CONTRIBUTION TO THE ERGODIC THEORY OF ROBUSTLY TRANSITIVE MAPS

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Abstract. In this article we intend to contribute in the understanding of the ergodic properties of the set $\mathcal{RT}$ of robustly transitive local diffeomorphisms on a compact manifold M without boundary. We prove that there exists a C^1 residual subset $\mathcal{R}_0 \subset \mathcal{RT}$ such that any $f \in \mathcal{R}_0$ has a residual subset of M with dense pre-orbits. Moreover, C^1 generically in the space of robustly transitive local diffeomorphisms with no splitting there are uncountably many ergodic expanding invariant measures with full support and exhibiting exponential decay of correlations. In particular, these results hold for an important class of robustly transitive maps considered in [7].

1. Introduction and Statement of the Main Results

In the last two decades many advances have been made to the study of robustly transitive diffeomorphisms, whose geometric properties are by now very well understood. In fact, it follows from Bonatti, Díaz, Pujals [1] that robustly transitive diffeomorphisms exhibit a weak form of hyperbolicity, namely, dominated splitting. The ergodic aspects of robustly transitive diffeomorphisms have called the attention of many authors recently. For instance, let us refer to the construction of SRB measures (see [3]) and maximal entropy measures (see [2]) for the class of DA-maps introduced by Mañé, and more recently, [12] proved intrinsic ergodicity(unique entropy maximizing measure) for partially hyperbolic diffeomorphisms homotopic to a hyperbolic one on the 3-torus.

The theory is much more incomplete in the non-invertible setting, in which case the study of robust transitivity has received far less attention. Since the negative iterates of an endomorphism are not easy to describe, the dynamics can be very hard to explain. Nevertheless, the first important contributions in this respect were given recently in [6, 7], where it is shown that there are robustly transitive local diffeomorphisms without any dominated splitting, and some necessary and sufficient conditions for robust transitivity of local diffeomorphisms are given. Our purpose here is to give a contribution to the better understanding of robustly transitive local diffeomorphisms and to give first results on their ergodic theory.

Let M be a compact Riemannian manifold. We say that an endomorphism $f : M \to M$ is *transitive* if there exists $x \in M$ such that its forward orbit by f, $\mathcal{O}_f^+(x) = \{f^n(x)\}_{n\geq 0}$, is dense in M. In the theory of differentiable dynamical systems, it is an important issue to know when a special feature is exhibited in all nearby systems (with respect to some topology), that is, a dynamical property is *robust* under perturbation. In particular, a map f is C^r *robustly transitive* $(r \geq 1)$

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if there exists a C^r open neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ is transitive.

We focus our attention to local diffeomorphisms, that is endomorphisms without critical points. Let us denote by $\mathcal{RT}$ the set of C^1 local diffeomorphisms on M that are robustly transitive. It is not hard to check that any endomorphism that admits a dense subset of points with dense pre-orbits is transitive. In our first result we address a sort of converse to the previous assertion for generic maps. More precisely,

Theorem A. *There exists a C^1 residual subset $\mathcal{R}_0 \subset \mathcal{RT}$ such that for any $f \in \mathcal{R}_0$ the following conditions hold:*

(1) *Periodic points are dense in M.*
(2) *All periodic orbits are hyperbolic.*
(3) *There exists a residual subset $\mathcal{D} \subset M$ of points such that for any $x \in \mathcal{D}$ the pre-orbit $\mathcal{O}_f^-(x) = \{w \in f^{-n}(x) : n \in \mathbb{N}\}$ is dense in M.*

In the remaining, our goal is to show that robustly transitive local diffeomorphisms are interesting from the ergodic theory point of view. For that discussion let us recall some necessary definitions. Given a compact forward invariant set Λ, we say that $f\mid_\Lambda$ has *no splitting in a C^1 robust way* if there exists a C^1 open neighborhood $\mathcal{U}(f)$ of f so that for all $g \in \mathcal{U}(f)$ the tangent space $T_\Lambda M$ does not admit non-trivial invariant subbundles. We denote by $\mathcal{RT}^* \subset \mathcal{RT}$ the open subset of C^1 robustly transitive local diffeomorphisms that have no splitting in a C^1 robust way. Moreover, an ergodic f-invariant probability measure μ is *expanding* if all the Lyapunov exponents are positive. Finally, we say that (f, μ) has *exponential decay of correlations* if there are constants $K, \alpha > 0$ and $\lambda \in (0,1)$ such that for all $\psi \in C^\alpha(M, \mathbb{R})$, $\varphi \in L^1(\mu)$ and $n \in \mathbb{N}$:

$$\Big| \int \psi\,(\varphi \circ f^n)\, d\mu - \int \psi\, d\mu \int \varphi\, d\mu \Big| \le K\lambda^n \|\psi\|_\alpha\, \|\varphi\|_{L^1(\mu)}.$$

Our next result illustrates that generically robustly transitive maps exhibit many ergodic measures with interesting dynamical meaning.

Theorem B. *There exists a C^1 residual subset $\mathcal{R}_1 \subset \mathcal{RT}^*$ such that for any $f \in \mathcal{R}_1$ there are uncountable many f-invariant, ergodic and expanding measures with full support and exponential decay of correlations.*

Some comments are in order. We note that, in opposition to the case of diffeomorphisms discussed in [1], there are open sets of local diffeomorphisms with robust non-existence of splitting (see Section 4 below). Moreover, there are open subsets of robustly transitive local diffemorphisms that do not admit invariant expanding measures, e.g. hyperbolic endomorphisms on $\mathbb{T}^n$. Clearly, these examples admit some non-trivial invariant subbundles robustly. The following proposition, which is interesting by itself, plays a key role for the proof of Theorem B.

