HOMOTOPY TYPE OF SPACES OF CURVES WITH CONSTRAINED CURVATURE ON FLAT SURFACES

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Abstract. Let S be a complete flat surface, such as the Euclidean plane. We determine the homeomorphism class of the space of all curves on S which start and end at given points in given directions and whose curvatures are constrained to lie in a given open interval, in terms of all parameters involved. Any connected component of such a space is either contractible or homotopy equivalent to an n-sphere, and every $n \geq 1$ is realizable. Explicit homotopy equivalences between the components and the corresponding spheres are constructed.

0. Introduction

This paper relies strongly on [4]. To understand some of the proofs presented here, it is recommended that the reader be familiar with the contents of sections $0, 1, 3$ and 5 therein.

Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$, $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$. Recall that $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$ denotes the set (furnished with the C^r topology, for some $r \geq 2$) of all regular curves $\gamma\colon [0,1] \to \mathbf{C}$ of class C^r satisfying:

(i) γ starts at $0 \in \mathbf{C}$ in the direction of $1 \in \mathbf{S}^1$ and ends at q in the direction of z;

(ii) The curvature κ_γ of γ satisfies $\kappa_1 < \kappa_\gamma(t) < \kappa_2$ for all $t \in [0,1]$.

This space can always be decomposed into the disjoint union of its subspaces $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q; \theta_1)$, where the latter contains those curves which have total turning θ_1, for $e^{i\theta_1} = z$. As shown in [4], each of these subspaces is either empty or a contractible connected component of $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$, except when $\kappa_1 < 0 < \kappa_2$ and $|\theta_1| < \pi$. To study what happens in this case, it may be assumed without loss of generality that $\kappa_1 = -1$ and $\kappa_2 = 1$, by Theorem 2.4 in [4]. For fixed $Q = (q, z)$ with $z \neq -1$, there is exactly one subspace $\mathcal{C}_{-1}^{+1}(Q; \theta_1)$ with $|\theta_1| < \pi$; it contains the curves in $\mathcal{C}_{-1}^{+1}(Q)$ of minimal total turning in absolute value. Let it be denoted by $\mathcal{M}(Q)$.

The central result of this work states that $\mathcal{M}(Q)$ is homotopy equivalent to $\mathbf{S}^n$ for some $n \in \{0, 1, \dots, \infty\}$, and allows one to obtain the value of n by means of a simple construction. In particular, any of the indicated values is possible (recall that $\mathbf{S}^\infty$ is contractible). See Figure 1.

Theorem. *Let $\theta_1 \in (-\pi, \pi)$, $z = e^{i\theta_1}$ and $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$. Then $\mathcal{M}(Q)$ is homeomorphic to $\mathbf{E} \times \mathbf{S}^{2k}$ or $\mathbf{E} \times \mathbf{S}^{2k+1}$ ($k \geq 0$) for q in the open region not intersecting the negative real axis bounded by the three circles*

$$\begin{cases} C_{4k+4}(iz - i) \ \text{ and } \ C_{4k+2}(\pm(i + iz)), & \text{or} \\ C_{4k+4}(i - iz) \ \text{ and } \ C_{4k+6}(\pm(i + iz)), & \text{respectively.} \end{cases}$$

If q does not lie in the closure of any of these regions, then $\mathcal{M}(Q) \approx \mathbf{E}$. If q lies on the boundary of one of them, then $\mathcal{M}(Q) \approx \mathcal{M}\big((q - \delta, z)\big)$ for all sufficiently small $\delta > 0$.

Here $\mathbf{E}$ denotes the separable Hilbert space, $C_r(a)$ denotes the circle of radius $r > 0$ centered at $a \in \mathbf{C}$ and $X \approx Y$ (resp. $X \simeq Y$) means that X is homeomorphic (resp. homotopy equivalent) to Y.

Corollary. *Let S be a complete flat surface, $\kappa_1 < \kappa_2$ and $u, v \in UTS$. Then each component of $\mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)$ is homeomorphic to $\mathbf{E} \times \mathbf{S}^n$, for some $n \in \{1, \dots, \infty\}$ depending upon the component.* □

The notation here is the same as that used in §8 of [4]. In particular, it should be assumed that $\kappa_2 = -\kappa_1 > 0$ if S is nonorientable. Actually, the theorem yields a complete determination of the homeomorphism class of $\mathcal{C}S_{\kappa_1}^{\kappa_2}(u, v)$ if an explicit description of S as a quotient of $\mathbf{C}$ by a group of isometries is known; see Proposition 8.3 of [4] for the details.

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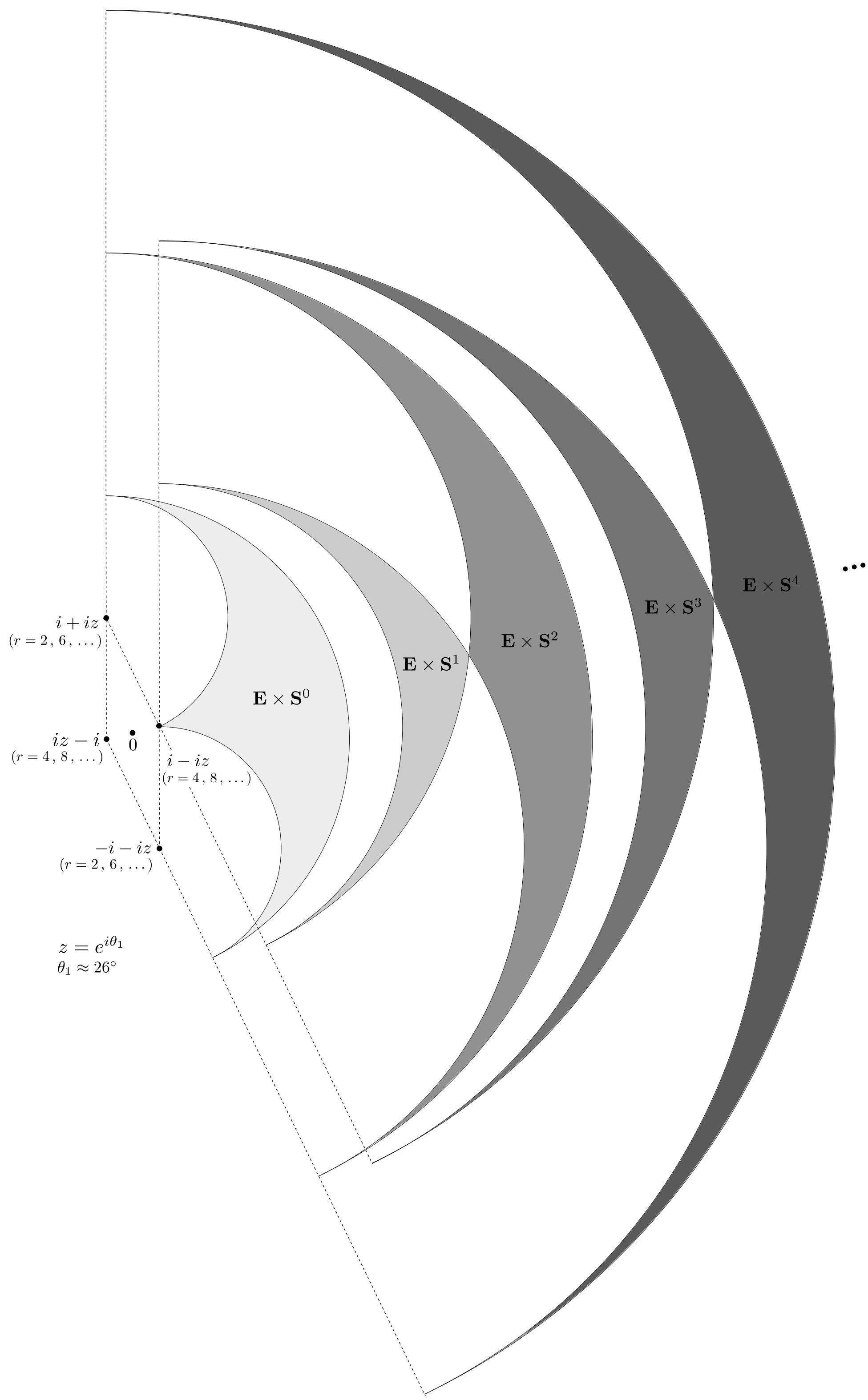

FIGURE 1. Let $\theta_1 \in (-\pi, \pi)$ be fixed, $z = e^{i\theta_1}$ and $Q = (q, z)$. This drawing to scale indicates the homeomorphism class of $\mathcal{M}(Q)$ in terms of the location of $q \in \mathbf{C}$. If q lies in the white region, then $\mathcal{M}(Q) \approx \mathbf{E}$ (the separable Hilbert space). The line segments are only auxiliary elements and do not bound any regions. The radii of the circles are indicated in parentheses near their respective centers.

Remark. Let S_n (resp. E) denote the set of all $Q \in UT\mathbf{C}$ for which $\mathcal{M}(Q)$ is homeomorphic to $\mathbf{E} \times \mathbf{S}^n$ (resp. $\mathbf{E}$). The theorem implies that S_n is bounded and neither open nor closed in $UT\mathbf{C}$, for all $n \in \mathbf{N}$. For $z = e^{i\theta_1}$ ($|\theta_1| < \pi$), let

$$S_n(z) = \big\{q \in \mathbf{C} : (q, z) \in S_k\big\} \ \ (k \in \mathbf{N}) \ \ \text{and} \ \ E(z) = \big\{q \in \mathbf{C} : (q, z) \in E\big\}.$$

Then $E(z)$ has infinite area for any $z \in \mathbf{S}^1 \smallsetminus \{-1\}$ and $\lim_{n\to\infty} \operatorname{Area}(S_n(z)) = +\infty$. The precise determination of $\operatorname{Area}(S_n(z))$ in terms of n and z will be left as an exercise.

Example. Let $Q_x = (x, 1) \in \mathbf{R} \times \mathbf{S}^1$. Then $\mathcal{M}(Q_x) \approx \mathbf{E}$ if $x \leq 0$ and

$$\mathcal{M}(Q_x) \approx \begin{cases} \mathbf{E} \times \mathbf{S}^{2k} & \text{if } \ \frac{x}{4} \in \big(\sqrt{k^2+k}\,,\, k+1\big] \\ \mathbf{E} \times \mathbf{S}^{2k+1} & \text{if } \ \frac{x}{4} \in \big(k+1\,,\, \sqrt{k^2+3k+2}\,\big] \end{cases} \qquad (k \in \mathbf{N},\ k \geq 0).$$

This is an immediate consequence of the theorem. Note that the size of the interval of the x-axis where $\mathcal{M}(Q_x) \approx \mathbf{E} \times \mathbf{S}^n$ approaches 2 as $n \in \mathbf{N}$ increases (see Figure 2).

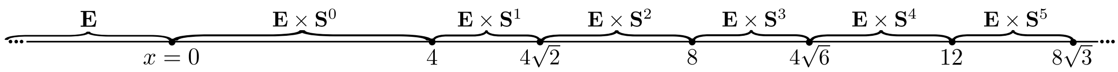

FIGURE 2. The homeomorphism class of $\mathcal{M}(Q_x)$ as a function of $x \in \mathbf{R}$. Curves in $\mathcal{M}(Q_x)$ start at the origin of $\mathbf{C}$ in the direction of $1 \in \mathbf{S}^1$ and end at x, with zero total turning.

Example. Suppose that $\mathcal{M}(Q) \simeq \mathbf{S}^1$. Then a generator of $\pi_1\mathcal{M}(Q)$ is represented by any family of curves $\gamma_s \in \mathcal{M}(Q)$ ($s \in [0, 1]$) such that:

(i) γ_s is condensed for $s \in [0, \frac{1}{4}) \cup (\frac{3}{4}, 1]$ and $\gamma_0 = \gamma_1$;
(ii) γ_s is diffuse for $s \in (\frac{1}{4}, \frac{3}{4})$;
(iii) γ_s is critical of type $+-$ when $s = \frac{1}{4}$ and critical of type $-+$ when $s = \frac{3}{4}$.

As $\pi_k\mathcal{M}(Q) = 0$ for $k > 1$, the resulting map $\mathbf{S}^1 \to \mathcal{M}(Q)$ is actually a weak homotopy equivalence, and hence a homotopy equivalence, since $\mathcal{M}(Q)$ is a Banach manifold (cf. Theorem 15 of [3]).

In particular, suppose that $4 < x \leq 4\sqrt{2}$ and let $Q_x = (x, 1) \in \mathbf{C} \times \mathbf{S}^1$, as in the preceding example. An explicit generator of $\pi_1\mathcal{M}(Q_x)$ can be visualized by completing Figure 3 to obtain a family $\gamma_s \in \mathcal{M}(Q_x)$ satisfying the preceding conditions. For $\gamma_{\frac{1}{2}}$ we may take the concatenation of a figure eight curve with $\gamma_0 = \gamma_1$, where the latter denotes the straight line segment from 0 to x.

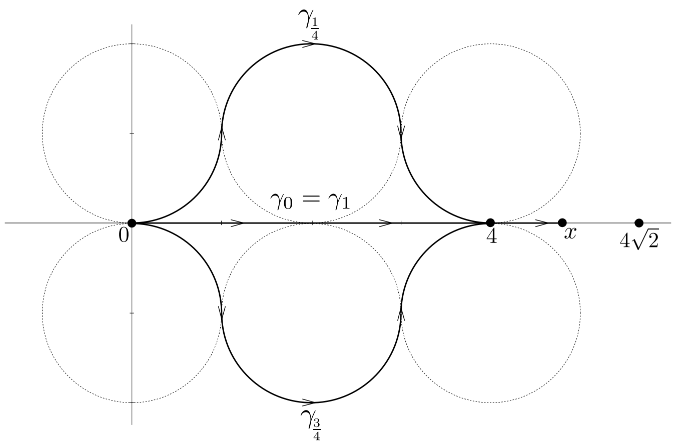

FIGURE 3. Constructing a generator of $\pi_1\mathcal{M}(Q)$ when $Q = (x, 1) \in \mathbf{R} \times \mathbf{S}^1$ and $4 < x \leq 4\sqrt{2}$.

Outline of the proof. Although the proof of the theorem is somewhat technical, the underlying idea is quite simple. See Definitions 3.2 and 5.2 of [4] to understand the terms used here.

For each sign string σ, such as $-+-$, we first define the concept of *quasicritical curves of type* σ. These form an open set $\mathcal{U}_\sigma \subset \mathcal{M}(Q)$ containing all critical curves of type σ in $\mathcal{M}(Q)$, with $\mathcal{U}_\sigma := \emptyset$ if there exists no curve of the latter type. The naive plan is to prove that $\mathcal{U}_c$, $\mathcal{U}_d$ and the $\mathcal{U}_\sigma$ (for σ ranging over all possible sign strings) form a good cover of $\mathcal{M}(Q)$, meaning that their k-fold intersections are either empty or contractible for any $k \geq 1$. Since $\mathcal{M}(Q)$ is a Banach manifold, its homeomorphism class is completely determined by the incidence data of this cover, which is equivalent either to that of the open cover of $\mathbf{R}^n \smallsetminus \{0\}$ by the half-spaces

$$U_{\pm k} = \big\{(x_1, \dots, x_n) \in \mathbf{R}^n : \pm x_k > 0\big\} \qquad (k = 1, \dots, n) \tag{1}$$

or else to the cover of $\mathbf{R}^n \smallsetminus$ ray obtained by excluding U_{-n}.

More precisely, let τ be a *top* sign string for $\mathcal{M}(Q)$, i.e., one having maximum length among those strings σ such that $\mathcal{M}(Q)$ contains critical curves of type σ. The fact that $\mathcal{U}_\tau \neq \emptyset$ immediately implies that $\mathcal{U}_\sigma \neq \emptyset$ if $|\sigma| < |\tau|$. The integer n appearing in (1) equals $|\tau|$ and the combinatorial equivalence between the cover of $\mathcal{M}(Q)$ and that in (1) is given by

$$\mathcal{U}_c \leftrightarrow U_{+1}, \quad \mathcal{U}_d \leftrightarrow U_{-1} \quad \text{and} \quad \mathcal{U}_\sigma \leftrightarrow U_{\sigma(1)|\sigma|} \quad (\mathcal{U}_\sigma \neq \emptyset). \tag{2}$$

where $\sigma(1)$ denotes the first sign of σ. Thus, $\mathcal{M}(Q)$ is contractible if $\mathcal{U}_{-\tau} = \emptyset$, and it has the homotopy type of $\mathbf{S}^{n-1}$ if $\mathcal{U}_{-\tau} \neq \emptyset$.

The determination of whether $\mathcal{M}(Q)$ contains critical curves of a certain type in terms of Q was already carried out in Proposition 5.3 of [4], and this is essentially what is depicted in Figure 1: For q in a gray region, $\mathcal{M}(Q)$ contains critical curves of types τ and $-\tau$, with darker shades corresponding to increasing $|\tau|$. For q in the white region, $\mathcal{M}(Q)$ does not contain critical curves of type $-\tau$, or does not admit a top sign string (this occurs if and only if $\mathcal{M}(Q)$ contains no critical curves at all).

Informally, $\gamma\colon [0,1] \to \mathbf{C}$ is quasicritical of type σ if it is possible to find $\varphi \in \mathbf{R}$ and $t_1 < \dots < t_{|\sigma|}$ such that the unit tangent vector $\mathbf{t}_\gamma$ to γ satisfies $\mathbf{t}_\gamma(t_k) \approx \sigma(k) i e^{i\varphi}$ for each $k = 1, \dots, |\sigma|$ and $\langle \mathbf{t}_\gamma, e^{i\varphi} \rangle > 0$ away from these points. Here $\sigma(k)$ denotes the k-th sign of σ. In words, γ is nearly vertical with respect to the "axis" $e^{i\varphi}$ near the points $\gamma(t_k)$, with directions prescribed by σ, but elsewhere its image is the graph of a function.

Unfortunately, the set of all $\varphi \in \mathbf{R}$ with respect to which a curve is quasicritical of type σ need not be an interval. Given a continuous family $K \to \mathcal{U}_\sigma$, $p \mapsto \gamma^p$, this makes it difficult to choose φ^p continuously so that each γ^p is quasicritical with respect to φ^p. To circumvent this, we work instead with a certain space $\mathcal{N}(Q) \subset \mathcal{M}(Q) \times \mathbf{R}$ of pairs (γ, φ). The strategy to understand the topology of $\mathcal{N}(Q)$ is exactly as described above: First an open cover $\mathfrak{V}$ of $\mathcal{N}(Q)$ by subsets $\mathcal{V}_c$, $\mathcal{V}_d$ and $\mathcal{V}_\sigma$ is defined, where roughly $\mathcal{V}_c$ and $\mathcal{V}_d$ are products of $\mathcal{U}_c$ and $\mathcal{U}_d$ with $\mathbf{R}$, and for each sign string σ, $\mathcal{V}_\sigma$ consists of pairs (γ, φ) such that γ quasicritical of type σ with respect to φ. It is then proved that these form a good cover of $\mathcal{N}(Q)$ whose combinatorics is determined by (2) when $\mathcal{U}$ is replaced by $\mathcal{V}$. Finally, it is established that the restriction to $\mathcal{N}(Q)$ of the projection $\mathcal{M}(Q) \times \mathbf{R} \to \mathcal{M}(Q)$ is a homotopy equivalence.

Outline of the sections. Given a sign string σ_2 and a substring σ_1 of σ_2, there are in general many ways to "embed" σ_1 into σ_2. For instance, if $\sigma_1 = -+$ and $\sigma_2 = -+-+$, then there are three substrings of σ_2 isomorphic to σ_1, namely, those determined by the pairs of coordinates $(1,2)$, $(1,4)$ and $(3,4)$. In §1 we consider certain subspaces of $\mathbf{R}^n$ determined by inequalities involving a set of strings $\sigma_1, \dots, \sigma_m$, each a substring of the next, which encode the purely combinatorial difficulties that arise in the study of the topology of $\mathcal{V}_{\sigma_1} \cap \dots \cap \mathcal{V}_{\sigma_m}$. The main result of the section states that the former subspaces are in fact all weakly contractible. In the case of two strings, we construct homeomorphisms from the resulting spaces onto Euclidean spaces, and for larger sets of strings we use induction and certain collapsing maps which are quasifibrations.

One of the tools in the proof that the cover $\mathfrak{V}$ of $\mathcal{N}(Q)$ is good is a procedure for "stretching" curves, illustrated in Figures 9 and 10, which generalizes the grafting construction of [4]. This procedure is explained in §2, along with some of its properties that are needed later.

The formal definitions of "quasicritical curve", the space $\mathcal{N}(Q)$ and its cover $\mathfrak{V}$ are contained in §3. Most of the results in this section concern basic properties of quasicritical curves, and how to continuously choose "stretchable" subarcs for a given family of such curves so that when the stretching construction is actually carried out, the resulting homotopy will preserve important properties of the original family, e.g., being condensed or simultaneously quasicritical of several types. It is also shown there that the projection $\mathcal{N}(Q) \to \mathcal{M}(Q)$ induces surjections on homotopy groups.

The combinatorics of the cover $\mathfrak{V}$ of $\mathcal{N}(Q)$ is determined in §4. It is very easy to see that $\mathcal{V}_c \cap \mathcal{V}_d = \emptyset$ and $\mathcal{V}_\sigma \cap \mathcal{V}_{-\sigma} = \emptyset$ for any sign string σ. On the other hand, given sign strings $\sigma_1, \dots, \sigma_m$ with $|\sigma_1| < \cdots < |\sigma_m|$, with some care one can deform a critical curve of type σ_m to make it simultaneously quasicritical of type σ_j for each j. Thus an intersection of nonempty elements of $\mathfrak{V}$ is empty if and only if it involves some "opposite pair", just as for the cover in (1).

The objective of §5 is to prove that $\mathfrak{V}$ is a good cover. Given a continuous family $(\gamma^p, \varphi^p) \in \mathcal{V}_{\sigma_1} \cap \cdots \cap \mathcal{V}_{\sigma_m}$, with p ranging over a compact space, each γ^p can be stretched to become nearly critical (as in Figure 10), and then deformed to a concatenation of circles and line segments of a special form (as in Figure 11) which is essentially determined by the slopes of the segments. The results of §1 can then be used to conclude that the resulting family is contractible.

The proof that $\mathcal{N}(Q)$ and $\mathcal{M}(Q)$ are homeomorphic is given in §6. Moreover, when $\mathcal{M}(Q) \simeq \mathbf{S}^{n-1}$, where $n = |\tau| \geq 2$ is as above, explicit homotopy inverses $f\colon \mathbf{S}^{n-1} \to \mathcal{M}(Q)$ and $g\colon \mathcal{M}(Q) \to \mathbf{S}^{n-1}$ are constructed. Let $\mathcal{C}_\tau$ denote the set of all critical curves of type τ in $\mathcal{M}(Q)$. Intuitively, the map g measures the failure of curves in $\mathcal{M}(Q)$ to belong to $\mathcal{C}_\tau$. If α is a generator of $H^*(\mathbf{S}^{n-1})$, then $g^*(\alpha)$ is (up to a sign) the "Poincaré dual" of $\mathcal{C}_\tau$, except that the latter is not really a submanifold of $\mathcal{M}(Q)$. The map f generates $\pi_{n-1}\mathcal{M}(Q)$ and admits the following description: Regard $\mathbf{S}^{n-1}$ as a CW complex with two k-cells $e^k_\pm$ for every $k = 0, \dots, n-1$. Then

$$f\big(e^{n-1}_+\big) \subset \mathcal{U}_d, \quad f\big(e^{n-1}_-\big) \subset \mathcal{U}_c,$$

and for each $k = 0, \dots, n-2$, f maps $e^k_\pm$ into the set of critical curves of type $\pm\sigma^{n-k}$ in $\mathcal{M}(Q)$, where σ^{n-k} denotes any of the two sign strings of length $n-k$. The actual definition of f is a bit different, but more precise; in particular, it shows that these inclusions can indeed be satisfied.

