A NOTE ON SYMPLECTIC TOPOLOGY OF b-MANIFOLDS

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ABSTRACT. A Poisson manifold (M^{2n}, π) is b-symplectic if $\bigwedge^n \pi$ is transverse to the zero section. In this paper we apply techniques native to Symplectic Topology to address questions pertaining to b-symplectic manifolds. We provide constructions of b-symplectic structures on open manifolds by Gromov's h-principle, and of b-symplectic manifolds with a prescribed singular locus, by means of surgeries.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

A Poisson structure on a manifold M can be described as a bivector $\pi \in \mathfrak{X}^2(M)$ which obeys the partial differential equation $[\pi, \pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket of multivector fields. The image of the induced bundle map $\pi^\sharp : T^*M \to TM$ is an involutive distribution, of possibly varying rank, each of whose integral submanifolds carries an induced symplectic form.

Symplectic structures are those Poisson structures whose underlying foliation has M^{2n} as its only leaf; equivalently, they are those Poisson structures for which $\pi^\sharp$ is invertible – i.e., for which $\bigwedge^n \pi$ does not meet the zero section.

In [9], this nondegeneracy condition has been relaxed in a very natural way, by demanding that $\bigwedge^n \pi$ be *transverse* to the zero section instead of avoiding it:

Definition 1.1. *A Poisson manifold (M^{2n}, π) is of b-**symplectic type** if $\bigwedge^n \pi$ is transverse to the zero section $M \subset \bigwedge^{2n} TM$.*

Such structures were first defined, in the case of dimension two, by Radko [16], who called them *topologically stable* Poisson structures. Poisson structures of b-symplectic type have also appeared under the name *log symplectic* [7, 3, 11, 12].

Symplectic structures are those Poisson structures of b-symplectic type whose **singular locus** $Z(\pi) := (\bigwedge^n \pi)^{-1} M \subset M$ is empty. Quite crucially for what follows is that general Poisson structures of b-symplectic type do not stay too far from being symplectic.

The transversality condition $\bigwedge^n \pi \pitchfork M$ ensures that the singular locus $Z = Z(\pi)$ is a codimension-one submanifold of M, which by the Poisson condition is itself foliated in codimension one by symplectic leaves of π. Hence, Z is a corank-one Poisson submanifold. Those vector fields $v \in \mathfrak{X}(M)$ which are tangent to Z form the space of all sections of a vector bundle ${}^bT(M, Z) \to M$, called the b-**tangent bundle** [14]. The bundle ${}^bT(M, Z)$ has a canonical structure of Lie algebroid, and

a Poisson structure of b-symplectic type π on M with singular locus Z can be described alternatively by a closed, nondegenerate section of $\bigwedge^2 \left({}^b T^*(M, Z) \right)$, in complete analogy with the symplectic case. This viewpoint motivates the nomenclature adopted in [9, 15] [1], and to which we adhere. Henceforth, we will refer to Poisson structures of b-symplectic type as b-**symplectic** structures. With this perspective, it is not surprising that many tools from Symplectic Topology can be adapted to this b-setting. In fact, the purpose of this paper is to use such tools to discuss the following existence problems in b-symplectic geometry:

(P1) Which manifolds M carry a structure of b-symplectic manifold?

(P2) Which corank one Poisson manifolds Z appear as singular loci of closed b-symplectic manifolds ?

For closed manifolds the answer to (P1) is unknown, even in the symplectic case. For *open* manifolds (i.e., whose connected components either have non-empty boundary or are non-compact), we show:

Theorem A. *An orientable, open manifold M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

In fact, the story here is completely analogous to the symplectic case: supporting a b-symplectic structure imposes restrictions on the de Rham cohomology of a closed manifold [3, 11], but these do not apply to open manifolds. There, we show the existence of b-symplectic structures is a purely homotopical question, which abides by a version of the h-principle of Gromov [6]. In some very special cases, the finer control granted by having an h-principle description allows one to prescribe the singular locus Z of the ensuing b-symplectic manifold. (However, in the case where Z bounds a compact region in M, these techniques break down completely.)

The singular locus Z a b-symplectic manifold is not just a general corank-one Poisson manifold, in that it can be defined by a **cosymplectic structure** $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$ (i.e., a pair of closed forms for which $\theta \wedge \eta^{n-1}$ is a volume form; cf. [9, Proposition 10] or Section 4). Thus, the Existence Problem (P2) is about describing which cosymplectic structures appear as the singular locus of a closed b-symplectic manifold.

Our first result says that (P2) can be rephrased as a problem of Symplectic Topology; namely, that of determining those (closed) cosymplectic manifolds which admit symplectic fillings (see Section 2).

Lemma 1.1. *A cosymplectic manifold (Z, η, θ) is the singular locus of a closed, orientable b-symplectic manifold if and only if (Z, η, θ) is symplectically fillable.*

Symplectic fillings of contact manifolds – and more generally symplectic cobordisms with concave/convex boundaries – are central to Symplectic Topology, whereas the case of cosymplectic (or flat) boundaries has received comparatively little attention. A notable exception is Eliashberg's result that 3-dimensional **symplectic mapping tori** (i.e., suspensions of a symplectomorphisms of surfaces) are symplectically fillable [4].

Our second result follows from observing that symplectic fillability of all cosymplectic 3-manifolds is a consequence of symplectic fillability of all symplectic mapping tori. This solves the cosymplectic existence problem in dimension 3:

Theorem B. *Any cosymplectic manifold of dimension 3 is the singular locus of orientable, closed, b-symplectic manifolds.*

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[1] Closed, nondegenerate sections of $\bigwedge^2 \left({}^b T^*(M, Z) \right)$ were introduced in [15] for $Z = \partial M$.

