

The letter T_2 gives us a new hyperplane configuration that was previously unknown and is illustrated in Figure 2b.

In proving Theorem 2.3, we will present a method to determine what critical hyperplanes and incidence relations will occur over any point in the \mathbb{R}^3 -Monster Tower. This method can be applied to the \mathbb{R}^n Monster Tower as well. In addition, this algorithm will tell us which fibers the Baby Monsters are born from.

We also want to point out that in Subsection 2.4 we will provide the basic definitions needed to understand the *RVT* coding system. Since we will be working exclusively with the $n = 2$ Monster Tower in this paper, we will just write \mathcal{P}^k for $\mathcal{P}^k(2)$.

2.3. Baby Monsters. One can apply prolongation to any analytic m -dimensional manifold F in place of \mathbb{R}^m . Start out with $\mathcal{P}^0(F) = F$ and take $\Delta_0^F = TF$. Then the prolongation of the pair (F, Δ_0^F) is $\mathcal{P}^1(F) = \mathbb{P}TF$ equipped with the rank m distribution $\Delta_1^F \equiv (\Delta_0^F)^1$. By iterating this process k times we end up with new the pair $(\mathcal{P}^k(F), \Delta_k^F)$, which is analytically diffeomorphic to $(\mathcal{P}^k(m-1), \Delta_k)$ ([CM12]).

Now, apply this process to the fiber $F_i(p_i) = \pi_{i,i-1}^{-1}(p_{i-1}) \subset \mathcal{P}^i(m-1)$ through the point p_i at level i . The fiber is an $(m-1)$ -dimensional integral submanifold for Δ_i . Prolonging, we see that $\mathcal{P}^1(F_i(p_i)) \subset \mathcal{P}^{i+1}(m-1)$, and $\mathcal{P}^1(F_i(p_i))$ has the associated distribution $\delta_i^1 \equiv \Delta_1^{F_i(p_i)}$; that is,

$$\delta_i^1(q) = \Delta_{i+1}(q) \cap T_q(\mathcal{P}^1(F_i(p_i)))$$

which is a hyperplane within $\Delta_{i+1}(q)$, for $q \in \mathcal{P}^1(F_i(p_i))$. When this prolongation process is iterated, we end up with the submanifolds

$$\mathcal{P}^j(F_i(p_i)) \subset \mathcal{P}^{i+j}(m-1)$$

with the hyperplane subdistribution $\delta_i^j(q) \subset \Delta_{i+j}(q)$ for $q \in \mathcal{P}^j(F_i(p_i))$.

Definition 2.2. A *Baby Monster* born at level i , is a sub-tower $(\mathcal{P}^j(F_i(p_i)), \delta_i^j)$, for $j \geq 0$ within the ambient Monster Tower. If $q \in \mathcal{P}^j(F_i(p_i))$ then we will say that a Baby Monster born at level i passes through q and that $\delta_i^j(q)$ is a *critical hyperplane* passing through q , which was born at level i .

Remark 2.7. In this paper $n = 3$, where our base is \mathbb{R}^3 , which results in our the distribution Δ_{i+j} being of rank 3 and that all of our critical hyperplanes δ_i^j being 2-planes within the distribution Δ_{i+j} . As a result, we will replace the term “hyperplane” by just “plane” for the rest of the paper.

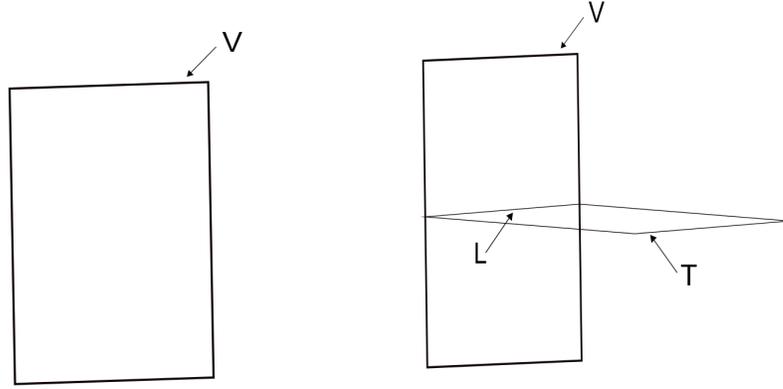
Definition 2.3. The *vertical plane* $V_k(q)$, often written as just $V(q)$, is the critical plane $\delta_k^0(q) = T_q(F_k(p_k))$. We note that it is always one of the critical planes passing through q .

2.4. RC coding of points.

Definition 2.4. A direction $\ell \subset \Delta_k(p_k)$, $k \geq 1$ is called a *critical direction* if there exists an immersed curve at level k that is tangent to the direction ℓ , and whose projection to level zero, meaning the base manifold, is a constant curve. If no such curve exists, then we call ℓ a *regular direction*. Note that while ℓ is technically a line we will by an abuse of terminology refer to it as a direction.

Definition 2.5. A point $p_k \in \mathcal{P}^k$, where $p_k = (p_{k-1}, \ell)$ is called a *regular or critical point* if the line ℓ is a regular direction or a critical direction.

Definition 2.6. For $p_k \in \mathcal{P}^k$, $k \geq 1$ and $p_i = \pi_{k,i}(p_k)$, we write $\omega_i(p_k) = R$ if p_i is a regular point and $\omega_i(p_k) = C$ if p_i is a critical point. Then the word $\omega(p_k) = \omega_1(p_k) \cdots \omega_k(p_k)$ is called the *RC code* for the point p_k . The number of letters within the *RC code* for p_k equals the level of the tower that the point lives in. Note that $\omega_1(p_k)$ is always equal to R , see [CH12].



(A) Above a regular point.

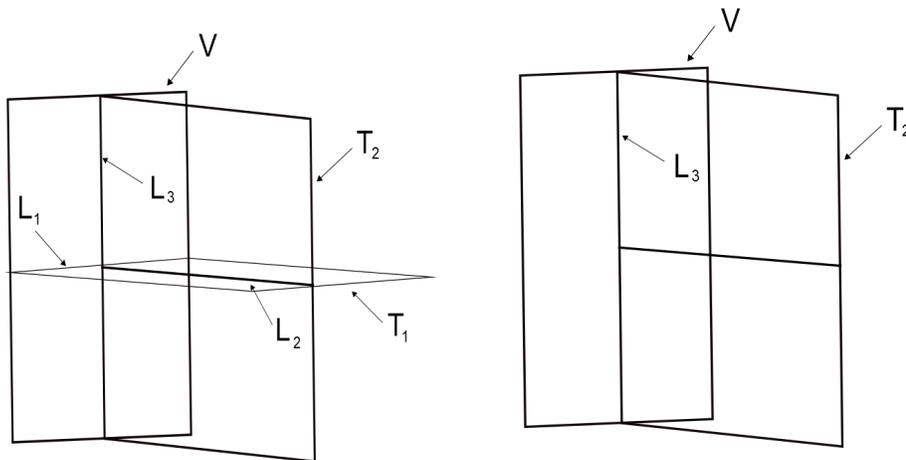
(B) Above a vertical or tangency point.

FIGURE 1. Arrangement of critical planes over the letters R , V , and T .

In the following section we will show that there is more than one kind of critical direction that can appear within the distribution Δ_k .

2.5. Arrangements of critical planes for $n = 2$. Over any point p_k , at the k -th level of the Monster Tower, there is a total of four possible plane configurations for Δ_k . These three configurations are shown in Figures 1a, 1b, 2a, and 2b.

Figure 1a is the picture for $\Delta_k(p_k)$ when the k -th letter in the RVT code for p_k is the letter R . This means that the vertical plane, labeled with a V , is the only critical plane sitting inside of $\Delta_k(p_k)$. Figure 1b is the picture for $\Delta_k(p_k)$ when the k -th letter in the RVT code is either the letter V or the letter T . This gives a total of two critical planes sitting inside of $\Delta_k(p_k)$ and one distinguished critical direction: one is the vertical plane and the other is the tangency plane, labeled by the letter T . The intersection of vertical and tangency plane gives a distinguished critical direction, which is labeled by the letter L . Now, Figure 2a describes the picture for $\Delta_k(p_k)$ when the k -th letter in the RVT code of p_k is the letter L . Figure 2a depicts this situation where there is now a total of three critical planes: one is the vertical plane, and two tangency planes, labeled as T_1 and T_2 . Now, because of the presence of these three critical planes we need to refine our notion of an L direction and add two more distinct L directions. These three directions are labeled as L_1 , L_2 , and L_3 . We will prove in this paper that this configuration will also persist for L_j for $j = 2, 3$, as well. The last figure given in Figure 2b is the critical plane configuration



(A) Above an L_j for $j = 1, 2, 3$ point.

