ON GEODESIC FLOWS MODELED BY EXPANSIVE FLOWS UP TO TIME-PRESERVING SEMI-CONJUGACY

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Abstract. Given a smooth compact surface without focal points and of higher genus, it is shown that its geodesic flow is semi-conjugate to a continuous expansive flow with a local product structure. Using the fact that the semi-conjugation preserves time-parametrization, it is concluded that the geodesic flow has a unique measure of maximal entropy.

Contents

Notations

$$M \xleftarrow{\ \pi\ } \sigma_t^{\bar\theta}\colon \tilde M \to \tilde M,\ H^\pm(\bar\eta)$$

$$\begin{array}{llll}
\theta = (p,v) \in T_1 M & & \bar\theta = (p,v) \in T_1\tilde M & \\
\phi_t\colon T_1M \to T_1M,\ \mathscr{F}^*(\theta) & \xleftarrow{\ \bar\pi\ } & \phi_t\colon T_1\tilde M \to T_1\tilde M,\ \tilde{\mathscr{F}}^*(\bar\theta) & * = s,u,cs,cu \\
\quad\downarrow \chi & & \quad\downarrow \bar\chi & \\
\psi_t\colon X \to X,\ W^*([\theta]) & \xleftarrow{\ \bar\Pi\ } & \bar\psi_t\colon \bar X \to \bar X,\ \tilde W^*([\bar\theta]) & * = ss,uu,cs,cu
\end{array}$$

1. Introduction

The resemblances between the geodesic flow of a compact surface of genus greater than one and without conjugate points and the geodesic flow of compact surface with constant negative curvature have been source of deep research in geometry and topological dynamics from the beginning of the 20th century. The first and perhaps the most influential work in the subject is the seminal work of Morse [35] showing that every globally minimizing geodesic in the universal covering of a compact surface of genus greater than one is "shadowed" by a geodesic in hyperbolic space (see Subsection 1.2). This beautiful result strongly suggested that the geodesic flow of a compact surface without conjugate points and of higher genus should be semi-conjugate to the geodesic flows of a constant negative curvature surface.

Recall that two continuous flows $\phi_t\colon Y \to Y$ and $\psi_t\colon X \to X$ acting on compact metric spaces Y and X are *semi-conjugate* if there exists a continuous, surjective map $\chi\colon Y \to X$ such that for each $p \in Y$ there exists a continuous, surjective reparametrization $\rho_p\colon \mathbb{R} \to \mathbb{R}$ such that

$$(\chi \circ \phi_t)(p) = (\psi_{\rho_p(t)} \circ \chi)(p).$$

The map χ is called a *semi-conjugacy* and when χ is a homeomorphism it is called a *conjugacy*. Observe that χ maps orbits to orbits. We say that ϕ_t is *time-preserving semi-conjugate* to ψ_t if ρ_p is the identity for every p. The existence of conjugacies to nearby flows in the C^1 topology characterizes structurally stable flows: Axiom A flows and, in particular, Anosov flows. Structural stability theory was developed in the 60's and 70's and the main ideas of the theory paved the path to study systems enjoying weaker forms of stability like topological stability. The existence of semi-conjugacies to nearby C^k systems defines C^k-topologically stable systems. Of course, C^k-structurally stable systems are C^k-topologically stable. But the converse is not true and there are many well known counterexamples, many of them in the category of expansive, non-hyperbolic systems.

Definition 1.1. A continuous flow $\psi_t\colon X \to X$ without singular points on a metric space (X,d) is *expansive* if for all $\varepsilon > 0$ and $x \in X$ and for each

$y \in X$ for which there exists a continuous surjective function $\rho: \mathbb{R} \to \mathbb{R}$ with $\rho(0) = 0$ satisfying

$$d(\psi_t(x), \psi_{\rho(t)}(y)) \leq \varepsilon$$

for every $t \in \mathbb{R}$ we have $\psi_{t(y)}(x) = y$ for some $|t(y)| < \varepsilon$.

The definition of expansive homeomorphism introduced by Bowen is previous to the above definition for flows (see also [7]), and Bowen's study of expansive homeomorphisms and systems showed how to find weak stability properties in non-hyperbolic geodesic flows using a dynamical approach rather than the global geometry approach suggested by Morse's ideas. Observe that, in fact, our definition is slightly stronger than in [7], however it appears naturally in the context of expansive geodesic flows (see, for example, [42, 43]). In the context of geodesic flows on compact manifolds without conjugate points both definitions are equivalent.

The application of both points of view to study stability has been quite fruitful so far. Bowen [6] and Bowen and Walters [7] pointed out that expansive systems with a local product structure are C^0 topologically stable (see also [50, 17] for flows); Lewowicz [32] and Hiraide [25] showed that expansive homeomorphisms of compact surfaces have an "almost" local product structure; Paternain [39] and Inaba and Matsumoto [23] extended Lewowicz's work to show that expansive geodesic flows of compact surfaces have local product structure and hence that they are topologically stable. Ruggiero [43] shows that expansive geodesic flows of compact manifolds without conjugate points have a local product structure, so topological stability holds as well.

The global geometry point of view of weak stability theory of geodesic flows enjoyed an extraordinary development as well in the 70's. We can mention the works of Eberlein [13] and Eberlein and O'Neill [15] about visibility manifolds extending Morse's work: quasi-geodesics in visibility manifolds are close to true geodesics. The theory of visibility manifolds went further and introduced a whole body of tools to study coarse hyperbolic geometry of manifolds. Thurston and Gromov [22] introduced the notion of hyperbolic groups and not only extended Eberlein's theory of visibility manifolds but created a rich theory to study coarse hyperbolic geometry in very general metric spaces. The so-called Gromov hyperbolic spaces satisfy as well Morse's shadowing property of quasi-geodesics by geodesics. All the above results suggested that the family of geodesic flows semi-conjugate to one with negative curvature could be much larger than geodesic flows of compact surfaces without conjugate points.

In the 80's, Ghys [18] proved the existence of a semi-conjugacy between the geodesic flow of a compact surface without conjugate points and of genus greater than one with a hyperbolic geodesic flow. However, in general, this semi-conjugacy is not time-preserving, that is, the reparametrization ρ_p is not the identity in general. While semi-conjugacies of discrete systems are time preserving by definition, the existence of such semi-conjugacies for flows is a much more delicate issue. Time-preserving semi-conjugacies are related to spectral ridigity problems in continuous dynamical systems and in particular in Riemannian geometry: two geodesic flows which are time-preserving semi-conjugate have the same length spectrum of periodic geodesics. The works of Otal [37], Croke [11], and Croke and Fathi [12], show that if this is the case of the geodesic flows of two compact surfaces without conjugate points and of genus greater than one then the surfaces are isometric. The main result result of the present article looks then surprising in the context of geodesic flows.

Theorem A. *Let (M,g) be a C^∞ compact connected boundaryless surface without focal points and genus greater than one. Let $\phi_t\colon T_1M\to T_1M$ be its geodesic flow.*

Then there exists a compact 3-manifold X diffeomorphic to T_1M and a continuous expansive flow $\psi_t\colon X\to X$ with a local product structure such that ϕ_t is time-preserving semi-conjugate to ψ_t.

So expansive flows arise as models of geodesic flows of compact surfaces without focal points and of genus greater than one up to time-preserving semi-conjugacies. Of course, the model flow ψ_t cannot be the geodesic flow of a compact surface. Indeed, if the model flow ψ_t was a geodesic flow then we would know that the corresponding surface has no conjugate points by the result of Paternain [39]. Therefore, we could apply the spectral rigidity results mentioned above to show that both surfaces would have to be isometric. But the initial geodesic flow might not be expansive, there might be flat strips (see Section 2) in the unit tangent bundle.

The existence of a time-preserving semi-conjugacy to an expansive flow has some interesting applications in topological dynamics. In fact, this expensive flow inherits most of the features of the topological hyperbolic dynamics which are found in the theory of expansive geodesic flows in manifolds without conjugate points (see Theorem 5.1 and compare [43]). These results play a crucial role in the proof of the following result which is related to the thermodynamical formalism and ergodic optimization.

Theorem B. *The geodesic flow of a C^∞ compact connected boundaryless surface without focal points and of genus greater than one has a unique invariant probability measure of maximal entropy.*

Theorem B is in fact close to Knieper's result [30] about the uniqueness of maximal entropy measures for rank one geodesic flows in manifolds with nonpositive curvature. But observe that there exists surfaces without focal points which admit some regions with positive curvature. Hence Theorem B generalizes Knieper's work (in the case of surfaces). Our approach to show Theorem B is closer to the classical one in thermodynamics that works very well for expansive systems. Since the geodesic flow in Theorem B might not be expansive, we combine Theorem A with a recent result by Buzzi et al. [10] which reduces the study of the maximal entropy measure of a so-called extension of an expansive system to the study of maximal entropy measures of expansive ones.

The main idea to prove Theorem A is in many senses a very natural one: the geodesic flow of a compact surface without conjugate points and of higher genus is not expansive in general. It may have regions of non-expansivity, that in the case of a surface without focal points consist of flat strips. So we define an equivalence relation in the unit tangent bundle of the surface that identifies points in the same strip (see Section 4). This relation induces naturally a flow which preserves the classes and the time of the initial geodesic flow. The quotient space then carries a flow without "non-expansive" orbits, and the hard part of the proof consists in showing that the quotient space has a good topological structure. We show that the quotient space is a topological 3-manifold, and hence carries a smooth manifold structure. Finally, we show that the quotient flow is expansive with respect to any metric defined on the quotient space.

Many of the steps of the proof hold for surfaces without conjugate points, however in some subtle steps the arguments do not extend to such surfaces.

2. Preliminaries

Standing assumption. Throughout the paper (M,g) will be a C^∞ compact connected Riemannian manifold without boundary. We shall always assume that M has *no conjugate points*, that is, that the exponential map is non-singular at every point. In particular, we will study the particular subclass of manifolds *without focal points*, that is, if $J(t)$ is a Jacobi field along a geodesic in M with $J(0)=0$ then $\|J(t)\|$ is strictly increasing in t.

Each vector $\theta \in TM$ determines a unique geodesic $\gamma_\theta(\cdot)$ such that $\gamma_\theta'(0) = \theta$. The geodesic flow $(\phi_t)_{t\in\mathbb{R}}$ acts by $\phi_t(\theta) = \gamma_\theta'(t)$. We shall study its restriction to the unit tangent bundle T_1M, which is invariant. All the geodesics will be parametrized by arc length.

We shall denote by $\tilde{M}$ the universal covering of M and endow it with the pullback $\tilde{g}$ of the metric g by the covering map $\pi\colon \tilde{M} \to M$ which gives the Riemannian manifold $(\tilde{M},\tilde{g})$. We shall also consider the geodesic flow of this manifold which acts on $T_1\tilde{M}$ and which we will also denote by $(\phi_t)_{t\in\mathbb{R}}$ (the domain of the flow ϕ_t is enough to specify the dynamical system under consideration). We will consider also the natural projection $\bar{\pi}\colon T_1\tilde{M} \to T_1M$. The distance associated to the Riemannian metric g will be denoted by d_g and the one associated to $\tilde{g}$ by $d_{\tilde{g}}$. We will omit the metric if there is no danger of confusion. The Riemannian metric on M lifts to the Sasaki metric on TM which we denote by d_S. We shall use the same notation for the Sasaki distances in $T_1\tilde{M}$ and $\widetilde{T_1M}$. For any $\theta \in TM$ we will consider the orthogonal decomposition of $T_\theta TM$ into horizontal and vertical parts $T_\theta TM = H_\theta \oplus V_\theta$, similar for $\bar{\theta} \in T_1\tilde{M}$.

Geodesics β and γ in $\tilde{M}$ are *asymptotic* (as $t \to \infty$) if $d(\beta(t),\gamma(t))$ is bounded as $t \to \infty$, that is, there exists $C > 0$ such that $d(\beta(t),\gamma(t)) \le C$

for all $t \geq 0$, and *bi-asymptotic* if $d(\beta(t),\gamma(t))$ is bounded as $t \to \pm\infty$, that is, the previous inequality holds for all $t \in \mathbb{R}$.

Given a metric space (X,d), and two subsets $Z_1, Z_2 \subset X$, let us denote by $d(Z_1,Z_2)$ the Hausdorff distance between Z_1 and Z_2.

2.1. **Horospheres and invariant submanifolds.** A very special property of manifolds with no conjugate points is the existence of the so-called Busemann functions: given $\bar\theta = (p,v) \in T_1\tilde M$, the *(forward and backward) Busemann function* $b^\pm_{\bar\theta}\colon \tilde M \to \mathbb{R}$ associated to $\bar\theta$ are defined by

$$b^+_{\bar\theta}(x) \coloneqq \lim_{t\to+\infty} d(x,\gamma_{\bar\theta}(t)) - t, \quad \text{and} \quad b^-_{\bar\theta}(x) \coloneqq \lim_{t\to+\infty} d(x,\gamma_{\bar\theta}(-t)) - t\,.$$

The level sets of the Busemann functions are the horospheres. We define the (level 0) *(positive and negative) horospheres* of $\bar\theta \in T_1\tilde M$ by

$$H^+(\bar\theta) \coloneqq (b^+_{\bar\theta})^{-1}(0) \quad \text{and} \quad H^-(\bar\theta) \coloneqq (b^-_{\bar\theta})^{-1}(0)\,.$$

The horospheres in $\tilde M$ lift naturally to horospheres in $T_1\tilde M$ as follows. Consider the gradient vector fields $\nabla b^\pm_{\bar\theta}$ and define the (positive and negative) horocycles $\tilde{\mathscr F}^{s/u}(\bar\theta)$ in $T_1\tilde M$ through $\bar\theta$ to be the restriction of $\nabla b^\pm_{\bar\theta}$ to $H^\pm(\bar\theta)$

$$\begin{aligned}\tilde{\mathscr F}^s(\bar\theta) &\coloneqq \left\{(q,-\nabla_q b^+_{\bar\theta})\colon q\in H^+(\bar\theta)\right\},\\ \tilde{\mathscr F}^u(\bar\theta) &\coloneqq \left\{(q,-\nabla_q b^-_{\bar\theta})\colon q\in H^-(\bar\theta)\right\}.\end{aligned}$$

Let us denote by $\sigma^{\bar\theta}_t\colon \tilde M \to \tilde M$ the integral flow of the vector field $-\nabla b^+_{\bar\theta}$. The orbits of this flow are the *Busemann asymptotes* of $\gamma_{\bar\theta}$.

We will list some basic properties of horospheres that we shall need (see for example [41, 16] for details).

Lemma 2.1. *Let (M,g) be a C^∞ compact connected boundaryless Riemannian manifold without conjugate points, then*

1. *$b^\pm_{\bar\theta}$ are C^1 functions for every $\bar\theta$.*
2. *The gradients $\nabla b^\pm_{\bar\theta}$ have norm equal to one at every point.*
3. *For e fixed $\theta \in T_1\tilde M$ the gradient vector field $\nabla b^\pm_{\bar\theta}$ is K-Lipschitz and has unit length. Each horosphere is a embedded submanifold of dimension $n-1$ tangent to a Lipschitz plane field (with K being a Lipschitz constant depending on curvature bounds).*
4. *The orbits of $\sigma^{\bar\theta}_t$ are geodesics which are everywhere perpendicular to the horospheres $H^+(\theta)$. In particular, the geodesic $\gamma_{\bar\theta}$ is an orbit of this flow and for every $t,s\in\mathbb{R}$ we have*

$$\sigma^{\bar\theta}_t(H^+(\bar\theta)) = H^+(\gamma_{\bar\theta}(t))\,.$$

5. *If the manifold has no focal points, then the Busemann functions are of class C^2 and every two bi-asymptotic geodesics in $\tilde M$ bound a flat strip.*

Item (4) in Lemma 2.1 implies that the horospheres are equidistant, that is, given any point $p \in H^+(\gamma_\theta(t))$ the distance $d(p, H^+(\gamma_\theta(s)))$ is equal to $|t-s|$. It is clear that $H^+(\gamma_\theta(t))$ varies continuously with $t \in \mathbb{R}$, however it is not known whether it varies continuously with θ.

2.2. **Morse's theorem, central foliations, and strips.** Let us recall some well known properties of the global dynamics of the geodesic flow of a compact surface without conjugate points. The *hyperbolic plane* $\mathbb{H}^2 = (\tilde{M}, \tilde{g}_{-1})$ is the Riemannian structure in $\tilde{M}$ with constant negative curvature -1, a *hyperbolic geodesic* is a geodesic of $(\tilde{M}, \tilde{g}_{-1})$.

First recall a famous result by Morse [35] which essentially states that every minimizing geodesic in $(\tilde{M}, \tilde{g})$ is "shadowed" by a geodesic in the hyperbolic plane. Recall that a A, B-*quasi geodesic* of $(\tilde{M}, \tilde{g})$ is a continuous rectifiable curve $\alpha\colon [0,1] \to \tilde{M}$ so that

$$\ell_{\tilde{g}}(\alpha[t,s]) \leq A\, d_{\tilde{g}}(\alpha(t), \alpha(s)) + B,$$

for every $t, s \in [0,1]$, where $\ell_{\tilde{g}}$ is the length in the metric $\tilde{g}$.

Theorem 2.2. *Given $A, B > 0$ there exists $Q = Q(A,B) > 0$ such that every A, B-quasi geodesic of $(\tilde{M}, \tilde{g})$ is contained in the Q-tubular neighborhood of a geodesic of $\mathbb{H}^2$.*

This above result was extended by Eberlein in [13] to visibility manifolds and by Gromov [22] to Gromov hyperbolic metric spaces.

Since every two Riemannian metrics in a compact surface are equivalent, minimizing geodesics in $(\tilde{M}, \tilde{g})$ are A, B-quasi geodesics of $\mathbb{H}^2$ for some A, B depending on g. So we get the following result.

