PORCUPINE-LIKE HORSESHOES. TOPOLOGICAL AND ERGODIC ASPECTS

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ABSTRACT. We introduce a class of topologically transitive and partially hyperbolic sets called porcupine-like horseshoes. The dynamics of these sets is a step skew product over a horseshoe. The fiber dynamics is given by a one-dimensional genuinely non-contracting iterated function system. We study this dynamics and explain how the properties of the iterated function system can be translated to topological and ergodic properties of the porcupines.

1. INTRODUCTION

1.1. **General setting.** In this paper we summarize and explain the main ingredients in the series of papers [14, 27, 10, 11, 12] about topological and ergodic properties of the so-called *porcupine-like horseshoes.* The naive image of these sets, that justifies their name, is the following: in $\mathbb{R}^3$ consider a horseshoe in $\mathbb{R}^2 \times \{0\}$ and for a dense subset of the horseshoe glue spines (segments) parallel to the direction $\{0^2\} \times \mathbb{R}$. This is done in such a way that the resulting set is partially hyperbolic (one-dimensional center parallel to the spines) and topologically transitive. These systems are step skew products over a horseshoe (hyperbolic part) whose central dynamics is defined in terms of a one-dimensional (non-contracting) iterated function system (IFS) of fiber maps.

The fiber IFS has finitely many generators, one of them reversing the orientation of the fibers. These systems exhibit topologically interesting features that contrast what has been studied in other contexts. For instance, iterated function systems with monotonically increasing fiber maps possess invariant sets being a finite union of graphs. As some examples we can name the famous Weierstrass function [19], pinched skew products over a minimal base dynamics with strange non-chaotic attractors [23], and skew-products over a shift space with a finite number of attractor-repeller pairs [25].

Let us outline some motivations of this line of research. The starting point are the results in [14] about bifurcations of hyperbolic homoclinic

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classes and their topological properties, followed by investigations of their ergodic properties and, in particular, their Lyapunov exponents in [27]. Second, the topological examples of recurrent sets of fibered systems over a shift space, so-called *bony attractors*, by Ilyashenko in [20]. These attractors have the following significant property: the map has an invariant manifold, and the intersection of the attractor with that manifold, called a *bone*, is much larger than the attractor of the restriction of the map to the invariant manifold. Bony attractors with one-dimensional bones were discovered in [26] and [21] provides further examples with higher-dimensional bones. Another motivation comes from the results about iterated function systems and skew product dynamics in [18].

Let us now say a few words about our examples of porcupines. For notational simplicity, in what follows we assume that the ambient space is three-dimensional. The porcupine horseshoe in [14, 27] is essentially hyperbolic with a two-dimensional contracting direction and has *attached* one hyperbolic saddle with one-dimensional stable direction, we say that this point is *exposed*. All the saddles in the transitive set different from the exposed one are hyperbolic with *unstable index* one (dimension of the unstable direction) and, in fact, such saddles are all *homoclinically related* (their invariant manifolds meet cyclically and transversely). Moreover, the porcupine is their *homoclinic class* (closure of the transverse intersections of the invariant manifolds of the saddle). This configuration has immediate implications for the ergodic properties: all ergodic measures are hyperbolic, the only measure with positive central exponent is the Dirac measure supported on the exposed saddle, the central spectrum (Lyapunov exponents corresponding to the central direction) forms an interval of the form $[a, 0] \cup \{b\}$, where $b > 0$ is the exponent of the exposed saddle. This is an example of a non-hyperbolic set which is hyperbolic in the sense that all its ergodic measures are hyperbolic (have non-vanishing Lyapunov exponents). Other examples of different nature can be found in [2, 7], but the novelty in [14, 27] is that measures have different type of hyperbolicity.

The examples in [10, 11, 12, 13] can be considered as extensions of [14] to genuinely non-hyperbolic settings. These extensions rely on a detailed study of the topological and ergodic properties of the underlying fibered iterated function system. The complexity of the dynamics is also related to the complexity of the exposed part of the porcupine (see Definition 1.2). In [10] we consider porcupines whose central spectrum contains a gap and an interval of positive and negative values. Here the exposed piece is also just a saddle. The examples in [11] are an extension of [10] exhibiting porcupines whose exposed pieces are horseshoes and deriving ergodic consequences of this fact (equilibrium states and special phase transitions for some geometric potentials). Finally, in [13] we consider more complicated exposed pieces that give raise to multiple gaps in the Lyapunov spectrum. A complete analysis of the Lyapunov spectrum and related properties is presented in [12].

Let us observe that porcupines are simple examples of non-hyperbolic transitive sets with rich dynamics. Many features of non-hyperbolic dynamics can be easily checked and explained. In general, when trying to understand the dynamics, the simplest examples to start with are iterated function systems and step skew products. The complexity increases for general skew products (where the fiber maps depend on the base point). In this context, the most complex case corresponds to partially hyperbolic dynamics with one-dimensional center. Note that under quite general and natural assumptions one can translate properties of general skew products to partially hyperbolic dynamical systems, see [24, 16].

Another motivation comes from the study of elementary pieces of dynamics that generalize the concept of basic sets in uniformly hyperbolic systems. In some cases homoclinic classes provide the right generalizing concept, see [6, Chapter 10]. Porcupines are significant examples of non-hyperbolic homoclinic classes.

Finally, the gaps in the spectrum explained in [27, 11, 13] provide examples of a new type of phase transitions for geometric potentials (in our case the potential $\varphi_c = -\log \|dF|_{E^c}\|$ associated to the derivative of the diffeomorphism in the central direction), that is, non-differentiability of the topological pressure. Pressure is a central object in ergodic theory as it joins various dynamical quantifiers such as Lyapunov exponents, dimension, entropy, decay of correlations, equilibrium states among others (see [30, 33]) and its regularity is of great interest. Such phase transitions come along with the co-existence of equilibriums states with positive entropy and with different exponents. Let us point out that an essential point in our examples is transitivity. Clearly when a system has several transitive components (though they could be intermingled) then this will be reflected dynamically (see [15] for an example of renormalizable unimodal maps). If, however, a dynamical system is transitive but there exist pieces that are "exposed" in a sense that dynamically and topologically they form extreme points then we still can observe the phenomenon of phase transitions and coexistence of equilibrium states.

1.2. **Porcupine-like horseshoes.** In this section we give some definitions and discuss more precisely some features of porcupines. We consider invariant sets having a dominated splitting with three non-trivial directions $E^s \oplus E^c \oplus E^u$, where E^s is uniformly contracting, E^u is uniformly expanding, and E^c is one-dimensional and non-hyperbolic. In this case we say that the splitting is *strongly partially hyperbolic.*

Definition 1.1 (Porcupines)**.** We call a compact F-invariant compact set Λ of a (local) diffeomorphism F a *porcupine-like horseshoe* or *porcupine* if

- Λ is the maximal invariant set in a compact set $\mathbf{C}$, *transitive* (existence of a dense orbit), and strongly partially hyperbolic,
- there is a subshift of finite type $\sigma\colon \Sigma \to \Sigma$ and a semiconjugation $\pi\colon \Lambda \to \Sigma$ such that $\sigma \circ \pi = \pi \circ F$, $\pi^{-1}(\xi)$ contains a continuum for

uncountably many $\xi \in \Sigma$ and a single point for uncountably many $\xi \in \Sigma$.