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1. On certain subspaces of Euclidean space determined by sign strings

A cell decomposition of $\mathbf{R}^n$. Throughout the article, the set $\{1, \dots, n\}$ will be denoted by $[n]$. Let $2 \leq n \in \mathbf{N}$, $m \in [n]$ and let $\emptyset \neq J_1, \dots, J_m \subset [n]$ satisfy $[n] = \bigsqcup_{j=1}^m J_j$. Define

$$W_{J_1,\dots,J_m} = \big\{x \in \mathbf{R}^n : x_k < x_{k'} \text{ if and only if } k \in J_j,\ k' \in J_{j'} \text{ for some } j < j' \in [m]\big\}.$$

It is easy to check that each *cell* $W_{J_1,\dots,J_m}$ is an m-dimensional convex cone. Furthermore, $\mathbf{R}^n$ is the disjoint union of all such cells. There is only one 1-cell $W_{[n]}$, which consists of the multiples of $(1, 1, \dots 1)$ in $\mathbf{R}^n$. At the other end, there are $n!$ cells of dimension n, each $W_{J_1,\dots,J_n}$ being determined by the permutation $\pi \in S_n$ such that $\pi(k)$ is the unique element of J_k. These n-cells are open in $\mathbf{R}^n$, while the k-cells for $1 < k < n$ are neither open nor closed. See Figure 4 (a) for an illustration of the case $n = 3$.

Remark. The k-cells in this decomposition are dual to the $(n-k)$-faces of the $(n-1)$-dimensional permutohedron. The total number of cells (faces) is given by the n-th ordered Bell number.

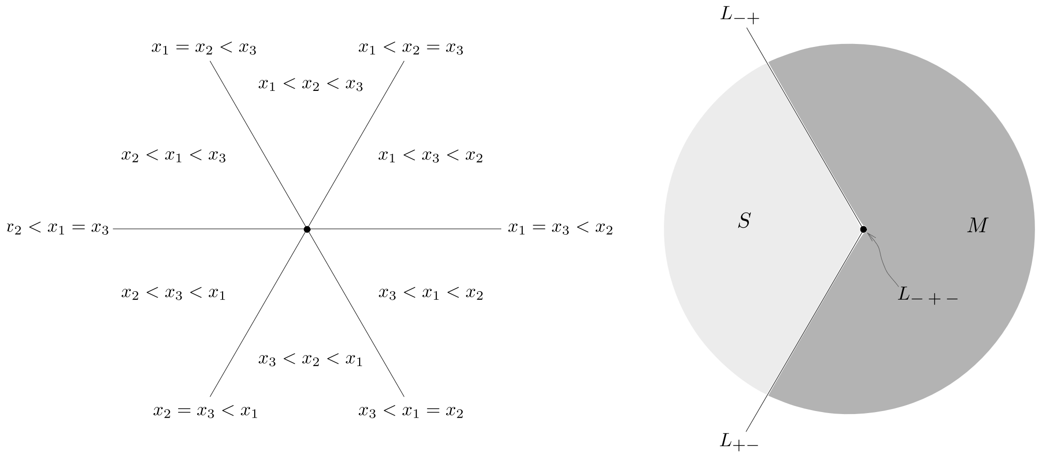
We are actually more interested in another decomposition of $\mathbf{R}^n$, obtained by comparing even and odd cordinates.

(1.1) Definition. Given $x = (x_1, \dots, x_n) \in \mathbf{R}^n$, let

$$t(x) = \min\big\{x_k - x_{k'} : k \text{ is odd and } k' \text{ is even, } k, k' \in [n]\big\}. \tag{3}$$

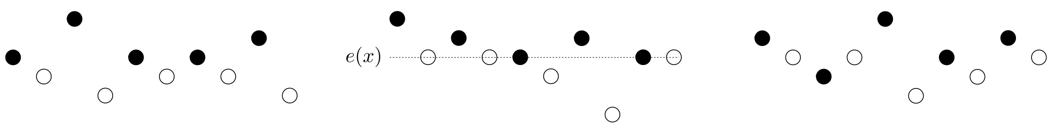
We call x *mixed*, *level* or *split* according as $t(x) < 0$, $t(x) = 0$ or $t(x) > 0$, respectively. Define

$$M = \big\{x \in \mathbf{R}^n : x \text{ is mixed}\big\}, \quad S = \big\{x \in \mathbf{R}^n : x \text{ is split}\big\}, \quad L = \big\{x \in \mathbf{R}^n : x \text{ is level}\big\}.$$



(a) (b)

FIGURE 4. The decomposition of $\mathbf{R}^3$ into the 13 cells W_* and into the sets M, S and L_σ, for $|\sigma| \geq 2$. More precisely, what is depicted here is the orthogonal projection of these sets onto the hyperplane $\{(x_1, x_2, x_3) \in \mathbf{R}^3 : x_1 + x_2 + x_3 = 0\}$.



(a) (b) (c)

FIGURE 5. Split, level and mixed points in $\mathbf{R}^{10}$, respectively, represented by beads (black for odd-indexed coordinates and white for even-indexed coordinates).

It is convenient to represent a point $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ as an ordered set of n beads, each of which is allowed to slide along a vertical line. The height of the k-th bead (above a certain fixed ground height) gives the value of x_k; see Figure 5.

An *interval* $J \subset [n]$ is a set of the form $(a, b) \cap [n]$ for some $a < b \in \mathbf{R}$. Given two intervals J_1, J_2, we write $J_1 < J_2$ if $k_1 < k_2$ whenever $k_1 \in J_1,\ k_2 \in J_2$.

(1.2) Definition. When $x \in \mathbf{R}^n$ is level, there exists a unique $e(x) \in \mathbf{R}$ satisfying $x_k = e(x) = x_{k'}$ for some odd k and even k' (see Fig. 5 (b); 'e' stands for "elevation"). For each integer $m \geq 2$, define

$$\sigma^m \colon [m] \to \{\pm\}\,,\ -\sigma^m \colon [m] \to \{\pm\} \quad \text{by} \quad \sigma^m(j) = (-1)^j \text{ and } -\sigma^m(j) = (-1)^{j+1}. \tag{4}$$

For example, σ^3 is represented by $-+-$, and $-\sigma^4$ by $+-+-$. A *sign string* σ is of the form σ^m for some $m \geq 2$, and we let $|\sigma|$ denote its *length* m. A level point $x = (x_1, \dots, x_n) \in \mathbf{R}^n$ is *of type* σ, for some sign string σ, if we can find nonempty intervals $J_1, \dots, J_{|\sigma|}$, such that:

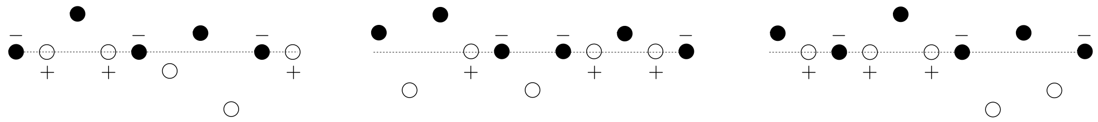
(i) $J_1 < J_2 < \cdots < J_{|\sigma|}$ and $[n] = \bigcup_{j=1}^{|\sigma|} J_j$.
(ii) If $k \in J_j$ and $x_k = e(x)$, then $(-1)^k = \sigma(j)$.

The set of all level points of type σ in $\mathbf{R}^n$ will be denoted by L_σ^n or simply L_σ.

In other words, to determine the type of a level point $x \in \mathbf{R}^n$, we assign a tag $-$ (resp. $+$) to each odd (resp. even) dot lying at height $e(x)$, and read off the sequence of signs at $e(x)$, omitting any repetitions; see Figure 6.

Observe that the sets M, S and L_σ are pairwise disjoint cones. Moreover, $L_{\sigma_n} = W_{[n]}$ and $L_\tau = \emptyset$ if $\tau = -\sigma_n$ or $|\tau| > n$. The sets M, S are open in $\mathbf{R}^n$, while the L_σ are neither open nor closed for $|\sigma| < n$. See Figure 4 (b) for the case $n = 3$. The proof of the following result (which will not be used anywhere) is a straightforward verification left to the reader.

(1.3) Proposition. *Each of the sets M, S and L_σ, for any sign string σ, is a union of cells W_* of $\mathbf{R}^n$. Equivalently, each cell of $\mathbf{R}^n$ is contained in one of these sets.* ☐



(a) (b) (c)

FIGURE 6. An element of $L^{10}_{\sigma^4}$ and two elements of $L^{11}_{-\sigma^4}$, respectively. According to (1.5), the latter space is homeomorphic to $\mathbf{R}^8$. In particular, the two points represented in (b) and (c) can be joined without leaving $L^{11}_{-\sigma^4}$.

(1.4) Definition. For an integer $m \geq 1$, let

$$H^m = \big\{(x_1, \dots, x_m) \in \mathbf{R}^m : x_m \geq 0\big\} \text{ and } -H^m = \big\{(x_1, \dots, x_m) \in \mathbf{R}^m : x_m \leq 0\big\}.$$

For a space $Y \approx H^m$, define ∂Y to consist of all $y \in Y$ such that the local homology $H_*(Y, Y \smallsetminus \{y\})$ at y is trivial. Note that ∂Y is exactly the image of $\mathbf{R}^{m-1} \times \{0\}$ under any homeomorphism $H^m \to Y$.

Our first goal is to prove the following result.

(1.5) Proposition. *For τ a sign string with $1 \leq |\tau| \leq n-1$, let*

$$\begin{aligned} Y_{\pm} &= \big\{(y_1, \dots, y_n) \in \mathbf{R}^n : \pm y_1 > 0\big\}, \\ Y_{\tau} &= \big\{(y_1, \dots, y_n) \in \mathbf{R}^n : y_k = 0 \text{ for all } k < |\tau| \text{ and } \tau(1) y_{|\tau|} > 0\big\}, \\ Y_{\sigma^n} &= \big\{(y_1, \dots, y_n) \in \mathbf{R}^n : y_k = 0 \text{ for all } k < n\big\}. \end{aligned}$$

Then there exists a homeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ such that $f(M) = Y_-$, $f(S) = Y_+$ and $f(L_\sigma) = Y_\sigma$ for all sign strings σ with $|\sigma| \leq n$, $\sigma \neq -\sigma^n$.

(1.6) Corollary. *Let M, S, $L^n_\sigma \subset \mathbf{R}^n$. Then $M \approx S \approx \mathbf{R}^n$, $\overline{M} \approx \overline{S} \approx H^n$, $L^n_\sigma \approx \mathbf{R}^{n+1-|\sigma|}$ and $\overline{L}^n_\sigma \approx H^{n+1-|\sigma|}$ ($|\sigma| < n$). Also, $L^n_{\sigma^n} = \overline{L}^n_{\sigma^n} \approx \mathbf{R}$ and $L^n_{-\sigma^n} = \emptyset$.* □

In particular, each set L_σ and $\overline{L}_\sigma$ is contractible. It is a good exercise to try to visualize a contraction using the representation by beads, as in Figure 6.

(1.7) Lemma. *Any homeomorphism $\partial H^k \to \partial H^k$ may be extended to a homeomorphism of H^k onto itself.* □

(1.8) Lemma. *Let $H_1 \approx H_2 \approx H^k$ be subspaces of some larger topological space and suppose that $\partial H_1 = \partial H_2 = H_1 \cap H_2$. Then there exists a homeomorphism $f\colon H_1 \cup H_2 \to \mathbf{R}^k$ such that $f(H_1) = H^k$ and $f(H_2) = -H^k$.*

Proof. Let $g_1\colon H_1 \to H^k$ and $g_2\colon H_2 \to -H^k$ be homeomorphisms. Then the restriction of $g_2 \circ (g_1)^{-1}$ to ∂H^k is a homeomorphism $\partial H^k \to \partial H^k$. Using (1.7), extend this to a homeomorphism $g\colon H^k \to H^k$. Now glue $g \circ g_1$ and g_2 along $\partial H_1 = \partial H_2$. □

(1.9) Lemma. *Let $H_1 \approx H_2 \approx H^k$ be subspaces of some larger topological space and suppose that $\partial H_i = C \cup D_i$, where $C \approx D_i \approx H^{k-1}$, $C \cap D_i = \partial C = \partial D_i$ and $H_1 \cap H_2 = C$ ($i = 1, 2$). Then $H_1 \cup H_2 \approx H^k$.*

Proof. Let $f_0\colon C \to H^{k-1}$ be a homeomorphism. Using (1.8), f_0 may be extended to a homeomorphism $f_1\colon C \cup D_1 \to \mathbf{R}^{k-1}$, and then since $\partial H_1 = C \cup D_1$, f_1 has an extension to a homeomorphism $g_1\colon H_1 \to H^k$, by (1.7). Finally, we compose g_1 with the homeomorphism

$$H^k \to Q_1 = \big\{(x_1, \dots, x_k) \in \mathbf{R}^k : x_{k-1} \geq 0 \text{ and } x_k \geq 0\big\},$$

obtained by taking the square root (in $\mathbf{C}$) of the last two coordinates (x_{k-1}, x_k) of points $x \in H^k$. The result is a homeomorphism $h_1\colon H_1 \to Q_1$ such that $h_1|_C = f_0$.

Repeating the argument for H_2, starting from f_0 again, we obtain a homeomorphism

$$h_2\colon H_2 \to Q_2 = \big\{(x_1, \dots, x_k) \in \mathbf{R}^k : x_{k-1} \geq 0 \text{ and } x_k \leq 0\big\},$$

with $h_2|_C = f_0$. Glueing h_1 and h_2 along C, we finally obtain the desired homeomorphism

$$h\colon H_1 \cup H_2 \to Q_1 \cup Q_2 = \big\{(x_1, \dots, x_k) \in \mathbf{R}^k : x_{k-1} \geq 0\big\} \approx H^k.$$ □

(1.10) Lemma. *Let M, S, $L \subset \mathbf{R}^n$ be as in (1.1). Then there exists a homeomorphism $g\colon \mathbf{R}^n \to L \times \mathbf{R}$ with $g(M) = L \times (-\infty, 0)$ and $g(S) = L \times (0, +\infty)$. In particular, $\overline{M} \cap \overline{S} = L$.*

Proof. Define a map $h\colon L \times \mathbf{R} \to \mathbf{R}^n$ by

$$h(x,t) = (x_1 + t, x_2 - t, \dots, x_n + (-1)^{n-1}t) \qquad (x = (x_1, \dots, x_n) \in L,\ t \in \mathbf{R}).$$

Given $x \in \mathbf{R}^n$, let $t(x)$ be as in eq. (3) and $\bar{t}(x) = \frac{1}{2}t(x)$. Let

$$g\colon \mathbf{R}^n \to L \times \mathbf{R}, \quad g(x) = \big((x_1 - \bar{t}(x), x_2 + \bar{t}(x), \dots, x_n + (-1)^n \bar{t}(x))\,,\, \bar{t}(x)\big).$$

Then g and h are inverse maps. Moreover, it is an immediate consequence of (1.1) that $g(M) = L \times (-\infty, 0)$ and $g(S) = L \times (0, +\infty)$, as claimed. □

(1.11) Lemma. *For any sign string σ, the closure $\overline{L}_\sigma$ of L_σ in $\mathbf{R}^n$ satisfies $\overline{L}_\sigma = L_\sigma \cup \bigcup_{|\tau| > |\sigma|} L_\tau$. In particular, $\overline{L}_{\sigma^m} \cap \overline{L}_{-\sigma^m} = \bigcup_{|\tau| \geq m+1} L_\tau$.*

Proof. Suppose first that τ is a sign string with $|\tau| \leq |\sigma|$. Let $x \in L_\tau$ and

$$\mu = \frac{1}{2} \min \big\{ \, |x_k - e(x)| \; : \, x_k \neq e(x),\ k \in [n] \big\}.$$

Then the set

$$U = \big\{ (y_1, \dots, y_n) \in \mathbf{R}^n \, : \, |y_k - x_k| < \mu \text{ for each } k \in [n] \big\}$$

defines a neighborhood of x with the property that $U \cap L_{\tau'} = \emptyset$ if τ' is not a substring of τ (see (1.13) for the formal definition of substring). Therefore, $\overline{L}_\sigma \subset L_\sigma \cup \bigcup_{|\tau| > |\sigma|} L_\tau$.

Conversely, if $|\tau| > |\sigma|$ and $x \in L_\tau$, choose indices $k_1 < \cdots < k_l$ such that:

(i) $x_{k_i} = e(x)$ for each $i \in [l]$;
(ii) If $k_1' < \cdots < k_r'$ are all the remaining indices such that $x_{k'} = e(x)$, then $r = |\sigma|$ and $(-1)^{k_j'} = \sigma(j)$ for each $j \in [r]$.

This is possible since σ is a substring of τ. Points in L_σ arbitrarily close to x can be obtained by moving the coordinates x_{k_i} away from $e(x)$. More precisely, for $s \in [0,1]$, let $x(s) = (x_1(s), \dots, x_n(s)) \in \mathbf{R}^n$ be defined by:

$$x_k(t) = \begin{cases} x_k & \text{if } k \neq k_1, \dots, k_l; \\ x_k + (-1)^{k-1}s & \text{if } k = k_i \text{ for some } i; \end{cases} \qquad (k \in [n]).$$

Then $x(0) = x$ and $x(s) \in L_\sigma$ for all $s > 0$ by construction. Hence $x \in \overline{L}_\sigma$. □

Proof of (1.5). By induction on n. If $n = 2$, then $L = L_{-+} = \big\{(x_1, x_2) \in \mathbf{R}^2 : x_1 = x_2\big\}$, while M (resp. S) consists of those points above (resp. below) this line. Thus, rotation by $\frac{\pi}{4}$ about the origin is the desired homeomorphism. (The case $n = 3$ follows from Figure 4(b).)

Let $n \geq 3$ and assume that the assertion has been proved for all dimensions smaller than n. The homeomorphism $\mathbf{R}^n \to \mathbf{R}^n$ will be constructed stepwise. We start with a homeomorphism $f\colon L^n_{\sigma^n} \to Y_{\sigma^n}$, which exists since both of these sets are lines in $\mathbf{R}^n$. Suppose that f has already been extended to a homeomorphism $f\colon \bigcup_{|\sigma| \geq m+1} L^n_\sigma \to \bigcup_{|\sigma| \geq m+1} Y_\sigma$ for some m satisfying $2 \leq m \leq n-1$.

Let $\phi\colon \mathbf{R}^n \to \mathbf{R}^{n-1}$ and $\lambda\colon \mathbf{R}^n \to [0, +\infty)$ be the maps which forget and recover the last coordinate:

$$\phi(x) = (x_1, \dots, x_{n-1}), \quad \lambda(x) = |x_n - e(x)| \qquad (x = (x_1, \dots, x_n) \in \mathbf{R}^n),$$

where $e(x)$ is as in (1.2). Let us suppose for concreteness that $m \equiv n \pmod 2$; the only difference in the other case is that the roles of $L^n_{\sigma^m}$ and $L^n_{-\sigma^m}$ are switched. A straightforward verification shows that

$$\phi \times \lambda \colon \overline{L}^n_{-\sigma^m} \to \overline{L}^{n-1}_{-\sigma^m} \times [0, +\infty)$$

is a homeomorphism, hence $\overline{L}^n_{-\sigma_m} \approx H^{n-m} \times [0, +\infty) \approx H^{n+1-m}$ by the induction hypothesis on n.

On the other hand, there is a decomposition $\overline{L}^n_{\sigma_m} = H_1 \cup H_2$, where

$$H_1 := \big\{ x \in \overline{L}^n_{\sigma^m} \, : \, \lambda(x) = 0,\ \phi(x) \in \overline{L}^{n-1}_{\sigma^{m-1}} \big\} \approx H^{n-m+1} \text{ via } \phi,$$

$$H_2 := \big\{ x \in \overline{L}^n_{\sigma^m} \, : \, \lambda(x) \geq 0,\ \phi(x) \in \overline{L}^{n-1}_{\sigma^m} \big\} \approx H^{n-m} \times [0, +\infty) \approx H^{n-m+1} \text{ via } \phi \times \lambda,$$

$$C := H_1 \cap H_2 = \big\{ x \in \overline{L}^n_{\sigma^m} \, : \, \lambda(x) = 0,\ \phi(x) \in \overline{L}^{n-1}_{\sigma^m} \big\} \approx H^{n-m} \text{ via } \phi,$$

by the induction hypothesis on n. Moreover, $\partial H_1 = C \cup D_1$ and $\partial H_2 = C \cup D_2$, where

$$D_1 = \big\{x \in \overline{L}^n_{\sigma^m} : \lambda(x) = 0,\ \phi(x) \in \overline{L}^{n-1}_{-\sigma^m}\big\} \approx H^{n-m} \text{ via } \phi,$$
$$D_2 = \big\{x \in \overline{L}^n_{\sigma^m} : \lambda(x) \geq 0,\ \phi(x) \in \overline{L}^{n-1}_{-\sigma^{m+1}} \cup \overline{L}^{n-1}_{\sigma^{m+1}}\big\} \approx \mathbf{R}^{n-m-1} \times [0, +\infty) \approx H^{n-m} \text{ via } \phi \times \lambda,$$

again by the induction hypothesis on n. Thus, we are in the setting of (1.9), and the conclusion is that $\overline{L}^n_{\sigma^m} = H_1 \cup H_2 \approx H^{n-m+1}$.

Now by (1.11),

$$\overline{L}^n_{\sigma^m} \cap \overline{L}^n_{-\sigma^m} = \bigcup_{|\tau| \geq m+1} \overline{L}^n_\tau .$$

Since by assumption we already have a homeomorphism from the latter set to $\bigcup_{|\tau|\geq m+1} \overline{Y}_\tau \approx \mathbf{R}^{n-m}$, (1.8) guarantees the existence of a homeomorphism

$$f\colon \bigcup_{|\tau|\geq m} \overline{L}^n_\tau \to \bigcup_{|\tau|\geq m} \overline{Y}_\tau \approx \mathbf{R}^{n+1-m}.$$

Continuing this down to $m = 2$, a homeomorphism $f\colon L \to \bigcup_{|\tau|\geq 2} \overline{Y}_\tau \approx \mathbf{R}^{n-1}$ taking each L_σ onto Y_σ is obtained. Finally, an application of (1.8) using (1.10) shows that this can be extended to a homeomorphism $f\colon \mathbf{R}^n \to \mathbf{R}^n$ having the required properties. □

Subspaces determined by nested strings. Let E, Y be topological spaces, $q\colon E \to Y$ be a (continuous) surjective map and for each $y \in Y$, let $F_y = q^{-1}(y)$ denote the fiber of y. Then q is a *quasifibration* if for any $k \geq 0$, $y \in Y$ and $e \in F_y$, the induced map $q_*\colon \pi_k(E, F_y, e) \to \pi_k(Y, y)$ on homotopy groups is an isomorphism.†

Thus, if $q\colon E \to Y$ is a quasifibration, then for any $y \in Y$ and $e \in F_y$, there is a long exact sequence

$$\cdots \to \pi_k(F_y, e) \xrightarrow{j_*} \pi_k(E, e) \xrightarrow{q_*} \pi_k(Y, y) \xrightarrow{\partial} \pi_{k-1}(F_y, e) \to \cdots \to \pi_0(E, e) \to 0 \tag{5}$$

which is obtained from the long exact sequence of the triple (E, F_y, e) by identifying $\pi_k(E, F_y, e)$ with $\pi_k(Y, y)$; here j is the inclusion $F_y \hookrightarrow E$. Just as for a Serre fibration, it can be shown that if Y is path-connected, then all fibers F_y have the same weak homotopy type.

