# PEANO CURVES WITH SMOOTH FOOTPRINTS

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ABSTRACT. We construct Peano curves $\gamma : [0,\infty) \to \mathbb{R}^2$ whose "footprints" $\gamma([0,t])$, $t > 0$, have $C^\infty$ boundaries and are tangent to a common continuous line field on the punctured plane $\mathbb{R}^2 \smallsetminus \{\gamma(0)\}$. Moreover, these boundaries can be taken $C^\infty$-close to any prescribed smooth family of nested smooth Jordan curves contracting to a point.

## 1. Introduction

A continuous map $\gamma : I \to \mathbb{R}^2$ defined on a nondegenerate interval $I \subseteq \mathbb{R}$ is called a *Peano curve* if its image has nonempty interior. A lot of water has run under the bridge since Peano established the existence of such curves in 1890. Many interesting problems concerning such curves are discussed in the book [7] by H. Sagan.

By Sard's theorem, Peano curves are non-differentiable. Nevertheless, they can have smooth "footprints", as a consequence of our main result:

**Theorem.** *There exist a Peano curve $\gamma : [0,\infty) \to \mathbb{R}^2$ and a continuous line field $\Lambda$ on the punctured plane $\mathbb{R}^2 \smallsetminus \{\gamma(0)\}$ such that for every $t > 0$, the boundary of the set $\gamma([0,t])$ is a $C^\infty$ curve $C_t$ containing the point $\gamma(t)$ and tangent to the line field $\Lambda$ at each point.*

*Moreover, it is possible to choose the Peano curve $\gamma$ so that each curve $C_t$ is $C^\infty$-close to the circle $x^2 + y^2 = t^2$.*

Let us state the "moreover" part formally: Given any upper semicontinuous function $k : (0,\infty) \to \mathbb{N}$ and any lower semicontinuous function $\varepsilon : (0,\infty) \to (0,\infty)$, we can choose the Peano curve $\gamma$ in the theorem with the following additional property: for each $t > 0$ the curve $C_t = \partial\gamma([0,t])$ is the image of a $C^\infty$ embedding $\beta_t$ of the circle $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ into $\mathbb{R}^2$ such that

$$\|\beta_t - \alpha_t\|_{k(t)} < \varepsilon(t), \tag{1.1}$$

where $\alpha_t : \mathbb{T} \to \mathbb{R}^2$ is the embedding $\theta \mapsto (t\cos\theta, t\sin\theta)$ and $\|\cdot\|_k$ is the usual $C^k$ norm; see § 2.1 for details.

Taking $k \equiv 2$ and a sufficiently small function $\varepsilon$, we can ensure that each curve $C_t$ has everywhere nonzero curvature, and so we obtain:

**Corollary.** *There exist Peano curves $\gamma : [0,1] \to \mathbb{R}^2$ such that each set $\gamma([0,t])$ is convex.*

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*Date*: July 22, 2014.

2010 *Mathematics Subject Classification.* 26A30; 26E10.

The first named author is partially supported by project Fondecyt 1140202 and project Anillo ACT1103 (Chile). The second named author is supported by FAPERJ (Brazil).

This result was first obtained by Pach and Rogers in [6], and independently by Vince and Wilson in [10]. It is inspired by the following question, attributed by Pach and Rogers to M. Mihalik and A. Wieczorek (see also [3, Problem A.37]):

*Question.* Is there a Peano curve $\gamma : I \to \mathbb{R}^2$ such that the image $\gamma(J)$ of each subinterval $J \subseteq I$ is a convex set?

To our knowledge, this question remains open. See [9] and [8, Chapter 6] for related information.

Coming back to our theorem, let us observe that the family of concentric circles can be replaced by an arbitrary smooth family of nested smooth Jordan curves contracting to a point. Indeed, it suffices to change coordinates by a suitable diffeomorphism of $\mathbb{R}^2$.

Despite the fact that the boundaries of the "footprints" $\gamma([0,t])$ are smooth, the line field $\Lambda$ is not. Indeed, $\Lambda$ is not even locally Lipschitz, because otherwise it would be uniquely integrable. It is also known that generic (in the sense of Baire) continuous line fields are uniquely integrable (see [2, pp. 121–123]), which shows that $\Lambda$ is quite pathological. Other highly non-uniquely integrable line fields are constructed in [1]; these are tangent to uncountably many $C^k$-foliations (where $1 \leqslant k < \infty$) and have the additional property of being Hölder continuous. It would be interesting to find what the optimal moduli of regularity of the line field $\Lambda$ and of the Peano curve $\gamma$ in our Theorem are – in particular, it is not clear whether they can be taken locally Hölder continuous on $\mathbb{R}^2 \smallsetminus \{\gamma(0)\}$ and $(0, \infty)$, respectively. Let us remark that the optimal Hölder coefficient of a general Peano curve is 1/2 (see e.g. [4, Prop. 2.3]).

It seems that it should be possible to extend the theorem to an arbitrary dimension $n \geqslant 2$, so that $\partial\gamma([0,t])$ is a $C^\infty$ hypersurface $C^\infty$-close to a sphere, and it is tangent to a continuous field of hyperplanes. Such construction should follow the same ideas of the $n = 2$ case, but since it would be considerably more technical, we will not dwell on it.

This paper is divided into two parts: the longer part, Section 2, is devoted to the proof of a local, more flexible version of the theorem, namely Proposition 2.3. In the shorter part, Section 3, we "glue" these local constructions in order to prove the theorem.

This paper is based on the master's dissertation [5] of the second named author, which is, in turn, inspired by ideas from [6, 10]. We thank the dissertation committee, especially Prof. Ricardo Sá Earp who posed questions that led to the improvement of the results of the dissertation, presented here.

## 2. Local construction

2.1. **Initial definitions and statement of the main proposition.** The aim of this section is to prove Proposition 2.3 below, which constructs special Peano curves whose footprints are *lunes*, objects that are defined as follows:

**Definition 2.1.** Let $f, g : [a,b] \to \mathbb{R}$ be $C^\infty$ functions such that:

(i) $f$ and $g$ and each of their derivatives coincide in $a$ and in $b$, that is, $f^{(k)}(a) = g^{(k)}(a)$ and $f^{(k)}(b) = g^{(k)}(b)$ for all integer $k \geqslant 0$;

(ii) $f(x) \leqslant g(x)$, for all $x \in [a,b]$.

The plane region

$$L = L(f, g) = \{(x, y) \mid a \leqslant x \leqslant b,\ f(x) \leqslant y \leqslant g(x)\}$$

is called a *lune* with *domain* $[a, b]$. The *support* of a lune $L(f, g)$ is the (open) set

$$S_L = \{x \in [a, b] \mid f(x) < g(x)\}.$$

A lune $L$ is said to be *simple* if its support is a nonempty interval. The *essential part* of the lune $L$ is defined as the closure of the interior of $L$:

$$L^{\text{ess}} = L^{\text{ess}}(f, g) = \overline{\text{int}(L)}\,.$$

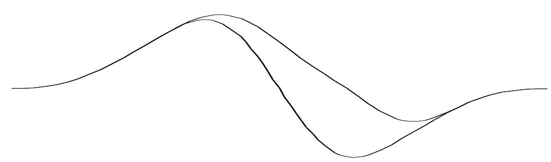


FIGURE 1. A typical lune in the construction.

**Lemma 2.2.** *For a simple lune $L$ whose support is $S_L = (c, d)$, we have*

$$L^{\text{ess}} = \{(x, y) \in L \mid c \leqslant x \leqslant d\}.$$

*In particular, $L^{\text{ess}}$ is itself a lune, and it is simple.*

*Proof.* If $L = L(f, g)$ has support $S_L = (c, d)$ then its interior is clearly

$$\{(x, y) \mid c < x < d, f(x) < y < g(x)\},$$

so that $\{(x, y) \in L \mid c \leqslant x \leqslant d\} = \overline{\text{int}(L)} = L^{\text{ess}}$. □

We let $C^\infty([a, b])$ denote the set of all $C^\infty$ functions $F : [a, b] \to \mathbb{R}$. The $C^0$ *norm* of $F \in C^\infty([a, b])$ is $\|F\|_0 = \sup_{x\in[a,b]} |F(x)|$. For $k \in \mathbb{N}$, the $C^k$ *norm* of $F$ is

$$\|F\|_k = \max\Big(\|F\|_0, \|F'\|_0, \ldots, \|F^{(k)}\|_0\Big)\,.$$

A *basic neighborhood* of $F$ is a set of the form

$$N(F, k, \varepsilon) = \big\{G \in C^\infty([a, b]) \mid \|G - F\|_k < \varepsilon\big\}.$$

We endow the space $C^\infty([a, b])$ with the topology generated by the basic neighborhoods, called the $C^\infty$ *topology*.