(B) Above a T_2 point.

FIGURE 2. Arrangement of critical planes over L_j for $j = 1, 2, 3$ and T_2 .

over the letter T_2 . It is worth pointing out that this is an unexpected result, at least to the authors of the paper, and gives us a new incidence relation that was previously unknown.

With the above pictures in mind, we can now refine our RC coding and define the RVT code for points within the Monster Tower. Take $p_k \in \mathcal{P}^k$ and if $\omega_i(p_k) = C$ then we look at the point $p_i = \pi_{k,i}(p_k)$, where $p_i = (p_{i-1}, \ell_{i-1})$. Then depending on which critical plane, or distinguished direction, contains ℓ_{i-1} , we replace the letter C by one of the letters V , T_i for $i = 1, 2$, or L_j for $j = 1, 2, 3$. We need to define the two distinguished tangency planes and the three distinguished directions that arise from the intersection of the three critical planes when the two tangency planes are present in Δ_k .

2.6. KR -Coordinates. For our work within the Monster Tower one needs to work with a suitable coordinate system. In Section 4.4 of [CH12] we detailed the basics of *Kumpe-Rubin coordinates*, or KR -coordinates for short, by determining the coordinates for the RVT class RVL . We will briefly summarize this construction to help the reader understand some of the basic properties of this coordinate system.

We begin with the pair $(\mathcal{P}^0, \Delta_0) = (\mathbb{R}^3, T\mathbb{R}^3)$ with the coframe $\{dx, dy, dz\}$ for $T\mathbb{R}^3$. We center our chart at (p_0, ℓ_0) where $p_0 = (0, 0, 0)$ and ℓ_0 is a direction in $T_{p_0}\mathbb{R}^3$ such that

$dx|_{\ell_0} \neq 0$. This allows us to introduce fiber affine coordinates $[dx : dy : dz] = [1 : \frac{dy}{dx} : \frac{dz}{dx}]$, where

$$u = \frac{dy}{dz} \quad v = \frac{dz}{dx},$$

and results in

$$\{dy - udx = 0, dz - vdx = 0\} = \Delta_1.$$

Now take the point $p_1 = (p_0, \ell_0)$ in the first level of the tower \mathcal{P}^1 and look at the line $\ell_1 \subset \Delta_1(p'_1)$ for p'_1 . Recall that the vertical plane is given by $V = \delta_1^0(p_1) = \text{span}\{\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\}$. Let us suppose $\ell_1 = \text{span}\{\frac{\partial}{\partial u}\}$ so that it is in the vertical plane. Then near ℓ_1 we have $[dx : du : dv] = [\frac{dx}{du} : 1 : \frac{dv}{du}]$ to give the affine coordinates

$$u_2 = \frac{dx}{du} \quad v_2 = \frac{dv}{du},$$

with

$$\Delta_2 = \{dy - udx = 0, dz - vdx = 0, \\ dx - u_2du = 0, dv - v_2du = 0\}.$$

Then the rank 3 distribution Δ_2 is coframed by $[du : du_2 : dv_2]$ with the vertical plane given by $du = 0$ and the tangency plane T given by $du_2 = 0$. The point $p_3 = (p_2, \ell)$ with ℓ being an L direction means that both $du|_{\ell} = 0$ and $du_2|_{\ell} = 0$. As a result, the only choice for local coordinates near p_3 is given by $[\frac{du}{dv_2} : \frac{du_2}{dv_2} : 1]$ to give the fiber coordinates

$$u_3 = \frac{du}{dv_2} \quad v_3 = \frac{du_2}{dv_2},$$

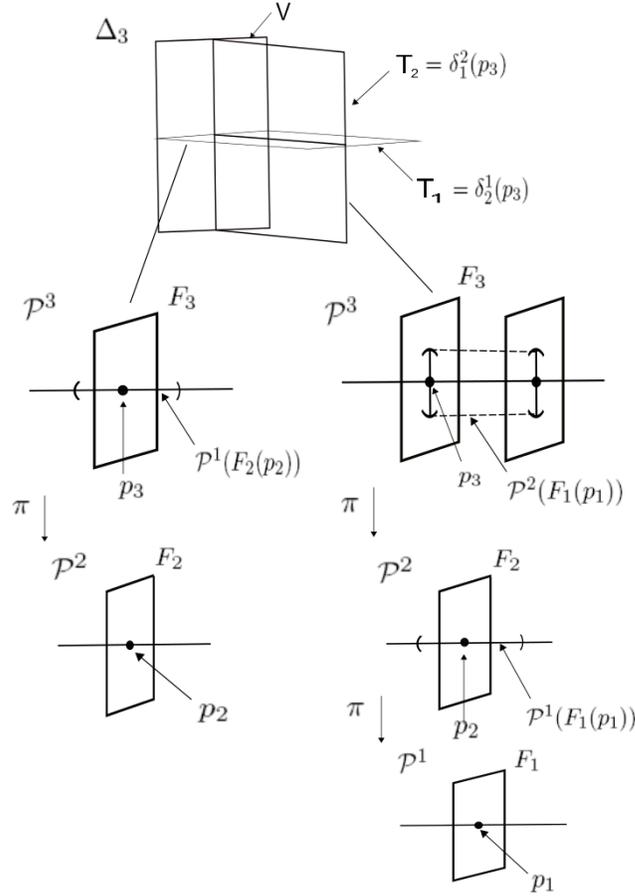
where the distribution Δ_3 is described by

$$\Delta_3 = \{dy - udx = 0, dz - vdx = 0, \\ dx - u_2du = 0, dv - v_2du = 0, \\ du - u_3dv_2 = 0, du_2 - v_3dv_2 = 0\}.$$

From here one can show that $T_1 = \delta_2^1(p_3)$ and $T_2 = \delta_1^2(p_3)$ in $\Delta_3(p_3)$ with p_3 given in KR -coordinates by $p_3 = (x, y, z, u, v, u_2, v_2, u_3, v_3) = (0, 0, 0, 0, 0, 0, 0, 0, 0)$.

Definition 2.7 (Definitions of the T_1 and T_2 critical planes). At any given level of the Monster Tower, say the k -th level, we can write the fiber space at the k -th level in the form $F_k(p_k) = \pi_{k,k-1}^{-1}(p_{k-1})$ and is written in KR -coordinates, see [CM12] for details, as $F_k(p_k) = (p_{k-1}, u_k, v_k)$, where p_{k-1} is a fixed point at the $(k-1)$ -st level. We can represent the vertical space V_k in a neighborhood of the point p_k as $\delta_k^0(p_k) = \text{span}\{\frac{\partial}{\partial u_k}, \frac{\partial}{\partial v_k}\}$. T_1 is the critical plane that intersects $\text{span}\{\frac{\partial}{\partial v_k}\}$. In local KR -coordinates for Δ_k it is given by $[df_k : du_k : dv_k] = [a : 0 : b]$ for $a, b \in \mathbb{R}$, with $a \neq 0$.¹

¹The covector df_k is known as the *uniformizing coordinate*, see [CM12] for more details.


 FIGURE 3. Critical plane configuration over $p_3 \in RVL$.

The T_2 directions are characterized as the directions which do not intersect the $\frac{\partial}{\partial v_k}$ component of the vertical space V_k . In KR -coordinates we have the T_2 critical plane is characterized by $[df_k : du_k : dv_k] = [a : b : 0]$ for $a, b \in \mathbb{R}$ and $a \neq 0$.