(1.12) Proposition ([1], Satz 2.2). *Let $q\colon E \to Y$ be a surjective map and suppose that $\mathfrak{U} = (U_\nu)_{\nu\in I}$ is an open cover of Y satisfying:*

(i) *For each $\nu \in I$, $q|_{q^{-1}(U_\nu)}\colon q^{-1}(U_\nu) \to U_\nu$ is a quasifibration;*
(ii) *If $y \in U_{\nu_1} \cap U_{\nu_2}$, then there exists ν such that $y \in U_\nu \subset U_{\nu_1} \cap U_{\nu_2}$ (for ν_1, ν_2, $\nu \in I$).*

Then q is a quasifibration. □

(1.13) Definition. Let $l \geq 2$. An *extended string* τ is a function $\tau\colon [l] \to \{\pm\}$. Thus, in contrast to sign strings, in an extended string some signs may be repeated. Given two extended strings $\tau_1\colon [l_1] \to \{\pm\}$, $\tau_2\colon [l_2] \to \{\pm\}$, τ_1 is a *substring* of τ_2, denoted $\tau_1 \preccurlyeq \tau_2$ (or $\tau_1 \prec \tau_2$ if in addition $\tau_1 \neq \tau_2$), if there is a strictly increasing $f\colon [l_1] \to [l_2]$ such that $\tau_1 = \tau_2 \circ f$. Note that if τ is an extended string, then there is a unique *reduced string* ϱ of maximal length such that $\varrho \preccurlyeq \tau$, obtained by omitting all repetitions in τ; e.g., the reduced sign string of $+--++$ is $+-+$.

(1.14) Definition. Let $1 \leq m \in \mathbf{N}$ and $\sigma_1 \preccurlyeq \ldots \preccurlyeq \sigma_m$, where σ_m is an extended string and the remaining σ_j are sign strings. Let $n = |\sigma_m|$. Define $X_{(\sigma_1,\ldots,\sigma_m)} \subset \mathbf{R}^n$ to consist of all $x = (x_1, \ldots, x_n)$ satisfying the following conditions:

(i) $\sigma_m(k)x_k \leq 0$ for all $k \in [n]$;
(ii) $|x_k| \leq m$ for all $k \in [n]$ and for each $j \in [m-1]$, if $k_1 < \cdots < k_l$ are all the indices in $[n]$ such that $|x_k| \leq j$, then σ_j is the reduced string of $\tau\colon [l] \to \{+-\}$, $\tau(i) = \sigma_m(k_i)$.

†See [1], Bemerkung 1.2 for the definition of $\pi_0(E, F_y, e)$. For $k = 0$, 1, when the set on the left side has no natural group structure, it should be understood that $q_*\colon \pi_k(E, F_y, e) \to \pi_k(Y, y)$ is a bijection.

Representing a point in $\mathbf{R}^n$ by beads, to determine whether it satisfies (ii) we assign a tag $\sigma_m(k)$ to its k-th bead for each $k \in [n]$, and read off the tags of those coordinates that lie at or below height j and at or above height $-j$; the resulting reduced string should coincide with σ_j for each $j \in [m-1]$, and $|x_k| \leq m$ should hold for all $k \in [n]$. See Figure 7. Note that if $m = 1$, then the resulting space is just an n-dimensional cube.

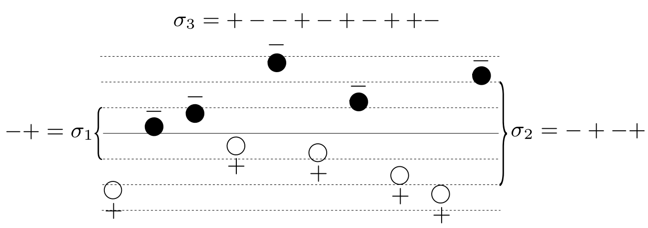


FIGURE 7. An element of $X_{(\sigma_1,\sigma_2,\sigma_3)}$ for σ_j as indicated in the figure. The value of the coordinate x_k is given by the height of the center of the k-th bead.

(1.15) Proposition. *Let $2 \leq m \in \mathbf{N}$ and $\sigma_1 \preccurlyeq \ldots \preccurlyeq \sigma_m$, where σ_m is an extended string and the remaining σ_j are sign strings. Then $X_{(\sigma_1,\ldots,\sigma_m)}$ is weakly contractible.*

We are only interested in the case where σ_m is also a sign string, but for the proof given below to work, this more general version is needed, as well as one last definition: Let σ_2 be an extended string and $\sigma_1 \preccurlyeq \sigma_2$ be a sign string, $|\sigma_2| = n$. Define a subspace $L_{\sigma_1}^{\sigma_2} \subset \mathbf{R}^n$ by declaring that $x = (x_1, \ldots, x_n) \in L_{\sigma_1}^{\sigma_2}$ if and only if it satisfies condition (i) above (with $m = 2$) together with:

(iii) $|x_k| \leq 1$ for all $k \in [n]$ and if $k_1 < \cdots < k_l$ are all the indices in $[n]$ such that $x_k = 0$, then σ_1 is the reduced string of $\tau\colon [l] \to \{+-\}$, $\tau(i) = \sigma_2(k_i)$.

(1.16) Lemma. *Let $\sigma_1 \preccurlyeq \sigma_2$ be a sign and an extended string, respectively. Then $L_{\sigma_1}^{\sigma_2}$ is contractible.*

Proof. Let $n = |\sigma_2|$ and

$$L_0 = \big\{(x_1, \ldots, x_n) \in L_{\sigma_1}^{\sigma_2} : x_k = x_{k+1} \text{ if } \sigma_2(k) = \sigma_2(k+1), \text{ for each } k \in [n-1]\big\}.$$

Let ϱ be the reduced string of σ_2, $r = |\varrho|$ and $J_1 < \cdots < J_r$ be the maximal intervals in $[n]$ such that $\sigma_2(J_i) = \{\varrho(i)\}$. Define a deformation retraction $f\colon [0,1] \times L_{\sigma_1}^{\sigma_2} \to L_{\sigma_1}^{\sigma_2}$ onto L_0 by:

$$f_k(s,x) = (1-s)x_k + s\mu_i(x) \text{ if } k \in J_i \ (k \in [n]), \text{ where}$$

$$\mu_i(x) = \begin{cases} \min\big\{x_j : j \in J_i\big\} & \text{if } \varrho(i) = -; \\ \max\big\{x_j : j \in J_i\big\} & \text{if } \varrho(i) = +. \end{cases}$$

No generality is lost in assuming that $\varrho = \sigma^r$ (instead of $-\sigma^r$), as defined in (4). Then L_0 is homeomorphic to the subspace of $L_{\sigma_1}^r$ consisting of those y for which $|y_i| \leq 1$ for all $i \in [r]$. But clearly, this subspace is a deformation retract of $L_{\sigma_1}^r$, hence $L_{\sigma_1}^{\sigma_2} \simeq L_{\sigma_1}^r$ is contractible by (1.6). $\square$

Proof of (1.15). By induction on m. For $m = 2$, define $f\colon [0,1] \times X_{(\sigma_1,\sigma_2)} \to X_{(\sigma_1,\sigma_2)}$ by:

$$f_k(s,x) = \begin{cases} x_k & \text{if } |x_k| \geq 1; \\ \min\{x_k - \sigma_2(k)s, 1\} & \text{if } |x_k| \leq 1 \text{ and } \sigma_2(k) = -; \\ \max\{x_k - \sigma_2(k)s, -1\} & \text{if } |x_k| \leq 1 \text{ and } \sigma_2(k) = +. \end{cases} \quad (k = 1, \ldots, n = |\sigma_2|).$$

Then f is a deformation retraction of $X_{(\sigma_1,\sigma_2)}$ onto the subspace

$$\big\{x \in X_{(\sigma_1,\sigma_2)} : |x_k| \geq 1 \text{ for all } k \in [n]\big\}.$$

It is easily verified that this is homeomorphic to $L_{\sigma_1}^{\sigma_2}$, hence $X_{(\sigma_1,\sigma_2)}$ is contractible by (1.16).

For $m \geq 3$, set

$$E = X_{(\sigma_1,\ldots,\sigma_m)}, \quad Y = L_{\sigma_{m-1}}^{\sigma_m} \quad \text{and} \quad n = |\sigma_m|.$$

Let $q\colon E \to Y$ be the map which collapses everything at height between $-(m-1)$ and $(m-1)$. To be precise, if $(x_1, \ldots, x_n) \in E$, then its image y under q has coordinates

$$y_k = -\sigma_m(k) \max\{|x_k| - (m-1), 0\} \quad (k \in [n]).$$

Although q is generally not a Serre nor a Dold fibration, we claim that it is a quasifibration.

For $y \in Y$, the fiber $F_y = q^{-1}(y)$ is homeomorphic to $X_{(\sigma_1,\dots,\sigma_{m-2},\tau)}$, where τ is the extended string formed by all coordinates y_k of y such that $|y_k| \le m-1$. Hence, by the induction hypothesis, F_y is weakly contractible.

Given $y \in Y$, let $\delta(y) = \min\big\{\, |y_k| \,:\, y_k \neq 0,\ k \in [n]\big\}$. Then the sets

$$U_{y,\delta} = \big\{(z_1,\dots,z_n) \in Y :\ |z_k - y_k| < \delta \text{ for each } k \in [n]\big\} \qquad \big(y \in Y,\ 0 < \delta < \delta(y)\big)$$

form an open cover $\mathfrak{U}$ of Y. Clearly, condition (ii) in (1.12) is satisfied by $\mathfrak{U}$. Each $U_{y,\delta} \in \mathfrak{U}$ is star-shaped with respect to y, hence contractible. Moreover, $q^{-1}(U_{y,\delta})$ deformation retracts onto F_y through $g\colon [0,1] \times q^{-1}(U_{y,\delta}) \to q^{-1}(U_{y,\delta})$, where

$$g_k(s,x) = \begin{cases} x_k & \text{if } \ |x_k| \le m-1 \\ (1-s)x_k + s\big[y_k - \sigma_m(k)(m-1)\big] & \text{if } \ |x_k| \ge (m-1) \end{cases} \qquad (k \in [n]).$$

Therefore, condition (i) in (1.12) is trivially satisfied: From the long exact sequence of homotopy groups of the pair $\big(q^{-1}(U_{y,\delta})\,,\, F_y\big)$, it follows that $\pi_i(q^{-1}(U_{y,\delta})\,,\, F_y\,,\, e)$ is trivial for all $i \ge 0$ and $e \in F_y$, and so is $\pi_i(U_{y,\delta}\,,\, y)$. Hence q is a quasifibration. By (1.16), Y is weakly contractible. Using exactness of (5) we conclude that E is weakly contractible. □

(1.17) Definition. Define $X_{(d,\sigma_1,\dots,\sigma_m)} \subset \mathbf{R}^n$ as in (1.14), but replacing (i) by:

(i_d) There exist $k_1, k_2 \in [n]$ with $\sigma_m(k_2) = -\sigma_m(k_1)$ and $\sigma_m(k_i)x_{k_i} > 0$.

(1.18) Proposition. *Then $X_{(d,\sigma_1,\dots,\sigma_m)}$ is weakly contractible.*

Proof. Analogous to the proof of (1.15): Use induction on m and the same collapsing map q as before to reduce to the case where $m = 1$. Then consider the map

$$p\colon X_{(d,\sigma_1)} \to L = \big\{x \in \mathbf{R}^n : x \text{ is level}\big\}, \quad p(x)_k = \begin{cases} x_k & \text{if } \sigma_m(k)x_k \le 0; \\ 0 & \text{if } \sigma_m(k)x_k \ge 0. \end{cases} \qquad (k \in [n]).$$

This is a quasifibration with convex fibers, and $L \approx \mathbf{R}^{n-1}$ is contractible by (1.5). □

(1.19) Remark. For the sake of simplicity, in condition (ii) of (1.14) the "heights" which are used in the inequalities were chosen to be $1,\dots,m$. However, we clearly could have required instead that the coordinates k satisfying $|x_k| \le \varepsilon_j$ should yield the reduced string σ_j, for some $0 < \varepsilon_1 < \dots < \varepsilon_m$. Furthermore, since only *weak* contractibility is asserted, it follows from this more general version of (1.15) and (1.18) that if some of the inequalities in (i) and (ii) are replaced by strict inequalities, then the resulting space is again weakly contractible.

2. Stretching

Stretching of functions. In this section we shall describe a procedure for "stretching" curves which generalizes the grafting construction of [4], as illustrated in Figure 9. We rely heavily on the results of §3 of [4] and retain the notation introduced there. The procedure is more clearly formulated in terms of real functions.

(2.1) Definition. Let $b > 0$, r_0, r_b, $A \in \mathbf{R}$ be fixed but otherwise arbitrary and $f\colon [0,b] \to \mathbf{R}$ be an absolutely continuous function with $\dot f \in L^2[0,b]$. Assume that f satisfies

(i) $\dot f(x) < \big[1 + f(x)^2\big]^{\frac{3}{2}}$ for almost every x in the domain of f;

(ii) $f(0) = r_0$, $f(b) = r_b$ and $\int f = A$.[†]

We say that f is *stretchable* if there exists a smooth function $[0,b] \to \mathbf{R}$ which satisfies (i), (ii) and which in addition has a zero in $[0,b]$.

Note that the property of being stretchable does not really concern f, only the numbers b, r_0, r_b and A. The reason for the terminology is that if a function is stretchable, then its domain can be gradually enlarged while respecting (i) and (ii); see Figure 8. The objective of this section is to

[†]For a (Lebesgue integrable) function $g\colon J \to \mathbf{R}$, $\int g$ denotes $\int_J g$. Compare conditions (i)–(iii) in Construction 3.8 of [4].

describe a construction which realizes this stretching, and to prove some regularity properties. The results are somewhat technical but very easy to prove.

For convenience, let us call $f\colon [0,b] \to \mathbf{R}$ *κ_0-stretchable* ($\kappa_0 \in (0,1)$) if it satisfies (ii),

(iii) $\dot f(x) \leq \kappa_0\big[1+f(x)^2\big]^{\frac{3}{2}}$ for almost every x in the domain of f

and if there exists a smooth function $[0,b] \to \mathbf{R}$ satisfying (ii), (iii), and having a zero in $[0,b]$. By Lemma 1.8 in [4], any stretchable function may be approximated by a smooth κ_0-stretchable function if $\kappa_0 \in (0,1)$ is large enough.

(2.2) Remark. The following assertions are all immediate consequences of (2.1).

(a) If $f\colon [0,b] \to \mathbf{R}$ satisfies (i) and has a zero in $[0,b]$, then it is stretchable. Similarly, if f satisfies the strict inequality in (iii) and has a zero in $[0,b]$, then it is κ_0-stretchable.
(b) Suppose that $\phi\colon [0,b] \to \mathbf{R}$ is stretchable and positive throughout $[0,b]$ and let $0 \leq \delta \leq \inf_{x\in[0,b]} \phi(x)$; then the function $x \mapsto \phi(x) - \delta$ is also stretchable.
(c) Let $M > 0$ be fixed. Then there exists $b_0 > 0$ such that if $b \geq b_0$, then any function $f\colon [0,b] \to \mathbf{R}$ satisfying $\sup_{x\in[0,b]} |f(x)| \leq M$ and (i) is stretchable.
(d) Let $c > b$ and $f\colon [0,c] \to \mathbf{R}$ satisfy (i). If $f|_{[0,b]}$ is stretchable, then so is f.

Let r_0, r_b, κ_0 be fixed throughout and let $g_\pm\colon \mathbf{R} \to \mathbf{R}$ and $h^b_\pm = h_\pm\colon \mathbf{R} \to \mathbf{R}$ be the functions in (24) and (25) of [4]. Since g_+ is strictly increasing and h^b_+ is strictly decreasing, the graphs of these functions either do not intersect, or do so at a single point. In the latter case, let $\lambda_+(b)$ denote the common value of g_+, h^b_+ at this point, and in the former, set $\lambda_+(b) = +\infty$. Let $\lambda_-(b)$ be defined analogously. (Notice that $g_\pm$ depends upon the values of κ_0, r_0 and $h^b_\pm$ depends upon the values of κ_0, r_b, even though this is not indicated explicitly in the notation. The same comment applies to the numbers $\lambda_\pm(b)$.)

(2.3) Remark. Let $c > b > 0$. Then $h^c_\pm$ is obtained from $h^b_\pm$ by a shift of the parameter through $c-b$, that is, $h^c_\pm(x) = h^b_\pm\big(x-(c-b)\big)$ for all $x \in \mathbf{R}$. The monotonicity of $h^b_\pm$ implies that $h^b_+ \leq h^c_+$ and $h^c_- \leq h^b_-$ throughout $\mathbf{R}$, hence $\lambda_-(c) \leq \lambda_-(b) \leq \lambda_+(b) \leq \lambda_+(c)$.

(2.4) Lemma. *Let $f\colon [0,b] \to \mathbf{R}$ be absolutely continuous and satisfy (ii) and (iii). Then*

$$\lambda_-(b) \leq \max\big\{g_-(x), h^b_-(x)\big\} \leq f(x) \leq \min\big\{g_+(x), h^b_+(x)\big\} \leq \lambda_+(b) \quad \text{for all } x \in [0,b]. \tag{6}$$

In particular, $\lambda_-(b) \leq 0 \leq \lambda_+(b)$ if there exists a function satisfying (ii), (iii) which has a zero.

Proof. The innermost inequalities were already established in (26) of [4]. The other two are immediate from the definition of $\lambda_\pm(b)$ and the monotonicity of $g_\pm$, $h^b_\pm$. □

(2.5) Lemma. *Let $f = f_1\colon [0,b] \to \mathbf{R}$ be absolutely continuous and satisfy (ii) and (iii), and let $f_0\colon [0,b] \to \mathbf{R}$ be as in Construction 3.8 and Remark 3.9 of* [4]*. Then $\mu_0 \in [\lambda_-(b), \lambda_+(b)]$.*

Proof. By eq. (27) of [4] and the definition of μ_0,

$$\inf_{x\in[0,b]} f(x) \leq \mu_0 \leq \sup_{x\in[0,b]} f(x).$$

Hence the lemma follows from (2.4). □

(2.6) Definition. For $b > 0$ and $\mu \in [\lambda_-(b), \lambda_+(b)] \cap \mathbf{R}$, define $\zeta_{(\mu,b)}\colon [0,b] \to \mathbf{R}$ by

$$\zeta_{(\mu,b)}(x) = \operatorname{median}\big(h^b_-(x)\,,\, g_-(x)\,,\, \mu\,,\, g_+(x)\,,\, h^b_+(x)\big) \quad (x \in [0,b]). \tag{7}$$

Notice that by monotonicity of $g_\pm$, $h^b_\pm$,

$$\inf_{x\in[0,b]} \zeta_{(\mu,b)}(x) = \min\{r_0,\, r_b,\, \mu\} \quad \text{and} \quad \sup_{x\in[0,b]} \zeta_{(\mu,b)}(x) = \max\{r_0,\, r_b,\, \mu\}\,. \tag{8}$$

(2.7) Lemma. *Let $b > 0$ be fixed and $\mu_1 < \mu_2 \in [\lambda_-(b), \lambda_+(b)] \cap \mathbf{R}$. Then $\zeta_{(\mu_1,b)}(x) \leq \zeta_{(\mu_2,b)}(x)$ for all $x \in [0,b]$ and strict inequality holds for at least one x. In particular, $\int \zeta_{(\mu,b)}$ is a strictly increasing function of $\mu \in [\lambda_-(b), \lambda_+(b)]$.*

Proof. The inequality is obvious from (7). Furthermore, if $x_0 \in [0,b]$ is such that $g_+(x_0) = h^b_+(x_0) = \lambda_+(b)$, then $\zeta_{(\mu_1,b)}(x_0) = \mu_1 < \mu_2 = \zeta_{(\mu_2,b)}(x_0)$. (Recall that the range of g_+ and h^b_+ were extended to $\mathbf{R} \cup \{\pm\infty\}$ by convention, so such an x_0 exists even if $\lambda_+(b) = +\infty$.) □

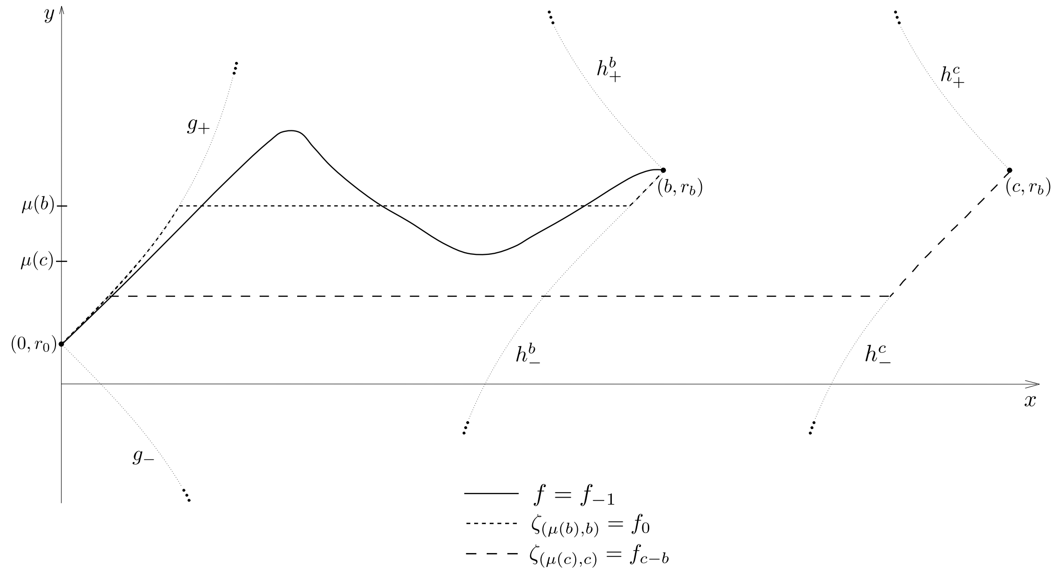


FIGURE 8.