Later on we will work with the space $C^\infty(\mathbb{T})$ of functions on the circle $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, which can be considered as $2\pi$-periodic functions on the line. The $C^k$ norms and the $C^\infty$ topology on this space are defined analogously.

If $F, G \in C^\infty([a, b])$ (or $C^\infty(\mathbb{T})$) are such that $F(x) \leqslant G(x)$ for all $x$, we write $F \leqslant G$.

We now state our main technical proposition:

**Proposition 2.3.** *Let $L = L(f, g)$ be a simple lune defined on an interval $[a, b]$. Then there exist:*

- *a Peano curve $\gamma : [0, 1] \to L$;*
- *a continuous map $t \in [0, 1] \mapsto F_t \in C^\infty([a, b])$;*
- *a continuous function $\psi : L \to \mathbb{R}$;*

*with the following properties:*

(i) $F_0 = f$, $F_1 = g$;
(ii) *If* $t \leqslant s$ *then* $F_t \leqslant F_s$;
(iii) *Writing* $\gamma(t) = (x(t), y(t))$, *we have* $y(t) = F_t(x(t))$.
(iv) $F_t'(x) = \psi(x, F_t(x))$;
(v) *for each* $t \in (0,1]$ *we have* $\gamma([0,t]) = L^{\mathrm{ess}}(f, F_t)$;
(vi) $\gamma(0) = (a, f(a))$, $\gamma(1) = (b, f(b))$.

Due to property (iii), the functions $F_t$ are called *ceiling functions*. By property (iv), their graphs are tangent to the line field $\Lambda(x,y)$ spanned by the vector field $(1, \psi(x,y))$. Note that for each $t \in (0,1]$, the point $\gamma(t)$ belongs to the boundary of $\gamma([0,t])$. Also, this boundary is everywhere tangent to the line field $\Lambda$, except for the two extreme points where it is not differentiable. Finally, note that $\gamma([0,1]) = L^{\mathrm{ess}}$.

Our construction actually yields simple lunes $L^{\mathrm{ess}}(f, F_t)$ for all $t \in (0,1]$, but since this fact is not needed we will not justify it.

2.2. **Lune subdivision processes.** The proof of Proposition 2.3 involves a limiting process on a sequence of subdivisions of the original lune. The basic subdivision processes on a lune are described here.

Throughout the remainder of this section, fix a $C^\infty$ function $\varphi : \mathbb{R} \to \mathbb{R}$ with the following properties:

(i) $0 \leqslant \varphi(x) \leqslant 1$ for all $x \in \mathbb{R}$;
(ii) $\varphi^{-1}(0) = (-\infty, 1/3]$;
(iii) $\varphi^{-1}(1) = [2/3, \infty)$.

For any $a, b \in \mathbb{R}, a < b$, let $\varphi_{a,b}(x) = \varphi\left(\frac{x-a}{b-a}\right)$. Notice that $\varphi_{0,1} = \varphi$, $\varphi_{a,b}^{-1}(0) = \left(-\infty, \frac{2a+b}{3}\right]$ and $\varphi_{a,b}^{-1}(1) = \left[\frac{a+2b}{3}, +\infty\right)$. Also, clearly

$$\sup_{x \in \mathbb{R}} |\varphi_{a,b}^{(k)}(x)| = \sup_{x \in [a,b]} |\varphi_{a,b}^{(k)}(x)|,$$

thus $\|\varphi_{a,b}\|_k$ is defined for all $k$ by taking the $C^k$ norm of $\varphi_{a,b}$ restricted to $[a,b]$. Moreover, since $\varphi_{a,b}^{(k)}(x) = \frac{1}{(b-a)^k} \cdot \varphi^{(k)}\left(\frac{x-a}{b-a}\right)$, it follows that

$$\|\varphi_{a,b}\|_k = \max_{i \leqslant k} \frac{\|\varphi^{(i)}\|_0}{(b-a)^i}.$$

Thus,

$$\|\varphi_{a,b}\|_k \leqslant \max(1, \|b-a\|^{-k}) \|\varphi\|_k.$$

The $C^k$ norm of a lune $L = L(f,g)$ is defined as $\|L\|_k = \|g - f\|_k$.

We have now set the stage for the definitions of the two basic subdivision processes:

**Definition 2.4** (Slicing)**.** Let $L = L(f,g)$ be a simple lune, and let $n \in \mathbb{N}$. For $j = 0, 1, \dots, n$ set $h_j = \left(1 - \frac{j}{n}\right) f + \frac{j}{n} g$, and for $i = 1, 2, \dots, n$ set $L_i = L(h_{i-1}, h_i)$. The set of lunes $\{L_1, L_2, \dots, L_n\}$ is called the *n-slicing* of $L$.

**Proposition 2.5.** *The* $n$*-slicing* $\{L_1, \dots, L_n\}$ *of a simple lune* $L$ *has the following properties:*

(a) $S_{L_i} = S_L$ *and, in particular,* $L_i$ *is a simple lune, for each* $i \in \{1, 2, \dots, n\}$;

(b) $L = L_1 \cup L_2 \cup \ldots \cup L_n$ *and* $L^{\text{ess}} = L_1^{\text{ess}} \cup L_2^{\text{ess}} \cup \ldots \cup L_n^{\text{ess}}$*;*
(c) $\|L_i\|_k = \frac{1}{n}\|L\|_k$ *for each* $i \in \{1, \ldots, n\}$*, and each* $k \in \mathbb{N}$*.*

*Proof.* Properties (a) and (b) are straightforward from the definition. Property (c) follows from the fact that $\|L_i\|_k = \|h_{i+1} - h_i\|_k = \frac{1}{n}\|g - f\|_k$. □

The second basic subdivision process is defined as follows:

**Definition 2.6** (Bipartition)**.** Let $L = L(f, g)$ be a simple lune defined on $[0, 1]$ with $S_L = (a, b)$, and set $h(x) = (1 - \varphi_{a,b}(x))g(x) + \varphi_{a,b}(x)f(x)$ for $0 \leqslant x \leqslant 1$. The pair of lunes $\{L(f, h), L(h, g)\}$ is called the *bipartition* of $L$.

**Proposition 2.7.** *The bipartition* $\{L_1, L_2\}$ *of a lune* $L$ *has the following properties:*
(a) $S_{L_1} = \left(a, \frac{a+2b}{3}\right)$ *and* $S_{L_2} = \left(\frac{2a+b}{3}, b\right)$*; in particular,* $L_1$ *and* $L_2$ *are simple lunes;*
(b) $L = L_1 \cup L_2$ *and* $L^{\text{ess}} = L_1^{\text{ess}} \cup L_2^{\text{ess}}$*;*
(c) $\max(\|L_1\|_k, \|L_2\|_k) \leqslant 2^k \|\varphi_{a,b}\|_k \|L\|_k$*,* $\forall k \in \mathbb{N}$*.*

*Proof.* Properties (a) and (b) are straightforward from the definition. As for (c), notice that

$$h^{(k)}(x) = g^{(k)}(x) + \sum_{i=0}^{k} \binom{k}{i} \varphi_{a,b}^{(k-i)}(x)(f - g)^{(i)}(x),$$

so that

$$\|h^{(k)} - g^{(k)}\|_0 \leqslant 2^k \|\varphi_{a,b}\|_k \|f - g\|_k$$

and

$$\|L_2\|_k \leqslant \max_{i \leqslant k}(2^i \|\varphi_{a,b}\|_i \|L\|_i) = 2^k \|\varphi_{a,b}\|_k \|L\|_k\,.$$

The estimate of $\|L_1\|_k$ is analogous. □

2.3. **A family of lunes.** Let us begin the proof of Proposition 2.3. Let a simple lune $L = L(f, g)$ with domain $[a, b]$ be given. Clearly it is sufficient to consider the case where $S_L = (a, b)$. By rescaling if necessary, we can assume that $a = 0$ and $b = 1$. The functions $f$ and $g$ will be fixed for the remainder of this section.

Our recursive construction is based on the following definition:

**Definition 2.8.** Let $\{m_j\}_{j \geqslant 1}$ be a sequence of positive integers. The *set of words* with respect to the sequence $\{m_j\}$ is the set

$$\Omega = \{\omega = (i_1, i_2, \ldots, i_n) \mid n \in \mathbb{N},\ i_j \in \{1, 2, \ldots, m_j\} \text{ for } j = 1, 2, \ldots, n\},$$

and its elements are called *words with respect to* $\{m_j\}_{j \geqslant 1}$, or simply *words.* The length of a word $\omega = (i_1, \ldots, i_n)$ is denoted as $|\omega|$ and equals $n$.

*The set of words with successor* with respect to $\{m_j\}$ is

$$\Omega^* = \{\omega = (i_1, i_2, \ldots, i_n) \in \Omega \mid i_n < m_n\}.$$

The *successor* of a word $\omega = (i_1, \ldots, i_n) \in \Omega^*$ is the word $\omega^+ = (i_1, \ldots, i_{n-1}, i_n + 1)$.

