SKEW PRODUCT CYCLES WITH RICH DYNAMICS: FROM TOTALLY NON-HYPERBOLIC DYNAMICS TO FULLY PREVALENT HYPERBOLICITY

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Abstract. Following the model in [DHRS] we introduce a two-parameter family of skew products $(G_{a,t})_{a>0,t\in[-\epsilon,\epsilon]}$ maps, where the parameter a models the central dynamics and the parameter t the unfolding of cycles (that occurs for $t=0$). The parameter a also measures the "central distortion" of the systems: for small a the distortion of the systems is small and it increases and goes to infinity as $a\to\infty$. The family $(G_{a,t})$ displays some of the main characteristic properties of the unfolding of heterodimensional cycles as intermingled homoclinic classes of different indices and secondary bifurcations via collision of hyperbolic homoclinic classes.

For $a\in(0,\log 2)$ the dynamics of $(G_{a,t})$ is always non-hyperbolic after the unfolding of the cycle. However, for $a>\log 4$ intervals of t-parameters corresponding to hyperbolic dynamics appear and turn into totally prevalent as $a\to\infty$ (the density of "hyperbolic parameters" goes to 1 as $a\to\infty$).

The dynamics of the maps $G_{a,t}$ is described using a family of iterated function systems modeling the dynamics in the one-dimensional central direction.

1. Introduction

In [BDV, Preface] there are discussed semi-local bifurcation mechanisms which are sources of persistent forms of non-hyperbolic dynamics. Based on these mechanisms two sorts of dynamics are presented:

(1) *Critical dynamics* whose paradigmatic examples are the quadratic and the Hénon families and whose genuine bifurcations are the *homoclinic tangencies* (the invariant manifolds of a saddle have a non-transverse intersection);
(2) *Non-critical dynamics* associated to the coexistence of intermingled hyperbolic sets having different unstable dimensions (*indices*) and exhibiting some weak form of hyperbolicity (partial hyperbolicity, a dominated splitting, for example). The genuine bifurcations in this sort of dynamics are the *heterodimensional cycles* (the invariant manifolds of a pair of hyperbolic saddles of different indices intersect cyclically).

In homoclinic bifurcations (tangencies) the one-dimensional quadratic family plays a key role and in some aspects it "models" the dynamics at these bifurcations: in very rough terms, the quadratic family is a *limit dynamics* and some of its

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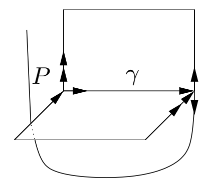

FIGURE 1. Heterodimensional cycle

properties can be translated to the dynamics of the bifurcating maps. See [PT2, Chapters 3 and 6] for details.

Still in the critical setting and for dynamics in surfaces, the Hénon family is a model that illustrates the transition from hyperbolic behavior to a non-hyperbolic one (boundary of hyperbolicity, see [BS, CLR]). In this transition some of the typical features of non-hyperbolic critical dynamics (as coexistence of infinitely many sinks [N], existence of strange attractors [BC, MV], and persistence of homoclinic tangencies [N], among others) are displayed. This family was intensively studied since the 80's and one of its appeals is its simplicity (a quadratic map depending on two parameters) which in some cases allows explicit calculations.

In the case of heterodimensional cycles there are no such "model families". The aim of this paper is to introduce a simple two-parameter family that displays most of the typical features of heterodimensional bifurcations (see the discussion below). This family has a "one-dimensional" model given by a system of iterated functions (IFS in what follows). Before going into the details we need a preliminary discussion.

Let us fix some notation. Consider two saddles P and Q of different indices of a diffeomorphism f which are involved in a heterodimensional cycle (in what follows just a cycle) such that $W^s(P) \cap W^u(Q)$ is a curve γ joining P and Q called a *connexion*[1] (see Figure 1). A heuristic principle is that the dynamics after the bifurcation is "essentially" determined by the restriction of f to γ (called *central dynamics*). Oversimplifying the analysis, the dynamics given by the "transition" from P to Q (called *cycle dynamics*) plays no relevant role (indeed, the only relevant point is if this transition preserves or not the "central orientation", but let us skip this technical point). When the "distortion" of the restriction $f_{|\gamma}$ is small the dynamics is robustly non-hyperbolic after the bifurcation, [D1, D2], while when this distortion is "big and nicely distributed" large parameter intervals of hyperbolic dynamics appear, [DR1]. Systems with "intermediate distortion" are poorly understood.

In the sequel of the unfolding of the cycle the homoclinic classes of P and Q (see definition below) explode and new homoclinic and heteroclinic points are generated. A key problem is to determine the dynamics of these two classes. A pre-requisite for hyperbolicity is the disjointness of these classes. Typical features displayed by the dynamics at these cycles include the following phenomena:

- robustly non-transitive sets and intermingled non-hyperbolic homoclinic classes, [D1, D2, DR2];

[1]In this case one speaks of a *connected cycle*.

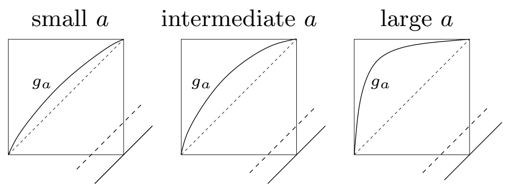

FIGURE 2. One-dimensional dynamics

- collisions of homoclinic classes (at the collision the homoclinic classes of P and Q are non-disjoint and their intersection is the orbit of a saddle-node), [DR3, DS];
- robust cycles (existence of two transitive hyperbolic sets of different indices whose invariant manifold intersect cyclically and robustly) [BD, BDK]; and
- hyperbolic dynamics, [DR1].

We present systems $(G_{a,t})_{a>0,t\in[\epsilon,\epsilon]}$ with a heterodimensional cycle at $t=0$ for every a. The parameter t describes the unfolding of the cycle while the parameter a measures the central distortion which goes from 0 to ∞ as a increases. The central dynamics of $G_{a,t}$ (any t) is given by a central map $g_a\colon [0,1/2]\to\mathbb{R}$ having a repelling point 0 and and attracting point 1/2 and the cycle map is a map $f_t\colon [0,1/2]\to\mathbb{R}$ (independent of a) with $f_0(1/2)=0$ and $f_t(1/2)=t$ (see Figure 2). Suitable compositions of these two maps lead to a two-parameter family of IFS's that determine the dynamics after the bifurcation (see the discussion below). An advantage of our model is that the maps g_a and f_t are given by explicit simple formulae which allows precise quantitative estimates.

We now discuss our results a bit more precisely. We follow the approach proposed in [PT1], we consider a skew product map[2] f with a cycle associated to a pair of "saddles" P and Q as above. We first fix a small *neighborhood V of this cycle* (i.e., an open set that contains the intersections $W^s(P)\cap W^u(Q)$ and $W^u(P)\cap W^u(Q)$ and the points P and Q). The goal is to describe the dynamics of perturbations g of f in the set V. Key objects in this description are the *relative homoclinic classes* of P and Q in V, denoted by $H_V(P,g)$ and $H_V(Q,g)$ (we omit the dependence of the saddles on the diffeomorphism). Recall that the class $H_V(R,g)$ is the closure of the transverse homoclinic points of R whose orbits remain in V, see Definition 2.5.

The strategy is to consider curves of diffeomorphisms $(f_t)_{t\in[\varepsilon,\varepsilon]}$ with $f_0=f$ unfolding the cycle. One pays special attention to those dynamical features which are displayed more frequently or with positive frequency after the bifurcation (say for $t>0$) by the diffeomorphisms f_t in the neighborhood V of the cycle. This leads to two main sets of parameters corresponding to non-hyperbolic and hyperbolic[3] dynamics:

$$
\begin{aligned}
N &\stackrel{\text{def}}{=} \{t\ge 0\colon H_V(P,f_t)\cap H_V(Q,f_t)\neq\emptyset\};\\
H &\stackrel{\text{def}}{=} \{t\ge 0\colon \text{the relative dynamics of } f_t \text{ in } V \text{ is hyperbolic}\}.
\end{aligned}
\tag{1.1}
$$

[2]The skew product maps that we consider have differentiable realizations. On the other hand, in the case of a diffeomorphism which has well defined strong stable and unstable foliations, the corresponding quotient map provides a skew product.

[3]by hyperbolicity we mean that the system is Axiom A and has no-cycles.

We are interested in the limits

$$
\begin{aligned}
\mathbf{N}^{+} &\stackrel{\text{def}}{=} \limsup_{s\to 0^+} \frac{|N\cap[0,s]|}{s}, \qquad \mathbf{N}^{-} \stackrel{\text{def}}{=} \liminf_{s\to 0^+} \frac{|N(s)\cap[0,s]|}{s},\\
\mathbf{H}^{+} &\stackrel{\text{def}}{=} \limsup_{s\to 0^+} \frac{|H(s)\cap[0,s]|}{s}, \qquad \mathbf{H}^{-} \stackrel{\text{def}}{=} \liminf_{s\to 0^+} \frac{|H(s)\cap[0,s]|}{s},
\end{aligned}
\tag{1.2}
$$

that measure the frequency of non-hyperbolic and hyperbolic dynamics at the bifurcation, respectively (here $|\cdot|$ stands for the Lebesgue measure). For each family $(G_{a,t})_{t\in[-\epsilon,\epsilon]}$ we define the sets $N_a(s)$ and $H_a(s)$ and the limit frequencies $\mathbf{N}_a^{\pm}$ and $\mathbf{H}_a^{\pm}$ as above.

The family $(G_{a,t})$ exhibits the known complex features associated to heterodimensional cycles mentioned above and has the property that the proportion of hyperbolicity after the bifurcation goes from 0 to 1:

- $a\in(0,\log 2)$, if t is small then $(0,t)\subset N_a(t)$ and thus $\mathbf{N}_a^{-}=1$ and $\mathbf{H}_a^{+}=0$,
- $\mathbf{H}_a^{-}\to 1$ as $a\to\infty$.

A much more complete description of the bifurcating diagram of the family $(G_{a,t})$ can be found in Theorem 2.6 that provides a picture of the bifurcation scenario for this family, that includes secondary cycles and collisions of homoclinic classes. Our methods allow us to consider parameters with “intermediate” distortion. However, there is a-parameter window with intermediate distortion for which the description of the dynamics is still embryonic.

We close this introduction with a brief discussion of the underlying IFS associated to the bifurcation which is interesting by its own. To study the dynamics of $G_{a,t}$ we fix an appropriate fundamental domain $D_{a,t}=(d_{a,t},g_a(d_{a,t})]$ of the central map g_a and consider the returns to this domain by (admissible) compositions of the maps g_a and f_t. This leads to a “return map” with resembles the Gauss map (although it preserves the orientation) with the following properties (see Figure 3):

(1) It has infinitely many branches and an asymptote at $x=d_{a,t}$;
(2) The derivative is positive and decreasing in each branch;
(3) The only branch that may not be onto is the one containing the extreme $g_a(d_{a,t})$.

The transition from systems with “small” to “big” distortion is illustrated as follows: (i) For maps with small distortion all branches are expanding. (ii) In the intermediate regime, a branch with contracting and expanding points appear. In some cases defining an induced map one may overpass this lack of expansion. (iii) For maps with “big” distortion the contracting points occupy a large proportion of the phase space.

This paper is organized as follows. In Section 2 we define the family $G_{a,t}$ and state the pertinent definitions and Theorem 2.6. Some terminology and general facts about skew products are presented in Section 3. In Section 4 we introduce the IFS associated to the skew product and study the dynamics of the this IFS. There are two different types of parameters, those whose returns are expanding (studied in Section 5) and those having a hyperbolic-like mixed behavior (studied in Section 6). In Section 7 we study the dynamics in a neighborhood of the cycle using the IFS. Finally, in Section 8 we prove our main result (Theorem 2.6).

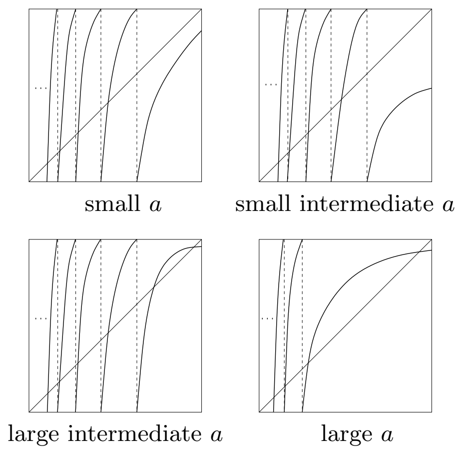

FIGURE 3. Return maps

2. The model family and statement of the main result

In this section, we describe the model family $G_{a,t}$ of one-step skew product maps that we consider and state precisely our main result. This family was motivated by the example of a skew product map with a cycle in [DHRS][4]

Consider the shift map σ defined on the set $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$ endowed with the standard metric. Denote an element $\alpha \in \Sigma_2$ by $\alpha = (\alpha_i)_{i\in\mathbb{Z}}$. By definition $\sigma(\alpha) = \bar{\alpha}$ where $\bar{\alpha}_i = \alpha_{i+1}$. We consider maps of the form

$$G_{a,t} \colon \Sigma_2 \times \left(\frac{-1}{2(e^a-1)}, 1 \right] \to \Sigma_2 \times \mathbb{R}, \quad G_{a,t}(\alpha; x) = \big(\sigma(\alpha); g_{\alpha_0,a,t}(x)\big). \tag{2.1}$$

We now define the fiber maps $g_{0,a,t}$ and $g_{1,a,t}$ for $a \in (0,\infty)$.

- The central *central map* $g_{0,a,t}$ are independent of t and defined[5] by
$$g_a(x) = \frac{x\, e^a}{2\, x\, e^a + (1-2\, x)}.$$
- The *cycle maps* $g_{1,a,t}$ are independent of a and defined by
$$g_{1,t}(x) = (x - 1/2) + t.$$

Remark 2.1 (Fixed points and cycle condition)**.** In $\big(-1/(2(e^a-1)), 1\big]$ the map g_a has two fixed points (independent of a): the repelling point 0 with $g_a'(0) = e^a > 1$ and the attracting point $1/2$ with $g_a'(1/2) = e^{-a} < 1$. The map $g_{1,0}$ maps the attracting point 0 into the repelling point 1, justifying the name cycle map.

Let $0^{\mathbb{Z}} \in \Sigma_2$ be the sequence consisting of 0's and $0^{-\mathbb{N}}.10^{\mathbb{N}} \in \Sigma_2$ be the sequence with $\alpha_0 = 1$ and $\alpha_i = 0$ if $i \neq 0$. Consider the points $Q = (0^{\mathbb{Z}}, 0)$ and $P = (0^{\mathbb{Z}}, 1/2)$. These points are fixed points of $G_{a,t}$ for every (a,t) and that

$$\{0^{\mathbb{Z}}\} \times (0, 1/2) \subset W^u(Q, G_{a,t}) \cap W^s(P, G_{a,t}),$$
$$(0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2) \in W^s(Q, G_{a,0}) \cap W^u(P, G_{a,0}).$$

[4]Let us observe that our example has a completely different nature: the cycle map in [DHRS] reverses the orientation, while in our case the orientation is preserved.

[5]Note that g_1 is the time-one map of the vector field $x' = 2\, x\, (1 - 2\, x)$.

This implies that the stable and unstable sets of P and Q intersect cyclically for every $G_{a,0}$. As the points P and Q have different central behavior (contracting and repelling, respectively), one can think of theses points as points with different indices and thus $G_{a,0}$ as a map with a heterodimensional cycle.

2.1. **Hyperbolicity, cycles, and homoclinic classes.** We next discuss succinctly how the notions of hyperbolicity, cycles, and homoclinic classes can be translated to the maps $G_{a,t}$ (this can be done in the general setting of skew-product maps). A more detailed discussion can be found in Section 3.

Definition 2.2 (Hyperbolicity)**.** A $G_{a,t}$-invariant set $K_{a,t}$ is *hyperbolic of contracting type* if there are constants $C > 0$ and $\lambda \in (0,1)$ such that for all $(\alpha, x) \in K_{a,t}$ and for all $n \geq 0$

$$|(g_{\alpha_n,a,t} \circ \cdots \circ g_{\alpha_0,a,t})'(x)| < C\,\lambda^{n+1}.$$

The set is *hyperbolic of expanding type* if

$$|(g_{\alpha_n,a,t} \circ \cdots \circ g_{\alpha_0,a,t})'(x)| > C\,\lambda^{-n-1}.$$

It follows that $Q = (0^{\mathbb{Z}}; 0)$ and $P = (0^{\mathbb{Z}}; 1/2)$ are hyperbolic fixed points of $G_{a,t}$ of expanding and contracting type, respectively.

Definition 2.3 (Heterodimensional cycle or cycle)**.** Two hyperbolic periodic points A and B of $G_{a,t}$ (of different type) have a *heterodimensional cycle* if their invariant sets intersect cyclically,

$$W^u(A, G_{a,t}) \cap W^s(B, G_{a,t}) \neq \emptyset \quad \text{and} \quad W^s(A, G_{a,t}) \cap W^u(B, G_{a,t}) \neq \emptyset.$$

Consider the sets

$$\begin{aligned} [.0^{\mathbb{N}}] &\stackrel{\text{def}}{=} \{\alpha = \cdots \alpha_{-n} \cdots \alpha_{-1}.0^{\mathbb{N}}, \text{ where } \alpha_{-i} \in \{0,1\}\}; \\ [0^{-\mathbb{N}}.] &\stackrel{\text{def}}{=} \{\alpha = 0^{-\mathbb{N}}.\alpha_0 \cdots \alpha_n \cdots, \text{ where } \alpha_i \in \{0,1\}\}. \end{aligned}.$$

Note that

$$[0^{-\mathbb{N}}.] \times \{1/2\} \subset W^u(P, G_{a,t}) \quad \text{and} \quad [.0^{\mathbb{N}}] \times \{0\} \subset W^u(Q, G_{a,t}).$$

This implies that $(0^{-\mathbb{N}}.10^{\mathbb{N}}, 1/2) \in W^s(Q, G_{a,0}) \cap W^u(P, G_{a,0})$. On the other hand, for all $a > 0$ and t it holds

$$\{0^{\mathbb{Z}}\} \times (0, 1/2) \subset W^s(P, G_{a,t}) \cap W^u(Q, G_{a,t}).$$

This implies that $G_{a,0}$ has a cycle associated to P and Q.