(2.8) Lemma. *Let $c > b$ and $\mu \in [\lambda_-(b), \lambda_+(b)] \cap \mathbf{R}$. Then $\int \zeta_{(\mu,c)} - \int \zeta_{(\mu,b)} = \mu(c-b)$ and $\zeta^{-1}_{(\mu,c)}(\{\mu\})$ is a closed interval of length at least $c - b$.*

Proof. Notice that $\zeta_{(\mu,c)}$ is defined because of (2.3). Let J be the inverse image of μ under $\zeta_{(\mu,b)}$. Then J is a (possibly degenerate) closed interval $[x_0, x_1]$. By (2.3),

$$\zeta_{(\mu,c)}(x) = \begin{cases} \zeta_{(\mu,b)}(x) & \text{if } x \in [0, x_0]; \\ \mu & \text{if } x \in [x_0, x_1 + (c-b)]; \\ \zeta_{(\mu,b)}(x - (c-b)) & \text{if } x \in [x_1 + (c-b), c]. \end{cases} \tag{9}$$

The assertions of the lemma are consequences of this expression. □

(2.9) Corollary. *Let $b > 0$, $\mu(b) \in [\lambda_-(b), \lambda_+(b)] \cap \mathbf{R}$ be fixed and $A = \int \zeta_{(\mu(b),b)}$. Suppose that $0 \in [\lambda_-(b), \lambda_+(b)]$. Then for each $c \geq b$, there exists a unique $\mu(c) \in \mathbf{R}$ such that $\int \zeta_{(\mu(c),c)} = A$. The resulting function $\mu\colon [b, +\infty) \to \mathbf{R}$, $c \mapsto \mu(c)$, is continuous and $|\mu(c)| \searrow 0$ as $c \to +\infty$.*

Proof. From $0 \in [\lambda_-(b), \lambda_+(b)]$ and (2.3), it follows that $0 \in [\lambda_-(c), \lambda_+(c)]$ for all $c \geq b$. No generality is lost in assuming that $\mu(b) \geq 0$. In this case, (2.7) and (2.8) yield:

$$\int \zeta_{(0,c)} = \int \zeta_{(0,b)} \leq A = \int \zeta_{(\mu(b),b)} \leq \int \zeta_{(\mu(b),b)} + (c-b)\mu(b) = \int \zeta_{(\mu(b),c)}. \tag{10}$$

Hence, by (2.7), there exists a unique $\mu(c) \in [0, \mu(b)]$ such that $\int \zeta_{(\mu(c),c)} = A$. Moreover, $\mu(c) > 0$ in case $\mu(b) > 0$. The same argument also shows that $\mu(d) \in [0, \mu(c)]$ whenever $d \geq c$. Thus, $\mu(c)$ is a decreasing function of c, $L = \lim_{c\to+\infty} \mu(c)$ exists and is nonnegative. The continuity of $c \mapsto \mu(c)$ follows from the explicit expression (9) for $\zeta_{(\mu,c)}$. Finally,

$$A = \int \zeta_{(\mu(c),c)} \geq \int \zeta_{(\lambda,c)} = L(c-b) + \int \zeta_{(\lambda,b)} \quad \text{for all } c \geq b.$$

Therefore $L = 0$. □

(2.10) Remark. If $\mu(b) > 0$, then the first inequality in (10) is strict by (2.7). The argument given in the preceding proof thus implies that $|\mu(c)|$ is a strictly decreasing function of $c \in [b, +\infty)$ if $\mu(b) \neq 0$. In particular, μ does not vanish under this assumption.

(2.11) Definition. Let $\phi\colon [0, b] \to \mathbf{R}$ satisfy conditions (ii) and (iii) above. The *flattening* of ϕ is the family $\phi_s\colon [0, b] \to \mathbf{R}$ ($s \in [0, 1]$) obtained by applying Construction 3.8 in [4] to $\phi_1 = \phi$.

Let $f_{-1} = f\colon [0,b] \to \mathbf{R}$ be a κ_0-stretchable function satisfying (ii). For $s \in [-1,0]$, let $f_s\colon [0,b] \to \mathbf{R}$ be the flattening of f_{-1}.† Then f_0 has the form $\zeta_{(\mu(b),b)}$ for some $\mu(b) \in \mathbf{R}$, and the hypothesis of (2.9) is satisfied, by (2.4). For $s \geq 0$, let $f_s = \zeta_{(\mu(b+s),b+s)}$, where $\mu\colon [b,+\infty) \to \mathbf{R}$ is as in (2.9). The *stretching* of $f = f_{-1}$ is the family $(f_s)_{(s\in[-1,+\infty))}$ so defined. See Figure 8.

(2.12) Lemma. *Let $f_{-1} = f\colon [0,b] \to \mathbf{R}$ be a smooth κ_0-stretchable function and $(f_s)_{(s\in[-1,+\infty))}$ be the stretching of f. Then:*

(a) *f_s is a piecewise smooth function for all s.*
(b) *$\sup|f_s|$ is a decreasing function of s.*
(c) *Let $A_\pm(s) = \int_{\{\pm f_s\geq 0\}} f_s$. Then A_- (resp. A_+) is an increasing (decreasing) function of s.*
(d) *If f does not change sign inside its domain, then none of the f_s do.*
(e) *For $s \in [-1,0]$, $\sup_{[0,b]} f_s$ (resp. $\inf_{[0,b]} f_s$) is a decreasing (resp. increasing) function of s.*
(f) *Let $f_s = \zeta_{(\mu(b+s),b+s)}$ $(s \geq 0)$ and let L_s denote the length of the interval*

$$\big\{x \in [0,b+s] \,:\, f_s(x) = \mu(b+s)\big\}.$$

Then $L_s \sim s$ (that is, $\lim_{s\to+\infty} \frac{L_s}{s} = 1$).

(g) *There exists $\varkappa_2 > 0$ such that*

$$|\mu(b+s)| \leq \frac{\varkappa_2}{s+1} \quad \text{for all } s \geq 0.$$

Moreover, if $f > 0$ over $[0,b]$, then there also exists $\varkappa_1 > 0$ such that

$$\frac{\varkappa_1}{s+1} \leq \mu(b+s) \quad \text{for all } s \geq 0.$$

(h) *Suppose that $f > 0$ over $[0,b]$. Then there exists a homotopy $s \mapsto g_s\colon [0,b] \to \mathbf{R}$ such that $g_0 = f$, g_s satisfies (ii) and (iii), $g_s \geq 0$ for each $s \in [0,1]$ and g_1 has a zero.*‡
(j) *f_s is κ_0-stretchable for all $s \in [-1,+\infty)$.*

Proof. The proof will be split into the corresponding parts.

(a): By definition, f_s is the median of a finite collection of smooth functions for all s.

(b): For $s \geq 0$, this follows from (2.9). For $s \in [-1,0]$, this follows from Corollary 3.12 of [4].

(c): For $s \geq 0$, $f_s = \zeta_{(\mu(s),b+s)}$ by definition. By (2.9), the function μ does not change sign in $[0,+\infty)$. Then (7) implies that $A_{-\operatorname{sign}(\mu(b))}(s)$ is a constant function of s; since $A_+(s) + A_-(s) = A$ is constant by construction, so is $A_{\operatorname{sign}(\mu(b))}(s)$. If $s \in [-1,0]$, then Lemma 3.11 of [4] states that $f_s(x)$ is an increasing (resp. decreasing) function of $s \in [-1,0]$ for all x lying in

$$S_- = \big\{x \in [0,b] \,:\, f(x) \leq \mu(b)\big\} \quad \text{or} \quad S_+ = \big\{x \in [0,b] \,:\, f(x) \geq \mu(b)\big\}, \quad \text{respectively.}^{\dagger\dagger}$$

Thus, if $\mu(b) \geq 0$, then $A_-(s)$ can only increase with s, and since $A_+(s) + A_-(s) = A$ is constant, $A_+(s)$ must decrease. Similarly, if $\mu(b) \leq 0$, then it follows that $A_+(s)$ decreases with s, so that $A_-(s)$ increases.

(d): No generality is lost in assuming that $f = f_{-1} \geq 0$ over $[0,b]$. Then, by Corollary 3.12 of [4],

$$0 \leq \inf_{x\in[0,b]} f_{-1}(x) \leq \inf_{x\in[0,b]} f_s(x) \quad \text{for all } s \in [-1,0].$$

Let $r_0 = f(0)$, $r_b = f(b)$; both are nonnegative by hypothesis. By (8),

$$\inf_{x\in[0,b+s]} f_s(x) = \inf_{x\in[0,b+s]} \zeta_{(\mu(b+s),b+s)}(x) = \min\{r_0,\, r_b,\, \mu(b+s)\} \quad \text{for all } s \geq 0.$$

Hence $\mu(b) \geq 0$, and (2.9) guarantees that $\mu(b+s) \geq 0$ for all $s \geq 0$.

(e): This was proved in Corollary 3.12 of [4].

(f): The functions $g_\pm$, $h^b_\pm$ all blow up to $\pm\infty$ in finite time (compare eqs. (24) and (25) of [4]). The length of $[0,b+s]$ is asymptotically equal to s, hence $L_s \sim s$ as well.

†More precisely, $f_s = \phi_{-s}\colon [0,b] \to \mathbf{R}$ $(s \in [0,1])$, where ϕ_{-s} is the flattening of $\phi_1 = f_{-1}$ as defined above.

‡Part (i) is skipped to avoid confusion with (i) of (2.1). Note that a topology on the set of functions satisfying (ii), (iii) has not been defined. Since we are only interested in applications of this result to sets of curves, which already come with a topology, we will ignore this. In any case, Construction 3.8 in [4], which is used in the proof, is so regular that it respects any reasonable topology.

††The number $\mu(b)$ is denoted by μ_0 in Lemma 3.11 of [4], since there b is kept fixed.

(g): Let $A = \int f = \int f_s$ (for all s). Assume without loss of generality that $\mu(b) \geq 0$. By (2.7),

$$A_0 = \int \zeta_{(0,b)} \leq \int \zeta_{(\mu(b),b)} = A.$$

As in (e), let L_s denote the length of $\big\{x \in [0, b+s] : f_s(x) = \mu(b+s)\big\}$. Then

$$L_s\,\mu(b+s) \leq A,$$

hence the first assertion follows from part (e). As in the proof (d), the assumption that $f_{-1} = f > 0$ implies that $f_0 > 0$ throughout $[0, b]$. Hence $\mu(b) > 0$, $A > A_0$, and

$$A \leq A_0 + L_s\,\mu(b+s) \ \text{ for every } s \geq 0.$$

The second assertion follows from this inequality and part (e).

(h): Start with a smooth function $g_2\colon [0, b] \to \mathbf{R}$ satisfying (ii), (iii) and having a zero in $[0, b]$. Construction 3.8 of [4] describes an explicit homotopy joining g_2 to $g_1 = \zeta_{(\mu(b),b)}$, which may be concatenated with the homotopy between g_1 and $g_0 = f$. Since $f > 0$, there exists a smallest $s_1 \in [0, 2]$ such that g_{s_1} has a zero. Restricting the deformation to $[0, s_1]$ finishes the proof.

(j): Because the property of being κ_0-stretchable only depends on b, r_0, r_b, the assertion is true by hypothesis for $s \in [-1, 0]$. Let $r_{b+s} = r_b$ for all $s \geq 0$. If $r_0 r_b \leq 0$ there is nothing to prove, for then every smooth function $[0, b+s] \to \mathbf{R}$ ($s \geq 0$) satisfying (ii) and (iii) has a zero. Thus, no generality is lost in assuming that $r_0, r_b > 0$. If $\mu(b) \leq 0$, then $\mu(b+s) \leq 0$ for all $s \geq 0$ by (2.9), hence again f_s itself has a zero for every $s \geq 0$. If $\mu(b) > 0$, then $f_0 > 0$ is a positive function. Let $g_0 > 0$ be a smooth approximation to f_0 and g_1 a smooth function which attains a negative value, both satisfying (ii), (iii). By Construction 3.8 of [4], there exists a homotopy $s \mapsto g_s$ such that each g_s satisfies (ii), (iii). Approximating this by a smooth homotopy, we can obtain a smooth $g\colon [0, b] \to \mathbf{R}$, $g \geq 0$, satisfying (ii), (iii) and having a zero $x_0 \in [0, b]$; then $\dot{g}(x_0) = 0$ as well. Define

$$h_s(x) = \begin{cases} g(x) & \text{if } x \in [0, x_0] \\ 0 & \text{if } x \in [x_0, x_0 + s] \\ g(x - s) & \text{if } x \in [x_0 + s, b + s] \end{cases} \qquad (x \in [0, b+s]).$$

Then h_s is continuously differentiable, has a zero and satisfies (ii), (iii). Also, it can be approximated by a smooth function which satisfies these conditions and still has a zero. ☐

Stretching of curves. Let $P = (p, w), Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ and $\gamma \in \mathcal{L}_{-1}^{+1}(P, Q)$. Suppose that $\psi \in \mathbf{R}$ is such that $\langle \mathbf{t}_\gamma, e^{i\psi} \rangle > 0$ throughout $[0, 1]$. After rotating $\mathbf{C}$ about the origin through $e^{i\psi}$ (and relabeling the x- and y-axes accordingly), γ may be reparametrized as $\gamma(x) = (x, y(x))$ for x in some closed interval, which may be assumed to be of the form $[0, b]$. Let $f = \dot{y}\colon [0, b] \to \mathbf{R}$. We call γ *stretchable* (*with respect to* ψ) if f is stretchable in the sense of (2.1). Similarly, γ is κ_0-*stretchable* (*with respect to* ψ) if f is κ_0-stretchable ($\kappa_0 \in (0, 1)$).

(2.13) Remark. In this context, $f(x) = \tan(\theta_\gamma(x) - \psi)$ for all $x \in [0, b]$. Condition (i) in (2.1) means that the curvature κ_γ of γ satisfies $|\kappa_\gamma| < 1$ almost everywhere. The numbers r_0, r_b in (ii) represent the slopes of w, z respectively, $A = \operatorname{Im}(q - p)$ and $b = \operatorname{Re}(q - p)$ (each of these with respect to the coordinate axes determined by $e^{i\psi}$ and $ie^{i\psi}$).

(2.14) Definition. Assume now that f is κ_0-stretchable for some $\kappa_0 \in (0, 1)$, and let $(f_s)_{s \in [-1, +\infty)}$ be the corresponding stretching of $f = f_{-1}$, as in (2.11). Define

$$(11) \qquad \gamma_s(x) = \Big(x\,,\, y(0) + \int_{J_s} f_s(u)du\Big) \text{ for } s \in [-1, +\infty), \text{ where } J_s = \begin{cases} [0, b] & \text{if } s \in [-1, 0]; \\ [0, b+s] & \text{if } s \geq 0. \end{cases}$$

The family $(\gamma_s)_{(s \in [-1, +\infty))}$ will be called the *stretching* of $\gamma = \gamma_{-1}$, and the family $(\gamma_s)_{(s \in [-1, 0])}$, the *flattening* of γ with respect to ψ. The *stretching by* M of γ is the family $(\gamma_s)_{(s \in [-1, M])}$, $M > 0$.

Note that $\gamma_s \in \mathcal{L}_{-1}^{+1}(P, Q_s)$ for $Q_s = (q_s, z) \in \mathbf{C} \times \mathbf{S}^1$, where $q_s = q$ for all $s \in [-1, 0]$ and $q_s = q + se^{i\psi}$ for $s \geq 0$. In vague but suggestive language, the family (γ_s) is obtained from γ by "stretching" it in the direction of $e^{i\psi}$. The details of the construction of the family (γ_s) may be safely forgotten. Only the properties stated in (2.12), or more accurately their interpretations in terms of γ_s, will be used.

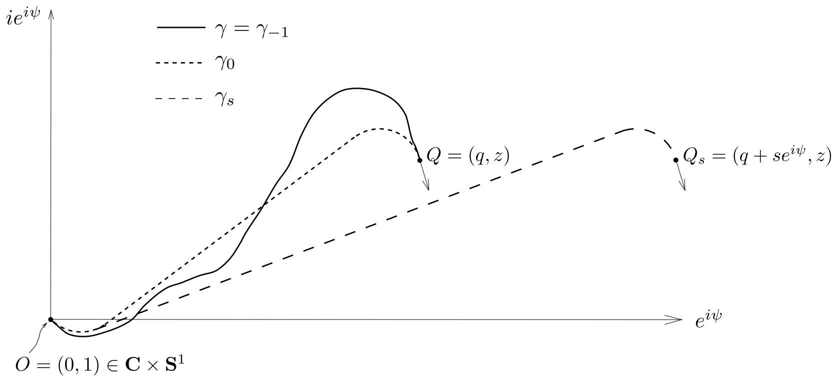


FIGURE 9. Flattening and stretching a curve γ in the direction of $e^{i\psi}$. Note that γ_s is a concatenation of an arc of circle of curvature $\pm\kappa_0$, a line segment and another arc of circle of curvature $\pm\kappa_0$ for any $s \geq 0$.

Exercise. Translate the assertions of (2.12) into statements about the curves γ_s.

For instance, part (b) states that $\sup_x |\theta_{\gamma_s}(x) - \psi|$ is a decreasing function of s.

Remark. Clearly, the stretching and flattening of a curve γ depend upon the chosen axis ψ. However, the curve γ_0 does not: It is the unique curve of shortest length in $\hat{\mathcal{L}}_{-\kappa_0}^{+\kappa_0}(P, Q)$, compare Remark 3.9 of [4]. (The space $\hat{\mathcal{L}}_{-\kappa_0}^{+\kappa_0}(P, Q)$ consists of curves having initial and final frames P, Q and curvature in the closed interval $[-\kappa_0, +\kappa_0]$. See Definition 1.16 and Lemma 1.17 of [4].)

(2.15) Remark. If $\gamma|_I$ is a line segment of length L, then there exists a subinterval I' such that $\gamma_s|_{I'}$ is a line segment of length greater than $L - 2\pi$ for all $s \in [-1, 0]$. This is immediate from [4], Construction 3.8.

(2.16) Remark. If γ is a line segment of length > 8 and $\kappa_0 > \frac{1}{2}$, then γ is κ_0-stretchable. This follows from Theorem 6.1 of [4]: $\mathcal{M}(P)$ is connected if $P = (x, 1) \in \mathbf{R} \times \mathbf{S}^1$ with $x > 4$.

3. Quasicritical curves

Notation. Throughout the rest of the paper, $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$ denotes a fixed element of $UT\mathbf{C}$ with $z \neq -1$. For our purposes, it is more convenient to work with the space $\mathcal{L}_{-1}^{+1}(Q)$ (see §1 of [4]) instead of the space $\mathcal{C}_{-1}^{+1}(Q)$ defined in the introduction; these are homeomorphic by Lemma 1.12 in [4]. Accordingly, $\mathcal{M}(Q)$ shall denote the unique subspace $\mathcal{L}_{-1}^{+1}(Q; \theta_1) \subset \mathcal{L}_{-1}^{+1}(Q)$ with $|\theta_1| < \pi$.

Let $\gamma \in \mathcal{M}(Q)$. As in [4], $\mathbf{t}_\gamma \colon [0, 1] \to \mathbf{S}^1$ denotes the unit tangent to γ, and $\theta_\gamma \colon [0, 1] \to \mathbf{R}$ the unique continuous function satisfying $e^{i\theta_\gamma} = \mathbf{t}_\gamma$ and $\theta_\gamma(0) = 0$. Also,

$$\bar{\varphi}^\gamma := \frac{1}{2}\Big(\max_{t \in [0,1]} \theta_\gamma(t) + \min_{t \in [0,1]} \theta_\gamma(t)\Big). \tag{12}$$

Finally, given $\varphi \in \mathbf{R}$, we let $\varphi_\pm := \varphi \pm \frac{\pi}{2}$.

Quasicritical curves. The central definition of this paper is the following generalization of the concept of critical curves.

(3.1) Definition. Let σ be a sign string, $n = |\sigma|$, $\gamma \in \mathcal{M}(Q)$, $\varphi \in \mathbf{R}$ and $\varepsilon \in (0, \frac{\pi}{4})$. Then γ is (φ, ε)-*quasicritical of type* σ if there exist closed intervals $J_1 < \cdots < J_n$ such that for each $k \in [n]$:

(i) $\theta_\gamma(J_k) \subset (\varphi_- + 2\varepsilon, \varphi_+ + \varepsilon)$ if $\sigma(k) = +$ and $\theta_\gamma(J_k) \subset (\varphi_- - \varepsilon, \varphi_+ - 2\varepsilon)$ if $\sigma(k) = -$;
(ii) $|\theta_\gamma(t) - \varphi| < \frac{\pi}{2} - 2\varepsilon$ for all $t \notin \operatorname{Int}\big(\bigcup_{k=1}^n J_k\big)$;
(iii) J_k contains at least one closed subinterval I_k such that $|\theta_\gamma(t) - \varphi_{\sigma(k)}| < \varepsilon$ for all $t \in I_k$ and $\gamma|_{I_k}$ is stretchable with respect to $\varphi_{\sigma(k)}$.

Condition (i) means that $\mathbf{t}_\gamma$ is far from $\mp ie^{i\varphi}$ throughout J_k if $\sigma(k) = \pm$, while (iii) states roughly that there should exist a subinterval of J_k where $\mathbf{t}_\gamma$ is vertical enough with respect to the axis $e^{i\varphi}$ to allow γ to be stretched in the direction of $\pm ie^{i\varphi}$. Outside of $\bigcup J_k$, $\mathbf{t}_\gamma$ is far from both $ie^{i\varphi}$ and $-ie^{i\varphi}$.

(3.2) Remark. The combination of (i) and (ii) in (3.1) implies that $\theta_\gamma([0,1]) \subset (\varphi_- - \varepsilon, \varphi_+ + \varepsilon)$.

(3.3) Remark. Being quasicritical of type σ is an open condition on $(\gamma, \varphi, \varepsilon)$. In fact, the same intervals J_k satisfy (i)–(iii) for the triple (η, ψ, δ) if the latter is close enough to $(\gamma, \varphi, \varepsilon)$.

(3.4) Lemma. *Let $\gamma \in \mathcal{M}(Q)$ be a critical curve of type σ. Then γ is $(\bar{\varphi}^\gamma, \varepsilon)$-quasicritical of type σ for all sufficiently small $\varepsilon > 0$.*

Proof. This is an easy consequence of the definition of critical curves, given in Definition 5.2 of [4], and of (2.2) (a). □

We will sometimes abuse the terminology by saying that I is a stretchable interval for γ if $\gamma|_I$ is stretchable (with respect to $\varphi_\pm$). Notice that there is a lot of freedom in the choice of the intervals J_k and their stretchable subintervals. The next two results compensate for this ambiguity.

(3.5) Lemma. *Let $\gamma \in \mathcal{M}(Q)$ be (φ, ε)-quasicritical of type σ, $n = |\sigma|$.*

(a) *Let $0 < \delta \leq 2\varepsilon$ and $W_\alpha \subset [0,1]$ $(\alpha \in A)$ be all the connected components of*
$$W = \big\{t \in [0,1] : \ |\theta_\gamma(t) - \varphi| > \tfrac{\pi}{2} - \delta\big\}.$$
Then there exists a decomposition $A = A_1 \sqcup \ldots \sqcup A_n$ such that for any choice of $J_1 < \cdots < J_n$ as in (3.1), $W_\alpha \subset J_k$ if and only if $\alpha \in A_k$ $(k \in [n])$.

(b) *Let $J_1^j < \cdots < J_n^j$, $J_k^j = [a_k^j, b_k^j]$, be as in (3.1) $(j \in [m])$. For each $k \in [n]$, set $a_k' = \max_j a_k^j$ and $b_k' = \min_j b_k^j$. Then the intervals $J_k' = [a_k', b_k']$ also satisfy (i)–(iii).*

(c) *Let $J_1' < \cdots < J_n'$ be as in (3.1) and $J_1 < \cdots < J_n$ be such that $J_k' \subset J_k$ for each $k \in [n]$. Then the J_k also satisfy (i)–(iii).*

Proof. The proof of each part will be given separately.