Proposition 1. *There exists a C^1 residual subset $\mathcal{R}_1 \subset \mathcal{RT}^*$ such that every $f \in \mathcal{R}_1$ admits a periodic source with dense pre-orbit.*

Concerning our results it is an interesting question to understand if there are robustly transitive local diffeomorphisms such that the set of periodic saddle points is dense while the set of periodic sources is non-empty.

The paper is organized as follows. In Section 2 we present some definitions involved and prove auxiliary lemmas. In Section 3 we prove the main results stated in Section 1. In Section 4 we present a large class of robustly transitive local diffeomorphisms which are not uniformly expanding and exhibit good ergodic properties. Finally, in Section 5 we do some further comments concerning the existence of relevant expanding measures assuming the presence of some kind of dominated splitting.

2. Robust transitivity and limit sets

In this section we prove some preliminary results relating robust transitivity and existence of dense pre-orbits that play a key role in the proof of the main results. For that purpose we shall introduce first some definitions. Given $\delta > 0$, we say that $U \subset M$ is *δ-dense* if $M \subset \bigcup_{x\in U} B_\delta(x)$, where $B_\delta(x)$ stands for the ball of radius δ around x. For any endomorphism $f : M \to M$ and $x \in M$, the *ω-limit set* of a point x, denoted by $\omega_f(x)$, is the set of points $y \in M$ such that there exists a sequence $(n_k)_{k\in\mathbb{N}}$ of positive integers such that $f^{n_k}(x) \to y$ when k goes to infinity. Analogously, the *α-limit set* of x, denoted by $\alpha_f(x)$, is the set of accumulation points $y \in M$ by the pre-orbit of x, that is, there exists a sequence $(x_{n_k})_{k\in\mathbb{N}}$ in $\mathcal{O}_f^-(x)$ satisfying $f^{n_k}(x_{n_k}) = x$ and such that $x_{n_k} \to y$ when k goes to infinity. Clearly, $\omega_f(x) = M$ if and only if the forward orbit of x is dense, and analogous statement also holds for pre-orbits. The following lemmas provide a dichotomy of the limit sets for continuous endomorphisms.

Lemma 2.1 (Dichotomy of Transitivity)**.** *If $f \in C^0(M, M)$ then only one of the following properties hold: either $\omega_f(x) \neq M$ for all $x \in M$, or the set $\{x \in M : \omega_f(x) = M\}$ is a residual subset of M.*

Proof. Let us suppose that there exists $p \in M$ such that $\mathcal{O}_f^+(p)$ is dense in M, otherwise we are done. Write $p_\ell = f^\ell(p)$. Given $n \geq 1$ consider the set $M_n = \{x \in M : \mathcal{O}_f^+(x) \text{ is } 1/n\text{-dense}\}$. By assumption, for each $\ell \in \mathbb{N}$ and $n \in \mathbb{N}$, there is some $k_{n,\ell}$ such that $\{p_\ell, \cdots, f^{k_{n,\ell}}(p_\ell)\}$ is $1/2n$-dense. Moreover, by continuity of f there exists $r_{n,\ell} > 0$ such that $f^j(B_{r_{n,\ell}}(p_\ell)) \subset B_{1/2n}(f^j(p_\ell))$ for all $0 \leq j \leq k_{n,\ell}$ and, consequently, for any $y \in B_{r_{n,\ell}}(p_\ell)$ it follows that the finite piece of orbit $\{y, \cdots, f^{k_{n,\ell}}(y)\}$ is $1/n$-dense. Therefore $\bigcup_{\ell\in\mathbb{N}} B_{r_{n,\ell}}(p_\ell) \subset M_n$ is a open and dense set. In particular this proves that $\bigcap_{n\in\mathbb{N}} \bigcup_{\ell\in\mathbb{N}} B_{r_{n,\ell}}(p_\ell)$ is a residual subset contained in $\bigcap_{n\in\mathbb{N}} M_n = \{x \in M : \omega_f(x) = M\}$, proving the lemma. $\square$

Given a continuous endomorphism $f \in C^0(M, M)$ we denote by $\mathcal{C}_f$ the set of critical points of f, that is, $x \in \mathcal{C}_f$ if for all $r > 0$ the restriction $f|_{B_r(x)}$ is not a homeomorphism. The next result relates forward and backward limit sets.

Lemma 2.2. *Let $f \in C^0(M, M)$ be such that the critical region $\mathcal{C}_f$ has empty interior. If f is transitive then $\{x \in M : \alpha_f(x) = \omega_f(x) = M\}$ is a residual subset of M.*

