

# WEAKLY HAMILTONIAN ACTIONS

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ABSTRACT. In this paper we generalize constructions of non-commutative integrable systems to the context of weakly Hamiltonian actions on Poisson manifolds. In particular we prove that abelian weakly Hamiltonian actions on symplectic manifolds split into Hamiltonian and non-Hamiltonian factors, and explore generalizations in the Poisson setting.

## 1. INTRODUCTION

An integrable system on a  $2n$ -dimensional symplectic manifold is given by  $n$  generically independent pairwise commuting functions. More generally, a non-commutative integrable system is determined by a set of  $2n - r$  integrals ( $r \leq n$ ), out of which  $r$  do pairwise commute. Integrable systems come with *infinitesimal* abelian actions which are Hamiltonian, in the sense that they have an equivariant *momentum map*.

Furthermore, under compactness assumption on the invariant sets these infinitesimal abelian actions integrate into a torus action for which there is a normal form (action-angle coordinates).

However, some discrete integrable systems [18] do not present commuting first integrals but rather commuting flows. Moreover, there are systems that become Hamiltonian after reduction by non-commutative symmetries. This justifies considering a more general framework where *weakly Hamiltonian actions* take over Hamiltonian actions. In this paper we look at (infinitesimal) actions of abelian Lie algebras on Poisson manifolds having first integrals, but that cannot be arranged into an equivariant momentum map. Our main purpose is discussing conditions under which one can still find “invariant subsets” (Poisson submanifolds) where the residual action is indeed Hamiltonian.

In the symplectic setting there is already a result of Souriau in this direction ([17], Theorem 13.15), which gives a conceptual explanation of a classical theorem of König on the decomposition of both the kinetic energy and the motion of a system. However, this is a result for actions of arbitrary Lie groups (the Galilean group in König’s theorem) and our main point here is to stress that in the abelian case one can go further and obtain a splitting not just of the symplectic manifold, but also of the action into a weakly Hamiltonian and a Hamiltonian factor. Moreover, in the much richer Poisson setting we will see how mild conditions on the action determine Poisson submanifolds, and how under stronger hypotheses a splitting for the Poisson structure can be found, and in some cases, even for the action. In fact, our global construction –when specialized to a local setting– has strong reminiscences of the classical Weinstein local splitting theorem in Poisson geometry [19], and we expect to be able to relate other local splitting results with our construction. Finally our techniques –which are different from Souriau’s– are also appropriate to draw global conclusions for actions of non-abelian Lie algebras.

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## 2. MOTIVATING EXAMPLES

We start by presenting three different types of (complete) weakly Hamiltonian abelian actions in symplectic and Poisson manifolds, and we close the section with a related example.

**2.1. Standard action by translations.** The paradigm of abelian symmetries by Hamiltonian vector fields, but not fitting into a Hamiltonian action, is that of a symplectic vector space  $(V, \sigma)$  on which  $V$  itself – seen as an abelian Lie group – acts by translations: to each vector  $v \in V$  we can assign a first integral which is the unique linear function  $H_u \in V^*$  such that  $dH_u = i_u \sigma$  (or rather, an affine one with that linear part). Since we have:

$$\{H_u, H_v\} = \sigma(u, v)$$

we will never be able to find a basis of first integrals in involution. Notwithstanding, any such choice of first integrals yields a weakly Hamiltonian action (indeed a non-commutative system with constant brackets).

**2.2. The Galilean group.** Let  $G(3)$  be the Galilean group and consider its standard representation (cf. (13.7) in [17]) on  $T^*\mathbb{R}^3$  with position and momentum coordinates  $q_i, p_i$  and particle mass  $m$ . Recall that  $G(3)$  is an extension of the Euclidean group  $E(3)$ , and the restriction of this representation to  $E(3)$  is Hamiltonian because it is the cotangent lift of its defining action on  $\mathbb{R}^3$ . However the Hamiltonian functions corresponding to the Galilean boosts  $mq_i$  and translations in the same direction  $p_i$  do not commute; indeed their Poisson bracket is the mass, and the corresponding cocycle is not exact, bringing in another example of weakly Hamiltonian action.