Finally, if $\omega_0 = (i_1, \ldots, i_k)$ and $\omega_1 = (j_1, \ldots, j_n)$ are two words, the word

$$\omega_0 * \omega_1 = (i_1, \ldots, i_k, j_1, \ldots j_n)$$

of length $k + n$ is the *concatenation* of $\omega_0$ and $\omega_1$.

Our first goal is to recursively define both a sequence $\{m_j\}_{j\geqslant 1}$ of integers and a family $\{L_\omega\}$ of lunes indexed by words with respect to this sequence.

Let us fix a sequence of positive real numbers $\{\varepsilon_n\}_{n\geqslant 2}$ such that

$$\sum \varepsilon_n < \infty\,.$$

Let $m_1 = 1$ and $L_{(1)} = L$. For $k \geqslant 2$:

Step 1. Assume we know the values of $m_1, m_2 \ldots, m_{k-1}$ and that $L_\omega$ is defined for every word $\omega$ of length $k-1$ with respect to the sequence $\{m_j\}_{j\geqslant 1}$ (although this sequence is not yet fully defined, the set of words of length $k-1$ depends only on the $k-1$ first integers in the sequence).

Step 2. For every word $\omega$ of length $k-1$, pick $n_\omega$ sufficiently large such that the lunes generated in the $n_\omega$-slicing of $L_\omega$, $\left\{L_\omega^1, L_\omega^2, \ldots, L_\omega^{n_\omega}\right\}$, satisfy:

$$\|L_\omega^j\|_k < \frac{\varepsilon_k}{2^k \|\varphi_{a,b}\|_k}, \quad j = 1, \ldots, n_\omega, \tag{2.1}$$

where $S_{L_\omega} = (a,b)$ (property (c) in Proposition 2.5 allows us to pick such an $n_\omega$).

Step 3. Set $n_k = \max\{n_\omega \mid \omega \in \Omega, |\omega| = k-1\}$ and $m_k = 2n_k$. Notice that, for every word $\omega$ of length $k-1$, the lunes generated in the $n_k$-slicing of $L_\omega$ (which is finer than the $n_\omega$-slicing) satisfy (2.1).

Step 4. For every $\omega$ of length $k-1$, consider the lunes $\left\{L_\omega^1, \ldots, L_\omega^{n_k}\right\}$ generated in the $n_k$-slicing of $L_\omega$. For $j = 1, 2, \ldots, n_k$, let $\{L_\omega^{j,1}, L_\omega^{j,2}\}$ be the bipartition of $L_\omega^j$, and set

$$L_{\omega*(2j-1)} = L_\omega^{j,1} \text{ and } L_{\omega*(2j)} = L_\omega^{j,2}$$

Step 5. We have defined $L_{\omega'}$ for every word $\omega'$ of length $k$. Moreover, by Proposition 2.7 and by inequality (2.1), $\|L_{\omega'}\|_k \leqslant \varepsilon_k$ whenever $|\omega'| = k$.

This subdivision process is illustrated by Figure 2. As a result of this construction, we obtain both a sequence $\{m_j\}_{j\geqslant 1}$ and a family of lunes $\{L_\omega\}_{\omega\in\Omega}$ indexed by words with respect to the aforementioned sequence. In what follows, let $L_\omega = L(f_\omega, g_\omega)$ for all $\omega \in \Omega$.

*Remark* 2.9. The following properties hold for the family $\{L_\omega\}_{\omega\in\Omega}$ of lunes:

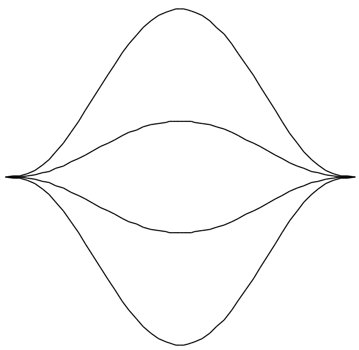
(a) $\|L_\omega\|_{|\omega|} \leqslant \varepsilon_{|\omega|}$;
(b) $f_{(1)} = f$, $g_{(1)} = g$;
(c) $f_{\omega*(1)} = f_\omega$, $g_{\omega*(m_{|\omega|+1})} = g_\omega$;
(d) $g_{\omega*(i)} = f_{\omega*(i+1)}$ for $i = 1, \ldots, m_{|\omega|+1} - 1$. In other words, $g_\omega = f_{\omega^+}$ for all $\omega \in \Omega^*$.

**Lemma 2.10.** *If a word $\omega \in \Omega$ has length $n$, then for each $1 \leqslant \ell \leqslant m_n$,*

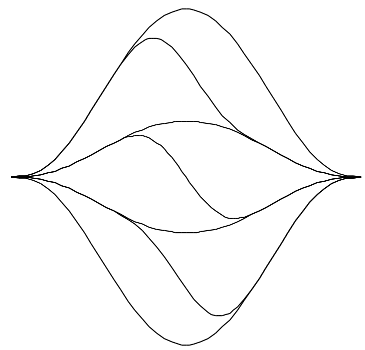
$$\|f_{\omega*(\ell)} - f_\omega\|_n < \varepsilon_{n+1} + \varepsilon_n$$

*Proof.* If $\ell$ is odd, then $f_{\omega*(\ell)}$ appeared after a slicing of $L_\omega$, hence

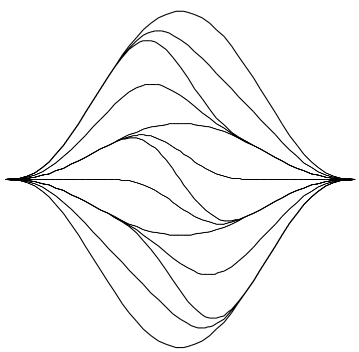
$$\|f_{\omega*(\ell)} - f_\omega\|_n \leqslant \|L_\omega\|_n < \varepsilon_n.$$



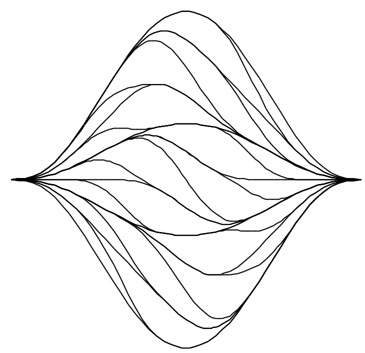
(A) A lune is first 3-sliced (here, schematically, $n_2 = 3$).



(B) Each slice is bipartitioned, completing the recursion for $k = 2$.



(C) Then, assuming $n_3 = 2$, each of the previous lunes is 2-sliced.



(D) Once again, the slices are bipartitioned, and the recursion with $k = 3$ is complete.

FIGURE 2. Illustration of two recursion steps for the subdivision of a lune. At the end of the recursion step for $k = 2$, we have a total of 6 lunes, $L_{(1,1)}, L_{(1,2)}, \ldots, L_{(1,6)}$. On the other hand, at the end of the recursion for $k = 3$ we have a total of 24 lunes, ranging from $L_{(1,1,1)}$ up to $L_{(1,6,4)}$; notice that $L_{(1,j,k)}$ is always the result of a subdivision of $L_{(1,j)}$.

If $\ell$ is even, then

$$\begin{aligned}
\|f_{\omega*(\ell)} - f_\omega\|_n &\leqslant \|f_{\omega*(\ell)} - f_{\omega*(\ell-1)}\|_n + \|f_{\omega*(\ell-1)} - f_\omega\|_n \\
&\leqslant \|g_{\omega*(\ell-1)} - f_{\omega*(\ell-1)}\|_n + \|f_{\omega*(\ell-1)} - f_\omega\|_n \\
&\leqslant \|L_{\omega*(\ell-1)}\|_{n+1} + \|f_{\omega*(\ell-1)} - f_\omega\|_n \\
&< \varepsilon_{n+1} + \varepsilon_n.
\end{aligned}$$

□

From this point on, we'll assume that from an initial lune $L = L(f, g)$ and a summable positive sequence $\{\varepsilon_n\}_{n\geqslant 2}$, we have obtained, though the procedure described here, a set of words $\Omega$ with respect to a sequence $\{m_k\}_{k\geqslant 1}$, and a family of lunes $\{L_\omega\}_{\omega\in\Omega}$, with all the properties that were mentioned.

2.4. **A helpful Cantor set.** Now that we have described the basic subdivision processes, we may begin to describe some auxiliary constructions that play an important part in the definition of a Peano curve (with some special properties) that will cover the initial lune $L$.

First, we will define a Cantor set $K$ through a family of closed intervals $\{J_\omega\}_{\omega\in\Omega}$ indexed by words with respect to $\{m_k\}_{k\geqslant 1}$. The open intervals that will be removed from $[0, 1]$, $\{G_\omega\}_{\omega\in\Omega^*}$, indexed by words with successor, will also play an important role.