(a): Let $J_1 < \cdots < J_n$, $J_1' < \cdots < J_n'$ be intervals as in (3.1). Set $A_k = \big\{\alpha \in A : W_\alpha \subset J_k\big\}$. Then $A = A_1 \sqcup \ldots \sqcup A_n$ since (ii) of (3.1) implies that any W_α must be completely contained in some J. We claim that $A_k' = A_k$ for each $k \in [n]$, where $A_k' = \big\{\alpha \in A : W_\alpha \subset J_k'\big\}$. This follows from the following simple observations (which also hold with A' in place of A):

- Each A_k is nonempty, by (iii) of (3.1).
- If $\alpha \in A_k$, $\alpha' \in A_{k'}$ with $k < k'$, then $W_\alpha < W_{\alpha'}$; indeed, $J_k < J_{k'}$.
- If $\alpha \in A_k$, then $\operatorname{sign}(\theta_\gamma(t) - \varphi) = \sigma(k)$ for all $t \in W_\alpha$, by (i) of (3.1).

Suppose that $\alpha \in A_1 \cap A_k'$ for some $k > 1$. Then the third observation implies that $k \geq 3$. Choose $\beta \in A_2'$. By the second observation, $W_\beta < W_\alpha$. Hence $\beta \in A_1 \cap A_2'$, contradicting the third observation. It follows that $A_1 = A_1'$. An entirely similar argument shows that if $A_j' = A_j$ for all $j \in [k]$, then $A_{k+1}' = A_{k+1}$ as well.

(b): Let $j_0, j_1 \in [m]$ be such that $a_k' = a_k^{j_0}$ and $b_k' = b_k^{j_1}$. By part (a), if $\alpha \in A_k$, then $W_\alpha \subset J_k^{j_0} \cap J_k^{j_1}$. In particular, $a_k' < b_k'$ and
$$[a_k', b_k'] = J_k^{j_0} \cap J_k^{j_1}.$$
Since the latter two intervals satisfy condition (i) by hypothesis, so does $[a_k', b_k']$. Set $\delta = 2\varepsilon$ in the definition of W. If I is a stretchable subinterval of $J_k^{j_0}$ as in (iii), then $I \subset W_\alpha$ for some $\alpha \in A_k$. By (a), $W_\alpha \subset J_k^{j_0} \cap J_k^{j_1} = [a_k', b_k']$, hence the latter satisfies (iii). To establish (ii), let $j_2 \in [m]$ be such that $a_{k+1}' = a_{k+1}^{j_2}$. As above, part (a) implies that $a_k^{j_2} < b_k'$ and $a_{k+1}' < b_{k+1}^{j_1}$. Hence
$$[b_k', a_{k+1}'] = \big[b_k', b_{k+1}^{j_1}\big] \cap \big[a_k^{j_2}, a_{k+1}'\big]. \tag{13}$$
Moreover,
$$\big[b_k', b_{k+1}^{j_1}\big] = \big[b_k', a_{k+1}^{j_1}\big] \cup J_{k+1}^{j_1} \quad \text{and} \quad \big[a_k^{j_2}, a_{k+1}'\big] = J_k^{j_2} \cup \big[b_k^{j_2}, a_{k+1}'\big].$$
By (i) and (ii) of (3.1), any $t \in \big[b_k', b_{k+1}^{j_1}\big]$ thus satisfies $|\theta_\gamma(t) - \varphi_{-\sigma(k+1)}| > 2\varepsilon$ and any $t \in \big[a_k^{j_2}, a_{k+1}'\big]$ satisfies $|\theta_\gamma(t) - \varphi_{-\sigma(k)}| > 2\varepsilon$. Together with (13), this implies that (ii) holds for the J_k'.

(c): This is an immediate consequence of (i)–(iii). □

Notation. In all that follows, K denotes a finite simplicial complex. (Actually, most of the time all that is required is that K be a compact Hausdorff topological space.)

(3.6) Lemma. *Let σ be a sign string of length n and*

$$p \mapsto \gamma^p \in \mathcal{M}(Q), \quad p \mapsto \varphi^p \in \mathbf{R}, \quad p \mapsto \varepsilon^p \in \mathbf{R}^+ \quad (p \in K)$$

be continuous maps such that γ^p is $(\varphi^p, \varepsilon^p)$-quasicritical of type σ for all $p \in K$. Then:

(a) *There exist continuous functions $a_k, b_k \colon K \to [0,1]$ such that for all $p \in K$, the intervals $J_k(p) = [a_k(p), b_k(p)]$ $(k \in [n])$ satisfy (3.1) when $(\gamma, \varphi, \varepsilon) = (\gamma^p, \varphi^p, \varepsilon^p)$.*

(b) *There exist an open cover $(U_i)_{i \in [l]}$ of K and real numbers $c_{i,k} < d_{i,k}$ $(i \in [l],\ k \in [n])$ such that for each $p \in \overline{U}_i$ and $k \in [n]$, $I_{i,k} := [c_{i,k}, d_{i,k}] \subset J_k(p)$, $\gamma^p|_{I_{i,k}}$ is stretchable with respect to $\varphi^p_{\sigma(k)}$ and*

$$\theta_{\gamma^p}\big(I_{i,k}\big) \subset \big(\varphi^p_{\sigma(k)} - \varepsilon^p, \varphi^p_{\sigma(k)} + \varepsilon^p\big). \tag{14}$$

Remark. The inclusion $I_{i,k} \subset J_k(p)$ in (b) is asserted to hold only when $p \in \overline{U}_i$. However, it will hold for such p independently of the choice of the $J_k(p)$ in (a).

It is generally impossible to obtain globally (and continuously) defined intervals $[c_k(p), d_k(p)]$ restricted to which γ^p is stretchable. The problem is similar to that of choosing points $t(p) \in [0,1]$ where a family $f^p \colon [0,1] \to \mathbf{R}$ of continuous functions attain their maxima.

Proof of (3.6). Let $p \in K$. Choose intervals $[a_1, b_1] < \cdots < [a_n, b_n]$ satisfying (i) and (ii) of (3.1) for $(\gamma, \varphi, \varepsilon) = (\gamma^p, \varphi^p, \varepsilon^p)$ and subintervals $[c_k, d_k] \subset [a_k, b_k]$ as in (iii). Since these conditions are open, they actually hold for the same choice of intervals for all q in the closure $\overline{U}_p$ of some neighborhood U_p of p. Let $(U_i)_{i \in [l]}$ be a finite subcover of the cover $(U_p)_{p \in K}$ so obtained, and let $a_{i,k},\ b_{i,k},\ c_{i,k},\ d_{i,k} \in [0,1]$ $(i \in [l], k \in [n])$ be the endpoints of the corresponding intervals.

Let $\rho_i \colon K \to [0,1]$ $(i \in [l])$ form a partition of unity subordinate to the cover (U_i),

$$a_k(p) := \sum_{i=1}^{l} \rho_i(p) a_{i,k}(p), \quad b_k(p) := \sum_{i=1}^{l} \rho_i(p) b_{i,k}(p) \quad \text{and} \quad J_k(p) := [a_k(p), b_k(p)] \quad (k \in [n]).$$

Because $a_{i,k} < b_{i,k} < a_{i,k+1}$ for each i and k by hypothesis, the definition of $J_k(p)$ makes sense and $J_1(p) < \cdots < J_n(p)$ holds for all $p \in K$. Now fix p and let $i_1, \dots, i_m \in [l]$ be all the indices i such that $\rho_i(p) > 0$. Set

$$a_k' := \max_{j \in [m]} a_{i_j,k}(p), \quad b_k' := \min_{j \in [m]} b_{i_j,k}(p) \quad (j \in [m]).$$

Then $[a_k', b_k'] \subset J_k(p)$, hence the combination of (b) and (c) of (3.5) shows that $J_k(p)$ satisfies (i)–(iii) for each $k \in [n]$. This proves (a).

Fix $i \in [l]$. By the choice of the intervals $I_{i,k} := [c_{i,k}, d_{i,k}]$, $\gamma^p|_{I_{i,k}}$ is stretchable with respect to $\varphi^p_{\sigma(k)}$ and (14) holds whenever $p \in \overline{U}_i$. Again by choice, $I_{i,k} \subset [a_{i,k}, b_{i,k}]$. Since the $[a_{i,k}, b_{i,k}]$ and the $J_k(p)$ satisfy (i)–(iii) provided that $p \in \overline{U}_i$, (3.5) (a) implies that $I_{i,k} \subset J_k(p)$ for such p and each $k \in [n]$. This proves (b). ☐

(3.7) Lemma. *In the situation of (3.6), it is possible to choose the cover $(U_i)_{i \in [l]}$ of K and the intervals $I_{i,k} = [c_k^i, d_k^i]$ so that:*

(a) *If $i < i'$ and $\overline{U}_i \cap \overline{U}_{i'} \neq \emptyset$, then for each $k \in [n]$, either $I_{i,k} \subset I_{i',k}$ or these two intervals are disjoint.*

(b) *For all $k \in [n]$, $i \in [l]$ and $p \in \overline{U}_i$, either $\big|\theta_{\gamma^p}(c_{i,k}) - \varphi^p_{\sigma(k)}\big| > \frac{1}{2}\varepsilon^p$ or $c_{i,k} = 0$, and either $\big|\theta_{\gamma^p}(d_{i,k}) - \varphi^p_{\sigma(k)}\big| > \frac{1}{2}\varepsilon^p$ or $d_{i,k} = 1$.*

(c) *If (19) below is satisfied, then for each $i \in [l]$ and $p \in \overline{U}_i$ there exist $k, k' \in [n]$ such that $\theta_{\gamma^p}(t) > \varphi^p_+$ for all $t \in I_{i,k}$ and $\theta_{\gamma^p}(t) < \varphi^p_-$ for all $t \in I_{i,k'}$.*

Remark. The purpose of part (a) is to guarantee that when $\gamma^p|_{I_{i,k}}$ is stretched for $p \in U_i \cap U_{i'}$, the "stretchability" of $\gamma^p|_{I_{i',k}}$ will not be affected. By (2.2) (d), this can be arranged simply by stretching these arcs successively for each $i = 1, \dots, l$. Similarly, part (b) will be used to ensure that stretching γ^p will not affect its property of being quasicritical of type τ for another sign string τ. Part (c) means that for diffuse curves one may always choose intervals $I_{i,k}$ exhibiting this property.

Proof. Let U_i be open sets as in (3.6), with associated stretchable intervals $I_k(U_i) := I_{i,k} \subset J_k(p)$, $k \in [n]$. We shall write $U_i \preccurlyeq U_{i'}$ if $\overline{U}_i \cap \overline{U}_{i'} = \emptyset$ or if $\overline{U}_i \cap \overline{U}_{i'} \neq \emptyset$ and for every $k \in [n]$, either $I_k(U_i) \subset I_k(U_{i'})$ or $I_k(U_i) \cap I_k(U_{i'}) = \emptyset$; it is not required that the same option hold for every k. (This is generally not a transitive relation.) The complement of a set W in K will be denoted by W^c. The rough idea behind the proof is to repeatedly apply the following procedure: If $\overline{U}_{i_1} \cap \cdots \cap \overline{U}_{i_\mu}$ is nonempty, then we excise it from each of the open sets U_{i_j} and add a new open set V to the cover which contains the intersection but is still sufficiently small. If $I_k(V)$ is taken to be a component of $\bigcup_j I_k(U_{i_j})$ for each k, then $U_i \preccurlyeq V$ for every $i = i_1, \dots, i_\mu$.

Let m be the largest integer for which there exist distinct $i_1, \dots, i_m \in [l]$ with $\overline{U}_{i_1} \cap \cdots \cap \overline{U}_{i_m} \neq \emptyset$. Note that there are only finitely many such m-tuples. Choose one of them, say $T_m = \{i_1, \dots, i_m\}$, and let V_{T_m} be an open set such that

$$\bigcap_{\mu=1}^{m} \overline{U}_{i_\mu} \subset V_{T_m} \subset \overline{V}_{T_m} \subset \bigcap_{i \neq i_j} \overline{U}_i^c.$$

Such a set exists because $\overline{U}_{i_1} \cap \cdots \cap \overline{U}_{i_m} \subset \overline{U}_i^c$ for every $i \neq i_j$, by maximality of m. Set

$$(\text{new})\, U_{i_j} := (\text{old})\, U_{i_j} \smallsetminus \bigcap_{\mu=1}^{m} \overline{U}_{i_\mu} \quad (j \in [m]).$$

For each $k \in [n]$, take $I_k\big(U_{i_j}\big)$ to be the same intervals as for the original sets U_{i_j} and $I_k\big(V_{T_m}\big)$ to be any connected component of $\bigcup_{j=1}^m I_k\big(U_{i_j}\big)$. Fix $k \in [n]$; if $p \in \bigcap_{\mu=1}^m \overline{U}_{i_\mu}$, then every interval $I_k(U_{i_j})$ ($j \in [m]$) satisfies the conditions stated in (3.6) (b). Therefore, by (2.2) (d), if V_{T_m} is sufficiently small, then $I_k\big(V_{T_m}\big)$ satisfies these conditions for all $p \in \overline{V}_{T_m}$. Further, by construction $I_k\big(V_{T_m}\big)$ either contains or is disjoint from $I_k(U_{i_j})$ for each j, k. Thus:

- The open sets U_i ($i \in [l]$) and V_{T_m} cover K.
- If $\overline{U}_i \cap \overline{V}_{T_m} \neq \emptyset$ then $i = i_j$ for some j. Hence, $U_i \preccurlyeq V_{T_m}$ for every $i \in [l]$.
- No new m-fold intersection has been created among the $\overline{U}_i$.

If there still exists an m-tuple $T'_m = \{i'_1, \dots, i'_m\}$ such that $\overline{U}_{i'_1} \cap \cdots \cap \overline{U}_{i'_m} \neq \emptyset$, the construction is repeated to excise the latter from each $U_{i'_j}$ and create an open set $V_{T'_m}$ such that

$$\bigcap_{\mu=1}^{m} \overline{U}_{i'_\mu} \subset V_{T'_m} \subset \overline{V}_{T'_m} \subset \bigcap_{i \neq i'_j} \overline{U}_{i'}^c \cap \big(\overline{V}_{T_m}\big)^c.$$

Such a set exists because there are no $(m+1)$-fold intersections among the $\overline{U}_i$ and $i'_j \notin \{i_1, \dots, i_m\}$ for at least one $j \in [m]$. By definition, $\overline{V}_{T_m} \cap \overline{V}_{T'_m} = \emptyset$, and $\overline{V}_{T'_m} \cap \overline{U}_i = \emptyset$ unless $i = i'_j$ for some $j \in [m]$. Again, let $I_k\big(V_{T'_m}\big)$ be a connected component of $\bigcup_{j=1}^m I_k\big(U_{i'_j}\big)$ for each k, so that $U_i \preccurlyeq V_{T'_m}$ for all $i \in [l]$. If $V_{T'_m}$ is sufficiently small, then all of the conditions in (b) are satisfied by the $I_k\big(V_{T'_m}\big)$ whenever $p \in \overline{V}_{T'_m}$. After finitely many iterations, there will be no more m-tuples of indices in $[l]$ for which the corresponding $\overline{U}_i$ intersect. Notice that by construction:

- $V_{T_m} \preccurlyeq V_{T'_m}$ for any $T_m \neq T'_m$, since their closures are disjoint.
- $U_i \preccurlyeq V_{T_m}$ for any T_m and $i \in [l]$.
- Every m-fold intersection among the $\overline{U}_i$ is empty.

Now the same procedure is carried out for $(m-1)$-fold intersections among the $\overline{U}_i$. Assume that $\overline{V}_{T^\nu_{m-1}}$ has been defined for all $\nu = 1, \dots, \nu_0 - 1$, where each $T^\nu_{m-1} \subset [l]$ has cardinality $m-1$, with $\overline{U}_i \cap \overline{V}_{T^\nu_{m-1}} \neq \emptyset$ only if $i \in T^\nu_{m-1}$. If $T^{\nu_0}_{m-1} = \{i_1, \dots, i_{m-1}\}$ is such that $\overline{U}_{i_1} \cap \cdots \cap \overline{U}_{i_{m-1}} \neq \emptyset$, choose a sufficiently small open set $V_{T^{\nu_0}_{m-1}}$ satisfying

$$\bigcap_{\mu=1}^{m-1} \overline{U}_{i_\mu} \subset V_{T^{\nu_0}_{m-1}} \subset \overline{V}_{T^{\nu_0}_{m-1}} \subset \bigcap_{i \neq i_j} \overline{U}_i^c \cap \bigcap_{\nu=1}^{\nu_0 - 1} \big(\overline{V}_{T^\nu_{m-1}}\big)^c,$$

excise $\bigcap_{\mu=1}^{m-1} \overline{U}_{i_\mu}$ from each U_{i_j} and let $I_k\big(V_{T^{\nu_0}_{m-1}}\big)$ be a connected component of $\bigcup_{j=1}^{m-1} I_k\big(U_{i_j}\big)$. The choice of $V_{T^{\nu_0}_{m-1}}$ is possible because by hypothesis there are no m-fold intersections among the $\overline{U}_i$

and for each $\nu \le \nu_0 - 1$, we have $i_j \notin T^{\nu}_{m-1}$ for at least one $j \in [m-1]$. At the end of this step we have sets U_i and V_T (with $|T| = m-1$ or m) covering K such that:

- $V_T \preccurlyeq V_{T'}$ whenever $|T| \le |T'|$.
- $U_i \preccurlyeq V_T$ for every $i \in [l]$ and every set V_T.
- There exists no nonempty $(m-1)$-fold intersection among the $\overline{U}_i$.

Continuing this down to twofold intersections, we obtain open sets V_T and U_i with $|T| = 2$ and $\overline{U}_i \cap \overline{U}_{i'} = \emptyset$ whenever $i \neq i'$. Finally, for each $i \in [l]$, set $V_{\{i\}} = U_i$. Then the sets V_T form an open cover of K and $V_T \preccurlyeq V_{T'}$ whenever $|T| \le |T'|$. To establish (a) we simply relabel the V_T in order of nondecreasing $|T|$, for $|T| = 1, \dots, m$.

By (2.2) (d), the original intervals $I_{i,k}$ given by (3.6) can always be enlarged so as to satisfy the condition on the enpoints stated in (b). Furthermore, if intervals $J_1, \dots, J_m$ satisfy (b), then so does any component of $\bigcup_{j=1}^m J_j$. Hence the proof of (a) preserves this property.

Similarly, suppose that (19) is satisfied, and let $t_0, t_0' \in [0,1]$ be such that $\theta_{\gamma^p}(t_0) > \varphi_+^p$ and $\theta_{\gamma^p}(t_0') < \varphi_-^p$. If $I = [c,d]$ is an interval containing t_0 such that $\theta_{\gamma^p}(c) = \varphi_+^p$ or $\theta_{\gamma^p}(d) = \varphi_+^p$, then I is automatically stretchable by (2.2) (a). Since being stretchable is an open condition, by reducing I it can actually be assumed that $\theta_{\gamma^p}(t) > \varphi_+^p$ for all $t \in I$ and q in a neighborhood of p, and similarly for $I' \ni t_0'$ and φ_-^p. This shows that the original intervals given by (3.6) may always be chosen to satisfy (c) (and (b) as well). Again, this property is compatible with the proof of (a) since if $J_1, \dots, J_m$ satisfy it, then so does any component of $\bigcup_{j=1}^m J_j$. □

(3.8) Lemma. *Let $\sigma_1 \prec \cdots \prec \sigma_m$ be sign strings and $\gamma \in \mathcal{M}(Q)$ be (φ, ε_j)-quasicritical of type σ_j for each $j \in [m]$. Then $\varepsilon_{j+1} > 2\varepsilon_j$ for each $j \in [m-1]$.*

Proof. Clearly, the lemma can be deduced from the special case where $m = 2$. Let $n = |\sigma_2|$, $l = |\sigma_1|$ and let $J_1 < \cdots < J_n$, $J_1' < \cdots < J_l'$ be intervals as in (3.1) for $(\sigma, \varepsilon) = (\sigma_2, \varepsilon_2)$ and $(\sigma_1, \varepsilon_1)$, respectively. For each $k \in [n]$, let $I_k \subset J_k$ be a subinterval where $|\theta_\gamma - \varphi| > \frac{\pi}{2} - \varepsilon_2$ throughout, as guaranteed by (iii). By (ii), if $t \notin \bigcup_{i \in [l]} J_i'$, then $|\theta_\gamma(t) - \varphi| \le \frac{\pi}{2} - 2\varepsilon_1$. Therefore, if $\varepsilon_2 \le 2\varepsilon_1$, then each I_k must be contained in a J_i'. Further, because $n > l$, there must exist $k \in [n-1]$, $i \in [l]$ such that $I_k \cup I_{k+1} \subset J_i'$. From $\sigma_2(k) = -\sigma_2(k+1)$ it follows that

$$\theta_\gamma(J_i') \cap \big(\varphi_+ - \varepsilon_2, \varphi_+ + \varepsilon_2\big) \neq \emptyset \quad \text{and} \quad \theta_\gamma(J_i') \cap \big(\varphi_- - \varepsilon_2, \varphi_- + \varepsilon_2\big) \neq \emptyset.$$

But this contradicts (i) of (3.1) (for $\sigma = \sigma_1$). Hence, $\varepsilon_2 > 2\varepsilon_1$. □

(3.9) Lemma. *Let σ be a sign string, $0 < \varepsilon < \varepsilon'$ and suppose that $\gamma \in \mathcal{M}(Q)$ is simultaneously (φ, ε)- and (φ, ε')-quasicritical of type σ. Then γ is (φ, δ)-quasicritical of type σ for any $\delta \in [\varepsilon, \varepsilon']$.*

Proof. Let $n = |\sigma|$ and $J_1 < \cdots < J_n$, $J_1' < \cdots < J_n'$ be as in (3.1), corresponding to $\varepsilon, \varepsilon'$, respectively. The inequalities $\varepsilon \le \delta \le \varepsilon'$ and (3.2) imply that the intervals J_k' still satisfy (i) and (ii) of (3.1) if ε' is replaced by δ. An argument similar to the proof of (3.5) (a) shows that if $I_k \subset J_k$ is any subinterval where $|\theta_\gamma - \varphi| > \frac{\pi}{2} - \varepsilon \ge \frac{\pi}{2} - \delta$ throughout, then $I_k \subset J_k'$. By (iii), for each $k \in [n]$, there exists such an I_k which, additionally, is stretchable. Hence the J_k' also satisfy (iii) if ε' is replaced by δ. □

(3.10) Remark. Let $0 < \delta \le \varepsilon$, γ be (φ, ε)-quasicritical of type σ, and J_k ($k \in [n]$) be intervals as in (3.1) for the pair (φ, ε). Suppose that $\theta_\gamma([0,1]) \subset (\varphi_- - \delta, \varphi_+ + \delta)$ and that each J_k contains a stretchable subinterval I_k where $\big|\theta_\gamma - \varphi_{\sigma(k)}\big| < \delta$ throughout. Then the J_k also satisfy (i)–(iii) of (3.1) for the pair (φ, δ), hence γ is (φ, δ)-quasicritical of type σ

(3.11) Lemma. *Let $\gamma \in \mathcal{M}(Q)$ be a critical curve of type σ. Let*

$$S = \big\{\varphi \in \mathbf{R} : \text{there exists } \varepsilon > 0 \text{ for which } \gamma \text{ is } (\varphi, \varepsilon)\text{-quasicritical of type } \sigma\big\}.$$

Then S is an open interval containing $\bar{\varphi}^\gamma$.