**2.3. Weakly Hamiltonian actions and nilpotent Lie algebras.** The symplectic form on the symplectic vector space  $(V, \sigma)$  can be interpreted as a 2-cocycle, and as such it gives rise to  $\mathfrak{g} = \mathbb{R} \oplus_{\omega} V$  a central extension of the abelian algebra  $V$ . This Heisenberg type-Lie algebra is nilpotent with one dimensional center. The coadjoint orbit corresponding to the affine hyperplane:

$$\{\alpha \in \mathfrak{g}^* \mid \alpha(1, 0) = 1\}$$

can be canonically identified with  $V$ , and the restriction of the coadjoint action to this orbit is the linear action by translations above (2.1).

More generally, let  $\mathfrak{g}$  be a nilpotent Lie algebra such that  $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{z}(\mathfrak{g})$  (a 2-step nilpotent Lie algebra). As brackets lie in the center, they become Casimirs as functions on  $\mathfrak{g}^*$ , and therefore constants on coadjoint orbits. Then any subspace of  $\mathfrak{g}$  intersecting trivially with the center provides an abelian Lie algebra acting in a weakly Hamiltonian fashion (which is not Hamiltonian provided that some of the brackets are non-trivial) on any coadjoint orbit (and in fact on the whole  $\mathfrak{g}^*$ ). We illustrate this with the following low dimensional example (additional ones can be found by inspecting the list of low dimensional nilpotent Lie algebras up to dimension 7 ([13], [14], [15])):

The nilpotent Lie algebra of dimension 6,  $A_{6,5}^a$  (for  $a \neq 0$ ) for which the non-vanishing relations on a base (see table III in [13]) are  $[e_1, e_3] = e_5$ ,  $[e_1, e_4] = e_6$ ,  $[e_2, e_3] = ae_6$ ,  $[e_2, e_4] = e_5$ . In this case, the symplectic foliation by coadjoint

orbits is given by  $e_6^*$  and  $e_5^*$ , thus defining a foliation with regular 4-dimensional symplectic leaves away from zero. The subspace spanned by  $e_1, e_2, e_3$  and  $e_4$  acts by commuting Hamiltonian vector fields but without momentum map, providing an example of weakly Hamiltonian action on a Poisson manifold.

**2.4. Related examples.** In [18], motivated by the study of discrete integrable systems, the “multi-time” Legendre transform is applied to multi-time Euler-Lagrange equations to obtain a system of commuting Hamiltonian flows. As observed in [18] this situation corresponds to having functions with constant Poisson brackets but these brackets are not necessarily zero<sup>1</sup>, thus providing an extra motivation to consider weakly Hamiltonian actions.

### 3. WEAKLY HAMILTONIAN ACTIONS AND REAL ANALYTIC FUNCTIONS

In this section we discuss how real analytic functions become a tool to study weakly Hamiltonian actions.

**Definition 1.** *Let  $(M, \pi)$  be a Poisson manifold and let*

$$\begin{aligned} \alpha : \mathfrak{g} &\longrightarrow \text{poiss}(M, \pi) \\ u &\longmapsto X_u \end{aligned}$$

*be an action by Poisson vector fields. The action is **weakly Hamiltonian** if the fundamental vector fields for the action are Hamiltonian vector fields. The action is called **complete** if all fundamental vector fields are complete.*

If  $\alpha : \mathfrak{g} \rightarrow \text{ham}(M, \pi)$  is a weakly Hamiltonian action, then it can always be lifted to a linear map:

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow C^\infty(M) \\ u &\longmapsto H_u. \end{aligned}$$

As it is well-known, the defect from the action being Hamiltonian is measured by the Casimir-valued 2-cocycle:

$$\begin{aligned} c : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \text{Cas}(M) \\ (u, v) &\longmapsto \{H_u, H_v\} - H_{[u, v]}. \end{aligned} \tag{1}$$

More precisely,  $c \in \Omega_{CE}^2(\mathfrak{g}; \text{Cas}(M))$  is a cocycle in the Chevalley-Eilenberg complex for the trivial representation of  $\mathfrak{g}$  on  $\text{Cas}(M)$ , which is:

- zero if and only if the chosen lift defines a Lie algebra morphism (this is equivalent to the mapping  $\rho$  being equivariant);
- exact if and only if there exists a choice of lift which is a morphism of Lie algebras.