We call $\pi^{-1}(\xi)$ a *spine* and say that it is *non-trivial* if it contains a continuum. The *spine of a point* $X \in \Lambda$ is the set $\pi^{-1}(\pi(X))$.

Given a saddle P of F, its *homoclinic class* $H(P,F)$ is the closure of the transverse intersections of the stable and unstable manifolds of the orbit of P. Two saddle points P and Q are *homoclinically related* if the invariant manifolds of their orbits meet cyclically and transversally. Note that there are points whose homoclinic classes are equal but are not homoclinically related. A homoclinic class is *non-trivial* if it contains at least two different orbits. Given a neighborhood U of P, we call the closure of the set of points R that are in the transverse intersections of the stable and unstable manifolds of P and whose orbit is entirely contained in U the *homoclinic class of P relative to U* and denote this set by $H_U(P,F)$. The homoclinic class of a saddle P is always a transitive set and the hyperbolic periodic points of the same unstable index as P are dense in the class. However, a homoclinic class may fail to be hyperbolic and may contain points of different unstable indices. Indeed, the porcupines illustrates this situation. We can define the (relative) homoclinic class of a transitive hyperbolic set similarly.

Definition 1.2 (Exposed piece)**.** We call a compact F-invariant subset E of a porcupine Λ an *exposed piece* if E is transitive and satisfies $H_{\mathbf{C}}(E) = E$, where $\mathbf{C}$ is a cube such that $\Lambda = \cap_{i\in\mathbb{Z}} F^i(\mathbf{C})$.

In some sense the exposed pieces cannot be "reached" by the remaining dynamics of the porcupine. This unattainability has two aspects. First, the orbits of the porcupine outside the exposed piece only accumulate on "one side" of it. Second, the existence of the spectral gap immediately implies that the dynamics does not satisfy the specification property [31]. Though such pieces do not break topological transitivity – which perhaps can be considered a concept to weak to capture the phenomena observed. Let us observe that in a slightly different context [3] provides examples of a chain recurrent class that properly contains a homoclinic class besides "exposed orbits" that correspond to heteroclinic intersections between saddles of different unstable indices.

So far we have studied three types of exposed pieces of porcupines:

- The exposed piece is a fixed point [14, 10]. This leads to a central spectrum of the form $[a,b]\cup\{c\}$, where $a<0\le b<c$. This spectrum provides a phase transition of the geometric potential φ_c at some critical parameter.
- The exposed piece is a horseshoe [11]. This leads to a central spectrum of the form $[a,b]\cup[c,d]$, where $a<0<b<c<d$. This spectrum provides a rich phase transition for the geometrical potential φ_c at some critical parameter.

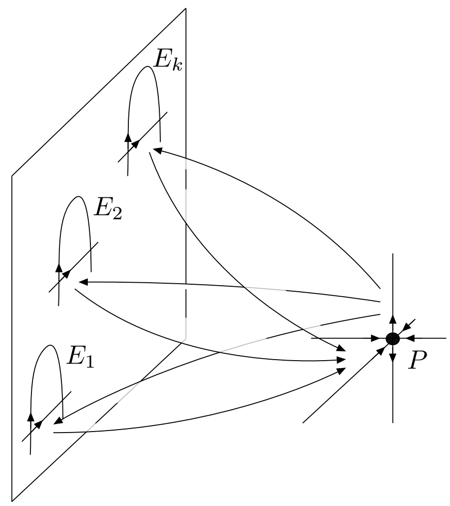

Figure 1. Exposed pieces and connections to a saddle P

- The exposed piece is composed by a finite number of disjoint horseshoes [13]. These horseshoes have disjoint central Lyapunov spectrum and appropriate entropies. This leads to a central spectrum of the form $[c_0, d_0] \cup [c_1, d_1] \cup \ldots \cup [c_k, d_k]$, where $c_0 < 0 < d_0 < c_1 < d_1 < \ldots < c_k < d_k$. This gives k rich phase transitions of the potential φ_c.

The exposed pieces mentioned above are connected by a heterodimensional cycles to one saddle in the porcupine (see Figure 4). This sort of configuration is non-generic. It was shown in [1] that C^1 generically the Lyapunov spectrum of a homoclinic class is convex. This implies immediately that spectra with gap(s) similar to our examples only can occur in non-generic situations.

Further, we would like to mention that in terms of an iterated function system the above mentioned heterodimensional cycle corresponds to a cycle condition for the fiber maps. This seems to play a role similar to the Misiurewicz property in one-dimensional dynamics as, for example, in the case of the quadratic map $f(x) = 1 - 2x^2$ or of Chebyshev polynomials [28, 29] where there is a periodic point that is "dynamically exposed" in the sense that it is immediately post-critical.

Above we focused mainly on topological aspects of porcupines. Let us observe that these topological properties have implications in the ergodic level. First, the fact that the central direction of the porcupine is one-dimensional implies that its topological entropy coincides with the one of its base [8]. Dynamics on the fibers does not generate additional entropy. Furthermore, this implies upper semicontinuity of the entropy function [9] which in turn is essential to guarantee existence of equilibrium states (measure of maximal entropy etc.) [33]. Observe also that porcupines are transitive sets of saddle-type containing intermingled hyperbolic sets of different unstable indices. Thus the one-dimensional central direction of the porcupine is contracting for some of these sets and expanding for others. Hence, a first question is to determine the dominating behavior. Indeed, intuitively it can be seen that it topologically behaves essentially contracting, this is illustrated by the fact that the maps of the underlying IFS map the interval $[0,1]$ into itself. We will support this observation by studying the ergodic properties of relevant measures on the porcupine.

The content of this paper is the following. In Section 2 we introduce the porcupines and explain the main properties of their fiber maps. In Section 3 we derive some properties for the underlying IFS of the porcupine. These properties can be summarized as follows: every point has forward and backward iterates which are uniformly expanding. An important consequence of this fact is the minimality of the IFS. Another consequence of this fact and the cycle property of the IFS is the existence of a spectral gap. The topological properties of the porcupines (transitivity, structure of the fibers, distribution of periodic poins) are studied in Section 4. In Section 5.1 we translate the spectral properties of the IFS in Section 3.6 to properties of the central spectrum of the porcupine. Finally, in Section 5.2 we study thermodynamical properties and see how the gaps in the spectrum lead to phase transitions of the geometrical potential φ_c.