First, set $J_{(1)} = [0, 1]$. For each $k \geqslant 2$, assume that $J_\omega$ has been defined for all $\omega \in \Omega$ with $|\omega| = k - 1$. For each such $\omega$, if $J_\omega = [\alpha, \beta]$ set

$$J_{\omega*(\ell)} = \left[\alpha + \frac{2\ell-2}{2m_k-1}(\beta-\alpha), \alpha + \frac{2\ell-1}{2m_k-1}(\beta-\alpha)\right], \quad 1 \leqslant \ell \leqslant m_k\ ,$$
$$G_{\omega*(\ell)} = \left(\alpha + \frac{2\ell-1}{2m_k-1}(\beta-\alpha), \alpha + \frac{2\ell}{2m_k-1}(\beta-\alpha)\right), \quad 1 \leqslant \ell < m_k\ .$$

Now set

$$K = \bigcap_{n\in\mathbb{N}} \bigcup_{\substack{\omega\in\Omega,\\ |\omega|=n}} J_\omega\ .$$

Notice that $K$ is a Cantor set, and $[0, 1] \smallsetminus K = \bigcup_{\omega\in\Omega^*} G_\omega$. The significance of this set $K$ will become apparent later on, but the basic idea is as follows: the curve $\gamma$ that we construct in this section will be such that $\gamma(J_\omega) = L_\omega^{\mathrm{ess}}$ (see Definition 2.1). However, usually $\gamma(\sup J_\omega) \neq \gamma(\inf J_{\omega^+})$, so we connect these "subcurves" using $G_\omega$.

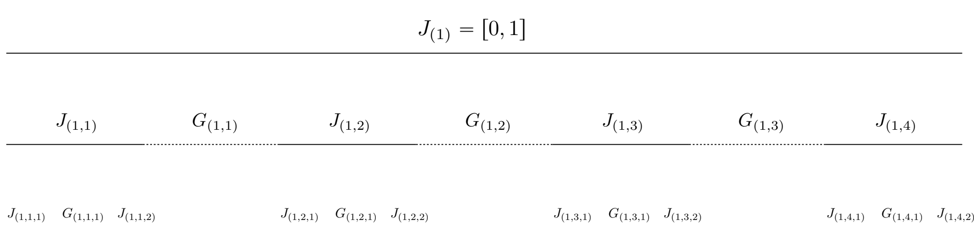



FIGURE 3. Illustration of the first few steps in the construction of $K$. Here $m_2 = 4$ and $m_3 = 2$.

2.5. **The family of ceiling functions.** A point $t \in [0, 1]$ belongs to the Cantor set $K$ if and only if there exists a sequence $\{\omega_n\}_{n\geqslant 1}$ of words with $\omega_1 = (1)$, $\omega_{n+1} = \omega_n * (\ell)$ for some $\ell \in \mathbb{N}$ and such that $t \in J_{\omega_n}$ for each $n$. Moreover, this sequence is unique, and we call it *the defining sequence of $t$ in $K$*. For what follows, recall the notation that $L_\omega = L(f_\omega, g_\omega)$.

**Lemma 2.11.** *Let $t \in K$ and let $\{\omega_n\}_{n\geqslant 1}$ be the defining sequence of $t$ in $K$. Then, for each $k \geqslant 0$, the sequences of functions $\{f^{(k)}_{\omega_n}\}_n$ and $\{g^{(k)}_{\omega_n}\}_n$ are uniformly Cauchy and*

$$\lim_{n\to\infty} f^{(k)}_{\omega_n} = \lim_{n\to\infty} g^{(k)}_{\omega_n}.$$

*Proof.* Using Lemma 2.10 and noticing that $|\omega_n| = n$, we see that whenever $n \geqslant k$ we have

$$\|f_{\omega_{n+1}} - f_{\omega_n}\|_k \leqslant \|f_{\omega_{n+1}} - f_{\omega_n}\|_n < \varepsilon_{n+1} + \varepsilon_n.$$

If $\varepsilon > 0$ is fixed and $N \geqslant k$ is such that $\sum_{n\geqslant N} \varepsilon_n < \frac{\varepsilon}{2}$, then if $m > n > N$,

$$\|f_{\omega_m} - f_{\omega_n}\|_k \leqslant \sum_{j=n}^{m-1} \|f_{\omega_{j+1}} - f_{\omega_j}\|_k < 2\sum_{j=n}^{m} \varepsilon_j < \varepsilon.$$

Since $\|h^{(k)}\|_0 \leqslant \|h\|_k$, it follows that $\{f^{(k)}_{\omega_n}\}_n$ is a uniformly Cauchy sequence. Notice that

$$\|g^{(k)}_{\omega_n} - f^{(k)}_{\omega_n}\|_0 \leqslant \|L_{\omega_n}\|_k < \varepsilon_n\ .$$

Hence, the sequence $\{g^{(k)}_{\omega_n}\}_n$ is also uniformly Cauchy, and both sequences have the same limit. □

**Definition 2.12.** For $t \in K$, let $\{\omega_n\}_{n\geqslant 1}$ be the defining sequence of $t$ in $K$. The *ceiling function at $t$* is

$$F_t = \lim_{n\to\infty} f_{\omega_n} = \lim_{n\to\infty} g_{\omega_n}.$$

A direct consequence of Lemma 2.11 is that, for each fixed $t$, $F_t$ is a $C^\infty$ function and $F^{(k)}_t = \lim_{n\to\infty} f^{(k)}_{\omega_n} = \lim_{n\to\infty} g^{(k)}_{\omega_n}$.

**Lemma 2.13.** *If $t, s \in K, t \leqslant s$, then $F_t \leqslant F_s$.*

*Proof.* Let $\{\omega_{t,n}\}_{n\geqslant 1}, \{\omega_{s,n}\}_{n\geqslant 1}$ be the defining sequences of $t$ and $s$ in $K$, respectively. Notice that $t < s$ if and only if for some $N > 1$, $\omega_{t,k} = \omega_{s,k}$ whenever $k < N$ but $\omega_{t,N} = \omega_{t,N-1} * (j), \omega_{s,N} = \omega_{s,N-1} * (\ell)$ with $j < \ell$. Now clearly, for each $n > N$,

$$f_{\omega_{t,n}} \leqslant g_{\omega_{t,N}} \leqslant f_{\omega_{s,N}} \leqslant f_{\omega_{s,n}},$$

because $L_{\omega_{t,n}}$ is a subdivision of $L_{\omega_{t,N}}$. □

**Proposition 2.14.** *The function $t \in K \mapsto F_t \in C^\infty([0,1])$ is continuous.*

*Proof.* Let $k \in \mathbb{N}$, fix $\varepsilon > 0$, and let $N > k$ be such that

$$\sum_{n=N}^{\infty} \varepsilon_n < \frac{\varepsilon}{4}.$$

Recall that the length of the interval $J_\omega$ is the same for every word $\omega$ such that $|\omega| = N$; call this length $\delta_N$. If $t, s \in K$, with $|t - s| < \delta_N$, then clearly $t, s \in J_\omega$ for some $\omega$ with $|\omega| = N$. Let $\omega_{t,n}, \omega_{s,n}$ be the defining sequences of $t$ and $s$ in $K$ (notice that $\omega_{t,N} = \omega_{s,N} = \omega$). Since by Lemma 2.10

$$\|f_{\omega_{t,n}} - f_\omega\|_k \leqslant \sum_{j=N}^{n-1} \|f_{\omega_{t,j+1}} - f_{\omega_{t,j}}\|_k < 2\sum_{j=N}^{n-1} \varepsilon_j$$

whenever $n > N$, it follows that by making $n \to \infty$ we have

$$\|F_t - f_\omega\|_k \leqslant 2 \sum_{j=N}^{\infty} \varepsilon_j < \frac{\varepsilon}{2}$$

and, analogously, $\|F_s - f_\omega\|_k < \frac{\varepsilon}{2}$. Therefore,

$$\|F_t - F_s\|_k \leqslant \|F_t - f_\omega\|_k + \|f_\omega - F_s\|_k < \varepsilon,$$

which completes the proof. □

**Lemma 2.15.** *Given $\omega \in \Omega^*$, suppose $G_\omega = (\alpha, \beta)$. Then $\alpha, \beta \in K$ and $F_\alpha = F_\beta$.*

*Proof.* Suppose $|\omega| = N$. By the definition of $K$, it is clear that $\alpha \in J_\omega, \beta \in J_{\omega^+}$. In fact, for each $n > N$ we have

$$\alpha \in J_{\omega*(m_{N+1}, m_{N+2}, \dots, m_n)}, \beta \in J_{\omega^+ * \underbrace{(0,0,\dots,0)}_{n-N}}.$$

Since

$$g_{\omega*(m_{N+1}, m_{N+2}, \dots, m_n)} = g_\omega = f_{\omega^+} = f_{\omega^+ * (0,0,\dots,0)},$$

it follows that

$$F_\alpha = \lim_{n\to\infty} g_{\omega_{\alpha,n}} = \lim_{n\to\infty} f_{\omega_{\beta,n}} = F_\beta,$$

where $\omega_{\alpha,n}$ and $\omega_{\beta,n}$ are the elements of the defining sequences of $\alpha$ and $\beta$ in $K$. □

What Lemma 2.15 implies is that the function $t \mapsto F_t$ can be extended to the interval $[0,1]$ in a natural way: for a point $t \notin K$, there exists a unique $G_\omega = (\alpha, \beta)$ such that $t \in G_\omega$. Then $F_t = F_\alpha = F_\beta$ is the ceiling function at $t$.