Our third result describes a class of symplectomorphisms φ which yield symplectically fillable symplectic mapping tori in arbitrary dimensions: namely, those built out of **Dehn twists** around parametrized Lagrangian spheres (see Definition 4.3) and their inverses:

Theorem C. *If Z is a symplectic mapping torus defined by a symplectomorphism which is Hamiltonian isotopic to a word on Dehn twists and their inverses, then Z is the singular locus of orientable, closed, b-symplectic manifolds.*

Remark 1.1. *Symplectic fillability of a symplectic mapping torus is a property that only depends on the Hamiltonian isotopy class of the symplectomorphism. The key result in Eliashberg's argument in [4] is that, for surfaces, the fillability of a symplectic mapping torus depends just on the symplectic isotopy class of the symplectomorphism. We do not know whether this is true in higher dimensions.*

While this project was being completed the authors learned of research by G. Cavalcanti which has some overlap with theirs. More precisely, the idea of constructing b-symplectic manifolds without boundary by gluing cosymplectic cobordisms appeared independently in [3].

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2. Cosymplectic cobordisms and b-symplectic structures

We summarize below basic facts and conventions about b-symplectic manifolds and cosymplectic cobordisms, and describe the relation between both structures. For a more detailed account we refer the reader to [7, 8, 9, 12, 14, 15].

2.1. **b-manifolds.** The Lie subalgebra $\mathfrak{X}(M, Z) \subset \mathfrak{X}(M)$ consisting of those vector fields v which are tangent to Z can be identified with the space of smooth sections of the **b-tangent bundle** ${}^{b}T(M, Z) \to M$. By its very construction, ${}^{b}T(M, Z)$ comes equipped with a bundle map ${}^{b}T(M, Z) \to TM$ covering id_M, which is the identity outside Z. Its restriction to Z defines an epimorphism ${}^{b}T(M, Z)|_Z \to TZ$, whose kernel ${}^{b}N(M, Z)$ has a canonical trivialization $\nu \in \Gamma(Z, {}^{b}N(M, Z))$: if one expresses Z locally as $x_1 = 0$ in a coordinate chart $(x_1, ..., x_n)$, then $x_1 \frac{\partial}{\partial x_1}$ is a no-where vanishing local section of ${}^{b}T(M, Z)$ independent of choices along Z.

The bundle dual to ${}^{b}T(M, Z)$ will be denoted by ${}^{b}T^*(M, Z)$; sections of its p-th exterior power will be called b-forms (of degree p) on (M, Z), and we write ${}^{b}\Omega^p(M, Z)$ for the space of all such forms.

Since $\mathfrak{X}(M, Z) \subset \mathfrak{X}(M)$ is a Lie subalgebra, ${}^{b}T(M, Z)$ has a natural structure of Lie algebroid, and as such, it carries a differential

$$ {}^{b}d : {}^{b}\Omega^p(M, Z) \to {}^{b}\Omega^{p+1}(M, Z) $$

given by the usual Koszul-type formula. Note that agrees with d outside Z, and that we have a short exact sequence of chain complexes:

$$ 0 \longrightarrow (\Omega^\bullet(M), d) \longrightarrow ({}^{b}\Omega^\bullet(M, Z), {}^{b}d) \overset{\flat}{\longrightarrow} (\Omega^{\bullet-1}(Z), d) \longrightarrow 0, \qquad (1) $$

where $\flat$ maps a b-form ω to its contraction with the canonical ν.

2.2. **Cosymplectic and b-symplectic structures.** Mimicking the usual terminology, $\omega \in {}^b\Omega^2(M, Z)$ will be called b-**symplectic** if ω^{top} is nowhere vanishing and ω is closed, ${}^b d\omega = 0$.

We recall from [9, Proposition 20] that there is a bijective correspondence between b-symplectic forms on (M, Z), and Poisson structures of b-symplectic type with singular locus Z. We can thus speak unambiguously of b-*symplectic manifolds*; that is, if (M, π) is a Poisson manifold of b-symplectic type, then $(\pi|_{M \setminus Z(\pi)})^{-1}$ extends to a b-symplectic form ω on the b-manifold $(M, Z(\pi))$.

Let then (M, ω) be a b-symplectic manifold, with singular locus $Z \subset M$. By the very definition, $\omega|_{M \setminus Z}$ is a symplectic manifold in the usual sense.

Before we describe the Poisson structure around points in the singular locus Z, we recall from [8] how *cosymplectic structures* appear in our context. Recall that a if $(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$ has the property that $\theta \wedge \eta^{\mathrm{top}-1}$ is a volume form, then to such a pair there corresponds a pair $(R, \nu) \in \mathfrak{X}(Z) \times \mathfrak{X}^2(Z)$, where R is the *Reeb vector field*

$$\iota_R \theta = 1, \ \iota_R \eta = 0,$$

and ν is characterized by:

$$\iota_\theta \nu = 0, \ \nu^\sharp \eta^\sharp + \theta \otimes R = \mathrm{id}_{TZ}.$$

Then:

$$d\theta = 0, \ d\eta = 0 \quad \Longleftrightarrow \quad [R, \nu] = 0, \ [\nu, \nu] = 0.$$

Hence from a Poisson-theoretic perspective, a cosymplectic structure (θ, η) is a pair (R, ν) consisting of a corank-one Poisson structure $\nu \in \mathrm{Poiss}(Z)$, together with a Poisson vector field $R \in \mathfrak{X}(Z)$ transverse to the leaves of ν.