2. The dynamics of the porcupine

We now introduce the class of step skew-product maps that we will consider. Given the square $\widehat{\mathbf{C}} = [0,1]^2$ of $\mathbb{R}^2$ and a planar diffeomorphism Φ having an affine horseshoe Γ in $\widehat{\mathbf{C}}$ that is conjugate to the full shift $\sigma\colon \Sigma_i \to \Sigma_i$ ($\Sigma_i \stackrel{\text{def}}{=} \{0,\dots,i-1\}^{\mathbb{Z}}$ for $i=2$ or 3 is the symbolic space) we consider the sub-cubes $\widehat{\mathbf{C}}_i$, of $\widehat{\mathbf{C}}$ given by the "first level" rectangles defined as the connected components of $\Phi^{-1}([0,1]^2)\cap[0,1]^2$. Let $\mathbf{C}=[0,1]^3$ and $\mathbf{C}_i = \widehat{\mathbf{C}}_i \times [0,1]$ and consider the map $F\colon \mathbf{C}\to\mathbb{R}^3$ defined by

$$F(X) \stackrel{\text{def}}{=} (\Phi(\widehat{x}), f_i(x)) \quad \text{if} \quad X=(\widehat{x},x)\in \mathbf{C}_i. \tag{2.1}$$

Here the maps $f_i\colon [0,1]\to[0,1]$ are C^1 injective maps satisfying the following two basic properties:

- (F0) The map f_0 is increasing and has exactly two hyperbolic fixed points in $[0,1]$, the point $q_0=0$ (repelling) and the point $p_0=1$ (attracting). Let $\beta_0 = f_0'(0) > 1 > \lambda_0 = f_0'(1) > 0$.
- (F1) The map f_1 is an affine contraction $f_1(x) \stackrel{\text{def}}{=} \gamma\,(1-x)$ where $\gamma\in(\lambda_0,1)$. We denote by p_1 the attracting fixed point of f_1. Note that f_1 satisfies the *cycle condition* $f_1(1)=0$.

Further properties of these maps (and also of f_2 when $i=3$) are specified in Section 3.1, compare Figure 2.

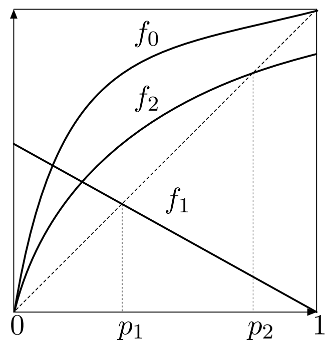

FIGURE 2. Fiber maps of the IFS

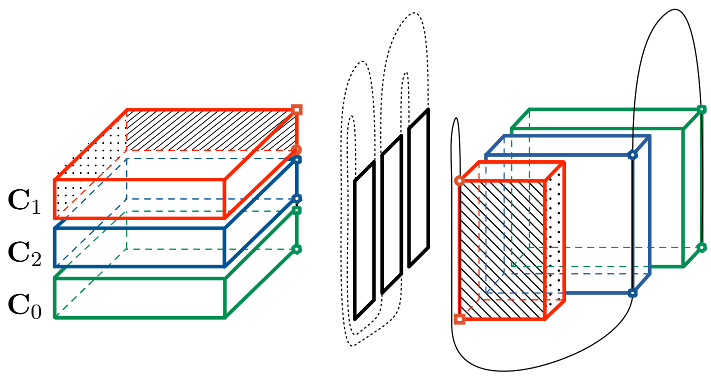

FIGURE 3. Level 1 rectangles and the dynamics of F

To complete the definition of F in $\mathbf{C}$ we will consider some appropriate C^1-continuation of F such that $F(\mathrm{int}(\mathbf{C} \setminus \bigcup_i \mathbf{C}_i)) \cap \mathbf{C} = \varnothing$.

We now briefly discuss some basic dynamical properties of the local diffeomorphism F, compare Figure 3. We focus on the dynamics of F on its maximal invariant set Λ in $\mathbf{C}$ defined by

$$\Lambda \stackrel{\text{def}}{=} \bigcap_{i \in \mathbb{Z}} F^i(\mathbf{C}). \tag{2.2}$$

By an appropriate choice of the horseshoe Γ, the set Λ is partially hyperbolic for F with central direction $E^c = \{(0^s, 0^u)\} \times \mathbb{R}$. Associated to the fixed points $\theta_0 \in \hat{\mathbf{C}}_0$ and $\theta_1 \in \hat{\mathbf{C}}_1$ of the horseshoe there are the following fixed points of Λ, $P_0 = (\theta_0, 1)$ and $P_1 = (\theta_1, p_1)$, with contracting central direction, and $Q_0 = (\theta_0, 0)$ with expanding central direction.

Conditions (F0) (implying that $f_0^i(x) \to 1$ and $f_0^{-i}(x) \to 0$ as $i \to \infty$ if $x \in (0,1)$) and (F1) (stating $f_1(1) = 0$) imply that F has a heterodimensional cycle associated to the the saddles P_0 and Q_0: the stable manifold of P_0 meets the unstable one of Q_0 and the unstable manifold of P_0 meets the stable one of Q_0, see Figure 3.

Under the additional condition (F01) the set Λ is transitive (actually it is a relative homoclinic class) where intermingled hyperbolic sets with different indices coexist. Indeed these hyperbolic sets are cyclically related. This fact generates a rich dynamics mixing hyperbolicity of different types.

This system also exhibits an interesting fiber structure. Recall that the horseshoe Γ in the base is conjugate to the shift $\sigma\colon \Sigma \to \Sigma$ (where $\Sigma = \Sigma_2$ or Σ_3, according to the case). Then there is a semiconjugation $\varpi\colon \Lambda \to \Sigma$ such that $\sigma \circ \varpi = \varpi \circ F$. We call a set $\varpi^{-1}(\xi) \subset \Lambda$ a *spine* and say that it is *non-trivial* if it contains a continuum. The spines are connected sets tangent to the central direction. We will see that the spines $\varpi^{-1}(\xi)$ are non-trivial for a uncountable dense subset of Σ and a single point for a residual subset of Σ.

Finally note that any point $X \in \Lambda$ is of the form $X = (X_\xi, x)$, $X_\xi \in \Gamma$ and $x \in [0,1]$, where $\xi = \varpi(X)$.

3. The underlying iterated function system

In this section we present a simplest case of our model and deduce some key properties of the IFS.

3.1. **The fiber maps.**

We now complete the description of the fiber maps $f_i\colon [0,1] \to [0,1]$, see Figure 2. For most topological conditions (as backward minimality and existence of forward expanding itineraries) it is enough to consider just the maps f_0 and f_1 and the properties (F0) and (F1) above and (F01) below. These properties provide the covering and expanding key properties for forward orbits in Lemma 3.2. These properties will imply the transitivity of Λ and that it is the relative homoclinic class of a saddle of unstable index two in $\mathbf{C}$.

3.1.1. *Forward expanding-like condition.* Recall the definitions of γ, λ_0, and β_0 above. We assume that f_0 and f_1 satisfy the following:

(F01) The derivative f_0' is decreasing in $[0,1]$ and satisfies

$$\gamma\, \lambda_0^3\, (1-\lambda_0)(1-\beta_0^{-1})^{-1} > 1.$$

Given $\gamma, \lambda_0 \in (0,1)$, condition (F01) holds if $\beta_0 > 1$ is sufficiently close to 1.