Definition 2.8 (Definition of the L_j directions for $j = 1, 2, 3$). The distinguished line L_1 is given by $V \cap T_1$, L_2 by $T_1 \cap T_2$, and L_3 by $V \cap T_2$.

Example 2.8 (Examples of RVT codes). The following are examples of RVT codes: $R \cdots R$, $RVVT$, $RVLT_2R$, and $RVLL_2$. The code RTL is not allowed because the letter T is preceded by the letter R and the code RLT_2 is not allowed because the letter L comes immediately after the letter R .

In the subsequent example we will redo the calculation for the critical planes over the L direction that was first presented in [CH12]. This time though we will be working with a different method which tells us where the various critical planes come from in the distribution Δ_k over an L point. This will help us avoid guessing which levels these critical

planes originate from. We will also be using this method in order to prove Theorem 2.3. Figure 3 illustrates which fibers these critical planes come from and our incidence relations.

Example 2.9 (Calculation of Incidence Relations for the Class RVL). We begin with the local KR -coordinates for a point p_3 in the third level of the \mathbb{R}^3 - Monster Tower, which is in the RVT class RVL . From the work done in Example 4.4 of [CH12] we have that Δ_3 in a neighborhood of $p_3 \in RVL$ is coframed by $[dv_2 : du_3 : dv_3]$. The vertical plane is characterized by $[0 : a : b]$ for $a, b \in \mathbb{R}$ with $a \neq 0$ in these coordinates. If there are any other critical planes then they are either of the form $[a : 0 : b]$ with $a \neq 0$ to give the critical plane T_1 or of the form $[a : b : 0]$ with $a \neq 0$ to give the T_2 critical plane. We will start with planes of this form and work backwards until we determine which fiber they must have come from. If we are unable to do this or arrive at a contradiction along the way (such as showing that there is a tangency plane above a regular point), then it implies that no such critical plane could have existed at that level.

We start with $[dv_2 : du_3 : dv_3] = [a : 0 : b]$ and work backwards. From looking at these projective coordinates it tells us that the coordinates v_2 and v_3 can't both be zero in a neighborhood of p_3 for the Baby Monster that gives rise to this critical plane, but u_3 must be identically zero. Then, one level down we have that Δ_2 is coframed in a neighborhood of $p_2 \in RV$ by $[du : du_2 : dv_2]$ and since we move in an L direction the coordinates on the third level are $u_3 = \frac{du}{dv_2}$ and $v_3 = \frac{du_2}{dv_2}$. This implies that the u coordinate must be identically zero on the Baby Monster and while the v_3 coordinate will be zero at p_3 it cannot be identically zero on the manifold. As a result, our Baby Monster one level down is given by $[du : du_2 : dv_2] = [0 : a : b]$. This tells us we can stop and do not need to go any farther down in the tower because our critical plane comes from the fiber space $F_2(p_2) = (p_1, u_2, v_2)$, implying that T_1 comes from the Baby Monster δ_2^1 .

The second tangency plane is given by $[dv_2 : du_3 : dv_3] = [a : b : 0]$. Again, these projective coordinates tell us that the coordinates $v_2 \neq 0$ and $u_3 \neq 0$ in a neighborhood of p_3 for our Baby Monster which gives rise to the T_2 tangency plane, but $v_3 \equiv 0$. One level down, in a neighborhood of p_2 , we have Δ_2 is coframed by $[du : du_2 : dv_2]$, since we moved in an L direction one level up and our plane is given by $[du : du_2 : dv_2] = [a : 0 : b]$. This implies that the $u_2 \equiv 0$ along the Baby Monster. Going down one more level we have Δ_1 in a neighborhood of p_1 is coframed by $[dx : du : dv]$. Since we moved in a V direction we end up with $[dx : du : dv] = [\frac{dx}{du} : 1 : \frac{dv}{du}]$ to give the coordinates. $u_2 = \frac{dx}{du}$ and $v_2 = \frac{dv}{du}$, which results in $[dx : du : dv] = [0 : a : b]$ along our Baby Monster. We can stop at this point and see that we do not need to go any farther down in the tower because our Baby Monster will arise from the fiber space $F_1(p_1) = (p_0, u, v)$, indicating that T_2 comes from the Baby Monster δ_1^2 , which again is in agreement with what we expect from Example 4.4 in [CH12].

3. THE CRITICAL PLANE METHOD

The above example helps to illustrate our algorithm for determining the various critical planes. Here is the overall idea of our algorithm: Take an RVT code of length k called ω . Say we are interested in understanding which critical letters we can add to the end of ω ,

and, at the same time, understanding where the critical planes originate from within the Monster Tower. Take a point $p_k \in \omega$ and determine what the local KR -coordinates are that describe the distribution $\Delta_k(p_k)$. From there we start with how the various critical planes are presented within the coframing for the distribution $\Delta_k(p_k)$, based upon Definition 2.7. Then we work backwards looking at the way these critical planes are represented in the coframing for each $\Delta_i(p_i)$ for $0 \leq i \leq k - 1$. If we end up with two fiber coordinates being nontrivial for our critical plane at some intermediate level, meaning u_j, v_j for some $1 \leq j \leq k - 1$ will both be nonvanishing for our critical plane, it tells us that our critical plane within $\Delta_k(p_k)$ will originate from the fiber space $F_j(p_j)$. If we can't find such a fiber space or if this critical plane is projected to a lower level and gives rise to a critical plane in $\Delta_i(p_i)$ for $1 \leq i \leq k - 1$ which can't appear in the distribution based upon our spelling rules, then it will imply that this critical plane can not appear within the distribution $\Delta_k(p_k)$.

While the above does not give an algorithm in the strict sense, it is a method for how one can determine if these critical planes exist or not within our distribution. We believe that it will also be much more illuminating to present the mechanics of how the above outline works through various examples such as the one above for the case of RVL and the ones that follow below.

3.1. The Critical Planes over the T_2 Plane. This section will be building upon the calculations done in Example 2.9. We show that the critical plane configuration over any point strictly in the code $RVLT_2$ in any critical plane is the vertical plane and the T_2 plane shown in Figure 5.

The distribution Δ_3 in a neighborhood of $p_3 \in RVL$ is coframed by $[dv_2 : du_3 : dv_3]$ and for $\ell \subset T_2$ we have the $dv_3 = 0$ and must have $dv_2 \neq 0$ $du_3 \neq 0$. This results in fiber coordinates $u_4 = \frac{du_3}{dv_2}$ and $v_4 = \frac{dv_3}{dv_2}$ with $u_4 \neq 0$ and $v_4 = 0$. For a direction ℓ to be completely in the T_2 plane it needs to be a linear combination of both the directions $\frac{\partial}{\partial v_2}$ and $\frac{\partial}{\partial u_3}$, i.e. in the $span\{\frac{\partial}{\partial v_2}, \frac{\partial}{\partial u_3}\}$. The distribution in a neighborhood of $p_4 \in RVLT_2$ is given by

$$\begin{aligned} \Delta_4 = \{ & dy - udx = 0, dz - vdx = 0, \\ & dx - u_2du = 0, dv - v_2du = 0 \\ & du - u_3dv_2 = 0, du_2 - v_3dv_2 = 0 \\ & du_3 - u_4dv_2 = 0, dv_3 - v_3dv_2 = 0\} \subset T\mathcal{P}^4. \end{aligned}$$

Our next step is to determine what the critical planes are below the point p_4 in the distribution $\Delta_4(p_4)$. In short, one can look at the prolongation of the fibers $F_3(p_3)$, $F_2(p_2)$, and $F_1(p_1)$ to see what the critical planes are and if they will appear in the distribution over the point p_4 . However, for a more general RVT code it is difficult and somewhat cumbersome to prolong each fiber space. Even in this case one would have to prolong 3 different fiber spaces in order to determine where the possible critical planes would come from. We will show, using the algorithm presented in Example 2.9, that the only critical

plane, other than the vertical one, over the point $p_4 \in RVLT_2$ is the T_2 critical plane that comes from the fiber $F_1(p_1)$.