Proof. The proof mimics the previous lemma with some care with the critical set $\mathcal{C}_f$. Since $M \setminus \mathcal{C}_f$ is open and dense, then $\bigcap_{j\geq 0} f^{-j}(M \setminus \mathcal{C}_f)$ is residual. It follows from the previous lemma that the intersection

$$\{x \in M : \omega_f(x) = M\} \cap \left(\bigcap_{j\geq 0} f^{-j}(M \setminus \mathcal{C}_f) \right) \neq \emptyset$$

is also a residual subset of M. Pick $p \in M$ such that $\omega_f(p) = M$ and $\mathcal{O}_f^+(p) \cap \mathcal{C}_f = \emptyset$, and write $p_j = f^j(p)$. Given $n \geq 1$ consider $A_n = \{x \in M : \mathcal{O}_f^-(x)$ is $1/n$-dense$\}$. Since $\mathcal{O}_f^+(p)$ is dense, there exists $k_n \in \mathbb{N}$ such that $\{p_0, \cdots, p_{k_n}\}$ is $1/2n$-dense. As $\{p_0, \cdots, p_{k_n}\} \subset \mathcal{O}_f^-(p_j)$ for all $j \geq k_n$, we get that $p_j \in A_n$ for every $j \geq k_n$. Now, using that $\mathcal{O}_f^+(p) \cap \mathcal{C}_f = \emptyset$, for each $j \geq k_n$ one can find $r_j > 0$ such that $f^j|_{B_{r_j}(p)}$ is a homeomorphism and $f^\ell(B_{r_j}(p)) \subset B_{\frac{1}{2n}}(f^\ell(p))$ for all $0 \leq \ell \leq j$. This proves that $\mathcal{O}_f^-(y)$ is $1/n$-dense for any $y \in f^j(B_{r_j}(p))$ and $j \geq k_n$. As $f^j(B_{r_j}(p))$ is an open neighborhood of p_j and $\{p_j; j \geq k_n\}$ is dense, then $\bigcup_{j \geq k_n} f^j(B_{r_j}(p)) \subset A_n$ is an open and dense subset of M. Therefore, $\bigcap_{n\in\mathbb{N}} \bigcup_{j\geq k_n} f^j(B_{r_j}(p)) \subset \bigcap_{n\in\mathbb{N}} A_n$ is a residual subset. Since $\bigcap_{n\in\mathbb{N}} A_n = \{x \in M : \alpha_f(x) = M\}$, we notice that $\{x \in M : \alpha_f(x) = M\} \cap \{x \in M : \omega_f(x) = M\}$ is a residual subset in M. This finishes the proof of the lemma. □

In particular we obtain the following immediate consequence:

Corollary 2.1. *If $f \in \mathcal{RT}$ then there exists a residual subset of points in M with dense orbit and pre-orbit.*

In fact, a converse result also holds obtaining that robust density of points with dense pre-orbit is equivalent to robust transitivity for local diffeomorphisms.

Lemma 2.3. *Let $\mathcal{U}$ be an open subset of the space of C^1 local diffeomorphisms and assume that every $f \in \mathcal{U}$ admits a dense set of points with dense pre-orbit. Then every $f \in \mathcal{U}$ is robustly transitive, that is, $\mathcal{U} \subset \mathcal{RT}$.*

Proof. Since the proof is simple we leave it as an easy exercise for the reader. □

Let us mention that expanding endomorphisms are not the only class of maps satisfying the assumptions of the previous lemma. In Section 4 we present a class of robustly transitive local diffeomorphisms that are not uniformly expanding but for which there exists a generic subset of points with dense pre-orbit.

3. Proof of the main results

This section is devoted to the proof of our main results.

3.1. **Proof of Theorem A.** Items (1) and (2) are a consequence of the C^1 closing lemma for local diffeomorphisms (see e.g. [4, 8, 10]) and Kupka-Smale theorem for local diffeomorphisms. Hence, there exists a residual subset $\mathcal{R}_0 \subset \mathcal{RT}$ such that for every $f \in \mathcal{R}_0$ holds that $\overline{\mathrm{Per}_h(f)} = \Omega(f) = M$, where $\mathrm{Per}_h(f)$ denotes the set of hyperbolic periodic points for f. So, we are left to prove the existence of dense pre-orbits for a generic subset of robustly transitive local diffeomorphisms. Using Corollary 2.1 it follows that every $f \in \mathcal{R}_0$ satisfies property (3). This finishes the proof of the theorem.

3.2. **Proof of Proposition 1.** Fix $f_0 \in \mathcal{RT}^*$. The first step is to recall that f_0 is volume expanding, that is, $|\det(Df_0)| > \sigma > 1$. This follows from adapting the arguments used by Bonatti, Díaz and Pujals [1] in the invertible setting, as we can see in the following theorem.

Theorem 3.1. [7, Theorem 4.3] *Let f be a C^1 local diffeomorphism and U open set in M such that $\Lambda_f(U) = \bigcap_{n\in\mathbb{Z}} f^n(\overline{U})$ is C^1 robustly transitive set and it has no splitting in a C^1 robust way. Then f is volume expanding.*

Now, we can proceed as in [1, Lemma 6.1] to prove that there exists a C^1 local diffeomorphism f arbitrarily close to f_0 and a periodic point $f^k(p) = p$ such that $Df^k(p)$ is an homothety. Since f_0 is robustly transitive with no splitting in a robust way it follows from the theorem above that f is volume expanding as well and, consequently, p is a repelling periodic point. Since this is a robust property we deduce that there is an open and dense subset $\mathcal{A} \subset \mathcal{RT}^*$ such that every $f \in \mathcal{A}$ has a repelling periodic point. In particular, if $\mathcal{R}_0$ is given by Theorem A then every map in the residual subset $\mathcal{R}_0 \cap \mathcal{A} \subset \mathcal{RT}^*$ has a dense set of hyperbolic periodic points and at least one periodic repelling point. Using recursively the Connecting Lemma [5, 11] and robust transitivity we get that there exists a residual subset $\mathcal{R}_1 \subset \mathcal{R}_0 \cap \mathcal{A}$ such that every $f \in \mathcal{R}_1$ has a periodic repelling point p such that the pre-orbit of p is dense, as claimed.