**Proposition 2.16.** *The function $t \in [0,1] \mapsto F_t \in C^\infty([0,1])$ is continuous.*

*Proof.* Given $k \in \mathbb{N}$, fix $\varepsilon > 0$ and let $N$ and $\delta_N$ be as in the proof of Proposition 2.14.

Suppose $t < s$ and $|t - s| < \delta_N$. If $t, s \in K$, we already know that $\|F_t - F_s\|_k < \varepsilon$. Otherwise, there exist $\alpha, \beta \in K$ such that $t \leqslant \alpha \leqslant \beta \leqslant s$ and $F_t = F_\alpha, F_s = F_\beta$. Since $|\alpha - \beta| \leqslant |t - s| < \delta_N$, we're done. □

## 2.6. **The function $\psi$ and the associated line field.**

**Lemma 2.17.** *Given $(x, y) \in L$, there exists $t \in [0,1]$ (not necessarily unique) such that $y = F_t(x)$. Moreover, if $y = F_{t_1}(x) = F_{t_2}(x)$, then $F'_{t_1}(x) = F'_{t_2}(x)$.*

*Proof.* Take $(x, y) \in L$, i.e., such that $x \in [0,1], f(x) \leqslant y \leqslant g(x)$. Since from the construction in § 2.3 we know that for each $n$,

$$\bigcup_{1 \leqslant i \leqslant m_n} L_{\omega*(i)} = L_\omega,$$

it follows that there exists a sequence $\{\omega_n\}_{n\geqslant 1}$ (not necessarily unique) such that $\omega_1 = (1), \omega_n = \omega_{n-1} * (i)$ for some $i \in \{1, 2, \dots, m_n\}$ with $f_{\omega_n}(x) \leqslant y \leqslant g_{\omega_n}(y)$. If $t$ is the only element in $\bigcap_{n\geqslant 1} J_{\omega_n} \subseteq K$, then

$$F_t(x) = \lim_{n\to\infty} f_{\omega_n}(x) = \lim_{n\to\infty} g_{\omega_n}(x) = y,$$

proving the first part.

Now suppose $t_1, t_2 \in [0,1]$ are such that $y = F_{t_1}(x) = F_{t_2}(x)$, and assume $t_1 \leqslant t_2$. By Lemma 2.13, $F_{t_1} \leqslant F_{t_2}$, so that $x$ is a local maximum of the function $F_{t_1} - F_{t_2}$. Therefore, $F'_{t_1}(x) - F'_{t_2}(x) = 0$. □

For each $(x,y) \in L$, take $t \in [0,1]$ such that $y = F_t(x)$, and set $\psi(x,y) = F_t'(x)$. By Lemma 2.17, this is well defined (not depending on the choice of $t$).

**Proposition 2.18.** *The function $\psi : L \to \mathbb{R}$ is continuous.*

*Proof.* Let $(x,y) \in L$ and let $(x_n,y_n) \in L$ be a sequence such that $(x_n,y_n) \to (x,y)$. By Lemma 2.17 and the previous paragraph, $y_n = F_{t_n}(x_n)$ for some $t_n$, and $\psi(x_n,y_n) = F'_{t_n}(x_n)$.

Suppose, by contradiction, that $\psi(x_n,y_n) \not\to \psi(x,y)$. In other words, $F'_{t_n}(x_n) \not\to F'_t(x)$, where $t$ is such that $y = F_t(x)$. By passing to a subsequence if necessary, we may assume that $|F'_{t_n}(x_n) - F'_t(x)| \geqslant \varepsilon$ for some $\varepsilon > 0$.

Let $\{t_{n_k}\}$ be a convergent subsequence of $\{t_n\}$, such that $t_{n_k} \to t^*$. By Proposition 2.16 and the continuity of $F_{t^*}$, the distance

$$\begin{aligned}|F_{t_{n_k}}(x_{n_k}) - F_{t^*}(x)| &\leqslant |F_{t_{n_k}}(x_{n_k}) - F_{t^*}(x_{n_k})| + |F_{t^*}(x_{n_k}) - F_{t^*}(x)| \\ &\leqslant \|F_{t_{n_k}} - F_{t^*}\|_0 + |F_{t^*}(x_{n_k}) - F_{t^*}(x)|\end{aligned}$$

can be made arbitrarily small as $k \to \infty$, so that $F_{t_{n_k}}(x_{n_k}) \to F_{t^*}(x)$ and $F_{t^*}(x) = F_t(x)$. By Lemma 2.17, $F'_{t^*}(x) = F'_t(x)$. If we pick $k$ sufficiently large such that $\|F_{t_{n_k}} - F_{t^*}\|_1 < \frac{\varepsilon}{2}$ and $|F'_{t^*}(x_{n_k}) - F'_{t^*}(x)| < \frac{\varepsilon}{2}$, we obtain

$$|F'_{t_{n_k}}(x_{n_k}) - F'_t(x)| < \varepsilon,$$

which is a contradiction. □

We then set $\Lambda(x,y)$ as the line whose direction vector is $(1, \psi(x,y))$, for each $(x,y) \in L$. By Proposition 2.18, this is a continuous line field.

2.7. **A sequence of curves.** We now proceed to the construction of a sequence $\gamma_n$ of curves that converges uniformly to a Peano curve $\gamma$, such that each $\gamma_n$ is tangent to the line field $\Lambda(x,y)$.

For an interval $I = [\alpha, \beta]$ and for $a, b \in \mathbb{R}$, let $\psi_{I,a,b} : I \to \mathbb{R}$ be a $C^\infty$ strictly monotone function such that

(i) $\lim\limits_{t \searrow \alpha} \psi_{I,a,b}(t) = a$, $\lim\limits_{t \nearrow \beta} \psi_{I,a,b}(t) = b$.

(ii) $\lim\limits_{t \searrow \alpha} \psi^{(k)}_{I,a,b}(t) = \lim\limits_{t \nearrow \beta} \psi^{(k)}_{I,a,b}(t) = 0$ for every $k \geqslant 1$.

Additionally, for any $h : [0,1] \to \mathbb{R}$ let $\Gamma_h$ be the graph of $h$, parametrized in the obvious way, i.e., $\Gamma_h(t) = (t, h(t))$ for $t \in [0,1]$.

For each $\omega \in \Omega$, write $S_{L_\omega} = (a_\omega, b_\omega)$, and let

$$\gamma_1(t) = \Gamma_{f_{(1)}}\left(\psi_{J_{(1)},a_{(1)},b_{(1)}}(t)\right)$$

for $t \in [0,1]$. Recursively, set for $n \geqslant 2$:

$$\gamma_n(t) = \begin{cases} \gamma_{n-1}(t), & \text{if } t \in \bigcup_{\omega \in \Omega^*, |\omega| < n} G_\omega, \\ \Gamma_{f_\omega}\left(\psi_{J_\omega,a_\omega,b_\omega}(t)\right), & \text{if } t \in J_\omega \text{ for some } \omega \in \Omega, |\omega| = n, \\ \Gamma_{g_\omega}\left(\psi_{G_\omega,b_\omega,a_{\omega^+}}(t)\right), & \text{if } t \in G_\omega \text{ for some } \omega \in \Omega^*, |\omega| = n. \end{cases}$$

**Lemma 2.19.** *If $|\omega| = n$, then*

(i) $\gamma_{n+1}(J_{\omega*(i)}) \subseteq L^{\text{ess}}_{\omega*(i)}$ *for any* $1 \leqslant i \leqslant m_{n+1}$;

(ii) $\gamma_{n+1}(G_{\omega*(i)}) \subseteq L^{\text{ess}}_{\omega*(i)}$ *for odd* $i, 1 \leqslant i < m_{n+1}$;

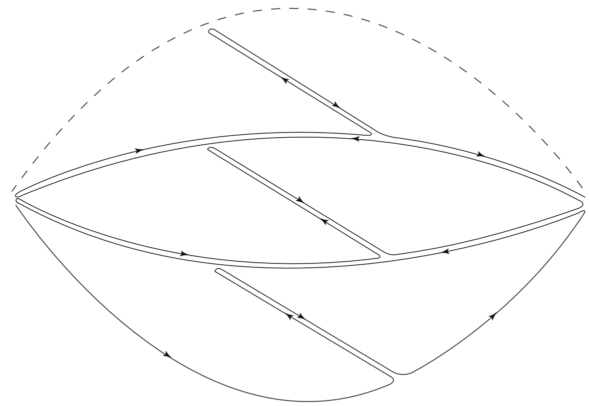


FIGURE 4. Schematic drawing of the curve $\gamma_2$, assuming $m_2 = 6$.