More generally, fixed (a,t), suppose that there are natural numbers $n_i \geq 1$, $i = 0, 1, \ldots, k$, such that

$$(g_{1,t} \circ g_a^{n_k}) \circ \cdots \circ (g_{1,t} \circ g_a^{n_2}) \circ (g_{1,t} \circ g_a^{n_1}) \circ g_{1,t}(1/2) = 0. \tag{2.2}$$

Then, by definition,

$$(0^{-\mathbb{N}}.10^{n_1}0^{n_2}1 \ldots 0^{n_k}10^{\mathbb{N}}; 1/2) \in W^u(P, G_{a,t}) \cap W^s(Q, G_{a,t}).$$

Hence $G_{a,t}$ has a cycle associated to P and Q.

A neighborhood of the cycle of $G_{a,0}$ associated to P and Q is an open set V that contains the set $\{0^{\mathbb{Z}}\} \times (0, 1/2) \subset W^s(P, G_{a,t}) \cap W^u(Q, G_{a,t})$ and the orbit of $\big(0^{-\mathbb{N}}.10^{\mathbb{N}}; 1/2\big) \in W^u(P, G_{a,0}) \cap W^s(Q, G_{a,0})$.

Definition 2.4 (Homoclinic class)**.** *The* homoclinic class *of hyperbolic point A for $G_{a,t}$, denoted by $H(A, G_{a,t})$, is the closure of the intersections of the invariant sets $W^s(A, G_{a,t})$ and $W^u(A, G_{a,t})$ of the orbit of A.*

Two hyperbolic periodic points A and B of $G_{a,t}$ are homoclinically related *if they are of the same type and the invariant sets of their orbits meet cyclically.*

Given a neighborhood U of the orbit of a periodic point A, the relative homoclinic class of A to U, *denoted by $H_U(A, G_{a,t})$, is the subset of $H(A, G_{a,t})$ of points whose orbit is contained in U.*

As in the case of differentiable dynamics, the homoclinic class of a periodic point coincides with the closure of the set of points (of the same type) homoclinically related to it. In the skew product context the transverse intersection condition on the invariant manifolds in the definition of homoclinic relations is not required (indeed it does not make sense). This is due to the fact that the dynamics in the central direction is non-critical and therefore all the intersections between invariant sets of hyperbolic periodic points of the same type behave as transverse ones. For details see Section 3.2.

Finally, note that homoclinic classes are transitive sets (existence of dense orbits) and they may fail to be hyperbolic (see items A and B of Theorem 2.6).

In what follows, we fix a small neighborhood V of the cycle and study the dynamics of $G_{a,t}$ in such a neighborhood (the dynamics relative to the set V). Note that as the periodic points P and Q are of different type if $H_V(P, G_{a,t}) = H_V(Q, G_{a,t})$ then $G_{a,t}$ is not hyperbolic.

Definition 2.5 (Relative dynamics)**.** We say that $G_{a,t}$ is *Axiom A relative to the neighborhood V of the cycle* if the non-wandering set of the restriction of $G_{a,t}$ to V is hyperbolic and coincides with the closure of the set of periodic points.

Similarly, a (heterodimensional) cycle relative to V associated to a pair of hyperbolic periodic points A and B of different type means that there are heterocilinic points $X \in W^u(A, G_{a,t}) \cap W^s(B, G_{a,t})$ and $Y \in W^s(A, G_{a,t}) \cap W^u(B, G_{a,t})$ whose orbits are contained in V.

2.2. **Main result.** Recall the definitions of the set of parameters $N_a(s)$, $H_a(s)$ in (1.1) and of frequencies $N_a^{\pm}$ and $H_a^{\pm}$ in (1.2). Recall also that a periodic point $(\alpha; a)$, $\alpha = (\alpha_0 \dots \alpha_k)^{\mathbb{Z}}$, is a *saddle-node* of $G_{a,t}$ if $(g_{\alpha_k,t} \circ \cdots \circ g_{\alpha_0,t})'(a) = 1$.

We are now ready to state our main result describing the dynamics and some typical bifurcations of the maps $G_{a,t}$.

Theorem 2.6. *Consider the family of skew product maps $G_{a,t}$ in (2.1) and the hyperbolic fixed points $Q = (0^{\mathbb{Z}}; 0)$ and $P = (0^{\mathbb{Z}}; 1/2)$ of different type. For every small neighborhood V of the cycle associated to P and Q the following holds:*

(A) Robustly non-hyperbolic dynamics: *for every $a \in (0, \log 2)$, $a \neq \log \frac{1+\sqrt{5}}{2}$, there is $t(a) > 0$ such that $(0, t(a)] \subset N_a$.*

(B) Persistence of non-hyperbolicity: *Let $a \in \big(\log 2, \log \frac{3+\sqrt{5}}{2}\big)$. Then*

(a) *either $G_{a,t_n(a)}$ has a (relative) cycle related to P and Q or there is a sequence $\alpha_n(a) \to 0^+$ such that*

$$\big[t_n(a) - \alpha_n(a), t_n(a)\big) \subset N_a.$$

(b) *There is $\zeta(a) \in (0, 1)$ with $\zeta(a) \to 1$ as $a \to \log 2$, such that for every n large enough*

$$\Big(t_{n+1}(a), t_{n+1}(a) + \zeta(a)\left(t_n(a) - t_{n+1}(a)\right)\Big) \subset N_a.$$

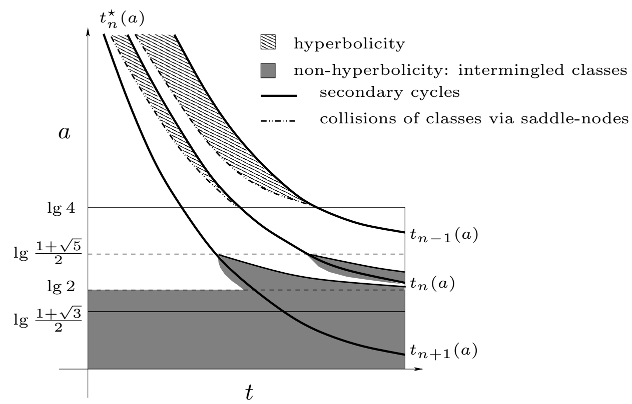

FIGURE 4. Bifurcation diagram

(C) Prevalent hyperbolicity and secondary bifurcations: *for every $a > \log 4$ there are sequences of parameters $t_n(a), t_n^\star(a) \searrow 0$, $t_n^\star(a) \in (t_{n+1}(a), t_n(a))$, such that:*

(a) Hyperbolicity: *It holds $(t_n^\star(a), t_n(a)) \subset H_a$ and for every $t \in (t_n^\star(a), t_n(a))$ the resulting non-wandering set of $G_{a,t}$ is the disjoint union of the relative homoclinic classes of P and Q in V. Moreover,*

$$\lim_{a\to\infty} \mathbf{H}_a^- = 1.$$

(b) Secondary heterodimensional cycles: *for every $t_n(a)$ the map $G_{a,t_n(a)}$ has a cycle relative to V associated to P and Q.*

(c) Collisions of homoclinic classes via saddle-noddes: *for every $t_n^\star(a)$ the intersection $H_V(P, G_{a,t_n^\star(a)}) \cap H_V(Q, G_{a,t_n^\star(a)})$ is the orbit of a saddle-node. Moreover, compact invariant subsets of these classes disjoint from the saddle-node are uniformly hyperbolic.*

The parameter $a = \frac{1+\sqrt{5}}{2}$ is exceptional (see Lemma 5.4) and corresponds to the appearance of branches of the IFS with contracting points. Similar results can be obtained for this parameter, but the proofs must consider different types of returns. We skip this technical discussion.

3. Skew product dynamics: homoclinic and heteroclinic points

In this section we state some properties of one-step skew product maps

$$G\colon \Sigma_2 \times \mathbb{K} \to \Sigma_2 \times \mathbb{K}, \quad G(\xi; x) = (\sigma(\xi), g_{\xi_0}(x)),$$

where $\mathbb{K}$ is a one-dimensional manifold (the circle, an interval, or the real line), $\sigma\colon \Sigma_2 \to \Sigma_2$ is the shift map, and $g_0, g_1 \colon \mathbb{K} \to \mathbb{K}$ are diffeomorphisms. We see how the notions of hyperbolicity and homoclinic and heteroclinic intersections are stated in the skew product context.

We use the cylinder notation for compositions of maps

$$g_{[\xi_0 \ldots \xi_m]} \stackrel{\text{def}}{=} g_{\xi_m} \circ \cdots \circ g_{\xi_0}$$

and the following notation for pre and/or post-periodic sequences

$$\xi = (\xi_i)_{i\in\mathbb{Z}} = \big((\rho_{-r} \cdots \rho_{-1})^{-\mathbb{N}}\, \eta_{-n} \cdots \eta_{-1} \,.\, \eta_0 \cdots \eta_k \,(\alpha_1 \cdots \alpha_m)^{\mathbb{N}}\big);$$

- $\xi_i = \eta_i$ if $i \in \{-n, \dots, 0, \dots, k\}$;
- $\xi_{k+s\,m+i} = \alpha_i$ for every $i \in \{1, \dots, m\}$ and $s \geq 0$;
- $\xi_{-n-s\,r-i} = \rho_{-i}$ for every $i \in \{1, \dots, r\}$ and $s \geq 0$.

We similarly define a periodic sequence $(\xi_0 \dots \xi_{\pi-1})^{\mathbb{Z}}$.

Given a sequence $\xi = (\xi_i)_{i\in\mathbb{Z}}$ we consider its positive and negative tails $\xi^+ \stackrel{\text{def}}{=} (\xi_i)_{i\geq 0}$ and $\xi^- \stackrel{\text{def}}{=} (\xi_i)_{i\leq 0}$.

3.1. **Hyperbolicity and continuations.** Given a hyperbolic fixed point p of $g_{[\xi_0\dots\xi_m]}$ consider its local invariant manifolds $W^{s,u}_{\text{loc}}(p, g_{[\xi_0\dots\xi_m]})$. If p is contracting (resp. expanding) then $W^u_{\text{loc}}(p, g_{[\xi_0\dots\xi_m]}) = \{p\}$ (resp. $W^s_{\text{loc}}(p, g_{[\xi_0\dots\xi_m]}) = \{p\}$). Associated to p there is the *"hyperbolic" periodic point* $P = ((\xi_0 \dots \xi_m)^{\mathbb{Z}}; p)$ of period $m+1$ of G (and vice-versa). We say that P is *contracting* (resp. expanding) is p is contracting (resp. expanding).

As in the differentiable case, hyperbolic points of skew product maps have well defined continuations. If F close to G then the cylinder map $f_{[\xi_0\dots\xi_m]}$ is close to $g_{[\xi_0\dots\xi_m]}$ and thus the continuation p_F of p is uniquely defined (p_F is close to p and $f_{[\xi_0\dots\xi_m]}(p_F) = p_F$). Then the *continuation* of P for F is $P_F = ((\xi_0 \dots \xi_m)^{\mathbb{Z}}; p_F)$.

The stable and unstable invariant sets of the periodic point P above are

$$W^s(P,G) = \left\{ (\eta; x) \colon \quad \begin{array}{l} \eta = \big(\cdots . \eta_0 \cdots \eta_k (\xi_0 \cdots \xi_m)^{\mathbb{N}} \big), \\ g_{[\eta_0\cdots\eta_k]}(x) \in W^s_{\text{loc}}(p, g_{[\xi_0\cdots\xi_m]}) \end{array} \right\};$$

$$W^u(P,G) = \left\{ (\eta; x) \colon \quad \begin{array}{l} \eta = \big((\xi_0 \cdots \xi_m)^{-\mathbb{N}}\, \eta_{-k} \cdots \eta_{-1} . \cdots \big), \\ g^{-1}_{[\eta_{-1}\cdots\eta_{-k}]}(x) \in W^u_{\text{loc}}(p, g_{[\xi_0\cdots\xi_m]}) \end{array} \right\}.$$

3.2. **Homoclinic and heteroclinic intersections.** A point $X \in W^u(P,G) \cap W^s(P,G)$ is called a *homoclinic point* X of P. Homoclinic points behave as the "transverse" ones in the differentiable case and have well defined continuations. To see why this is so suppose, for instance, that P is contracting. Since $X = (\eta; x) \in W^u(P,G) \cap W^s(P,G)$, after replacing X by some iterate if necessary, we can assume that

$$X = (\eta; x) = \big((\xi_0 \dots \xi_m)^{-\mathbb{N}} . \eta_0 \dots \eta_r\, (\xi_0 \dots \xi_m)^{\mathbb{N}}; x \big).$$

As $W^u(p, g_{[\xi_0\dots\xi_m]}) = \{p\}$ one has that $x = p$ and thus $X = (\eta; p)$. Note that

$$G^{r+1}(X) = (\hat\eta; \hat x) = (\hat\eta; g_{[\eta_0\dots\eta_r]}(p)), \quad \text{where } \hat\eta = \big(\cdots . (\xi_0 \dots \xi_m)^{\mathbb{N}}\big).$$

Since $X \in W^s(P,G)$, after replacing $\hat x$ by some iterate of it of the form $g^{\ell}_{[\xi_0\dots\xi_m]}(\hat x)$ we can assume that $\hat x \in (p-\delta, p+\delta) \subset W^s_{\text{loc}}(p, g_{[\xi_0\dots\xi_m]})$.

Consider a map F close to G and the continuation $P_F = (\xi; p_F)$ of P for F. If F is close enough to G then $(p-\delta, p+\delta) \subset W^s_{\text{loc}}(p_F, f_{[\xi_0\dots\xi_m]})$. Consider the point $X_F = (\eta; p_F)$. Note that by construction $X_F \in W^u(P_F, F)$ and $F^{r+1}(X_F) = (\hat\eta, f_{[\eta_0\dots\eta_r]}(p_F))$. As F is close to G then $f_{[\eta_0\dots\eta_r]}(p_F)$ is close to $g_{[\eta_0\dots\eta_r]}(p)$, thus

$$f_{[\eta_0\dots\eta_r]}(p_F) \in (p-\delta, p+\delta) \subset W^s_{\text{loc}}(p_F, f_{[\xi_0\dots\xi_{m-1}]}).$$

This implies that $X_F \in W^s(P_F, F)$. Thus $X_F \in W^u(P_F, F) \cap W^s(P_F, F)$ and it is a homoclinic point of P_F, called the *continuation* of X.

The following conditions for the existence of homoclinic and heteroclinic intersections are consequences of the arguments in the discussion above.

Corollary 3.1 (Homoclinic and heteroclinic intersections)**.** *Consider hyperbolic periodic points $A = \big((\xi_0 \dots \xi_m)^{\mathbb{Z}}; a\big)$ and $B = \big((\nu_0 \dots \nu_\ell)^{\mathbb{Z}}; b\big)$ of contracting and expanding type of G, respectively.*

(1) If there is a finite sequence $\beta_0 \dots \beta_r$ such that

$$g_{[\beta_0 \dots \beta_r]}(a) \in W^s_{\mathrm{loc}}(a, g_{[\xi_0 \dots \xi_m]})$$

then $\big((\xi_0 \cdots \xi_m)^{-\mathbb{N}} \,.\, \beta_0 \cdots \beta_r \, (\xi_0 \cdots \xi_m)^{\mathbb{N}}, a\big)$ is a homoclinic point of A.

(2) If there are a sequence $\alpha_{-r} \dots \alpha_{-1}$ and $z \in W^u_{\mathrm{loc}}(b, g_{[\nu_0 \dots \nu_\ell]})$ with

$$g_{[\alpha_{-r} \dots \alpha_{-1}]}(z) = b.$$

Then $\big((\nu_0 \cdots \nu_\ell)^{-\mathbb{N}} \, \alpha_{-r} \cdots \alpha_{-1} \,.\, (\nu_0 \cdots \nu_\ell)^{\mathbb{N}}, b\big)$ is a homoclinic point of B.

(3) If there is a finite sequence $\gamma_0 \dots \gamma_k$ such that

$$g_{[\gamma_0 \dots \gamma_k]}(a) = b$$

then $\big((\xi_0 \cdots \xi_m)^{-\mathbb{N}} \,.\, \gamma_0 \cdots \gamma_r \, (\nu_0 \cdots \nu_\ell)^{\mathbb{N}}, a\big) \in W^u(A, G) \cap W^s(B, G)$.

(4) If there are $x \in W^u_{\mathrm{loc}}(b, g_{[\nu_0 \dots \nu_\ell]})$ and a finite sequence $\tau_0 \dots \tau_j$ such that

$$g_{[\tau_0 \dots \tau_j]}(x) \in W^s_{\mathrm{loc}}(a, g_{[\xi_0 \dots \xi_\ell]})$$

then $\big((\nu_0 \cdots \nu_\ell)^{-\mathbb{N}} \,.\, \tau_0 \cdots \gamma_r \, (\xi_0 \cdots \xi_m)^{\mathbb{N}}; x\big) \in W^u(B, G) \cap W^s(A, G)$.

4. One-dimensional dynamics. Iterated function systems

In this section we introduce the iterated function system associated to the maps $G_{a,t}$ and the unfolding of the cycle. This IFS describes the *central dynamics* in the sequel of the bifurcation.