**Definition 2.** *To any complete weakly Hamiltonian action  $\rho : \mathfrak{g} \rightarrow C^\infty(M)$ , we assign its **flow evaluation map**:*

$$\begin{aligned} \zeta : \mathfrak{g} \times \mathfrak{g} \times M &\longrightarrow C^\infty(\mathbb{R}) \\ (u, v, x) &\longrightarrow H_u(\phi_v^s(x)), \end{aligned} \tag{2}$$

where  $\phi_v^s$  denotes the flow of  $X_v$ .

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<sup>1</sup>As observed by the author in [18] indeed the vanishing of these Poisson brackets is equivalent to Lagrangian 1-form employed in the construction being closed on the solutions of the Euler-Lagrange equations.

Note that if the action  $\rho$  is Hamiltonian with momentum map  $\mu$ , then the flow evaluation map is the result of pulling back via the momentum map the flow evaluation map for the coadjoint action: indeed, because the momentum map is  $\mathfrak{g}$ -equivariant we have

$$\zeta_{u,v,x}(s) = H_u(\phi_v^s(x)) = H_u(\phi_v^s \cdot x) = \langle \text{Ad}^*(\Phi_v^s)(\mu(x)), u \rangle \quad (3)$$

where  $\Phi_v^s \in G(\mathfrak{g})$  – the simply connected Lie group integrating  $\mathfrak{g}$  – and  $\text{Ad}^*(\Phi_v^s)(\mu(x))$  is the corresponding coadjoint flow of  $v$  starting at  $\mu(x)$ . Therefore, the  $\zeta_{u,v,x}$ 's generalize the linear projections of the coadjoint flow to the complete weakly Hamiltonian case.

The coadjoint flow is real analytic, and each of its projections (3) is a real analytic function which extends to an entire function of exponential type (growth). For example, this can be checked by noting that to analyze the coadjoint action it suffices to use matrix groups. For complete weakly Hamiltonian actions this property still holds (cf. [6], lemma 6, where real analyticity is studied in the setting of quasi-representations on symplectic manifolds):

**Proposition 1.** *For any triple  $(u, v, x) \in \mathfrak{g} \times \mathfrak{g} \times M$  the corresponding flow evaluation  $\zeta_{u,v,x}$  is real analytic with expansion at zero:*

$$\zeta_{u,v,x}(s) = \sum_{j=0}^{\infty} H_{\text{ad}^j(v)(u)}(x) \frac{s^j}{j!} + \sum_{j=1}^{\infty} c_{\text{ad}^{j-1}(v)(u),v}(x) \frac{s^j}{j!}. \quad (4)$$

Moreover, it extends to an entire function of exponential type.

*Proof.* The fundamental theorem of calculus yields:

$$\begin{aligned} \zeta_{u,v,x}(s) &= \zeta_{u,v,x}(0) + \int_0^s \zeta'_{u,v,x}(t) dt = \zeta_{u,v,x}(0) + \int_0^s dH_u(X_v(\phi_v^t(x))) dt \\ &= \zeta_{u,v,x}(0) + \int_0^s \{H_u, H_v\}(\phi_v^t(x)) dt = \zeta_{u,v,x}(0) + \int_0^s H_{[u,v]}(\phi_v^t(x)) dt + c_{u,v}(x)s. \end{aligned}$$

The formula in (4) follows by induction. In order to check convergence of the power series expansion fix any metric in  $\mathfrak{g}$ , and pick a neighborhood of the origin  $W \subset \mathfrak{g}$  so that

$$|[u, v]| \leq C|u||v|, \forall u, v \in W.$$

The linear maps  $u \mapsto H_u(y)$  are continuous, and also vary continuously with  $y$ . The bilinear map  $(u, v) \mapsto c_{u,v}(x)$  is also continuous. Therefore the norm of the remainder of the Taylor expansion can be bounded in any compact subset as follows:

$$\begin{aligned} \left| \int_0^{s_1} \cdots \int_0^{s_{j+1}} H_{\text{ad}^j(v)(u)}(\phi^{s_{j+1}}(c)) ds_{j+1} \cdots ds_1 + c_{\text{ad}^{j+1}(v)(u),v}(x) \frac{s^{j+1}}{(j+1)!} \right| \leq \\ K_1 |u| \frac{(C|v|s)^{j+1}}{(j+1)!} + K_2 |u| \frac{(C|v|s)^{j+1}}{(j+1)!} \end{aligned}$$

Then uniform convergence on any compact subset is straightforward.