Theorem 2.3. *Let (M, g) be a C^∞ compact connected boundaryless surface without focal points and of genus greater than one. Then there exists $Q > 0$ such that each minimizing geodesic in $(\tilde{M}, \tilde{g})$ is contained in the Q-tubular neighborhood of a certain hyperbolic geodesic.*

Given a point $\bar{\theta} \in T_1\tilde{M}$, its *center stable set* and its *center unstable set* are defined by

$$\tilde{\mathscr{F}}^{cs}(\bar{\theta}) := \bigcup_{t \in \mathbb{R}} \tilde{\mathscr{F}}^{s}\big(\phi_t(\bar{\theta})\big) \quad \text{and} \quad \tilde{\mathscr{F}}^{cu}(\bar{\theta}) := \bigcup_{t \in \mathbb{R}} \tilde{\mathscr{F}}^{u}\big(\phi_t(\bar{\theta})\big),$$

respectively. The images of $\tilde{\mathscr{F}}^{s/u}(\bar{\theta})$ by the natural projection $\bar{\pi}\colon T_1\tilde{M} \to T_1M$ are $\mathscr{F}^{s/u}(\theta)$, the stable and the unstable leaf of $\theta = \bar{\pi}(\bar{\theta})$, respectively. Likewise, we denote by $\mathscr{F}^{cs/cu}(\theta)$ the *center stable and center unstable sets of* θ being the natural projections of $\tilde{\mathscr{F}}^{cs/cu}(\bar{\theta})$.

We list now some of the most important basic properties of the center stable and unstable sets.

Theorem 2.4. *Let (M, g) be a C^∞ compact connected boundaryless surface without conjugate points. Then the following assertions hold:*

(1) *The family of sets*

$$\tilde{\mathscr{F}}^s := \bigcup_{\bar\theta\in T_1\tilde M} \tilde{\mathscr{F}}^s(\bar\theta), \quad \tilde{\mathscr{F}}^{cs} := \bigcup_{\bar\theta\in T_1\tilde M} \tilde{\mathscr{F}}^{cs}(\bar\theta)$$

are collections of C^0 submanifolds. In each union they either are disjoint or coincide.

(2) *The sets $\tilde{\mathscr{F}}^{s/u}(\bar\theta)$, $\tilde{\mathscr{F}}^{cs/cu}(\bar\theta)$ depend continuously on $\bar\theta \in T_1\tilde M$ and hence the collections $\tilde{\mathscr{F}}^{s/u}$, $\tilde{\mathscr{F}}^{cs/cu}$ are continuous foliations of $T_1\tilde M$ by C^0 leaves.*

(3) *If M has genus greater than one, there exists a positive constant L such that for every $\bar\theta = (p,v) \in T_1\tilde M$, $\bar\eta = (q,w) \in \tilde{\mathscr{F}}^s(\bar\theta)$, we have*

$$d(\gamma_{\bar\theta}(t),\gamma_{\bar\eta}(t)) \le L\, d(p,q) + Q$$

for all $t \ge 0$, where $Q > 0$ is the constant in Theorem 2.3. The same inequality holds for $\bar\eta \in \tilde{\mathscr{F}}^u(\bar\theta)$ with $t \le 0$.

(4) *Given $p \in \tilde M$, there exists a homeomorphism*

$$\Psi_p \colon \tilde M \times \tilde V_p \to T_1\tilde M$$

such that

$$\Psi_p(\tilde M \times \{(p,v)\}) = \tilde{\mathscr{F}}^{cs}(p,v).$$

In particular, $\tilde{\mathscr{F}}^{cs}$ and $\mathscr{F}^{cs}$ are continuous foliations and the space of leaves of $\mathscr{F}^{cs}$ is homeomorphic to the vertical fiber $\tilde V_p$ for any $p \in \tilde M$.

(5) *(Eschenburg [16]) If (M,g) has no focal points, then the sets $\tilde{\mathscr{F}}^s(\bar\theta)$, $\tilde{\mathscr{F}}^{cs}(\bar\theta)$ are C^1 submanifolds and the foliations $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^{cs}$ are continuous in the C^1, compact open topology, that is, the set of tangent unit vectors at p.*

The foliations $\tilde{\mathscr{F}}^s$, $\tilde{\mathscr{F}}^{cs}$ are called the *stable* and *the central stable foliation*, respectively. Of course, analogous statements hold for the corresponding so-called *unstable* and *central unstable foliation* $\tilde{\mathscr{F}}^u$, $\tilde{\mathscr{F}}^{cu}$, respectively. The image of such foliations by the covering map $\bar\pi \colon T_1\tilde M \to T_1M$ form foliations $\mathscr{F}^s$, $\mathscr{F}^{cs}$, $\mathscr{F}^u$, $\mathscr{F}^{cu}$ that we shall call by the same names. Observe that in the case when the surface is the torus it must be flat by the work of Hopf [26]. So Theorem 2.4 is trivial in this case.

Item (3) of Theorem 2.4 tells us essentially that the geodesic flow of a compact surface $(\tilde M,\tilde g)$ without conjugate points and of genus greater than one behaves like an expansive flow up to $2Q$-tubular neighborhoods of geodesics. Let us give a precise description of this fact.

Definition 2.5 (Strip)**.** Let (M,g) be a C^∞ compact connected boundaryless Riemannian surface without conjugate points. Given $\bar\theta = (p,v) \in T_1\tilde M$, the *strip* $F(\bar\theta) \subset \tilde M$ is a maximal subset of geodesics $\gamma_{\bar\eta} \subset \tilde M$ such that $\gamma_{\bar\eta}$ is an orbit of both the vector fields $-\nabla b^+_{\bar\theta}$ and $-\nabla b^-_{\bar\theta}$.

The following is a consequence of Theorem 2.3.

Lemma 2.6. *The width of any strip is bounded above by* $2Q$*, where* Q *the constant in Theorem 2.3.*

Proof. By item (3) of Theorem 2.4, two geodesics in the same strip are bi-asymptotic. Hence they are shadowed by a certain hyperbolic geodesic γ which has to be unique since there are no bi-asymptotic hyperbolic geodesics. So both of them are within a distance Q from γ according to Theorem 2.3. Hence the claim follows from the triangle inequality. □

Lemma 2.7. *Given* $\bar{Q} > Q$*, there exists* $T = T(\bar{Q}) > 0$ *such that for every* $\bar{\theta} = (p, v) \in T_1\tilde{M}$ *and every Busemann asymptote* $\gamma_{\bar{\eta}}(t)$ *of* $\gamma_{\bar{\theta}}$ *with* $\bar{\eta} = (q, w)$ *and* $d(q, \gamma_{\bar{\theta}}) > Q$ *we have* $d(\gamma_{\bar{\eta}}(t), \gamma_{\bar{\theta}}) \geq \bar{Q}$ *for every* $t \leq T$*.*

Proof. This is a consequence of item (3) in Theorem 2.4. □

2.3. **Visibility manifolds and topological dynamics.** The universal covering $(\tilde{M}, \tilde{g})$ of a compact surface without conjugate points (M, g) belongs to a special class of manifolds without conjugate points called visibility manifolds.

Definition 2.8. Let (M, g) be a complete simply connected Riemannian manifold without conjugate points. We say that (M, g) is a *visibility manifold* if for every $\varepsilon > 0$, $p \in M$ there exists $T = T(\varepsilon, p) > 0$ such that for every two unit speed geodesic rays γ, β with $\gamma(0) = p = \beta(0)$, if the distance from p to every point of the geodesic joining $\gamma(t)$ to $\beta(s)$, $0 \leq s \leq t$, is larger than T then the angle formed by $\gamma'(0)$ and $\beta'(0)$ is less than ε. When T does not depend on p we say that (M, g) is a *uniform visibility manifold*.

Eberlein [13] shows that if M is compact and $(\tilde{M}, \tilde{g})$ is a visibility manifold, then $(\tilde{M}, \tilde{g})$ is a uniform visibility manifold. Moreover, if (M, h) is another Riemannian structure without conjugate points in M, then $(\tilde{M}, \tilde{h})$ is a visibility manifold, too. Ruggiero in [44] shows that whenever (M, g) is a compact Riemannian manifold without conjugate points, then $(\tilde{M}, \tilde{g})$ is a visibility manifold if, and only if, $(\tilde{M}, \tilde{g})$ is a Gromov hyperbolic space and geodesic rays diverge. So there is a close relationship between visibility and coarse hyperbolic geometry. This result applied to compact surfaces without conjugate points, together with the divergence of geodesic rays in the universal covering of such surfaces proved by Green [20], provides a proof of the visibility property for compact surfaces of genus greater than one (Eberlein [13] gives a direct proof of this fact checking the visibility condition directly). Compact manifolds without conjugate points and with the so-called bounded asymptote condition [16] whose fundamental groups are Gromov hyperbolic are as well visibility manifolds. Theorem 2.4 is true for compact manifolds without focal points, but its extension to any manifold without conjugate points is still an open problem. Combining Gromov hyperbolicity and divergence of geodesic rays, Theorem 2.4 can be extended to visibility manifolds without further assumptions on the curvature (see [45] for instance).

The geodesic flow of a compact manifold whose universal covering is a visibility manifold shares many important dynamical properties with Anosov geodesic flows. The following results are proved by Eberlein [13] (see [14] for visibility manifolds with nonpositive curvature), but they extend to visibility universal coverings of compact manifolds without conjugate points applying Gromov hyperbolic theory [22].

Recall that a continuous flow $\psi_t\colon X \to X$ is *topologically mixing* if for any two open sets $U, V \subset X$ there exists $T > 0$ such that for $|t| \geq T$, $\psi_t(U) \cap V \neq \varnothing$.

Theorem 2.9. *Let (M, g) be a compact Riemannian manifold without conjugate points such that $(\tilde{M}, \tilde{g})$ is a visibility manifold. Then:*

1. *The foliations $\mathscr{F}^s$, $\mathscr{F}^u$ are minimal.*
2. *The geodesic flow of (M, g) is topologically mixing and, in particular, topologically transitive.*

Theorem 2.9 will be very important for the sequel. Observe that it holds not only for surfaces but for any compact manifold without conjugate points whose universal covering is a visibility manifold. The two-dimensional character of surfaces plays an important role in the next section.

3. The geometry of strips

From now on we will additionally assume that (M, g) is a compact surface without focal points and of genus greater than one.

In this section we survey geometric properties of strips in manifolds without focal points. Most of them are already well-known and extend also to strips in the so-called surfaces with bounded asymptote (see [16]). The starting point is the following Flat Strip Theorem (see Eberlein-O'Neill [15, Proposition 5.1], Pesin [41, Theorem 7.3]).

Theorem 3.1. *Any two bi-asymptotic geodesics in $(\tilde{M}, \tilde{g})$ are contained in a flat strip.*

The main basic geometric property of flat strips is upper semi-continuity.

Lemma 3.2. *Let $(\bar{\theta}_n)_{n\geq 1} \subset T_1\tilde{M}$ be a sequence converging to $\bar{\theta}$. Then the outer limit of the strips $F(\bar{\theta}_n)$ is contained in the strip $F(\bar{\theta})$.*

As a consequence of Lemma 3.2, in the case of surfaces without focal points the width of strips defined below is a measurable upper semi-continuous function. Let us introduce some notations.

Definition 3.3 (Non-/trivial strips)**.** By Theorem 3.1, any set $F(\bar{\theta})$ is an isometric embedding $f\colon \mathbb{R}\times[a, b] \to \tilde{M}$ of an Euclidean strip into the universal covering. If $[a, b]$ is chosen to be maximal with this property we call $b - a$ the *width of the strip* $F(\bar{\theta})$. Let

$$A_n := \left\{F(\bar{\theta})\colon F(\bar{\theta}) \text{ is strip of width} \geq n^{-1}\right\}.$$

If the strip $F(\bar{\theta})$ consists of a single geodesic, then the point $\bar{\theta} \in T_1\tilde{M}$ is called *expansive point* and the strip is called *trivial strip*. Otherwise, we call $F(\bar{\theta})$ *nontrivial.*

Lemma 3.4. *For each $n > 0$ the set A_n has empty interior.*

Proof. By contradiction, suppose that there is $n > 0$ such that A_n has nonempty interior. Let $\bar{\theta}$ be in the interior of A_n and let $V_{\bar{\theta}}$ be the vertical fiber of $\bar{\theta}$ in $T_1\tilde{M}$. There exists an open subset $I(\bar{\theta})$ of $V_{\bar{\theta}}$ in A_n such that the set of geodesic rays

$$\{\gamma_{\bar{\eta}}(t): t \geq 0, \bar{\eta} \in I(\bar{\theta})\}$$

cover an open cone in $\tilde{M}$ whose curvature vanishes everywhere since it is covered by strips of width at least n^{-1}. Since the geodesic rays in the boundary of the cone diverge as $t \to \infty$, the cone contains a fundamental domain of the surface (M, g) which is clearly a contradiction since the genus of M is at least one and therefore it does not support a Riemannian metric with vanishing curvature. □

The next remark was already made by Burns [8, Lemma 2.1].

Lemma 3.5. *If the orbit of a point in T_1M by the geodesic flow is dense, then none of its lifts to $T_1\tilde{M}$ has a nontrivial strip.*

Proof. By contradiction, suppose that a lift belongs to A_n. By density and upper semi-continuity of strips we get that every point in $T_1\tilde{M}$ belongs to A_n. But this is impossible in view of Lemma 3.4. □

Lemma 3.5 combined with Theorem 2.9 yield the following interesting consequence. Since there is a dense orbit and the central foliations are minimal, the set of points in $T_1\tilde{M}$ with nontrivial strip is dense in T_1M. So the geodesic flow is in some sense "generically" expansive.

Corollary 3.6. *Let (M, g) be a compact surface without focal points and of genus greater than one.*

Then the set of expansive points is dense in the unit tangent bundle.

3.1. **Recurrent points.** Our next result improves Lemma 3.5.

Lemma 3.7. *Let $\theta \in T_1M$ be a recurrent point. Then for every lift $\bar{\theta} \in T_1\tilde{M}$ of θ the strip $F(\bar{\theta})$ is the only strip of the form $F(\bar{\eta})$ with $\bar{\eta} \in \tilde{\mathscr{F}}^s(\bar{\theta})$. In particular, if the strip $F(\bar{\theta})$ is trivial then every point in $\tilde{\mathscr{F}}^s(\bar{\theta})$ is an expansive point.*

Proof. Let $\theta \in T_1M$ be a positively recurrent point. Then there is a point θ accumulated by a sequence of points $(\theta_n)_{n \geq 1}$ given by $\theta_n = \phi_{t_n}(\theta)$. Consider a lift $\bar{\theta} \in T_1\tilde{M}$ of θ and its strip $F(\bar{\theta})$.

Suppose that there exists another strip $F(\bar{\eta})$ with $\bar{\eta} \in \tilde{\mathscr{F}}^s(\bar{\theta})$ of positive width. Since the geodesics in the strip $F(\bar{\eta})$ are asymptotic to the geodesic

$\gamma_{\bar\theta}$, there exists $B > 0$ such that

$$d(\gamma_{\bar\theta}(t), \gamma_{\bar\zeta}(t)) \le B$$

for every $t \ge 0$, and every geodesic $\gamma_{\bar\zeta}$ in $F(\bar\eta)$. We can pick a sequence of lifts $(\bar\theta_n)_n$ of the sequence $(\theta_n)_n$ which is converging to $\bar\theta$. Since the central stable sets $\tilde{\mathscr{F}}^{cs}(\bar\theta_n)$ are all isometric to $\tilde{\mathscr{F}}^{cs}(\bar\theta)$, we observe the following facts:

- Since θ_n lies in the orbit of θ, the strips $F(\bar\theta_n)$ are isometric copies of $F(\bar\theta)$.
- For each n there is an isometric copy $F(\bar\eta_n)$, $\bar\eta_n \in \tilde{\mathscr{F}}^s(\phi_{-t_n}(\bar\theta))$, of the strip $\bar F(\bar\eta)$ whose geodesics are asymptotic to $\gamma_{\bar\theta_n}$. In particular,
$$d(\gamma_{\bar\theta_n}(t), \gamma_{\bar\zeta}(t)) \le B$$
for every geodesic $\gamma_{\bar\zeta}$ in $F(\bar\eta_n)$ and for every $t \ge -t_n$.

Given $T \in \mathbb{R}$, let

$$F^T(\bar\zeta) \coloneqq \bigcup_{t\ge T} \left\{ \gamma_{\bar\xi}(t) \colon \gamma_{\bar\xi} \subset F(\bar\zeta) \right\}.$$

Each $(2Q+B)$-tubular neighborhood around $\gamma_{\bar\theta_n}$ contains the strip $F(\bar\theta_n)$ and the "half" strip of rays $F^{-t_n}(\bar\eta_n)$. By the semi-continuity of strips, the outer limit of the sets

$$F(\bar\theta_n) \cup F^{-t_n}(\bar\eta_n)$$

is contained in a strip of geodesics which are bi-asymptotic to $\gamma_{\bar\theta}$ and whose width is at least the width of $F(\bar\theta)$ plus the width of $F(\bar\eta)$ (which is positive). This contradicts the definition of $F(\bar\theta)$, being the maximal strip of geodesics which are bi-asymptotic to $\gamma_{\bar\theta}$. The contradiction implies that such $\bar\eta$ cannot exist, and thus the strip $F(\bar\theta)$ must be unique among the strips of points in its center stable leaf. □

Combining Lemma 3.7 and Corollary 3.6 we have the following.

Corollary 3.8. *Let (M, g) be a compact surface without focal points and of genus greater than one.*

Then the set of expansive points in $T_1\tilde M$ is open. If θ has a dense orbit under the geodesic flow and $\bar\theta$ is any of its lifts, then every point in $\tilde{\mathscr{F}}^{cs}(\bar\theta) \cup \tilde{\mathscr{F}}^{cu}(\bar\theta)$ is an expansive point.

3.2. **Wandering points.** Let us now study strips of points in the central stable leaf of a wandering point. First observe that Lemma 3.7 applies to any wandering point in the stable leaf of a recurrent point. However, for a wandering orbit the limit set might be more complicated, and there might be more than one strip in its central leaf.