4.1. Preliminary calculations.

The definitions of g_a and $g_{1,t}$ provide explicit formulae for their iterations, compositions, and derivatives. We list some identities (that follow from the definitions of g_a and $g_{1,t}$) that we will use throughout the text.

For $x \in (0, 1/2)$ a straightforward calculation gives,

$$g_a^n(x) = \frac{x\, e^{n\,a}}{2\,x\,e^{n\,a} + (1 - 2\,x)}, \qquad (g_a^n)'(x) = \frac{e^{-n\,a}}{x^2}\,(g_a^n(x))^2\,. \tag{4.1}$$

Fixed $a > 0$, consider sequence of parameters $t_n(a)$ given by

$$g_a^n(t_n(a)) = 1/2 - t_n(a) \quad \Longleftrightarrow \quad g_{1,t_n(a)} \circ g_a^n \circ g_{1,t_n(a)}(1/2) = 0. \tag{4.2}$$

An immediate consequence of Corollary 3.1 is the following:

Remark 4.1. *Given $a > 0$, the map $G_{a,t_n(a)}$ has a cycle associated to P and Q for every parameter $t_n(a)$.*

From (4.1) after a simple calculation we get

$$e^{-na} = \frac{(2\,t_n(a))^2}{(1 - 2\,t_n(a))^2}. \tag{4.3}$$

In particular, $t_n(a) \searrow 0^+$. We also have the following relation between the parameters $t_n(a)$,

$$\frac{t_{n+1}(a)}{t_n(a)} = e^{-\frac{a}{2}}\,\frac{(1 - 2\,t_{n+1}(a))}{(1 - 2\,t_n(a))} > e^{-\frac{a}{2}}, \qquad \lim_{n\to+\infty} \frac{t_{n+1}(a)}{t_n(a)} = e^{-\frac{a}{2}}. \tag{4.4}$$

Using (4.3), we can rewrite (4.1) as follows,

$$\begin{aligned} g_a^n(x) &= \frac{x(1-2\,t_n(a))^2}{2\,x(1-2\,t_n(a))^2+(1-2\,x)\,(2\,t_n(a))^2}; \\ (g_a^n)'(x) &= \left(\frac{2\,t_n(a)(1-2\,t_n(a))}{2x(1-2\,t_n(a))^2+(1-2\,x)(2\,t_n(a))^2}\right)^2 \simeq \left(\frac{t_n(a)}{x}\right)^2. \end{aligned} \tag{4.5}$$

4.2. **Returns and iterated function systems.** Given n define the interval of parameters

$$I_n(a) \stackrel{\text{def}}{=} (t_{n+1}(a), t_n(a)].$$

For each $t \in I_n(a)$ define $d_{a,t} \in (0, 1/2)$ by

$$g_{1,t} \circ g_a^n(d_{a,t}) = 0 \tag{4.6}$$

and consider the fundamental domain of g_a given by

$$D_{a,t} \stackrel{\text{def}}{=} \big(d_{a,t}, g_a(d_{a,t})\big]. \tag{4.7}$$

This domain varies continuously with t in $I_n(a)$. For $x \in D_{a,t}$ and $k \geq 0$ one has

$$0 < g_{1,t} \circ g_a^{n+k}(x) < g_{1,t}(1/2) \leq t \leq d_{a,t}.$$

Thus for each $k \geq 0$ there is exactly one $i_k(x) \geq 1$ with

$$g_a^{i_k(x)} \circ g_{1,t} \circ g_a^{n+k}(x) \in D_{a,t}.$$

Bearing this in mind, for every pair $j, k \geq 0$ we define the following subsets of $D_{a,t}$,

$$D_{a,t}^{(j,k)} \stackrel{\text{def}}{=} \{x \in D_{a,t} : i_k(x) = j\}.$$

Note that some of these subsets may be empty. By the monotonicity of the maps $g_{1,t}$ and g_a one has that $D_{a,t}^{(j,k)}$ is either empty or

$$D_{a,t}^{(j,k)} \stackrel{\text{def}}{=} \big(d_{a,t}^{-,(j,k)}, d_{a,t}^{+,(j,k)}\big].$$

Finally, by definition, for each $k \geq 0$ one has

$$D_{a,t} = \bigcup_{j\geq 1} D_{a,t}^{(j,k)}, \quad D_{a,t}^{(j,k)} \cap D_{a,t}^{(m,k)} = \emptyset \quad \text{if } j \neq m. \tag{4.8}$$

Define now the *return maps*

$$\Gamma_{a,t}^{(j,k)} \colon D_{a,t}^{(j,k)} \to D_{a,t}, \quad \Gamma_{a,t}^{(j,k)}(x) \stackrel{\text{def}}{=} g_a^j \circ g_{1,t} \circ g_a^{n+k}(x), \; t \in I_n(a). \tag{4.9}$$

We also consider the compositions of the maps $\Gamma^{(j,k)}$,

$$\Gamma_{a,t}^{\mathbf{b}} \stackrel{\text{def}}{=} \Gamma_{a,t}^{(j_n,k_n)} \circ \cdots \circ \Gamma_{a,t}^{(j_1,k_1)} \colon D_{a,t}^{\mathbf{b}} \to D_{a,t}, \quad \mathbf{b} = (j_n,k_n)\cdots(j_1,k_1), \tag{4.10}$$

where

$$D_{a,t}^{\mathbf{b}} \stackrel{\text{def}}{=} (d_{a,t}^{-,\mathbf{b}}, d_{a,t}^{+,\mathbf{b}}]$$

is the maximal set where the return map $\Gamma_{a,t}^{\mathbf{b}}$ is defined (this set may be empty). Note that $(\Gamma_{a,t}^{\mathbf{b}})'$ is strictly decreasing in $D_{a,t}^{\mathbf{b}}$.

We close this section with some estimates for $d_{a,t}$. Note that, by definition, if $t \in (t_{n+1}(a), t_n(a))$ then

$$g_{1,t} \circ g_a^n(t_n(a)) < g_{1,t_n(a)} \circ g_a^n(t_n(a)) = 0.$$

Thus

$$d_{a,t_n(a)} = t_n(a) \quad \text{and} \quad d_{a,t} > t_n(a) > t \quad \text{if } t \in (t_{n+1}(a), t_n(a)). \tag{4.11}$$

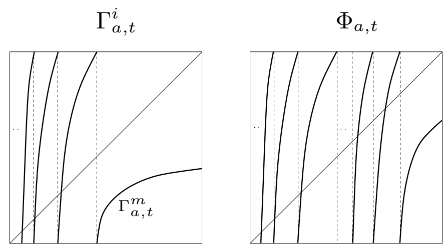

FIGURE 5. The return map $\Phi_{a,t}$ ($m = m_{a,t}$).

Write $t = t_n(a)\,(1+\mu)$, $\mu \le 0$. From the definition of $d_{a,t}$ and (4.5) one gets

$$d_{a,t} = \frac{\big(1-2\,t_n(a)\,(1+\mu)\big)\,t_n(a)}{2\,\big(1-2\,t_n(a)(1+\mu)\big)\,t_n(a) + (1+\mu)\,(1-2\,t_n(a))^2}. \tag{4.12}$$

It follows that for large n one has

$$d_{a,t} \simeq \frac{t_n(a)}{1+\mu} \quad \text{and} \quad g_a(d_{a,t})) \simeq \frac{e^a\,t_n(a)}{1+\mu}, \quad t = t_n(a)(1+\mu). \tag{4.13}$$

5. EXPANDING RETURN MAPS FOR $a \in \left(0, \log \frac{1+\sqrt{5}}{2}\right)$

In this section, for appropriate pairs of parameters (a,t), we will construct an expanding return map $\Phi_{a,t}$ as follows: (i) the map $\Phi_{a,t}$ is a composition of maps $\Gamma^{(i,0)}_{a,t}$, has infinitely many discontinuities (a countable number) and is uniformly expanding in each domain of continuity, (ii) the discontinuity points are mapped to the right extreme of $D_{a,t}$.

5.1. **The expanding return map $\Phi_{a,t}$.** In this section we only consider maps $\Gamma^{(i,j)}_{a,t}$ with $j = 0$. Thus for notational simplicity let us write $\Gamma^i_{a,t} \stackrel{\text{def}}{=} \Gamma^{(i,0)}_{a,t}$ and $D^i_{a,t} \stackrel{\text{def}}{=} D^{(i,0)}_{a,t}$.

Let us explain the definition of $\Phi_{a,t}$. This map is obtained using compositions of the maps $\Gamma^i_{a,t}$. Let $m_{a,t}$ be the first $j \ge 1$ such that $D^j_{a,t} \ne \emptyset$. We will see that the maps $\Gamma^j_{a,t}$, $j \ge 4$, are uniformly expanding in $D^j_{a,t}$ (Lemma 5.6). On the other hand, when $m_{a,t} < 4$ the map $\Gamma^{m_{a,t}}_{a,t}$ may fail to be expanding. This is the reason we need to replace the map $\Gamma^{m_{a,t}}_{a,t}$ by some map of the form $\Gamma^k_{a,t} \circ \Gamma^{m_{a,t}}_{a,t}$. In some cases this new map is expanding. We now go into the details of this construction.

For each point $x \in D^{m_{a,t}}_{a,t}$ we define $r(x) = r_{a,t}(x)$ by the condition

$$\Gamma^{m_{a,t}}_{a,t}(x) \in D^{r(x)}_{a,t}. \tag{5.1}$$

Noting that $D_{a,t} = \bigcup_{j \ge m_{a,t}} D^j_{a,t}$ we define the following induced return map (see Figure 5):

$$\Phi_{a,t}\colon D_{a,t} \to D_{a,t}, \quad \Phi_{a,t}(x) \stackrel{\text{def}}{=} \begin{cases} \Gamma^i_{a,t}(x) & \text{if } x \in \displaystyle\bigcup_{i \ge m_{a,t}+1} D^i_{a,t}, \\ \Gamma^{r(x)}_{a,t} \circ \Gamma^{m_{a,t}}_{a,t}(x) & \text{if } x \in D^{m_{a,t}}_{a,t}. \end{cases} \tag{5.2}$$

In Sections 5.4, 5.5, and 5.6 we will see three different cases where the return map $\Phi_{a,t}$ is uniformly expanding (for some pair of parameters (a,t)).

Remark 5.1 (Discontinuities of $\Phi_{a,t}$)**.** Given a pair of parameters (a,t) we denote by $\mathcal{D}(\Phi_{a,t})$ the set of discontinuities of $\Phi_{a,t}$. Write $m = m_{a,t}$ and define $\ell_{a,t} = \ell$ by the condition $\Gamma^m_{a,t}(g_a(d_{a,t})) \in D^\ell_{a,t}$. For each $j \geq \ell$ there is a unique $d^{m,j}_{a,t} \in D^m_{a,t}$ such that $\Gamma^m_{a,t}(d^{m,j}_{a,t}) = d^j_{a,t}$. Then

$$\mathcal{D}(\Phi_{a,t}) = \{d^i_{a,t},\, i \geq m\} \cup \{d^{m,j}_{a,t}, j \geq \ell\}.$$

Note that $d^{m,j}_{a,t} \to d^m_{a,t}$ as $j \to \infty$.

Observe that, by definition, $g_{1,t} \circ g^n(d_{a,t}) = 0$. Using this fact and the definition of the discontinuity set we get the following:

Remark 5.2 (Images of the discontinuities of $\Phi_{a,t}$)**.**

- $g_{1,t} \circ g_a^{n-1} \circ \Gamma^{i+1}_{a,t}(d^i_{a,t}) = 0$;
- $g_{1,t} \circ g_a^{n-1} \circ \Gamma^j_{a,t} \circ \Gamma^{m_{a,t}}_{a,t}(d^{m_{a,t},j}_{a,t}) = 0$.

5.2. **The sets $D^i_{a,t}$.** The analysis of the dynamics of the map $\Phi_{a,t}$ crucially involves the extremes of the sets $D^i_{a,t}$. Define

$$d^i_{a,t} \in (0,1/2) : \; g^i_a \circ g_{1,t} \circ g^n_a(d^i_{a,t}) = d_{a,t}, \quad \text{where } t \in (t_{n+1}(a), t_n(a)]. \tag{5.3}$$

Note that $D^j_{a,t} = (d^j_{a,t}, d^{j-1}_{a,t}] \neq \emptyset$ if $j > m_{a,t}$ and $D^{m_{a,t}}_{a,t} = \big(d^{m_{a,t}}_{a,t}, g_a(d_{a,t})\big]$.

Lemma 5.3. *Consider $a > 0$ and $t = (1+\mu)\, t_n(a) \in (t_{n+1}(a), t_n(a)]$. Then for every n sufficiently big*

$$d^i_{a,t} \simeq t_n(a)\, K_i(a,\mu), \quad \textit{where} \quad K_i(a,\mu) \overset{\text{def}}{=} \frac{e^{i\,a}\,(1+\mu)}{e^{i\,a}\,(1+\mu)^2 - 1}.$$

Proof. Consider the extension of $\Gamma^i_{a,t}$ to $[0,1/2]$ given by $g^i_a \circ g_{1,t} \circ g^n_a$. With some abuse of notation we will denote this extension also by $\Gamma^i_{a,t}$. Fix a and write $t_n = t_n(a)$ and $d^i_t = d^i_{a,t}$. Using the first part of (4.5) and (4.13), we rewrite equation (5.3) as follows,

$$\Gamma^i_{a,t}(d^i_t) \simeq e^{i\,a} \left((1+\mu)\, t_n - \frac{1}{2} + \frac{d^i_t\,(1-2\,t_n)^2}{2\,d^i_t(1-2\,t_n)^2 + (1-2\,d^i_t)\,(2\,t_n)^2} \right) \simeq \frac{t_n}{1+\mu}.$$

Simplifying the central term of the equation above we get

$$t_n \left(\frac{e^{i\,a}\,(1+\mu) - \frac{1}{1+\mu}}{e^{i\,a}} \right) \simeq \frac{(1-2\,d^i_t)\,(2\,t_n)^2}{4\,d^i_t\,(1-2\,t_n)^2 + 2\,(1-2\,d^i_t)\,(2\,t_n)^2}.$$

Or equivalently,

$$\frac{e^{i\,a}}{t_n\,\left(e^{i\,a}\,(1+\mu) - \frac{1}{1+\mu}\right)} \simeq \frac{4\,d^i_t\,(1-2\,t_n)^2 + 2\,(1-2\,d^i_t)\,(2\,t_n)^2}{(1-2\,d^i_t)\,(2\,t_n)^2}. \tag{5.4}$$

With the notation in the lemma it follows

$$d^i_t \simeq \frac{K_i(a,\mu)\,t_n - 2\,t_n^2}{2\,t_n\,K_i(a,\mu) + 1 - 4\,t_n} = t_n \left(\frac{K_i(a,\mu) - 2\,t_n}{2\,t_n\,K_i(a,\mu) + 1 - 4\,t_n} \right) \simeq t_n\,K_i(a,\mu).$$

This completes the proof of the lemma. $\square$

Next lemma states some relations between $m_{a,t}$ and a.

Lemma 5.4 (Values of $m_{a,t}$)**.**

(1) Let $a \in (0, \log 2)$. Then $m_{a,t} > 1$ for every t small enough.

(2) Let $a \in \left(0, \log \frac{1+\sqrt{5}}{2}\right)$. Then $m_{a,t} > 2$ for every t small enough.
(3) Let $a > \log 2$. Then $m_{a,t_n(a)} = 1$ for every $t_n(a)$ small enough.

Proof. For $t \in (t_{n+1}(a), t_n(a)]$ write $t = (1+\mu)\, t_n(a)$.

To get that $m_{a,t} > 1$ (i.e., $D^1_{a,t} = \emptyset$) it is enough to see that $g_a(d_{a,t}) < d^1_{a,t}$. By Lemma 5.3 and (4.13),

$$g_a(d_{a,t}) \simeq \frac{e^a\, t_n(a)}{1+\mu} < t_n(a)\, \frac{e^a\,(1+\mu)}{e^a\,(1+\mu)^2 - 1} \simeq d^1_{a,t}.$$

This inequality is equivalent to

$$1 < \frac{(1+\mu)^2}{e^a\,(1+\mu)^2 - 1}$$

which holds for all $a \in (0, \log 2)$ and $1+\mu \in (e^{-a/2}, 1]$.

To prove item (2) it is enough to see that $D^2_{a,t} = \emptyset$ (or equivalently that $g_a(d_{a,t}) < d^2_{a,t}$) for every t small enough. As above

$$g_a(d_{a,t}) \simeq \frac{e^a\, t_n(a)}{1+\mu} < t_n(a)\, \frac{e^{2\,a}\,(1+\mu)}{e^{2\,a}\,(1+\mu)^2 - 1} \simeq d^2_{a,t}.$$

Thus it is enough to see that

$$1 < \frac{e^a\,(1+\mu)^2}{e^{2\,a}\,(1+\mu)^2 - 1}.$$

Note that for $1+\mu \in (e^{-a/2}, 1]$ one has

$$\frac{e^a}{e^{2\,a} - 1} < \frac{e^a\,(1+\mu)^2}{e^{2\,a}\,(1+\mu)^2 - 1}.$$

Thus it is enough to see that $e^{2\,a} - 1 < e^a$, where this inequality holds for all $0 < a < \log \frac{1+\sqrt{5}}{2}$.