A natural way cosymplectic structures occur in Symplectic Geometry is as hypersurfaces transverse to symplectic vector fields. The setting is the following: one is given a hypersurface X in a symplectic manifold (W, Ω), and we assume that on an open neighborhood $U \subset W$ of X there exists a symplectic vector field $v \in \mathfrak{X}(U)$ transverse to X. Denote by

$$\varphi : U \times \mathbb{R} \supset \mathrm{dom}(\varphi) \longrightarrow W, \quad \frac{d}{dt}\varphi_t(x) = v \circ \varphi_t(x)$$

the local flow of v, and consider the induced map:

$$c : X \times \mathbb{R} \cap \mathrm{dom}(c) \longrightarrow W, \quad c(x, t) := \varphi_t(x).$$

Then the *adapted collar* c embeds in W the open neighborhood

$$V_\varepsilon \subset X \times \mathbb{R}, \quad V_\varepsilon := \{(x, t) | -\varepsilon(x) < t < \varepsilon(x)\}$$

for some small enough positive function $\varepsilon : U \to \mathbb{R}_+$, and c pulls Ω back to:

$$c^* \Omega = dt \wedge \theta + \eta, \tag{2}$$

where $(\theta, \eta) := (c^* \iota_v \Omega|_X, c^* \Omega|_X)$ and we have abused notation and omitted the first projection map on $V_\varepsilon \subset X \times \mathbb{R}$.

More to the point, a description similar to (2) exists for a b-symplectic structure ω in a neighborhood of its singular locus $Z(\omega)$. When $Z(\omega)$ is coorientable, it goes as follows: start with any collar $c : V_\varepsilon \hookrightarrow M$ extending $\mathrm{id}_{Z(\omega)}$ and regard c as a b-embedding $c : (V_\varepsilon, Z(\omega)) \hookrightarrow (M, Z(\omega))$. If we let θ stand for $\flat(c^*\omega)$, then $\widetilde{\eta} := c^*\omega - d\log|t| \wedge \theta$ is an honest closed two-form $\widetilde{\eta} \in \Omega^2(V_\varepsilon)$; in particular, $\eta := \widetilde{\eta}|_{Z(\omega)}$ makes sense as a closed two-form on $Z(\omega)$, and note that (θ, η) defines a cosymplectic structure on $Z(\omega)$.

Define on V_ε the b-symplectic form

$$\omega_0 := d\log|t| \wedge \theta + \eta \tag{3}$$

and observe that the path of b-forms $\omega_t := \omega_0 + t(c^*\omega - \omega_0)$ is b-symplectic on $V_{\varepsilon'}$, for some $0 < \varepsilon' < \varepsilon$. Now, $c^*\omega - \omega_0 = d\alpha$, for some $\alpha \in \Omega^1(V_{\varepsilon'})$ which vanishes along $Z(\omega)$. Hence the time-dependent b-vector field v_t defined by $\iota_{v_t}\omega_t + \alpha = 0$ vanishes along $Z(\omega)$, and hence the local flow of v_t is defined up to time one on some $V_{\varepsilon''}$ for some $0 < \varepsilon'' < \varepsilon'$, and satisfies $\phi_t^*\omega_t = \omega_0$. Therefore, we obtain the *adapted collar*:

$$\bar{c} := c\phi_1 : V_{\varepsilon''} \hookrightarrow M$$

pulling ω back to ω_0. Note that in formula (4) the one-form θ is intrinsically defined, while η depends on the collar, this choice not affecting its restriction to the symplectic distribution $\ker \theta$. Observe also that the inverse $\pi_0 := \omega_0^{-1} \in \mathrm{Poiss}(M)$ is given by:

$$\pi_0 := t\frac{\partial}{\partial t} \wedge R + \nu, \tag{4}$$

where (R, ν) is the pair determined by (θ, η). In particular, on its singular locus $Z(\omega)$, a b-symplectic form ω determines a corank-one Poisson structure ν which comes from a cosymplectic structure (θ, η) on $Z(\omega)$. This is still the case when $Z(\omega)$ is not coorientable in M (as in the example below), where a $\mathbb{Z}_2$-equivariant version of the normal form (4) can be proved for the pullback of ω to the orientation covering of M.

Example 2.1 (Radko's sphere)**.** *The b-form $\omega = \frac{1}{h}dh \wedge d\theta$ on $\mathbb{S}^2$, where h, θ stand for cylindrical coordinates, is b-symplectic. Its symplectic leaves are either points in the equator $\mathbb{S}^1 \subset \mathbb{S}^2$, or components of $\mathbb{S}^2 \backslash \mathbb{S}^1$. This b-form is invariant under the antipodal map on the sphere and, thus, induces a b-symplectic form on the projective plane $\mathbb{RP}^2$ which is a non-orientable b-symplectic manifold.*
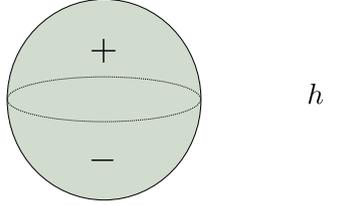

FIGURE 1. Radko sphere $\mathbb{S}^2$ with the equator as critical hypersurface and the upper and lower hemisphere as positive and negative symplectic leaf, respectively.

2.3. **Cobordisms.** An alternative perspective on b-symplectic manifolds is that they arise by gluing certain cobordisms in the symplectic category.

Definition 2.1. *A **cosymplectic cobordism** is a compact symplectic manifold (M, ω), together with a cosymplectic structure of the form $(\theta, \omega|_{\partial M})$ on its boundary ∂M.*

As pointed out in the previous subsection, the cosymplectic structure $(\theta, \omega|_{\partial M})$ is induced by a symplectic vector field v defined in a neighborhood of ∂M and transverse to it. A connected component X of ∂M is called *incoming* or *outgoing* acording to whether a such defining transverse symplectic vector field v points into M or out of it, respectively.