Given a wandering point $\theta \in T_1M$ and one of its lifts $\bar\theta \in T_1\tilde M$, the collection of strips in the central leaf is a disjoint union of strips $F(\bar\theta_n)$ of $\bar\theta$, where n belongs to a certain set of indices Γ. Let

$$I(\bar\theta_n) \coloneqq F(\bar\theta_n) \cap H^+(\bar\theta), \quad J(\bar\theta) \coloneqq \{I(\bar\theta_n) \colon n \in \Gamma\}$$

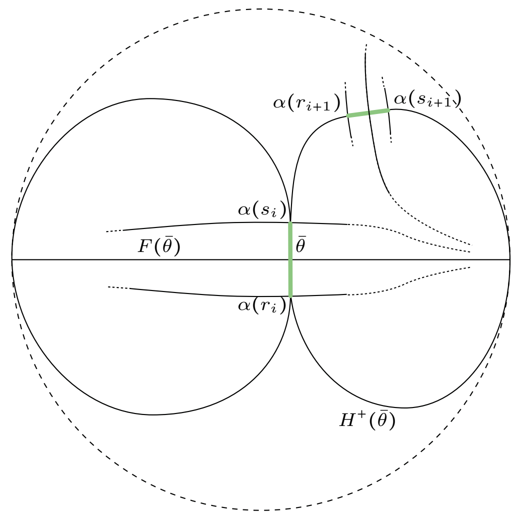

FIGURE 1. Strips (thick segments) in the horocycles $J(\bar\theta)$

(compare Figure 1).

Lemma 3.9. *There exists $L > 0$, depending only on the constant Q in Theorem 2.2, such that for every wandering point $\theta \in T_1 M$ and for every of its lifts $\bar\theta \in T_1\tilde{M}$ the Lebesgue measure of $J(\bar\theta)$ in $H^+(\bar\theta)$ is bounded from above by L.*

Proof. Let $\bar\theta = (p, v)$, and let $\alpha\colon \mathbb{R} \to H^+(\bar\theta)$ be an arc length parametrization of $H^+(\bar\theta)$ satisfying $\alpha(0) = p$. Let $U(m)$ be the union of $\alpha([-m, m])$ with the subset of $J(\bar\theta)$ which intersects $\alpha([-m, m])$. Observe that $U(m)$ is a connected curve whose length, by Morse's Theorem 2.2, differs at most by $2Q$ from the length of $\alpha([-m, m])$.

Again, by Morse's Theorem, there exists $T_m > 0$ such that for every $t \geq T_m$ we have

$$\sigma_t^{\bar\theta}(U(m)) \subset V(\gamma_{\bar\theta}, 2Q),$$

where $V(\gamma_{\bar\theta}, 2Q)$ is the tubular neighborhood of radius $2Q$ of the geodesic $\gamma_{\bar\theta}$, and $\sigma_t^{\bar\theta}$ is the Busemann flow of $\bar\theta$. Since the strips are flat, $\sigma_t^{\bar\theta}$ restricted to a strip is an isometry and hence the Lebesgue measure of $J(\bar\theta) \cap U(m)$ is the same as the Lebesgue measure of $\sigma_t^{\bar\theta}(J(\bar\theta) \cap U(m))$.

By Lemma 2.1 the horospheres have bounded geometry in $(\tilde{M}, \tilde{g})$. This yields that there exists a constant $L > 0$ such that for every horosphere $H^+(\bar\eta)$, the length of the curve obtained by intersecting $H^+(\bar\eta)$ with $V(\gamma_{\bar\eta}, 2Q)$ is bounded above by L. Notice that $\sigma_t^{\bar\theta}(U(m))$ is a subset of $H^+(\phi_t(\bar\theta))$ for every $t \geq T_m$. Let $\ell_t(c)$ be the Lebesgue measure of a curve c contained in $H^+(\phi_t(\bar\theta))$. Then we get

$$\ell_0(J(\bar\theta) \cap U(m)) \leq \ell_t\big(\sigma_t^{\bar\theta}(U(m)\big) \leq L$$

for every $m > 0$. Since $J(\bar\theta)$ is the increasing union in m of $J(\bar\theta) \cap U(m)$ this yields the statement. □

Corollary 3.10. *Let $L > 0$ be as in Lemma 3.9. Let $\theta \in T_1M$ be a wandering point, let $\bar\theta \in T_1\tilde M$ be any of its lifts. Let $A_n(\bar\theta)$ be the set of strips in the center stable leaf of $\bar\theta$ whose width is at least n^{-1}. Then*

(1) *The number of strips in $A_n(\bar\theta)$ is bounded above by nL, in particular, the set of strips in each center stable leaf is countable.*
(2) *Let α be an arc length parametrization of $H^+(\bar\theta)$ and let $\alpha([r_k, s_k])$, $k = 1, 2, \dots, K_n$ be the collection of connected components of $A_n(\bar\theta)$ in $H^+(\bar\theta)$. Then every point of the form $(\alpha(s), -\nabla_{\alpha(s)} b^+_{\bar\theta})$ for $s \in (s_k, r_{k+1})$ is an expansive point, for every $k = 1, 2, \dots, K_n - 1$.*

Proof. Let Δ_n be the set of indices m for which $I(\bar\theta_m)$ is contained in $A_n(\bar\theta)$. Then

$$\frac{1}{n}\#\Delta_n \le \ell_0(A_n(\bar\theta)) \le L$$

according to Lemma 3.9, so

$$K_n = \#\Delta_n \le nL\,.$$

Since the set of strips in the center stable leaf of $\bar\theta$ meets $H^+(\bar\theta)$ in the union for $n \in \mathbb{N}$ of the $A_n(\bar\theta)$, the collection of such strips is a countable union of finite collections of sets, which implies that the set of strips is countable. This proves item (1).

To show item (2), suppose that the interior of the set of expansive points in $\alpha([s_k, r_{k+1}])$ is empty. Then the set of flat strips would meet $\alpha([s_k, r_{k+1}])$ in a dense set. This would imply that the set

$$S = \left\{\sigma_t^{\bar\theta}(\alpha([s_k, r_{k+1}])) \colon t \in \mathbb{R}\right\}$$

would be totally flat, and hence their boundary geodesics would have to be parallel in $\tilde M$. But this would yield that S is a flat strip properly containing the flat strips of $\alpha([r_k, s_k])$ and $\alpha([r_{k+1}, s_{k+1}])$, contradicting the fact that these two strips are disjoint connected components of the set of strips. This finishes the proof of item (2). □

Remark 3.11. Item (2) in Corollary 3.10 might not be true for surfaces without conjugate points if we drop the condition that there are no focal points.

4. Quotient space and the model flow

Let us define the following equivalence relation in T_1M.

Definition 4.1. Two points θ and $\eta \in T_1M$ are related if, and only if,

- $\eta \in \mathscr{F}^s(\theta)$
- if $\bar\theta$ is any lift of θ and $\bar\eta$ a lift of η satisfying $\bar\eta \in \tilde{\mathscr{F}}^s(\bar\theta)$, then the geodesics $\gamma_{\bar\theta}$ and $\gamma_{\bar\eta}$ are bi-asymptotic.

It is straightforward to check that this relation is indeed an equivalence relation. It "collapses" each strip into a single curve. Given $\theta \in T_1M$, we denote by $[\theta]$ the equivalence class which contains θ and by X the set of all equivalence classes.

We consider the flow $\psi_t \colon X \to X$ defined by

$$\psi_t([\theta]) := [\phi_t(\theta)] .$$

As the geodesic flow preserves the foliation $\mathscr{F}^s$ and asymptoticity, this flow is well defined. Moreover, it is time-preserving semi-conjugate to the geodesic flow by its very definition. Namely, considering the quotient map

$$\chi \colon T_1M \to X \colon \theta \mapsto [\theta], \qquad \text{we have} \qquad \psi_t \circ \chi = \chi \circ \phi_t .$$

The equivalence relation on T_1M induces naturally an equivalence relation in $T_1\tilde{M}$, with quotient map $\bar{\chi} \colon T_1\tilde{M} \to \bar{X}$. Let us denote by $[\bar{\theta}]$ the corresponding equivalence class of $\bar{\theta} \in T_1\tilde{M}$. Let $\bar{\psi}_t \colon \bar{X} \to \bar{X}$ be the corresponding quotient flow.

Notice that the absence of focal points was not needed in the above construction and extends to surfaces without conjugate points.

The quotient flow ψ_t is continuous in the quotient topology. The present section will be devoted to study the topological properties of the quotient space and this flow. The main result of the section is the following.

Theorem 4.2. *Let (M, g) be a compact surface without focal points. Then the quotient space X is a compact topological 3-manifold. In particular, X admits a smooth 3-dimensional structure where the quotient flow ψ_t is continuous.*

Recall that the quotient topology is the topology generated by the sets $U \subset X$ such that $\chi^{-1}(U)$ is an open set of T_1M. Theorem 4.2 is not at all obvious and requires an careful analysis of the quotient topology and its relationship with the dynamics of the geodesic flow. The absence of focal points will be crucial in some subtle steps of the proof. The main idea of its proof is to exhibit a special basis for the quotient topology, whose construction will be made in several steps. The proof will be concluded at the end of this section.

4.1. **A family of cross sections for the quotient topology.** As first step to obtain a basis for the quotient topology we construct a special family of cross sections for the quotient flow from which we shall obtain a basis by shifting them by the geodesic flow.

Given $\bar{\theta} \in T_1\tilde{M}$, let

$$\mathcal{I}(\bar{\theta}) := \tilde{\mathscr{F}}^s(\bar{\theta}) \cap \tilde{\mathscr{F}}^u(\bar{\theta}).$$

This set is a lift of $I(\bar{\theta})$ to $T_1\tilde{M}$, where the set $\tilde{\mathscr{F}}^{cs}(\bar{\theta}) \cap \tilde{\mathscr{F}}^{cu}(\bar{\theta})$ contains an isometric copy of the flat strip of the geodesic $\gamma_{\bar{\theta}}$.

Let us now choose the local cross section. Given a point $\bar{\theta}$, let us construct a smoothly embedded closed two-dimensional disc $\Sigma = \Sigma_{\bar{\theta}}(\varepsilon, \delta) \subset T_1\tilde{M}$ which

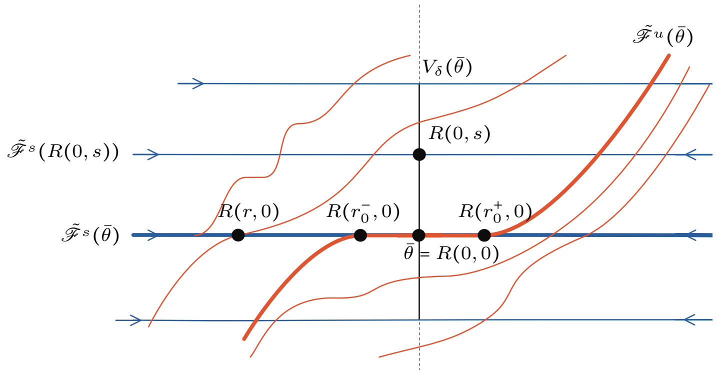

FIGURE 2. Parametrization of the disk $\Sigma_{\bar\theta}(\varepsilon,\delta)$

is transverse to the geodesic flow and contains $\bar\theta$. This disk will be foliated by the leaves of $\tilde{\mathscr{F}}^s$. To begin the construction, let $\varepsilon>0$, $\delta>0$ be sufficiently small, let $V_\delta(\bar\theta)$ be the δ-tubular neighborhood of $\bar\theta$ in its vertical fiber, and let

$$R\colon (r_0^- - \varepsilon, r_0^+ + \varepsilon)\times(-\delta,\delta)\to\Sigma \tag{1}$$

be the homeomorphism with the following properties:

(1) $R(0,0)=\bar\theta$,
(2) $R(0,s)$ with $s\in(-\delta,\delta)$ is the arc length parametrization of $V_\delta(\bar\theta)$ in the Sasaki metric.
(3) $R(r,0)$ is the arc length parametrization of the ε-tubular neighborhood of $\mathcal{I}(R(0,0))=\mathcal{I}(\bar\theta)$ in $\tilde{\mathscr{F}}^s(\bar\theta)$, $R(r_0^-,0)$ and $R(r_0^+,0)$ being the endpoints of $\mathcal{I}(\bar\theta)$;
(4) For each $s\in(-\delta,\delta)$, $r\mapsto R(r,s)$ is the arc length parametrization of the continuous curve in $\tilde{\mathscr{F}}^s(R(0,s))$.

Since the foliation $\tilde{\mathscr{F}}^s$ is a continuous foliation by Lipschitz curves, by Brower's Open Mapping Theorem the image of R is a two-dimensional section that we will denote by $\Sigma_{\bar\theta}(\varepsilon,\delta)$. Clearly, this section depends on the parameters ε and δ, we shall omit this dependence in the notation as well but keep it in mind. Let us consider open neighborhoods of $\bar\theta$ in $T_1\tilde M$ of the form

$$B_{\bar\theta}(\varepsilon,\delta,\tau) := \big\{\phi_t\big(\Sigma_{\bar\theta}(\varepsilon,\delta)\big)\colon |t|<\tau\big\} \tag{2}$$

where ϕ_t is the geodesic flow of $T_1\tilde M$ (we use the same notation for the geodesic flows in T_1M and $T_1\tilde M$, the domain of the flow ϕ_t is enough to specify the one under consideration). We shall omit the point $\bar\theta$ in the index unless we change it.

Let

$$\Pi_\Sigma\colon B(\varepsilon,\delta,\tau)\to\Sigma$$

be the projection onto Σ by the geodesic flow of $T_1\tilde M$.

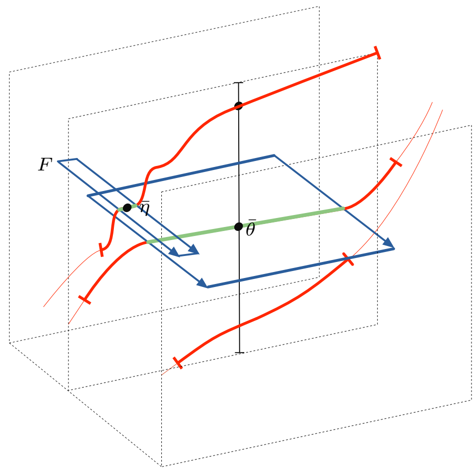

FIGURE 3. Flat strips in $B_{\bar{\theta}}(\varepsilon, \delta, \tau)$ and their projections (green) to $\Sigma_{\bar{\theta}}(\varepsilon, \delta)$

We have the following important result.

Lemma 4.3. *Given a strip F which intersects $B(\varepsilon, \delta, \tau)$, there exists $\bar{\eta} \in \Sigma$ such that $\Pi_\Sigma(F \cap B(\varepsilon, \delta, \tau))$ is a connected component of $\mathcal{I}(\bar{\eta})$. In particular, if $\mathcal{I}(\bar{\eta}) \subset B(\varepsilon, \delta, \tau)$ then $\Pi_\Sigma(F) = \mathcal{I}(\bar{\eta})$ and*

$$F \cap B(\varepsilon, \delta, \tau) = \left\{\phi_t(\mathcal{I}(\bar{\eta})) : |t| < \tau\right\}.$$

Proof. Observe that, by the construction of Σ, each strip F intersects Σ in a connected component of some $\mathcal{I}(\bar{\eta})$, $\bar{\eta} \in \Sigma$. Moreover, the geodesic flow preserves the sets $\mathcal{I}$, that is, for every $t \in \mathbb{R}$ we have $\phi_t(\mathcal{I}(\bar{\eta})) = \mathcal{I}(\phi_t(\bar{\eta}))$ for every $\bar{\eta} \in T_1\tilde{M}$, compare Figure 3. This concludes the proof. □

4.2. **A basis for the quotient topology.** Following the notations of the previous subsection, let us denote

$$\Sigma^+ := \{R(r,s) : r \in (r_0^- - \varepsilon, r_0^+ + \varepsilon), s \in (0, \delta)\} \subset \Sigma,$$
$$\Sigma^- := \{R(r,s) : r \in (r_0^- - \varepsilon, r_0^+ + \varepsilon), s \in (-\delta, 0)\} \subset \Sigma.$$

For each point $\bar{\eta} \in B(\varepsilon, \delta, \tau)$, denote by

$$B_{\bar{\eta}}^{cs}(\varepsilon, \delta, \tau) \subset \tilde{\mathscr{F}}^{cs}(\bar{\eta}) \cap B(\varepsilon, \delta, \tau), \quad B_{\bar{\eta}}^{cu}(\varepsilon, \delta, \tau) \subset \tilde{\mathscr{F}}^{cu}(\bar{\eta}) \cap B(\varepsilon, \delta, \tau)$$

the connected components of the intersections of the central sets of $\bar{\eta}$ with $B(\varepsilon, \delta, \tau)$ which contain $\bar{\eta}$. Given $\bar{\eta} \in \Sigma$ let

$$\begin{aligned} W_\Sigma^s(\bar{\eta}) &:= \Pi_\Sigma\big(B_{\bar{\eta}}^{cs}(\varepsilon, \delta, \tau)\big) = B_{\bar{\eta}}^{cs}(\varepsilon, \delta, \tau)) \cap \Sigma_{\bar{\theta}}(\varepsilon, \delta), \\ W_\Sigma^u(\bar{\eta}) &:= \Pi_\Sigma\big(B_{\bar{\eta}}^{cu}(\varepsilon, \delta, \tau)\big). \end{aligned} \tag{3}$$

Note that, in fact, by the definition of the map R, for every $\bar{\eta} \in \Sigma$ there exist parameters r, s such that $W_\Sigma^s(\bar{\eta}) = R(r,s)$. However, in general $W_\Sigma^u(\bar{\eta})$ might not satisfy such a property.