To prove item (3) note that condition $D^1_{a,t_n(a)} \neq \emptyset$ is equivalent to $d_{a,t} < d^1_{a,t} < g_a(d_{a,t})$. As above

$$d_{a,t_n(a)} \simeq t_n(a) < d^1_{a,t_n(a)} \simeq t_n(a)\, \frac{e^a}{e^a - 1} < g_a(d_{a,t_n(a)} \simeq e^a\, t_n(a).$$

That is

$$1 < \frac{e^a}{e^a - 1} < e^a,$$

which is satisfied for $a > \log 2$. The proof of the lemma is now complete. □

We close this subsection with an extension of the first part of Lemma 5.4 that will be used in Section 5.5. Observe that $D^1_{a,t} = \emptyset$ (or $m_{a,t} > 1$) when $d^1_{a,t} > g_a(d_{a,t})$. By Lemma 5.3 and equation (4.13), for sufficiently large n, this occurs when

$$t_n(a) \left(\frac{e^a\,(1+\mu)}{e^a\,(1+\mu)^2 - 1} \right) > e^a \left(\frac{t_n(a)}{1+\mu} \right) \quad \Longleftrightarrow \quad \frac{(1+\mu)^2}{e^a\,(1+\mu)^2 - 1} > 1.$$

For $a > \log 2$ define $\nu(a) \in (e^{-a/2} - 1, 0)$ by the condition

$$\frac{(1+\nu(a))^2}{e^a\,(1+\nu(a))^2 - 1} = 1.$$

Observe that $\nu(a) \to (e^{-a/2} - 1)$ as $a \to \infty$.

Note also that for $a > \log 2$ we have $D^2_{a,t} \neq \emptyset$ (or $m_{a,t} \leq 2$). This is equivalent to

$$t_n(a) \left(\frac{e^{2\,a}\,(1+\mu)}{e^{2\,a}\,(1+\mu)^2 - 1} \right) < e^{2\,a} \left(\frac{t_n(a)}{1+\mu} \right) \quad \Longleftrightarrow \quad \frac{e^{a}\,(1+\mu)^2}{e^{2\,a}\,(1+\mu)^2 - 1} < 1,$$

where the last inequality holds for every μ with $e^{-a/2} < (1+\mu) \leq 1$ if $a > \log 2$.

Define for $a > \log 2$

$$I_n(\nu(a)) \stackrel{\text{def}}{=} \big(t_{n+1}(a), (1+\nu(a))\,t_n(a)\big) \subset I_n(a). \tag{5.5}$$

The choice of $\nu(a)$ and the discussion above imply the following lemma.

Lemma 5.5. *Consider $a > \log 2$ and $\nu(a)$. Then there is $n_0(a)$ such that for every $n \geq n_0(a)$ and $\mu \in \big(e^{-a/2} - 1, \nu(a)\big)$ it holds*

$$m_{a,t} = 2 \quad \text{for every } t \in I_n(\nu(a)) \subset I_n(a).$$

5.3. **Lower bounds for $\big(\Gamma^j_{a,t}\big)'$.** The next step in the construction of the expanding returns is to get a lower bound for $\big(\Gamma^j_{a,t}\big)'$. In what follows, for the extremes of $D^j_{a,t}$ the expressions $\big(\Gamma^j_{a,t}\big)'(d^{j-1}_{a,t})$ and $\big(\Gamma^j_{a,t}\big)'(d^j_{a,t})$ mean the derivatives to the left and to the right, respectively.

Lemma 5.6. *Let $t = t_n(a)\,(1+\mu) \in I_n(a)$.*

(1) For every n big enough it holds

$$\left(\Gamma^j_{a,t}\right)'(x) \geq e^{(j-2)\,a}\,(1+\mu)^2, \quad \text{for all} \quad x \in D^j_{a,t}.$$

(2) Let $a > \log \frac{1+\sqrt{5}}{2}$. Then there is $\tau(a) > 1$ such that for every n big enough and $j \geq 3$ it holds

$$\left(\Gamma^j_{a,t}\right)'(x) \geq \tau(a) \quad \text{for all,} \quad x \in D^j_{a,t}.$$

Proof. Consider $x \in D^j_{a,t}$ where $t = t_n(a)\,(1+\mu) \in I_n(a)$. From the monotonicity of g'_a and $(g^j_a)'(0) = e^{j\,a}$ we get

$$\left(\Gamma^j_{a,t}\right)'(x) \geq \left(\Gamma^j_{a,t}\right)'(d^j_{a,t}) \geq \big(g^j_a \circ g_{1,t} \circ g^n_a\big)'(d^j_{a,t}) \simeq e^{j\,a}\,(g^n_a)'(d^j_{a,t}). \tag{5.6}$$

Using the monotonicity of the derivative of g_a we get

$$\left(\Gamma^j_{a,t}\right)'(x) \geq \big(g^j_a \circ g_{1,t} \circ g^n_a\big)'(g_a(d_{a,t})) \simeq e^{j\,a}\,(g^n_a)'(g_a(d_{a,t})). \tag{5.7}$$

Noting that $g_a(d_{a,t}) \simeq e^a\,d_{a,t}$ and that $d_{a,t} \simeq t_n(a)/(1+\mu)$ (see (4.13)), from equation (4.5) one gets

$$(g^n_a)'(g_a(d_{a,t})) \simeq e^{-2a}\,(1+\mu)^2.$$

The first item of the lemma now follows from (5.7).

For the second item of the lemma, using equation (4.5), $d^3_{a,t} > d^j_{a,t}$ for all $j \geq 3$, and Lemma 5.3 we get for $j \geq 3$

$$\left(\Gamma^j_{a,t}\right)'(x) \geq e^{j\,a} \left(\frac{t_n}{d^j_{a,t}} \right)^2 \geq e^{3\,a} \left(\frac{t_n}{d^3_{a,t}} \right)^2 \simeq e^{3\,a} \left(\frac{t_n}{t_n \frac{e^{3\,a}\,(1+\mu)}{e^{3\,a}\,(1+\mu)^2 - 1}} \right)^2$$

As $e^{-a/2} < (1+\mu) \leq 1$ we have that

$$\left(\Gamma^j_{a,t}\right)'(x) \geq e^{3\,a} \left(\frac{e^{3\,a}\,e^{-a}-1}{e^{3\,a}\,e^{-a/2}}\right)^2 = e^{-2\,a}\,(e^{2\,a}-1)^2 = \tau(a) > 1.$$

Where the last inequality follows from $a > \log\left(\frac{1+\sqrt{5}}{2}\right)$. □

5.4. **Expanding returns for $a \in (0, \log 2)$.**

Theorem 5.7 (Expanding induced map $\Phi_{a,t}$)**.** *Let $a \in (0, \log 2)$ with $a \neq \log \frac{1+\sqrt{5}}{2}$. There is $\kappa(a) > 1$ such that for every small t*

$$\Phi'_{a,t}(x) \geq \kappa(a) \quad \textit{for all } x \in D_{a,t}.$$

Proof. Take $t = (1+\mu)\,t_n(a) \in I_n(a)$, $\mu \leq 0$. By Lemma 5.6 and $(1+\mu) > e^{-a/2}$ (recall (4.3)) for sufficiently large n we get

$$\left(\Gamma^j_{a,t}\right)'(x) \geq e^{(j-2)\,a}(1+\mu)^2 > e^a > 1, \text{ if } x \in D^j_{a,t}, \text{ and } j \geq 4. \tag{5.8}$$

This estimate and the definition of $\Phi_{a,t}$ imply the theorem for points in $D^j_{a,t}$ with $j \geq 4$ (note that if $m_{a,t} \geq 4$ this inequality also implies the expansion of $\Phi_{a,t}$ in $D^{m_{a,t}}_{a,t}$).

Recall that $D^1_{a,t} = \emptyset$ ($m_{a,t} > 1$) for $a \in (0, \log 2)$ (item (1) in Lemma 5.4). Thus to prove the theorem it remains to estimate the derivative of $\Phi_{a,t}$ in $D^2_{a,t} \cup D^3_{a,t}$. Note that in this case $m_{a,t} = 2$ or 3. The expansion of $\Phi_{a,t}$ comes from the next proposition.

Proposition 5.8. *There is $\rho(a) > 1$ such that $\Phi'_{a,t}(x) > \rho(a)$ for every $x \in D^3_{a,t} \cup D^2_{a,t}$.*

Proof. We need the following lemma.

Lemma 5.9. *Let $x \in D^{m_{a,t}}_{a,t}$. Then $\Phi'_{a,t}(g_a(d_{a,t})) \geq (e^a-1)^{-2} > 1$.*

Proof. Let $m_{a,t} = m$ and note that $g_a(d_{a,t}) \in D^m_{a,t}$. By the monotonicity of g'_a it is enough to see that $\Phi'_{a,t}(g_a(d_{a,t})) > 1$. To see why this is so define

$$d'_{a,t} \stackrel{\text{def}}{=} \Gamma^m_{a,t}(g_a(d_{a,t})) = g^m_a \circ g_{1,t} \circ g^{n+1}_a(d_{a,t}) \simeq e^{m\,a}\,(1-e^{-a})\,(1+\mu)\,t_n, \tag{5.9}$$

where the last identity follows from $g^{n+1}_a(d_{a,t}) \simeq 1/2 - e^{-a}\,t$ (this follows from $g^n_a(d_{a,t}) = 1/2 - t$) and hence $g_{1,t} \circ g^n_a(g_a(d_{a,t})) \simeq (1-e^{-a})\,t$.

On the other hand, by equations (4.5), (5.6), and (5.9)

$$\left(\Gamma^j_{a,t}\right)'(d'_{a,t}) \simeq e^{j\,a}\,\frac{t_n^2}{e^{2\,\ell\,a}\,(1-e^{-a})^2(1+\mu)^2 t_n^2} = \frac{e^{j\,a}}{e^{2\,\ell\,a}(1-e^{-a})^2(1+\mu)^2}.$$

This identity, Lemma 5.6, and $r = r(g_a(d_{a,t})) \geq m$ (definition of $m = m_{a,t}$), imply that

$$\Phi'_{a,t}(g_a(d_{a,t})) = (\Gamma^r_{a,t})'(d'_{a,t})\,(\Gamma^m_{a,t})'(g_a(d_{a,t})) \geq \frac{e^{m\,a}\,e^{(m-2)\,a}\,(1+\mu)^2}{e^{2\,m\,a}(1-e^{-a})^2(1+\mu)^2}.$$

For $m = 2$ or 3 we get

$$\Phi'_{a,t}(g_a(d_{a,t})) \geq \frac{1}{e^{2\,a}\,(1-e^{-a})^2} = \frac{1}{(e^a-1)^2} > 1.$$

This ends the proof of the lemma. □

We are now ready to conclude the proof of the proposition. If $D^2_{a,t}=\emptyset$ the proposition follows immediately from Lemma 5.9 and $(e^a-1)^{-2}>1$ for $a\in(0,\log 2)$.

If $D^2_{a,t}\neq\emptyset$ then $m_{a,t}=2$ and $a>\log\frac{1+\sqrt5}{2}$ (see item (2) in Lemma 5.4). The expansion for points in $D^2_{a,t}$ follows from Lemma 5.9. For points in $D^3_{a,t}$ note that item (2) in Lemma 5.6 implies that $\big(\Phi_{a,t}\big)'(x)\geq\big(\Gamma^3_{a,t}\big)'(x)>\tau(a)>1$.

To conclude the proof in this case just take $\rho(a)=\min\{\tau(a),(e^a-1)^{-2}\}>1$. $\square$

The theorem follows taking $\kappa(a)=\min\{e^a,\rho(a)\}>1$. $\square$

5.5. **Expanding returns for $a\in\big[\log 2,\log\frac{3+\sqrt5}{2}\big)$ close to $(t_n(a))^-$.** For $a\in\big[\log 2,\log\frac{3+\sqrt5}{2}\big)$ we select values of μ such that for every n big enough and every $t=(1+\mu)\,t_n(a)$ the following properties hold:

(i) $m_{a,t}=2$ (i.e. $D^1_{a,t}=\emptyset$ and $D^2_{a,t}\neq\emptyset$) and (ii) $\Gamma^2_{a,t}(g_a(d_{a,t}))\notin D^2_{a,t}$.

We will select parameters (a,t) where the map $\Phi_{a,t}$ is uniformly expanding (see Theorem 5.12).

Given $a>\log 2$, take $\nu(a)$ as in Lemma 5.5 and recall the definition of $I_n(\nu(a))$ in (5.5). Then (i) and (ii) hold for $t\in I_n(\nu(a))$ and $n\geq n_0(a)$.

Lemma 5.10. *There is a continuous map $\xi\colon\big[\log 2,\log\frac{3+\sqrt5}{2}\big)\to(e^{-a/2}-1,0)$ with $\xi(a)\to 0$ as $a\to\log\frac{3+\sqrt5}{2}$ such that*

$$\Gamma^2_{a,t}(g_a(d_{a,t}))\in\bigcup_{i\geq 3}D^i_{a,t}$$

for all $t=t_n(a)(1+\mu)\in I_n(a)$ with $1+\mu<1+\xi(a)$ and n big enough.

Recalling the definition of $r_{a,t}(x)=r(x)$ for $x\in D^{m_{a,t}}_{a,t}=D^2_{a,t}$ (i.e., $\Gamma^2_{a,t}(x)\in D^{r(x)}_{a,t}$) this lemma immediately implies the following:

Corollary 5.11. *For every $t=t_n(a)(1+\mu)\in I_n(a)$ with $1+\mu<1+\xi(a)$ and n big enough it holds $r_{a,t}(x)\geq 3$ for every $x\in D^2_{a,t}$.*

Proof of Lemma 5.10. We need to select parameters such that $\Gamma^2_{a,t}(g_a(d_{a,t}))<d^2_{a,t}$. Arguing as in the proof of Lemma 5.3 and recalling the approximation of $g_a(d_{a,t})$ in (4.13) we get

$$\begin{aligned}\Gamma^2_{a,t}(g_a(d_{a,t}))\simeq\Gamma^2_{a,t}\left(\frac{e^a\,t_n}{1+\mu}\right)\simeq e^{2\,a}\left((1+\mu)\,t_n-\frac{t_n^2}{\frac{e^a\,t_n}{1+\mu}}\right)=\\ =e^a\,t_n\,(1+\mu)\,(e^a-1).\end{aligned}\tag{5.10}$$

Recalling the definition of $d^3_{a,t}$ and Lemma 5.3 it is enough to see that

$$e^a\,t_n\,(1+\mu)\,(e^a-1)<t_n\,\frac{e^{2\,a}\,(1+\mu)}{e^{2\,a}\,(1+\mu)^2-1}\quad\Longleftrightarrow\quad(e^a-1)<\frac{e^a}{e^{2\,a}\,(1+\mu)^2-1}.$$

Note that for $1+\mu=e^{-a/2}$ this inequality is equivalent to $(e^a-1)^2<e^a$ that holds for all $a\in\big[\log 2,\log\frac{3+\sqrt5}{2}\big)$. We define a continuous map $\xi(a)\colon\big[\log 2,\log\frac{3+\sqrt5}{2}\big)\to\big(e^{-a/2}-1,0\big)$ such that

$$(e^a-1)<\frac{e^a}{e^{2\,a}\,(1+\mu)^2-1}\quad\text{for every }\mu\text{ with }e^{-a/2}<1+\mu<1+\xi(a).$$

By definition, $\xi(a) \to 0$ as $a \to \log((3+\sqrt{5})/2$. The proof of the lemma is now complete. $\square$

Define the map $\eta(a) \stackrel{\text{def}}{=} \min\{\nu(a), \xi(a)\}$, where ν and ξ are defined in Lemmas 5.10 and 5.5.

Theorem 5.12 (Expanding induced map $\Phi_{a,t}$)**.** *Consider $a \in \big(\log 2, \log \frac{3+\sqrt{5}}{2}\big)$ and $\mu \in (e^{-a/2}, 1+\eta(a))$. Then for every n sufficiently big and $t = t_n(a)\,(1+\mu) \in I_n(a)$ there is $\kappa(a,t) > 1$ such that $(\Phi_{a,t})'(x) > \kappa(a,t)$ for all $x \in D_{a,t}$.*

Proof. We now estimate the derivative of $\Phi_{a,t}$. The result for points in the complement of $D^2_{a,t}$ follows exactly as in Theorem 5.7 (see equation (5.8) that does not depend on the choice of a). We now consider points $x \in D^2_{a,t}$. As in previous cases, it is enough to estimate the derivative in $g_a(d_{a,t})$. From (4.13) and (4.5) we get

$$\big(\Gamma^2_{a,t}\big)'(g_a(d_{a,t})) \simeq e^{2\,a}\,(g_a^n)'\left(\frac{e^a\,t_n}{1+\mu}\right) = e^{2\,a}\left(\frac{t_n}{\frac{e^a\,t_n}{1+\mu}}\right)^2 = (1+\mu)^2. \tag{5.11}$$

Arguing as in Lemma 5.3 we get

$$\begin{aligned}\Gamma^2_{a,t}(g_a(d_{a,t})) &\simeq \Gamma^2_{a,t}\left(\frac{e^a\,t_n}{1+\mu}\right) \simeq e^{2\,a}\left((1+\mu)\,t_n - \frac{t_n^2}{\frac{e^a\,t_n}{1+\mu}}\right) = \\ &= e^a\,t_n\,(1+\mu)\,(e^a-1).\end{aligned} \tag{5.12}$$

Using (5.10), for $r \geq 3$ we have that[6]

$$\begin{aligned}\big(\Gamma^r_{a,t}\big)'\big(\Gamma^2_{a,t}(g_a(d_{a,t}))\big) &\simeq \big(\Gamma^r_{a,t}\big)'\big(e^a\,t_n\,(1+\mu)\,(e^a-1)\big) \simeq \\ &\simeq e^{r\,a}\,(g_a^n)'\big(e^a\,t_n\,(1+\mu)\,(e^a-1)\big) \geq \\ &\geq e^{3\,a}\left(\frac{t_n}{e^a\,t_n\,(1+\mu)\,(e^a-1)}\right)^2 = \\ &= \frac{e^a}{(1+\mu)^2\,(e^a-1)^2}.\end{aligned} \tag{5.13}$$

Consider $x \in D^2_{a,t}$. By Corollary 5.11 we have that $r(x) = r \geq 3$. Equations (5.11) and (5.13) imply that

$$\begin{aligned}(\Phi_{a,t})'(x) &\geq \big(\Gamma^r_{a,t}\big)'\big(\Gamma^2_{a,t}(g_a(d_{a,t}))\big)\,\big(\Gamma^2_{a,t}\big)'(g_a(d_{a,t}) \geq \\ &\geq \frac{e^a}{(1+\mu)^2\,(e^a-1)^2}\,(1+\mu)^2 = \frac{e^a}{(e^a-1)^2} \stackrel{\text{def}}{=} \tau(a) > 1,\end{aligned}$$

where the last inequality follows from $a \in \big[\log 2, \log \frac{3+\sqrt{5}}{2}\big)$. This completes the proof of the theorem. $\square$

5.6. **Expanding returns for $a \in \big[\log 2, \log \frac{3+\sqrt{5}}{2}\big)$ close to $(t_n(a))^+$.** In the previous two subsections we have constructed expanding returns for parameters such that $D^1_{a,t} = \emptyset$. In this section we consider the case when $D^1_{a,t} \neq \emptyset$. Recall that item (3) of Lemma 5.4 implies that $D^1_{a,t_n(a)} \neq \emptyset$ every $a > \log 2$ and every n large enough. Note also that Lemma 5.5 implies that we must consider parameters close to $t_n^+(a)$ since otherwise $D^1_{a,t} = \emptyset$ (recall that $a \in \big[\log 2, \log \frac{3+\sqrt{5}}{2}\big)$).