Nearly the same proof shows that the power expansion has coefficients whose norm is dominated by those of an exponential, and therefore the desired result follows.  $\square$

One can roughly rephrase Proposition 1 by saying that a complete weakly Hamiltonian action on a Poisson manifold gives a large supply of very special analytic functions (non-trivial except for abelian Hamiltonian actions).

**Remark 1.** If  $\rho$  is a complete weakly Hamiltonian action on a symplectic manifold, then the momentum map becomes  $G(\mathfrak{g})$ -equivariant for the (affine) correction of the coadjoint action by the integration of the cocycle (1) ([17], [10, 11] and [5]). The flow evaluation map becomes again the pullback of the linear projections for this corrected action, for which formula (4) can readily be obtained. Indeed it can be easily checked that in the symplectic setting the following formula is obtained for the flow evaluation map where  $\theta$  is the 1-cocycle associated to the action,

$$\zeta_{u,v,x}(s) = \left\langle \mu(x), Ad_{\exp(-sv)}^* u \right\rangle + \langle \theta(\exp(sv)), u \rangle.$$

We chose not to “embed” weakly Hamiltonian actions into the equivariant framework both to make this note self-contained and because this cannot be done in general for Poisson manifolds.

**Remark 2.** In [8] it is shown that any action of a compact group on a symplectic manifold is equivalent to an analytic one (also the symplectic form being analytic). Thus if the action is weakly Hamiltonian one obtains automatically that the functions  $\zeta_{u,v,x}$  are analytic. Our result for symplectic manifolds is more general in the sense that it only requires a complete Lie algebra action, and it makes clear the dependence of the expansion of  $\zeta_{u,v,x}$  in terms of the Lie algebra structure and the cocycle  $c$ .

#### 4. SPLITTING OF THE STRUCTURES

König’s theorem [7] establishes that the kinetic energy of a system is the sum of the kinetic energy associated to the movement of the center of mass and the kinetic energy associated to the movement of the particles relative to the center of mass (the so-called internal energy). In particular, the motion of a particle can be decomposed as the motion of its center of mass (which by the conservation of linear momentum moves uniformly in a line) and the motion of the system around its center of mass. König’s theorem is interpreted in [17], paragraph 13.25, as a decomposition theorem associated to the normal abelian subgroup of the Galilean group corresponding to fixing the identity matrix in  $SO(3, \mathbb{R})$  in its matrix representation.

In this section we generalize Souriau’s decomposition theorem in several directions. On the one hand, we discuss actions on Poisson manifolds. On the other hand, because we only discuss abelian Lie algebras, we can find sufficient conditions to split also the action into a translational standard factor and a Hamiltonian factor. This makes precise the idea that in some systems after reducing by translations (or restricting to appropriate slices to the translation action), what we get is a Hamiltonian system (and sometimes a completely integrable one).

**4.1. The Poisson case.** Let  $\rho : \mathfrak{g} \rightarrow C^\infty(M)$  be a weakly Hamiltonian action with cocycle  $c$ . Let us consider  $\mathfrak{g} \times M \rightarrow M$  as a trivial vector bundle over  $M$ . For simplicity, let us assume that  $M$  is connected. A Poisson manifold is naturally endowed with a symplectic foliation  $\mathcal{F}$  whose tangent distribution is generated by the set of all Hamiltonian vector fields. As such it has an associated leaf space  $M/\mathcal{F}$ .

**Definition 3.** We say that  $F \rightarrow M$  is a sub-bundle of  $\mathfrak{g} \times M$  over the leaf space if  $F \rightarrow M$  is a sub-bundle such that the fibers  $F_x$  and  $F_y$  are equal whenever  $x, y \in M$  are in the same symplectic leaf.