Observe that if $(M_0, \omega_0, \theta_0)$ and $(M_1, \omega_1, \theta_1)$ are cosymplectic cobordisms, and there exists a diffeomorphism:

$$\varphi : \partial M_0 \xrightarrow{\sim} \partial M_1, \quad \varphi^*\theta_1 = \theta_0, \quad \varphi^*(\omega|_{\partial M_1}) = \omega|_{\partial M_0},$$

then a b-symplectic manifold (M_{01}, ω_{01}) and embeddings $\varphi : M_i \hookrightarrow M$ exist, with:

- $Z(\omega_{01}) = \varnothing$ and
$$\varphi_0 M_0 \cup \varphi_1 M_1 = M_{01}$$
  if ∂M_0 is incoming and ∂M_1 is outgoing, or ∂M_0 is outgoing and ∂M_1 is incoming;
- $Z(\omega_{01}) \simeq \partial M_0 \simeq \partial M_1$ and outside a neighborhood $U \simeq Z(\omega_{01}) \times (-1, 1)$ of $Z(\omega_{01})$, we have:
$$\varphi_0 M_0 \coprod \varphi_1 M_1 = M \smallsetminus U$$
  if $\partial M_0, \partial M_1$ are both incoming or both outgoing.

In the first case one constructs collars as in (4) and glues the result into a symplectic manifold; in the second case the orientations do not match, and so one mediates the previous gluing by a collar of the form:
$$(Z(\omega) \times [-1, 1], df \wedge \theta + \eta),$$
where $f = f(t)$ is a monotone function with $df = d \log |t|$ around $t = 0$ and $df = \pm dt$ around $t = \pm 1$.

Recall that the **double** $M \# \overline{M}$ of an orientable manifold with boundary can be given, up to isomorphism, a unique smooth structure in which $M \hookrightarrow M \# \overline{M}$ and $\overline{M} \hookrightarrow M \# \overline{M}$ are smooth embeddings. An illustration of the natural way b-symplectic structures occur when trying to pass from compact symplectic manifolds to closed ones is given by the following result:

Corollary 2.1. *If (M, ω, θ) is a cosymplectic cobordism, then its double $M \# \overline{M}$ is a b-symplectic manifold.*

Of course, when the boundary orientations are opposite, the singular locus of the ensuing b-symplectic manifold is empty, and it is a consequence of Moser' s argument that its symplectomorphism type does not depend on the choice of collars.

For example, Radko's sphere is the *b-symplectic double* of any closed symplectic disk $(\mathbb{D}^2, \omega)$ endowed with a nowhere zero 1-form on $\mathbb{S}^1$ of period 2π.

Remark 2.1. *Just as in the symplectic case, a b-symplectic version of Moser's argument* [9] *shows that the b-symplectic type of a b-symplectic manifold obtained by gluing symplectic cobordisms with equal boundary orientations does not depend on the choices of collars.*

3. h-PRINCIPLE

In this section, we use standard h-principle arguments to provide a complete answer to the Existence Problem (P1) under the additional assumption that M is open.

A necessary condition for a manifold M^{2n} to be symplectic is that it carry a nondegenerate two-form, or, equivalently, an almost-complex structure. If M is compact, we have a further necessary condition, namely, that there be a degree-two cohomology class $\tau \in H^2(M)$ with $\tau^n \neq 0$.

For *open* manifolds M a classical theorem of Gromov [6] states that the sole obstruction to the existence of a symplectic structure is that M be almost-complex. More precisely, given any non-degenerate two-form ω_0 and a cohomology class $\tau \in H^2(M)$, there is a path $\omega : [0, 1] \to \Omega^2(M)$ of nondegenerate two-forms connecting ω_0 to a closed two-form ω_1 representing $[\omega_1] = \tau$.

We consider now the case of b-symplectic structures. Recall that b-symplectic manifolds need *not* be oriented as usual manifolds, so in particular they may fail to be almost-complex. However:

Lemma 3.1. *If an orientable M admits a b-symplectic structure ω, then $M \times \mathbb{C}$ is almost-complex.*

The proof is a straightforward adaptation of the argument in [1, §4], which we omit.

Just as in the symplectic case, if we demand that M be compact, the existence of a b-symplectic structure is obstructed:

(1) there exists a cohomology class $\tau \in H^2(M)$ with $\tau^{n-1} \neq 0$ [11];
(2) furthermore, if M is orientable, a non-trivial $\vartheta \in H^2(M)$ must exist squaring to zero [3].

None of these obstructions appear when M is open, so one wonders if, in that case, $M \times \mathbb{C}$ being almost-complex is sufficient to ensure that M carries a b-symplectic structure. We answer the question in the affirmative:

Theorem A. *Let M be an orientable, open manifold. Then M is b-symplectic if and only if $M \times \mathbb{C}$ is almost-complex.*

We need to introduce the analogs of nondegenerate two-forms. To do that, observe that a bivector $\pi \in \mathfrak{X}^2(M^{2n})$ whose top exterior power $\bigwedge^n \pi$ is transverse to the zero section defines a b-manifold $(M, Z(\pi))$, $Z(\pi) := (\bigwedge^n \pi)^{-1}M \subset M$. Since b-bivectors $\mathfrak{X}^2(M, Z(\pi))$ sit inside the space of all bivectors $\mathfrak{X}^2(M)$, it makes sense to require π to be a b-bivector in the b-manifold it defines (a simple coordinate check shows this is not true in general).

Definition 3.1. *A bivector $\pi \in \mathfrak{X}^2(M)$ is b-serious if it is transversally nondegenerate and a b-bivector in $(M, Z(\pi))$.*

In the sequel we show that b-serious bivectors can be homotoped into b-symplectic ones, provided that the manifold be open:

Theorem 3.1. *On an* open *manifold M, a b-serious bivector π_0 is homotopic through b-serious bivectors to a Poisson bivector π_1. Moreover, one can arrange that $Z(\pi_1)$ be non-empty if $Z(\pi_0)$ is non-empty.*

This statement is a result of checking that 1-jets of Poisson bivectors of exact b-symplectic type form a microflexible differential relation, invariant under the pseudogroup of local diffeomorphisms of M, cf. [6]. We opted instead to follow the somewhat more visual scheme of proof of [5].