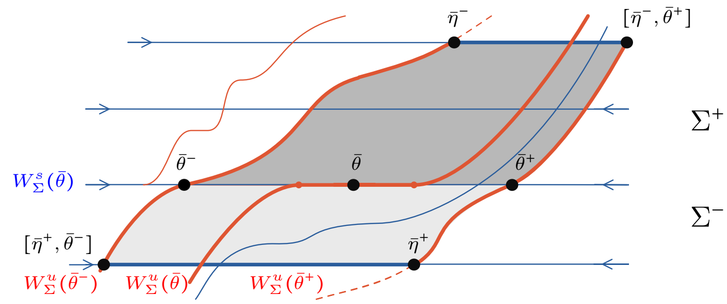

FIGURE 4. Region defined by expansive points $\bar\theta^\pm$ and $\bar\eta^\pm$, contained in the region $\Sigma_{\bar\theta}(\varepsilon,\delta)$ split into Σ^+ and Σ^-, and containing the open set $U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ (shaded region)

If the section Σ is sufficiently narrow in the vertical direction being close enough to $\mathcal{I}(\bar\theta)$ then every two different points in Σ are heteroclinically related. Given $\bar\theta$, choose $\delta_{\bar\theta}$ small such that $\tilde{\mathscr{F}}^{cu}(\bar\theta)$ intersects $\tilde{\mathscr{F}}^s(R(0,\pm\delta_{\bar\theta}))$. Moreover, for $\delta\in(0,\delta_{\bar\theta})$ there exist $\varepsilon_0=\varepsilon_0(\bar\theta,\delta)>0$ such that for every $\varepsilon\in(0,\varepsilon_0)$ for every $r\in(r_0^- -\varepsilon,r_0^-)\cup(r_0^+,r_0^+ +\varepsilon)$ and every s with $|s|<\delta$ we have

$$W^u_\Sigma(R(r,0))\cap W^s_\Sigma(R(0,s))\subset\Sigma_{\bar\theta}(\varepsilon,\delta)\,.$$

Definition 4.4. As usual in the context of dynamical systems (e.g. [27, Chapter 6.4]), given $\bar\eta,\bar\xi\in\Sigma_{\bar\theta}(\varepsilon,\delta)$, we denote

$$[\bar\eta,\bar\xi\,]:=W^s_\Sigma(\bar\eta)\cap W^u_\Sigma(\bar\xi)\,. \tag{4}$$

The basis we shall construct is in many respects a "blow up" of the classical local product neighborhood of hyperbolic dynamics.

By Lemma 3.7 and Corollary 3.10, given $\varepsilon_0=\varepsilon_0(\bar\theta,\delta)$ and $\varepsilon\in(0,\varepsilon_0)$ there exist expansive points of the form

$$\bar\theta^-=R(r_0^- -\rho^-,0),\quad \bar\theta^+=R(r_0^+ +\rho^+,0),$$

for some numbers $\rho^-,\rho^+\in(0,\varepsilon)$. The sets $W^u_\Sigma(\bar\theta^\pm)$ are curves which are disjoint from $W^u_\Sigma(\bar\theta)$. Moreover, all such curves bound a region in $\Sigma_{\bar\theta}(\varepsilon,\delta)$ homeomorphic to a rectangle whose relative interior contains $\mathcal{I}(\bar\theta)$ (compare Figure 4). The set $W^s_\Sigma(\bar\theta)$ divides $W^u_\Sigma(\bar\theta^-)$ into two parts: one in Σ^+ and one in Σ^-. By the above, for expansive points

$$\bar\eta^-\in W^u_\Sigma(\bar\theta^-)\cap\Sigma^+\quad\text{ and }\quad\bar\eta^+\in W^u_\Sigma(\bar\theta^+)\cap\Sigma^-$$

the intersections $[\bar\eta^-,\bar\theta^+]$ and $[\bar\eta^+,\bar\theta^-]$ are nonempty. Note that this intersection can contain a nontrivial compact curve homeomorphic to an interval, and by the previous remarks we can assume that such intersections are in $\Sigma_{\bar\theta}(\varepsilon,\delta)$. Given such points $\bar\eta^-$ and $\bar\eta^+$, let us denote by $U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ the open two-dimensional region in $\Sigma_{\bar\theta}(\varepsilon,\delta)$ whose boundary is formed by the above pieces of stable and unstable arcs. Notice that the choice of the expansive points $\bar\eta^-, \bar\eta^+$ depends on δ and that there exist such points for every $\delta > 0$. The region $U_{\bar\theta}(\varepsilon, \delta, \bar\theta^-, \bar\theta^+, \bar\eta^-, \bar\eta^+)$ clearly contains $\mathcal{I}(\bar\theta)$ (compare Figure 4).

Lemma 4.5. *Given* $\bar\theta \in T_1\tilde M$, $\delta \in (0, \delta_{\bar\theta})$, $\varepsilon_0 = \varepsilon_0(\bar\theta, \delta)$, $\varepsilon \in (0, \varepsilon_0)$, *and expansive points* $\bar\theta^- = R(r_0^- - \rho^-, 0)$, $\bar\theta^+ = R(r_0^+ + \rho^+, 0)$ *with* $\rho^\pm \in (0, \varepsilon)$, *and* $\bar\eta^\pm \in W^u_\Sigma(\bar\theta^\pm) \cap \Sigma^\mp$, *the above constructed region* $U = U_{\bar\theta}(\varepsilon, \delta, \bar\theta^-, \bar\theta^+, \bar\eta^-, \bar\eta^+)$ *in the section* $\Sigma_{\bar\theta}(\varepsilon, \delta)$ *has the following properties:*

(1) *We have* $\bar\chi^{-1}(\bar\chi(U)) = U$ *and hence the set* $\bar\chi(U)$ *is an open neighborhood of* $\bar\chi(\bar\theta)$ *in the quotient topology restricted to* $\bar\chi(\Sigma_{\bar\theta}(\varepsilon, \delta))$.

(2) *For every positive numbers* $t', \delta', \varepsilon'$ *we can choose* δ, ε *such that the above considered region* $U = U_{\bar\theta}(\varepsilon, \delta, \bar\theta^-, \bar\theta^+, \bar\eta^-, \bar\eta^+)$ *for every* t *with* $|t| \le t'$ *satisfies*

$$U_{\phi_t(\bar\theta)}\big(\varepsilon, \delta, \phi_t(\bar\theta^-), \phi_t(\bar\theta^+), \phi_t(\bar\eta^-), \phi_t(\bar\eta^+)\big) \subset \Sigma_{\phi_t(\bar\theta)}(\varepsilon', \delta'),$$

and moreover,

$$\phi_t(U) \subset U_{\phi_t(\bar\theta)}\big(\varepsilon, \delta', \phi_t(\bar\theta^-), \phi_t(\bar\theta^+), \phi_t(\bar\eta^-), \phi_t(\bar\eta^+)\big).$$

(3) *The set* $\bar\chi(U)$ *is a (topological) local cross section of the quotient flow in* $\bar X$, *namely there exists an open set containing* $\bar\chi(\bar\theta)$ *such that the quotient flow restricted to this set intersects* $\bar\chi(U)$ *in just one point.*

Proof. Given $\bar\xi \in U$, we have $\bar\chi^{-1}(\bar\chi(\bar\xi)) = \mathcal{I}(\bar\xi) \subset \Sigma_{\bar\theta}(\varepsilon, \delta)$. By the construction of $\Sigma_{\bar\theta}(\varepsilon, \delta)$, the curve $\mathcal{I}(\bar\xi)$ meets the boundary C of U if, and only if, $\bar\xi$ is in C already. Because this boundary is made of pieces of center stable and center unstable leaves, if a strip through a point $\bar\xi \in \Sigma_{\bar\theta}(\varepsilon, \delta)$ meets C then the whole set $\mathcal{I}(\bar\xi)$ must be included in one of the pieces of center stable and center unstable leaves used to construct C. So we conclude that $\bar\chi^{-1}(\bar\chi(\bar\xi))$ is in U for every $\bar\xi \in U$, and obviously $\bar\chi^{-1}(\bar\chi(\bar\xi))$ contains U.

By the definition of the quotient topology restricted to $\Sigma_{\bar\theta}(\varepsilon, \delta)$, we have that $\bar\chi(U)$ is a relative open neighborhood of $\bar\chi(\bar\xi)$ in $\bar\chi(\Sigma_{\bar\theta}(\varepsilon, \delta))$ thus proving item (1) in the lemma.

The proof of item (2) follows from the construction of U. In fact, all the dynamical objects involved in its construction, that is, stable leaves and heteroclinic intersections, are invariant by the geodesic flow. The constants ε and δ may vary a little since they are geometric quantifiers of compact pieces of stable leaves which contain strips. The size of a strip does not change under the action of the geodesic flow but the size of a neighborhoods of it changes continuously. From the above statements is straightforward to conclude item (2).

Item (3) follows from the construction by the definition of equivalence relation within any flat. □

Lemma 4.6. *Given* $\bar\theta \in T_1\tilde M$, $\delta \in (0, \delta_{\bar\theta})$, $\varepsilon_0 = \varepsilon_0(\bar\theta, \delta)$, $\varepsilon \in (0, \varepsilon_0)$, *and expansive points* $\bar\theta^- = R(r_0^- - \rho^-, 0)$, $\bar\theta^+ = R(r_0^+ + \rho^+, 0)$ *with* $\rho^\pm \in (0, \varepsilon)$, *and* $\bar\eta^\pm \in$

$W^u_\Sigma(\bar\theta^\pm)\cap\Sigma^\mp$, consider the above constructed region $U = U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ in the section $\Sigma_{\bar\theta}(\varepsilon,\delta)$ for $\tau>0$ consider the set

$$A = A_{\bar\theta}(\tau,\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+) := \bigcup_{|t|<\tau}\phi_t(U) \tag{5}$$

satisfies $\bar\chi^{-1}(\bar\chi(A)) = A$. Hence, the collection of such sets

$$\Big\{\bar\chi\big(A_{\bar\theta}(\tau,\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)\big)\Big\}$$

forms a basis for the quotient topology of $\bar X$.

Proof. First observe that A by definition is homeomorphic to $(-\tau,\tau)\times U$. Therefore, Brower's open mapping theorem implies that each such set is open in $T_1\tilde M$.

Moreover, by Lemma 4.5 item (2) we have

$$A = \bigcup_{|t|<\tau} U_{\phi_t\left(\bar\theta\right)}\big(\varepsilon,\delta,\phi_t(\bar\theta^-),\phi_t(\bar\theta^+),\phi_t(\bar\eta^-),\phi_t(\bar\eta^+)\big)\big)$$

for τ small enough. Applying Lemma 4.5 item (1) to the above union of sets we deduce that $\bar\chi^{-1}(\bar\chi(A)) = A$ as claimed.

This yields that the family of sets $A_{\bar\theta}(\tau,\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ specified as in the statement is a family of open sets in the quotient $\bar X$ since, by definition, an open set for the quotient topology is any set whose preimage by the quotient map $\bar\chi$ is an open set in $T_1\tilde M$. To see that they provide a basis for the quotient topology observe that the family of open sets $B_{\bar\theta}(\varepsilon,\delta,\tau')$ defined in (2) is a basis for the family of open neighborhoods of the sets $\mathcal{I}(\bar\theta)$, and that $A_{\bar\theta}(\tau,\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ is contained in $B_{\bar\theta}(\varepsilon,\delta,\tau')$. So let $\bar\chi(\bar\theta)$ be the equivalence class of $\bar\theta$ and V an open neighborhood of $\bar\chi(\bar\theta)$ in the quotient $\bar X$. The set $\bar\chi^{-1}(V)$ is an open neighborhood of $\mathcal{I}(\bar\theta)$ which contains some $B_{\bar\theta}(\varepsilon,\delta,\alpha)$ for some parameters ε,δ,τ'. Hence the family of sets $A_{\bar\theta}(\tau,\rho,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ provide a basis, too. □

4.3. **Smooth manifold structure.** The main result of the subsection is the following.

Proposition 4.7. *Given $\bar\theta\in T_1\tilde M$, let $U = U_{\bar\theta}(\varepsilon,\delta,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+)$ be the local cross section of the geodesic flow as defined in the previous section and consider the corresponding set A as defined in (5) for some $\tau>0$. There exist numbers $a<a'$, $b<b'$ depending on $\bar\theta,\delta,\varepsilon,\bar\theta^-,\bar\theta^+,\bar\eta^-,\bar\eta^+$ and a homeomorphism*

$$f\colon (a,a')\times(b,b')\times(-\tau,\tau)\to\bigcup_{|t|<\tau}\bar\psi_t\big(\bar\chi(U)\big) = \bar\chi(A)$$

for every $\tau>0$.

In particular, the quotient spaces $\bar X$ and X are topological 3-manifolds.

Proof. The proof relies essentially on the transitivity of the geodesic flow and the minimality of central foliations. By Theorem 2.9, each stable leaf is dense in U. In particular, this holds for the central stable leaf of a dense orbit which has no nontrivial strips. This happens as well with the intersections of center unstable leaves without nontrivial strips.

Let $\theta^* \in T_1M$ be a point whose orbit is dense in T_1M, and let $\bar{\theta}^*$ one of its lifts in $T_1\tilde{M}$. Suppose that $\bar{\theta}^* \in U$ and that $d_S(\bar{\theta}^*, \bar{\theta}) < \delta$. The lifts of the stable set and the unstable set of the orbit of $\bar{\theta}^*$ in $T_1\tilde{M}$ will both be dense in $\Sigma_{\bar{\theta}}$.

Consider the arc length parametrizations

$$R^s\colon (a,a') \to U \quad \text{with} \quad R^s(0) = \bar{\theta}^*$$
$$R^u\colon (b,b') \to U \quad \text{with} \quad R^u(0) = \bar{\theta}^*$$

of the arc of the intersection of the stable leaf $\tilde{\mathscr{F}}^s(\bar{\theta}^*)$ within U and of the arc of intersection of the center unstable leaf $\tilde{\mathscr{F}}^{cu}(\bar{\theta}^*)$ within U, respectively. Since we can choose $\bar{\theta}^*$ as closed to $\bar{\theta}$ as we wish, we can choose $a' - a$ very close to the length of the connected component of the stable set of $\bar{\theta}$ in U containing $\bar{\theta}$. Analogously, $b' - b$ can be chosen very close to the length of the connected component of the unstable set of $\bar{\theta}$ in U containing $\bar{\theta}$.

Claim. *For every $r \in (a,a')$, $s \in (b,b')$ the point $[R^u(r), R^s(s)]$ is contained in U. Moreover, the map*

$$h\colon (a,a') \times (b,b') \to \bar{\chi}(U)\colon (r,s) \mapsto \bar{\chi}\big([R^u(r), R^s(s)]\big)$$

is a homeomorphism. In particular, $\bar{\chi}(U)$ is a topological 2-manifold.

Proof. Recall the definition of $W^{s/u}_\Sigma(\cdot)$ in (3). The set U is foliated by

$$\{W^s_\Sigma(\bar{\eta}) \cap U \colon \bar{\eta} \in U\} \quad \text{and by} \quad \{W^u_\Sigma(\bar{\eta}) \cap U \colon \bar{\eta} \in U\}$$

in a trivial way. Both families of sets induce a product foliation of U by embeddings of open segments. Therefore, each of these sets splits its complement in the closure of U into two disjoint open regions (see Figure 5). Let $c^s(\bar{\eta}^-)$ denote the curve that is formed by the piece of $W^s_\Sigma(\bar{\eta}^-)$ bounded by $\bar{\eta}^-$ and $[\bar{\eta}^-, \bar{\eta}^+]$ and let $c^s(\bar{\eta}^+)$ be the curve that is formed by $W^s_\Sigma(\bar{\eta}^+)$ bounded by $[\bar{\eta}^+, \bar{\eta}^-]$ and $\bar{\eta}^+$. Analogously, let $c^u(\bar{\eta}^-)$ denote the curve that is formed by the piece of $W^u_\Sigma(\bar{\eta}^-)$ bounded by $\bar{\eta}^-$ and $[\bar{\eta}^+, \bar{\eta}^-]$ and let $c^u(\bar{\eta}^+)$ be the curve that is formed by $W^u_\Sigma(\bar{\eta}^+)$ bounded by $\bar{\eta}^+$ and $[\bar{\eta}^-, \bar{\eta}^+]$. Each set $W^s_\Sigma(\bar{\eta})$ splits the closure of U into two open disjoint regions, one of them containing $c^s(\bar{\eta}^-)$ and the other one containing $c^s(\bar{\eta}^+)$. Analogously, each set $W^u_\Sigma(\bar{\eta})$ splits the closure of U into two open disjoint regions, one of them containing the curve $c^u(\bar{\eta}^-)$ and the other one the curve $c^u(\bar{\eta}^+)$.

Now, it is easy to show that the intersections $[R^u(r), R^s(s)]$ are all contained in U. Indeed, each curve $W^s_\Sigma(R^u(r))$ contains points of both $c^u(\bar{\eta}^-)$ and $c^u(\bar{\eta}^+)$, so for every $s \in (b,b')$ by the Jordan curve theorem and the continuity of the stable foliation it has to cross $W^u_\Sigma(R^s(s))$. Since by Lemma 4.5 each intersection of the form $W^s_\Sigma(\bar{\eta}) \cap W^u_\Sigma(\bar{\eta})$ for $\bar{\eta} \in U$ is a class in a strip that is contained in U, we get that $[R^u(r), R^s(s)] \in U$.