[6]Here there is a slight abuse of notation, if $\Gamma^2_{a,t}(g_a(d_{a,t}) \notin D^r_{a,t}$ we consider the extension of this map given by $g_a^r \circ g_{1,t} \circ g_a^n \colon [0,1/2] \to \mathbb{R}$.

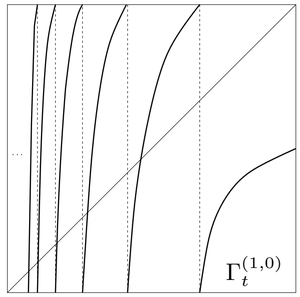

FIGURE 6. $m_{a,t_n(a)} = 1$.

Lemma 5.13. *Consider $a \in \left[\log 2, \log \frac{3+\sqrt{5}}{2}\right)$ and n sufficiently big. Then*

$$\Gamma^1_{a,t_n(a)}(g_a(d_{a,t_n(a)})) \notin D^1_{a,t_n(a)}.$$

Proof. Fix a and write $t_n = t_n(a)$. To prove the lemma it is enough to see that

$$\Gamma^1_{a,t_n}(g_a(d_{a,t_n})) = g_a \circ g_{1,t_n} \circ g_a^n(g_a(d_{a,t_n}) \leq d^1_{a,t_n}. \tag{5.14}$$

Recall that $g_a(d_{a,t_n}) \simeq e^a\, t_n$ (see (4.13)). Arguing as in the proof of Lemma 5.3, for sufficiently large n we get

$$\Gamma^1_{a,t_n}(g_a(t)) \simeq e^a \left(t_n - \frac{1}{2} + \frac{e^a\, t_n\,(1-2\,t_n)^2}{2\,e^a\,t_n\,(1-2\,t_n)^2 + (1-2\,e^a\,t_n)\,(2\,t_n)^2} \right) \simeq$$
$$\simeq e^a\,\left(1 - e^{-a}\right)\right)\, t_n.$$

By Lemma 5.3 it is enough to see that for large n

$$e^a\,\left(1-e^{-a}\right)\right)\,t_n < K_1(0)\,t_n = \frac{e^a}{(e^a-1)}\,t_n.$$

This inequality is equivalent to

$$1 - e^{-a} < (e^a - 1)^{-1} \qquad \Longleftrightarrow \qquad e^{2\,a} - 3\,e^a < -1.$$

The lemma now follows from $a \in \left[\log 2, \log \frac{3+\sqrt{5}}{2}\right)$. $\square$

Lemma 5.13 implies that $r_{a,n}(x) = r(x) \geq 2$ for every $x \in D^1_{a,t_n(a)}$, recall that $r(x)$ is given by the condition $\Gamma^1_{a,t_n(a)}(x) \in D^{r(x)}_{a,t_n(a)}$.

Theorem 5.14 (Expanding induced map $\Phi_{a,t_n(a)}$)**.** *Let $a \in \left[\log 2, \log \frac{3+\sqrt{5}}{2}\right)$. Then for every n big enough*

1. *either $\Gamma^1_{a,t_n(a)}(g_a(t_n(a)) = d^1_{a,t_n}$ and then $\Phi'_{a,t_n}(g_a(t_n)) = 1$;*
2. *or there is $\kappa_n(a) > 1$ such that $(\Phi_{a,t_n(a)})'(x) > \kappa_n(a)$ for all $x \in D_{a,t_n(a)}$.*

For t close enough to $t_n(a)^-$, $t < t_n(a)$, we can define the induced return map $\Phi_{a,t}$ as above and obtain the following:

Corollary 5.15. *Let $a \in \left[\log 2, \log \frac{3+\sqrt{5}}{2}\right)$ and big n such that $\Gamma^1_{a,t_n(a)}(g_a(t_n(a)) \neq d^1_{a,t_n}$. Then there are $\bar{\kappa}_n(a) > 1$ and $\alpha_n(a) > 0$ such that $(\Phi_{a,t})'(x) > \bar{\kappa}_n(a)$ for all $x \in D_{a,t}$ and $t \in [t_n(a) - \alpha_n(a), t_n(a)]$.*

Proof of Theorem 5.14. For simplicity write $t_n = t_n(a)$. There are three possibilities for a point $x \in D_{a,t_n}$: (i) $x \in D^3_{a,t_n}$ with $j \geq 3$, (ii) $x \in D^2_{a,t_n}$, and (iii) $x \in D^1_{a,t_n}$. In Case (i), by Lemma 5.6, $\Phi'_{a,t_n}(x) \geq e^a > 1$.

We now consider Case (ii). Note that by symmetry of g_a, if $x \in (0, 1/2)$ and $k \geq 0$ satisfy $g_a^k(x) = 1/2 - x$ then $(g_a^k)'(x) = 1$ and if $g_a^k(x) < 1/2 - x$ then $(g_a^k)'(x) > 1$. This fact and $g_a^{n+2}(g_a^{-1}(t_n)) = 1/2 - g_a^{-1}(t_n)$ (definition of t_n) imply

$$(g_a^n)'(g_a(t_n))\,(g_a^2)'(g_a^{-1}(t_n)) = 1, \qquad (g_a^n)'(g_a(t_n)) = \frac{1}{(g_a^2)'\left(g_a^{-1}(t_n)\right)}. \tag{5.15}$$

Lemma 5.16. *For every t_n sufficiently small there is $\tau_n = \tau_n(a) > 1$ such that*

$$\left(\Gamma^2_{a,t_n}\right)'(x) \geq \tau_n \quad \text{for all } x \in D^2_{a,t_n}.$$

Proof. If $x \in D^{(2,0)}_{a,t_n}$ then $g_{1,t_n} \circ g_a^n(x) < g_a^{-1}(t_n)$. Thus, by the monotonicity of g'_a,

$$(g_a^2)'(g_{1,t_n} \circ g_a^n(x)) > (g_a^2)'(g_a^{-1}(t_n)).$$

Since $x < g_a(t_n)$ (recall item 3 in Lemma 5.4), the previous inequality, the monotonicity of g'_a, and (5.15) imply that

$$\left(\Gamma^2_{a,t_n}\right)'(x) > (g_a^2)'(g_{1,t_n} \circ g_a^n(x))\,(g_a^n)'(g_a(t_n)) \geq \frac{(g_a^2)'(g_a^{-1}(t_n))}{(g_a^2)'(g_a^{-1}(t_n))} = 1.$$

This concludes the proof of the lemma □

To conclude the proof of the theorem it remains to consider points in D^1_{a,t_n}. It is enough to prove the following lemma:

Lemma 5.17. *For every $x \in D^1_{a,t_n}$ it holds $\Phi'_{a,t_n}(x) \geq 1$. Moreover,*

(1) $\Phi'_{a,t_n}(x) > 1$ for every $x \neq g_a(t_n)$;
(2) $\Phi'_{a,t_n}(g_a(t_n)) \geq 1$ and $\Phi'_{a,t_n}(g_a(t_n)) = 1$ if, and only if, $\Gamma^1_{a,t_n}(g_a(t_n)) = d^1_{a,t_n}$.

Proof. We need to estimate derivatives of compositions $\left(\Gamma^r_{a,t_n} \circ \Gamma^1_{a,t_n(a)}\right)'(x)$ with $x \in D^1_{a,t_n}$ and $r \geq 2$. Since these derivatives are lower bounded by the minimum of $\left(\Gamma^2_{a,t_n} \circ \Gamma^1_{a,t_n(a)}\right)'$ it is enough to estimate this last derivative. Note that

$$\Gamma^1_{a,t_n}(g_a(t_n)) \leq d^1_{a,t_n} \quad \text{and} \quad \Gamma^2_{a,t_n}\left(d^1_{a,t_n}\right) = g_a(t_n).$$

The monotonicity of g_a implies that

$$\Gamma^2_{a,t_n} \circ \Gamma^1_{a,t_n}(g_a(t_n)) \leq \Gamma^2_{a,t_n}\left(d^1_{a,t_n}\right) = g_a(t_n).$$

Consider now the auxiliary map h_{a,t_n} defined by

$$h_{a,t_n}(x) \stackrel{\text{def}}{=} g_{1,t_n} \circ g_a^{n+1}(x). \tag{5.16}$$

Using the map h_{a,t_n} we get

$$g_a^2 \circ h^2_{a,t_n}(t_n) \leq g_a(t_n). \tag{5.17}$$

This implies that

$$h^2_{a,t_n}(t_n) \leq g_a^{-1}(t_n) \text{ and } h_{a,t_n}(t_n) \leq (h_{a,t_n})^{-1}(g_a^{-1}(t_n)), \quad (k \geq 1). \tag{5.18}$$

From (5.15), $h^2_{a,t_n}(t_n) \leq g_a^{-1}(t_n)$, and the monotonicity of g'_a we get

$$(g_a^2)'(h^2_{a,t_n}(t_n)) \geq (g_a^2)'(g_a^{-1}(t_n)) = \left((g_a^n)'(g_a(t_n))\right)^{-1}. \tag{5.19}$$

We now write the equality in (5.17) as follows

$$\Gamma^2_{a,t_n} \circ \Gamma^1_{a,t_n}(g_a(t_n)) = g_a^2 \circ h^2_{a,t_n}(t_n) = g_a^2 \circ h_{a,t_n} \circ g_{1,t_n} \circ g_a^n(g_a(t_n)).$$

From (5.19) we get

$$\begin{aligned}\left(\Gamma^2_{a,t_n} \circ \Gamma^1_{a,t_n}\right)'(g_a(t_n)) &= (g_a^2)'\big(h^2_{a,t_n}(t_n)\big)\,(h_{a,t_n})'\big(h_{a,t_n}(t_n)\big)\,(g_a^n)'\big(g_a(t_n)\big) \geq \\ &\geq (g_a^2)'\big(g_a^{-1}(t_n)\big)\,(h_{a,t_n})'\big(h_{a,t_n}(t_n)\big)\,(g_a^n)'\big(g_a(t_n)\big) = \\ &\geq (h_{a,t_n})'\big(h_{a,t_n}(t_n)\big),\end{aligned}$$

where this inequality is not strict if, and only if, $\Gamma^{(1,0)}_{a,t_n}(g_a(t_n)) = d^{+,(2,0)}_{a,t_n}$.

Thus to prove the lemma it is enough to see that

Claim 5.18. $(h_{a,t_n})'\big(h_{a,t_n}(t_n)\big) \geq 1$.

Proof. We have the following:

- $h_{a,t_n}(t_n) \leq h^{-1}_{a,t_n}(g_a^{-1}(t_n))$ (see (5.18));
- $(g_a^2)'(h^2_{a,t_n}(t_n)) \geq \big((g_a^n)'(g_a(t_n))\big)^{-1}$ (see (5.19)); and
- Arguing as in the proof of (5.15) one gets

$$h'_{a,t_n}(h_{a,t_n}(t_n)) = \big(h'_{a,t_n}(h^{-1}_{a,t_n}(g_a^{-1}(t_n)))\big)^{-1}.$$

Using the monotonicity of the derivative of h_{a,t_n} and putting together these inequalities we get

$$h'_{a,t_n}(h_{a,t_n}(t_n)) \geq h'_{a,t_n}(h^{-1}_{a,t_n}(g_a^{-1}(t_n)) = \big(h'_{a,t_n}(h_{a,t_n}(t_n))\big)^{-1}.$$

This implies that $(h'_{a,t_n}(h_{a,t_n}(t_n)))^2 \geq 1$. Therefore, as $(h_{a,t_n(a)})' > 0$ we get $h'_{a,t_n}(h_{a,t_n}(t_n)) \geq 1$, ending the proof of the claim. □

The proof of Lemma 5.17 is now complete. □

The proof of Theorem 5.14 is now complete. □

5.7. **Covering properties for expanding returns.**

Definition 5.19 (Expanding pair of parameters)**.** We say that (a,t) is an *expanding pair* of parameters if $\Phi_{a,t}$ satisfies the hypotheses either in Theorem 5.7, or in Theorem 5.12, or in Corollary 5.15.

Proposition 5.20 (Covering property)**.** *Let (a,t) an expanding pair of parameters with $t \in I_n(a) = (t_{n+1}(a), t_n(a)]$. Then for every open interval $\emptyset \neq U \subset D_{a,t}$ there are k (arbitrarily large), i, and $x \in U$ such that*

$$g_{1,t} \circ g_a^{n-1} \circ \Gamma^i_{a,t} \circ \Phi^k_{a,t}(x) = 0.$$

Recalling the definition of $\Phi_{a,t}$ we get the following:

Corollary 5.21. *Let (a,t) an expanding pair of parameters with $t \in I_n(a) = (t_{n+1}(a), t_n(a)]$. Then for every open set $U \neq \emptyset \subset D_{a,t}$ there are arbitrarily large r, a sequence of pairs $(n_r, \ell_r) \dots (n_1, \ell_1)$, and $x \in U$ such that*

$$g_{1,t} \circ g_a^{n-1} \circ \Gamma^{(n_r,\ell_r)\cdots(n_1,\ell_1)}_{a,t}(x) = 0.$$

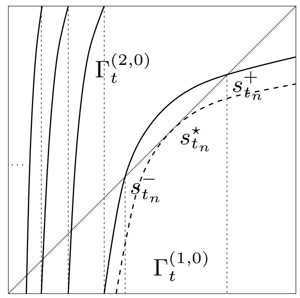

FIGURE 7. The maps $\Gamma_t^{(i,0)}$.

Proof of Proposition 5.20. Let $\kappa > 1$ an expansion constant of $\Phi_{a,t}$ (i.e., $(\Phi_{a,t})'(x) > \kappa$ for all $x \in D_{a,t}$). By Remark 5.2 it is enough to see that there is ℓ such that $\Phi^\ell_{a,t}(U)$ contains a discontinuity of $\Phi_{a,t}$. If this is not the case then $\Phi^\ell_{a,t}(U)$ is contained in an interval where $\Phi_{a,t}$ is continuous and therefore $|\Phi^\ell_{a,t}(U)| > \kappa^\ell\,|U|$. Since the intervals of continuity of $\Phi_{a,t}$ are bounded this is a contradiction. □

6. FULL HYPERBOLIC DYNAMICS FOR $a = \infty$

In this section we consider parameters $a > \log 4$ and their associated parameter intervals $(t_{n+1}(a), t_n(a)]$, (large n). The main result in this section is Proposition 6.4, to state it we need a preliminary result. Consider the parameter

$$t_n^\star(a) = t_n^\star \stackrel{\text{def}}{=} \frac{2\,t_n(a)}{(1-2\,t_n(a))e^{\frac{a}{2}} + 2\,t_n(a)} \in (t_{n+1}(a), t_n(a)). \tag{6.1}$$

In what follows, the parameter $a > \log 4$ remains fixed and thus it will be omitted (although in some cases we will keep a for clearness).

Lemma 6.1 (Saddle-node bifurcations)**.** *For every $t \in (t_n^\star, t_n)$ the map $\Gamma_t^{(1,0)}$ has a pair of fixed points s_t^- (expanding) and s_t^+ (contracting) in $D_t^{1,0}$, $s_t^- < s_t^+$, colliding to a saddle-node $s_{t_n^\star} = s^-_{t_n^\star} = s^+_{t_n^\star}$ for $t = t_n^\star$:*

$$\begin{aligned}
&\Gamma_{t_n^\star}^{(1,0)}(s_{t_n^\star}) = s_{t_n^\star} \quad \textit{and} \quad \big(\Gamma_{t_n^\star}^{(1,0)}\big)'(s_{t_n^\star}) = 1;\\
&\Gamma_t^{(1,0)}(s_t^+) = s_t^+ \quad \textit{and} \quad \big(\Gamma_t^{(1,0)}\big)'(s_t^+) \in (0,1), \quad \textit{if } t \in (t_n^\star, t_n);\\
&\Gamma_t^{(1,0)}(s_t^-) = s_t^- \quad \textit{and} \quad \big(\Gamma_t^{(1,0)}\big)'(s_t^-) > 1, \quad \textit{if } t \in (t_n^\star, t_n).
\end{aligned}$$

Proof. To prove the first part of the lemma, for $t \in [t_n^\star, t_n)$ define the map

$$h_t \colon g_t^{-1}(D_t) \cup D_t \to [0, 1/2], \qquad h_t(x) = g_{1,t} \circ g_a^{n+1}(x).$$

Claim 6.2. *The point*

$$z_n^\star \stackrel{\text{def}}{=} \frac{t_n}{(1-2\,t_n)e^{\frac{a}{2}} + 2\,t_n} \in g_a^{-1}(D_{a,t_n^\star}).$$

is a saddle-node fixed point of $h_{t_n^\star}$.