Proof. Take $\pi_0 \in \mathfrak{X}^2_{\overline{\square}}(M)$ b-serious, $\pi_0 \in \mathfrak{X}(M, Z_0)$, and let $\omega_0 \in {}^b\Omega^2(M, Z_0)$ the corresponding b-symplectic form.

Observe that the b-differential ${}^bd : {}^b\Omega^p(M, Z_0) \longrightarrow {}^b\Omega^{p+1}(M, Z_0)$ can be factored as a composition ${}^bd = \widetilde{\mathrm{symb}}({}^bd) \circ j_1$, where j_1 denotes the 1-jet map

$$j_1 : \Gamma(M, \bigwedge^p {}^bT^*(M, Z_0)) \longrightarrow \Gamma(M, J_1 \bigwedge^p {}^bT^*(M, Z_0))$$

and

$$\widetilde{\mathrm{symb}}({}^bd) : \Gamma(M, J_1 \bigwedge^p {}^bT^*(M, Z_0)) \longrightarrow \Gamma(M, \bigwedge^{p+1} {}^bT^*(M, Z_0))$$

is induced by a bundle map

$$\mathrm{symb}({}^bd) : J_1 \bigwedge^p {}^bT^*(M, Z_0) \longrightarrow \bigwedge^{p+1} {}^bT^*(M, Z_0).$$

As one easily checks, $\mathrm{symb}({}^bd)$ is an epimorphism with contractible fibres; in particular, we can lift ω_0 to $\widetilde{\omega}_0 \in \Gamma(M, J_1 \bigwedge^p {}^bT^*(M, Z_0))$.

Now, since M is an open manifold, there exists a a subcomplex K of a smooth triangulation of M, of positive codimension, with the property that, for an arbitrarily small open $U \subset M$ around K, there exists an isotopy of open embeddings $g_t : M \hookrightarrow M$, $g_0 = \mathrm{id}_M$, with $g_1(M) \subset U$ and $g_t|_K = \mathrm{id}_K$. We will refer to K as a core of M, and say that g_t compresses M into U. Note in passing that one can always find a core K of M meeting Z_0.

Fix then a core K of M, and a compression of M into an open U around K. The Holonomic Approximation theorem of [5] then says that we can find

- an isotopy h_t of M mapping K into U;
- an open $V \subset U$ around $h_1(K)$;
- a section $\alpha \in \Gamma(V, {}^bT^*(M, Z_0))$

such that $j_1\alpha$ is so C^0-close to $\widetilde{\omega}_0$ that we can find a homotopy

$$\widetilde{\omega}(t) \in \Gamma(V, J_1{}^bT^*(M, Z_0)),$$

connecting $\widetilde{\omega}_0|_V$ to $j_1\alpha$, and with $\widetilde{\mathrm{symb}}({}^bd)\widetilde{\omega}_t$ nondegenerate b-forms on V. Moreover, the scheme of proof in [5] shows that one can require in addition that $h_1(K)$ also meet Z_0.

Now regard the compression g_t as a smooth family of b-maps

$$g_t : (M, Z_t) \longrightarrow (M, Z_0), \quad Z_t := g_t^{-1}Z_0,$$

and set $\omega_1 := {}^bd(g_1^*\alpha) \in {}^b\Omega^2(M, Z_1)$. Observe now that $\widehat{\omega}_t^1 := g_t^*\widetilde{\omega}_0$ connects $\widetilde{\omega}_0$ to $g_1^*(\widetilde{\omega}_0|_V)$, and $\widehat{\omega}_t^2 := g_1^*\widetilde{\omega}(t)$ connects $g_1^*(\widetilde{\omega}_0|_V)$ to a lift of ω_1. Let $\widehat{\omega}_t$ denote the concatenation of $\widehat{\omega}_t^1$ and $\widehat{\omega}_t^2$:

$$\widehat{\omega}_t := \begin{cases} \widehat{\omega}_{2t}^1 & 0 \leqslant t \leqslant 1/2, \\ \widehat{\omega}_{2t-1}^2 & 1/2 \leqslant t \leqslant 1. \end{cases}$$

Then $t \mapsto \pi_t := \widehat{\omega}_t^{-1} \in \mathfrak{X}(M, Z_t)$ defines a homotopy of b-serious bivectors between π_0 and a Poisson π_1. $\square$

A few remarks are in order:

- If ω_0 could be C^0-approximated by a closed $\omega_1 \in {}^b\Omega^2(M, Z_0)$, we would be done. However, such an approximation is severely obstructed, in that it would imply that Z_0 admits a structure of cosymplectic manifold.(For compact Z, the existence of a cosymplectic structure implies e.g. that it has non-trivial cohomology in all possible degrees).
- We get around this problem by changing the topology of Z_0 rather drastically; observe in particular that Z_1 may be disconnected even if Z_0 is connected. One should perhaps think of Z_1 as Z_0 with those places 'blown to infinity' where ω_0 cannot be approximated by closed b-forms.
- Of course, when M is itself almost-complex, Gromov's theorem allows us to produce an honest symplectic structure.
- If in the statement of Theorem 3.1 we further assume that:
  - $Z = Z(\pi_0)$ is a regular fibre $f^{-1}(0)$ of a proper Morse function $f : M \to \mathbb{R}$, unbounded from above and from below, and
  - ω_0 is already ${}^bd$-closed around Z,

  then one can impose that the homotopy π_t above be stationary around Z [5, Theorem 7.2.4].