Proof. Let $\bar{\varphi} = \bar{\varphi}^\gamma$ be as in (12). By (3.3), S is open and by (3.4), $\bar{\varphi} \in S$. Suppose that γ is (φ, ε)-quasicritical of type σ; no generality is lost in assuming that $\bar{\varphi} \le \varphi$. Since γ is critical, $\inf_{t \in [0,1]} \theta_\gamma(t) = \bar{\varphi}_-$. Hence, by (3.2),

$$\varepsilon > \varphi - \bar{\varphi}. \tag{15}$$

Let $\psi \in [\bar{\varphi}, \varphi]$, $\delta = \varepsilon - (\varphi - \psi)$ and let $J_1 < \cdots < J_n$ be as in (3.1) for the pair (φ, ε). We claim that these intervals also satisfy (i)–(iii) for the pair (ψ, δ).

Notice that $\theta_\gamma([0,1]) = [\bar{\varphi}_-, \bar{\varphi}_+] \subset (\psi_- - \delta, \psi_+ + \delta)$, as a consequence of (15). It is easy to check that

$$\psi_+ - 2\delta > \varphi_+ - 2\varepsilon \quad \text{and} \quad \psi_- + 2\delta < \varphi_- + 2\varepsilon.$$

Consequently, the J_k satisfy (i), (ii) of (3.1) for the pair (ψ, δ).

Let $t_1 < \cdots < t_n$ be such that $\theta_\gamma(t_k) = \bar{\varphi}_{\sigma(k)}$. By (15), each t_k must be contained in an interval $J_{k'}$ with $\sigma(k') = \sigma(k)$. Therefore, no two of the t_k can be contained in the same J, so that $t_k \in J_k$ for all $k \in [n]$. Since $\bar{\varphi}_- \leq \psi_-$, if $\sigma(k) = -$, then J_k must contain some t such that $\theta_\gamma(t) = \psi_-$. In particular, by (2.2) (a), condition (iii) of (3.1) is satisfied by J_k for the pair (ψ, δ) whenever $\sigma(k) = -$. If $\sigma(k) = +$, let $I \subset J_k$ be an interval as in (iii) for the pair (φ, ε). By (2.2) (b), this interval is also stretchable with respect to ψ_+. Moreover,

$$\psi_+ - \delta = \varphi_+ - \varepsilon < \theta_\gamma(t) \leq \bar{\varphi}_+ < \psi_+ + \delta \quad \text{for all } t \in I;$$

hence J_k also satisfies (iii) for the pair (ψ, δ) in case $\sigma(k) = +$. □

(3.12) Definition. Let $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$, $z \neq -1$. Let $R(Q)$ denote the open interval of size $\pi - |\theta_1|$ centered at $\frac{\theta_1}{2}$, where $e^{i\theta_1} = z$ and $|\theta_1| < \pi$. Let $\mathcal{U}_c$, $\mathcal{U}_d$ be the open subsets of $\mathcal{M}(Q)$ consisting of all all condensed (resp. diffuse) curves. Define

$$\mathcal{V}_d := \mathcal{U}_d \times R(Q);$$
$$\mathcal{V}_c := \big\{(\gamma, \varphi) \in \mathcal{M}(Q) \times R(Q) : \varphi_- < \inf_{t\in[0,1]} \theta_\gamma(t) < \sup_{t\in[0,1]} \theta_\gamma(t) < \varphi_+\big\}.$$

If $\mathcal{M}(Q)$ does not contain critical curves of type σ, set $\mathcal{V}_\sigma := \emptyset$. Otherwise, define

$$\mathcal{V}_\sigma := \big\{(\gamma, \varphi) \in \mathcal{M}(Q) \times R(Q) : \gamma \text{ is } (\varphi, \varepsilon)\text{-quasicritical of type } \sigma \text{ for some } \varepsilon \in (0, \tfrac{\pi}{4})\big\}.$$

The union of $\mathcal{V}_c$, $\mathcal{V}_d$ and all the $\mathcal{V}_\sigma$ will be denoted by $\mathcal{N}(Q)$, and the cover of $\mathcal{N}(Q)$ by these sets will be denoted by $\mathfrak{V}$. Note that each $\mathcal{V}_*$ is an open subset of $\mathcal{M}(Q) \times \mathbf{R}$, hence so is $\mathcal{N}(Q)$. For sign strings $\sigma_1 \prec \cdots \prec \sigma_m$, the intersection $\mathcal{V}_{\sigma_1} \cap \cdots \cap \mathcal{V}_{\sigma_m}$ will be denoted by $\mathcal{V}_{(\sigma_1,\ldots,\sigma_m)}$. Similarly, $\mathcal{V}_{(c,\sigma_1,\ldots,\sigma_m)} := \mathcal{V}_c \cap \mathcal{V}_{(\sigma_1,\ldots,\sigma_m)}$ and $\mathcal{V}_{(d,\sigma_1,\ldots,\sigma_m)} := \mathcal{V}_d \cap \mathcal{V}_{(\sigma_1,\ldots,\sigma_m)}$.

Remark. Observe that $R(Q) = \big(\theta_1 - \frac{\pi}{2}, \frac{\pi}{2}\big)$ if $\theta_1 \geq 0$ and $R(Q) = \big(-\frac{\pi}{2}, \theta_1 + \frac{\pi}{2}\big)$ if $\theta_1 \leq 0$. In either case, it consists of all $\varphi \in \mathbf{R}$ such that $\varphi_- < 0\,, \theta_1 < \varphi_+$.

(3.13) Lemma. *Let* $\mathrm{pr}\colon \mathcal{N}(Q) \to \mathcal{M}(Q)$ *be the restriction of the projection* $\mathcal{M}(Q) \times \mathbf{R} \to \mathcal{M}(Q)$. *Let* K *be any compact space and* $g\colon K \to \mathcal{M}(Q)$ *a continuous map. Then there exists* $\tilde{g}\colon K \to \mathcal{N}(Q)$ *such that* $\mathrm{pr} \circ \tilde{g} = g$.

Proof. Let $g\colon p \mapsto \gamma^p \in \mathcal{M}(Q)$ and $\bar{\varphi}^p := \bar{\varphi}^{\gamma^p}$, as in (12). Let $\omega(p)$ denote the amplitude of γ^p. Since $\frac{\theta_1}{2}$ always lies in $R(Q)$, if γ^p is diffuse, then $\big(\gamma^p, \frac{\theta_1}{2}\big) \in \mathcal{V}_d$. Similarly, if γ^p is condensed, then $\bar{\varphi}^p \in R(Q)$ and $(\gamma^p, \bar{\varphi}^p) \in \mathcal{V}_c$. Finally, if γ^p is critical, then $\bar{\varphi}^p \in \overline{R(Q)}$.

Using (3.11) and compactness of K, choose $s_0 \in (0,1]$ and $\delta > 0$ so small that:

- γ^p is (ψ, ε)-quasicritical of type σ (for some σ and $\varepsilon > 0$, whose values are irrelevant) for $\psi = (1 - s_0)\bar{\varphi}^p + s_0\frac{\theta_1}{2}$ whenever $|\omega(p) - \pi| \leq 2\delta$;
- $(\gamma^p, \psi) \in \mathcal{V}_d$ for $\psi = (1 - s)\bar{\varphi}^p + s\frac{\theta_1}{2}$, whenever $s \in [s_0, 1]$ and $\pi + \delta \leq \omega(p) \leq \pi + 2\delta$;
- $(\gamma^p, \psi) \in \mathcal{V}_c$ for $\psi = (1 - s)\bar{\varphi}^p + s\frac{\theta_1}{2}$, whenever $s \in [0, s_0]$ and $\pi - 2\delta \leq \omega(p) \leq \pi - \delta$.

Let $s\colon \mathbf{R} \to [0,1]$ be an increasing continuous function satisfying:

$$s(u) = \begin{cases} s_0 & \text{if } |u - \pi| \leq \delta; \\ 1 & \text{if } u \geq \pi + 2\delta; \\ 0 & \text{if } u \leq \pi - 2\delta; \end{cases}$$

and set $\varphi^p := [1 - s(\omega(p))]\bar{\varphi}^p + s(\omega(p))\frac{\theta_1}{2}$. Then $\tilde{g}(p) = (\gamma^p, \varphi^p) \in \mathcal{N}(Q)$ for all $p \in K$. □

(3.14) Corollary. *If* $\mathcal{N}(Q)$ *is contractible, then so is* $\mathcal{M}(Q)$.

Proof. Indeed, $\mathrm{pr}\colon \mathcal{N}(Q) \to \mathcal{M}(Q)$ induces surjections on homotopy groups and a weakly contractible Hilbert manifold is contractible. □

(3.15) Lemma. *Let $p\colon X \to Y$ be a continuous map between topological spaces. Suppose that $X \simeq \mathbf{S}^n$ for some $n \in \mathbf{N}$ and that given any compact space K and any map $g\colon K \to Y$, there exists $\tilde{g}\colon K \to X$ such that $p\tilde{g} = g$. Then Y is either weakly contractible or a homology n-sphere.*

Proof. The hypothesis immediately implies that Y is a Moore space $M(\mathbf{Z}/(k), n)$ for some $k \in \mathbf{N}$. Let K be a CW complex obtained by attaching an $(n+1)$-cell to $\mathbf{S}^n$ via a map of degree k. Let $g\colon K \to Y$ be such that $g_*\colon H_*(K) \to H_*(Y)$ is an isomorphism. By hypothesis, g factors through X. Since $H_n(X) \simeq \mathbf{Z}$, this implies that either $k = 0$ or $k = 1$. □

The homotopy type of $\mathcal{M}(Q)$ will be determined as follows. If $\mathcal{M}(Q)$ contains no critical curves, then it is homotopy equivalent to $\mathbf{S}^0$ by the results of [4]. Otherwise, let n denote the greatest length $|\sigma|$ among those sign strings σ for which $\mathcal{V}_\sigma \neq \emptyset$. In §4 the cover $\mathfrak{V}$ will be shown to have the same combinatorics as that in (1), and in §5 it will be shown that $\mathfrak{V}$ is a good cover of $\mathcal{N}(Q)$. Then (3.15) will imply that either $\mathcal{M}(Q)$ is contractible or it has the homotopy type of $\mathbf{S}^{n-1}$. Finally, if $\mathcal{N}(Q) \simeq \mathbf{S}^{n-1}$, then $\mathcal{M}(Q) \simeq \mathbf{S}^{n-1}$ as well, because in this case a non-nullhomotopic map $\mathbf{S}^{n-1} \to \mathcal{M}(Q)$ can be constructed explicitly; this is done in §6.

(3.16) Lemma. *Let $\sigma_1 \prec \cdots \prec \sigma_m$ be sign strings and $f\colon K \to \mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$, $p \mapsto (\gamma^p, \varphi^p)$, be a continuous map. Then there exist continuous $\varepsilon_j\colon K \to \mathbf{R}^+$, $p \mapsto \varepsilon_j^p$, such that for each $p \in K$, γ^p is $(\varphi^p, \varepsilon_j^p)$-quasicritical of type σ_j. Moreover, $\varepsilon_{j+1} > 2\varepsilon_j$ for each $j \in [m-1]$ throughout K.*

Proof. By (3.3), such functions can be defined on a neighborhood of every $p \in K$. Globally defined $\varepsilon_j\colon K \to \mathbf{R}^+$ ($j \in [m]$) are obtained by using partitions of unity; this works because of (3.9). The last assertion is an immediate consequence of (3.8). □

(3.17) Definition. Let $(\gamma, \varphi) \in \mathcal{V}_\sigma$, $n = |\sigma|$, and let J_k ($k \in [n]$) be intervals satisfying the conditions in (3.1) for some $\varepsilon \in (0, \frac{\pi}{4})$. Define $h\colon \mathcal{V}_\sigma \to \mathbf{R}^n$ by:

$$h_k(\gamma,\varphi) = \begin{cases} \sup_{t\in J_k}\{\theta_\gamma(t) - \varphi_+\} & \text{if } \sigma(k) = +; \\ \inf_{t\in J_k}\{\theta_\gamma(t) - \varphi_-\} & \text{if } \sigma(k) = -; \end{cases} \quad (k \in [n]). \tag{16}$$

(3.18) Remark. Even though ε and the J_k are not uniquely determined, (3.5) (a) implies that h is well-defined. Furthermore, it is continuous. Indeed, by (3.3), for (η, ψ) sufficiently close to (γ, φ), we may choose the same intervals J_k in (3.1) for (η, ψ) as for (γ, φ); but for fixed $J_k \subset [0, 1]$, it is clear that (16) depends continuously upon (γ, φ).

Given intervals $I_1, \dots, I_n$, let $I_1 * \cdots * I_n$ denote the smallest closed interval containing $I_1 \cup \cdots \cup I_n$.

(3.19) Lemma. *Let $\sigma_1 \prec \sigma_2$ be sign strings and suppose that $\gamma \in \mathcal{M}(Q)$ is (φ, ε_j)-quasicritical of type σ_j, $j = 1, 2$. Let $|\sigma_1| = l$, $|\sigma_2| = n$ and $J_1 < \cdots < J_n$ be intervals as in (3.1) for the pair $(\sigma_2, \varepsilon_2)$. Then there exist intervals $J'_1 < \cdots < J'_l$ satisfying (3.1) for $(\sigma_1, \varepsilon_1)$ such that:*

(a) *Each J'_i has the form $J_k * J_{k'}$, for some $k \le k' \in [n]$ depending on $i \in [l]$.*
(b) *If $k \in [n]$ is such that $|h_k(\gamma, \varphi)| \le 2\varepsilon_1$, then $J_k \subset J'_i$ for some $i \in [l]$.*
(c) *For each $i \in [l]$, there exists $k \in [n]$ such that $|h_k(\gamma, \varphi)| < \varepsilon_1$ and $J_k \subset J'_i$.*

Proof. Let $k_1 < \cdots < k_m$ be all the indices $k \in [n]$ such that $|h_k(\gamma, \varphi)| \le 2\varepsilon_1$. Define $\tau\colon [m] \to \{\pm\}$ by $\tau(j) = \sigma_2(k_j)$. For each $j \in [m]$, choose $t_{k_j} \in J_j$ such that $\theta_\gamma(t_{k_j}) = \varphi_{\sigma_2(k_j)} + h_{k_j}(\gamma, \varphi)$. Let $J''_1 < \cdots < J''_l$ be any intervals as in (3.1) for the pair $(\sigma_1, \varepsilon_1)$. Then:

- Each t_{k_j} must be contained in some J''_i with $\sigma_2(k_j) = \sigma_1(i)$. This follows immediately from condition (ii) of (3.1) for the pair $(\sigma_1, \varepsilon_1)$.
- For each $i \in [l]$, J''_i must contain one of the t_{k_j}. Indeed, by (iii) of (3.1), for any i there exists $s_i \in J''_i$ such that $|\theta_\gamma(s_i) - \varphi_{\sigma_1(i)}| < \varepsilon_1$. By (3.8), $2\varepsilon_1 < \varepsilon_2$, hence $s_i \in J_k$ for some k, which forces $|h_k(\gamma, \varphi)| < \varepsilon_1$. Therefore $k = k_j$ for some j, and it follows that t_{k_j} must be contained in J''_i.

Let ϱ be the reduced string of τ. The first assertion implies that ϱ is a substring of σ_1, while the second one implies that it cannot be a proper substring. Consequently $\varrho = \sigma_1$.

Thus, there exists a decomposition of $\{k_1, \dots, k_m\}$ as the disjoint union of nonempty sets $S_1 < \cdots < S_l$ with $\sigma_2(k) = \sigma_1(i)$ whenever $k \in S_i$. Set $J'_i = \ast_{k\in S_i} J_k$. Then $J'_1 < \cdots < J'_l$, and parts (a) and (b) hold by construction. Moreover, $|\theta_\gamma(t) - \varphi| < \frac{\pi}{2} - 2\varepsilon_1$ if $t \notin \operatorname{Int}\big(\bigcup_i J'_i\big)$: If $t \notin \bigcup_k J_k$, then this is obvious from (ii) of (3.1), since $\varepsilon_2 > 2\varepsilon_1$ by (3.8); if $t \in J_k$ for some k, then necessarily $|h_k(\gamma,\varphi)| > 2\varepsilon_1$, hence again the inequality holds. This proves that condition (ii) of (3.1) is satisfied by the J_i'. Condition (i) is also easily verified using that $\varepsilon_2 > 2\varepsilon_1$.

Since γ is (φ,ε_1)-quasicritical, there exist intervals $I_1 < \dots < I_l$ such that I_i is stretchable and

$$\left|\theta_\gamma(t) - \varphi_{\sigma_1(i)}\right| < \varepsilon_1 \quad \text{for all } t \in I_i \text{ and } i \in [l]. \tag{17}$$

The inequality implies that each of these intervals must be contained in some J', and no two subsequent intervals may be contained in the same J'. Hence $I_i \subset J_i'$ for each $i \in [l]$. This proves that condition (iii) of (3.1) is satisfied by the J'. Since $\varepsilon_1 < \varepsilon_2$, (17) also implies that each I_i must be contained in some J_k with $|h_k(\gamma,\varphi)| < \varepsilon_1$, so that $J_k \subset J_i'$ by the definition of the J'. This proves part (c). □

4. Incidence data of the cover of $\mathcal{N}(Q)$

Good covers of Hilbert manifolds. An open cover $\mathfrak{U} = (U_\nu)_{\nu\in I}$ of a space is *good* if for any finite $J \subset I$, the intersection $\bigcap_{\nu\in J} U_\alpha$ is either empty or contractible. Let $\mathfrak{V} = (V_\nu)_{\nu\in I}$ be a good cover of another space, indexed by the same set I. Then $\mathfrak{U}$ and $\mathfrak{V}$ will be called (combinatorially) *equivalent* when for any finite $J \subset I$, $\bigcap_{\nu\in J} U_\nu = \emptyset$ if and only if $\bigcap_{\nu\in J} V_\nu = \emptyset$. Recall that the *nerve* $K_{\mathfrak{U}}$ of an open cover $\mathfrak{U}$ of a space is a simplicial complex whose n-simplices correspond bijectively to the nonempty $(n+1)$-fold intersections of distinct elements of $\mathfrak{U}$, for each $n \in \mathbf{N}$.

(4.1) Lemma. *If $\mathfrak{U}$ is a good cover of a paracompact space X, then X is homotopy equivalent to the nerve $K_{\mathfrak{U}}$.*

Proof. See [2], Corollary 4G.3 or [5], p. 141. □

Because the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(P,Q)$ are closed submanifolds of the separable Hilbert space $\mathbf{E}$ (see Definition 1.6 of [4]), they are second-countable and metrizable. It follows that they are also paracompact. It will be tacitly assumed below that all Hilbert manifolds are separable and metrizable.

(4.2) Corollary. *If two Hilbert manifolds $\mathcal{M}$ and $\mathcal{N}$ admit equivalent good covers, then $\mathcal{M} \approx \mathcal{N}$.*

Proof. Let $\mathfrak{U}$ and $\mathfrak{V}$ be equivalent good covers of $\mathcal{M}$ and $\mathcal{N}$, respectively. Let K be the nerve of $\mathfrak{U}$, which is homeomorphic to the nerve of $\mathfrak{V}$ by hypothesis. By (4.1), there exist homotopy equivalences $\mathcal{M} \to K$ and $K \to \mathcal{N}$. The corollary thus follows from the fact that a homotopy equivalence between two Hilbert manifolds is homotopic to a homeomorphism, see [4], Lemma 1.7(b). □

(4.3) Corollary. *If a Hilbert manifold $\mathcal{M}$ and a finite-dimensional manifold N admit equivalent good covers, then $\mathcal{M} \approx \mathbf{E} \times N$.* □

Incidence data of the cover of $\mathcal{N}(Q)$. The purpose of this subsection is to determine which of the open sets $\mathcal{V}_* \subset \mathcal{N}(Q)$ described in (3.12) intersect each other.

(4.4) Lemma. *Suppose that $\gamma \in \mathcal{M}(Q)$ is simultaneously (φ,ε)-quasicritical of type σ and (φ,ε')-quasicritical of type σ', for some $\varphi \in \mathbf{R}$, $\varepsilon,\varepsilon' \in (0,\frac{\pi}{4})$ and sign strings σ,σ'. Then $\sigma' \neq -\sigma$.*

Proof. No generality is lost in assuming that $\varepsilon \le \varepsilon'$. Let $n = |\sigma|$, $l = |\sigma'|$ and $J_1 < \dots < J_n$, $J_1' < \dots < J_l'$ be intervals as in (3.1), for the pairs (σ,ε) and (σ',ε'), respectively. For each $k \in [n]$, choose an interval $I_k \subset J_k$ such that

$$\theta_\gamma(I_k) \subset (\varphi_{\sigma(k)} - \varepsilon, \varphi_{\sigma(k)} + \varepsilon).$$

Then for each $k \in [n]$, I_k must be contained in some J_i' with $\sigma(k) = \sigma'(i)$. In particular, I_k and I_{k+1} are not contained in the same J' for any k. Therefore, either $l > n$ or $l = n$ and $\sigma' = \sigma$. □

(4.5) Lemma. *Let σ_k $(2 \le k \le n)$ be sign strings satisfying $|\sigma_k| = k$. Then there exist intervals $R_2 \subset \dots \subset R_n = [n]$, $|R_k| = k$, such that for each $k = 2,\dots,n$, if $R_k = \{r_1 < \dots < r_k\}$, then $\sigma_n(r_i) = \sigma_k(i)$ for all $i \in [k]$.* □

In words, we can find nested copies of each σ_k inside of σ_n by an appropriate choice of the R_k. The proof is an easy induction which will be left to the reader.

(4.6) Lemma. *Let $\kappa_1 \in (0,1)$. Suppose that $\alpha \in \mathcal{L}_{-1}^{+1}(P,Q)$ is condensed, $\mathbf{t}_\alpha(0) = \mathbf{t}_\alpha(1)$ and $\kappa_\alpha([0,1]) \subset [-\kappa_1 + \kappa_1]$, but α is not a line segment. Then for all sufficiently small $\varepsilon > 0$, there exists a homotopy $s \mapsto \alpha_s \in \mathcal{L}_{-1}^{+1}(P,Q)$ $(s \in [0,1])$ with $\alpha_1 = \alpha$ and $\omega(\alpha_1) - \omega(\alpha_0) = \varepsilon$.*[†]

Proof. Let $\kappa_0 \in (\kappa_1, 1)$ and H be as in Proposition 3.4 of [4]. Then $u \mapsto \alpha_u = H(u,\alpha)$ $(u \in [0,1])$, the flattening of $\alpha = \alpha_1$ with curvature κ_0, is a deformation within $\mathcal{L}_{-1}^{+1}(P,Q)$ such that $\omega(\alpha_u)$ is an increasing function of u. Moreover, $\delta = \omega(\alpha_1) - \omega(\alpha_0) > 0$ by Lemma 3.16 of [4] and the hypotheses on α. Hence, for any $\varepsilon \in (0,\delta]$, there exists $u_0 \in [0,1]$ such that $\omega(\alpha_1) - \omega(\alpha_{u_0}) = \varepsilon$. □

(4.7) Lemma. *Let σ_k $(2 \le k \le n)$ be sign strings satisfying $|\sigma_k| = k$. Suppose that $\mathcal{M}(Q)$ contains critical curves of type σ_n. Then $\mathcal{V}_{(c,\sigma_2,\dots,\sigma_n)}$ and $\mathcal{V}_{(d,\sigma_2,\dots,\sigma_n)}$ are nonempty.*

Proof. Write $Q = (q,z) \in \mathbf{C} \times \mathbf{S}^1$. By Proposition 5.3 of [4], the region

$$R_{\sigma_n} = \big\{p \in \mathbf{C} : \mathcal{M}(P) \text{ contains critical curves of type } \sigma_n,\ P = (p,z)\big\}$$

is open in $\mathbf{C}$. Hence, there exists $\kappa_1 \in (0,1)$ such that $\kappa_1 q \in R_{\sigma_n}$. Let $\tilde{Q} = (\kappa_1 q, z)$. If $\tilde{\eta} \in \mathcal{M}(\tilde{Q})$ is a critical curve of type σ_n, then the dilated curve $\eta = \frac{1}{\kappa_1}\tilde{\eta}$ is a critical curve of type σ_n in $\mathcal{M}(Q)$ whose curvature takes values in $[-\kappa_1, +\kappa_1]$. A quasicritical curve of the required type can be obtained by modifying η in neighborhoods of the points $\eta(t)$ where $\theta_\eta(t) = \bar{\varphi}_\pm^\eta$.