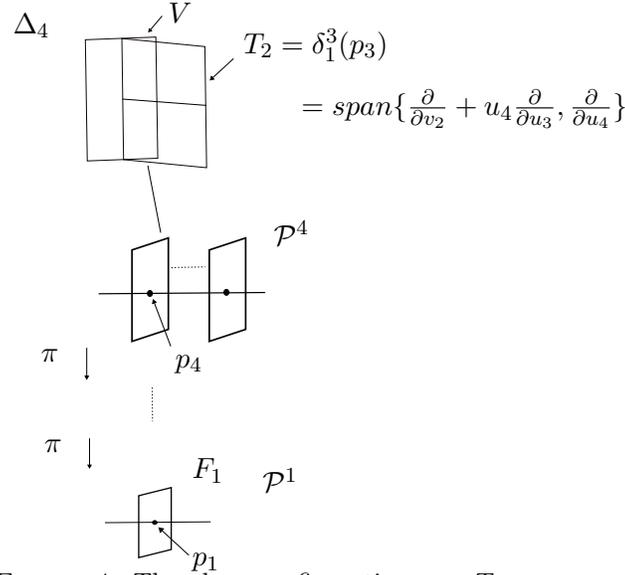
3.1.1. *Showing that $T_2 = \delta_1^3(p_4)$.* We begin with the coframing for Δ_4 in a neighborhood of the point p_4 given in KR -coordinates by $[dv_2 : du_4 : dv_4]$. If the T_2 plane is to exist it must be of the form $[dv_2 : du_4 : dv_4] = [a : b : 0]$ for $a, b \in \mathbb{R} - 0$. If there exists a Baby Monster that gives rise to this T_2 critical plane it must have both $v_2 \neq 0$ and $u_4 \neq 0$ in a neighborhood of p_4 on this submanifold. Going one level down we have Δ_3 is coframed by $[dv_2 : du_3 : dv_3]$ and since $u_4 = \frac{du_3}{dv_2}$ and $v_4 = \frac{dv_3}{dv_2}$ it implies that our Baby Monster must have $u_3 \neq 0$ in a neighborhood of p_4 , but v_3 will vanish. Moving another level down and looking at the KR -coordinates for Δ_2 gives $[du : du_2 : dv_2]$ with $u_3 = \frac{du}{dv_2}$ and $v_3 = \frac{dv_2}{dv_2}$ it implies that our Baby Monster will have $u \neq 0$ and $u_2 \equiv 0$. Then moving down one last level we have Δ_1 is coframed by $[dx : du : dv]$ with $u_2 = \frac{dx}{du}$ and $v_2 = \frac{dv}{du}$, since u_2 is trivial it forces x to be trivial on our Baby Monster, but since $v_2 \neq 0$ it implies $v \neq 0$ can't be trivial as well. This means that both the coordinates $u \neq 0$ and $v \neq 0$ in a neighborhood of p_4 and hence we do not need to go any father down in the tower and conclude that our T_2 critical plane will come from prolonging the fiber space $F_1(p_1) = \pi_{1,0}^{-1}(p_0) = (0, 0, 0, u, v)$ and that we will end up with $T_2 = \delta_1^3(p_4)$ in $\Delta_4(p_4)$. In KR -coordinates we will have that δ_1^3 is given by $\text{span}\{\frac{\partial}{\partial v_2} + u_4 \frac{\partial}{\partial u_3}, \frac{\partial}{\partial u_4}\}$.

3.1.2. *Showing that T will not appear in Δ_4 .* Now, let's show that the T critical plane will not appear in $\Delta_4(p_4)$. Suppose that the T plane did appear in $\Delta_4(p_4)$, then in our coframing for the distribution it would be of the form $[dv_2 : du_4 : dv_4] = [a : 0 : b]$ with $a, b \in \mathbb{R}$ with $a \neq 0$. Then one level down we would have in the coframing for Δ_3 , in neighborhood of p_3 , that $[dv_2 : du_3 : dv_3]$ with $u_4 = \frac{du_3}{dv_2}$ and $v_4 = \frac{dv_3}{dv_2}$ which implies that Baby Monster must be of the form $[dv_2 : du_3 : dv_3] = [a : 0 : b]$ with $a, b \in \mathbb{R}$ and $a \neq 0$. However we would have $u_4 = 0$ at p_4 , which creates a contradiction to what was established above about the coordinate $u_4 \neq 0$ at the point p_4 .

In summary, there are two critical planes over the T_2 point: one that is the vertical plane and the other comes from a vertical plane, prolonged 3 times, that results in a T_2 critical plane. In Figure 4 we see how fibers are prolonged to give us our critical plane T_2 .

Remark 3.1 (Continuing in the T_2 direction). From using the above techniques in Subsection 3.1, it is not hard to show that if we continue moving in the T_2 directions and look at the RVT code $RVLT_2 \cdots T_2$ that we will end up with the exact same critical plane configuration and that the T_2 critical plane will come from the same source, the fiber space $F_1(p_1)$. This is illustrated in Figure 6 where the coordinates u_4, \dots, u_k cannot be zero when we evaluate at the point p_k .

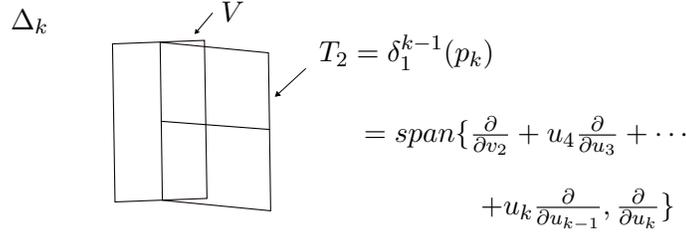
3.2. The Critical Planes over the L_2 Direction. In this section we will show what the critical planes are over the L_2 directions for the case of $RVLL_2$. Notice that this is the very first time that the L_2 direction can occur within any RVT code. We will show that the two tangency planes will be given by $T = \delta_2^2(p_4)$ and $T_2 = \delta_1^3(p_4)$. Initially we thought that the critical plane configuration above a L_2 point should be the same as a T


 FIGURE 4. The plane configuration over T_2 .

point or maybe a T_2 point, since it is contained in the intersection of those two planes. It is worth noting that because $L_2 = \text{span}\{\frac{\partial}{\partial v_2}\}$ and when we prolong to the fourth level we have $[dv_2 : du_3 : dv_3] = [1 : \frac{du_3}{dv_2} : \frac{dv_3}{dv_2}] = [1 : 0 : 0]$ with $u_4 = 0$ and $v_4 = 0$ at the point p_4 . This is different than the case above for the T_2 plane. In that case the coordinate $u_4 \neq 0$ and resulted in “blocking” the T_1 critical plane from appearing in the distribution.

We begin by looking at what the conditions are at each level with the KR -coordinates in relationship to the RVT code. Let $p_4 \in RVLL_2$. The first level: $[dx : dy : dz] = [1 : \frac{dy}{dx} : \frac{dz}{dx}]$ to give an R direction. The second level: $[dx : du : dv] = [\frac{dx}{du} : 1 : \frac{dv}{du}]$ to give a V direction. The third level: $[du : du_2 : dv_2] = [\frac{du}{dv_2} : \frac{du_2}{dv_2} : 1]$ to give an L direction. The L_2 direction in $\Delta_3(p_3)$ is given by $L_2 = \text{span}\{\frac{\partial}{\partial v_2}\}$. The fourth level is then given by $[dv_2 : du_3 : dv_3] = [1 : \frac{du_3}{dv_2} : \frac{dv_3}{dv_2}]$. Now, Δ_4 in a neighborhood of p_4 will be coframed by dv_2 , du_4 , and dv_4 . Let’s now look at what the possible critical planes are over the L_2 point. We have $V = \text{span}\{\frac{\partial}{\partial u_4}, \frac{\partial}{\partial v_4}\}$. Now, if a T_1 tangency plane is going to be present, then it will be of the form $[dv_2 : du_4 : dv_4] = [a : 0 : b]$ and if a T_2 tangency plane is going to be present then it will be of the form $[dv_2 : du_4 : dv_4] = [a : b : 0]$. We will work backwards to see which fiber produces these tangency critical planes. If we can’t find such a fiber, meaning one doesn’t exist, or a contradiction occurs, then no such tangency plane will exist.