This condition provides a fundamental domain $J = [f_0^{-1}(b), b] \in (0,1)$ of f_0 (b close to 0) and a large N such that $f_1 \circ f_0^N(J) \subset (0,b)$, and the restriction of $f_1 \circ f_0^N$ to J is uniformly expanding. This is the key for defining expanding itineraries (see Section 3.3).

3.1.2. *Backward expanding-like condition.* In some cases we consider an extra hypotheses to get a more accurate description of the dynamics or a dynamics with a richer structure. We introduce two types of conditions. A blender-like condition implying some expansion for backward iterates is the following:

(F_B) $f_0'(x) \in (0,1)$ for all $x \in [\gamma, 1] = [f_1(0), 1]$.

3.1.3. *Lateral horseshoe condition.* To get an exposed set containing a horseshoe we need to consider the following condition:

(F2) The map f_2 is increasing and has two hyperbolic fixed points in $[0,1]$, the point $q_2 = 0$ (repelling) and the point $p_2 \in (0,1)$ (attracting). Let $\beta_2 = f_2'(0) > 1$. This map is close to f_0.

This condition implies that the set Λ contains a horseshoe Λ_{02} in $(\widehat{C}_0 \cup \widehat{C}_1) \times \{0\}$ conjugate to the full shift of two symbols. The set of central Lyapunov exponent of this horseshoe is the closed interval of extremes $\log \beta_2$ and $\log \beta_0$. This condition will imply the rich phase transitions in Section 5.2.

3.2. **Admissible sequences. Notation.** In what follows, for the IFS associated to $\{f_0, f_1, f_2\}$ we use the standard cylinder notation, given *finite* sequences $(\xi_0 \dots \xi_n)$ and $(\xi_{-m} \dots \xi_{-1})$, $\xi_i \in \{0,1,2\}$, we let

$$f_{[\xi_0 \dots \xi_n]} \stackrel{\text{def}}{=} f_{\xi_n} \circ \dots \circ f_{\xi_1} \circ f_{\xi_0} \colon [0,1] \to [0,1]$$

and

$$f_{[\xi_{-m} \dots \xi_{-1}.]} \stackrel{\text{def}}{=} (f_{\xi_{-1}} \circ \dots \circ f_{\xi_{-m}})^{-1} = (f_{[\xi_{-m} \dots \xi_{-1}]})^{-1}.$$

Note that the maps $f_{[\xi_{-m} \dots \xi_{-1}.]}$ are in general only defined on a closed subinterval of $[0,1]$. The finite sequence $(\xi_{-m} \dots \xi_{-1})$ is *admissible* for $x \in [0,1]$ if the map $f_{[\xi_{-m} \dots \xi_{-1}.]}$ is well-defined at x (i.e., $f_{[\xi_{-j} \dots \xi_{-1}.]}(x) \in [0,1]$ for all $j = 1, \dots, m-1$).

Every sequence $\xi = (\dots \xi_{-1}.\xi_0 \xi_1 \dots) \in \Sigma$ is of the form $\xi = \xi^-.\xi^+$, where $\xi^+ \in \Sigma^+$ and $\xi^- \in \Sigma^-$ ($\Sigma^\pm$ are defined in the obvious way). A one-sided infinite sequence $\xi^- = (\dots \xi_{-2}\xi_{-1}.) \in \Sigma^-$ is *admissible* for x if $(\xi_{-m} \dots \xi_{-1})$ is admissible for x for all $m \geq 1$. Note that admissibility of a sequence ξ does not depend on ξ^+. By writing (ξ, x) we always suppose that ξ^- is admissible for x.

3.3. **Expanding itineraries and covering property.** We begin by stating a simple (but important) expansion property of the IFS generated by f_0 and f_1.

Lemma 3.1 (Expansion property)**.** *Assume that conditions (F0), (F1), and (F01) hold. Then there are $\kappa > 1$ and $b > 0$ close to 0 such that for any interval $J \subset [f_0^{-2}(b), b]$ there is a number $n(J)$ (uniformly bounded) such that for every $x \in J$*

$$\big(f_1 \circ f_0^{n(J)}\big)(x) \in (0,b] \quad \text{and} \quad |(f_1 \circ f_0^{n(J)})'(x)| \geq \kappa$$

Proof. For simplicity assume linearity of f_0 close to 0 and 1. Note that there are small $b > 0$ and large n such that (recall the definitions of β_0 and λ_0)

$$f_0^n([\beta_0^{-1} b, b]) = [1-b, 1-\lambda_0\, b],$$

From the monotonicity of f_0' and $J \subset [\beta_0^{-2} t, t]$ we get that, for all $x \in J$,

$$(f_0^n)'(x) \ge \frac{\lambda_0^2 (1-\lambda_0)}{1-\beta_0^{-1}}.$$

Let $n(J) = n$ if $f_0^n(J) \subset [1-b, 1]$ and $n(J) = n+1$ otherwise. This implies that for all $x \in J$ we have

$$(f_0^{n(J)})'(x) \ge \frac{\lambda_0^3 (1-\lambda_0)}{1-\beta_0^{-1}}.$$

Thus, by (F01),

$$|(f_1 \circ f_0^{n(J)})'(x)| \ge \gamma \left(\frac{\lambda_0^3 (1-\lambda_0)}{1-\beta_0^{-1}} \right) > \kappa > 1.$$

Noting that $f_0^{n(J)}(J) \subset [1-b, 1-\lambda_0^2 b]$ we get $f_1 \circ f_0^{n(J)}(J) \subset (0, \gamma b] \subset (0, b]$, proving the lemma. □

We now deduce some consequences from the above lemma. To each interval $J \subset [f_0^{-2}(b), b]$ we associate the sequence $\xi(J) = (0^{n(J)}\, 1\, 0^{m(J)})$, where $m(J)$ is the first positive integer such that $(f_0^k \circ f_1 \circ f_0^{n(J)})(J)$ intersects $[f_0^{-1}(b), b)]$. Note that $m(J) \ge 0$ and that the restriction of $f_0^{m(J)} \circ f_1 \circ f_0^{n(J)}$ to J is uniformly expanding. Moreover, there is κ independent of J such that

$$|f_{[\xi(J)]}(J)| = |f_{[0^{n(J)} 1 0^{m(J)}]}(J)| \ge \kappa |J|.$$

We say that $\xi(J)$ is the *expanding return* of J to $[f_0^{-2}(b), b]$ and that $f_{[\xi(J)]}(J)$ is the *expanded successor* of J.