By Corollary 5.4 of [4], the set of all $\varphi \in \mathbf{R}$ such that $\mathcal{M}(\tilde{Q})$ contains a critical curve $\tilde{\eta}$ with $\bar{\varphi}^{\tilde{\eta}} = \varphi$ is an open interval. Hence it may be assumed that $0, \theta_1 \in (\bar{\varphi}_-^\eta, \bar{\varphi}_+^\eta)$, so that

$$\mu = \min\big\{|\bar{\varphi}_\pm^\eta|,\ |\bar{\varphi}_\pm^\eta - \theta_1|, \tfrac{\pi}{4}\big\} > 0, \tag{18}$$

where $\theta_1 = \theta_\eta(1)$ is the unique number in $(-\pi,\pi)$ such that $e^{i\theta_1} = z$. Since

$$C = \big\{t \in [0,1] : \theta_\eta(t) = \bar{\varphi}_\pm^\eta\big\}$$

is compact, it intersects only finitely many components $V_1 < \dots < V_l$ of

$$V = \big\{t \in [0,1] : |\theta_\eta(t) - \varphi_\eta| > \tfrac{\pi}{2} - \mu\big\}.$$

Observe that V_i is an open subinterval of $(0,1)$ for each $i \in [l]$, and either the maximum or the minimum of $\theta_\eta|_{\overline{V}_i}$ is attained at both endpoints. Let

$$\lambda = \frac{\pi}{2} - \sup\big\{\, |\theta_\eta(t) - \bar{\varphi}^\eta| \ : t \notin \textstyle\bigcup_i V_i\big\} > 0.$$

By grafting η at points of C if necessary (see Definition 4.13 and Figure 9 of [4]), it may be assumed that for each i, $\eta|_{\overline{V}_i}$ contains a line segment of some larg length $L > 0$ where $\theta_\eta = \bar{\varphi}_\pm^\eta$.

Let $\alpha_i = \eta|_{\overline{V}_i}$. Then each α_i satisfies the hypothesis of (4.6). Hence there exists δ, $0 < 2\delta < \min\{\lambda,\mu\}$, such that for each $i \in [l]$, if $\varepsilon_i \le \delta$ then α_i can be deformed (keeping initial and final frames fixed) to a curve β_i such that $\omega(\alpha_i) - \omega(\beta_i) = \varepsilon_i$. We claim that an appropriate choice of the ε_i yields a curve of the required type.

Since η is a critical curve of type σ_n, there is a partition of $[l]$ into sets $A_1 < \dots < A_n$ such that for every $k \in [n]$ and $i \in A_k$, there exists $t \in V_i$ for which $\theta_\eta(t) = \bar{\varphi}_{\sigma_n(k)}^\eta$. In view of (4.5), no generality is lost in assuming that $\sigma_k = \sigma_n|_{[k]}$ for each $k = 2,\dots,n$. Set $\varepsilon_i = 0$ if $i \in A_1 \cup A_2$ and $\varepsilon_i = 4^{k-n}\delta$ if $i \in A_k$ for $k > 2$. Let γ be the curve which results by deforming each α_i to β_i, as described above. Notice that $\bar{\varphi}^\gamma = \bar{\varphi}^\eta$. Furthermore:

(a) γ is not condensed, because for every $i_1 \in A_1$, $i_2 \in A_2$, there exist $t_1 \in V_{i_1}$, $t_2 \in V_{i_2}$ such that $\theta_\gamma(t_i) = \bar{\varphi}_{\sigma_2(i)}^\gamma$ $(i = 1,2)$.

(b) γ is not diffuse, since η is not diffuse and each β_i was obtained from α_i by a deformation which decreases amplitude.

(c) γ is $\big(\bar{\varphi}^\gamma, 4^{k-n}\delta\big)$-quasicritical of type σ_k for each $k = 2,\dots,n$ by construction. Indeed, setting $J_k = \ast_{i \in A_k} V_i$ (the smallest closed subinterval containing these V_i) for each $k \in [n]$, it is straightforward to verify that $J_1 < \dots < J_k$ satisfy (i) and (ii) of (3.1) for $\varepsilon = 4^{k-n}\delta$ $(k \ge 2)$. Condition (iii) is a consequence of (2.2)(c) and our assumption that each arc $\eta|_{V_i}$ contains a line segment of some large length L where $\theta_\eta = \bar{\varphi}_\pm^\eta$.

[†]Recall that $\omega(\gamma) = \sup\theta_\gamma - \inf\theta_\gamma$ denotes the amplitude of γ.

Therefore, by (18), $(\gamma, \bar{\varphi}^\gamma) \in \mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$. By Proposition 5.1 of [4], the boundaries of $\mathcal{U}_c$ and $\mathcal{U}_d$ in $\mathcal{M}(Q)$ are both equal to the set of all critical curves in $\mathcal{M}(Q)$. Therefore, by (3.3), a slight perturbation of γ yields a curve $\tilde{\gamma}$ such that $(\tilde{\gamma}, \bar{\varphi}^\gamma) \in \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ or $(\tilde{\gamma}, \bar{\varphi}^\gamma) \in \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$. □

Let us say that τ is a *top* sign string for $\mathcal{M}(Q)$ if the latter contains critical curves of type τ, but does not contain critical curves of type τ' for any sign string τ' with $|\tau'| > |\tau|$. Set $n = |\tau|$. Proposition 5.3 of [4] determines whether $\mathcal{M}(Q)$ contains critical curves of type σ in terms of Q, for any sign string σ. Notice in particular that $\mathcal{M}(Q)$ always admits a top sign string τ, except in case it does not contain critical curves at all.

(4.8) Proposition. *Let τ be a top sign string for $\mathcal{M}(Q)$, $n = |\tau|$, $\mathfrak{V}$ be the cover of $\mathcal{N}(Q)$ described in (3.12) and $\mathfrak{U} = \{U_{\pm k}\}_{k\in[n]}$, where $U_{\pm k} \subset \mathbf{R}^n$ are as in* (1)*.*

(a) *If $\mathcal{M}(Q)$ contains critical curves of type $-\tau$, then* (2) *defines a combinatorial equivalence between $\mathfrak{V}$ and the cover $\mathfrak{U}$ of $\mathbf{R}^n \smallsetminus \{0\}$.*
(b) *If $\mathcal{M}(Q)$ does not contain critical curves of type $-\tau$, then* (2) *defines a combinatorial equivalence between $\mathfrak{V}$ and the cover $\mathfrak{U} \smallsetminus \{U_{-n}\}$ of $\mathbf{R}^n \smallsetminus \{(0,\dots,0,x_n) : x_n \leq 0\}$.*

Proof. It is clear that $\mathcal{V}_c \cap \mathcal{V}_d = \emptyset$, and by (4.4), $\mathcal{V}_\sigma \cap \mathcal{V}_{-\sigma} = \emptyset$ for any sign string σ. On the other hand, (4.7) implies that an intersection of nonempty sets in $\mathfrak{V}$ is empty only if it involves one such pair. The combinatorics of $\mathfrak{V}$ is thus the same as that of $\mathfrak{U}$, as asserted. □

5. Topology of the cover of $\mathcal{N}(Q)$

(5.1) Proposition. *Let $\sigma_1 \prec \cdots \prec \sigma_m$ be sign strings. Then the subspaces $\mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$, $\mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ and $\mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$ of $\mathcal{N}(Q)$ are either empty or contractible.*

Let $\mathcal{V}$ denote any of these subspaces. Since $\mathcal{V}$ is a Hilbert manifold, it suffices to prove that it is either empty or weakly contractible. Given a family $(\gamma^p, \varphi^p) \in \mathcal{V}$, for p ranging over a compact space, the idea is to stretch each γ^p in the direction of $\pm ie^{i\varphi^p}$ so that it becomes nearly critical (see Figure 10), and then flatten it piecewise to obtain a concatenation of circles and line segments of a special form (see Figure 11). The results of §1 are then used to conclude that the resulting family is contractible. The proof is quite technical since the conditions in (3.1) need to be verified at each step; it will be split into several lemmas. The first one reduces the case of $\mathcal{V} = \mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$ to that of $\mathcal{V} = \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$.

(5.2) Lemma. *Let $K \to \mathcal{V}$, $p \mapsto (\gamma_0^p, \varphi^p)$ be a continuous map, where $\mathcal{V} = \mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$ or $\mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$. Then there exists a homotopy $(s,p) \mapsto (\gamma_s^p, \varphi^p) \in \mathcal{V}$ such that $\gamma^p := \gamma_1^p$ satisfies*

$$\inf_{t\in[0,1]} \theta_{\gamma^p}(t) < \varphi_-^p \quad \text{and} \quad \varphi_+^p < \sup_{t\in[0,1]} \theta_{\gamma^p}(t) \quad \text{for all } p \in K. \tag{19}$$

Thus, by deforming γ_0^p they can be made not only diffuse but "diffuse with respect to φ^p".

Proof. Assume first that $\mathcal{V} = \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$ and that K consists of a single point p. Then there exist $t, t' \in [0,1]$ such that $\theta_{\gamma^p}(t') = \pi + \theta_{\gamma^p}(t)$. Define a homotopy $(s,p) \mapsto \gamma_s^p$ ($s \in [0, \frac{1}{2}]$) by grafting straight line segments having directions $\mathbf{t}_{\gamma^p}(t)$, $\mathbf{t}_{\gamma^p}(t')$ and length greater than 4 at $\gamma(t)$ and $\gamma(t')$ (see [4], Definition 4.13). Note that $(\gamma_s^p, \varphi^p) \in \mathcal{V}$ for all $s \in [0, \frac{1}{2}]$, since $\theta_{\gamma_s^p}$ is essentially the same function as $\theta_{\gamma_0^p}$. Extend the homotopy to all of $[0,1]$ by deforming each of these segments to create a "belly" (see Figure 10 of [4]) so that (19) is satisfied for $s = 1$. This is possible because $\mathcal{M}(P)$ is connected if $P = (x, 1) \in \mathbf{R} \times \mathbf{S}^1$ with $x > 4$, by Theorem 6.1 of [4]. For a general finite simplicial complex K, the same idea works if partitions of unity are used. The details will be omitted since a similar construction (for deforming segments into eight curves, instead of bellies) was already carried out in Lemmas 4.15 and 4.16 of [4].

Now take $\mathcal{V} = \mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$. By Corollary 1.11 of [4], it may be assumed that each γ^p is smooth and that all of its derivatives depend continuously upon $p \in K$. Choose $\kappa_0 \in (\frac{1}{2}, 1)$ such that $\kappa_{\gamma^p}([0,1]) \subset (-\kappa_0, +\kappa_0)$ for every $p \in K$. Assume first that $K = \{p\}$. Let $J_k(p)$ be intervals satisfying (3.1) for the sign string σ_m and some $\varepsilon > 0$, and choose stretchable intervals $I \subset J_k(p)$, $I' \subset J_{k'}(p)$ with $\sigma_m(k) = +$ and $\sigma_m(k') = -$. By choosing a larger $\kappa_0 \in (\frac{1}{2}, 1)$ if necessary, it may be assumed that the restriction of γ^p to each of I, I' is κ_0-stretchable with respect to $\varphi^p_{\sigma_m(k)}$. Define a homotopy $(s,p) \mapsto \gamma_s^p$ by stretching each of $\gamma_0^p|_I, \gamma_0^p|_{I'}$ in the direction of $\pm ie^{i\varphi^p}$ by more than $4+2\pi$, linearly with $s \in [0,\frac{1}{2}]$. Extend this to $[0,1]$ by choosing straight line segments of length greater than 4 within each of $\gamma_{\frac{1}{2}}^p|_I, \gamma_{\frac{1}{2}}^p|_{I'}$ and deforming them to create bellies as above, so that (19) is satisfied for $s=1$. For a general finite simplicial complex K, use partitions of unity, (3.6) and (3.7) (a). $\square$

(5.3) Lemma. *Let $K \to \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$, $p \mapsto (\gamma^p,\varphi^p)$ be continuous and suppose that* (19) *holds. Then given $\delta_0 > 0$, there exists a homotopy $(s,p) \mapsto (\gamma_s^p,\varphi^p) \in \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$ such that $\gamma_0^p = \gamma^p$ and*

$$\varphi_-^p - \delta_0 < \inf_{t\in[0,1]} \theta_{\gamma_1^p}(t) < \varphi_-^p \quad \text{and} \quad \varphi_+^p < \sup_{t\in[0,1]} \theta_{\gamma_1^p}(t) < \varphi_+^p + \delta_0 \quad \text{for every } p \in K. \tag{20}$$

Moreover, the homotopy is obtained by stretching subarcs of γ^p in the direction of $\pm ie^{i\varphi^p}$.

Proof. By Corollary 1.11 of [4], no generality is lost in assuming that γ^p is smooth for every $p \in K$, and that its derivatives depend continuously on p. In particular, there exists $\kappa_0 \in (0,1)$ such that $\kappa_{\gamma^p}([0,1]) \subset (-\kappa_0,+\kappa_0)$ for all $p \in K$. Fix p and let

$$W_p = \left\{t \in [0,1] : \left|\theta_{\gamma^p}(t) - \varphi^p\right| > \tfrac{\pi}{2}\right\}, \quad C_p = \left\{t \in [0,1] : \left|\theta_{\gamma^p}(t) - \varphi^p\right| \geq \tfrac{\pi}{2} + \tfrac{\delta_0}{2}\right\}.$$

By (2.2) (a), the closure of any component of W_p is a κ_0-stretchable interval for γ^p. Moreover, C_p is compact, hence it intersects only finitely many of the components of W_p. Choose disjoint intervals $[c_k,d_k]$ ($k \in [n]$, $n = n(p) \in \mathbf{N}$), such that:

- $C_p \subset \bigcup_{k=1}^n [c_k,d_k]$;
- $\gamma^p|_{[c_k,d_k]}$ is κ_0-stretchable with respect to $\varphi_\pm^p$ for every $k \in [n]$;
- $\left|\theta_{\gamma^p} - \varphi^p\right| > \frac{\pi}{2}$ throughout $[c_k,d_k]$;
- $\left|\theta_{\gamma^p}(c_k) - \varphi^p\right| < \frac{\pi}{2} + \frac{\delta_0}{2}$ and $\left|\theta_{\gamma^p}(d_k) - \varphi^p\right| < \frac{\pi}{2} + \frac{\delta_0}{2}$.

Let $U_p \subset K$ be a neighborhood of p such that these conditions still hold if p is replaced by any $q \in \overline{U}_p$. Cover K by finitely many such open sets U_i ($i \in [l]$), with associated stretchable intervals $[c_k^i,d_k^i] \subset [0,1]$, $k \in [n(i)]$. By the argument used in the proof of (3.7) (a), it may be assumed that if $i < i'$ and $\overline{U}_i \cap \overline{U}_{i'} \neq \emptyset$, then for each $k \in [n(i)]$ and $k' \in [n(i')]$, either $[c_k^i,d_k^i] \subset [c_{k'}^{i'},d_{k'}^{i'}]$ or these two intervals are disjoint. Let $(\rho_i)_{i\in[l]}$, $\rho_i\colon K \to [0,1]$, be a partition of unity subordinate to $(U_i)_{i\in[l]}$. Let $m_\pm(i)$ denote the cardinality of

$$S_\pm(i) = \left\{k \in [n(i)] : \pm \operatorname{sign}\left(\theta_{\gamma^p}(t) - \varphi^p\right) > 0 \text{ for all } t \in [c_k^i,d_k^i]\right\}.$$

Observe that $m_+(i)$, $m_-(i) \geq 1$ by (19). Let $C > 0$ and for each $i = 1,\dots,l$ successively, let γ_s^p ($s \in [\frac{i-1}{l},\frac{i}{l}]$) be obtained by stretching

$$\gamma_{\frac{i-1}{l}}^p|_{[c_k^i,d_k^i]} \text{ by } \begin{cases} m_-(i)\rho_i(p)C & \text{if } k \in S_+(i) \\ m_+(i)\rho_i(p)C & \text{if } k \in S_-(i) \end{cases} \text{ for each } k \in [n(i)]. \tag{21}$$

The factors $m_\pm(i)$ are incorporated here to guarantee that $\gamma_s(1) = q$ for all $s \in [0,1]$. By (2.12) (b), (d) and (j), for each $p \in K$, the conditions above remain valid for γ_s^p ($s \in [0,1]$), so that this deformation is well-defined. Further, by (2.12) (g), if C is large enough, then the resulting curves γ_1^p will satisfy (20) for all $p \in K$. $\square$

For the sake of convenience, a curve $\gamma \in \mathcal{M}(Q)$ will be called *of type cl* if it is the concatenation of an arc of circle of amplitude $< \pi$ and a line segment, where either of these may degenerate to a point and the circle has radius $\frac{1}{\kappa_0}$; the value of κ_0 will be clear from the context. The analogous abbreviation for a more general word on $\{c,l\}$ will also be used.

(5.4) Lemma. *Let $g_0\colon K \to \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$, $g_0(p) \mapsto (\gamma_0^p,\varphi^p)$, be a continuous map, where K is a finite simplicial complex. Then for all sufficiently large $C > 0$, there exists a homotopy $g_s\colon K \to \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ ($s \in [0,m]$), $g_s(p) = (\gamma_s^p,\varphi^p)$, such that for each $p \in K$ and $j \in [m]$:*

(i) *γ_m^p is (φ^p,δ_j)-quasicritical of type σ_j, where*

$$\delta_j = \operatorname{arccot}\left(C^{2(m-j)+1}\right); \tag{22}$$

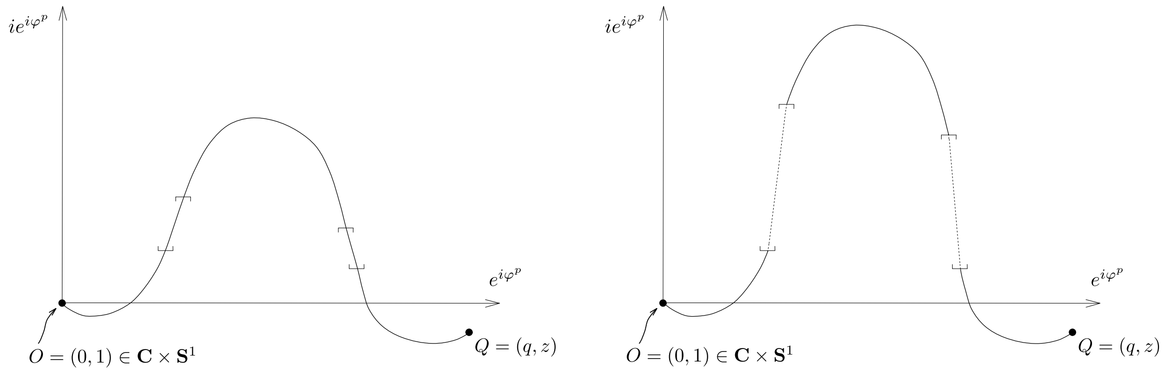


FIGURE 10. Streching a curve in the direction of $\pm ie^{i\varphi^p}$.

(ii) *If $J_{j,1}(p) < \cdots < J_{j,|\sigma_j|}(p)$ are intervals satisfying (3.1) for the quadruple $(\gamma^p_m, \varphi^p, \delta_j, \sigma_j)$, then for each $k \in [|\sigma_j|]$, there exists an interval $I \subset J_{j,k}(p)$ such that $\left|\theta_{\gamma^p_m}(t) - \varphi^p_{\sigma_j(k)}\right| < \delta_j$ for all $t \in I$ and $\gamma^p_m|_I$ is a line segment of length greater than $\cot(\delta_j)$.*

Proof. Let γ^p_0 be denoted simply by γ^p. By Corollary 1.11 of [4], it may be assumed that each γ^p is smooth and that all of its derivatives depend continuously on $p \in K$. Let $R > 0$ be such that the image of γ^p is contained in the open disk of radius R centered at the origin for all $p \in K$. Take $\kappa_0 \in (\frac{1}{2}, 1)$ large enough so that $\kappa_{\gamma^p}([0,1]) \subset [-\kappa_0, +\kappa_0]$ for every $p \in K$. For each $j \in [m]$, let $\varepsilon_j \colon K \to \mathbf{R}^+$ $(j \in [m])$ be as in (3.16), $n_j = |\sigma_j|$ and let $J_{j,k}(p)$, $I_{i,j,k}$ be the intervals corresponding to σ_j as in (3.6) and (3.7), for $k \in [n_j]$ and some open cover $(U_{i,j})_{i \in [l_j]}$ of K. For each $j \in [m]$, let $\rho_{i,j} \colon K \to [0,1]$ $(i \in [l_j])$ form a partition of unity subordinate to the cover $(U_{i,j})_{i \in [l_j]}$. By choosing a larger $\kappa_0 \in (\frac{1}{2}, 1)$ if necessary, it may be assumed that $\gamma^p|_{I_{i,j,k}}$ is κ_0-stretchable with respect to $\varphi^p_{\sigma_j(k)}$ for each $p \in \overline{U}_{i,j}$ and $k \in [n_j]$. Again by compactness, there exists $\varkappa_1 > 0$ such that the inequality in (2.12) (g) is satisfied by the function corresponding to the stretching of $\gamma^p|_{I_{i,j,k}}$ for all $j \in [m]$, $i \in [l_j]$, $k \in [n_j]$ and $p \in \overline{U}_{i,j}$. Take $C > 0$ to be so large that

$$C > 8 \max\left\{R + \pi\kappa_0^{-1}\,,\; m\varkappa_1^{-1}\,,\; \sup_{p \in K} \cot(\varepsilon^p_1)\right\}. \tag{23}$$

For the sake of simplicity, it will be assumed that $I_{i,j,k} \cap I_{i',j,k} = \emptyset$ whenever $i \neq i'$ and $U_{i,j} \cap U_{i',j} \neq \emptyset$. The only difference if this did not occur is that it would be necessary to stretch the restriction of γ^p to these intervals one i at a time; see (3.7) and the remark following it.

Define a homotopy $(s,p) \mapsto \gamma^p_s$ $(s \in [0,m])$ inductively as follows (compare Figure 10). For $s \in [j-1, j]$, let γ^p_s be obtained from γ^p_{j-1} by stretching its restriction to $I_{i,j,k}$ linearly with s in the direction of $\exp\left(i\varphi^p_{\sigma_j(k)}\right)$ by:†

$$l_j \rho_{i,j}(p)\, C^{2(m+1-j)} \quad \text{for each } i \in [l_j],\ k \in [n_j]. \tag{24}$$

Actually, if n_j is odd, then one must introduce different constants into the above formula for k even and for k odd to guarantee that $\left\langle \gamma_s(1), ie^{i\varphi^p} \right\rangle$ is constant for $s \in [j-1, j]$; compare eq. (21). It is easy to check that the estimates below all hold in this case as well, so for simplicity it will be assumed that all n_j are even.