3.2.1. *Showing $T_1 = \delta_2^2(p_4)$.* We said if the T_1 direction exists then it will be given in KR -coordinates by $[dv_2 : du_4 : dv_4] = [a : 0 : b]$. In local coordinates we know, at least in a neighborhood of p_4 , that the v_2 and v_4 coordinate will not be identically zero, but u_4 must be. This tells us the following information about the local coordinate structure of the Baby Monster. It should be of the form: $(x, y, z, \dots, u_4, v_4) = (0, 0, 0, \dots, \dots, v_2, \dots, 0, v_4)$, where v_4 will be zero at p_4 . The blanks mean that we don’t have enough information to

FIGURE 5. Critical plane configuration over $p \in RVLT_2 \cdots T_2$.

determine if these coordinates are zero or nonzero yet in a neighborhood of p_4 . Working backwards we have that $u_4 = \frac{du_3}{dv_2}$ and $v_4 = \frac{dv_3}{dv_2}$. Since $u_4 = 0$ in a neighborhood of the point p_4 then u_3 must be as well, but since v_4 is not identically zero in a neighborhood of p_4 we must have v_3 and v_2 are not identically zero as well. This tells us more information about the coordinate structure, so we can fill in some of the above blanks to give $(0, 0, 0, -, \dots, -, v_2, 0, v_3, 0, v_4)$, where both v_3 and v_4 will be zero at p_4 . Then, at the next level down we had $[du : du_2 : dv_2]$ with $u_3 = \frac{du}{dv_2}$ and $v_3 = \frac{dv_2}{dv_2}$. Since $u_3 = 0$ in a neighborhood of the point, then $u = 0$ will be as well, but v_3 is not identically zero in a neighborhood and means that u_2 and v_2 are not identically zero in a neighborhood of p_4 and we are able to control what values they take. As a result, we end up with the coordinate structure $(0, 0, 0, -, -, u_2, v_2, 0, v_3, 0, v_4)$. This information then points to the fact that the T_1 critical plane comes from the fiber $F_2(p_2) = (p_1, u_2, v_2)$, where p_1 is fixed and hence $T_1 = \delta_2^2(p_4)$.

3.2.2. *Showing $T_2 = \delta_1^3(p_4)$.* The T_2 tangency plane, if it is going to appear, must be of the form $[dv_2, du_4 : dv_4] = [a : b : 0]$. Using the same techniques as the above, we see that in local coordinates, in a neighborhood of p_4 , it is given by $(x, y, z, \dots, u_4, v_4) = (0, 0, 0, -, \dots, -, v_2, -, -, u_4, 0)$. Then one level down we must have $[dv_2 : du_3 : dv_3]$ and since $u_4 = \frac{du_3}{dv_2}$ and $v_4 = \frac{dv_3}{dv_2}$, we have the u_4 is not identically zero and so it implies that both u_3 and v_2 can't be identically zero as well, but since $v_4 = 0$ in a neighborhood of p_4 it implies v_3 will be as well. This gives $(0, 0, 0, -, \dots, -, v_2, u_3, 0, u_4, 0)$. The next level down gives $[du : du_2 : dv_2]$ with $u_3 = \frac{du}{dv_2}$ and $v_3 = \frac{du_2}{dv_2}$, where u_3 is not identically equal to zero and implies that u and v_2 are not as well. Then because $v_3 = 0$ in a neighborhood of p_4 we have the $u_2 = 0$ in a neighborhood as well. We now have the following form $(0, 0, 0, u, -, 0, v_2, u_3, 0, u_4, 0)$. The next level down gives $[dx : du : dv]$ with $u_2 = \frac{dx}{du}$, $v_2 = \frac{dv}{du}$, where $u_2 = 0$ in a neighborhood of p_4 and implies $x = 0$, but since v_2 is not identically equal zero we have v is not identically zero as well. This tells us that the coordinates are given by $(0, 0, 0, u, v, 0, v_2, u_3, 0, u_4, 0)$, which is enough information

and tells us we can stop and say that the T_2 critical plane in $\Delta_4(p_4)$ will come from the prolongation of the $F_1(p_1) = (0, 0, 0, u, v)$ fiber space.

4. PROOFS

Now we are in a position to prove the two theorems of this paper. In this section we will first present the proof of Theorem 2.3, but we will only present it for two general cases. In particular, we will only present the cases of $R\omega L_3$ and $R\omega T_2$. This is due to the fact that the techniques and procedure for the way the other cases are proved in an almost identical manner and the cases of $R\omega L_3$ and $R\omega T_2$ are illustrative examples. Once we prove Theorem 2.3 we move on to Theorem 2.2. Similar to the presentation of Theorem 2.3, we will again only present a few illustrative examples that demonstrate the techniques used to prove the rest of the cases. Specifically we will show the cases L_2 and T_2 .

4.1. The Proof of Theorem 2.3. In the various examples above we have already introduced the techniques and algorithm that we will use to prove this theorem. We will provide part of the details of the proof for the cases of L_3 and T_2 points.

In addition, while all of the sources for the various critical planes were found using our algorithm of working backwards, some of the cases below like with the class $R\omega L_2 T_2$ in Subsection 4.3.1, are easier to present if we work from the level of the fiber space and then prolong them to higher levels of the Monster Tower. This is done to help the reader follow along with the subtleties of the KR -coordinate system is built, which might be somewhat difficult to derive going from the last letter of a particular RVT code and then working backwards.

4.2. Over an L_3 point. In this section we will show that the L_3 direction has the same behavior as an L direction, meaning that the plane configuration over an L_3 point will look the same as an L direction. While the plane configuration will look like an L direction, the sources of the critical planes will be different and will depend on which letters came before the L_3 direction. Let $p_k \in R\omega L_3$ and let this be the first occurrence of the L_3 direction in the RVT code. This implies that we can say that the code actually has to be of the form $R\omega LL_3$ ². In fact we can say a little more because of the RVT spelling rules we can say

$$\text{that } R\omega LL_3 = \begin{cases} R\omega VT^m LL_3 & \text{for } m \geq 0 \\ R\omega LLL_3 & \end{cases}$$

Now, from looking at Table 2 one can see that the last 4 cases involve the letter T_2 (these are $R\omega' LLT_2 L_3, \dots, R\omega' VT^m L^s L_2 T_2 L_3$ for $m \geq 0, s \geq 0$). For the sake of clarity we will only present the above situation and leave these calculations to the reader, since the process used in those cases will be very similar to the following presentation.

We begin by showing that there is always at least one tangency plane over the point L_3 which is the $\delta_{k-1}^1(p_k)$ for $p_k \in R\omega LL_3$. We start with the fiber space $F_{k-1}(p_{k-1}) =$

²We should technically write $R\omega LL_3$ as $R\omega' LL_3$, but by an abuse of notation we will just keep using ω to signify an undetermined block of RVT code. We will also continue to do this in subsequent cases.

$(p_{k-2}, u_{k-1}, v_{k-1})$ and use prolongation to show that it will give the $T = \delta_{k-1}^1(p_k)$ critical plane. The first prolongation of the fiber space is given by

$$\begin{aligned} \mathcal{P}^1(F_{k-1}(p_{k-1})) &= (p_{k-2}, u_{k-1}, v_{k-1}, [dv_{k-2} : du_{k-1} : dv_{k-1}]) \\ &= (p_{k-2}, u_{k-1}, v_{k-1}, [0 : a : b]) = (p_{k-2}, u_{k-1}, v_{k-1}, [0 : 1 : \frac{b}{a}]) \\ &= (p_{k-2}, u_{k-1}, v_{k-1}, 0, v_k). \end{aligned}$$

Then when we evaluate at p_k we have that $\delta_{k-1}^1(p_k)$ will be given in the coframing for $\Delta_k(p_k)$ as $[du_{k-1} : du_k : dv_k] = [a : 0 : b]$, which gives us the T critical plane.

Now, a second tangency plane may show up depending on the previous letters that appear in the RVT code $R\omega LL_3$.