Lemma 3.2 (Covering and expanding properties)**.** *Assume that conditions (F0), (F1), and (F01) hold. Then for every interval $J \subset [f_0^{-2}(b), b]$ there is a finite sequence $\eta(J) = (0^{n_0}\, 1\, 0^{m_0} \dots 0^{n_\ell}\, 1\, 0^{m_\ell})$ such that*

1. *$f_{[\eta(J)]}$ is uniformly expanding in J,*
2. *$f_{[\eta(J)]}(J)$ contains $[f_0^{-2}(b), f_0^{-1}(b)]$.*

This above lemma follows from considering recursively the expanding returns and the expanded successors of J. Let $J = J_0$ and inductively define J_{j+1} to be the expanded successor of $J_j \subset [f_0^{-2}(b), b]$. Since

$$|J_{j+1}| \ge \kappa\, |J_j| \ge \kappa^{j+1}\, |J_0|,$$

and $\kappa > 1$ there is smallest number $\ell = \ell(J)$ such that $J_{\ell+1}$ is not contained in $[f_0^{-2}(b), b]$ and thus contains $[f_0^{-2}(b), f_0^{-1}(b)]$. Now it is enough to consider the concatenation $\eta(J)$ of the expanding returns $\xi(J_0), \xi(J_1), \dots, \xi(J_\ell)$.

An immediate consequence of the previous lemma is the following.

Corollary 3.3. *Assume that conditions (F0), (F1), and (F01) hold. Consider any closed interval $J \subset [f_0^{-2}(b) f_0^{-1}(b)]$ and its sequence $\eta(J)$. Then the map $f_{[\eta(J)]}$ has an expanding fixed point $q_J^* \in J$ whose unstable manifold $W^u(q_J^*, f_{[\eta(J)]})$ contains $[f_0^{-2}(b), f_0^{-1}(b)]$.*

An important property of the previous construction is that it *only* involves iterations in an interval that does not contain the point 0.

Remark 3.4 (Backward covering property)**.** Condition (F_B) guarantees that for every point $x \in [0,1]$

$$\max\big\{|(f_0^{-1})'(x)|, |(f_1^{-1})'(x)|\big\} \geq \varrho > 1.$$

This implies that every point $x \in [0,1]$ has an admissible negative sequence $\eta^- = (\dots, \eta_{-n} \dots \eta_{-1})$ satisfying for every n

$$|f'_{[\eta_{-n}\dots\,\eta_{-1}.]}(x)| > \varrho^n.$$

This property is used to define backward expanding successors in the same spirit as before. Arguing as in [5], we get that the pre-orbit of any open interval $J \subset [0,1]$ contains the point $(1-\gamma)$ and therefore the points $0 = f_1^{-1}(1-\gamma)$ and $1 = f_1^{-1}(0)$. In particular, since $W^u(1, f_1) \supset (0,1]$, the backward orbit of J covers the whole interval $(0,1]$. This covering property for backward iterations is similar to the one in Lemma 3.2 (for forward iterations).

3.4. **Minimality.** We now study the minimality of the IFS generated by the fiber maps. Recall that a map $f\colon X \to X$ defined on a metric space X is *minimal* if any closed proper subset Y of X with $f(Y) \subset Y$ is empty. Consider now the IFS by $\{f_0, f_1\}$, this IFS is *forward minimal* if each closed proper subset $Y \subset [0,1]$ such that $f_{[\xi_0\dots\,\xi_k]}(Y) \subset Y$ for all finite sequence $(\xi_0 \dots \xi_k)$ is empty.

Given a point $x \in [0,1]$ consider the set of all forward images of x,

$$\mathcal{O}^+(x) \stackrel{\text{def}}{=} \{f_{[\eta_0\dots\,\eta_n]}(x)\colon \eta_i \in \{0,1\},\, n \geq 0\}$$

The forward minimality of the IFS is equivalent to the density of $\mathcal{O}^+(x)$ in $[0,1]$ for every x.

We can also study backward minimality of this IFS considering the inverse maps f_0^{-1}, f_1^{-1} and a characterization using dense backward orbits. Observe that not all backward concatenations are possible and that we need to focus on the admissible ones. Consider the set of all pre-images of x under the IFS.

$$\mathcal{O}^-(x) \stackrel{\text{def}}{=} \{f_{[\eta_{-n}\dots\,\eta_{-1}.]}(x)\colon \eta^- \text{ is admissible for } x \text{ and } n \geq 1\}.$$

Note that in our case $\mathcal{O}^-(1) = \{1\}$ and $\mathcal{O}^-(0) = \{0,1\}$ and thus the IFS is not backward minimal. However, next proposition claims that these two points are the only points of exception.

Proposition 3.5 (Minimality)**.** *Assume that (F0), (F1), (F01), and (F_B) are satisfied. Then the set $\mathcal{O}^+(x)$ is dense in $[0,1]$ for every $x \in [0,1]$ and the set $\mathcal{O}^-(x)$ is dense in $[0,1]$ for every $x \in (0,1)$.*

The density of $\mathcal{O}^+(x)$ follows from Remark 3.4 that implies that for every point $x \in (0,1]$ and every open set $U \subset [0,1]$ there are a open set $V \subset U$ and a finite sequence $(\eta_{-n} \dots \eta_{-1})$ such that $x \in f_{[\eta_{-n}\dots\,\eta_{-1}.]}(V)$. Thus

$f_{[\eta_{-1}\dots\eta_{-n}]}(x) \in V$. This implies the density of $\mathcal{O}^+(x)$ for all $x \in (0,1]$. For $x = 0$ just note that $f_1(0) \in (0,1)$ and thus the forward orbit of 0 is also dense in $[0,1]$.

To get the density of the pre-orbits $\mathcal{O}^-(x)$ it is enough to see that for every open interval $J \subset [0,1]$ and every point $p \in (0,1)$ there is a sequence $(\eta_0 \dots \eta_m)$ with $p \in f_{[\eta_0\dots\eta_m]}(J)$. This is guaranteed by the next lemma.

Lemma 3.6. *Assume that (F0), (F1), and (F01) hold. Then for every $p \in (0,1)$ and every interval $J \subset (0,1)$ there is a finite sequence $(\eta_0 \dots \eta_m)$ such that*

$$f_{[\eta_0\dots\eta_m]}(J) \supset [p - \delta/2, p + \delta/2].$$

An important fact in this lemma (in order to control derivatives) is that the covering property holds in some sense uniformly: if $p \in (\Delta, 1-\Delta)$, $J \subset (\Delta, 1-\Delta)$, and $|J| \geq \tau$ then the length of the finite sequence $(\eta_0 \dots \eta_m)$ can be bounded uniformly by some number depending only on Δ and τ.