This last comment can be regarded as a sufficient condition to realize a given cosymplectic structure on Z on a *given* manifold M:

Corollary 3.1. *Any given cosymplectic structure on the regular fibre Z of a proper Morse function $f : M \to \mathbb{R}$ can be realized as the singular locus of a b-symplectic structure, provided f is unbounded from above and from below.*

Proof of Theorem A. It remains to show that the existence of an almost-complex structure on $M \times \mathbb{C}$ ensures the existence of a b-serious bivector. But according to [2], the former guarantees the existence of a folded symplectic form $\phi \in \Omega^2(M)$, namely, a closed two-form, whose top power is transverse to the zero section, thus defining a smooth folding locus $Z = (\phi^n)^{-1}M \subset M$, along which ϕ^{n-1} does not vanish. For every one-form $\theta \in \Omega^1(Z)$ satisfying $\theta \wedge \phi|_Z^{n-1} \neq 0$ everywhere, one can construct an open embedding

$$c : C \hookrightarrow M, \quad c^*\phi = d\left(\frac{t^2}{2}\theta\right) + \phi|_Z,$$

following the recipe in [1, Theorem 1], where $C \subset Z \times \mathbb{R}$ is an open neighborhood of Z.

Fix a Riemannian metric g on M with $c^*g := g_Z + dt^2$, and denote by π the bivector on M which is g-dual to ϕ; then

$$c^*\pi = t\frac{\partial}{\partial t} \wedge R + \nu,$$

where

$$R = g^\flat(\theta), \quad \nu = g^\flat\left(\phi|_Z + t^2/2d\theta\right)$$

has no $\frac{\partial}{\partial t}$-component. Therefore, π is b-serious, and has singular locus Z. $\square$

4. Prescribing the singular locus of a b-symplectic manifold

The Existence problem (P2) asks for the description of those corank 1 Poisson manifolds that appear as singular locus of a b-symplectic manifold:

Definition 4.1. *A cosymplectic manifold is the singular locus of a b-symplectic manifold (M, ω), if it is diffeomorphic to $Z(\omega)$, endowed with the cosymplectic structure induced by some adapted collar.*

Thus, our problem splits into two questions: Firstly, deciding which corank-one Poisson structures come from cosymplectic structures, a matter discussed in [8]. Secondly, describing which (connected) cosymplectic manifolds are singular loci of b-symplectic manifolds, which is the question we will tackle.

Note that the question is trivial if either:

- the b-symplectic manifold is not required to have empty boundary (e.g., a closed, adapted collar associated to a cosymplectic manifold (Z, θ, η) would provide such a realization), or
- if we do not require that the cosymplectic manifold be the entire singular locus of (M, ω) (e.g., the double of the adapted collar is a b-symplectic manifold without boundary, whose singular locus consists of *two* copies of (Z, θ, η)).

We shall therefore assume that all cosymplectic manifolds appearing henceforth are compact and connected.

Definition 4.2. *Let (Z_0, η_0, θ_0), (Z_1, η_1, θ_1) be cosymplectic manifolds. We say that Z_0 is **cosymplectic cobordant** to Z_1 if there exists a cosymplectic cobordism (M, ω, θ) and diffeomorphisms of cosymplectic manifolds*

$$\varphi_0 : (\partial_{\text{in}}M, \omega|_{\partial_{\text{in}}M}, \theta) \xrightarrow{\sim} (Z_0, \eta_0, \theta_0), \quad \varphi_1 : (\partial_{\text{out}}M, \omega|_{\partial_{\text{out}}M}, \theta) \xrightarrow{\sim} (Z_1, \eta_1, \theta_1).$$

A cosymplectic manifold (Z, η, θ) will be called **symplectically fillable** if it is cosymplectic cobordant to the empty set.

Lemma 4.1. *A cosymplectic manifold is the singular locus of a closed, oriented b-symplectic manifold if and only if it is symplectically fillable.*

Proof. If (Z, θ, η) is the singular locus of (M, ω), then after removing an open, adapted collar inducing the cosymplectic structure, we obtain a cosymplectic cobordism with two connected components, each of which is a cosymplectic cobordism from (Z, θ, η) to the empty set.

Conversely, if we have a cosymplectic cobordism from (Z, θ, η) to the empty set, then (Z, θ, η) is the singular locus of the double of the cobordism. $\square$

Since cosymplectic cobordisms can be composed, any cosymplectic structure cobordant to a fillable one is also fillable. Thus, it is important to discuss some basic constructions of cosymplectic cobordisms.

Lemma 4.2. *Two cosymplectic structures (η_0, θ_0), (η_1, θ_1) on Z are cobordant if there is a homotopy (η_t, θ_t) of cosymplectic structures joining them, and $[\eta_t]$ is constant.*

In particular, among cosymplectic structures on Z symplectic fillability is stable under rescaling of the 1-form by a positive scalar, small C^0-perturbations of the 1-form, and under replacement of the 2-form by a cohomologous one.

Proof. Subdivide $[0, 1]$ into $0 = t_0 < t_1 < \cdots < t_N = 1$ so that $\theta_{t_{i+1}}|_{\ker \eta_t} > 0$, for all $t \in [t_i, t_{i+1}]$. It suffices to show that $(Z, \eta_{t_i}, \theta_{t_i})$ is cobordant to $(Z, \eta_{t_{i+1}}, \theta_{t_{i+1}})$ for each i, so we may as well assume that $N = 1$. Now, $\theta_t \wedge dt + \eta_0$ is then a symplectic form on $M = Z \times [0, 1]$ defining a cosymplectic cobordism between (Z, η_0, θ_0) and (Z, η_0, θ_1). Hence we may assume without loss of generality $\theta_0 = \theta_1$.

Let $\mathcal{F}$ denote the codimension-two foliation on M which is the product of the foliation of the interval by points and the cosymplectic foliation on Z. We employ a suitable adaptation of Thurston's trick for $\mathcal{F}$.