A framing for a sub-bundle over the leaf space is a global trivialization  $\mathfrak{b} : F \rightarrow \mathbb{R}^l$  which is constant over the symplectic leaves.

We are interested in sub-bundles  $F \rightarrow M$  over the leaf space which are symplectic relative to the cocycle  $c \in \Omega_{CE}^2(\mathfrak{g}; \text{Cas}(M))$ . By this we mean that for all  $x \in M$  the constant 2-form  $c(x)$  restricts to a symplectic form along  $F_x$ .

Such sub-bundles give rise to an interesting structure on  $(M, \pi)$ :

**Theorem 1.** *Let  $\rho: \mathfrak{g} \rightarrow C^\infty(M)$  be a complete weakly Hamiltonian action of an abelian Lie algebra and let  $c$  be its associated cocycle. If  $F \rightarrow M$  is a vector sub-bundle over the leaf space which is symplectic relative to  $c$ , then it determines:*

- (1) *A Poisson-Dirac submanifold  $N$ .*
- (2) *A diffeomorphism*

$$\Phi : M \rightarrow F|_N$$

*such that the inverse image of the fiber  $F_x$ ,  $x \in N$ , is a symplectic submanifold  $L_x$  of the leaf through  $x$  and*

$$\Phi : (L_x, \pi^{-1}|_{L_x}) \rightarrow (F_x, c(x))$$

*is a symplectomorphism.*

*Moreover, if  $F \rightarrow M$  has a framing  $\mathfrak{b}$  over the leaf space, then the diffeomorphism has target a product manifold (trivial bundle)*

$$\Phi : M \rightarrow \mathbb{R}^{2d} \times N, \quad (5)$$

*and  $\Phi^{-1}(N \times \{u\})$ ,  $u \in \mathbb{R}^{2d}$ , is a Poisson-Dirac submanifold of  $(M, \pi)$ .*

*Proof.* We define

$$N = \{x \in M \mid H_u(x) = 0, \forall u \in F_x\}.$$

If  $x \in N$ , we fix  $\mathfrak{b}$  a local framing around  $x$  which is constant on the leaf space. Note that this is always possible as we can start with  $v_1, \dots, v_{2d}$  a basis of  $F_x$  and then project orthogonally into nearby fibers using a fixed inner product on  $\mathfrak{g}$ .

Using this framing we define the map:

$$\psi_{\mathfrak{b}} : U \subset M \rightarrow \mathbb{R}^{2d}, \quad x \mapsto (H_{\mathfrak{b}^{-1}(e_1)}(x), \dots, H_{\mathfrak{b}^{-1}(e_{2d})}(x)),$$

where  $e_1, \dots, e_{2d}$  is the canonical basis. By linearity of the representation  $\rho$ ,  $N \cap U = \psi_{\mathfrak{b}}^{-1}(0)$ , so it suffices to show that 0 is a regular value.

For  $u \in F_x$ , we pick  $v \in F_x$  so that  $c_{u,v}(x) \neq 0$ . Because

$$H_u(X_v) = \omega(X_u, X_v) = \{H_u, H_v\} = \{H_u, H_v\} - H_{[u,v]} = c_{u,v},$$

$\psi_{\mathfrak{b}}$  is a submersion restricted to each symplectic leaf.

We also need to prove that  $N$  is non-empty. Given  $y \in M$ , we look for  $v \in F_y$  such that  $\phi_v^1(y) \in N$ . This is equivalent to:

$$0 = H_u(\phi_v^1(y)) = \zeta_{u,v,y}(1), \quad \forall u \in F_y. \quad (6)$$

As in the abelian case the formula for the flow evaluation (4) becomes

$$\zeta_{u,v,y}(s) = H_u(y) + c_{u,v}(y)s, \quad (7)$$

and formula (6) becomes the linear system of equations:

$$H_u(y) + c_{u,v}(y) = 0$$

By non-degeneracy of  $c_{u,v}(y)$ , there exist a unique solution of this system. Therefore  $N$  is a non-empty submanifold of  $M$  intersecting each leaf transversely.