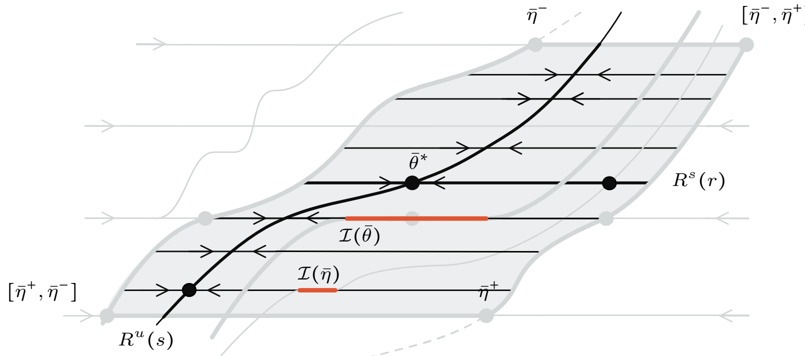

FIGURE 5. Parametrization based on an expansive point in U (shaded region). All points in each flat (e.g. $\mathcal{I}(\bar\eta)$) have *one* common pair (r,s) of parameters

Moreover, the parameterizations R^s, R^u induce continuous parameterizations of their quotients

$$\bar\chi \circ R^s\colon (a,a') \to \bar\chi(U), \quad \bar\chi \circ R^u\colon (b,b') \to \bar\chi(U),$$

because strips of orbits in the center stable and unstable sets of $\bar\theta^*$ are trivial. Therefore, the map

$$h\colon (a,a') \times (b,b') \to \bar\chi(U),$$
$$h(r,s) := \bar\chi([R^u(r), R^s(s)]) = [(\bar\chi \circ R^u)(r), (\bar\chi \circ R^s)(s)]$$

gives us a homeomorphism from an open rectangle onto $\bar\chi(U)$. Indeed, the map h is already a bijection restricted to the dense subset of intersections between the center stable and center unstable sets of dense orbits intersecting U. But taking the quotient we have that the map h is continuous and injective in its image because each set of the form $\bar\chi([R^u(r), R^s(s)])$ is just a point in the quotient. By construction, the inverse of h is also continuous. So we have a homeomorphism of a rectangle onto $\bar\chi(U)$. By the Brower's open mapping theorem, the image of h is an open 2-dimensional subset, which shows the claim. □

To conclude the proof of the proposition we now apply the quotient flow to the section $\bar\chi(U)$ which is a topological surface. Indeed, by the construction each set of the form

$$\bigcup_{|t|<\tau} \tilde\psi_t(\bar\chi(U))$$

is continuously bijective to the product $(a,a') \times (b,b') \times (-\tau,\tau)$. By the Claim this correspondence is a homeomorphism and therefore $\bar\chi(U)$ is an open 3-dimensional set. This finishes the proof of the proposition. □

Proof of Theorem 4.2. Proposition 4.7 implies that each point in $\bar{X}$ has an open set that is continuously parametrized by an open subset of $\mathbb{R}^3$ which characterizes a topological 3-manifold. Now, a celebrated result due to Bing [3] and Moise [34] tells us that the space $\bar{X}$ has a smooth structure compatible with the quotient topology. Since the quotient X is locally homeomorphic to $\bar{X}$, the above assertions extend to X. □

The fact that the quotient space X is a compact manifold is very important in many respects. Any smooth manifold admits a Riemannian metric and hence there exists a distance $d: X \times X \to \mathbb{R}$ which endows X with a structure of a complete metric space. By the definitions of $\bar{X}$ and X, it is straightforward to see that the map

$$\bar{\Pi}: \bar{X} \to X, \quad \bar{\Pi}(\bar{\chi}(\bar{\theta})) = \chi(\bar{\pi}(\bar{\theta}))$$

is a covering map, where $\bar{\pi}: T_1\tilde{M} \to T_1M$ is the natural projection and $\chi: T_1M \to X$ and $\bar{\chi}: T_1\tilde{M} \to \bar{X}$ are the quotient maps inducing the quotient spaces. The pullback $\bar{d}$ of d to $\bar{X}$ by $\bar{\Pi}$ provides a structure of a complete metric space $(\bar{X}, \bar{d})$ locally isometric to (X, d). The metric d is continuous and hence the quotient flow is continuous with respect to d. We observe that the projection $\bar{\Pi}$ of the basis $\bar{\chi}(A_{(\cdot)}(\cdot))$ defined in Lemma 4.6 naturally defines a basis in X.

5. The dynamics of the quotient flow

We start this section by recalling some general definitions. Given a general complete continuous flow $\psi_t: X \to X$ acting on a complete metric space (X, d), the *strong stable set* $W^{ss}(x)$ of a point $x \in X$ is the set of points $y \in X$ such that

$$\lim_{t \to +\infty} d(\psi_t(y), \psi_t(x)) = 0.$$

The *strong unstable set* of a point $x \in X$ is defined to be the strong stable set of x with respect to ψ_{-t} and denoted by $W^{uu}(x)$. The *center stable set* $W^{cs}(x)$ of a point $x \in X$ is the set of points $y \in X$ such that

$$d(\psi_t(y), \psi_t(x)) \leq C$$

for some $C > 0$ and every $t \geq 0$. The *center unstable set* is defined to be the center stable set of x with respect to ψ_{-t} and denoted by $W^{cu}(x)$. For $x \in X$ and $\varepsilon > 0$ let

$$W^{cs}_\varepsilon(x) := \{y \in X: d(\psi_t(y), \psi_t(x)) \leq \varepsilon \text{ for every } t \geq 0\},$$
$$W^{cu}_\varepsilon(x) := \{y \in X: d(\psi_{-t}(y), \psi_{-t}(x)) \leq \varepsilon \text{ for every } t \geq 0\}.$$

The flow ψ_t is said to have *local product structure*[1] if for each sufficiently small $\varepsilon > 0$ there is $\delta > 0$ such that for every $x, y \in X$ with $d(x, y) \leq \delta$ there is a unique $\tau = \tau(x, y)$ with $|\tau| \leq \varepsilon$ satisfying $W^{cs}_\varepsilon(\psi_\tau(x)) \cap W^{cu}_\varepsilon(y) \neq \varnothing$.

[1]In other references such as, for example [5, 50], this property is called *canonical coordinates* which we avoid because of its similarity to geometric properties.

Recall that the flow ψ_t is *topologically transitive* if for any open sets U_1 and U_2 there is $t > 0$ such that $\psi_t(U_1) \cap U_2 \neq \varnothing$ or, equivalently, of there exists a dense orbit. The flow is *topologically mixing* if for any open sets U_1 and U_2 there is $t_0 > 0$ such that $\psi_t(U_1) \cap U_2 \neq \varnothing$ for every $t \geq t_0$.

In the remainder of this section we consider the quotient flow $\psi_t\colon X \to X$ on the quotient space (X, d) defined in Section 4 and we describe the dynamical properties of this flow. The following theorem is the main result of this section.

Theorem 5.1. *The quotient flow $\psi_t\colon X \to X$ has the following properties:*

1. *The flow is expansive.*
2. *For each $\chi(\theta) \in X$ its center stable set (center unstable set) is the quotient of the center stable set (center unstable set) of θ with respect to the geodesic flow.*
3. *For each $\chi(\theta) \in X$ the strong stable set (strong unstable set) is the quotient of $\mathscr{F}^s(\theta)$ (of $\mathscr{F}^u(\theta)$).*
4. *If $(p, v) \neq (q, -w)$ then the center stable set of $[(q, w)]$ intersects the center unstable set of $[(p, v)]$ at a single orbit of ψ_t. In particular, the flow has a local product structure.*
5. *The flow is topologically transitive.*
6. *Each strong stable set (strong unstable set) is dense.*
7. *The flow is topologically mixing.*

We shall prove Theorem 5.1 in several steps and complete its proof at the end of this section. In the forthcoming sections we shall give some interesting applications of Theorem 5.1 and we shall continue exploring the regularity of the quotient space X.

Theorem 2.9 asserts many density properties of dynamical objects associated to the geodesic flow of (M, g). Since in the quotient topology an open set is a set whose pre-image under the quotient map is open in T_1M, all such properties are inherited by the quotient flow ψ_t in a straightforward way.

Let $\mathrm{Isom}(\bar{X})$ be the group of isometries of $(\bar{X}, \bar{d})$, which contains a representation Γ of the fundamental group $\pi_1(X)$. Notice that for every covering isometry T defined in $T_1\tilde{M}$ the composition $\bar{\chi} \circ T$ induces a deck transformation $\bar{T}\colon \bar{X} \to \bar{X}$ that satisfies $\bar{\chi} \circ T = \bar{T} \circ \bar{\chi}$ and is an element of Γ.

5.1. **Expansiveness.** Intuitively, expansiveness is a property we should expect since the quotient collapses strips which are the only "obstructions" to it.

We will need the following basic result of the theory of metric spaces.

Lemma 5.2. *Let Y be a smooth manifold which admits a complete metric space structure (Y, D). Suppose that there exists a sequence of compact sets $(K_n)_{n\geq 1}$ such that*

1. *for every $n \geq 1$ there exists an open neighborhood V_n of K_n contained in K_{n+1},*

(2) $\bigcup_n K_n = Y$.

Then for every $p \in Y$ *and every sequence* $(x_n)_{n\geq 1}$ *of points* $x_n \in K_{n+1} \setminus K_n$, *we have that* $D(p, x_n) \to +\infty$ *if* $n \to +\infty$.

Proof. Suppose, by contradiction, that there exists $L > 0$ such that $d(p, x_n) \leq L$ for every $n \geq 1$. Since (Y, D) is complete and the closed ball $B(p, L)$ of radius L centered at p is compact, the sequence $(x_n)_n$ has a subsequence $(x_{n_k})_k$ converging to a point $q \in B(p, L)$. By item (2), there exists $m \in \mathbb{N}$ such that $q \in K_m$. By item (1), $q \in V_m \subset K_{m+1}$ where V_m is an open set. So there is $k_0 > 0$ such that $p_{n_k} \in V_m$ for every $k \geq k_0$. This contradicts the choice of the sequence $(p_{n_k})_k$, since $p_{n_k} \in K_{n_k+1} \setminus K_{n_k}$ for every k and $V_m \subset K_{m+1} \subset K_k$ for every $k \geq m+1$. □

Now we show that $\bar{X}$ has a sequence of compact sets K_n satisfying the assumptions of Lemma 5.2.

Lemma 5.3. *If* $K \subset \bar{X}$ *is compact then* $\bar{\chi}^{-1}(K) \subset T_1\tilde{M}$ *is compact.*

Proof. We know that $\bar{\chi}^{-1}(K)$ is closed by continuity of $\bar{\chi}$. By Lemma 4.6 the sets $\bar{\chi}(A)$ for $A = A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+)$ with $\bar{\theta} \in T_1\tilde{M}$ and $\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+$ as in Lemma 4.6 form a basis for the quotient topology. Here each set of the basis satisfies

$$\bar{\chi}^{-1}\big(\bar{\chi}(A)\big) = A.$$

Each set A is an open subset with compact closure. Since K is compact, it can be covered by a finite collection $\{\bar{\chi}(A_i)\}_{i=1,\ldots,m}$ of sets in this basis. By Lemma 4.6, this implies that $\bar{\chi}^{-1}(K)$ is covered by the finite collection A_i of open sets. Since the union of the closures of the sets A_i is a compact set, $\bar{\chi}^{-1}(K)$ is a closed subset of a compact set. Thus $\bar{\chi}^{-1}(K)$ is compact. □

Lemma 5.4. *Let* $\bar{\theta} \in T_1\tilde{M}$, *and consider the family of balls*

$$B_{d_S}(\bar{\theta}, r) \coloneqq \big\{\bar{\eta} \in T_1\tilde{M} \colon d_S(\bar{\theta}, \bar{\eta}) \leq r\big\},$$

where d_S *is the induced Sasaki distance. Let* $Q > 0$ *be the constant defined in Theorem 2.2. Then the sequence of compact sets* $(K_n)_{n\geq 1}$ *in the space* $\bar{X}$ *given by*

$$K_n \coloneqq \chi\big(B_{d_S}(\bar{\theta}, 3Qn)\big)$$

satisfies the hypotheses of Lemma 5.2.

Proof. By Lemma 2.6, the width of a flat strip is bounded above by $2Q$. So the length of the equivalence class of every point in $T_1\tilde{M}$ is bounded above by $2Q$ since the flat strips in $\tilde{M}$ are isometric to the strips of bi-asymptotic orbits of the geodesic flow of $T_1\tilde{M}$. Let $C_r \subset T_1\tilde{M}$ be the union of all classes of points in $B_{d_S}(\bar{\theta}, r)$. By the triangular inequality, C_r is a compact connected set whose diameter is at most $r + 2Q$. Thus, since the interior of C_{r+3Q} contains the open ball of radius $r + 3Q$ centered at $\bar{\theta}$, the set C_{r+3Q} contains C_r. Moreover, there exists an open cover $\mathcal{U}_r$ of C_r by open sets taken from the family constructed in Lemma 4.6 which is contained in the interior of C_{r+3Q}. This is because we can cover an equivalence class with a set in this family that is arbitrarily close to the class.

Consider the sets

$$K_n := \chi(C_{3Qn}) = \chi(B_{d_S}(\bar{\theta}, 3Qn))$$

for $n \in \mathbb{N}$. Each such set is compact because it is the continuous image of a compact set. Clearly, the union of the C_{3Qn} over n is the whole $T_1\tilde{M}$, so the union over n of the K_n is $\bar{X}$. By the choice of the radius $3Qn$, we have that $C_{3Qn} \subset C_{3(n+1)Q}$ for every $n \in \mathbb{N}$, and hence $K_n \subset K_{n+1}$ for every integer n. This yields item (2) of Lemma 5.2. Moreover, by the previous paragraph the open neighborhood $\chi(\mathcal{U}_{3nQ})$ of $\chi(C_{3Qn}) = K_n$ is in the interior of K_{n+1} for each integer n. This implies item (1) of Lemma 5.2. □

Lemma 5.5. *The flow $\bar{\psi}_t$ is expansive.*

Proof. By contradiction, suppose that there exist two points $[\bar{\theta}], [\bar{\eta}] \in \bar{X}$ whose orbits have bounded Hausdorff distance, say, bounded by $L > 0$. Up to some reparametrization we can assume that $\bar{d}([\bar{\theta}], [\bar{\eta}]) \le L$ and that there is some increasing homeomorphism $s(t)$ of $\mathbb{R}$ satisfying $s(0) = 0$ such that for every $t \in \mathbb{R}$ we have

$$\bar{d}\big(\bar{\psi}_t([\bar{\theta}]), \bar{\psi}_{s(t)}([\bar{\eta}])\big) \le L\,.$$

We need the following intermediate result.

Claim. *There exists $\bar{L} > 0$ such that*

$$d_S(\phi_t(\bar{\theta}), \phi_{s(t)}(\bar{\eta})) \le \bar{L}$$

for every t, where d_S is the Sasaki distance.

Proof. Fix some fundamental domain $\mathcal{D}$ containing $\bar{\theta}$, let K be the diameter of $\mathcal{D}$ of T_1M in $T_1\tilde{M}$. Given $t \in \mathbb{R}$, let $T_t\colon T_1\tilde{M} \to T_1\tilde{M}$ be a covering isometry such that $T_t(\phi_t(\bar{\theta})) \in \mathcal{D}$. Hence, we have

$$d_S(\bar{\theta}, T_t(\phi_t(\bar{\theta}))) \le K$$

for every t.

By contradiction, suppose that for every $n > 0$ there would exist some $t_n \in \mathbb{R}$ such that the infimum of the Sasaki distance d_S from $\phi_{t_n}(\bar{\theta})$ to the set $\{\phi_\tau(\bar{\eta})\colon \tau \in \mathbb{R}\}$, is attained at $\tau = \alpha(t_n)$ where

$$d_S(\phi_{t_n}(\bar{\theta}), \phi_{\alpha(t_n)}(\bar{\eta})) \ge 3Qn + K + 2Q\,.$$

This would imply that $T_{t_n}(\phi_\tau(\bar{\eta})) \notin C_{3Qn}$ for every $\tau \in \mathbb{R}$ by the definition of C_{3Qn}. Taking the quotient we would get

$$(\chi \circ T_{t_n} \circ \phi_{t_n})(\bar{\theta}) \in \chi(\mathcal{D}) \quad \text{and} \quad (\chi \circ T_{t_n} \circ \phi_\tau)(\bar{\eta}) \notin K_n$$

for every $n \ge 1$ and every $\tau \in \mathbb{R}$. By Lemma 5.2 and Lemma 5.4 we would get that for each $\tau \in \mathbb{R}$

$$\bar{d}\big((\chi \circ T_{t_n} \circ \phi_{t_n})(\bar{\theta}), (\chi \circ T_{t_n} \circ \phi_\tau)(\bar{\eta})\big) \to \infty$$

if $n \to \infty$. But this would imply

$$\sup_{\tau\in\mathbb{R}} \bar{d}\big((\chi \circ T_{t_n} \circ \phi_{t_n})(\bar\theta), (\chi \circ T_{t_n} \circ \phi_\tau)(\bar\eta)\big) = \sup_{\tau\in\mathbb{R}} \bar{d}\big(\tilde{T_{t_n}}([\phi_{t_n}(\bar\theta)], \tilde{T_{t_n}}([\phi_\tau(\bar\eta)])\big)$$

As we remarked above, the induced deck transformation $\bar{T}_t$ is an isometry with respect to $\bar{d}$. Hence, the above supremum is equal to

$$\sup_{\tau\in\mathbb{R}} \bar{d}\big([\phi_{t_n}(\bar\theta)], [\phi_\tau(\bar\eta)]\big) = \sup_{\tau\in\mathbb{R}} \bar{d}\big(\bar\psi_{t_n}([\bar\theta], \bar\psi_\tau([\bar\eta])\big).$$

But the latter term is bounded from above by L for all n. This contradiction shows that such a sequence $(t_n)_n$ does not exists. This proves the claim. □

Let us consider the orbits of $\bar\theta$ and $\bar\eta$ under the flow ϕ_t in $T_1\tilde{M}$. Observe that the Claim implies that the strips $F(\bar\theta)$ and $F(\bar\eta)$ are within a distance of $\bar{L} + \bar{Q}$, where $\bar{Q}$ depends on the constant Q in Morse's Theorem 2.2. Thus, taking the canonical projection from $T_1\tilde{M}$ onto $\tilde{M}$ we get that the strips of the geodesics $\gamma_{\bar\theta}$, $\gamma_{\bar\eta}$ are within a distance $\bar{L} + \bar{Q}$ (remind that the canonical projection is a Riemannian submersion). This can only happen when both geodesics are bi-asymptotic and hence they are in the same strip. So $F(\bar\theta) = F(\bar\eta)$ and therefore their quotients coincide in $\bar{X}$.