We show that there will be at least be two tangency planes in the distribution over L_3 in $R\omega VT^m LL_3$ for $m \geq 0$. Let ω be an RVT block of length r . We will show that the T_2 plane will be given by $\delta_{r+1}^{m+3}(p_{r+m+4})$ within $\Delta_{r+m+4}(p_{r+m+4})$. Again, we will suppose that such a critical plane exists and work backwards until we arrive at the fiber space where it was born. We begin with the coframing $[du_{r+m+3} : du_{r+m+4} : dv_{r+m+4}]$ in a neighborhood of p_k for Δ_{r+m+4} and see that the T_2 critical plane will be of the form $[a : b : 0]$ with $a \neq 0$. Moving one level down we have that since $L_3 = \text{span}\{\frac{\partial}{\partial u_{r+m+3}}\}$, $\Delta_{r+m+3}(p_{r+m+3})$ is coframed by $[dv_{r+m+2} : du_{r+m+3} : dv_{r+m+3}]$ and our Baby Monster would be given by $[a : b : 0]$ in this coframing, telling us that v_{r+m+2} and u_{r+m+2} will not vanish in a neighborhood of the point p_{r+m+2} on our Baby Monster, but v_{r+m+2} must vanish. One more level down the Baby Monster in our coframing for Δ_{r+m+2} is given by $[du_{r+1} : du_{r+m+2} : dv_{r+m+2}] = [a : 0 : b]$ since we move in an L direction with $u_{r+m+3} = \frac{du_{r+1}}{dv_{r+m+2}}$ and $v_{r+m+3} = \frac{dv_{r+m+2}}{dv_{r+m+2}}$. Then after this level we go backwards m steps, where at each of those levels we moved in T directions. This gives that our Baby Monster in $\Delta_{r+2}(p_{r+2})$ will be given by $[du_{r+1} : du_{r+2} : dv_{r+2}] = [a : 0 : b]$. Moving one last level down we have our that our Baby Monster in $\Delta_{r+1}(p_{r+1})$ is coframed by $[df_{r+1} : du_{r+1} dv_{r+1}] = [0 : a : b]$ since we moved in a V direction, which gives $u_{r+2} = \frac{df_{r+1}}{du_{r+1}}$ and $v_{r+2} = \frac{dv_{r+1}}{du_{r+1}}$, which means we can stop at this point and see that our T_2 critical plane comes from the fiber space $F_{r+1}(p_{r+1}) = (p_r, u_{r+1}, v_{r+1})$ and at level $k = r + m + 4$ is given by $\delta_{r+1}^{m+3}(p_{r+m+4})$.

When we look at the $R\omega LLL_3$ case, we can show, using a similar argument, that for $p_k \in R\omega LLL_3$ the other critical plane comes from the prolongation of the fiber space $F_{k-3}(p_{k-3})$. Prolonging 3 times gives $\delta_{k-3}^3(p_k) = \text{span}\{\frac{\partial}{\partial u_{k-1}}, \frac{\partial}{\partial u_k}\}$, which gives the T_2 critical plane in $\Delta_k(p_k)$.

4.3. Over a T_2 point.

In this section we show that there are two critical planes over T_2 points. These are given by the vertical plane and the T_2 tangency plane. Using $R\omega T_2$ as a starting point, we determine the origin of the T_2 critical plane. This will then help us see where the plane is situated and how it is prolonged to higher levels of the tower. Based on our knowledge of the spelling rules, and the fact that we are assuming that no other T_2 occurs in $R\omega T_2$,

the RVT codes look like either $R\omega LT_2$, $R\omega L_2 T_2$, or $R\omega L_3 T_2$, which is based upon the fact that we are assuming only one occurrence of each of the L_j 's for $j = 1, 2, 3$ occurring in the code as well. Start with the case of $R\omega LT_2$. We have

$$R\omega LT_2 = \begin{cases} R\omega VT^m LT_2 & \text{for } m \geq 0 \\ R\omega LLT_2 & \end{cases}$$

For the classes of the form $R\omega L_2 T_2$ there will be two possible forms, where

$$R\omega L_2 T_2 = \begin{cases} R\omega VT^m LL_2 T_2 & \text{for } m \geq 0 \\ R\omega VT^m L^s L_2 T_2 & \text{for } m \geq 0, s \geq 2 \end{cases}$$

Classes of the form $R\omega L_3 T_2$ will be of the following form

$$R\omega L_3 T_2 = \begin{cases} R\omega VT^m LL_3 T_2 & \text{for } m \geq 0 \\ R\omega VT^m L^s L_3 T_2 & \text{for } m \geq 0, s \geq 2 \end{cases}$$

4.3.1. The class $R\omega T_2 = R\omega LT_2$.

We start with $R\omega VT^m LT_2$ with $m \geq 0$. Before we begin we want to point out that we can actually look just at the RVT code $RVT^m LT_2$ instead of $R\omega VT^m LT_2$ because of which fiber space we need to prolong in order to obtain the T_2 critical plane. In short, we do not need to go any farther back than the level where the last V appears. This will also ease the amount of notation needed.

We want to show for the code $RVT^m LT_2$ for $m \geq 0$, that T_2 comes from the critical plane given by $T_2 = \delta_1^{m+3}(p_{m+4})$. We begin by looking in a neighborhood of the point p_{m+4} and we have that T_2 , if it exists, in Δ_{m+4} will be of the form $[dv_{m+2} : du_{m+4} : dv_{m+4}] = [a : b : 0]$ with $a \neq 0$. Two levels down we have $\delta_1^{m+2}(p_{m+3})$ will be given in $\Delta_{m+3}(p_{m+3})$ by $[du : dv_{m+2} : dv_{m+2}] = [a : 0 : b] = [\frac{a}{b} : 0 : 1]$ to give a L direction.

The fiber $F_1(p_1)$ gives the vertical plane $[dx : du : dv] = [0 : a : b]$ within $\Delta_1(p_1)$. After moving in a V direction the critical plane $\delta_1^1(p_s)$ will be given in the coframing for $\Delta_2(p_2)$ by $[du : du_2 : dv_2] = [a : 0 : b] = [1 : 0 : \frac{a}{b}]$ to give a T direction. If one were to continue to move in strictly T directions $(m-1)$ more times that this pattern will persist and we will have $\delta_1^{m+1}(p_{m+2})$ will be given by $[du : du_{m+1} : dv_{m+1}] = [a : 0 : b] = [\frac{a}{b} : 0 : 1]$, moving in an L direction. This shows that the T_2 critical plane will indeed come from the $F_1(p_1)$ fiber.

The next step is to show that over T_2 the only spelling rules are R , V , and T_2 . Suppose that $\ell \subseteq \Delta_{m+3}(p_{m+3})$ and is strictly in the T_2 plane. This means for a direction ℓ to lie in this plane it will be of the form $\ell = a \frac{\partial}{\partial v_{m+2}} + b \frac{\partial}{\partial u_{m+3}}$ with $a, b \in \mathbb{R}$ with $a \neq 0$. The KR -coordinates for the $(m+4)$ -th level of the tower will be $[dv_{m+2} : du_{m+3} : dv_{m+3}] = [1 : \frac{du_{m+3}}{dv_{m+2}} : \frac{dv_{m+3}}{dv_{m+2}}]$. We will have $u_{m+4} = \frac{du_{m+3}}{dv_{m+2}}$ and $v_{m+4} = \frac{dv_{m+3}}{dv_{m+2}}$ where u_{m+4} cannot be equal to zero at the point p_{m+4} or else it will not be a T_2 point.

In summary, the above argument shows that the T critical plane will not appear in the distribution over points $p_k \in RVT^m LT_2$, since it would force $u_{m+4} = 0$ at the point p_{m+4} , and hence in the distribution over points $p_k \in R\omega VT^m LT_2$. We would like to point out that the above argument can be repeated in a very similar fashion to show that the T_1

critical planes will not appear over any of these points. In order to avoid being repetitive we will omit presenting this justification for the absence of the T critical plane again and again for each of the following cases below.