Let $\eta_0 - \eta_t = d\alpha_t$, and choose $\tilde{\alpha} \in \Omega^1(M)$ such that $\tilde{\alpha}_{|Z \times \{t\}} = \alpha_t$. Define $\omega' = \eta_0 - d\tilde{\alpha}$ which is symplectic on the leaves of $\mathcal{F}$. Since M is compact, for $K > 0$ large enough, the form

$$\omega = \omega' + K\theta_0 \wedge dt$$

is symplectic and restricts to η_i on Z_i. Hence (M, ω, θ), with $\theta_{|Z \times \{i\}} = \theta_i$, $i = 0, 1$, is the desired cobordism from (η_0, θ_0) to (η_1, θ_1). Reversing the direction of the path (η_t, θ_t) gives a cobordism in the opposite direction. $\square$

Next, we discuss symplectic fillability for the C^0-dense subset of cosymplectic structures given by 1-forms with rational periods.

Symplectic mapping tori and symplectic fillings. We shall regard a mapping torus as a foliated bundle with base $\mathbb{S}^1$. The corresponding holonomy representation is generated by a diffeomorphism $\varphi \in \text{Diff}(F)$. Conversely, the suspension any such φ defines a mapping torus $Z(\varphi)$ with fiber diffeomorphic to F.

As F is assumed to be compact, mapping tori can be defined as fibrations $Z \to \mathbb{S}^1$ with an Ehresmann connection. We shall also identify $\mathbb{S}^1$ with $\mathbb{R}/\mathbb{Z}$.

A **symplectic mapping torus** is a bundle over the circle, whose total space is endowed with a closed two-form η which is symplectic on each fiber. The kernel of η defines an Ehresmann connection, and its holonomy φ preserves the symplectic structure of the fiber, i.e., $\varphi \in \text{Symp}(F, \sigma)$. Conversely, the suspension of any $\varphi \in \text{Symp}(F, \sigma)$ canonically defines a symplectic mapping torus $(Z(\varphi), \eta_\varphi)$.

A symplectic mapping torus becomes a cosymplectic manifold upon the choice of a defining closed 1-form for the fibration; this is equivalent to the choice of a period $\lambda > 0$, as it is convened that the pullback of the oriented generator of $H^1(\mathbb{S}^1; \mathbb{Z})$ has period 1.

We shall abuse notation and regard a symplectic mapping torus as a cosymplectic manifold $(Z(\varphi), \eta_\varphi, \theta_\varphi)$ by declaring θ_φ to have period 1. Having this convention

in mind, Lemma 4.2 implies that a cosymplectic manifold with compact foliation is symplectically fillable if and only if its associated symplectic mapping torus is symplectically fillable.

Let $\varphi_0, \varphi_1 \in \mathrm{Symp}(F, \sigma)$. A symplectic isotopy φ_t between φ_0 and φ_1 corresponds to an isotopy of symplectic vector fields given by the normalized kernel of the closed 2-forms η_{φ_t}. It is well-known that the isotopy can be modified to a Hamiltonian one iff the vector fields are Hamiltonian, i.e. if the cohomology class $[\eta_{\varphi_t}]$ is independent of t. Hence by Lemma 4.2 the symplectic fillability of $(Z(\varphi), \eta_\varphi)$ only depends on the Hamiltonian isotopy class of φ. Lastly, since changing the identification of the fiber (F, σ) by a symplectomorphism results in conjugating the holonomy by that symplectomorphism, we see that whether $(Z(\varphi), \eta_\varphi)$ is symplectically fillable or not φ only depends on the Hamiltonian isotopy class of its conjugacy class.

In what follows, we discuss a family of Hamiltonian isotopy classes of diffeomorphisms yielding fillable symplectic mapping tori.

Dehn twists. There is a class of symplectomorphisms φ which we can 'cap off': Dehn twists. We briefly recall the construction of those maps, and refer the reader to [17] for further details.

The norm function $\mu : (T^*\mathbb{S}^{n-1} \setminus \mathbb{S}^{n-1}) \to \mathbb{R}$, $\mu(\xi) = \|\xi\|$, associated to the round metric on $\mathbb{S}^{n-1}$, is the moment map of a Hamiltonian $\mathbb{S}^1$-action on $(T^*\mathbb{S}^{n-1} \setminus \mathbb{S}^{n-1})$. Upon identifying $T^*\mathbb{S}^{n-1} \subset \mathbb{R}^n \times \mathbb{R}^n$ as $T^*\mathbb{S}^{n-1} = \{(u, v) : \langle u, v \rangle = 0, \|u\| = 1\}$, we can write

$$e^{2\pi i t} \cdot (u, v) = \left(\cos(2\pi t)u + \sin(2\pi t)v\|v\|^{-1}, \cos(2\pi t)v - \sin(2\pi t)\|v\|u \right),$$

Then $e^\pi \cdot (u, v) = (-u, -v)$ extends by the antipodal map to a symplectomorphism $T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$.

Choose now a function $r : \mathbb{R} \to \mathbb{R}$, satisfying:

(1) $r(t) = 0$ for $|t| \geqslant C > 0$;
(2) $r(t) - r(-t) = t$ for and $|t| \ll 1$,

and let ϕ^t denote the flow of the Hamiltonian vector field of $r(\mu)$.

Observe that $\phi^{2\pi}$ extends to a symplectomorphism $\psi : T^*\mathbb{S}^{n-1} \to T^*\mathbb{S}^{n-1}$, supported on the compact subspace $T(\varepsilon) \subset T^*\mathbb{S}^{n-1}$ the subspace of cotangent vectors of length $\leqslant C$. This is called a **model Dehn twist**.