3.3. **Proof of Theorem B.** In this section we use the notion of zooming times from [9] to deduce the existence of interesting measures. More precisely, we prove the following:

Theorem 3.2. *If a C^1 local diffeomorphism f has a periodic source with dense pre-orbit then there are uncountable many invariant, ergodic and expanding measures with full support and exponential decay of correlations.*

Since we deal with C^1 local diffeomorphisms, we need to relax the $C^{1+\alpha}$ condition required in Proposition 9.3 and Theorem 5 of [9].

Definition 3.1 (Zooming contraction). *A sequence $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$ of functions $\alpha_n : [0, +\infty) \to [0, +\infty)$ is called a* zooming contraction *if it satisfies:*

- $\alpha_n(r) < r$, *for every* $r > 0$ *and* $n \geq 1$*;*
- $\alpha_n(r) \leq \alpha_n(\widetilde{r})$, *for every* $0 \leq r \leq \widetilde{r}$ *and* $n \geq 1$*;*
- $\alpha_n \circ \alpha_m(r) \leq \alpha_{n+m}(r)$, *for every* $r > 0$ *and* $n, m \geq 1$*;*
- $\sup_{0\leq r\leq 1} \Big(\sum_{n=1}^{\infty} \alpha_n(r) \Big) < \infty.$

Observe that an exponential backward contraction is an example of a zooming contraction, $\alpha_n(r) = \lambda^n r$ with $0 < \lambda < 1$. Let $\alpha = \{\alpha_n\}_n$ be a zooming contraction and δ a positive constant.

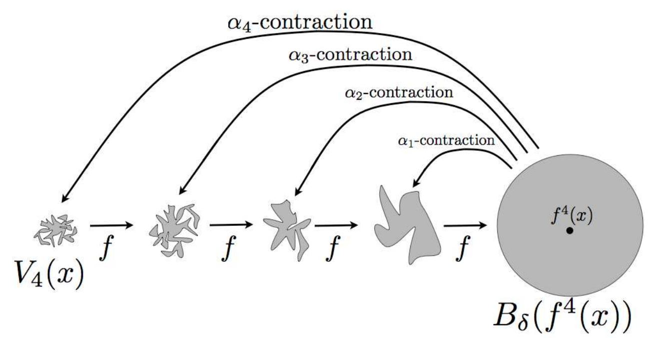

FIGURE 1. A zooming time for $x \in \mathrm{Z}_4(\alpha, \delta, f)$

Definition 3.2 (Zooming times)**.** *We say that* $n \geq 1$ *is a* (α, δ)-zooming time *for* $p \in M$*, with respect to* f*, if there is a neighborhood* $V_n(p)$ *of* p *such that* f^n *sends* $\overline{V_n(p)}$ *homeomorphically onto* $\overline{B_\delta(f^n(p))}$ *and for all* $x, y \in V_n(p)$ *and* $0 \leq j < n$

$$\operatorname{dist}(f^j(x), f^j(y)) \leq \alpha_{n-j}\big(\operatorname{dist}(f^n(x), f^n(y))\big).$$

The ball $B_\delta(f^n(p))$ is called a *zooming ball* and the set $V_n(p)$ is called a *zooming pre-ball.* Denote by $\mathrm{Z}_n(\alpha, \delta, f)$ the set of points of X for which n is an (α, δ)-zooming time. A point $x \in M$ is called (α, δ)-zooming (with respect to f) if

$$\limsup_{n\to\infty} \frac{1}{n} \#\big\{1 \leq j \leq n\,; x \in \mathrm{Z}_j(\alpha, \delta, f)\big\} > 0. \tag{3.1}$$

Moreover, a positively invariant set $\Lambda \subset X$ is called a (α, δ)-*zooming set*, with respect to f, if every $x \in \Lambda$ is (α, δ)-zooming.

Lemma 3.1. *If* p *is a periodic repeller with* $\alpha_f(p) = M$ *then there are* $\ell \in \mathbb{N}$, $\delta > 0$ *and* $\lambda > 1$ *such that* $\mathcal{O}_f^-(p)$ *is a* (α, δ)*-zooming set with respect to* $\widetilde{f} := f^\ell$*, where* $\alpha = \{\alpha_n\}_n$ *is given by* $\alpha_n(x) = (1/8)^n x$*. Furthermore, there exists* $p' \in \mathcal{O}_f^+(p)$ *such that*

$$\Big\{ y \in \mathcal{O}_f^-(p') \colon \; \#\{j \in \mathbb{N}\,;\, y \in Z_j(\alpha, \delta, \widetilde{f}\,) \text{ and } \widetilde{f}^j(y) = p'\} = \infty \Big\}$$

is dense in a neighborhood of p'*.*

Proof. Let $\gamma = \text{period(p)}$. Since $\mathcal{O}_f^+(p)$ is a finite set there exists $n_0 \geq 1$ large so that $\log(\|(Df^{n\,\gamma}(q))^{-1}\|^{-1}) > \log 32$ for all $n \geq n_0$ and every $q \in \mathcal{O}_f^+(p)$. Let $\delta > 0$ be small enough such that, for every $q \in \mathcal{O}_f^+(p)$ there is a neighborhood $W(q)$ of the point q satisfying $f^{n_0\gamma}(W(q)) = B_\delta(q)$ and $(f^{n_0\gamma}|_{W(q)})^{-1}$ is a $(e^{-\lambda_0})$-contraction, where $\lambda_0 = \log 16$.