Remark 4.1 (Continuing to Move in T_2 Directions). From the code $RVT^m LT_2$ it can be shown that if we continue to move in direction within the plane T_2 then the RVT code will have the form $RVT^m LT_2^s$ and the T_2 will have the form $\delta_1^{m+s+1}(p_{m+s+2}) = \text{span}\left\{\frac{\partial}{\partial v_{m+2}} + u_{m+5}\frac{\partial}{\partial u_{m+4}} + \cdots + u_{m+s+2}\frac{\partial}{\partial u_{m+s+1}}\frac{\partial}{\partial u_{m+s+2}}\right\}$.

Now we present the case of $R\omega'LLT_2$. We assume the total length of this code is k and the coframing for the the distribution Δ_k in a neighborhood of points in $R\omega'LLT_2$ is given in KR -coordinates by $[dv_{k-2} : du_k : dv_k]$ and if the T_2 plane exists it is given by $[a : b : 0]$ in this coframing. We aim to show that in Δ_k that $T_2 = \delta_{k-3}^3$. Going one level down, this plane projects to $[a : b : 0]$ in the coframing $[dv_{k-2} : du_{k-1} : dv_{k-1}]$ for Δ_{k-1} . Again, projecting this plane another level down to Δ_{k-2} we have $[dv_{k-3} : du_{k-2} : dv_{k-2}] = [a : 0 : b]$. Doing this one last time, the plane is given in the coframing for Δ_{k-3} by $[df_{k-4} : du_{k-3} : dv_{k-3}] = [0 : a : b]$, which indicates that the T_2 plane in Δ_k does in fact come from the fiber space F_{k-3} and is equal to δ_{k-3}^3 . We can also show that the T plane does not exist in Δ_k by using a similar argument as the above case for $R\omega V T^m LT_2$ because the coordinate $u_k \neq 0$ at any point in $R\omega'LLT_2$.

4.3.2. The class $R\omega T_2 = R\omega L_2 T_2$.

In this case we will start with $R\omega V T^m LL_2 T_2$. We will again, for ease of notation, just look at the code $RVT^m LL_2 T_2$, where one can see how it would generalize for the case of $R\omega V T^m LL_2 T_2$. We show that the T_2 critical plane that appears over the points in $RVT^m LL_2 T_2$ will come from the Baby Monster δ_1^{m+4} . We begin with the fiber space $F_1(p_1)$ and see that the vertical plane $\delta_1^0(p_1)$ will be given in $\Delta_1(p_1)$ as $[dx : du : dv] = [0 : a : b]$ for $a, b \in \mathbb{R}$ with $a \neq 0$. Then δ_1^1 will be given in a neighborhood of the point p_2 in Δ_2 by $[du : du_2 : dv_2] = [a : 0 : b] = [1 : 0 : \frac{b}{a}]$ to give a T direction. Repeating this $m - 1$ times more gives δ_1^m in a neighborhood of p_{m+1} by $[du : du_{m+1} : dv_{m+1}] = [a : 0 : b] = [1 : 0 : \frac{b}{a}]$ to give the last T direction. For the L direction we have that δ_1^{m+1} will be $[du : du_{m+2} : dv_{m+2}] = [a : 0 : b] = [\frac{a}{b} : 0 : 1]$. For the L_2 direction we have that δ_1^{m+2} will be $[dv_{m+2} : du_{m+3} : dv_{m+3}] = [a : b : 0] = [1 : \frac{b}{a} : 0]$. Then for δ_1^{m+3} we have $[dv_{m+2} : du_{m+4} : dv_{m+4}] = [a : b : 0] = [1 : \frac{b}{a} : 0]$ to give the T_2 direction. Lastly, δ_1^{m+4} will be given in Δ_{m+5} within a neighborhood of p_{m+4} by $[dv_{m+2} : du_{m+5} : dv_{m+5}] = [a : b : 0]$, which gives the desired result.

The next case is when $R\omega L_2 T_2 = R\omega V T^m L^s L_2 T_2$ for $m \geq 0$ and $s \geq 2$. We will look at the RVT code $RVT^m L^s L_2 T_2$ and show that the T_2 critical plane in $\Delta_{m+s+4}(p_{m+s+4})$ is given by the Baby Monster $\delta_{m+s}^4(p_{m+s+4})$. Start with local coordinates for Δ_{m+s} in a neighborhood of $p_{m+s} \in RVT^m L^{s-2}$ for $p_{m+s+4} \in RVT^m L^s L_2 T_2$, which is given by $[dv_{m+s-1} : du_{m+s} : dv_{m+2}]$ and when we prolong the fiber space $F_{m+s}(p_{m+s})$ we will get in a neighborhood of p_{m+s+1} that δ_{m+s}^1 is $[dv_{m+s} : du_{m+s+1} : dv_{m+s+1}] = [a : 0 : b]$. After going in an L direction we have locally that δ_{m+s}^2 is given by $[dv_{m+s+1} : du_{m+s+2} :$

$dv_{m+s+2}] = [a : b : 0]$. Then after going in the $L_2 = \text{span}\{\frac{\partial}{\partial v_{m+s+1}}\}$ direction, δ_{m+2}^3 is given by $[dv_{m+s+1} : du_{m+s+3} : dv_{m+s+3}] = [a : b : 0]$ and after moving in a T_2 direction, where we divide by dv_{m+s+1} , we get δ_{m+s}^4 in a neighborhood of p_{m+s+4} is given in the KR -coordinates for Δ_{m+s+4} by $[dv_{m+s} : du_{m+s+4} : dv_{m+s+4}] = [a : b : 0]$, which gives us the desired T_2 critical plane.

4.3.3. The class $R\omega L_3 T_2$.

The argument for this case is similar to the previous two cases presented in Sections 4.3.1 and 4.3.2 above. As a result, we will omit this case.

4.4. The Proof of Theorem 2.2. In this section we will show using induction how the spelling rules hold in the general case. We understand how the spelling rules work for the cases of R, V, T , and L . We want to completely determine the spelling rules for the letters T_2, L_2 , and L_3 . We have already shown what the spelling rules are for the first occurrence of each of these letters in a given RVT code in Table 2. We need to take an arbitrary RVT code that ends in either a T_2, L_2 , or L_3 and show that these spelling rules will persist. We do this by induction on the number of T_2 's, L_2 's, and L_3 's in a given RVT code. We pause for a moment to explain this method of induction on the total number of T_2 's, L_2 's, and L_3 's through a few examples. If $n = 4$, then $RVLL_2 T_2 L_3 L_2$, $RRVL_2^4$, $RVLT_2 T_2 L_2 L_2$, and $RVT^7 LL_3 L_2 L_3 L_2$ are examples. However, $RVLL_2 L_3 T_2 T_2 L_3$ and $RVTL_2 T_2 R$ are not examples, since there is a total of 5 T_2 's, L_2 's, and L_3 's in the first example and the second example only has a total of 2 of these letters.

The base case is $n = 1$, proved above in Section 4.1, and is detailed in Table 2. The inductive hypothesis is that the spelling rules hold true for any RVT code that has a total number of n T_2 's, L_2 's, or L_3 's in it. Now suppose we have an RVT code ω that contains a total of $n + 1$ of these letters in it and suppose that the last letter of ω is one of these three possible letters. We show that the spelling rules will hold for the cases of $\omega = \omega' L_2$ and $\omega = \omega' T_2$, where the terminal letter in the code is L_2 and T_2 respectively, from which one can see how the case of $\omega = \omega' L_3$ will follow.

4.5. The case of the terminal letter L_2 .

Our code is $\omega' L_2$ and we want to show that the letters $R, V, T = T_i$ for $i = 1, 2$ and L_j for $j = 1, 2, 3$ exist over this L_2 . This means we need to look at the possible planes that appear in $\Delta_k(q)$, where k equals the length of the code ω and q is a point in a neighborhood of $p \in \omega$. By the inductive hypothesis we have the following refinement of our code

$$\omega' L_2 = \begin{cases} \tilde{\omega} L_1 L_2 \\ \tilde{\omega} L_2 L_2 \\ \tilde{\omega} L_3 L_2 \end{cases}$$

A coframing for our distribution $\Delta_k(q)$ is given by $[df_{k-1} : du_k : dv_k]$. Also, we will show that the spelling rules hold for the case of $\tilde{\omega} L L_2$, since the cases of $\tilde{\omega} L_j L_2$ for $j = 2, 3$ are almost identical.