(iii) $\gamma_{n+1}(G_{\omega*(i)}) \subseteq L^{\rm ess}_{\omega*(i-1)} \cup L^{\rm ess}_{\omega*(i)}$ *for even* $i, 1 \leqslant i < m_{n+1}$.

*As a consequence,* $\gamma_m(J_\omega) \subseteq L^{\rm ess}_\omega$ *for all* $m \geqslant n$.

*Proof.* If $t \in J_{\omega*(i)}$, then

$$\gamma_{n+1}(t) = \Gamma_{f_{\omega*(i)}}(\psi_{J_{\omega*(i)},a_{\omega*(i)},b_{\omega*(i)}}(t)).$$

Since $a_{\omega*(i)} \leqslant \psi_{J_{\omega*(i)},a_{\omega*(i)},b_{\omega*(i)}}(t) \leqslant b_{\omega*(i)}$, Lemma 2.2 implies property (i).

If $t \in G_{\omega*(i)}$ with odd $i$, then $L_{\omega*(i)}$ is the first lune in a bipartition of one of the slices of $L_\omega$, hence $b_{\omega*(i)} = (a_\omega + 2b_\omega)/3$ and $a_{\omega*(i+1)} = (2a_\omega + b_\omega)/3$ (see Proposition 2.7). Since

$$\gamma_{n+1}(t) = \Gamma_{g_{\omega*(i)}}(\psi_{G_{\omega*(i)},b_{\omega*(i)},a_{\omega*(i+1)}}(t)),$$

property (ii) follows.

If $t \in G_{\omega*(i)}$ with even $i$ and $i < m_{n+1}$, $L_{\omega*(i)}$ is the second lune in a bipartition, hence $b_{\omega*(i)} = b_\omega$ and $a_{\omega*(i+1)} = a_\omega = a_{\omega*(i-1)}$. Therefore, if the value $\psi_{G_{\omega*(i)},b_{\omega*(i)},a_{\omega*(i+1)}}(t)$ is greater than or equal to $a_{\omega*(i)}$ then $\gamma_{n+1}(t) \in L_{\omega*(i)}$; on the other hand, if that value is less than or equal to $a_{\omega*(i)} \leqslant b_{\omega*(i-1)}$ then $\gamma_{n+1}(t) \in L_{\omega*(i-1)}$ (recall that if $x \leqslant a_{\omega*(i)}$, $g_{\omega*(i)}(x) = f_{\omega*(i)}(x) = g_{\omega*(i-1)}(x)$), thus property (iii) follows.

To see the consequence, notice that the three properties imply that for any $\omega \in \Omega$ with $|\omega| = n$, $\gamma_{n+1}(J_{\omega*(i)}), \gamma_{n+1}(G_{\omega*(i)}) \subseteq L^{\rm ess}_\omega$ for each $i$. In particular, for any $\omega' \in \Omega$ with $|\omega'| = k$, $\gamma_{n+k+1}(J_{\omega*\omega'*(i)}), \gamma_{n+k+1}(G_{\omega*\omega'*(i)}) \subseteq L^{\rm ess}_{\omega*\omega'} \subseteq L^{\rm ess}_\omega$. Since for each $k$ we have

$$J_\omega = \bigcup_{\substack{\omega' \in \Omega \\ |\omega'| = k}} J_{\omega*\omega'} \cup \bigcup_{\substack{\omega' \in \Omega^* \\ |\omega'| = k}} G_{\omega*\omega'},$$

it follows that $\gamma_{n+k}(J_\omega) \subseteq L^{\rm ess}_\omega$. □

2.8. **The Peano curve.** We now wish to show that the curves $\gamma_n$ defined above converge uniformly to some curve $\gamma$.

**Lemma 2.20.** *There exists a sequence* $D_n \to 0$ *such that for each* $\omega \in \Omega$, *the diameter* $D_\omega$ *of* $L^{\rm ess}_\omega$ *satisfies* $D^2_\omega \leqslant D_{|\omega|}$.

*Proof.* Let $(x_1, y_1), (x_2, y_2) \in L_\omega^{\text{ess}}$, and assume without loss of generality that $y_1 \leqslant y_2$. By Lemma 2.2, $a_\omega \leqslant x_1, x_2 \leqslant b_\omega$ and $f_\omega(x_1) \leqslant y_1 \leqslant y_2 \leqslant g_\omega(x_2)$. Hence,

$$\begin{aligned}\text{dist}((x_1, y_1), (x_2, y_2))^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &\leqslant (b_\omega - a_\omega)^2 + (g_\omega(x_2) - f_\omega(x_1))^2 \\ &\leqslant (b_\omega - a_\omega)^2 + (g_\omega(x_2) - f_\omega(x_2) + f_\omega(x_2) - f_\omega(x_1))^2 \\ &\leqslant (b_\omega - a_\omega)^2 + \left(|g_\omega(x_2) - f_\omega(x_2)| + \left|\int_{x_1}^{x_2} f'_\omega(t)dt\right|\right)^2 \\ &\leqslant (b_\omega - a_\omega)^2 + (\|L_\omega\|_0 + \|f_\omega\|_1(b_\omega - a_\omega))^2 .\end{aligned}$$

Suppose $|\omega| = n$. Clearly $\|L_\omega\|_0 \leqslant \|L_\omega\|_n < \varepsilon_n$, by the construction of the family of lunes. Moreover, also by construction (and by Proposition 2.7), $b_\omega - a_\omega = (2/3)^n(b_{(1)} - a_{(1)}) \leqslant (2/3)^n$. Finally, let $M = \sup_{t\in[0,1]} \|F_t\|_1$ (which is finite by Proposition 2.16). Since

$$f_\omega = \lim_{k\to\infty} f_{\omega*\underbrace{(1,1,\ldots,1)}_{k}} = F_t$$

for some $t \in [0, 1]$, it follows that $\|f_\omega\|_1 \leqslant M$. In other words, if we take

$$D_n = \left(\frac{2}{3}\right)^{2n} + \left(\varepsilon_n + M\left(\frac{2}{3}\right)^n\right)^2 \to 0,$$

we're done. □

**Lemma 2.21.** *The curves $\gamma_n$ form a uniformly Cauchy sequence, and in particular converge uniformly to a continuous curve $\gamma$.*

*Proof.* Let $\varepsilon > 0$ and take $N$ such that $D_N < \varepsilon$. We wish to show that whenever $m, n \geqslant N$, we have $|\gamma_m(t) - \gamma_n(t)| < \varepsilon$ for each $t \in [0, 1]$.

In fact, if $t \in \bigcup_{\omega\in\Omega*,|\omega|\leqslant N} G_\omega$ this is clear, because $\gamma_k(t) = \gamma_N(t)$ for all $k \geqslant N$. On the other hand, if $t \in J_\omega$ for some $\omega, |\omega| = N$, then Lemma 2.19 shows that $\gamma_m(t), \gamma_n(t) \in L_\omega^{\text{ess}}$, thus $|\gamma_m(t) - \gamma_n(t)| \leqslant D_N < \varepsilon$. □

We now wish to relate the footprint $\gamma([0, t])$ with the ceiling function $F_t$. We need a few results first:

**Lemma 2.22.** *Let $\omega \in \Omega$, $(x, y) \in L_\omega^{\text{ess}}$. There exists a sequence $\{\omega_n\}_{n\geqslant|\omega|}$ such that $\omega_{|\omega|} = \omega$, $\omega_{n+1} = \omega_n * (\ell)$ for some $\ell \in \mathbb{N}$ and $(x, y) \in L_{\omega_n}^{\text{ess}}$ for each $n$.*

*Moreover, there exists $t \in J_\omega \cap K$ such that the defining sequence $\{\omega_{t,n}\}_{n\geqslant1}$ of $t$ satisfies $\omega_{t,n} = \omega_n$ whenever $n \geqslant |\omega|$.*

*Proof.* The first part is immediate from the fact that

$$L_{\omega_n}^{\text{ess}} = \bigcup_{1\leqslant i\leqslant m_{n+1}} L_{\omega_n*(i)}^{\text{ess}}.$$

For the second part, we take $t$ to be the single element of $\bigcap_{n\geqslant|\omega|} J_{\omega_n}$. □

The sequence in Lemma 2.22 is not necessarily unique, and we call every such sequence a *defining sequence* of $(x, y)$ in $L_\omega$.