Given any $y \in \mathcal{O}_f^-(p)$ there exists a natural number a such that $f^a(y) = p$. Write $a = (n_0\gamma)\, k_0 + r_0$ with $0 \leq r_0 \leq (n_0\gamma) - 1$ and $k_0 \geq 0$. Let $q_y = f^{n_0\gamma - r_0}(p) \in \mathcal{O}_f^+(p)$. Thus,

$$f^{(k_0+1)n_0\gamma}(y) = f^{n_0\gamma - r_0}(f^{k_0\, n_0\gamma + r_0}(y)) = f^{n_0\gamma - r_0}(p) = q_y.$$

We proceed to construct zooming neighborhoods of the point y. Pick an open neighborhood W_0 of y so that $f^{(k_0+1)n_0\gamma}|_{W_0}$ is a diffeomorphism onto a neighbourhood of q_y. Note that since $(f^{n_0\gamma}|_{W(q_y)})^{-1}$ is a $(e^{-\lambda_0})$-contraction, there exists $j_0 \in \mathbb{N}$ such that for all $j \geq j_0$ it holds

$$(f^{n_0\gamma}|_{W(q_y)})^{-j}(B_\delta(q_y)) \subset f^{(k_0+1)n_0\gamma}(W_0).$$

For every $j \geq j_0$, let

$$U_{n_0\gamma(j+k_0+1)}(y) := (f^{(k_0+1)n_0\gamma}|_{W_0})^{-1} \circ (f^{n_0\gamma}|_{W(q_y)})^{-j}(B_\delta(q_y)).$$

Therefore, there is some $j_1 \geq j_0$ such that $(f^{n_0\gamma(j+k_0+1)}|_{U_{n_0\gamma(j+k_0+1)}(y)})^{-1}$ is a $(e^{-\lambda_1})^{(j+k_0+1)}$-contraction for every $j \geq j_1$, where $\lambda_1 = \log 8$. By construction

$$f^{n_0\gamma(\,j+k_0+1)}(U_{n_0\gamma(j+k_0+1)}(y)) = B_\delta(q_y) = B_\delta(f^{n_0\gamma(j+k_0+1)}(y)), \tag{3.2}$$

and, consequently, $y \in Z_{j+k_0+1}(\alpha, \delta, f^{n_0\gamma})$ for all $j \geq j_1$, where the zooming sequence $\alpha = \{\alpha_n\}_n$ is given by $\alpha_n(x) = e^{-\lambda_1\, n}x = (1/8)^n x$. This proves the first part of the lemma.

From now on set $\ell = n_0\gamma$. We obtained that $y \in Z_j(\alpha, \delta, f^\ell)$ for all $j \geq j_1+k_0+1$ and, by (3.2), it follows that

$$\#\{j \in \mathbb{N}\,;\, y \in Z_j(\alpha, \delta, \widetilde{f}\,) \text{ and } \widetilde{f}^j(y) = q_y\} = \infty, \tag{3.3}$$

where $\widetilde{f} = f^\ell$.

Consider the function $\varphi : \mathcal{O}_f^-(p) \to \mathcal{O}_f^+(p)$ defined as $\varphi(y) = q_y$, where $q_y \in \mathcal{O}_f^+(p)$ is chosen to satisfy (3.3). Since $\mathcal{O}_f^-(p)$ is dense in M and one can write $\mathcal{O}_f^-(p) = \varphi^{-1}(p) \uplus \cdots \uplus \varphi^{-1}(f^{\gamma-1}(p))$ as a disjoint union, there is $p' \in \mathcal{O}_f^+(p)$ such that $\varphi^{-1}(p')$ is dense in some open set $U \subset M$. Let $x_0 \in \varphi^{-1}(p') \cap U$ and $b \geq 0$ such that $\widetilde{f}^b(x_0) = p'$. Since f is a local homeomorphism, $\widetilde{f}^b(U)$ is a neighborhood of p' and using that $\varphi^{-1}(p') \subset \mathcal{O}_f^-(p)$ then it is also dense in $\widetilde{f}^b(U)$. This finishes the proof of the lemma.

□

So in the remaining of this section we describe how to construct uncountable many ergodic and expanding measures with full support and exponential decay of correlations.

Proof of Theorem 3.2. Since the proof follows closely the one of [9, Proposition 9.3] we give an outline of the proof and focus on the main ingredients. Assume that f is a C^1 local diffeomorphism and p is a periodic source with dense pre-orbit $\mathcal{O}_f^-(p)$. Then, by the previous lemma there exist $\ell \in \mathbb{N}$ and $\delta > 0$, such that $\mathcal{O}_f^-(p)$ is a (α, δ)-zooming set with respect to $\widetilde{f} := f^\ell$ (in particular $\mathcal{O}_{\widetilde{f}}^-(p)$ is also a (α, δ)-zooming set for $\widetilde{f}$), where $\alpha = \{\alpha_n\}_n$ is the zooming sequence given by $\alpha_n(x) = (1/8)^n x$. Moreover, changing p for some $p' \in \mathcal{O}_f^+(p)$ if necessary, we have that $\mathcal{O}^Z(p) := \big\{y \in \mathcal{O}_{\widetilde{f}}^-(p) \;;\; \#\{j \in \mathbb{N}\,;\, y \in Z_j(\alpha, \delta, \widetilde{f}\,) \text{ and } \widetilde{f}^j(y) = p\} = \infty\;\big\}$ is dense in a neighborhood of p. As $\sum_n \alpha_n(r) < r/4$, let $0 < r < \delta/4$ be small such that $B_r(p) \subset \overline{\mathcal{O}^Z(p)}$.