This implies that the the quotient flow $\bar\psi_t$ is expansive. □

Lemma 5.6. *The flow ψ_t is expansive.*

Proof. Observe that $\bar{X}$ covers X. Thus, by compactness of X, there exists $r > 0$ such that the covering map restricted to any ball of radius r is a homeomorphism. By contradiction, suppose that ψ_t would not be expansive and given two points $[\theta], [\eta] \in X$ on distinct orbits, there would exist a homeomorphism $s(t)$ of $\mathbb{R}$ satisfying $s(0) = 0$ such that

$$d\big(\psi_t([\theta]), \psi_{s(t)}([\eta])\big) \le r$$

for every $t \in \mathbb{R}$. Then we could lift these orbits to a pair of orbits in $\bar{X}$ which would stay within a distance r from each other. Therefore, by Lemma 5.5, these orbits would coincide. Thus, the orbits of $[\theta]$ and $[\eta]$ would coincide, which gives a contradiction. □

5.2. **Invariant sets and heteroclinic relation.** We will consider the following invariant sets of the quotient flows $\bar\psi_t$ and ψ_t. Recall the definition of the projction map $\bar\chi: T_1\tilde{M} \to \bar{X}$ in Section 4. Given $\theta \in T_1M$ and one of its lifts $\bar\theta \in T_1\tilde{M}$, let

$$\tilde{W}^*([\bar\theta]) := \bar\chi\big(\tilde{\mathscr{F}}^*(\bar\theta)\big), \quad W^*([\theta]) := \chi\big(\mathscr{F}^*(\theta)\big), \quad * = cs, cu,$$
$$\tilde{W}^{ss}([\bar\theta]) := \bar\chi\big(\tilde{\mathscr{F}}^s(\bar\theta)\big), \quad W^{ss}([\theta]) := \chi\big(\mathscr{F}^s(\theta)\big),$$
$$\tilde{W}^{uu}([\bar\theta]) := \bar\chi\big(\tilde{\mathscr{F}}^u(\bar\theta)\big), \quad W^{uu}([\theta]) := \chi\big(\mathscr{F}^u(\theta)\big).$$

In this way, we get that X is the union of the sets $W^{cs}([\theta])$, $[\theta] \in X$, which strongly indicates that X should be a foliated space. Analogously, X is the union of the sets $W^*([\theta])$, $[\theta] \in X$, for $* = ss, cu, su$ respectively. In this section we will study the dynamical properties of these sets and will establish that indeed each set $W^{ss}([\theta])$ is a strong stable set of the quotient flow as defined above, justifying our notation.

Definition 5.7. We call $\bar\theta, \bar\eta \in T_1\tilde M$ *heteroclinically related* if the intersections

$$\tilde{\mathscr{F}}^{cs}(\bar\theta) \cap \tilde{\mathscr{F}}^{cu}(\bar\eta) \quad \text{and} \quad \tilde{\mathscr{F}}^{cs}(\bar\eta) \cap \tilde{\mathscr{F}}^{cu}(\bar\theta)$$

both are nonempty.

The following statement can be shown as an application of Morse Theorem 2.2. It can be viewed as a sort of coarse product structure (in general there is no global product structure for the geodesic flow of the universal covering of compact surfaces without conjugate points and of higher genus).

Proposition 5.8. *Let (M,g) be a compact surface without conjugate points and of genus greater than one. Then every two points $\bar\theta = (p,v)$ and $\bar\eta$ in $T_1\tilde M$ satisfying $\bar\eta \notin \tilde{\mathscr{F}}^{cs}(p,-v)$ are heteroclinically related. These sets may contain a non-trivial strip.*

Lemma 5.9. *Given $D > 0$ there exists $D' > 0$ such that for every $\bar\theta \in T_1\tilde M$, and every pair of points $[\bar\tau], [\bar\eta] \in \tilde{\mathscr{F}}^s(\bar\theta)$ such that $\bar d([\bar\tau],[\bar\eta]) \le D$, we have that*

$$\bar d\big(\bar\psi_t([\bar\tau]), \bar\psi_t([\bar\eta])\big) \le D'$$

for every $t \ge 0$.

Proof. By the assumption we have $\bar\tau, \bar\eta \in \tilde{\mathscr{F}}^s(\bar\theta)$. The distance between their orbits is non-increasing since we consider a surface (M,g) without focal points. So let $L(D) > 0$ be such that for every pair of points $\bar\theta_1, \bar\theta_2 \in T_1\tilde M$ with $\bar d([\bar\theta_1],[\bar\theta_2]) \le D$ we have $d_S(\bar\theta_1,\bar\theta_2) \le L(D)$. The constant $L(D)$ exists by Lemma 5.3, the preimage by $\bar\chi$ of the ball of radius D centered at $[\bar\theta_1]$ is a compact subset of $T_1\tilde M$. So for $D := \bar d([\bar\tau],[\bar\eta])$ we have $d_S(\bar\tau,\bar\eta) \le L(D)$ and hence

$$d_S(\phi_t(\bar\tau), \phi_t(\bar\eta)) \le L(D)$$

for every $t \ge 0$. By co-compactness of $T_1\tilde M$, given a fundamental domain $\mathcal{D}$ there exist representatives $T_t(\phi_t(\bar\tau)), T_t(\phi_t(\bar\eta))$ of $\bar\tau, \bar\eta$ by isometries T_t of the fundamental group of T_1M such that

(1) $T_t(\phi_t(\bar\tau)) \in \mathcal{D}$ for each $t > 0$,
(2) $d_S\big(\bar\tau,\bar\eta\big) = d_S\big(T_t(\phi_t(\bar\tau)), T_t(\phi_t(\bar\eta))\big)$ for every $t > 0$.

Since the image by $\bar\chi$ of a compact set is compact, there exists a constant $D' = D'(D)$ such that

$$\bar d\big([T_t(\phi_t(\bar\tau))], [T_t(\phi_t(\bar\eta))]\big) \le D'.$$

But by definition, for each covering isometry T acting on $T_1\tilde M$ we have an induced isometry $\bar T$ of $\bar d$ which acts as

$$\bar T([\bar\theta]) = \bar T \circ \bar\chi(\bar\eta) = \bar\chi \circ T(\bar\eta) = [T(\bar\theta)].$$

So we get

$$\bar{d}\big([T_t(\phi_t(\bar{\tau}))],[T_t(\phi_t(\bar{\eta}))]\big)=\bar{d}\big(\bar{\psi}_t([\bar{\tau}]),\bar{\psi}_t([\bar{\eta}])\big)\leq D'$$

for every $t\geq 0$, thus concluding the proof. □

Lemma 5.10. *The quotient flow $\bar{\psi}_t$ is uniformly contracting on the stable sets $\tilde{W}^{ss}([\bar{\theta}])$ in the following sense: for every $D,\varepsilon>0$, there exists $t_0=t_0(D,\varepsilon)>0$ such that for every $\bar{\theta}\in T_1\tilde{M}$ and every two points $[\bar{\eta}],[\bar{\xi}]\in\tilde{W}^{ss}([\bar{\theta}])$ satisfying $\bar{d}([\bar{\eta}],[\bar{\xi}])\leq D$ we have*

$$\bar{d}\big(\bar{\psi}_t([\bar{\xi}]),\bar{\psi}_t([\bar{\eta}])\big)\leq\varepsilon$$

for every $t\geq t_0$.

Proof. The proof follows from a standard argument of expansive dynamics. We just sketch it. By contradiction, suppose that there exist numbers $D,\varepsilon>0$ and sequences $([\bar{\eta}_n])_n,([\bar{\xi}_n])_n\subset\tilde{W}^{ss}([\bar{\theta}_n])$ such that for every $n\geq 1$

$$\bar{d}\big([\bar{\eta}_n],[\bar{\xi}_n]\big)\leq D \quad\text{and}\quad \bar{d}\big(\bar{\psi}_n([\bar{\xi}_n]),\bar{\psi}_n([\bar{\eta}_n])\big)\geq\varepsilon\,. \tag{6}$$

By Lemma 5.9 there exists $D'>0$ such that $\bar{d}(\bar{\phi}_t([\bar{\eta}_n]),\bar{\phi}_t([\bar{\xi}_n]))\leq D'$ for every $t\geq 0$.

Again, by the co-compactness of $\bar{X}$ we get a convergent subsequences of points $([\bar{\eta}_{n_k}])_k,([\bar{\xi}_{n_k}])_k$ such that

$$\varepsilon\leq\bar{d}\big(\bar{\psi}_t([\bar{\eta}_{n_k}]),\bar{\psi}_t([\bar{\xi}_{n_k}])\big)\leq D'$$

for every $t\geq -n_k$. Denote by $[\bar{\eta}_\infty]$ and $[\bar{\xi}_\infty]$ the limits of these subsequences and observe that they are distinct points whose orbits are bi-asymptotic. On the other hand, by Lemma 5.6 the flow $\bar{\psi}_t$ is expansive, so these two orbits coincide. But this contradicts the second property of the sequences $[\bar{\xi}_n],[\bar{\eta}_n]$ in (6). This proves the lemma. □

Another important property of the invariant sets of the quotient flows is the following.

Lemma 5.11. *For each $*=cs,ss,cu,uu$, the set $\tilde{W}^*([\bar{\theta}])$ varies continuously in $[\bar{\theta}]\in\bar{X}$, uniformly on compact sets, with respect to the Hausdorff topology.*

Proof. Recall that, by Theorem 2.4, each family of invariant sets constitutes a continuous foliation by C^1 leaves. So they vary continuously on compact sets with respect to the Hausdorff topology. In fact, in $T_1\tilde{M}$ the invariant foliations are continuous with respect to the C^1 compact open topology, that is stronger than continuity with respect to the Hausdorff topology. The quotient preserves continuity properties, so the same holds for the quotient invariant sets. □

5.3. **Further properties.**

Lemma 5.12. *Given $\bar\theta = (p,v) \in T_1\tilde M$, for every $\bar\eta = (q,w) \in T_1\tilde M$ such that $(q,-w) \neq (p,v)$ the set $\tilde W^{cs}([\bar\eta]) \cap \tilde W^{cu}([\bar\theta])$ consists of a single orbit of $\bar\psi_t$.*

Proof. By Proposition 5.8, the intersection $\tilde{\mathscr{F}}^{cs}(\bar\eta) \cap \tilde{\mathscr{F}}^{cu}(\bar\theta)$ is nonempty and consists of a strip of bi-asymptotic orbits. The quotient map preserves this intersection. Observing that each strip becomes a single orbit in the quotient space, we deduce the statement. □

Proof of Theorem 5.1. Lemma 5.6 shows item (1). The results in Subsection 5.2 imply items (2) and (3). Lemma 5.12 proves item (4).

What remains to show are items (5)–(7). We will sketch their proof using the basis of sets $\bar\chi(A_{(\cdot)}(\cdot))$ constructed in Lemma 4.6 and their projection to X. As observed in the end of Section 4, the projection $\bar\Pi\colon \bar X \to X$ defines a local homeomorphism and hence the basis $\bar\chi(A_{(\cdot)}(\cdot))$ naturally projects to a basis in X.

Item (5) claims the transitivity of the quotient flow. This follows from the transitivity of the geodesic flow of (M,g) (Theorem 2.9 item (2)) and the fact that the family of sets $A_{\bar\theta}(\cdot)$ established in Lemma 4.6 is a family of open neighborhoods of sections of strips $\mathcal{I}(\bar\theta) \subset T_1\tilde M$. Indeed, each dense orbit must intersect any open set and hence each dense orbit intersects any set A from the basis specified in Lemma 4.6. Hence, the quotient of a dense orbit must intersect any open set $\bar\chi(A)$. This proves the density of the quotient of any dense orbit in the quotient topology.

Item (6) claims the minimality of the quotient of each strong stable set $W^{ss}([\theta])$ and each strong unstable set $W^{uu}([\theta])$. By Theorem 2.9 item (1), the horocycle foliations of T_1M are minimal. The same argument applied in the previous paragraph shows that the quotient of any dense set of T_1M is dense in the quotient X.

Item (7) claims that the quotient flow ψ_t is topologically mixing. This follows from Theorem 2.9 item (2) and the fact that the sets from the basis specified in Lemma 4.6 form a family of open neighborhoods in $T_1\tilde M$. Hence, the mixing property is verified in particular in these sets. Taking the quotient, this property is verified as well by the quotient flow.

This finishes the proof. □

Proof of Theorem A. The theorem is a consequence of Proposition 4.7 and Theorem 5.1. □

6. Measures of maximal entropy

In this setting we study the entropy of a flow. We first establish some general results and finally provide the proof of Theorem B.

6.1. **Expansive flows with local product structure.** In this subsection we start by considering a general continuous flow $\psi_t\colon X \to X$ without singular points on a compact metric space X. Of course, we have in mind the quotient flow defined in Section 4.

Given positive numbers a and δ we call a pair of sequences $(x_k)_{k=k_0}^{k_1}$ of points $x_k \in X$ and numbers $(\tau_k)_{k=k_0}^{k_1}$ ($k_0 = -\infty$ or $k_1 = \infty$ are permitted), a (δ, a)*-pseudo orbit* for the flow $(\psi_t)_t$ if for every k we have $\tau_k \geq a$ and $d(\psi_{\tau_k}(x_k), x_{k+1}) < \delta$. Denote $s_0 = 0, s_k = \tau_0 + \ldots + \tau_{k-1}$, and $s_{-k} = \tau_{-k} + \ldots + \tau_{-1}$. A (δ, a)-pseudo orbit $(x_k, \tau_k)_k$ is ε*-traced* by a true orbit $(\psi_t(y))_{t\in\mathbb{R}}$ if there is some increasing homeomorphism $\alpha\colon\mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$, such that for every for every $k = 0, 1, \ldots$ for every $t \geq 0$ satisfying $s_k \leq t < s_{k+1}$ we have

$$d(\psi_{\alpha(t)}(y), \psi_{t-s_k}(x_k)) \leq \varepsilon$$

and for every $k = 1, 2, \ldots$ for every $t \leq 0$ satisfying $-s_{-k} \leq t \leq -s_{-k+1}$

$$d(\psi_{\alpha(t)}(y), \psi_{t+s_{-k}}(x_{-k})) \leq \varepsilon$$

The flow $(\psi_t)_t$ is said to have the *pseudo orbit tracing property with respect to time* $a > 0$ if for every $\varepsilon > 0$ there is $\delta_0 > 0$ such that every $\delta \in (0, \delta_0)$ for every (δ, a)-pseudo orbit is ε-traced by a true orbit of $(\psi_t)_t$. For $a = 1$ we simply speak of the *pseudo orbit tracing property.*

Recall the definition of local product structure in Section 5.

Proposition 6.1 ([50, Theorem 7.1])**.** *Every continuous expansive flow without singular points on a compact metric space which has local product structure has the pseudo orbit tracing property.*

We say that the flow $(\psi_t)_t$ has the *periodic orbit specification property*[2] if for every $\varepsilon > 0$ there is a positive number $T = T(\varepsilon)$ such that for any integer $n \geq 2$, any collection of points $x_0, \ldots, x_n \in X$, and any sequence of real numbers $t_0 < t_1 < \cdots < t_{n+1}$ satisfying $t_{k+1} - t_k \geq T$ for every $k = 0, \ldots, n$, there is a sequence of numbers $r_0, \ldots, r_{n+1}$ having the following properties: $r_0 = 0$ and for every $k = 0, \ldots, n$

1) r_{k+1} is determined by $x_0, \ldots, x_{k+1}$ and by $t_0, \ldots, t_{k+1}$,
2) $|r_{k+1} - r_k| < \varepsilon$,
3) there is a periodic point $y \in X$ with period τ satisfying

$$|\tau - (t_{k+1} - t_0)| \leq (k+1)\varepsilon$$

and satisfying for every $\ell = 0, \ldots, k$ and for every $t \in [t_\ell, t_{\ell+1} - T]$

$$d(\psi_{t+r_\ell}(y), \psi_{t-t_\ell}(x_\ell)) < \varepsilon\,.$$

Observe that condition 1) is slightly stronger than in usual definitions (for example in [17]). It requires that for the existence of the periodic point in any intermediate construction step of solely considering a subset of points $\{x_0, \ldots, x_{k+1}\} \subset \{x_0, \ldots, x_n\}$ the numbers r_k are, in fact, defined recursively.

[2] We follow the definition in [38] which corrects some results on uniqueness of equilibrium states in [17] under stronger hypotheses.

Proposition 6.2. *Every continuous expansive topologically mixing flow without singular points on a compact metric space having the pseudo orbit tracing property has the periodic orbit specification property.*

We will need the following auxiliary result (see [50, Proposition 3.2]).

Lemma 6.3. *Let $(\psi_t)_t$ be a continuous expansive flow without singular points on a compact metric space which has local product structure. For every $\varepsilon > 0$, there exists $\delta_p = \delta_p(\varepsilon) > 0$ such that for every $x, y \in X$, every interval $[T_1, T_2]$ containing 0, and for every increasing homeomorphism $\alpha: \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ and $d(\psi_{\alpha(t)}(x), \psi_t(y)) \le \delta$ for every $t \in [T_1, T_2]$, then for every $t \in [T_1, T_2]$ we have*

$$|\alpha(t) - t| < \varepsilon\,.$$

Proof of Proposition 6.2. Let $\varepsilon > 0$. Let $\varepsilon_0 > 0$ be an expansivity constant. Let $\varepsilon_2 := \min\{\varepsilon, \varepsilon_0/2\}$. Let $\delta_p = \delta_p(\varepsilon)$ be as in Lemma 6.3. Let $\delta_0 = \delta_0(\min\{\delta_p, \varepsilon_0/2\})$ be the number given by the pseudo orbit tracing property. Let $\delta \in (0, \min\{\delta_0, \varepsilon_2/2\})$.

Take a covering of X by finitely many open sets U_i satisfying $\operatorname{diam} U_i \le \delta$. Since the flow is topologically mixing, there exists $T > 0$ such that for every i, j we have

$$\psi_t(U_i) \cap U_j \neq \varnothing \quad \text{for every } |t| \ge T/2\,. \tag{7}$$

Let $x_0, \ldots, x_n \in X$ be a sequence of points and $t_0 < t_1 < \ldots < t_{n+1}$ a sequence of numbers satisfying $t_{k+1} - t_k \ge T$ for every $k = 0, \ldots, n$. For every $k = 0, \ldots, n$ let

$$y_k := \psi_{t_{k+1} - t_k - T/2}(x_k)\,.$$

For every $k = 0, \ldots, n$ there are indices j_k and i_k such that

$$x_k \in U_{j_k}, \quad y_k \in U_{i_k}\,.$$

It follows from (7) that for every $k = 0, \ldots, n-1$ there is $z_k \in U_{i_k} \subset B(y_k, \delta)$ such that $\psi_{T/2}(z_k) \in U_{j_{k+1}}$ and that for every $k = 0, \ldots, n$ there is $w_k \in U_{i_k}$ such that $\psi_{T/2}(w_k) \in U_{j_0}$.