Now we define the map:

$$\Phi : M \rightarrow F|_N, \quad y \mapsto (y, \phi_v^1(y)),$$

where  $v$  is the unique solution of (6). This map is clearly a bijection. It is smooth because for  $z$  in a small neighborhood of  $y$ , the point  $\phi_v^1(z)$  belongs to a small neighbourhood of  $\phi_v^1(y)$ , where the local trivialization (over the leaf space) can be used to correct  $v$  into the appropriate vector  $v(z)$  (and the system we have to solve varies smoothly with the point  $z$ ). The smoothness of the inverse

$$\Phi^{-1}(u, x) = \phi_{-u}^1(x)$$

is clear.

It remains to prove Poisson theoretic properties of  $\Phi$ . The foliation defined by the fibers of  $F|_N$  is transferred by  $\Phi$  to a foliation with leaves  $L_x$ ,  $x \in N$ . This leaf can be alternatively described as the orbit through  $x$  of the action of  $F_x \subset \mathfrak{g}$ . In particular, for any  $u \in F_x$  the Hamiltonian vector field  $X_u|_{L_x}$  is tangent to  $L_x$ , and a basis of  $F_x$  provides a framing of the tangent bundle  $TL_x$ . By construction at  $y \in L_x$ :

$$\pi^{-1}(X_u, X_v) = c_{u,v}(x),$$

thus proving item (2).

Recall from [1] that  $N \subset (M, \pi)$  is a Poisson-Dirac submanifold if

$$T_x N \cap \pi^\#(T_x N^0) = \{0\}, \quad (8)$$

and the induced bi-vector  $\pi_N \in \mathfrak{X}^2(N)$  happens to be smooth.

We showed that  $T_x N$  is the common kernel of  $\{dH_u | u \in F_x\}$ , or, conversely, that the latter subspace of covectors is the annihilator  $T_x N^0$ .

As  $\pi^\#(dH_u) = X_u \in T_x L_x$ , equation (8) holds.

The smoothness of  $\pi_N$  follows from the smoothness of the distribution defined by the symplectic leaves  $L_x$ ,  $x \in N$ , because its annihilator at points of  $N$  can be identified with  $T^*N$ , and  $\pi_N^\#$  is the restriction of  $\pi^\#$  to this smooth sub-bundle [1].

Finally, if we have a framing  $\mathfrak{b}$  of  $F \rightarrow N$  over the leaf space, then  $F$  becomes trivial and we get a diffeomorphism:

$$\Phi : M \rightarrow \mathbb{R}^{2d} \times N.$$

The same arguments as above show that  $\Phi^{-1}(\{u\} \times N) = \psi_{\mathfrak{b}}^{-1}(u)$  is a Poisson-Dirac submanifold of  $(M, \pi)$ .  $\square$

If  $F \rightarrow M$  is a framed sub-bundle over the leaf space and it is symplectic relative to the cocycle  $c$ , there is no reason why the framing should trivialize as well the fiberwise linear symplectic forms. If that is the case, then  $(M, \pi)$  is in fact a product Poisson manifold:

**Theorem 2.** *Let  $\rho: \mathfrak{g} \rightarrow C^\infty(M)$  be a complete weakly Hamiltonian action of an abelian Lie algebra and let  $c$  be its associated cocycle. If  $F \rightarrow M$  is a vector sub-bundle over the leaf space which is symplectic relative to  $c$ , then a framing over the leaf space compatible with the fiberwise symplectic structure determines a splitting of  $M$  into a symplectic vector space and a Poisson submanifold:*

$$\Phi : (M, \pi) \rightarrow (\mathbb{R}^{2d} \times N, \omega^{-1} \oplus \pi|_N).$$

*Proof.* By Theorem 1 we have a diffeomorphism

$$\Phi : (M, \pi) \rightarrow \mathbb{R}^{2d} \times N. \quad (9)$$

A vector  $u \in \mathbb{R}^{2d}$  acts on the right hand side of (9) by translations of the first factor. The diffeomorphism  $\Phi$  conjugates this translation to the time one flow of the the Hamiltonian vector field of the function  $H_{\mathfrak{b}^{-1}(u)}$ . This, together with the fact that the function by hypothesis  $\Phi|_{L_x} c(x)$  is a constant symplectic form  $\omega$  on  $\mathbb{R}^{2d}$ , implies that  $\Phi$  sends  $\pi$  into the product Poisson structure  $\omega^{-1} \oplus \pi|_N$ .  $\square$