So, the (α, δ)-zooming nested ball with respect to $\widetilde{f}$, $\Delta = B_r^*(p)$, is an open neighborhood of p contained in $B_r(p)$ (see Definition 5.9 and also Lemma 5.12 in [9] for more details). Furthermore, there is a dense set of points in Δ (the pre-orbit $\mathcal{O}^Z(p) \cap \Delta$) returning by $\widetilde{f}$ to Δ in a (α, δ)-zooming time. In consequence, it follows from [9, Corollary 6.6] that there exists collection $\mathcal{P}$ of open connected subsets of Δ and an induced map $F : \Delta \to \Delta$ given by $F(x) = \widetilde{f}^{R(x)}(x)$ $(= f^{\ell\, R(x)}(x))$, with $\{R > 0\} = \bigcup_{P \in \mathcal{P}} P$, such that R is "the first (α, δ)-zooming return time" to Δ (see Definition 6.2 and 6.3 of [9]).

The function $R : \Delta \to \mathbb{N}$ is constant on elements of $\mathcal{P}$ and F satisfies the Markov property that $F(P) = \Delta$, $F\mid_P$ is a C^1-diffeomorphism and $DF|_P > 8$ for all $P \in \mathcal{P}$.

Now, for any sequence $a = (a_P)_{P \in \mathcal{P}}$ of real numbers satisfying $0 < a_P < 1$, $\sum_{P \in \mathcal{P}} a_P = 1$ and $\sum_{P \in \mathcal{P}} a_P R(P) < \infty$, let ν_a denote the Bernoulli measure which is defined on elements of the partition $\mathcal{P}^{(n)} = \bigvee_{j=0}^{n-1} F^{-j}\mathcal{P}$ by

$$\nu_a(P_0 \cap F^{-1}P_1 \cap \cdots \cap F^{-(n-1)}P_{n-1}) = \prod_{j=0}^{n-1} a_{P_j}$$

for all $n \geq 1$. It is not hard to check that ν_a is a F-invariant and ergodic probability measure and ν_a has constant Jacobian on cylinders (in fact $J_{\nu_a}F\mid_P = a_P$ for all

$P \in \mathcal{P}$). Now, using that $\int R \, d\nu_a = \sum_{P\in\mathcal{P}} a_P R(P) < \infty$ then

$$\mu_a = \frac{1}{\ell}\sum_{j=0}^{\ell-1} f_*^j\Big(\frac{1}{\int R\, d\nu_a}\sum_{P\in\mathcal{P}}\sum_{j=0}^{R(P)-1}\tilde f_*^j(\nu_a\mid_P)\Big)$$

defines an f-invariant and ergodic probability measure. Moreover, $\mu_a\mid_\Delta \ll \nu_a$ and using that ν_a gives positive weight to open subsets of Δ then $P \subset \operatorname{supp}\mu_a$. Since f is transitive every positive invariant set with non empty interior is dense. Thus, μ_a has dense support and by compactness, the support is the whole manifold.

Furthermore, since each probability measure μ_a is ergodic and two ergodic measures either coincide or are mutually singular, we deduce that there are uncountably many ergodic measures with full support, and those measures are expanding. Indeed, ν_a-almost every x is (α,δ)-zooming, because $\nu_a\big(\bigcap_{j\ge 0} F^{-j}(\{R>0\})\big) = 1$ and we have

$$\limsup_n \frac{1}{n}\sum_{j=0}^{n-1}\log\big(\|(D\tilde f(\tilde f^j(x)))^{-1}\|^{-1}\big) > \log 8 > 0$$

for every $x \in \bigcap_{j\ge0} F^{-j}(\{R>0\})$. So, $\lim_n \frac{1}{n}\sum_{j=0}^{n-1}\log\big(\|(D\tilde f(\tilde f^j(x)))^{-1}\|^{-1}\big) > \log 8 > 0$ for μ_a-almost every x, since μ_a is f-invariant (and so, $\tilde f$-invariant). This implies that all Lyapunov exponents with respect to $\tilde f$ (and also to f) are positive for μ_a-almost every point. Finally, we notice that by [13], μ_a has exponential decay of correlations provided that

$$\nu_a(R\ge n) = \sum_{k\ge n}\sum_{R(P)=k} a_P$$

has exponential decay in n. Since the later property is satisfied for an uncountable many $(a_P)_{P\in\mathcal{P}}$, this finishes the proof of the theorem. □

4. Examples

4.1. **Existence of expanding measures with exponential decay.** We shall consider now an important class of robustly transitive local diffeomorphisms introduced in [6, 7]. Take $n \ge 2$ and $r \ge 1$. The following result holds:

Theorem 4.1. [7, Main Theorem] Let $f \in E^r(\mathbb{T}^n)$ be a volume expanding map such that $\{w \in f^{-k}(x) : k \in \mathbb{N}\}$ is dense for every $x \in \mathbb{T}^n$ and satisfies the properties:

(1) There is an open set U_0 in $\mathbb{T}^n$ such that $f|_{U_0^c}$ is expanding and $diam(U_0)<1$;
(2) There exists $0 < \delta_0 < diam_{int}(U_0^c)$ and there exists an open neighborhood U_1 of $\overline{U}_0$ such that for every arc γ in U_0^c with diameter larger than δ_0, there is a point $y \in \gamma$ such that $f^k(y) \in U_1^c$ for any $k \ge 1$; and
(3) For every $z \in U_1^c$, there exists $\bar z \in U_1^c$ such that $f(\bar z) = z$.