We can graft this construction onto manifolds using Weinstein's Lagrangian neighborhood theorem. If $l : \mathbb{S}^{n-1} \hookrightarrow (F, \sigma)$ embeds $\mathbb{S}^{n-1}$ as a Lagrangian sphere, there are neighborhoods $\mathbb{S}^{n-1} \subset U \subset T^*\mathbb{S}^{n-1}$ and $l(\mathbb{S}^{n-1}) \subset V \subset F$ and a symplectomorphism $\varphi : (U, \omega_{\mathrm{can}}) \to (V, \omega)$ extending l. If ψ is a model Dehn twist, supported inside U, we produce a symplectomorphism $\tau : (F, \sigma) \to (F, \sigma)$, supported in V, by

$$\tau(x) := \begin{cases} \varphi \circ \psi \circ \varphi^{-1}(x) & \text{if } x \in V; \\ x & \text{if } x \in F \setminus V. \end{cases}$$

Definition 4.3. *A symplectomorphism of the form above will be called a **Dehn twist** around $l := l(\mathbb{S}^{n-1})$, and it will be denoted by τ_l.*

We also recall that any two Dehn twists around a parametrized Lagrangian sphere l are Hamiltonian isotopic if $n > 2$, and symplectically isotopic if $n = 2$ [17].

Proof of Theorem C. By Lemma 4.1, is it enough to show that $Z(\varphi)$ is symplectically fillable.

Assume that $l : \mathbb{S}^{n-1} \hookrightarrow Z(\varphi)$, $n > 2$, is a parametrized Lagrangian sphere sitting inside a leaf of $Z(\varphi)$, and observe that the normal bundle to l in $Z(\varphi)$ is trivial and carries a canonical framing. The cobordism M obtained from $Z(\varphi) \times [0, 1]$ by

attaching a n-handle along $l \times 1$ carries the structure of a cosymplectic cobordism, as follows from [13, Proposition 4], with $\partial_{\mathrm{in}} M \simeq Z(\varphi)$ and $\partial_{\mathrm{out}} M \simeq Z(\tau_l^{-1}\varphi)$ according to [13, Theorem 3]. We call this the *trace* of a positive Lagrangian surgery along l. Negative Lagrangian surgeries can be similarly defined, by attaching a n-handle to $l \times 1$ according to the opposite of the canonical framing, and one obtains a cosymplectic cobordism M with $\partial_{\mathrm{in}} M \simeq Z(\varphi)$ and $\partial_{\mathrm{out}} M \simeq Z(\varphi \tau_l)$. Note that if, after performing negative Lagrangian surgery, we modify the identification of F by the Dehn twist, we also obtain $\partial_{\mathrm{out}} M \simeq Z(\tau_l \varphi)$.

By hypothesis, φ is –up to conjugacy by a symplectomorphism– Hamiltonian isotopic to $\tau_{l_1}^{\varepsilon_1} \cdots \tau_{l_m}^{\varepsilon_m}$, where $l_i : \mathbb{S}^{n-1} \hookrightarrow (F, \sigma)$, are parametrized Lagrangian spheres and $\varepsilon_i = \pm 1$, $i = 1, \ldots, m$.

By the discussion above,

$$Z(\varphi), \quad Z(\tau_{l_1}^{-\varepsilon_1}\varphi), \quad \cdots, \cdots, Z(\tau_{l_1}^{-\varepsilon_1} \cdots \tau_{l_m}^{-\varepsilon_m}\varphi) = Z(\mathrm{id}_F)$$

are all cosymplectic cobordant, and $Z(\mathrm{id}_F)$ is symplectically fillable, since it bounds $(F \times \mathbb{D}^2, \sigma + dy_1 \wedge dy_2)$.

This shows that $Z(\varphi)$ is symplectically fillable, and hence by Lemma 4.1 it is the singular locus of (M, ω) a closed, oriented b-symplectic manifold. The construction of (M, ω) non-orientable deferred to the end of the section. $\square$

Proof of Theorem B. Let (Z, η, θ) be a 3-dimensional cosymplectic manifold. By Lemma 4.1 all we must show is that (Z, η, θ) is symplectically fillable (again, the case of (M, ω) non-orientable is deferred to the end of the section). By Lemma 4.2, we may assume without loss of generality that (Z, η, θ) is a symplectic mapping torus. But, according to Eliashberg [4], all 3-dimensional symplectic mapping tori are symplectically fillable. $\square$

Remark 4.1 (A comment of Y. Mitsumatsu). *One can give an alternative proof of Theorem B without using the full strength of Eliashberg's result as follows: every symplectic transformation on a closed surface Σ is symplectically isotopic to a word on Dehn twists [10]. By the proof of Theorem C, our starting mapping torus is cobordant to one whose monodromy is symplectically isotopic to the identity. This means that we arrived at the trivial mapping torus $\Sigma \times \mathbb{S}^1$, but with*

$$\eta = \sigma + \theta \wedge \beta,$$

where β is a closed 1-form on Σ. By Lemma 4.2, we may assume without loss of generality that β has integral periods. As Y. Mitsumatsu has pointed out to us, one can get rid of the summand $\theta \wedge \beta$ by first applying a positive Lagrangian surgery, followed by a negative Lagrangian surgery, around the Poincaré dual to β (and rescaling the original η).

Remark 4.2 (The non-orientable case). *A few words about the non-orientable case are in order. The normal bundle to the singular locus of a non-orientable b-symplectic manifold is classified by a nontrivial $\tau \in H^1(Z; \mathbb{Z}_2)$, which determines a two-sheeted covering $Z_\tau \to Z$. It is not difficult to see that a cosymplectic manifold (Z, θ, η) is the singular locus of a non-orientable (M, ω), with normal bundle given by $\tau \in H^1(Z; \mathbb{Z}_2)$, if and only if the covering $(Z_\tau, \theta_\tau, \eta_\tau)$ is the singular locus of an orientable b-symplectic manifold.*

This allows one to prove version of Theorems B and C when the ambient b-symplectic manifold is non-orientable.

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