4.5.1. Case of $\tilde{\omega}LL_2$.

We can refine our coframing and say that our uniformizing coordinate is $df_{k-1} = dv_{k-2}$. This is because in Δ_{k-2} we move in an L direction to give $[df_{k-3} : du_{k-2} : dv_{k-2}] = [\frac{df_{k-3}}{dv_{k-2}} : \frac{du_{k-2}}{dv_{k-2}} : 1]$ and then in a L_2 direction $[dv_{k-2} : du_{k-1} : dv_{k-1}] = [1 : \frac{du_{k-1}}{dv_{k-2}} : \frac{dv_{k-1}}{dv_{k-2}}]$ in Δ_{k-1} .

Now, if the T critical plane exists in $\Delta_k(q)$ then it must be of the form $[dv_{k-2} : du_k : dv_k] = [a : 0 : b]$. When we project one level down we have $[dv_{k-2} : du_{k-1} : dv_{k-1}] = [dv_{k-2} : du_{k-1} : dv_{k-1}] = [a : 0 : b]$ in Δ_{k-1} which is the T plane. This implies that the T critical plane will exist in $\Delta_k(q)$ and comes from the prolongation of the T plane one level below.

If the T_2 critical plane is to exist in $\Delta_k(q)$ then it is of the form $[dv_{k-2} : du_n : dv_n] = [a : b : 0]$. Projecting one level down we have $[dv_{k-2} : du_{k-1} : dv_{k-1}] = [a : b : 0]$, which is the T_2 critical plane, which must exist in $\Delta_{k-1}(q)$ since we are over an L point.

Since both tangency critical planes exist above ωLL_2 we end up with the distinguished directions L_j for $j = 1, 2, 3$ existing as well in $\Delta_k(q)$ and hence the spelling rules persist.

4.6. The case of the terminal letter T_2 .

We want to show that from the code $\omega'T_2$ that the letters R, V, T_2 , and L_3 are the only possibilities. We again need to look at the possible planes that appear in $\Delta_k(q)$. By the

inductive hypothesis we have the following refinement of our code $\omega'T_2 = \begin{cases} \tilde{\omega}LT_2 \\ \tilde{\omega}L_2T_2 \\ \tilde{\omega}L_3T_2 \\ \tilde{\omega}T_2T_2 \end{cases}$ We

will show that the spelling rules hold for $\tilde{\omega}LT_2$ and $\tilde{\omega}T_2T_2$, since the cases of $\tilde{\omega}L_jT_2$ for $j = 2, 3$ will be similar.

4.6.1. Case of $\tilde{\omega}LT_2$.

We look at the code $\tilde{\omega}LT_2$ of length k , where Δ_k is coframed by $[df_{k-1} : du_k : dv_k] = [dv_{k-2} : du_k : dv_k]$. This is because when we go two levels down we have that Δ_{k-2} is coframed by $[df_{k-3} : du_{k-2} : dv_{k-2}] = [\frac{df_{k-3}}{dv_{k-2}} : \frac{du_{k-2}}{dv_{k-2}} : 1]$ to give an L direction. Notice that when we move in a T_2 direction in Δ_{k-1} we have $[dv_{k-2} : du_{k-1} : dv_{k-1}] = [a : b : 0] = [1 : \frac{b}{a} : 0]$. This results in $u_k \neq 0$ and v_k is identically zero for points in $\tilde{\omega}T_2$, where $u_k = \frac{du_{k-1}}{dv_{k-2}}$ and $v_k = \frac{dv_{k-1}}{dv_{k-2}}$.

For T_2 to be present in Δ_k we would have $[dv_{k-2} : du_k : dv_k] = [a : b : 0]$ and projecting one level down gives $[dv_{k-2} : du_{k-1} : dv_{k-1}] = [a : b : 0]$, the T_2 critical plane in Δ_{k-1} and again we get that T_2 will exist in Δ_k .

We now show that no T critical plane can exist in Δ_k . Suppose that such a plane did exist, then it would be given by $[dv_{k-2} : du_k : dv_k] = [a : 0 : b]$. Projecting one level down gives $[dv_{k-2} : du_{k-1} : dv_{k-1}] = [a : 0 : b]$. When we move in a T_2 direction we have $[1 : 0 : \frac{b}{a}]$ and gives $u_k = 0$ for points within $\omega'T_2$, but this creates a contradiction to $u_k \neq 0$. Hence T is not present in Δ_k .

Then since L_3 comes from the intersection of the vertical plane and the T_2 plane, we have shown the spelling rules over the last T_2 .

4.6.2. Case of $\tilde{\omega}T_2T_2$.

From the code $\omega'T_2T_2$ we have the coframing $[df_{k-1} : du_k : dv_k] = [df_{k-3} : du_k : dv_k]$ for Δ_k , since in Δ_{k-2} has $[df_{k-3} : u_{k-2} : dv_{k-2}] = [1 : \frac{du_{k-2}}{df_{k-3}} : \frac{dv_{k-2}}{df_{k-3}}] = [1 : u_{k-1} : v_{k-1}]$ with $u_{k-1} \neq 0$ and v_k vanishing at points in $\omega'T_2$ and similarly in Δ_{k-1} we have $[df_{k-2} : du_{k-1} : dv_{k-1}] = [df_{k-3} : du_{k-1} : dv_{k-1}] = [1 : \frac{du_{k-1}}{df_{k-2}} : \frac{dv_{k-1}}{df_{k-2}}] = [1 : u_k : v_k]$, with $u_k \neq 0$ and $v_k = 0$ at points in $\omega'T_2T_2$.

For T_2 to be present in Δ_k we would have $[df_{k-3} : du_k : dv_k] = [a : b : 0]$ for $a, b \in \mathbb{R}$ for $a \neq 0$. Projecting one level down gives $[df_{k-3} : du_{k-1} : dv_{k-1}] = [a : b : 0]$, which is again the T_2 critical plane. Therefore the T_2 critical plane will exist in Δ_k .

Lastly, we show that T_1 cannot be present within the distribution Δ_k . Suppose that it was, then it would be given by $[df_{k-3} : du_k : dv_k] = [a : 0 : b]$ and we project one level down to get $[df_{k-3} : du_{k-1} : dv_{k-1}] = [a : 0 : b] = [1 : 0 : \frac{b}{a}] = [1 : u_k : v_k]$, but would imply that $u_k = 0$, which creates a contradiction.

5. CONCLUSION

In this work we have completed the spelling rules for the RVT code in the \mathbb{R}^3 - Monster Tower and a method which determines the level at which the critical planes are born from within the rank 3 distribution.

It is also worth pointing out that this algorithm can be applied to a more general problem of understanding the incidence relations that occur within the \mathbb{R}^n -Monster Tower. In this case $n \geq 4$, as was pointed out in [PS12], there will be even more critical planes existing, which makes the RVT class spelling rules even more complicated. The algorithm presented in this paper serves as a tool to help understand what incidence relations occur in this more general case as well.

One of the main motivations for this work was not only to understand the geometry of the Monster Tower, but also to understand the relationship between the RVT coding system and Mormul's EKR coding system. The authors first studied this relationship in [CH12] in order to answer some basic questions concerning connections between the RVT coding system and the EKR system that Mormul uses to stratify Goursat Multi-Flags [Mor04]. Recent work done by Pelletier and Slayman in [PS12] displays not only the connection between the two coding systems up to level 4 of the Monster Tower, but also establishes how the EKR coding system relates to the constraints that are placed on the various configurations of the articulated arm system. Pelletier and Slayman present a beautiful result, Proposition 5.1, that displays the relationship between the EKR codes and these restrictions placed on the articulated arms. In addition, they also raise an important question about how certain EKR classes give rise to more than one RVT class. An example of this can be seen with the EKR class 121 splitting into the RVT classes RVR and RVT . Further examples can be found in [CH12] and [How13]. Pelletier and Slayman point out that it would be interesting to see if the articulated arm model could explain this

splitting between the RVT and EKR systems. Taking these two things into consideration, it would be interesting to use these connections to further explore the relationship between the geometry of the Monster Tower and the articulated arm system.

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