**Lemma 2.23.** *For each $\omega \in \Omega$, $\gamma(J_\omega) = \gamma(J_\omega \cap K) = L_\omega^{\text{ess}}$.*

*Proof.* Lemma 2.19 implies that $\gamma(J_\omega) \subseteq L^{\mathrm{ess}}_\omega$. For the other direction, take for each $(x,y) \in L^{\mathrm{ess}}_\omega$, $\{\omega_n\}_{n \geqslant |\omega|}$ and $t \in J_\omega \cap K$ as in Lemma 2.22. Since $\gamma(t) \in L^{\mathrm{ess}}_{\omega_n}$ for each $n$ (and this is a sequence of nested compact sets), it follows that $\gamma(t) = (x,y)$. Therefore, $\gamma(J_\omega) \subseteq L^{\mathrm{ess}}_\omega \subseteq \gamma(J_\omega \cap K) \subseteq \gamma(J_\omega)$ and we're done. □

**Lemma 2.24.** *For each* $t \in [0,1]$*,* $\gamma([0,t]) = \gamma(K \cap [0,t])$*.*

*Proof.* We need to show that for each $t \notin K$, there exists $s < t, s \in K$ with $\gamma(s) = \gamma(t)$. Take $t \notin K$, so that $t \in G_{\omega * (i)}$ for some $\omega \in \Omega^*$, $|\omega| = n$, $i < m_{n+1} - 1$. By Lemma 2.19, $\gamma(t) = \gamma_{n+1}(t) \in L^{\mathrm{ess}}_{\omega*(i-1)} \cup L^{\mathrm{ess}}_{\omega*(i)}$ (or simply $L^{\mathrm{ess}}_{\omega*(i)}$ if $i = 1$). By Lemma 2.23, $\gamma(J_{\omega*(i-1)} \cap K) \cup \gamma(J_{\omega*(i)} \cap K) = L^{\mathrm{ess}}_{\omega*(i-1)} \cup L^{\mathrm{ess}}_{\omega*(i)}$, and this yields the result. □

**Lemma 2.25.** *For* $t \in [0,1]$*, if* $\gamma(t) = (x(t), y(t))$*, then* $y(t) = F_t(x(t))$*.*

*Proof.* If $t \notin K$, then $t \in G_\omega = (\beta_\omega, \alpha_{\omega^+})$ for some $\omega$. By the definition of $\gamma$, $y(t) = g_\omega(x(t))$, because $\gamma(t)$ is in the graph of $g_\omega$. Since $F_t = F_{\beta_\omega} = g_\omega = f_{\omega^+} = F_{\alpha_{\omega^+}}$, the result follows.

If $t \in K$ and $\omega_n$ is the defining sequence of $t$, we know by Lemma 2.23 that $(x(t), y(t)) \in L^{\mathrm{ess}}_{\omega_n}$. By Lemma 2.2, $f_{\omega_n}(x(t)) \leqslant y(t) \leqslant g_{\omega_n}(x(t))$. Since both bounding sequences converge to $F_t(x(t))$, we're done. □

**Proposition 2.26.** *For each* $t \in (0,1]$*,* $\gamma([0,t]) = L^{\mathrm{ess}}(f, F_t)$*.*

*Proof.* It suffices to show the result for $t \in K$, for if $t \notin K$ and $t_K = \max\{s \in K \mid s \leqslant t\}$, then $F_t = F_{t_K}$ and $\gamma([0,t]) = \gamma([0,t] \cap K) = \gamma([0, t_K])$.

Let $t \in K$. We need to show that $\gamma([0,t]) = \gamma(K \cap [0,t]) = L^{\mathrm{ess}}(f, F_t)$. To see that $L^{\mathrm{ess}}(f, F_t) \subseteq \gamma(K \cap [0,t])$, take $(x,y) \in \mathrm{int}(L(f, F_t))$, so that $f(x) < y < F_t(x)$. Let $\{\omega_n\}_{n \geqslant 1}$ be a defining sequence of $(x,y)$ in $L_{(1)} = L$, which is also the defining sequence of some $s \in K$. Then $\gamma(s) = (x,y)$. Since $y = \lim f_{\omega_n}(x) = \lim g_{\omega_n}(x) = F_s(x)$, it follows that $F_s(x) < F_t(x)$. Lemma 2.13 implies that we necessarily have $F_s \leqslant F_t$ and thus $s < t$. Consequently, $\mathrm{int}(L(f, F_t)) \subseteq \gamma(K \cap [0,t])$. Since $\gamma(K \cap [0,t])$ is a compact set (thus closed), $L^{\mathrm{ess}}(f, F_t) = \overline{\mathrm{int}(L(f, F_t))} \subseteq \gamma(K \cap [0,t])$.

To see that $\gamma(K \cap [0,t]) \subseteq L^{\mathrm{ess}}(f, F_t)$, write $J_\omega = [\alpha_\omega, \beta_\omega]$ for each $\omega \in \Omega$. Take $s \in K, 0 < s \leqslant t$, and suppose initially that $s = \alpha_{\hat{\omega}}$ for some $\hat{\omega}$. Since $s > 0$, there exists a unique $\omega$ such that $s = \alpha_{\omega^+}$. Since

$$
\begin{aligned}
\gamma(\alpha_{\omega^+}) &= \lim_{t \nearrow \alpha_{\omega^+}} \gamma(t) \\
&= \lim_{t \nearrow \alpha_{\omega^+}} (\psi_{G_\omega, b_\omega, a_{\omega^+}}(t), g_\omega(\psi_{G_\omega, b_\omega, a_{\omega^+}}(t))) \\
&= \lim_{t \nearrow \alpha_{\omega^+}} (\psi_{G_\omega, b_\omega, a_{\omega^+}}(t), F_{\alpha_{\omega^+}}(\psi_{G_\omega, b_\omega, a_{\omega^+}}(t)))
\end{aligned}
$$

is a limit point of the closed set $L^{\mathrm{ess}}(f, F_s)$, it follows that $\gamma(s) \in L^{\mathrm{ess}}(f, F_s) \subseteq L^{\mathrm{ess}}(f, F_t)$.

If, on the other hand, $s \notin \{\alpha_\omega\}_{\omega \in \Omega}$ and $\{\omega_n\}$ is the defining sequence for $s$, then $\alpha_{\omega_n} \nearrow s$, thus $\gamma(\alpha_{\omega_n}) \to \gamma(s)$ and $\gamma(\alpha_{\omega_n}) \in L^{\mathrm{ess}}(f, F_{\alpha_{\omega_n}}) \subseteq L^{\mathrm{ess}}(f, F_t)$. □

We have thus concluded the proof of Proposition 2.3. To summarize, the continuity of $\gamma$, $F_t$, and $\psi$ are established in Lemma 2.21, Proposition 2.16, and Proposition 2.18, respectively; property (i) comes directly from the definition; properties (ii) and (iii) are respectively Lemmas 2.13 and 2.25, property (iv) holds by definition, property (v) is Proposition 2.26, and property (vi) is also true by construction.

## 3. Proof of the theorem

The first is step is to obtain a cylindrical version of Proposition 2.3. Recall that $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, and that $C^\infty(\mathbb{T})$ can be regarded as the space of $C^\infty$ $2\pi$-periodic functions $\mathbb{R} \to \mathbb{R}$, endowed with the $C^\infty$ topology (see §2.1 for details).

**Proposition 3.1.** *Given intervals* $[t_0, t_1]$ *and* $[c, d]$*, there exist:*

- *a continuous map* $\gamma : [t_0, t_1] \to \mathbb{T} \times [c, d]$*;*
- *a continuous map* $t \in [t_0, t_1] \mapsto F_t \in C^\infty(\mathbb{T})$*;*
- *a continuous function* $\psi : \mathbb{T} \times [c, d] \to \mathbb{R}$*;*

*with the following properties:*

(i) $F_{t_0} \equiv c$, $F_{t_1} \equiv d$*;*
(ii) *If* $t \leqslant s$ *then* $F_t \leqslant F_s$*;*
(iii) *Writing* $\gamma(t) = (x(t), y(t))$*, we have* $y(t) = F_t(x(t))$*.*
(iv) $F_t'(x) = \psi(x, F_t(x))$*;*
(v) *for each* $t \in (t_0, t_1]$*, the image* $\gamma([t_0, t])$ *equals the closure of the interior of* $\{(x, y) \mid x \in \mathbb{T},\ c \leqslant y \leqslant F_t(x)\}$*;*
(vi') $\gamma(t_0) = (0, c)$, $\gamma(t_1) = (\pi, d)$*;*