Then, for every g C^r-close enough to f, $\{w \in g^{-k}(x) : k \in \mathbb{N}\}$ is dense for every $x \in \mathbb{T}^n$. In particular, f is C^r-robustly transitive.

These latter theorem essentially means that robust transitivity is obtained for local diffeomorphisms whose pre-orbits are dense, if it is uniformly expanding in a definite region of the ambient space and for every sufficiently large arc in this expanding region there exists a point whose forward orbit remains in the expanding region. In fact, it follows from [7] that hypotheses (b) and (c) above assure the existence of a locally maximal expanding invariant set Λ_f which has a topological property of 'separation'. Roughly, every open set intersects Λ_f after a finite number of iterates which implies for future iterates the internal radius growth(IRG property):

Lemma 4.1. [7, Lemma 2.29] *There exist $\mathcal{V}_2(f)$ and $R > 0$ such that for every $g \in \mathcal{V}_2(f)$, if there is $x \in M$ such that $g^n(x) \notin U_0$ for every $n \geq 0$, then there is $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $\mathbb{B}_R(g^N(x)) \subset g^N(\mathbb{B}_\varepsilon(x))$.*

Note that Lemma 4.1 proves the robustness of IRG property, which is fundamental to prove the density of the pre-orbit of any point under the perturbed map. For further details, see [7]. After the discussion above, we are now in condition to present a large class of examples that illustrate our main results. Let us consider $\mathcal{F}$ the class of C^r endomorphisms f in the n-dimensional torus $\mathbb{T}^n$ satisfying the following properties:

(1) (volume expanding) There exists $\sigma > 1$ such that $|det(Df(x))| \geq \sigma$ for all $x \in \mathbb{T}^n$;
(2) There exists a dense subset $D \subset \mathbb{T}^n$ of points with dense pre-orbit;
(3) There is an open set U_0 in $\mathbb{T}^n$ such that $f|_{U_0^c}$ is expanding and $diam(U_0) < 1$;
(4) There exists a locally maximal expanding invariant set Λ_f with the topological property of 'separation' above.

Note that property (4) above is a consequence of the hypotheses of Theorem 4.1. Hence, every map satisfying the assumptions in Theorem 4.1 belong to $\mathcal{F}$. In particular, every $f \in \mathcal{F}$ has at least one periodic source in Λ_f with dense pre-orbit. Thus Theorem 3.2 yields *for any $f \in \mathcal{F}$ there are uncountable many ergodic, invariant and expanding measures with full support and exponential decay of correlations.*

4.2. **Example of robustly non-existence of splitting.** In this subsection we provide examples exhibiting the phenomenon of robust non-existence of splitting. It is sufficient to have periodic points with complex eigenvalues, since these mix any invariant directions that could exist.

Fix $n \geq 2$. Consider $\mathcal{E} : \mathbb{T}^n \to \mathbb{T}^n$ a linear expanding endomorphism, with degree $N = \deg(\mathcal{E})$ large enough. Therefore there exists a Markov partition and it has N elements, denote by R_i these elements, with $1 \leq i \leq N$. Recall that each R_i is closed, $int(R_i)$ is nonempty and $int(R_i) \cap int(R_j) = \emptyset$ for $i \neq j$. Choose $\psi : \mathbb{T}^n \to \mathbb{T}^n$ isotopic to the identity to deform the elements of the initial Markov partition, in order to get a new partition which elements are not all of the same size (it could contains some very small elements and others very big) and denote by $\widehat{R}_i = \psi(R_i)$ for every i. Now, pick U_0 an open set in $\mathbb{T}^n$ such that if $\widetilde{U}$ is the convex hull of the lift of U_0, then $\widetilde{U} \cap [0,1]^n$ is contained in the interior of $[0,1]^n$, i.e. $diam(U_0) < 1$. Note that there exists $\widehat{R}_j$ such that $\widehat{R}_j \cap U_0$ is nonempty. We also request that there are some $\widehat{R}_i$ contained in U_0^c, observe that this condition is feasible since the initial map has several elements in the partition.

Define $f_0 : \mathbb{T}^n \to \mathbb{T}^n$ by $f_0 = \psi \circ \mathcal{E}$. Assume that there exist $p \in U_0$ and $q_i \in U_0^c$ fixed points for f_0, with $1 \leq i \leq n-1$. This is possible because we may start with an expanding map which has as many fixed points as we need. Let us suppose also that p and q_i are expanding fixed points for f_0. Pick $\varepsilon > 0$ small enough such that

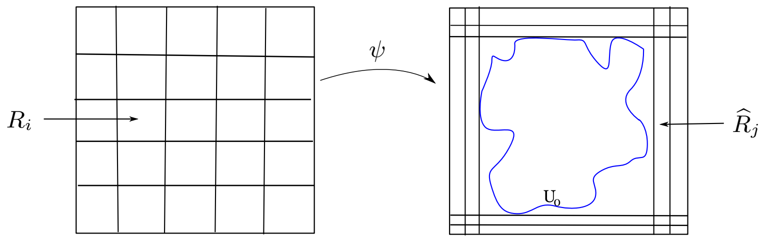

FIGURE 2. *Deforming the initial Markov Partition*

$\mathbb{B}_\varepsilon(q_i) \cap U_0 = \emptyset$ and $\mathbb{B}_\varepsilon(q_i) \cap \mathbb{B}_\varepsilon(q_j) = \emptyset$ for all $i \neq j$. Denote the decomposition of the tangent space as follows

$$T_x(\mathbb{T}^n) = \mathbb{E}_1^u \prec \mathbb{E}_2^u \prec \cdots \prec \mathbb{E}_{n-1}^u \prec \mathbb{E}_n^u,$$

where $\prec$ denotes that $\mathbb{E}_i^u$ dominates the expanding behavior of $\mathbb{E}_{i-1}^u$. Next we deform f_0 by a smooth isotopy supported in $U_0 \cup (\bigcup \mathbb{B}_\varepsilon(q_i))$ in such a way that:

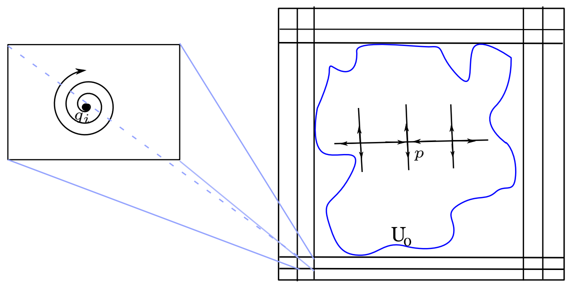

FIGURE 3. *f isotopic to f_0*

(1) the continuation of p goes through a pitchfork bifurcation, giving birth to two periodic points r_1, r_2, such that both are repeller and p becomes a saddle point. But the new map f still expands volume in U_0;
(2) two expanding eigenvalues of q_i become complex expanding eigenvalues in a way that the two expanding subbundles of $T_{q_i}(\mathbb{T}^n)$ corresponding to $\mathbb{E}_i^u(q_i)$ and $\mathbb{E}_{i+1}^u(q_i)$ are mixed obtaining $T_{q_i}(\mathbb{T}^n) = \mathbb{E}_1^u \prec \mathbb{E}_2^u \prec \cdots \prec \mathbb{F}_i^u \prec \cdots \prec \mathbb{E}_n^u$, where $\mathbb{F}_i$ is two dimensional and correspond to the complex eigenvalues;
(3) f coincides with f_0 in the complement of $U_0 \cup (\bigcup \mathbb{B}_\varepsilon(q_i))$;
(4) f is expanding in U_0^c; and
(5) there exists $\sigma > 1$ such that $|det(Df(x))| > \sigma$ for every $x \in \mathbb{T}^n$.

Note that the existence of these periodic points with complex eigenvalues prevent any non-trivial invariant subbundle, and this construction is robust. Let us stress that the expanding region in these examples can be taken as small as desired.

5. FURTHER COMMENTS

In this section, we address the problem of existence of relevant expanding measures for robustly transitive local diffeomorphisms that admit some non-trivial invariant subbundle. First we introduce some notions. Given a local diffeomorphism

$f \in \text{Diff}_{\text{loc}}(M)$ and a compact forward invariant set $\Lambda \subset M$ we say that Λ admits a *dominated splitting* if there exists a continuous splitting $T_\Lambda M = E^1 \oplus E^2$ and constants $C, a > 0$ and $\lambda \in (0,1)$ such that for all $x \in \Lambda$ and $n \in \mathbb{N}$:

- $Df(x)\, E^1_x = E^1_{f(x)}$ (E^1 is Df-invariant); and
- the cone $\mathcal{C}^2_x = \{u + v \in E^1_x \oplus E^2_x : \|u\| \leq a\|v\|\}$ satisfies the invariance condition $Df(x)(\mathcal{C}^2_x) \subset \mathcal{C}^2_{f(x)}$, and for all $v \in E^1_x \setminus \{0\}$ and $w \in \mathcal{C}^2_x \setminus \{0\}$

$$\frac{\|Df^n(x)v\|}{\|Df^n(x)w\|} \leq C\lambda^n.$$

Since our previous results hold for maps whose tangent bundle does not admit invariant subbundles we now discuss the existence of expanding measures with full support in the presence of dominated splittings that are robust by C^1-perturbations.

We say that a dominated splitting $T_\Lambda M = E^1 \oplus E^2$ is of *expanding type*, namely if the subbundle E^1 satisfies $\|(Df^n \mid_{E^1_x})^{-1}\| \leq C\lambda^n$ for all $x \in \Lambda$ and $n \geq 1$. This implies that f is uniformly expanding and so, by the theory developed by Sinai-Ruelle-Bowen, there are uncountable many f-invariant, ergodic and expanding measures with full support and exponential decay of correlations.

In a dual way, we say that a dominated splitting $T_\Lambda M = E^1 \oplus E^2$ is of *contracting type* if the subbundle E^1 satisfies $\|Df^n \mid_{E^1_x}\| \leq C\lambda^n$ for all $x \in \Lambda$ and $n \geq 1$. Note that if $T_\Lambda M = E^1 \oplus E^2$ is a dominated splitting of contracting type then expanding measures cannot exist due to the existence of invariant stable direction with uniform contraction along the orbits. Hence, one could ask wether there are uncountable ergodic and hyperbolic measures with total support and exponential decay of correlations. The same strategy to prove Theorem B could answer the previous question provided the existence of Markovian induced schemes for maps with a dense non-uniformly hyperbolic set, which is an open question.

Finally, it remains to consider the case where E^1 is a center bundle with non-uniform expanding or contracting behavior and dominated by a a subbundle E^2 with uniform expansion. Examples illustrating this situation and where there exists a periodic source with dense pre-orbit can be found in [7, subsection 5.3]. In particular such class of maps admit uncountable many invariant, ergodic and expanding probability measures with full support and exponential decay of correlations. We expect an analogous result as Theorem B to hold for these type of maps.

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