Lemma 3.6 immediately follows from Lemma 3.2 when p and J are contained in $[f_0^{-2}(b), f_0^{-1}(b)]$. In the general case, note that for ℓ sufficiently large $f_0^\ell(J)$ is close to 1 and thus $f_1 \circ f_0^\ell(J) \subset (0, f^{-2}(b))$. Considering further iterations by f_0 we get $m > 0$ such that (after shrinking J if necessary)

$$J_0 = f_0^m \circ f_1 \circ f_0^\ell(J) \subset [f^{-2}(b), f_0^{-1}(b)].$$

We can now consider forward expanding returns of J_0 to eventually cover $[f^{-2}(b), f_0^{-1}(b)]$. This provides a finite sequence $(\xi_0 \dots \xi_k)$ such that

$$[f_0^{-2}(b), f_0^{-1}(b)] \subset f_{[\xi_0\dots\xi_k]}(J).$$

As $[f_0^{-2}(b), f_0^{-1}(b)]$ is a fundamental domain of f_0 and the forward orbit of b by f_0 accumulates to 1 this implies that

$$[f_0^{-2}(b), 1) \subset \bigcup_{i\geq 0} f_{[\xi_0\dots\xi_k 0^i]}(J).$$

The cycle property $f_1(1) = 0$ and the fact that b is close to 0 imply that

$$(0,1) \subset \Big(\bigcup_{i\geq 0} f_{[\xi_0\dots\xi_k 0^i]}(J)\Big) \cup \Big(\bigcup_{i\geq 0} f_{[\xi_0\dots\xi_k 0^i 1]}(J)\Big),$$

which implies the lemma.

3.5. **Creation of periodic points.** An uniform version of Lemma 3.6 implies that given any interval $[p-\delta, p+\delta] \subset (0,1)$, small $\delta > 0$, there is a sequence $(\eta_0 \dots \eta_m)$ with $f_{[\eta_0\dots\eta_m]}([p-\delta, p+\delta]) \supset [p-\delta, p+\delta]$. This provides a fixed point q of $f_{[\eta_0\dots\eta_m]}$ close to p. If the sequence $(\eta_0 \dots \eta_m)$ mainly consists of segments corresponding to expanding returns one gets an expanding point. Much more interesting is that it is possible to choose (with some constrains, see Section 3.6) the value of $f'_{[\eta_0\dots\eta_m]}(q)$. For this one considers combinations of expanding returns (providing expansion) and consecutive iterates of f_0 (providing contraction). Of course, bounded variation of the derivatives and control of the distortion are involved in these calculations.

3.6. **Spectral gap.** We now discuss the existence of an spectral gap. First, given a point $x \in [0,1]$ and a sequence $\xi \in \Sigma$, the *upper (forward) Lyapunov exponent* of x with respect to ξ is defined by

$$\overline{\chi}^+(\xi,x) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{1}{n} \log |(f_{[\xi_0\ldots\xi_{n-1}]})'(x)|.$$

The lower exponent $\underline{\chi}^+$ is defined analogously taking lim inf instead of lim sup. We can also define backward exponents $\overline{\chi}^-$ and $\underline{\chi}^-$ in a similar way. In this case we can only consider admissible pairs (ξ, x) When the lim inf and the lim sup are equal we speak simply of forward and backward Lyapunov exponents $\chi^\pm$.

Assume, for instance, that $\Sigma = \Sigma_3$ and consider the set $\mathcal{E}$ of *exceptional points* defined by

$$\begin{aligned}(3.1)\quad \mathcal{E} \stackrel{\text{def}}{=} &\big\{(\xi,0)\colon \xi \in \{0,2\}^{\mathbb{Z}}\big\}\cup\\ &\big\{\big((0^{-\mathbb{N}}.0^k\,1\,\xi^+),1\big),\big((0^{-\mathbb{N}}\,1\,\xi_k.\xi^+),0\big)\colon k\geq 0, \xi_k \in \{0,2\}^k, \xi^+\big\}\end{aligned}$$

where $\xi^+ \in \{0,2\}^{\mathbb{N}}$. That is, the set $\mathcal{E}$ codes all points in the "full lateral horseshoe" Λ_{02} (recall Section 3.1.3) together with its stable manifold.

Let us assume that $\log\beta_2$ is close to but smaller than $\log\beta_0$. For points in the exceptional set $\mathcal{E}$ the exponents are in $[\log\beta_2, \log\beta_0]$. We claim that the points in $\Lambda \setminus \Lambda_{02}$ have exponent less than some number $\log\widetilde{\beta} < \log\beta_2$. The heuristic argument is the following. Due to monotonicity of the maps f_0, f_2 and their derivatives, the only way that a point $x \in \Lambda\setminus\Lambda_{02}$ could have a central Lyapunov exponent close to $[\log\beta_2, \log\beta_0]$ is when its orbit would stay close to Λ_{02} for a long time. By the cycle configuration, such an orbit previously visited a small neighborhood of 1 for a long time (a comparable amount of time). The latter introduces a contraction that compensates the expansion corresponding to the iterates close to Λ_{02}. Consequently the central Lyapunov exponent of these points is smaller than some number $\log\widetilde{\beta} < \log\beta_2$.

Proposition 3.7 (Spectral gap)**.** *Under hypotheses (F0), (F1), (F2), and (F01),*

$$\exp\Big(\sup\big\{\overline{\chi}^+(\xi,x)\colon (\xi,x)\notin\mathcal{E}\big\}\Big) \stackrel{\text{def}}{=} \widetilde{\beta} < \beta_2.$$

This result holds also for the case of two generators (in such a case (F2) is superfluous). Indeed, in [12] it is proved that under quite natural conditions the central spectrum has exactly one gap. More precisely we have the following.

Theorem 3.8. *Under assumptions (F0), (F1), (F01), and (F_{B}) the IFS satisfies the following: There is $0 < \widetilde{\beta} < \beta_0$ such that*

Forward spectrum:

(i) *For every $\chi \in [0, \log\widetilde{\beta}\,]$ the set of points y such that there exists $\xi^+ \in \Sigma^+$ with $\chi_+(y,\xi^+) = \chi$ is dense in $[0,1]$.*

(ii) *For every* $\chi \in [\log\lambda, 0]$ *there exists* $\xi^+ \in \Sigma^+$ *with* $\chi_+(y,\xi^+) = \chi$ *for every* $y \in [0,1]$.

Backward spectrum:

(iii) *For every* $\chi \in [0, \log\widetilde{\beta}\,]$ *and for every* $y \in (0,1)$ *there exists* $\xi^- \in \Sigma^-$ *with* $\chi_-(y,\xi^-) = \chi$.

(iv) *For every* $\chi \in [\log\lambda, 0]$ *the set of points* y *for which there exists* $\xi^- \in \Sigma^-$ *with* $\chi_-(y,\xi^-) = \chi$ *is dense in* $[0,1]$.

In view of these constructions, the problem of the existence of examples whose spectrum has several gaps raises in a natural way. In [13] there are given examples where the lateral horseshoe splits into finitely many disjoint horseshoes Λ_j coded by a subshift H_j, $j = 1, \ldots, \ell$, such that

$$\big\{\underline{\chi}(\xi,0), \overline{\chi}(\xi,0) \colon \xi \in H_\ell\big\} = [\alpha_j^-, \alpha_j^+],$$

where $\alpha_j^+ < \alpha_{j+1}^-$. This construction leads to a Lyapunov spectrum with gaps $(\alpha_\ell^+, \alpha_{\ell-1}^-), \ldots, (\alpha_2^-, \alpha_1^+)$.