By these choices, for every $k = 1, \ldots, n$ consider the pair of finite sequences of points $(\widehat{x}_0, \ldots, \widehat{x}_{2k+3}) := (x_0, z_0, x_1, z_1, \ldots x_{k-1}, z_{k-1}, x_k, w_k, x_0)$ and of numbers

$$(\widehat{\tau}_1, \ldots, \widehat{\tau}_{2k+2}) := (t_1 - t_0 - \frac{T}{2}, \frac{T}{2}, t_2 - t_1 - \frac{T}{2}, \frac{T}{2}, \ldots, t_{k+1} - t_k - \frac{T}{2}, \frac{T}{2})$$

that we concatenate infinitely many times and thus define a periodic $(\delta, T/2)$-pseudo orbit. By our assumption, this pseudo orbit is δ_p-traced by a true orbit $(\psi_t(y))_t$, that is, there is a point y and some increasing homeomorphism $\alpha: \mathbb{R} \to \mathbb{R}$ satisfying $\alpha(0) = 0$ such that for every k and for every t satisfying $s_k \le t < s_{k+1}$

$$d(\psi_{\alpha(t)}(y), \psi_t(\widehat{x}_k)) \le \min\{\delta_p, \varepsilon_0/2\}\,.$$

Since the flow is expansive, by the choice of ε_0 this bi-infinite tracing orbit $(\psi_t(y))_t$ is uniquely determined. As the pseudo orbit is periodic, the tracing orbit must be closed, that is, we have $\psi_\tau(y) = y$ for some $\tau > 0$. By Lemma 6.3 and the choice of δ_p, as the shadowing orbit $(\psi_{\alpha(t)}(y))_t$ is close to pieces of orbits $(\psi_t(x_k))_t$, the period τ of the tracing periodic orbit must satisfy property 3) in the definition of the periodic orbit specification property. This finishes the proof. □

Corollary 6.4. *If a continuous expansive topologically mixing flow without singular points on a compact metric space has local product structure then the flow has the periodic orbit specification property.*

6.2. **Entropy.** A measure μ is said to be *invariant* under the flow $(\psi_t)_t$ if it is ψ_t-invariant for every t. Let $\mathcal{M}$ denote the set of all invariant probability measures μ. A set $A \subset X$ is *ψ-invariant* if $\psi_t(A) = A$ for every t. A measure $\mu \in \mathcal{M}$ is said to be *ergodic* for ψ if for every ψ-invariant set $A \subset X$ we have either $\mu(A) = 0$ or 1. Given $\mu \in \mathcal{M}$, we denote by $h_\mu(\psi_t)$ the *metric entropy* with respect to the time-t map ψ_t (see [54] for its definition). By Abramov's formula [1], we have

$$h_\mu(\psi_t) = |t|\, h_\mu(\psi_1) \qquad \text{for every } t.$$

One calls $h_\mu(\psi) = h_\mu(\psi_1)$ the *metric entropy* of μ with respect to the flow.

Given $\varepsilon > 0$, $T > 0$, and $x \in X$, define

$$B_T(x,\varepsilon) := \{y\colon d(\psi_s(y),\psi_s(x)) \le \varepsilon \text{ for every } s \in [0,T]\}\,.$$

Two points x, y are called *(T,ε)-separated* for $(\psi_t)_t$ if $y \notin B_\tau(x,\varepsilon)$, $E \subset X$ is *(T,ε)-separated* for $(\psi_t)_t$ if every pair of elements in E is. Given an invariant set $Z \subset X$, let $M(T,\varepsilon,Z)$ denote the maximum cardinality of any (T,ε)-separated subset of Z. The *topological entropy* of Z with respect to the flow $\psi = (\psi_t)_t$ is defined by

$$h(\psi, Z) := \lim_{\varepsilon\to 0}\limsup_{T\to\infty}\frac{1}{\tau}\log M(T,\varepsilon,Z).$$

We simply write $h(\psi) = h(\psi, X)$ for the topological entropy of the flow $(\psi_t)_t$. One can verify Abramov's formula also in the case of topological entropy

$$h(\psi_1, Z) = \frac{1}{|t|}h(\psi_t, Z) \qquad \text{for every } t\,.$$

(see [24] or [4, Proposition 21]) and, together with the variational principle (see [54]) denoting by $\mathcal{M}(\psi_1, Z) \subset \mathcal{M}$ the set of all ψ_1-invariant measures μ for which $\mu(Z) = 1$, one has

$$h(\psi, Z) = \sup_{\mu\in\mathcal{M}(\psi_1,Z)} h_\mu(\psi_1) = h(\psi_1, Z)\,. \tag{8}$$

In the following we will only work with the entropy of the time-1 map. A set of points E is called *(n,ε)-spanning* with respect to ψ_1 if $x, y \in E, x \ne y$, implies $d(\psi_{kt}(x_i), \psi_{kt}(x_j)) > \varepsilon$ for some $k \in \{0, \dots, n-1\}$. Given $Z \subset X$, let $N(n,\varepsilon,Z)$ be the maximal cardinality of a (n,ε)-spanning set $\{x_i\}_i \subset Z$. Recall that

$$h(\psi_1, Z) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon,Z)\,.$$

6.3. **Maximal entropy measures.** An ergodic measure μ is a *measure of maximal entropy* if it realizes the supremum in (8). Notice that such a measure always exists provided the entropy map $\nu\mapsto h_\nu(\psi_1)$ is upper semi-continuous [54]. This is the case if the flow is smooth [36] or if the map ψ_1 is h-expansive [54].

For any axiom A flow the measure of maximal entropy is unique [6]. In general, the continuous flows we want to study are not hyperbolic, so we have to rely on a more general result due to Franco [17] (see also Oka [38]) considering a continuous expansive flow. The measure of maximal entropy can be constructed explicitly from the distribution of closed orbits as follows [5]. Let $\varepsilon>0$ be an expansivity constant. For every T there is only a finite number of closed orbits for ψ with minimal period between $T-\varepsilon$ and $T+\varepsilon$, denote this family by $\mathrm{Per}(\psi,T-\varepsilon,T+\varepsilon)$. Denote by # the cardinality. Consider

$$\widehat{\nu}_{\psi,T} := \frac{\sum_\gamma \nu_{\psi,\gamma}}{\#\,\mathrm{Per}(\psi,T-\varepsilon,T+\varepsilon)} \tag{9}$$

with summation taken over all $\gamma\in\mathrm{Per}(\psi,T-\varepsilon,T+\varepsilon)$ and $\nu_{\psi,\gamma}$ denoting the flow invariant probability measure supported on γ. Note that

$$\#\,\mathrm{Per}(\psi,T-\varepsilon,T+\varepsilon)\le M(T,\varepsilon,X) \tag{10}$$

(compare also the proof of [7, Theorem 5]). We also consider

$$\nu_{\psi,T} := \frac{\sum_\gamma \nu_{\psi,\gamma}}{\#\,\mathrm{Per}(\psi,T)}$$

which is defined analogously considering instead of periodic orbits with minimal period between $T-\varepsilon$ and $T+\varepsilon$ ones in the set $\mathrm{Per}(\psi,T)$ of periodic orbits with minimal period less than or equal to T.

Proposition 6.5 ([17]). *Let $\psi_t\colon X\to X$ be a continuous expansive flow on a compact metric space satisfying the periodic orbit specification property.*

Then $\widehat{\nu}_{\psi,T}$ and $\nu_{\psi_1,T}$ both converge in the weak$\ast$ topology to the unique measure of maximal entropy as $T\to\infty$. In particular,

$$\lim_{T\to\infty}\frac{1}{T}\log\#\,\mathrm{Per}(\psi,T-\varepsilon,T+\varepsilon) = \lim_{T\to\infty}\frac{1}{T}\log\#\,\mathrm{Per}(\psi,T) = h(\psi)$$

Corollary 6.4 hence implies the following.

Corollary 6.6. *If a continuous expansive topologically mixing flow without singular points on a compact metric space has local product structure then $\widehat{\nu}_{\psi,T}$ and $\nu_{\psi,T}$ both converge in the weak$\ast$ topology to the unique measure of maximal entropy as $T\to\infty$.*

Consider a continuous flow $\phi_t\colon Y \to Y$ acting on a compact metric space Y being time-preserving semi-conjugate to the continuous flow $\psi_t\colon X \to X$ via a semi-conjugacy $\chi\colon Y \to X$ (of course we have in mind the extension introduced in Section 4). For $y \in Y$ consider the equivalence class

$$[y] := \{z \in Y\colon \chi(z) = \chi(y)\} = \chi^{-1}(\chi(y)).$$

The class $[y]$ is compact for every y. If $\chi(x)$ is a periodic point for ψ with period ℓ and $\gamma = \{\psi_t(\chi(x))\colon 0 \le t \le \ell\}$ its orbit then $[\beta] = \chi^{-1}(\gamma)$ is compact and ϕ-invariant and has period ℓ. Hence we can pick a probability measure $\nu_{\phi,[\beta]}$ supported on $[\beta]$ invariant with respect to the flow ϕ. We define

$$\mu_{\phi,T} := \frac{\sum_{[\beta]} \nu_{\phi,[\beta]}}{\#\operatorname{Per}(\phi,T)},$$

with summation taken over all $[\beta] \in \operatorname{Per}(\phi,T)$. By the same arguments as above, the probability measure $\nu_{\phi,T}$ is ϕ-invariant and, in particular, ϕ_1-invariant.

We can now formulate [10, Theorem 1.5] in our setting.

Theorem 6.7. *Let $\psi_t\colon X \to X$ be a continuous flow of a compact metric space that has a unique measure of maximal entropy ν_ψ. Let $\phi_t\colon Y \to Y$ be a continuous flow time-preserving semi-conjugate to ψ_t through some continuous surjective map $\chi\colon Y \to X$ and assume that the following conditions are satisfied:*

(1) $h(\phi_1,[y]) = 0$ *for every* $y \in Y$,
(2) $\nu_\psi(\{\chi(y)\colon [y] = \{y\}\}) = 1$.

Then $\mu_{\phi,T}$ converges in the weak$\ast$ topology as $T \to \infty$ to the unique ergodic probability measure of maximal entropy for $(\phi_t)_t$.

6.4. **Small entropy on strips.** We first prove an auxiliary result.

Proposition 6.8. *Let $\psi_t\colon X \to X$ be a continuous flow satisfying the periodic orbit specification property. Let $\mathcal{H} \subset X$ be a compact invariant set. If there exists a compact invariant set with positive topological entropy in $X \smallsetminus \mathcal{H}$ then $h(\psi_1,\mathcal{H}) < h(\psi_1)$.*

Proof. By the hypothesis, $h(\psi) > 0$. Hence, the statement is trivial if $h(\psi_1,\mathcal{H}) = 0$. So we consider the case that $h(\psi_1,\mathcal{H}) > 0$. Let $E \subset X \smallsetminus \mathcal{H}$ be a compact invariant set with entropy $h(\psi_1,E) =: \eta > 0$.

By contradiction, assume that $h(\psi_1,\mathcal{H}) = h(\psi) =: h$. By definition of the entropy, given $\tau \in (0,\min\{h,E\}/4)$, there is $\varepsilon_0 = \varepsilon_0(\tau)$ such that for every $\varepsilon \in (0,\varepsilon_0)$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \log M(n,\varepsilon/2,\mathcal{H}) \in \big(h-\tau,h+\tau\big)$$
$$\limsup_{n\to\infty} \frac{1}{n} \log M(n,\varepsilon/2,E) \in \big(\eta-\tau,\eta+\tau\big).$$

Hence, there exists $n_0 = n_0(\varepsilon)$ such that there exists a sequence $(n_k)_k$ of increasing numbers $n_k \ge n_0$ such that for every $k \ge 1$ we have

$$e^{n_k(h-2\tau)} \le M(n_k, \varepsilon/2, \mathcal{H}) \le e^{n_k(h+2\tau)}\,,$$
$$e^{n_k(\eta-2\tau)} \le M(n_k, \varepsilon/2, E) \le e^{n_k(\eta+2\tau)}\,.$$

For simplicity, let us assume $n_k = k$ for every k. We will produce an appropriate separated set for $\mathcal{H} \cup E$.

Given $\varepsilon > 0$, choose $\varepsilon' \in (0, \varepsilon)$ sufficiently small such that $d(\psi_t(z), z) < \varepsilon/4$ for every $z \in X$ and every $|t| < \varepsilon'$. Let $T = T(\varepsilon'/4)$ be provided by the periodic orbit specification property. Let $n_0 = n_0(2\varepsilon)$. Possibly after increasing n_0 we can assume that for every $n \ge n_0$ we have

$$e^{-\tau(n+2)} > \frac{e^{Th}}{1 + e^{(n+2)(\eta-h)}}\,. \tag{11}$$

For every $n \ge n_0$ consider sequences of $(n+2, 2\varepsilon)$-separated points in $\{x_{i,n}\}_i \subset \mathcal{H}$ and in $\{y_{i,n}\}_i \subset E$ each of maximal cardinality. Let $\{z_{i,n}\}_i = \{x_{i,n}\}_i \cup \{y_{i,n}\}_i$. Observe that this set has $M(n+2, 2\varepsilon, \mathcal{H}) + M(n+2, 2\varepsilon, E)$ elements.

By the periodic orbit specification property of the flow ψ_t, for every k-tuple of points $z_{i_1}, \ldots, z_{i_k}$, for the numbers $t_0 = 0, t_1 = n + 2 + T, \ldots, t_k = k(n+2+T)$ there are numbers $s_1, \ldots, s_k$ such that $|s_1| < \varepsilon/4, |s_j - s_{j-1}| < \varepsilon'/4$, $j = 2, \ldots, k$, and there is a periodic orbit $\widehat{z}_{i_1 \ldots i_k}$ of period $\tau_{i_1 \ldots i_k}$ satisfying

$$|\tau_{i_1 \ldots i_k} - k(2n + 2 + T)| \le k\varepsilon'$$

and satisfying

$$d(\psi_{t+s_j}(\widehat{z}_{i_1 \ldots i_k}), \psi_{t-(n+T)}(z_{i_j})) < \varepsilon'/4$$

for every $t \in [(j-1)(n+2+T), j(n+2+T)]$, $j = 1, \ldots, k$ (here we considered orbit pieces of length $n + 2$ instead of just n in order to take into consideration the fact that $n + T + s_1$ is not necessarily an integer). In particular, $\{\psi_\ell(\widehat{z}_{ij})\}_{\ell=1}^n$ first $\varepsilon/2$-shadows the orbit piece $\{\psi_\ell(z_i)\}_{\ell=1}^n$ and then, up to some time shift along the orbit which corresponds to a displacement of distance at most $\varepsilon/4$, the orbit piece $\{\psi_{\ell+n+T}(\widehat{z}_{i_1 \ldots i_k})\}_\ell$ $\varepsilon/2$-shadows the orbit piece $\{\psi_\ell(z_j)\}_\ell$. Hence, the points $\widehat{z}_{i_1 \ldots i_k}$ are pairwise (N, ε)-separated with

$$N = k(n + 2 + T)\,.$$

Hence, we have

$$\begin{aligned} M(k(n+2+T), \varepsilon, \mathcal{H} \cup E) &\ge \big(M(n+2, 2\varepsilon, \mathcal{H}) + M(n+2, 2\varepsilon, E)\big)^k \\ &\ge \big(e^{(n+2)(h-\tau)} + e^{(n+2)(\eta-\tau)}\big)^k \end{aligned}$$

and therefore

$$\begin{aligned} h \ge h(\psi_1, \mathcal{H} \cup E) &= \lim_{\varepsilon \to 0} \limsup_{m \to \infty} \frac{1}{m} \log M(m, \varepsilon, \mathcal{H} \cup E) \\ &\ge \limsup_{k \to \infty} \frac{k}{k(n+2+T)} \log\big(e^{(n+2)(h-\tau)} + e^{(n+2)(\eta-\tau)}\big)\,. \end{aligned}$$

By our choice of τ in (11) the latter term is bigger than h which is a contradiction. Hence, $h(\psi_1, \mathcal{H}) < h$. $\square$

We now return to our setting in the rest of the paper. By construction, the geodesic flow ϕ_t is time-preserving semi-conjugate to the quotient flow ψ_t from Definition 4.1. We will need the following lemma.

Lemma 6.9. *$h(\psi_1, [\theta]) = 0$ for any $\theta \in T_1M$.*

Proof. The result is trivial for every θ for which $[\theta] = \{\theta\}$.

Assume that $[\theta]$ contains a point $\eta = (q, w) \neq \theta$. Then $\eta \in \mathscr{F}^{ss}(\theta)$. If $\tilde{\mathscr{F}}^{ss}(\bar\theta)$ is any lift of $\mathscr{F}^{ss}(\theta)$ in $T_1\tilde M$ and $\bar\eta \in \tilde{\mathscr{F}}^{ss}(\bar\theta)$ a lift of η then $\gamma_{\bar\theta}, \gamma_{\bar\eta}$ are bi-asymptotic and by item (5) in Lemma 2.1 bound a flat strip in $\tilde M$ and stay in constant distance for every $t \in \mathbb{R}$. Hence, for every $\delta > 0$ a $(1, \delta)$-spanning set of $[\theta]$ is (n, δ)-spanning for every n. Hence, in particular $N(n, \delta, [\theta]) \leq N(1, \delta, [\theta])$ and thus $h(\psi_1, [\theta]) = 0$. $\square$

From [4, Theorem 17] and Lemma 6.9 we conclude the following.