Proof. From equations (4.1) and (4.3) we get

$$g_{1,t_n} \circ g_a^{n+1}(z_n^\star) = \frac{1-2\,t_n}{2\,(1-2\,t_n) + 4\,t_n\,e^{-\frac{a}{2}}} - \frac{1}{2} + t_n = \frac{t_n\big(-1 + (1-2\,t_n)e^{\frac{a}{2}} + 2\,t_n\big)}{(1-2\,t_n)e^{\frac{a}{2}} + 2\,t_n}.$$

After a straightforward calculation we get $h_{a,t_n^\star}(z_n^\star) = z_n^\star$.

Another simple calculation using (4.1) implies that for $t \in (t_{n+1}, t_n)$ it holds

$$(h_t)'(z_n^\star) = (g_{1,t_n} \circ g_a^{n+1})'(z_n^\star) = (g_a^{n+1})'(z_n^\star) = 1,$$

ending the proof of the claim. □

Note that the map $t \mapsto h_t(z_n^\star)$ is increasing and thus $h_t(z_n^\star) > z_n^*$ for $t \in (t_n^\star, t_n)$. Since h_t' is strictly decreasing, for $t \in (t_n^*, t_n)$ the map h_t has a pair of fixed points z_t^- (expanding) and z_t^+ (contracting) with $z_t^- < z_t^+$. Let

$$s_t^\pm \stackrel{\text{def}}{=} g_a(z_t^\pm) \quad \text{for } t \in (t_n^\star, t_n) \text{ and } \quad s_{t_n^\star} \stackrel{\text{def}}{=} g_a(z_n^\star).$$

Note that $h_t(D_t)$ is at the left of D_t, thus we have that $h_t(d_t) < d_t$. Thus from $h_t(z_{t_n^\star}) > z_{t_n^\star}$ we get that $z_t^+ \in (z_{t_n^\star}, d_t) \subset g_a^{-1}(D_t)$. These observations imply that $s_t^+ \in D_t$ and thus $s_t^+ \in D_t^{(1,0)}$.

We consider the (natural) extension of $\Gamma_t^{(1,0)}$ to D_t and, with a slight abuse of notation, denote it also by $\Gamma_t^{(1,0)}$. Note that by definition

$$\Gamma_t^{(1,0)}(s_t^\pm) = g_a \circ g_{1,t} \circ g_a^n(s_t^\pm) = g_a \circ g_{1,t} \circ g_a^{n+1}(z_t^\pm) = g_a \circ h_t(z_t^\pm) = g_a(z_t^\pm) = s_t^\pm.$$

A similar property holds for $s_{t_n^\star}$.

The fact that $s_t^- \in D_t^{(1,0)}$ follows from $\Gamma_t^{(1,0)}(d_t^{-,(1,0)}) \le d_t^{-,(1,0)}$.

The fact that s_t^- is expanding and s_t^+ is contracting for $\Gamma_t^{(1,0)}$, $t \in (t_n^\star, t_n)$, follows from the similar properties for z_t^- and z_t^+ for h_t above. A similar argument implies that $s_{t_n^\star}$ is a saddle-node for $\Gamma_{t_n^\star}^{(1,0)}$. This completes the proof of the lemma. □

For small $t > 0$ consider the set of *hyperbolic parameters*[7] in $[0, t]$ defined by

$$H_a(t) \stackrel{\text{def}}{=} [0, t] \cap \Big(\bigcup_n (t_n^\star(a), t_n(a)) \Big). \tag{6.2}$$

Lemma 6.3 (Density of hyperbolic parameters)**.**

$$\liminf_{t \to 0^+} \frac{|H_a(t)|}{t} \ge h(a) > 0, \qquad \textit{where } h(a) \to 1 \textit{ as } a \to \infty.$$

Proof. Recalling the relation between $t_n(a)$ and $t_{n+1}(a)$ in (4.3) and the definition of $t_n^\star(a)$ in (6.1) it follows

$$\lim_{n \to +\infty} \frac{t_n(a) - t_n^\star(a)}{t_n(a) - t_{n+1}(a)} = \lim_{n \to +\infty} \frac{t_n(a)\left(1 - \frac{t_n^\star(a)}{t_n(a)}\right)}{t_n(a)\left(1 - \frac{t_{n+1}(a)}{t_n(a)}\right)} = \frac{e^{\frac{a}{2}} - 2}{e^{\frac{a}{2}} - 1} \stackrel{\text{def}}{=} h(a).$$

It is obvious that $h(a) \to 1$ as $a \to \infty$. □

To state the next result we need a definition. A point $x \in D_{a,t}$ has a *return of type* (i,j) if $x \in D_{a,t}^{(i,j)}$, i.e., $\Gamma_{a,t}^{(i,j)}(x) \in D_{a,t}$. Note that a point have infinitely many different types of returns.

Proposition 6.4 (Contracting and expanding returns)**.** *For every n sufficiently big and every $t \in [t_n^*, t_n)$ consider the points s_t^-, s_t^+, and $s_{t_n^\star}$ in Lemma 6.1 and the associated partition of the fundamental domain D_t of g_a given by*

$$L_t^u = \left[d_t, s_t^-\right], \quad L_t^c = \left(s_t^-, s_t^+\right), \quad \textit{and} \quad L_t^s = \left[s_t^+, g_a(d_t)\right]$$

[7]Proposition 6.4 justifies this name

(note that $L^c_{t^\star_n}$ is empty). This partition satisfies the following invariance and expansion/contraction properties for parameters $t \in [t^\star_n, t_n)$:

The contracting and trapping region L^s_t:

(s1) Every return of a point of L^s_t to D_t is of type $(1,r)$ for some $r \geq 0$ and $\Gamma_t^{(1,r)}(L^s_t) \subset L^s_t$ for every $r \geq 0$;

(s2) if $t \in (t^\star_n, t_n)$ there is $\lambda_t \in (0,1)$ such that $0 < \big(\Gamma_t^{(1,r)}\big)'(x) < \lambda_t$ for all $x \in L^s_t \cap D_t^{(1,r)}$ and $r \geq 0$;

(s3) $0 < (\Gamma_{t^\star_n}^{(1,r)})'(x) < 1$ for all $x \in L^s_{t^\star_n} \cap D^{(1,r)}_{t^\star_n}$ with $x \neq s_{t^\star_n}$ and $\Gamma_{t^\star_n}^{(1,0)})'(s^*_{t_n}) = 1$;

(s4) $\Gamma_t^{(1,k)}(D_t) \subset L^s_t$ for all $k \geq 1$.

The wandering region L^c_t:

(w1) Every return of a point of L^c_t to D_t is of type $(1,r)$, $r \geq 0$. Moreover, $\Gamma_t^{(1,0)}(L^c_t) = L^c_t$ and $\Gamma_t^{(1,0)}(x) > x$ for all $x \in L^c_t$.

The expanding region L^u_t:

(u1) Every return of a point of L^u_t to D_t is of type $(r,0)$ with $r \geq 1$ or $(1,j)$ with $j \geq 1$;

(u2) if $t \in (t^\star_n, t_n)$ there is $\sigma_t > 1$ such that $\big(\Gamma_t^{(r,0)}\big)'(x) > \sigma_t$ for all $x \in L^u_t \cap D_t^{r,0}$ and $r \geq 0$;

(u3) $(\Gamma_{t^\star_n}^{(1,r)})'(x) > 1$ for all $x \in L^u_{t^\star_n} \cap D^{(0,r)}_{t^\star_n}$ with $x \neq s_{t^\star_n}$ and $\Gamma_{t^\star_n}^{(1,0)})'(s^*_{t_n}) = 1$.

Remark 6.5 (Coding orbits)**.** Using Proposition 6.4 we code the forward orbits of points in D_t by the IFS $\Gamma_t^{(i,j)} : D_t^{(i,j)} \to D_t$. To each $x \in D_t$ and each orbit of it we associate a sequence $(j_i)_{i \geq 0}$, $j_i \in \{s, c, u\}$, as follows: write $x = x_0$ and list $x_i \in D_t$ the successive iterates of x by the IFS, if $x_i \in L^k_t$ we let $j_i = k$. By Proposition 6.4, if $j_i = s$ then $j_\ell = s$ for all $\ell \geq i$ and if $j_i = c$ then $j_\ell \in \{s, c\}$ for all $\ell \geq i$.

Proof of Proposition 6.4. To prove the assertions for the interval L^s_t note that

$$\Gamma_t^{(1,0)}(L^s_t) = \big[\Gamma_t^{(1,0)}(s_t^+), \Gamma_t^{(1,0)}(g_a(d_t))\big] \subset [s_t^+, g_a(d_t)] = L^s_t.$$

For $r \geq 1$ the set $\Gamma_t^{(1,r)}(D_t)$ is at the right of $\Gamma_t^{(1,0)}(L^s_t)$ and contained in D_t, thus

$$\Gamma_t^{(1,r)}(D_t) \subset L^s_t, \quad \text{for all } t \in [t^*_n, t_n) \text{ and } r \geq 1.$$

This proves the inclusion property in (s1) and the trapping property (s4).

To see the first part of (s1) (points of L^s_t have only returns of type $(1,r)$) note that if $i \geq 2$ then for every j one has

$$\Gamma_t^{(i,j)}(x) \geq \Gamma_t^{(2,0)}(x) \geq g_a(s_t^+) > g_a(d_t),$$

thus $\Gamma_t^{(i,j)}(x) \notin D_t$. This prevents points in L^s_t to have returns of type (i,j) with $i \geq 2$. This completes the proof of (s1).

To get (s2) note that the monotonicity of the derivatives implies that if $x \in L^s_t$ then

$$0 < \big(\Gamma_t^{(1,r)}\big)'(x) \leq (\Gamma_t^{(1,0)}\big)'(s_t^+) = \lambda_t < 1.$$

To obtain (s3) one argues similarly, obtaining $0 < \big(\Gamma_{t^*_n}^{(1,r)}\big)'(x) \leq (\Gamma_{t^*_n}^{(1,0)}\big)'(s_{t^\star_n}) = 1$, where the equality only holds for $x = s^\star_{t_n}$

This ends the proof of claims concerning the contracting and trapping region.

The proofs of the assertions for the regions L^c_t an L^u_t (conditions (w1) and (u1)–(u3)) follow similarly and are omitted. □

7. Dynamics of the maps $G_{a,t}$ in a neighborhood of the cycle

We consider a neighborhood of the cycle associated to the points $P = \big(0^{\mathbb{Z}}; 1/2\big)$ (contracting) and $Q = \big(0^{\mathbb{Z}}; 0\big)$ (expanding) of $G_{a,t}$ (see (2.1)) for $t = 0$. We study the dynamics of $G_{a,t}$ in such a cycle. This analysis relies on the results about the iterated function systems in Sections 5 and 6, see Proposition 7.10 and Remark 7.13.

We use the cylinder notation for compositions of the maps g_a and $g_{1,t}$,

$$g_{[\xi_0\ldots\xi_m],a,t} \stackrel{\text{def}}{=} g_{\xi_m,t} \circ \cdots \circ g_{\xi_0,t}, \qquad \text{where } g_{0,t} = g_a \text{ and } \xi_i \in \{0,1\}.$$

Given $r, m \geq 0$, the *cylinder* $[\xi_{-r} \cdots . \xi_0 \cdots \xi_m]$ is the subset of Σ_2 given by

$$[\xi_{-r} \cdots . \xi_0 \cdots \xi_m] \stackrel{\text{def}}{=} \big\{(\eta_i)_{i\in\mathbb{Z}} \colon \eta_i = \xi_i \text{ for all } i \in \{-r, \ldots, m\}\big\}.$$

These cylinders define a basis of the topology of Σ_2.

7.1. **Choice of a neighborhood of the cycle.**

Consider the heteroclinic set

$$\gamma = \{0^{\mathbb{Z}}\} \times (0, 1/2) \subset W^u(Q, G_{a,t}) \cap W^s(P, G_{a,t}), \quad \text{for all } a, t$$

and the (ϵ, k)-neighborhood of its closure given by

$$V^{\epsilon,k}_{[0,1/2]} \stackrel{\text{def}}{=} \big[0^{-k}.0^k\big] \times \left(-\epsilon, \frac{1}{2} + \epsilon\right). \tag{7.1}$$

Consider the point

$$Z \stackrel{\text{def}}{=} \Big((0^{-\mathbb{N}}.10^{\mathbb{N}}); 1/2)\Big).$$

By Corollary 3.1,

$$Z = \Big((0^{-\mathbb{N}}.10^{\mathbb{N}}); 1/2)\Big) \in W^u(P, G_{a,0}) \cap W^s(Q, G_{a,0}).$$

Note also that

$$\{G^{\ell+1}_{a,0}(Z), G^{-\ell}_{a,0}(Z),\, \ell \geq k\} \subset V^{\epsilon,k}_{[0,1/2]}.$$

Pick small $\delta = \delta(a) \in (0, \epsilon)$ and $t_0 = t(a) \in (0, \epsilon)$ such that for all $t \in [0, t_0]$ it holds

$$\Big(g_a^{-k}([1/2 - \delta, 1/2 + \delta])\Big) \bigcup \Big(g_a^{k+1} \circ g_{1,t}([1/2 - \delta, 1/2 + \delta])\Big) \subset (0 - \epsilon, 1/2 + \epsilon)$$

and consider the neighborhood of Z defined by

$$V^{\delta,k}_Z \stackrel{\text{def}}{=} \big[0^{-2k-2}.10^{2k+1}\big] \times \left(\frac{1}{2} - \delta, \frac{1}{2} + \delta\right). \tag{7.2}$$

These choices imply that for every $t \in [0, t_0]$ one has

$$\text{closure}\Big(G^{k+1}_{a,t}(V^{\delta,k}_Z) \cup G^{-k-1}_{a,t}(V^{\delta,k}_Z)\Big) \subset V^{\epsilon,k}_{[0,1/2]}.$$

Remark 7.1 (Choice of parameters)**.** We can assume that $t(a) > 0$, ϵ, and δ are small enough satisfying

$$g^2_{1,t}(1/2) = 2\,t - 1/2 < -\epsilon \quad \text{and} \quad t + d_{a,t} < 1/2 - \epsilon, \quad \text{for all } t \in [0, t(a)].$$

Finally, consider the (δ, k, ϵ)-neighborhood of the cycle[8] given by

$$V^{\delta,\epsilon,k} \stackrel{\text{def}}{=} V^{\epsilon,k}_{[0,1/2]} \cup \left(\bigcup_{i=-k}^{k+1} G^i_{a,0}(V^{\delta,k}_Z)\right). \tag{7.3}$$

[8]This set contains the elements in the cycle: the two fixed points P and Q and the orbits of the heteroclinic point Z and the heteroclinic connexion γ.

From now on δ, ϵ, k will remain fixed and they will be omitted, thus we will use the notation $V \stackrel{\text{def}}{=} V^{\delta,\epsilon,k}$, $V_Z \stackrel{\text{def}}{=} V_Z^{\delta,k}$, and $V_{[0,1/2]} \stackrel{\text{def}}{=} V_{[0,1/2]}^{\epsilon,k}$. In what follows we study the dynamics of $G_{a,t}$, small t, in the neighborhood V of the cycle.

Relevant sets in our context are the maximal invariant sets of $G_{a,t}$ in the set V

$$\Lambda^+_{a,t} \stackrel{\text{def}}{=} \bigcap_{i\le 0} G^i_{a,t}(V), \qquad \Lambda^-_{a,t} \stackrel{\text{def}}{=} \bigcap_{i\ge 0} G^i_{a,t}(V), \qquad \Lambda_{a,t} \stackrel{\text{def}}{=} \Lambda^+_{a,t} \cap \Lambda^-_{a,t}. \tag{7.4}$$

We also consider the *non-wandering set of $G_{a,t}$ relative to V* denoted by $\Omega_{a,t}$. A point $X \in V$ belongs to $\Omega_{a,t}$ if for every neighborhood $U \subset V$ there are $Y \in U$ and $j > 0$ such that $G^j_{a,t}(Y) \in U$ and the segment of orbit $Y, \dots, G^j_{a,t}(Y)$ is contained in V. Note that $\Omega_{a,t} \subset \Lambda_{a,t}$.

Remark 7.2. If $X = (\xi; x) \in V$ then $x \in [-\epsilon, 1/2 + \epsilon]$.

We close this subsection with the following consequence of Corollary 3.1.

Remark 7.3. Consider a finite sequence $\alpha = \alpha_1 \dots \alpha_m$ such that

$$(g_{[\alpha_{m-i}\dots\alpha_m],a,t})^{-1}(0) \in [0, 1/2) \quad \text{for all } i = 0, \dots, m-1.$$

Then $Y_\alpha = \big((0^{-\mathbb{N}}\alpha_1 \dots \alpha_m.0^{\mathbb{N}}); 0\big)$ is a homoclinic point of Q contained in $\Lambda_{a,t}$.

7.2. **A reference domain and its returns.** Given $X = (\xi; x) \in \Lambda_{a,t}$, for $i \ge 1$ we write

$$X_i \stackrel{\text{def}}{=} G^i_{a,t}(X) \stackrel{\text{def}}{=} \big(\sigma^i(\xi); g_{[\xi_0\dots\xi_{i-1}],a,t}(x_0)\big) = (\sigma^i(\xi); x_i).$$

Recall the definition of the fundamental domain $D_{a,t} = (d_{a,t}, g_a(d_{a,t})] \subset [t, 1/2)$ of g_a, see (4.7) and (4.11), and consider its associated *reference cube* $\Delta_{a,t}$

$$\Delta_{a,t} \stackrel{\text{def}}{=} \big[0^{-k}.0^k\big] \times D_{a,t} = \{X = (\xi; x) \in V : x \in D_{a,t}\} \subset V. \tag{7.5}$$

To study of the relative dynamics of $G_{a,t}$ in V we analyze the returns of points in $\Delta_{a,t}$ to $\Delta_{a,t}$.