4. Topological dynamics

We next discuss aspects of the topological dynamics of the porcupine horseshoes. These properties are consequence of the properties of the IFS in the previous section. In the next result we assume that the porcupine is fibered over a horseshoe conjugate to Σ_2 (i.e., we just consider the maps f_0 and f_1). A similar result can be stated for porcupines fibered over Σ_3.

Theorem 4.1. *Assume that conditions (F0), (F1), and (F01) hold and let* F *be the associated skew-product diffeomorphism. Then the maximal invariant set* Λ *of* F *in* $\mathbf{C}$*, see* (2.2)*, satisfies the following properties:*

- *The set* Λ *is the relative homoclinic class in* $\mathbf{C}$ *of a saddle of unstable index two. In particular,* Λ *is transitive and the set of periodic points of unstable index two is dense in* Λ*.*
- *The set of periodic points of unstable index one is dense in* Λ*.*
- *There is an uncountable and dense subset of sequences* $\xi \in \Sigma$ *such that* $\varpi^{-1}(\xi)$ *is non-trivial. The set of non-trivial spines is dense in* Λ*.*
- *There is a residual subset of sequences* $\xi \in \Sigma$ *such that* $\varpi^{-1}(\xi)$ *is trivial. The set of trivial spines is dense in* Λ*.*

4.1. **Transitivity.** We now sketch the proof of the transitivity of Λ. Let $q^* = q_J^*$ be the expanding point of $f_{[\eta(J)]}$ in Corollary 3.3. Consider the periodic sequence $(\eta(J))^{\mathbb{Z}}$ and the periodic point Q^* of F associated to it whose central coordinate is q^*. This point has unstable index two. We see that Λ is the homoclinic class of Q^* relative to $\mathbf{C}$.

Condition $[f_0^{-2}(b), f_0^{-1}(b)] \subset W^u(q^*, f_{[\eta(J)]})$ implies that the unstable manifold of the orbit of Q^* transversely intersects every stable segment of the form $[0,1] \times \{(x^u, x)\}$ with $x \in (0,1)$. Similarly, the stable manifold

$W^s(Q^*, F)$ of the orbit of Q^* transversely intersects any center-unstable disk $\{x^s\} \times [0,1] \times J$ such that $J \subset (0,1)$ is a fundamental domain of f_0.

An *inner point* of Λ is a point $X = (x^s, x^u, x) \in \Lambda$ such that for all $i \in \mathbb{Z}$ the central central coordinate x_i of $F^i(X) = (x^s_i, x^u_i, x_i)$ belongs to $(0,1)$. Let us just explain why inner points are in in the (relative) homoclinic class of Q^*. This is the main step of the proof of the transitivity of Λ.

Given any small $\delta > 0$ consider the stable segment centered at X,

$$\Delta^s_\delta(X) \stackrel{\text{def}}{=} [x^s - \delta, x^s + \delta] \times \{(x^u, x)\}.$$

Using that F^{-1} uniformly expands the x^s-direction, we get some n such that

$$F^{-n}(\Delta^s_\delta(X)) \supset [0,1] \times \{(x^u_n, x_n)\}.$$

By the previous comments, $\Delta^s_\delta(X)$ transversely intersects the unstable manifold $W^u(Q^*, F)$ of the orbit of Q^*. Let

$$X_\delta \stackrel{\text{def}}{=} (x^s_\delta, x^u, x) \in \Delta^s_\delta(X) \pitchfork W^u(Q^*, F)$$

and for small $\varepsilon > 0$ consider the center-unstable rectangle

$$\Delta^{cu}_\varepsilon(X_\delta) \stackrel{\text{def}}{=} \{x^s_\delta\} \times [x^u - \varepsilon, x^u + \varepsilon] \times [x - \varepsilon, x + \varepsilon] \subset W^u(Q^*, F).$$

We claim that $\Delta^{cu}_\varepsilon(X_\delta)$ intersects $W^s(Q^*, F)$ transversely. This immediately implies $\Delta^{cu}_\varepsilon(X(\delta))$ contains a transverse homoclinic Y point of Q^*. As δ and ε can be chosen arbitrarily small, the point Y can be taken arbitrarily close to X. Moreover, since we just consider iterations in $\mathbf{C}$, the whole orbit of Y is contained in $\mathbf{C}$. Thus Y is a transverse homoclinic point relative to $\mathbf{C}$ and hence X is in the relative homoclinic class of Q^*.

To prove our claim, $\Delta^{cu}_\varepsilon(X_\delta) \pitchfork W^s(Q^*, F)$, note first that (due to uniform expansion in the x^u-direction) after positive iterations by F of $\Delta^{cu}_\varepsilon(X(\delta))$ one gets a "large disk" of the form $\{y^s\} \times [0,1] \times I$, where I is an interval in $(0,1)$. Using the heterodimensional cycle associated to P_0 and Q_0, after further forward iterates, one gets a disk $\{z^s\} \times [0,1] \times J$, where $J \subset (f_0^{-2}(b), b)$. We can now apply Lemma 3.2 to the interval J to get a finite sequence $(\xi_0 \dots \xi_m)$ such that $f_{[\xi_0 \dots \xi_m]}(J)$ covers a fundamental domain D of f_0 in $(0,1)$. Putting together these comments we get large N such that $F^N(\Delta^{cu}_\varepsilon(X(\delta)))$ contains a center-unstable rectangle of the form $\{x^s\} \times [0,1] \times D$. As D is a fundamental domain of f_0 the comments above imply that $W^s(Q^*, F)$ transversely meets $\Delta^{cu}_\varepsilon(X(\delta))$.

This type of arguments can be used to prove that every pair of interior saddles of unstable index two are homoclinically related. Note that the interior assumption is essential: the exposed saddle Q_0 of unstable index two is not related to other saddles.

Finally, let us observe that if condition (F_B) is satisfied the same kind or arguments implies that Λ is the relative homoclinic class of P_0 (indeed of any saddle of unstable index one in Λ).

4.2. **Structure of the spines.** First observe that any spine containing a saddle of unstable index two (thus expanding in the central direction) is non-trivial. This immediately follows from $f_i([0,1]) \subset [0,1]$, that implies that the extremes of a periodic spine are either attracting or neutral in the central direction. As the periodic points of unstable index two are dense in Λ, the set of non-trivial spines is also dense in Λ.

To describe the spines, given a sequence $\xi = \xi^- \, \xi^+$, $\xi^- = (\ldots \xi_{-m} \ldots \xi_{-1})$ we define the pre-spine of m-th generation by $I_{[\xi_{-m} \cdots \, \xi_{-1} .]}$ as the domain of definition of $f_{[\xi_{-m} \cdots \, \xi_{-1} .]}$. These sets are non-trivial sub-intervals of $[0,1]$ forming a decreasing nested sequence. Note that if $-m-1 = 0$ then $I_{[\xi_{-m-1} \xi_{-m} \cdots \, \xi_{-1} .]} = I_{[\xi_{-m} \cdots \, \xi_{-1} .]}$. We define

$$I_{[\xi]} \stackrel{\text{def}}{=} \bigcap_{m \geq 1} = I_{[\xi_{-m} \cdots \, \xi_{-1} .]}.$$

Note that the positive of a tail of ξ plays no role in the definition of $I_{[\xi]}$. By construction, we obtain the spine $\varpi^{-1}(\xi) = \{X_\xi\} \times I_{[\xi]}$.