**Remark 3.** *A special instance where Theorem 2 applies is when there exists  $V$  a subspace of  $\mathfrak{g}$  which intersects trivially all  $\{\ker c|_F\}_{F \in \mathcal{F}_\pi}$ . We obtain a Poisson splitting:*

$$(M, \pi) \cong (V \times N, c|_V^{-1} \times \pi|_N),$$

*given by the action of  $V$ .*

*It is worth pointing out that this splitting result can be seen as a global version of Weinstein splitting theorem. The original proof of the Weinstein's splitting theorem amounts to constructing a weakly Hamiltonian abelian action of maximal rank with symplectic cocycle.*

*It is also possible to reinterpret other splitting results in the Poisson setting using this language. For example, in [12, 3] an equivariant Weinstein splitting theorem is proved at a fixed point of a Poisson for an action of a compact Lie group. For a Hamiltonian action of compact  $G$ , this result can be restated saying that the (local) Hamiltonian action of  $\mathfrak{g}$  can be extended to a weakly Hamiltonian action of an extension  $\mathfrak{g} \ltimes \mathbb{R}^{2d}$ , where  $2d$  is the rank of the Poisson structure at  $x$  (the cocycle has kernel  $\mathfrak{g}$ ). From this perspective, the equivariant Poisson splitting follows trivially.*

As for induced actions, the sub-bundle  $F \rightarrow M$  in Theorem 1 can be regarded as a groupoid, and as such it acts on  $M$ . As we prefer not to go to the greater generality of groupoid actions on Poisson manifolds, we confine ourselves to the situation in which the sub-bundle is framed. This produces an action of  $\mathbb{R}^{2d}$  on  $(M, \pi)$  with cocycle the pullback of  $c|_F$  under the framing (very far from being Hamiltonian). This is the action we used in the proof of Theorem 2.

We are interested in sufficient conditions for the existence of “invariant Hamiltonian submanifolds”:

**Theorem 3.** *Let  $\rho: \mathfrak{g} \rightarrow C^\infty(M)$  be a complete weakly Hamiltonian action of an abelian Lie algebra and let  $c$  be its associated cocycle. If  $\ker c \rightarrow M$  admits a framing  $\mathfrak{b}'$  over the leaf space, then an inner product on  $\mathfrak{g}$  determines a Poisson-Dirac submanifold  $N$  and a residual Hamiltonian action*

$$\rho_{\mathfrak{b}'}: \mathbb{R}^k \rightarrow C^\infty(N)$$

*Proof.* Because  $\ker c \rightarrow M$  is a bundle over the leaf space, so is  $\ker c^\perp \rightarrow M$ . By applying Theorem 1 to  $\ker c^\perp \rightarrow M$  we obtain a Poisson-Dirac submanifold  $N \subset (M, \pi)$ . The action on  $N$  is induced by the framing  $\mathfrak{b}'$  exactly as in the proof of Theorem 2. As the corresponding cocycle is the restriction of  $c$  to  $\ker c \rightarrow N$ , it follows that the action is Hamiltonian.  $\square$

**4.2. The symplectic case.** As a consequence of the theorem above applied in the symplectic context, a slightly improved version of Souriau's splitting theorem is obtained and the action can be split into a Hamiltonian factor and a weakly Hamiltonian action corresponding to the first motivating example in this paper:

**Corollary 1.** *Let  $\rho: \mathfrak{g} \rightarrow C^\infty(M)$  be a complete weakly Hamiltonian action of an abelian Lie algebra on a symplectic manifold and let  $c \in \wedge^2 \mathfrak{g}^*$  be its associated cocycle. Any subspace  $V \subset \mathfrak{g}$  complementary to the kernel of  $c$  determines a symplectic splitting*

$$\Phi: (M, \omega) \rightarrow (V \times N, c|_V \oplus \omega|_N)$$

*and a splitting of the action into a standard translational action on  $(V, c|_V)$  and a Hamiltonian one on  $(N, \omega|_N)$ .*