Definition 7.4 (Returns to $\Delta_{a,t}$)**.** Let $X \in \Delta_{a,t}$. The *sequence of return times* $\varrho_i(X)$ of X to $\Delta_{a,t}$, $\varrho_i(X) < \varrho_{i+1}(X)$, is defined as follows: $\varrho_0(X) = 0$, $G^{\varrho_j(X)}_{a,t}(X) \in \Delta_{a,t}$, and $G^i_{a,t}(X) \in (V \setminus \Delta_{a,t})$ for every $i \in \big(\varrho_{j-1}(X), \varrho_j(X)\big) \cap \mathbb{N}$.

We let $X_{[i]} \stackrel{\text{def}}{=} X_{\varrho_i(X)} = G^{\varrho_i(X)}_{a,t}(X)$ the *i-th return to* $\Delta_{a,t}$ of X and $I(X)$ the maximal interval in $\mathbb{Z}$ such that $\varrho_i(X)$ is defined for $i \in I(X)$.

Note that the sequence of return times of a point X may be finite and that if $\varrho_i(X) > 0$ (resp. < 0) is defined then $G^j_a(X) \in V$ for all $0 \le j \le \varrho_i(X)$ (resp. $\varrho_i(X) \le j \le 0$).

Next proposition claims that "most" points of $\Lambda_{a,t}$ have iterates in $\Delta_{a,t}$.

Proposition 7.5. *Consider any $t \in (0, t(a)]$. Then every $X \in \Lambda_{a,t} \setminus \{P, Q\}$, has some iterate by $G_{a,t}$ in $\Delta_{a,t}$.*

To prove this proposition we need two preparatory lemmas.

Lemma 7.6. *Consider small $t > 0$ and $X = (\xi; x) \in V$ such that $X_0, \dots, X_i \in V$ for some $i \ge 0$. If $g^i_a(x) \in [0, \frac{1}{2} + t - \epsilon)$ then $\xi_0 = \dots = \xi_{i-1} = 0$.*

Proof. Note that $[0, \frac{1}{2}+t-\epsilon] \subset [0, \frac{1}{2})$. As $g_a(x) \ge x$ for all $x \in [0, 1/2]$ and $g_a(0) = 0$ one has that $g^j_a(x) \in [0, \frac{1}{2}+t-\epsilon)$ for all $j = 0, \dots, i$. Assume, by contradiction, that there is a first $j \in \{0, \dots, i-1\}$ with $\xi_j = 1$. Then $g_{1,t} \circ g^j_a(x) < g_{1,t}(\frac{1}{2}+t-\epsilon) = -\epsilon$. This implies that $X_{j+1} \notin V$, a contradiction. $\square$

Lemma 7.7. *Consider $t \in (0, t(a))$ and $X = (\xi; x) \in \Lambda_{a,t}$. Then $x_i \in [0, 1/2]$ for every $i \in \mathbb{Z}$.*

Proof. By Remark 7.2 the second coordinate x_i of X_i satisfies $x_i \in [-\epsilon, 1/2 + \epsilon]$. Thus, after replacing X by some iterate, we can assume that $x_0 = x_i \in [-\epsilon, 1/2+\epsilon]$.

We claim that $x_0 \notin [-\epsilon, 0)$. Arguing by contradiction, assume that $x_0 \in [-\epsilon, 0)$. If $\xi_i = 0$ for all $i \geq 0$ then there is $m \geq 0$ with $x_{m+1} = g_{[\xi_0 \ldots \xi_m],a,t}(x_0) = g_a^{m+1}(x_0) < -\epsilon$, contradicting Remark 7.2. Thus there is a first $m \geq 0$ with $\xi_m = 1$. Then

$$x_{m+1} = g_{[\xi_0 \ldots \xi_m],a,t}(x_0) = g_{1,t} \circ g_a^m(x_0) < g_{1,t}(0) = t - 1/2 < -\epsilon,$$

contradicting Remark 7.2.

The case $x_0 \in (1/2, 1/2 + \epsilon]$ follows identically considering negative iterates. □

Proof of Proposition 7.5. Consider $X = (\xi; x) \in \Omega_{a,t} \setminus \{P, Q\}$. By Lema 7.7, $x \in [0, 1/2]$. We first consider the case $x \in (0, 1/2)$. If the sequence $\xi = 0^{\mathbb{Z}}$ we are done since $g_a^i(x) \to 1/2$ as $i \to +\infty$ and $g_a^i(x) \to 0$ as $i \to -\infty$ and thus the sequence $(g_a^i(x))_{i\in\mathbb{Z}}$ necessarily contains some point in the fundamental domain $D_{a,t}$ of g_a in $(0, 1/2)$. Thus it remains to consider the case where ξ contains some 1. After replacing X by some iterate we can assume that $\xi_0 = 1$. As $\{g_{1,t}(x), x\} \subset [0, 1/2]$ we have $g_{1,t}(x) = x_1 \in [0, t] \subset [0, d_{a,t}]$, recall (4.11). If $\xi_i = 0$ for all $i \geq 1$ then there is $j \geq 1$ with $x_{j+1} = g_a^j(x_1) \in (d_{a,t}, g_a(d_{a,t})] = D_{a,t}$. Thus as $X_{j+1} \in V$ and by the definition of $\Delta_{a,t}$ one has that $X_{j+1} \in \Delta_{a,t}$.

We are left to consider the case where there is a first $j \geq 1$ with $\xi_j = 1$.

Claim 7.8. $x_j = g_a^{j-1}(x_1) > d_{a,t}$

Proof. If the claim does not hold, by Remark 7.1 we have

$$x_{j+1} = g_{1,t} \circ g_a^{j-1}(x_1) \leq g_{1,t}(d_{a,t}) \leq d_{a,t} + t - 1/2 < -\epsilon,$$

contradicting Lemma 7.7. □

Since $x_1 \in [0, t] \subset [0, d_{a,t}]$ Claim 7.8 implies that $x_i \in D_{a,t}$ for some $i \leq j$ and thus $X_i \in \Delta_{a,t}$. This completes the proof of the lemma when $x \in (0, 1)$.

We now consider the case $x = 1/2$. As $X \neq P$ this implies that the sequence ξ contains some 1. Since $g_{1,t}(x) \neq 1/2$ for all $x \leq 1/2$, one has that ξ^- consists only of 0's. Thus there is a first $j > 0$ with $\xi_j = 1$. Then

$$x_{j+1} = g_{[\xi_0 \ldots \xi_j],a,t} = g_{1,t} \circ g_a^j(1/2) = g_{1,t}(1/2) = t \in (0, 1/2).$$

Thus we are in the previous case and we are done.

The case $x = 0$ is analogous and thus omitted. The proof of the proposition is now complete. □

One can easily prove the following version of [DR1, Lemma 7.1] straightforwardly adapted to our setting.

Lemma 7.9. *Consider small $t > 0$. Given $X \in \Delta_{a,t} \cap \Lambda_{a,t}$ the following holds:*

(1) $X \in W^s(P, G_{a,t}) \cup W^s(Q, G_{a,t})$, if and only if, X has only finitely many forward returns $\varrho_i(X)$, $i > 0$;
(2) $X \in W^u(P, G_{a,t}) \cup W^u(Q, G_{a,t})$, if and only if, X has only finitely many backward returns $\varrho_i(X)$, $i < 0$; and
(3) X has infinitely many forward and backward returns $\varrho_i(X)$.

7.3. **Itineraries and return maps.** In this section, we associate an itinerary to points in $\Delta_{a,t}$ having returns to $\Delta_{a,t}$ (Proposition 7.10) and determine the fiber dynamics of these returns in terms of the maps $\Gamma^{(i,j)}_{a,t}$ in Section 4.2 (Remark 7.13).

Proposition 7.10. *Let $t \in (t_{n+1}(a), t_n(a)]$. Consider $X = (\xi; x) \in \Delta_{a,t}$ with a sequence of forward returns $\varrho_i(X)$ to $\Delta_{a,t}$. Then*

$$\xi_0\xi_1 \dots \xi_{\varrho_i(X)-1} = 0^{k_1^- + n} 1 0^{k_1^- + n + k_2^+} 1 0^{k_2^- + n + k_3^+} 1 \dots \dots 1 0^{k_{i-1}^- + n + k_i^+} 1 0^{k_i^+},$$

where $k_j^- \geq 0$ and $k_j^+ \geq 1$ for every $j = 1, \dots, i$.

If X has only i forward returns (i.e., $I(X) = \{0, \dots, i\}$) and $X \in \Lambda^+_{a,t}$ then

$$\xi^+ = 0^{k_1^- + n} 1 0^{k_1^+ + n + k_2^-} 1 \cdots 1 0^{k_{i-1}^- + n + k_i^+} 1 0^\infty.$$

Proof. The first step of the proof is the following lemma.

Lemma 7.11. *Let $X = (\xi; x) \in \Delta_{a,t}$ with a first return $\varrho_1 = \varrho_1(X)$ to $\Delta_{a,t}$. Then*

$$\xi_0 \dots \xi_{\varrho_1 - 1} = 0^{k_0} 1 0^{k_1}, \quad \textit{where } k_0 \geq n \textit{ and } k_1 \geq 1.$$

Proof. By definition of a first return time, $G^i_{a,t}(X) \in V$ for all $i = 0, \dots, \varrho_1$. The monotonicity of g_a implies that $\xi_0 \dots \xi_{\varrho_1 - 1} \neq 0^{\varrho_1}$. Thus there are $j \geq 1$, $m_1, \dots, m_j \geq 1$, $k_1, \dots, k_{j-1} \geq 1$, and $k_0, k_j \geq 0$ such that

$$\xi_0 \dots \xi_{\varrho_1 - 1} = 0^{k_0} 1^{m_1} 0^{k_1} \dots 0^{k_j - 1} 1^{m_j} 0^{k_j}.$$

To see that $m_1 = 1$ note that $g_{[0^{k_0} 1], a,t}(x) < g_{1,t}(1/2) = t \leq d_{a,t}$. Hence, if $m_1 \geq 2$ we have

$$x_{k_0+2} = g_{[0^{k_0} 1^2], a,t}(x) < g^2_{1,t}(1/2) = g_{1,t}(t) \leq g_{1,t}(d_{a,t}) < -\epsilon.$$

By Remark 7.2 this implies that $X_{k_0+2} \notin V$, which is a contradiction.

Note that $g_{[0^{k_0} 1], a,t}(x) \leq d_{a,t}$ implies that $k_1 \geq 1$.

Claim 7.12. $g_{[0^{k_0} 1 0^{k_1}], a,t}(x) \in D_{a,t}$

This claim immediately implies that $\varrho_1 = 0^{k_0} 1 0^{k_1}$.

Proof of Claim 7.12. Note that $g_{[0^{k_0} 1], a,t}(x) \leq d_{a,t}$ implies that $g_{[0^{k_0} 1 0^{k_1}], a,t}(x) \leq g_a(d_{a,t})$, otherwise we get $\ell < k_1$ with $g_{[0^{k_0} 1 0^{\ell}], a,t}(x) \in (d_{a,t}, g_a(d_{a,t})]$. Then X has a return to $\Delta_{a,t}$ for $k_0 + 1 + \ell < \varrho_1$, contradicting that ϱ_1 is a first return of X. Thus $g_{[0^{k_0} 1 0^{k_1}], a,t}(x) \leq d_{a,t}$ and $\xi_0 \dots \xi_{k_0+1+k_1+1} = 0^{k_0} 1 0^{k_1} 1$. From Remark 7.1 it follows

$$x_{k_0+1+k_1+1} = g_{[0^{k_0} 1 0^{k_1} 1], a,t}(x) \leq g_{1,t}(d_{a,t}) < d_{a,t} - 1/2 + t < -\epsilon,$$

contradicting $X_{k_0+1+k_1+1} \in V$. □

It remains to check that $k_0 \geq n$. Assume, by contradiction, that $k_0 < n$. The definition of $(t_{n+1}(a), t_n(a)]$ implies that if $t \in (t_{n+1}(a), t_n(a)]$ then $g_{1,t}(g_a^{k_0}(x)) \leq 0$. This implies that the orbit of X must have some iterate outside V before returning to $\Delta_{a,t}$, contradicting the definition of a first return. This ends the proof of the lemma. □

In view of Lemma 7.11 we use the following notation

$$\xi_0 \dots \xi_{\varrho_1 - 1} = 0^{k_1^- + n} 1 0^{k_1^+}, \quad k_1^- \geq 0 \text{ and } k_1^+ \geq 1.$$

Concatenating consecutive forward returns and applying recursively Lemma 7.11, we get

$$\xi_0\xi_1\dots\xi_{\varrho_i(X)-1} = 0^{k_1^-+n}10^{k_1^++k_2^-+n}10^{k_2^++k_3^-+n}1\dots\dots10^{k_i^-+n}10^{k_i^+}.$$

Finally, using the arguments above it is not difficult to see that if $X\in\Lambda^+_{a,t}$ has only i forward returns then

$$X=(\xi;x)=(\cdots.0^{k_1^-+n}10^{k_1^++k_2^-+n}1\cdots10^{k_{i-1}^++k_i^-+n}10^\infty;x).$$

The proof of the proposition is now complete. □

Using Proposition 7.10 to the return $\varrho_i(X)$ of X we associate a chain o pairs

$$\mathfrak{r}_i(X)=\mathfrak{r}_i\stackrel{\text{def}}{=}(k_1^+,k_1^-)\cdots(k_i^+,k_i^-),\quad k_i^-\geq 0,\, k_i^+\geq 1.$$

Using the maps $\Gamma^{(i,j)}_{a,t}$ in (4.9) we get the following:

Remark 7.13 (Returns and the IFS $\Gamma^{(i,j)}_{a,t}$)**.** Consider $X=(\xi;x)\in\Delta_{a,t}$ with an i-th return $\varrho_i(X)$ with associated chain of pairs $\mathfrak{r}_i(X)=\mathfrak{r}_i$. Then

$$X_{[i]}=(\sigma^{\varrho_i(X)}(\xi),x_{[i]}),\quad\text{where}\quad x_{[i]}=\Gamma^{\mathfrak{r}_i}_{a,t}(x)=\Gamma^{(k_i^+,k_i^-)}_{a,t}\circ\cdots\circ\Gamma^{(k_1^+,k_1^-)}_{a,t}(x).$$

7.4. **Wandering points.**

Proposition 7.14. *For every $a>\log 4$ and $t\in(t_n^*(a),t_n(a))$, large n, it holds*

$$\{0^{\mathbb{Z}}\}\times(0,1/2)\cap\Omega_{a,t}=\emptyset.$$

Proof. Let us omit the parameter a. First, recall that by Lemma 6.1 for every $t\in(t_n^*,t_n)$ there are uniquely defined fixed points $s_t^-,s_t^+\in D_{a,t}$, $s_t^-<s_t^+$, of the map $\Gamma^{(1,0)}_{a,t}$. Given $X\in\{0^{\mathbb{Z}}\}\times(0,1/2)$, after replacing it by some backward iterate we can assume that $X=(0^{\mathbb{Z}};x)$ with $0<x<s_t^+$.

Consider the reference cube $\widehat{\Delta}_{a,t}$ containing $\Delta_{a,t}$ defined by

$$\widehat{\Delta}_{a,t}\stackrel{\text{def}}{=}\{X=(\xi;x)\in V\text{ with }x\in[0,g_a(d_{a,t})]\}.$$

To complete the proof of the proposition we need the next lemma follows using arguments similar to the ones in Lemma 7.11, thus we just sketch its proof.

Lemma 7.15. *Consider $Y=Y_0=(\eta;y)\in\Delta_{a,t}$ with $y\geq s_t^+$ such that $Y_1,\dots,Y_{k-1}\in(V\setminus\widehat{\Delta}_{a,t})$ and $Y_k\in\widehat{\Delta}_{a,t}$. Then $Y_k=(\sigma^k(\eta);y_k)$ where $y_k\geq s_t^+$.*

Sketch of the proof. Note first that $k\geq n$ and $\eta_0=\cdots\eta_{n-1}=0$, otherwise the point Y is mapped to the left of 0. As the point Y has some iterate in $\widehat{\Delta}_{a,t}$ there is a first $\eta_i=1$. As in Lemma 7.11 we have that $\eta_{i+1}=0$. From $y>s_t^+$, $i\geq n$, and the definition of s_t^+ it follows that

$$y_{i+2}=g_{[\eta_0\dots\eta_{i+1}],a,t}(y)=g_{[0^i10],a,t}(y)>g_a\circ g_{1,t}\circ g_a^n(s_t^+)=s_t^+.$$

This immediately implies that forward iterates of Y in $\widehat{\Delta}_{a,t}$ are to the left of s_t^+. This ends the sketch of the proof. □

Lemma 7.15 implies that every point $X=(0^{\mathbb{Z}},x)$ with $0<x<s_t^+$ does not belong to $\Omega_{a,t}$: points close to X in $\widehat{\Delta}_{a,t}$ only can return to $\widehat{\Delta}_{a,t}$ to the right of s_t^+ and thus to the right of x. The proof of the proposition is now complete. □

8. Proof of Theorem 2.6

8.1. **Non-hyperbolic dynamics.** In this subsection we prove items (A) and (B) of Theorem 2.6. For the next result recall the definitions of an expanding pair of parameters (a,t) (Definition 5.19), relative homoclinic class $H_V(Q, G_{a,t})$ (Definition 2.4), and the set N_a of non-hyperbolic parameters (Equation (1.1)).