*Proof.* It is sufficient to consider the case $[t_0, t_1] = [c, d] = [0, 1]$, since the general case follows by rescaling. Let $g : \mathbb{R} \to [0, 1]$ be a $2\pi$-periodic $C^\infty$ function such that $g(0) = 0$, $g(\pi) = 1$, $g^{(k)}(0) = g^{(k)}(\pi) = 0$ for every $k \geqslant 1$, and which is strictly monotone in each of the intervals $[0, \pi]$ and $[\pi, 2\pi]$. Consider the following four functions:

$$
\begin{aligned}
&f_1, g_1 : [0, 2\pi] \to \mathbb{R} \quad \text{given by} \quad f_1 \equiv 0, \qquad\quad g_1 = g|_{[0,2\pi]}, \\
&f_2, g_2 : [\pi, 3\pi] \to \mathbb{R} \quad \text{given by} \quad f_2 = g|_{[\pi,3\pi]},\ g_2 \equiv 1,
\end{aligned}
$$

and the corresponding lunes $L_1 = L(f_1, g_1)$ and $L_2 = L(f_2, g_2)$. Applying Proposition 2.3 to the lune $L_i$ we obtain a Peano curve $\gamma_i$, a family of ceiling functions $F_{i,t}$, and a function $\psi_i$. Let $p : L_1 \cup L_2 \to \mathbb{T} \times [0, 1]$ be the bijective map defined by $p(x, y) = (x \bmod 2\pi, y)$; set

$$
\gamma^*(t) = \begin{cases} \gamma_1(3t), & 0 \leqslant t < 1/3; \\ (3\pi(1-t), g\left(3\pi(1-t)\right)), & 1/3 \leqslant t \leqslant 2/3; \\ \gamma_2(3t-2), & 2/3 < t \leqslant 1; \end{cases}
$$

and $\gamma = p \circ \gamma^*$; set $\psi = \psi^* \circ p^{-1}$ for $\psi^* : L_1 \cup L_2 \to \mathbb{R}$ such that $\psi^*|L_1 = \psi_1$ and $\psi^*|L_2 = \psi_2$; finally, for $x \in \mathbb{T}$, let

$$
F_t(x) = \begin{cases} F_{1,3t}(x), & 0 \leqslant t < 1/3; \\ g(x), & 1/3 \leqslant t \leqslant 2/3; \\ F_{2,3t-2}(x), & 2/3 < t \leqslant 1. \end{cases}
$$

Then $\gamma$, $\psi$ and $F_t$ have the desired properties. □

We improve the previous proposition by controlling the derivatives:

**Proposition 3.2.** *Given intervals* $[t_0, t_1]$, $[c, d]$ *and numbers* $k_0 \in \mathbb{N}$, $\delta_0 > 0$*, there exist maps* $\gamma$, $F_t$*, and* $\psi$ *satisfying properties* (i)–(v) *in Proposition 3.1 and, in addition, the following ones:*

(vi) $\gamma(t_0) = (0, c)$, $\gamma(t_1) = (0, d)$*;*

(vii) $\|F_t - c_t\|_{k_0} < \delta_0$ *for every* $t$, *where* $c_t$ *is the constant* $\frac{c(t_1-t)+d(t-t_0)}{t_1-t_0}$.

*Proof.* Again, it is sufficient to consider the case $[t_0, t_1] = [c, d] = [0, 1]$. Let $\hat{\gamma}$, $\hat{F}_t$, and $\hat{\psi}$ be given by the previous proposition. By compactness, we have $\|\hat{F}_t\|_{k_0} \leqslant C$ for some finite $C$ independent of $t$. Fix an odd integer $n > (C+1)/\delta_0$. By rescaling and translating we obtain $\hat{\gamma}_j$, $\hat{F}_{j,t}$, and $\hat{\psi}_j$ for the cylinders $\mathbb{T} \times [j/n, (j+1)/n]$, where $j = 0, 1, \dots n-1$, with the extra property that if $t \in [j/n, (j+1)/n]$ then $\|\hat{F}_{j,t} - j/n\|_{k_0} \leqslant C/n < \delta_0 - 1/n$. Finally, we rotate the cylinders so that everything glues: in other words, for each $t \in [0,1]$ we let $j = \lfloor nt \rfloor$ and define:

$$\gamma(t) = \hat{\gamma}_j(nt) + (j\pi, 0)\,, \quad F_t(x) = \hat{F}_{j,nt-j}(x + j\pi)\,, \quad \psi(x,t) = \hat{\psi}_j(x + j\pi, t)\,.$$

It is clear that these maps have the required properties (i)–(v) and (vi). To see property (vii), notice that $c_t = t$ so letting $j = \lfloor nt \rfloor$ we have

$$\|F_t - c_t\|_{k_0} \leqslant \|F_t - j/n\|_{k_0} + 1/n < \delta_0\,. \qquad \square$$

Finally, we explain how the previous proposition allows us to conclude:

*Proof of the theorem.* Let $k : (0,\infty) \to \mathbb{N}$ be upper semicontinuous and $\varepsilon : (0,\infty) \to (0,\infty)$ be lower semicontinuous. Then there exist two-sided sequences $\{t_n\}_{n\in\mathbb{Z}}$, $\{k_n\}_{n\in\mathbb{Z}}$, and $\{\varepsilon_n\}_{n\in\mathbb{Z}}$ taking values in $(0,\infty)$, $\mathbb{N}$, and $(0,\infty)$ respectively, such that $\{t_n\}$ is monotonically increasing, $\lim_{n\to-\infty} t_n = 0$, $\lim_{n\to+\infty} t_n = \infty$, and

$$t \in [t_n, t_{n+1}] \ \Rightarrow \varepsilon(t) \geqslant \varepsilon_n \text{ and } k(t) \leqslant k_n.$$

For each $n$, let $\delta_n = \varepsilon_n / 2^{k_n}$, and apply Proposition 3.2 with both intervals equal to $I_n = [t_n, t_{n+1}]$, thus obtaining a Peano curve $\gamma_n : I_n \to \mathbb{T} \times I_n$, a family of ceiling functions $F_{n,t}$ and a continuous function $\psi_n$ defined on $\mathbb{T} \times I_n$ satisfying all seven properties. Also, notice that $c_t = t$ in part (vii).

Define a diffeomorphism $P \colon \mathbb{T} \times (0,\infty) \to \mathbb{R}^2 \smallsetminus \{(0,0)\}$ by $P(\theta, r) = (r\cos\theta, r\sin\theta)$. Recall that $\alpha_t(\theta) = P(\theta, t)$. We now construct maps $\gamma^* : (0,\infty) \to \mathbb{T} \times (0,\infty)$, $F_t^* : \mathbb{T} \to (0,\infty)$ and $\psi^* : \mathbb{T} \times (0,\infty) \to \mathbb{R}$ by setting for $(x,t) \in \mathbb{T} \times [t_n, t_{n+1}]$:

$$\gamma^*(t) = \gamma_n(t), \quad F_t^*(x) = F_{n,t}(x), \quad \psi(x,t) = \psi_n(x,t)\,.$$

Next, we define the Peano curve $\gamma$ by $\gamma(0) = 0$ and $\gamma(t) = P \circ \gamma^*(t)$ for $t > 0$. Let $\Lambda^*$ be the line field on $\mathbb{T} \times (0,\infty)$ spanned by the vector field $\frac{\partial}{\partial\theta} + \psi^*(r,\theta)\frac{\partial}{\partial r}$; by pushing it forward by the derivative of $P$, we obtain a line field $\Lambda$ on $\mathbb{R}^2 \smallsetminus \{(0,0)\}$.

Notice that, for each $t > 0$, $\gamma^*([0,t]) = \{(x,y) : x \in \mathbb{T}, 0 < y \leqslant F_t(x)\}$. It follows then that $\beta_t : \theta \in \mathbb{T} \mapsto P(\theta, F_t(\theta)) \in \mathbb{R}^2$ is a smooth embedding whose image is $\partial\gamma([0,t])$. By property (iv) of Proposition 3.1, $\beta_t$ is tangent to the line field $\Lambda$.

To conclude the proof we check that the proximity condition (1.1) is satisfied. Since $\beta_t(\theta) = F_t(\theta)\alpha_1(\theta)$ and $\alpha_t(\theta) = t\alpha_1(\theta)$, we have, for each $k \in \mathbb{N}$,

$$\|\beta_t^{(k)}(\theta) - \alpha_t^{(k)}(\theta)\| \leqslant \sum_{i=0}^{k} \binom{k}{i} \left|(F_t - t)^{(i)}(\theta)\right| \underbrace{\left|\alpha_1^{(k-i)}(\theta)\right|}_{1} \leqslant 2^k \|F_t - t\|_k\,.$$

Given $t > 0$, let $n$ be such that $t \in I_n$. Then

$$\|\alpha_t - \beta_t\|_{k(t)} \leqslant \|\alpha_t - \beta_t\|_{k_n} \leqslant 2^{k_n}\|F_t - t\|_{k_n} < 2^{k_n}\delta_n = \varepsilon_n \leqslant \varepsilon(t). \qquad \square$$

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