In the symplectic setting it is useful to consider the correction of the weakly Hamiltonian action by the 1-cocycle. Denote by  $\mu: M \rightarrow \mathfrak{g}^*$  the associated momentum map. In this case the symplectic manifold  $N$  can be interpreted as the Marsden-Weinstein reduction associated to the  $V$ -action with moment map  $\tilde{\mu} = \pi \circ \mu$  where  $\pi$  is the projection  $i^*$  with  $i: V \hookrightarrow \mathfrak{g}$ . This identification can be done by considering the pre-image of any element in  $V^*$  by  $\tilde{\mu}$  which by construction intersects each orbit of the action at a single point therefore its quotient by the action of  $V$  is  $N$ .

Corollary 1 is a generalization of a (local) result in [16]. There, one assumes the existence of what is called a semicanonical system of functions in a  $2n$  dimensional symplectic manifold. In our language these is a weakly Hamiltonian action of an  $n$ -dimensional abelian Lie algebra, so that the differentials of the associated Hamiltonian functions are linearly independent. It turns out that neither hypothesis on the number of functions nor on the rank of differentials of the functions in involution are necessary.

**Remark 4.** *If  $(M, \pi)$  is a Poisson manifold supporting only trivial Casimirs (constants), then the symplectic splitting theorem holds word by word. As examples of Poisson manifolds with trivial Casimirs we may consider, for instance, the Reeb foliation of  $S^3$  with leafwise area form, compact cosymplectic manifolds with non-compact leaves endowed with natural Poisson structures [4], and other Poisson manifolds constructed out of them via products, surgeries, etc.*

## 5. AN APPLICATION TO NILPOTENT ACTIONS

We end up this paper with an application of Proposition 1 to study of nilpotent actions which we consider only in the symplectic context for the sake of simplicity.

The real analyticity of the flow evaluation map is rather powerful, and it has been used (under a different guise) to draw consequences on complete actions of nilpotent and semisimple Lie algebras on compact manifolds [2].

Here we want to point out yet another global results for nilpotent actions:

**Corollary 2.** *Let  $\rho: \mathfrak{g} \rightarrow C^\infty(M)$  be a complete weakly Hamiltonian effective representation of a nilpotent non-abelian Lie algebra on a (non-compact) symplectic manifold. Let  $v$  be a vector not in the center of  $\mathfrak{g}$ . Then the any periodic of  $X_v$  must be contained in the level set of a non-constant function (in general different from  $H_v$ ).*

*Proof.* By our assumptions we can find  $u \in \mathfrak{g}$  such that according to (4) for all  $x \in M$ ,  $\zeta_{u,v,x}(s)$  is a polynomial with linear coefficient  $H_{[v,u]}(x) - c_{u,v}$ , which is a non-constant function on  $x$ . Hence if we have a periodic orbit of  $X_v$ , by compactness  $\zeta_{u,v,x}(s)$  must be constant and therefore must be in the zero subset of  $H_{[v,u]}(x) - c_{u,v}$  (and also in the zero set of functions corresponding to the coefficients of higher order in (4)).  $\square$

As for an analog of the splitting theorem for abelian complete weakly Hamiltonian actions, the situation in the nilpotent case is much more complicated. One may take a basis of  $\mathfrak{g}$  and use the corresponding Hamiltonians to arrange a function to Euclidean space, and one can obtain the following information: for a given point  $x$  and  $u, v \in \mathfrak{g}$ :

- either  $H_{\text{ad}^j v(u)}(x) - c_{\text{ad}^{j-1} v(u), v} = 0$  for all  $j$ , in which case the orbit of  $X_v$  through  $x$  will either not intersect  $H_u^{-1}(a)$ ,  $a \in \mathbb{R}$ , or be contained in it;
- or some  $H_{\text{ad}^j v(u)}(x) - c_{\text{ad}^{j-1} v(u), v} \neq 0$ , in which case for all  $a \in \mathbb{R}$  the orbit of  $X_v$  through  $x$  intersects  $H_u^{-1}(a)$  a finite number of times.

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