Lemma 6.10. *For any $E \subset T_1M$ we have $h(\phi_1, E) = h(\psi_1, \chi(E))$.*

We now extend [30, Corollary 6.2] shown for compact rank one surfaces (in fact, for manifolds of any dimension). Let $\mathcal{R}$ be the set of points which belong to a trivial strip,

$$\mathcal{R} := \left\{\theta \in T_1M : [\theta] = \{\theta\}\right\}, \quad \text{and let} \quad \mathcal{H} := T_1M \smallsetminus \mathcal{R}\,.$$

Based on the semi-conjugacy, using the above introduced notation for $T > 0$ the set $\mathrm{Per}_{\mathcal{R}}(T) := \mathrm{Per}(\phi, T) \cap \mathcal{R}$ is the set of primitive periodic orbits γ in $\mathcal{R}$ of period $\ell(\gamma) \leq T$. In the case of nontrivial flats in each flat there is a continuum of parallel periodic orbits all of the same length, and $\mathrm{Per}_{\mathcal{H}}(T) := \mathrm{Per}(\phi, T) \cap \mathcal{H}$ is the set of representatives of each homotopy class of primitive periodic orbits γ of $\mathcal{H}$ of period $\ell(\gamma) \leq T$. Observe that by [30] for a compact rank 1 surface we have $h(\phi_1, \mathcal{H}) = 0$. But it is unclear if this holds also true for a surface without focal points and of higher genus. However, we have the following.

Corollary 6.11. *Let $\phi_t : T_1M \to T_1M$ be the geodesic flow of a C^∞ compact connected boundaryless surface (M, g) without focal points and of genus greater than one.*

We have $h(\phi_1, \mathcal{H}) < h(\phi_1)$. In particular, there exists $\varepsilon > 0$ and $T > 0$ such that for all $t \geq T$ we have

$$e^{\varepsilon t} \# \mathrm{Per}_{\mathcal{H}}(t) \leq \# \mathrm{Per}_{\mathcal{R}}(t) \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{t} \log \# \mathrm{Per}_{\mathcal{R}}(t) = h(\phi_1)\,.$$

Proof. Recall that ϕ_t preserves the Liouville measure m. Observe that $\mathcal{R}$ is open and invariant. Let $\widetilde{m}$ be the Liouville measure m restricted to $\mathcal{R}$ and normalized to obtain a probability measure. Ergodicity of $\widetilde{m}$ was proved in [41]. Moreover, $\lambda(\widetilde{m}) > 0$ by [41], where $\lambda(\cdot)$ denotes the positive Lyapunov exponent provided by the Oseledec decomposition. Since $\widetilde{m}$

is absolutely continuous and has positive density throughout $\mathcal{R}$, and thus $h_{\widetilde{m}}(\phi_1) = \lambda(\widetilde{m}) > 0$ by Pesin's formula. By Katok's horseshoe approximation (see [27, Supplement S.5] and [28, Theorem 4.1] for a related result and indications of modifications needed in the case of a flow) there is a compact invariant locally maximal hyperbolic set E of topological entropy arbitrarily close to $h_{\widetilde{m}}(\phi_1)$ (and hence, in particular, of positive entropy). By hyperbolicity we must have $E \subset \mathcal{R}$.

By construction and by Theorem 5.1, the geodesic flow ϕ_t is time-preserving semi-conjugate to the quotient flow ψ_t from Definition 4.1. By Lemma 6.10 we have $h(\phi_1, E) = h(\psi_1, \chi(E)) > 0$. Hence, applying Proposition 6.8 and observing that $\chi(\mathcal{H}) \subset X \smallsetminus \chi(\mathcal{R})$ we obtain

$$h(\phi_1, \mathcal{H}) = h(\psi_1, \chi(\mathcal{H})) < h(\psi_1) = h(\phi_1)\,. \tag{12}$$

The flow ψ_t is a continuous expansive flow with local product structure $\psi_t\colon X \to X$. By Theorem 5.1 this flow is topologically mixing. By Corollary 6.6 the flow ψ_t hence has a unique measure ν_ψ of maximal entropy which is obtained as the weak$*$ limit of the measures $\nu_{\psi,T}$ defined in (9) and for sufficiently small α we have

$$\lim_{T\to\infty} \frac{1}{T} \log \# \operatorname{Per}(\psi, T-\alpha, T+\alpha) = h(\psi_1)$$

By (10), for α sufficiently small and T sufficiently large we have

$$\operatorname{Per}(\psi, T-\alpha, T+\alpha) \cap \chi(\mathcal{H}) \le M(T, \alpha, \chi(\mathcal{H}))\,.$$

For ε small, (12) implies

$$\operatorname{Per}(\psi, T-\alpha, T+\alpha) \cap \chi(\mathcal{H}) \le e^{T(h(\psi_1,\chi(\mathcal{H}))+\varepsilon)} < e^{T(h(\psi_1)-\varepsilon)} \tag{13}$$

and hence

$$\lim_{T\to\infty} \frac{1}{T} \log \# \operatorname{Per}(\psi, T) \cap \chi(\mathcal{R}) = h(\psi_1)\,.$$

Since χ is a homeomorphism on $\mathcal{R}$ preserving parametrization, this immediately implies

$$\lim_{T\to\infty} \frac{1}{T} \log \# \operatorname{Per}_{\mathcal{R}}(T) = h(\psi_1) = h(\phi_1)\,.$$

Moreover, from (13) we also obtain

$$e^{\varepsilon T} \# \operatorname{Per}_{\mathcal{H}}(T) \le \# \operatorname{Per}_{\mathcal{R}}(T)\,.$$

This proves the corollary. □

We are now ready to give the proof of Theorem B.

Proof of Theorem B. Observe that

$$\nu_\psi(\chi(\mathcal{H})) = \lim_{T\to\infty} \frac{\sum_\gamma \nu_{\psi,\gamma}(\chi(\mathcal{H}))}{\# \operatorname{Per}(\psi, T)} = \lim_{T\to\infty} \frac{\# \operatorname{Per}(T) \cap \chi(\mathcal{H})}{\# \operatorname{Per}(T-\alpha, T+\alpha)} = 0\,.$$

This proves hypothesis (2) in Theorem 6.7. Finally Lemma 6.9 show hypothesis (1), which finishes the proof. □

Appendix A. The quotient space is a Seifert fibered bundle

The purpose of this section is to show the following result:

Theorem A.1. *Let (M, g) be a compact surface without conjugate points and of genus greater than one. Then the quotient space X admits a $\widetilde{PSL(2,\mathbb{R})}$-structure. In particular, X is a Seifert fibered space.*

Theorem A.1 will follow from the following proposition combined with the geometrization theory of 3-manifolds.

Proposition A.2. *Let (M, g) be a compact surface without focal points and of genus greater than one. Then the map $\chi: T_1M \to X$ is a homotopy equivalence. In particular, the fundamental group of the quotient X is isomorphic to a cocompact subgroup of $PSL(2,\mathbb{R})$.*

We shall subdivide the proof in several steps

A.1. **Approaching closed curves.** First recall the notations of the quotient space $\chi: T_1M \to X$ and the induced quotient $\bar{\chi}: T_1\tilde{M} \to \bar{X}$ we are dealing with. Let us also introduce the space $\widetilde{X}$ and the quotient $\tilde{\chi}: \widetilde{T_1M} \to \widetilde{X}$

Let $\widetilde{\pi}: \widetilde{T_1M} \to T_1M$ be the universal covering map. The geodesic flow $\phi_t: T_1M \to T_1M$ lifts to the geodesic flow $\widetilde{\phi_t}: \widetilde{T_1M} \to \widetilde{T_1M}$.

Given a point $\theta \in T_1M$ and $\hat{\theta} \in \widetilde{T_1M}$ such that $\hat{\pi}(\hat{\theta}) = \theta$, the set $\hat{\mathscr{F}}^s(\hat{\theta})$ is the pre-image by $\hat{\pi}$ of the set $\mathscr{F}^s(\theta)$ which contains $\hat{\theta}$. The set $\hat{\mathscr{F}}^u(\hat{\theta})$ is the pre-image of $\mathscr{F}^u(\theta)$ which contains $\hat{\theta}$. Their saturates by the geodesic flow $\widetilde{\phi}_t$ are respectively $\hat{\mathscr{F}}^{cs}(\hat{\theta})$ and $\hat{\mathscr{F}}^{cu}(\hat{\theta})$. All such sets form foliations which are invariant by the geodesic flow.

The intersections

$$\hat{I}(\hat{\theta})) = \hat{\mathscr{F}}^s(\hat{\theta}) \cap \hat{\mathscr{F}}^u(\hat{\theta}),$$
$$\hat{F}(\hat{\theta}) = \hat{\mathscr{F}}^{cs}(\hat{\theta}) \cap \hat{\mathscr{F}}^{cu}(\hat{\theta})$$

are the natural extensions of the notion of flat strip and its perpendicular sections already defined for T_1M in Section 2.

The space $\widetilde{X}$ is the collection of equivalence classes of $\widetilde{T_1M}$ by the following equivalence relation: two points are equivalent if and only if they belong to some $\hat{I}(\hat{\theta})$. The class of $\hat{\theta}$ will be denoted by $[\hat{\theta}]$.

The space $\widetilde{X}$ inherits a quotient flow $\hat{\psi}_t: \widetilde{X} \to \widetilde{X}$ given by $\hat{\psi}_t([\hat{\theta}]) = [\tilde{\phi}_t(\hat{\theta})]$ since the equivalence classes are preserved by the geodesic flow. The fundamental group of T_1M acts on $\widetilde{X}$ of course since the covering maps are isometries which preserve the invariant foliations and the strips.

Since there is a natural covering map $\Gamma: \widetilde{X} \to X$ defined by the identification of all classes in $\tilde{X}$ in the same orbit of the action of the fundamental group of T_1M, Theorem 4.2 implies that $\widetilde{X}$ admits a smooth manifold structure.

The main result of this subsection is the following.

Lemma A.3. *Let (M, g) be a compact surface without focal points and of genus greater than one.*

Given a closed smooth curve $\alpha: S^1 \to X$ there exists a closed continuous curve β in T_1M such that $\chi(\beta)$ is homotopic to α. Hence, the map $\chi_*: \pi_1(T_1M) \to \pi_1(X)$ between the first homotopy groups is surjective. The same holds for closed continuous curves in $\widetilde{X}$, hence the map $\tilde{\chi}_*: \pi_1(\widetilde{T_1M}) \to \widetilde{X}$ is also surjective.

Proof. Let us prove the lemma for the quotient X, the proof for $\widetilde{X}$ is analogous. The proof is a construction, for each closed continuous curve $\alpha \subset X$ we shall construct the curve β. Notice that if the quotient map χ was a local homeomorphism we could try to get the curve β just by lifting locally the curve α, as it is usually made with topological coverings. However, the map χ is not a local homeomorphism, it does not have the lifting property.

The main idea of the proof of Lemma A.3 is nevertheless based on the theory of covering maps: we shall try to obtain a sort of "lift" of a curve that is close to α (and hence homotopic to it) using the basis for the quotient topology found in Section 4.

Let $\{\chi(A_{\bar{\theta}}(\tau, \varepsilon, \delta, \bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+))\}$ be the basis for the quotient topology exhibited in Subsection 4.2. To facilitate notation, in the following we will suppress the choices of expansive points $\bar{\theta}^\pm, \bar{\eta}^\pm$. Notice that the lift of this basis by the covering map Γ provides an analogous basis for the topology of $\tilde{X}$ with the same properties claimed in Subsection 4.2.

There exist $\tau_0 > 0$, $\delta_0 > 0$, $\varepsilon_0 > 0$ such that every $\chi(A_{(\cdot)}(\tau, \varepsilon, \delta))$ with $\tau < \tau_0$, $\delta < \delta_0$, $\varepsilon < \varepsilon_0$, is simply connected. Moreover, we can choose the parameters in a way that every continuous closed curve $\beta: [0, 1] \to X$, $\beta(0) = \beta(1)$, satisfying

$$\beta \in \bigcup_{t \in [0,1]} \chi(A_{\alpha(t)}(\tau_0, \varepsilon_0, \delta_0))$$

is homotopic to α^n for some $n \in \mathbb{Z}$. We shall restrict ourselves to the collection of sets $S_{a,b}$ for $0 < b < a$ small, in the basis $\chi(A_{(\cdot)}(\tau, \varepsilon, \delta))$ such that the diameter of each set in S_a is bounded above by a and the inner diameter of each set of S_a is bounded below by b (such a, b exist for $\tau_0, \varepsilon_0, \delta_0$ and appropriate choices of $\bar{\theta}^-, \bar{\theta}^+, \bar{\eta}^-, \bar{\eta}^+$ for each $\bar{\theta}$).

Let us choose a sequence of points $\alpha(t_i)$, $t_0 = 0$, in α in the following way: let $U_0 \in S_{a,b}$ be an open set containing $\alpha(0)$. Let $(-s_0, t_1)$ be the maximal open interval such that $\alpha(t) \in U_0$ for every $t \in (-s_0, t_1)$. Let $U_1 \in S_{a.b}$ be such that $\alpha(t_1) \in U_1$. By the choice of t_1, the maximal interval (s_1, t_2) such that $\alpha(t) \in U_2$ satisfies $t_2 > t_1$. Since the inner diameter of U_2 is at least $b > 0$, the distance from $\alpha(t_2)$ to $\alpha(t_1)$ is bounded above by b. Then, define recursively intervals (s_{k-1}, t_k), sets $U_k \in S_{a,b}$ such that $\alpha(t_k) \in U_k$ and (s_{k-1}, t_k) is the maximal interval such that $\alpha(t) \in U_{k-1}$ for every $t \in (s_{k-1}, t_k)$. Let $N_{a,b} > 0$ be the minimum number sets U_k which covers all α. We get sequence of points $\alpha(t_k)$, $k = 0, 1, \ldots, N_{a,b} - 1$, and a covering of α by open sets of the basis U_k such that

(1) $t_{k+1} > t_k$, for every $k < N_{a,b} - 1$.
(2) $d(\alpha(t_{N_{a,b}} - 1), \alpha(0)) < a$,
(3) $U_k \cap U_{k+1} \neq \varnothing$ for every $k < N_{a,b}$ and $U_{N_{a,b}-1} \cap U_0 \neq \varnothing$,
(4) The sets U_k are simply connected.

Now, let us consider the pre-images $\chi^{-1}(U_k) = A_{\alpha(t_k)}(\tau_0, \varepsilon_0, \delta_0)$ By the construction

$$A_{\alpha(t_k)}(\tau_0, \varepsilon_0, \delta_0) \cap A_{\alpha(t_{k+1})}(\tau_0, \varepsilon_0, \delta_0)$$

is a nonempty open set for every k so let us choose points $x_0 \in \chi^{-1}(\alpha(0))$, $x_k \in A_{\alpha(t_k)}(\tau_0, \varepsilon_0, \delta_0) \cap A_{\alpha(t_{k-1})}(\tau_0, \varepsilon_0, \delta_0)$ for every $k = 1, 2, \ldots, N_{a,b} - 1$, $x_{N_{a,b}} = x_0$, and let us consider continuous curves $\beta_k : [0,1] \to T_1M$ such that

(1) $\beta_0(0) = x_0$,
(2) $\beta_{k+1}(0) = \beta_k(1) = x_{k+1}$ for every $k = 0, 1, .., N_{a,b} - 2$,
(3) $\beta_{N_{a,b}-1}(1) = x_0$,
(4) $\beta_k[0,1] \subset A_{\alpha(t_k)}(\tau_0, \varepsilon_0, \delta_0)$ for every k.

Let $\beta : [0,1] \to T_1M$, $\beta(0) = \beta(1)$ be the sum of the curves $\beta_0 * \beta_1 * \ldots * \beta_{N_{a,b}-1}$ (where the sum $*$ is given as usual in homotopy theory) reparametrized in a way that $\beta(0)\alpha(0)$, $\beta(t_k) = \alpha(t_k)$ for every $k = 1, 2, \ldots, N_{a,b} - 1$.

The closed curve $\chi(\beta)$ is a closed continuous curve formed by the union of the compact curves $\chi(\beta_i)$ which by construction, is homotopic to α. The curve β might be considered as a sort of lift of α by the quotient map χ. Thus, the image of the homotopy class of β by the map $\chi_* : \pi_1(T_1M) \to \pi_1(X)$ is the homotopy class of α, and since α may be any closed continuous curve in X the lemma follows. □

A.2. **A universal covering map.** In this section we show that the fundamental group of X contains a subgroup that is isomorphic to $\pi_1(T_1M)$. Actually, Lemma A.3 is meaningful if the fundamental group of X is nontrivial. One of the consequences of Lemma A.3 is the following.

Lemma A.4. *The space $\widetilde{X}$ is simply connected. In particular, $\widetilde{X}$ is irreducible, that is, every sphere in $\widetilde{X}$ bounds a ball.*

Proof. The space $\widetilde{X}$ is clearly connected since it is the continuous image of a connected set. Then Lemma A.3 tells us that each homotopy class in $\widetilde{X}$ is the image of a homotopy class of $\widetilde{T_1M}$ by a group homomorphism. But the fundamental group of $\widetilde{T_1M}$ is the identity, and the image of the identity by a group homomorphism is the identity. So the fundamental group of $\widetilde{X}$ is the identity, thus proving that $\widetilde{X}$ is simply connected. The irreducibility follows from the proof of the Poincaré conjecture by Perelman [40]. □

Corollary A.5. *The map $\Gamma : \widetilde{X} \to X$ is the universal covering map of X. In particular, the fundamental group of X is isomorphic to the fundamental group of T_1M.*

Proof. The corollary follows from the basic theory of covering spaces, Lemma A.4, and the definition of the map Γ. □

Proof of Proposition A.2. By Corollary A.5 the fundamental group of X is isomorphic to $\pi_1(T_1M)$. By Lemma A.3 the homomorphism $\chi_*: \pi_1(T_1M) \to \pi_1(X)$ is surjective. Since the image of a group by a homomorphism is a subgroup of the target group then $\chi_*(\pi_1(T_1M))$ is a subgroup isomorphic to $\pi_1(X)$, therefore, its kernel is trivial and hence the map χ_* is an isomorphism. □

Proof of Theorem A.1. The proof follows from Corollary A.5 and the well known classification results concerning Seifert fibered bundles by Scott [48]. Namely, Seifert fibered bundles are classified by their fundamental groups, if the fundamental group of a compact boundaryless manifold is isomorphic to the fundamental group of a Seifert fibered space then the manifold is diffeomorphic to the Seifert fibered space. In particular, X is actually diffeomorphic to T_1M. □

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