Theorem 8.1. *Consider (a,t) an expanding pair of parameters. Then $\Lambda_{a,t} \subset H_V(Q, G_{a,t})$. In particular, $t \in N_a$.*

The first part of the theorem implies that if (a,t) is a expanding pair then $H_V(P, G_{a,t}) \subset H_V(Q, G_{a,t})$ and thus $t \in N_a$. Note that:

- Theorems 8.1 and 5.7 imply item (A) in Theorem 2.6;
- Theorems 8.1 and 5.14 imply item (B)(a) in Theorem 2.6;
- Theorems 8.1 and 5.12 imply item (B)(b) in Theorem 2.6.

The proof of Theorem 8.1 follows from the arguments in [BD], see also [DG]. Thus we will skip some details. In what follows the parameter a remains fixed and thus it is omited.

To prove the theorem it is enough to see that any $X \in \Lambda_t$, $X \neq P, Q$, belongs to $H_V(Q, G_t)$. By Proposition 7.5, every point $X \in \Lambda_t$, $X \neq P, Q$, has some iterate in Δ_t, thus replacing X by some iterate we can assume that $X \in \Delta_t$. Let us also assume that X has infinitely many forward and backward returns to Δ_t. The case where the number of forward or backwards returns is finite is similar, indeed simpler, thus it will be omitted. Let $t \in (t_{n+1}, t_n]$. The condition on the forward returns implies that $X = (\xi; x)$ where

$$\xi^+ = 0^{k_1^- + n} 1 0^{k_1^+ + n + k_2^-} 1 0^{k_2^+ + n + k_3^-} 1 \cdots 1 0^{k_j^- + n + k_j^+} 1 \cdots, \quad \text{where } k_j^+ \geq 1,\ k_j^- \geq 0.$$

Next claim implies the theorem.

Claim 8.2. $X \in H_V(Q, G_t)$.

Proof. Fix an small neighborhood U of X, we see that it contains a homoclinic point of Q whose orbit is contained in V. Note that by the definition of X there are m and a neighborhood $J = J(x)$ of x in $(0, 1/2)$ such that

$$U_{m,J} \stackrel{\text{def}}{=} [\xi_{-m} \dots \xi_{-1} . 0^{k_1^- + n} 1 0^{k_1^+ + n + k_2^-} 1 \cdots 1 0^{k_{m-1}^+ + n + k_m^-} 1 0^{k_m^+}] \times J \subset U.$$

After shrinking J if necessary, this allows us to consider the composition

$$\Gamma_t^{(k_m^+, k_m^-) \cdots (k_1^+, k_1^-)}(J) = g_{[0^{k_1^- + n} 1 0^{k_1^+ + n + k_2^-} 1 \cdots 1 0^{k_{m-1}^+ + n + k_m^-} 1 0^{k_m^+}]a,t}(J) = I, \tag{8.1}$$

where by construction I is contained in $D_{a,t}$.

Recalling the definition of $\Gamma_{a,t}^{(i,j)}$ for $t \in (t_{n+1}, t_n]$, Corollary 5.21 provides a number r and a sequence of pairs $(n_r, \ell_r) \cdots (n_1, \ell_1)$ such that

$$0 \in g_1 \circ g_a^{n-1} \circ (g_a^{n_r} \circ g_{1,t} \circ g_a^{n+\ell_r}) \circ \cdots \circ (g_a^{n_1} \circ g_{1,t} \circ g_a^{n+\ell_1})(I). \tag{8.2}$$

Consider the finite sequence $\alpha = \alpha_{-\ell} \dots \alpha_{-1}$ associated to the concatenation of the maps g_a and $g_{1,t}$ in equations (8.1) and (8.2), that is

$$\alpha \stackrel{\text{def}}{=} 0^{k_1^- + n} 1 0^{k_1^+ + n + k_2^-} 1 \cdots 1 0^{k_m^- + n + k_m^+} 0^{\ell_1 + n} 1 0^{n_1 + \ell_2 + n} 1 \cdots 0^{n_{r-1} + \ell_r + n} 1 0^{n_r} 0^{n-1} 1.$$

The inclusion in (8.2) and the definition of $U_{m,J}$ imply that

$$Y_\alpha = \left(0^{-\mathbb{N}} \alpha_{-\ell} \dots \alpha_{-1} . 0^{\mathbb{N}}; 0\right) \in G_{a,t}^{\ell}(U_{m,J}).$$

The choice of the itinerary α implies that

$$(g_{[\alpha_{-1}\dots\alpha_{-\ell}],a,t})^{-1}(0) \in [0,1/2) \quad \text{for all } k=1,\dots,\ell.$$

By Remark 7.3, Y_α is a homoclinic point of Q whose orbit is (by construction) contained in V. This completes the proof of the claim. $\square$

The proof of the theorem is now complete.

8.2. **Hyperbolic dynamics.** In this subsection we prove part (C) of Theorem 2.6 about the occurrence of hyperbolic dynamics. The proof of this result follows adapting some arguments in [DR1]. The proof has two parts: we first split the set Ω_t into two disjoint parts and using these sets we code the itinerary of the orbits (Theorem 8.4), thereafter we see that these two sets are hyperbolic relative homoclinic classes (Theorem 8.10).

We consider $a > \log 4$ and parameters $t \in (t_n^\star(a), t_n(a))$ for large n (in what follows the dependence on a will be omitted). Associated to the intervals L_t^k, $k = s, c, u$, in Proposition 6.4 we define the subsets of the reference cubes Δ_t

$$\Delta_t^i = \{(\xi, x) \in \Delta_t \text{ such that } x \in L_t^i\}.$$

Definition 8.3. *Consider $X \in \Lambda_t \cap \Delta_t$ with returns $\{\varrho_j(X)\}_{j\in I(X)}$. The s,c,u-*itinerary of X *is the sequence $\{i_j(X)\}_{j\in I(X)}$, $i_j(X) \in \{s,c,u\}$, defined by*

$$i_j(X) = k \quad \textit{if, and only if,} \quad X_{[j]} = G_t^{\varrho_j(X)}(X) \in \Delta_t^k.$$

If $\mathfrak{r}_i(X)$ is the sequence of pairs associated to the returns of X then the condition in Definition 8.3 is equivalent to

$$i_j(X) = k \quad \text{if, and only if,} \quad \Gamma_t^{\mathfrak{r}_j(X)}(x) \in L_t^k. \tag{8.3}$$

For $k = s, c, u$ consider the following subsets of the non-wandering set Ω_t of G_t relative to the neighborhood V of the cycle:

$$\begin{aligned}
\widetilde{\Omega}_t^k &\stackrel{\text{def}}{=} \{X \in \Omega_t \cap \Delta_t \text{ such that } i_j(X) = k \text{ for every } j \in I(X)\};\\
\Omega_t^s &\stackrel{\text{def}}{=} \{P\} \cup \{X \in \Omega_t \text{ such that } G_t^k(X) \in \widetilde{\Omega}_t^s \text{ for some } k\ \};\\
\Omega_t^u &\stackrel{\text{def}}{=} \{Q\} \cup \{X \in \Omega_t \text{ such that } G_t^k(X) \in \widetilde{\Omega}_t^u \text{ for some } k\};\\
\Omega_t^c &\stackrel{\text{def}}{=} \{X \in \Omega_t \text{ such that } G_t^k(X) \in \widetilde{\Omega}_t^c \text{ for some } k\}.
\end{aligned}$$

By (8.3) we have that

$$\widetilde{\Omega}_t^k = \{X = (\xi; x) \in \Omega_t \cap \Delta_t \text{ such that } \Gamma_t^{\mathfrak{r}_j}(x) \in L_t^k \text{ for every } j \in I(X)\}. \tag{8.4}$$

Theorem 8.4. *Let $t \in (t_n^*, t_n)$. Then for every n big enough it holds $\Omega_t = \Omega_t^s \cup \Omega_t^u$.*

This theorem follows from the next proposition.

Proposition 8.5. *Let $t \in (t_n^*, t_n)$ and $X \in \Omega_t \cap \Delta_t$. Then $X \in \widetilde{\Omega}_t^s \cup \widetilde{\Omega}_t^u$.*

To get the theorem just recall the definitions of Ω_t^s and Ω_t^u and that every point $X \in \Omega_t \setminus (\{P, Q\})$ has some iterate in Δ_t (Proposition 7.5).

To prove the Proposition 8.5 we need some preparatory results. For $i, j \in \{s, c, u\}$, let $\Omega_t^{i,j} \stackrel{\text{def}}{=} \Omega_t^i \cup \Omega_t^j$ and $\Delta_t^{i,j} \stackrel{\text{def}}{=} \Delta_t^i \cup \Delta_t^j$.

Lemma 8.6. *Consider $t \in (t_n^*, t_n)$.*

- *If $X \in \Omega_t \cap \Delta_t^{s,c}$ then $i_j(X) \in \{c, s\}$ for all $j \in I(X)^+$;*

- *If $X \in \Omega_t \cap \Delta_t^s$ then $i_j(X) \in \{s\}$ for all $j \in I(X)^+$.*

Proof. Just note that by (8.3) and (s1), (s2), (s4), and (w1) in Proposition 6.4 if $X = X_{[0]} \in \Delta_t^{s,c}$ then all forward returns $X_{[j]}$ of X are of type $(1,r)$, $r \geq 0$, and $X_{[j]} \in \Delta_t^{s,c}$, thus $i_j(X) \in \{s,c\}$. The second item of the lemma follows analogously. □

Lemma 8.7. *Let $t \in (t_n^*, t_n)$ and $X \in \Omega_t \cap \Delta_t^u$. Then $X \in \widetilde{\Omega}_t^u$.*

Proof. We need to see that $i_j(X) = u$ for all $j \in I(X)$. The proof is by contradiction. By Lemma 8.6, if $X_{[j]} \in \Delta_t^{s,c}$ for some j then $X_{[\ell]} \in \Delta_t^{s,c}$ for all $\ell > j$. This allows us to construct a neighborhood of X consisting of points whose forward returns (for sufficiently large iterates) are in $\Delta_t^{s,c}$ and thus separated from $X \in \Delta_t^u$. This prevents the point to be non-wandering.

The previous argument also implies that the backward returns of X are in Δ_t^u: if $X_{[-\ell]} \in \Delta_t^{s,c}$, $\ell > 0$, then Lemma 8.6 implies that $X_{[0]} \in \Delta_t^{sc}$, a contradiction. □

Note that the argument in the proof of Lemma 8.7 implies that an itinerary of a point in $X \in \Delta_t \cap \Omega_t$ is constant. Combining this fact, Lemmas 8.6 and 8.7 we get

Corollary 8.8. *Let $t \in (t_n^*, t_n)$ and $j \in \{s,c,u\}$. If $X \in \Omega_t \cap \Delta_t^j$ then $X \in \widetilde{\Omega}_t^j$.*

Lemma 8.9. *Let $t \in (t_n^*, t_n)$. Then $\widetilde{\Omega}_t^c = \emptyset$.*

Proof. Assume by contradiction that $\widetilde{\Omega}_t^c \neq \emptyset$. Take any $X \in \widetilde{\Omega}_t^c$, after replacing X by some iterate of it we can assume that $X = X_{[0]} = (\xi_{[0]}; x_{[0]}) \in \Delta_t^c \cap \Omega_t$. By Corollary 8.8, $X_{[i]} = (\xi_{[i]}; x_{[i]}) \in \Delta_t^c$ for all $i \geq 1$. By Proposition 6.4 $x_{[i+1]} = \Gamma_t^{(1,0)}(x_{[i]}) = \big(\Gamma_t^{(1,0)}\big)^{i+1}(x_{[0]}) \in L_t^c$ for all $i \geq 0$ and $(\Gamma_t^{(1,0)})^i(x) \to s_t^+$ as $i \to \infty$. As above, this prevents the point X to be non-wandering, getting a contradiction. □

Proof of Proposition 8.5. Just note that Corollary 8.8 and Lemma 8.9 imply that

$$\Omega_t \cap \Delta_t = \widetilde{\Omega}_t^s \cup \widetilde{\Omega}_t^c \cup \widetilde{\Omega}_t^u = \widetilde{\Omega}_t^s \cup \widetilde{\Omega}_t^u.$$

This ends the proof of the proposition. □

Consider now the point $s_{t_n^\star}$ in Lemma 6.1 with $\Gamma_{t_n^\star}^{(1,0)}(s_{t_n^\star}) = s_{t_n^\star}$ and $\big(\Gamma_{t_n^\star}^{(1,0)}\big)'(s_{t_n^\star}) = 1$ and the periodic sequence $\eta^\star = (0^n 10)^{\mathbb{Z}}$. By construction, the point $S_{t_n^\star} \stackrel{\text{def}}{=} (\eta^\star; s_{t_n^\star})$ is a saddle-node of $G_{a,t_n^\star}$ (its central derivative is equal to one).

Theorem 8.10. *Let $t \in [t_n^\star, t_n)$. Then*

1. *$\Omega_t^s = H_V(P, G_t)$ and $\Omega_t^u = H_V(Q, G_t)$.*
2. *If $t \neq t_n^\star$ then Ω_t^s and Ω_t^u are uniformly hyperbolic (of central contracting and of central expanding types, respectively).*
3. *If $t = t_n^\star$ then $\Omega_t^s \cap \Omega_t^u$ is the orbit of the saddle-node $S_{t_n^\star}$. Moreover, every invariant compact subset of Ω_t^s (resp. Ω_t^u)) disjoint from $S_{t_n}^\star$ is uniformly hyperbolic of contracting type (reps. expanding type).*

Items (1) and (2) in Theorem 8.10 imply item (C)(a) of Theorem 2.6, where the density of hyperbolic parameters follows from Lemma 6.3. Item (3) implies item (C)(c) of Theorem 2.6.

Proof of Theorem 8.10. Let us first observe that Ω_t^s and Ω_t^u are compact sets which are G_t invariant. Moreover, they are disjoint for $t \in (t_n^\star, t_n)$.

The first item in the theorem follows from the two claims below:

Claim 8.11. *$\Omega_t^u \subset H_V(Q, G_t)$ and $\Omega_t^s \subset H_V(P, G_t)$.*

Proof. We prove $\Omega_t^u \subset H_V(Q, G_t)$ (the case $\Omega_t^s \subset H_V(P, G_t)$ is analogous). By the G_t-invariance of a homoclinic class, it is enough to see that every point $X \in \widetilde{\Omega}_t^u$ belongs to $H_V(Q, G_t)$. This proof follows considering an argument similar to the ones in Claim 8.2, so we just sketch this proof.

Let us consider the case when $X = (\xi; x)$ has infinitely many forward returns (the general case is similar). Take a small neighborhood I containing x and contained in L_t^u. Since the returns of x to $D_{a,t}$ are in L_t^u (Lemma 8.7), Proposition 6.4 implies that these returns are uniformly expanding. This implies that some return of I contains a boundary point of L_t^u. If this extreme is $d_{a,t}$ arguing exactly as in Claim 8.2 we get a homoclinic point of Q. If the extreme is s_t^+ we can add a tail of returns of type $(1, 0)$. Since s_t^+ is the unique fixed point of $\Gamma_t^{(1,0)}$ in L_t^u and is expanding these iterations of I contains $d_{a,t}$ and we are in the previous case. □

Claim 8.12. *$H_V(Q, G_t) \subset \Omega_t^u$ and $H_V(P, G_t) \subset \Omega_t^s$.*

Proof. We prove $H_V(Q, G_t) \subset \Omega_t^u$ (the case $H_V(P, G_t) \subset \Omega_t^s$ is analogous). Take a point X having a dense orbit in $H_V(Q, G_t)$. Lemma 7.9 implies this point has infinitely many iterates in Δ_t, thus we can assume that $X \in \Delta_t$. If $X \in \widetilde{\Omega}_t^j$ then its whole orbit X belongs to Ω_t^j and its closure is also contained in Ω_t^j. If $X \in \widetilde{\Omega}_t^u$ we are done. Otherwise, $X \in \widetilde{\Omega}_t^s$ and hence $H_V(Q, G_t) \subset \Omega_t^s$. The inclusion in Claim 8.11 implies that $\Omega_t^u \subset H_V(Q, G_t) \subset \Omega_t^s$, which is a contradiction. This completes the proof of the theorem. □

The hyperbolicity of Ω_t^u in (2) follows from the expansion properties (u2) and (u3) in Proposition 6.4. To get the hyperbolicity of Ω_t^s in (2) we use (s2) and (s3) in Proposition 6.4. These arguments are similar to the one in [DR1].

To prove (3) note that by construction the point $S_{t_n^\star} \in \Omega_{t_n^\star}^s \cap \Omega_{t_n^\star}^u$. By construction, the intersection $\Omega_{t_n^\star}^s \cap \Omega_{t_n^\star}^u$ is the finite orbit of $S_{t_n}^\star$. The hyperbolicity-like properties follow from (s2), (s3), (u2), and (u3) in Proposition 6.4 arguing as above. □

It remains to prove item (C)(b) of Theorem 2.6. This follows from the next lemma:

Lemma 8.13. *The map $G_{a,t_n(a)}$ has a cycle relative to V associated to P and Q.*

Proof. Recall that the definition of t_n in (4.2) implies that

$$g_{1,t_n} \circ g_a^n(t_n) = 0.$$

By item (4) in Corollary 3.1 this implies that $\big(0^{-\mathbb{N}}.0^n 10^{\mathbb{N}}; 1/2\big) \in W^u(P, G_{t_n}) \cap W^s(Q, G_{t_n})$ and that orbit of this intersection point is contained in the neighborhood of the cycle. Since $\{0^{\mathbb{Z}}\} \times (0, 1/2) \subset W^s(P, G_{t_n}) \cap W^s(Q, G_{t_n})$ this implies that G_{t_n} has a heterodimensional cycle associated to P and Q (relative to V). □

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