Note that for each m the map $\xi \mapsto |I_{[\xi_{-m} \cdots \, \xi_{-1} .]}|$ is continuous. This implies that the map $\xi \mapsto |I_{[\xi]}|$ is upper semi-continuous. Also it is not difficult to see that the set of ξ whose spines have length less than $1/n$ is open and dense in Σ. These two facts immediately implies that the set of sequences with trivial spines is a residual subset of Σ.

Roughly speaking, to get a trivial spine it is enough to consider sequences whose negative tail contains long sequences of consecutive 1's, that corresponds to backward expansions. This implies that after some negative iterations a significant part of the m-th spine will be mapped outside $[0,1]$ and thus be removed. Conversely, to get non-trivial spines one considers sequences whose negative part contains a huge amount of $0's$.

5. Ergodic properties

5.1. **Central spectrum. Lyapunov exponents.** Note that the porcupine Λ is partially hyperbolic with a dominated splitting $E^s \oplus E^c \oplus E^u$, where E^c is parallel to $\{0^2\} \times \mathbb{R}$. In particular, for every Lyapunov regular point $R \in \Lambda$ this splitting coincides with the Oseledec one provided by the multiplicative ergodic theorem. Thus the *Lyapunov exponent associated to the central direction* E^c at such regular point R is well-defined and equal to the Birkhoff average of the continuous map $R \mapsto \log \|dF|_{E^c_R}\|$,

$$\chi_c(R) \stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{n} \log \|dF^n|_{E^c_R}\| = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \|dF|_{E^c_{F^k(R)}}\|.$$

Write $R = (r^s, r^u, r) \in \Lambda$ and consider the sequence $\xi = (\ldots \xi_{-1} . \xi_0 \xi_1 \ldots) = \pi(R)$. Since the central dynamics is determined by the fiber IFS we have that

$$\chi_c(R) = \lim_{n\to\infty} \frac{1}{n} \log |(f_{[\xi_0 \ldots \, \xi_{n-1}]})'(r)|.$$

This formula estates the relation between the Lyapunov exponents of the IFS studied in Section 3.6 and the central Lyapunov exponents of the porcupine.

Finally, observe that the remaining exponents of the porcupine are associated to the stable and the unstable hyperbolic directions, thus they are uniformly bounded away from zero.

Let us observe that as a consequence of our results and the methods for constructing non-hyperbolic ergodic measures with large support introduced in [18] and further developed in [4] one can construct ergodic measures with full support in the porcupine having a central exponent in $[\log \lambda_0, \log \widetilde{\beta}\,)$.

5.2. **Thermodynamic formalism.** Let us finally explain the thermodynamic consequences of the spectral gaps. Given a continuous function $\varphi\colon \Lambda \to \mathbb{R}$, the topological pressure $P(\varphi)$ of φ is defined in purely topological terms, but can be expressed in measure-theoretic terms via the variational principle [33]

$$P(\varphi) = \sup_{\mu \in \mathcal{M}(\Lambda)} \Big(h(\mu) + \int \varphi \, d\mu \Big), \tag{5.1}$$

where $h(\mu)$ denotes the entropy of the measure μ and the supremum is considered in the set $\mathcal{M}(\Lambda)$ of ergodic F-invariant measures supported on Λ. A measure μ is called *equilibrium state* if it attains the supremum in (5.1). By [9, 33] such states exist.

Non-differentiability of P is closely related to the coexistence of equilibrium states. The question of existence and uniqueness of equilibrium states has been studied in several contexts already. In [32] in the context of uniformly expanding C^2 circle maps uniqueness was derived for any φ satisfying $P(\varphi) > \sup \varphi$. This condition is essential.

It is an immediate consequence of (5.1) that for every $t \in \mathbb{R}$ we have $P(t\varphi_c) \geq \sup(t\varphi_c)$. Observe that to each measure μ the graph of the map $t \mapsto h(\mu) - t \int \varphi_c \, d\mu$ gives a line below the graph of the function $P(t) \stackrel{\text{def}}{=} P(t\varphi_c)$. Hence, by (5.1) the function $P(t)$ is convex. Recall that $P(t)$ is real analytic if Λ is a basic set [30]. The pressure function $P(t)$ exhibits a *phase transition* at a characteristic parameter t_* if it fails to be real analytic at t_*.

In the case that $\sup(-\varphi_c)$ is attained in a single point $R \in \Lambda$ (which is the case in one of our models, where R is the exposed saddle) then exactly one of the following two cases is true: either $P(t)$ converges to the line $t \mapsto t\varphi_c(R)$ as $t \to -\infty$ (in which case there would be ergodic measures with central exponent converging to $-\varphi_c(R)$) or we have $P(t) = t\varphi_c(R)$ for every $t \leq t_*$ for some $t_* < 0$ (in which case $P(t)$ is not analytic at t_*). As we showed that there is a gap in the central Lyapunov spectrum, we are in the second case. In fact, it implies that there is a *first order phase transition* at t_*, that is, $P(t)$ fails to be differentiable at t_* and there exist (at least two) ergodic equilibrium states with exponents equal to the negative of the left/right derivatives at t_*. We call this transition *rich* if there exist (at least

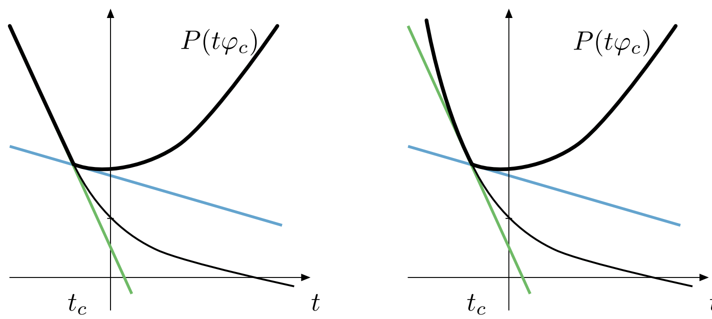

FIGURE 4. First order and rich first order phase transitions

two) equilibrium states with positive entropy. This is exactly what happens in the case when the exposed piece is a horseshoe.

Two different types of phase transitions we obtain are depicted in Figure 4 (compare also the examples of different types of exposed pieces discussed in Section 1.2).

Studying in [11] extensions of Bernoulli measures in the lateral horseshoe, it can be concluded that the transitive set containing intermingled hyperbolic sets of different indices is in a sense essentially contracting: the part with non-positive central exponent carries larger entropy. The right derivative of the pressure function at $t = 0$ is non-negative (there is a measure of maximal entropy with non-positive central Lyapunov exponent). This has as a consequence that the maximal entropy that can be obtained from an embedded hyperbolic horseshoe with unstable index 2 is smaller than the one obtained from horseshoes with unstable index 1.

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