It is an immediate consequence of (2.12) (d) that $(\gamma^p_s, \varphi^p) \in \mathcal{V}_c$ for all $s \in [0,m]$ since this is true when $s = 0$. Let δ_j be as in (22). We claim that for each $p \in K$ and $j \in [m]$:

(a) If $s \in [0, j]$, then γ^p_s is $(\varphi^p, \varepsilon^p_j)$-quasicritical of type σ_j.

(b) If $s \in [j, m]$, then γ^p_s satisfies (i) and (ii) (with s in place of m).

In particular, $(\gamma^p_s, \varphi^p) \in \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ for all $s \in [0,m]$ as claimed.

To establish (a), we prove by induction on $j' \in [j]$ that the intervals $J_{j,k}(p)$ $(k \in [n_j])$ satisfy the conditions in (3.1) for the quadruple $(\gamma^p_s, \varphi^p, \varepsilon^p_j, \sigma_j)$ for any $s \in [j'-1, j']$. By hypothesis, this is true when $s = 0$. By (3.8), $\varepsilon^p_{j'} \leq \varepsilon^p_j$ for all $p \in K$. Hence, by (3.5) (a), for any $i' \in [l_{j'}]$ and $k' \in [n_{j'}]$,

†When appearing in an exponential, the letter i denotes the imaginary unit, not an index.

the interval $I_{i',j',k'}$ is contained in some $J_{j,k}(p)$ whenever $p \in U_{i',j'}$. It follows immediately from (2.12) (b) that the $J_{j,k}(p)$ satisfy (i) and (ii) of (3.1) for $(\gamma_s^p, \varphi^p, \varepsilon_j^p, \sigma_j)$ and all $s \in [j'-1, j']$. If $J_{j,k}(p)$ contains $I_{i',j',k'}$ for some $i' \in [l_{j'}]$ with $p \in U_{i',j'}$, then condition (iii) of (3.1) is satisfied by $I_{i',j',k'}$ for all $s \in [j'-1, j']$ by (2.12) (j). If not, then $J_{j,k}(p)$ is disjoint from $I_{i',j',k'}$ whenever $p \in U_{i',j'}$, so that $\theta_{\gamma_s^p}(t) = \theta_{\gamma_{j'-1}^p}(t)$ for all $t \in J_{j,k}(p)$ and $s \in [j'-1, j']$. In particular, if $I \subset J_{j,k}(p)$ satisfies (iii) for $(\gamma_s^p, \varphi^p, \varepsilon_j^p, \sigma_j)$ when $s = j'-1$, then I is not affected by the stretching, hence it satisfies (iii) for this quadruple for all $s \in [j'-1, j']$. This completes the proof of the induction step and of claim (a).

Now write $I_{i,j,k} = [c_{i,j,k}, d_{i,j,k}]$. If $i \in [l_j]$ is such that $\rho_{i,j}(p) \geq \frac{1}{l_j}$, then

$$\begin{aligned} &\big\langle \gamma_j^p(d_{i,j,k}) - \gamma_j^p(c_{i,j,k})\,,\, \exp(i\varphi^p_{\sigma_j(k)})\big\rangle > C^{2(m+1-j)} - 2R, \text{ while} \\ &\big|\big\langle \gamma_j^p(d_{i,j,k}) - \gamma_j^p(c_{i,j,k})\,,\, \exp(i\varphi^p)\big\rangle\big| < 2R. \end{aligned} \tag{25}$$

The first inequality is immediate from (24) and the hypothesis that the image of γ^p is contained in the open disk $B_R(0)$, and the second one follows from this hypothesis and (2.12) (c). By the definition of stretching, $\gamma_j^p|_{I_{i,j,k}}$ is a curve of type *clc*. Using (23) we conclude that $I_{i,j,k} \subset J_{j,k}(p)$ contains a subinterval I such that $\gamma_j^p|_I$ is a line segment of length greater than

$$C^{2(m+1-j)} - 2R - 2\pi\kappa_0^{-1} > C^{2(m-j)+1}$$

and slope greater in absolute value than

$$\frac{1}{2R}\big(C^{2(m+1-j)} - 2R - 2\pi\kappa_0^{-1}\big) > \frac{1}{4R}C^{2(m+1-j)} > C^{2(m-j)+1} = \cot(\delta_j).$$

Hence $\big|\theta_{\gamma_j^p}(t) - \varphi^p_{\sigma_j(k)}\big| < \delta_j$ throughout I, and by (3.10), γ_j^p is (φ^p, δ_j)-quasicritical of type σ_j. This proves (b) when $s = j$.

We now establish (b) for all $s \in [j, m]$. Fix $p \in K$. Observe first that no $t \in [0,1]$ can belong to two intervals $I_{i',j',k'}$ and $I_{i'',j'',k''}$ with $\rho_{i',j'}(p) > 0$, $\rho_{i'',j''}(p) > 0$ and $\sigma_{j'}(k') = -\sigma_{j''}(k'')$. Moreover, if j' is the smallest index such that $t \in I_{i',j',k'}$ and $\rho_{i',j'}(p) > 0$ for some $i' \in [l_{j'}]$ and $k' \in [n_{j'}]$, then

$$\big|\tan\big(\theta_{\gamma_s^p}(t) - \varphi^p_{\sigma_{j'}(k')}\big)\big| \geq \frac{\varkappa_1}{mC^{2(m+1-j')}} > \frac{4}{C^{2(m+1-j')+1}} = 4\tan(\delta_{j'-1}) \quad \text{for all } s \in [0, m].$$

Here (23) has been used; the factor m in the denominator of the second term comes from the fact that t belongs to at most $(m+1-j') \leq m$ such intervals. Since $4\tan x > \tan(2x)$ for $x \in \big(0, \frac{\pi}{8}\big)$,

$$\big|\theta_{\gamma_s^p}(t) - \varphi^p_{\sigma_{j'}(k')}\big| > 2\delta_{j'-1} \quad \text{for all } s \in [0, m]. \tag{26}$$

Now suppose that $t \in J_{j,k}(p)$ for some $k \in [n_j]$. There are three possibilities:

- If t does not belong to any $I_{i',j',k'}$ with $\rho_{i',j'}(p) > 0$, then $\theta_{\gamma_s^p}(t) = \theta_{\gamma_0^p}(t)$ for all $s \in [0, m]$ by construction, hence
$$\big|\theta_{\gamma_s^p}(t) - \varphi^p_{-\sigma_j(k)}\big| > 2\varepsilon_j^p > 2\delta_j \quad \text{for all } s \in [0, m].$$
- If $t \in I_{i',j',k'}$ with $\rho_{i',j'}(p) > 0$ and $\sigma_{j'}(k') = \sigma_j(k)$, then (2.12) (b) implies that
$$\big|\theta_{\gamma_s^p}(t) - \varphi^p_{-\sigma_j(k)}\big| > \frac{\pi}{2} > 2\delta_j \quad \text{for all } s \in [0, m].$$
- If $t \in I_{i',j',k'}$ with $\rho_{i',j'}(p) > 0$ and $\sigma_{j'}(k') = -\sigma_j(k)$, then $j' \geq j+1$, otherwise the inequality $\varepsilon_{j'}^p \leq \varepsilon_j^p$ would immediately yield a contradiction. Hence, by (26),
$$\big|\theta_{\gamma_s^p}(t) - \varphi^p_{-\sigma_j(k)}\big| > 2\delta_j \quad \text{for all } s \in [0, m].$$

Thus, in any case condition (i) of (3.1) is satisfied by $(\gamma_s^p, \varphi^p, \delta_j, \sigma_j)$ for all $s \in [0, m]$. Similarly, if $t \notin \operatorname{Int}\big(\bigcup_{k=1}^{n_j} J_{j,k}(p)\big)$, then either t does not belong to any $I_{i',j',k'}$ with $\rho_{i',j'}(p) > 0$ or $t \in I_{i',j',k'}$ with $\rho_{i',j'}(p) > 0$ for some $j' \geq j+1$. Then, by the same reason as in the first and third possibilities above,

$$\big|\theta_{\gamma_s^p}(t) - \varphi^p_{\pm}\big| > 2\delta_j \quad \text{for all } s \in [0, m].$$

This proves that condition (ii) of (3.1) is also satisfied for all $s \in [0, m]$. Finally, we shall prove by induction on j' ($j \leq j' \leq m$) that condition (iii) holds for all $s \in [j, j']$. For $j' = j$, this was established in the preceding paragraph; let $I \subset J_{j,k}(p)$ be as described there and assume that

$j' > j$. By (3.8), $\varepsilon^p_{j'} > 2\varepsilon^p_j$, hence (3.7) (b) implies that if $\rho_{i',j'}(p) > 0$ and $k' \in [n_{j'}]$, then either $I \subset [c_{i',j',k'}, d_{i',j',k'}]$ or these two intervals are disjoint. If I is disjoint from any such interval, then $\theta_{\gamma^p_s}(t) = \theta_{\gamma^p_{j'-1}}(t)$ for all $t \in I$ and $s \in [j'-1, j']$. Hence $I \subset J_{j,k}(p)$ satisfies condition (iii) of (3.1) for all such s, since by the induction hypothesis this is true when $s = j'-1$. Suppose then that $I \subset [c_{i',j',k'}, d_{i',j',k'}]$ for some i', k' with $\rho_{i',j'}(p) > 0$. Using (2.15) and reducing I if necessary, it can be assumed that $\gamma^p_s|_I$ is a line segment for all $s \in [j'-1, j']$. Let $s_0 \in [j'-1, j']$ correspond to the instant where the flattening deformation ends and the stretching begins. The same estimates as in (25) show that the slope of $\gamma^p_{s_0}|_I$ is greater than $\cot(\delta_j)$. Since this is also true when $s = j'-1$ by the induction hypothesis, it follows from the monotonicity of $\theta_{\gamma^p_s}(t)$ with respect to $s \in [j'-1, s_0]$ (see Lemma 3.11 of [4]) that this holds for all $s \in [j'-1, s_0]$. For $s \in [s_0, j-1]$ the same conclusion holds by (2.9). □

(5.5) Lemma. *Let $g\colon [0,m] \times K \to \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ be as in (5.4) and $n = |\sigma_m|$. Then g admits an extension to $[0, m+2n] \times K$, $g_s(p) = (\gamma^p_s, \varphi^p)$, such that γ^p_{m+2n} is of the form*

$$c\,\underbrace{lc\dots lc}_{n}$$

and each of its straight subarcs has length greater than 8, for all $p \in K$.

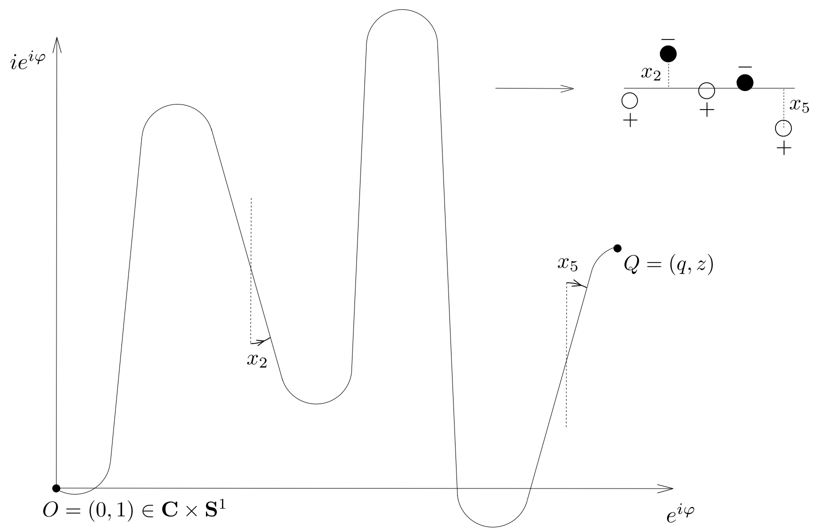


FIGURE 11. An illustration of a curve obtained by the homotopy in (5.5).

Proof. We retain the notation of the proof of (5.4). Let $J_k(p) = [a_k(p), b_k(p)]$ ($k \in [n]$) be intervals satisfying the conditions of (5.4) for the quadruple $(\gamma^p_0, \varphi^p, \varepsilon^p_m, \sigma_m)$, and hence the same conditions for $(\gamma^p_m, \varphi^p, \delta_m, \sigma_m)$. Set $t_0(p) = 0$, $t_{2n}(p) = 1$,

$$t_{2k-1}(p) = \frac{1}{2}[a_k(p) + b_k(p)] \quad (k \in [n]) \quad \text{and} \quad t_{2k}(p) = \frac{1}{2}[b_k(p) + a_{k+1}(p)] \quad (k \in [n-1]). \tag{27}$$

Notice that $t_{2k-1}(p) \in J_k(p)$ for all $k \in [n]$ and $J_k(p) < t_{2k}(p) < J_{k+1}(p)$ for all $k \in [n-1]$. For each $\nu \in [2n]$, let $I_\nu(p) = [t_{\nu-1}(p), t_\nu(p)]$. Since $I_\nu(p)$ intersects exactly one $J_k(p)$ for all ν, the amplitude

$$\omega\big(\gamma^p_m|_{I_\nu(p)}\big) = \sup_{t \in I_\nu(p)} \big\{\theta_{\gamma^p_m}(t)\big\} - \inf_{t \in I_\nu(p)} \big\{\theta_{\gamma^p_m}(t)\big\}$$

is less than π for all $\nu \in [2n]$. Thus, $\gamma^p_m|_{I_\nu(p)}$ can be flattened; let

$$\psi^p_\nu = \frac{1}{2}\bigg(\sup_{t \in I_\nu(p)} \big\{\theta_{\gamma^p_m}(t)\big\} + \inf_{t \in I_\nu(p)} \big\{\theta_{\gamma^p_m}(t)\big\}\bigg) \quad (\nu \in [2n]).$$

Extend the homotopy of (5.4) to $[0, m+1] \times K$ by letting $\gamma^p_{m+s}|_{I_\nu(p)}$ be the flattening of $\gamma^p_m|_{I_\nu(p)}$ in the direction of $e^{i\psi^p_\nu}$ ($s \in [0,1]$). It follows immediately from (2.12) (e) that $(\gamma^p_{m+s}, \varphi^p) \in \mathcal{V}_c$ for all $s \in [0,1]$.

Again, it needs to be verified that $(\gamma^p_{m+s},\varphi^p)\in\mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$. We claim that γ^p_{m+s} is (φ^p,δ_j)-quasicritical of type σ_j for all $s\in[0,1]$ and $j\in[m]$. Fix $p\in K$ and let

$$J_{j,1}(p)<\dots<J_{j,n_j}(p)\quad(j\in[m])$$

be intervals as in (5.4) for σ_j. Using (3.19), it may be assumed that any such interval has the form $J_{k_1}(p)*J_{k_2}(p)$ for some $k_1,k_2\in[n]$. (It is however unnecessary to assume that the endpoints of $J_{j,k}(p)$ depend continuously on p, since no constructions using them will be carried out.)

Let $\nu\in[2n]$. As $I_\nu(p)$ intersects exactly one $J_k(p)$, it intersects at most one $J_{j,r}(p)$. Thus, if $I_\nu(p)\cap J_{j,r}(p)\neq\emptyset$, then $\big|\theta_{\gamma^p_{m+s}}(t)-\varphi^p_{-\sigma_j(r)}\big|>2\delta_j$ for $s=0$ and all $t\in I_\nu(p)$. By (2.12)(e), this inequality holds for all $s\in[0,1]$. Since $\bigcup_\nu I_\nu(p)=[0,1]$, we conclude that

$$\big|\theta_{\gamma^p_{m+s}}(t)-\varphi^p_{-\sigma_j(r)}\big|>2\delta_j\ \text{ for all } s\in[0,1],\ t\in J_{j,r}(p). \tag{28}$$

Now let $I\subset J_{j,r}(p)$ be an interval as in (ii) of (5.4). Let $\nu\in[2n]$ be such that $I\subset I_\nu(p)\cup I_{\nu+1}(p)$. Then the restriction of γ^p_m to one of $I\cap I_\nu(p)$ or $I\cap I_{\nu+1}(p)$ has length equal to at least half the length of $\gamma^p_m|_I$. Suppose without loss of generality that the former occurs. Let $R>1$ and C be as in the proof of (5.4). Then

$$\begin{aligned}\big|\big\langle\gamma^p_m(t_{\nu(p)})-\gamma^p_m(t_{\nu-1}(p))\,,\,i\exp(i\varphi^p)\big\rangle\big|&>\frac{1}{2}C^{2(m+1-j)}-2R,\text{ while}\\ \big|\big\langle\gamma^p_m(t_{\nu(p)})-\gamma^p_m(t_{\nu-1}(p))\,,\,\exp(i\varphi^p)\big\rangle\big|&<2R.\end{aligned}$$

Recall that by the definition of flattening, $\gamma^p_{m+s}(t_\nu(p))=\gamma^p_m(t_\nu(p))$ for all $s\in[0,1]$, and similarly at $t_{\nu-1}(p)$. Moreover, $\gamma^p_{m+1}|_{I_\nu(p)}$ is of type *clc*. Its subarc which is a line segment must thus have slope greater in absolute value than

$$\frac{1}{2R}\Big(\frac{1}{2}C^{2(m+1-j)}-2R-2\pi\kappa_0^{-1}\Big)>\frac{1}{8R}C^{2(m+1-j)}>C^{2(m-j)+1}=\cot(\delta_j).$$

Let $I'\subset I\cap I_\nu(p)$ be an interval such that $\gamma^p_{m+s}|_{I'}$ is a line segment of length >8 for all $s\in[0,1]$, as guaranteed by (2.15). Then $\gamma^p_{m+s}|_{I'}$ is κ_0-stretchable by (2.16). Further, the above estimate implies that $\big|\theta_{\gamma^p_{m+s}}(t)-\varphi^p_{\sigma_j(k)}\big|<\delta_j$ throughout I' when $s=1$, and this is also true when $s=0$ by (5.4). By the monotonocity of $\theta_{\gamma^p_{m+s}(t)}$ with respect to s (see Lemma 3.11 of [4]), this inequality holds for all $s\in[0,1]$. Thus, condition (iii) of (3.1) is satisfied.

To verify condition (ii), let ν_0,ν_1 be the greatest (resp. smallest) index satisfying $t_{\nu_0}(p)<J_{j,k}(p)<t_{\nu_1}(p)$. Since $t_{\nu_i}(p)\notin\bigcup_k J_k(p)$ by definition,

$$\big|\theta_{\gamma_{m+s}}(t_{\nu_i}(p))-\varphi^p\big|=|\theta_{\gamma_m}(t_{\nu_i}(p))-\varphi^p|<\frac{\pi}{2}-2\varepsilon^p_m\ \text{ for } i=0,1 \text{ and all } s\in[0,1].$$

Therefore, it is possible to enlarge $J_{j,r}(p)$ to a subinterval of $\big(t_{\nu_0}(p),t_{\nu_1}(p)\big)$ so that

$$\big|\theta_{\gamma^p_{m+s}}(t)-\varphi^p\big|<\frac{\pi}{2}-2\delta_j\ \text{ for all } t\in\big[t_{\nu_0}(p),t_{\nu_1}(p)\big]\smallsetminus J_{j,r}(p) \text{ and } s\in[0,1].$$

If this enlargement is carried out for each $k\in[n_j]$, then condition (ii) of (3.1) will be satisfied by the $J_{j,r}(p)$. It is easily verified that the validity of conditions (i) and (iii) is not affected.

Now γ^p_{m+1} is of the form

$$\underbrace{(clc)(clc)\dots(clc)}_{n}\ \text{for all } p\in K.$$

To prove the lemma, it thus suffices to reduce subarcs of type *cclc* to arcs of type *clc*. Let L^p denote the length of γ^p_{m+1}; no generality is lost in assuming that $\gamma^p_{m+1}\colon[0,1]\to\mathbf{C}$ is parametrized proportionally to arc-length for all p. Set

$$A_\nu(p):=\Big[t_\nu(p)-\frac{\pi}{\kappa_0L^p}\,,\,t_{\nu+1}(p)\Big] \tag{29}$$

For each $\nu=1,\dots,2n-1$ in turn, let $\gamma^p_{m+\nu+1}$ be obtained from $\gamma^p_{m+\nu}$ by flattening the arc $\gamma^p_{m+\nu}|_{A_\nu(p)}$ in the direction of

$$\frac{1}{2}\bigg(\sup_{t\in A_\nu(p)}\big\{\theta_{\gamma^p_{m+\nu}}(t)\big\}+\inf_{t\in A_\nu(p)}\big\{\theta_{\gamma^p_{m+\nu}}(t)\big\}\bigg).$$

Using estimates similar to the preceding ones, it is not hard to check that $\gamma^p_s\in\mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ for all $s\in[m+1,m+2n]$. Moreover, γ^p_{m+2n} has the desired form for all $p\in K$ by construction. □

We now establish a version of (5.4) and (5.5) for $\mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$. The proof is a repetition of the arguments used to establish these results, aside from a preliminary deformation which is needed to guarantee that γ_s^p will remain diffuse throughout the deformation.

(5.6) Lemma. *Let $g\colon K \to \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$, $g(p) \mapsto (\gamma^p, \varphi^p)$, be a continuous map, where K is a finite simplicial complex. Then there exists a homotopy $[0,5] \times K \to \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$, $g_s(p) = (\gamma_s^p, \varphi^p)$, such that $\gamma_0^p = \gamma^p$ and γ_5^p is of the form*

$$c\,\underbrace{lc\dots lc}_{n}$$

and each of its straight subarcs has length greater than 8, for all $p \in K$.

Proof. Let the notation be as in the first paragraph of (5.4) and let $\theta^p := \theta_{\gamma^p}$ and $\theta_s^p := \theta_{\gamma_s^p}$ (where γ_s^p is to be defined below). Since $1 \in R(Q)$ by definition (see (3.12)), $\cos\varphi^p = \langle 1, e^{i\varphi^p}\rangle > 0$ for all $p \in K$. By (5.2), it may be assumed that (19) holds. Given p, choose $u_j \in [0,1]$ $(j = 1,2)$ such that

$$(-1)^j\big(\theta^p(u_j) - \varphi^p\big) > \frac{\pi}{2}$$

and the origin $0 \in \mathbf{C}$ lies in the interior of the triangle whose vertices are 1, $e^{i\theta^p(u_1)}$ and $e^{i\theta^p(u_2)}$. These conditions are still satisfied for all q in a neighborhood U_p of p. Let $(U_i)_{i\in[l_0]}$ be a subcover of the resulting cover of K and $u_{i,j} \in [0,1]$ be the corresponding numbers. Then we can write

$$0 = a_{i,0}(p) + a_{i,1}(p)e^{i\theta^p(u_{i,1})} + a_{i,2}(p)e^{i\theta^p(u_{i,2})} \quad \text{for } a_{i,j}(p) > 0,\ p \in U_i.$$

Moreover, $a_{i,j}\colon U_i \to \mathbf{R}^+$ can be chosen to depend continuously on p and as large as desired for each $j = 0,1,2$. Let $\rho_i\colon K \to [0,1]$ be a partition of unity subordinate to $(U_i)_{i\in[l_0]}$. Set $\gamma_0^p := \gamma^p$ and define a homotopy $[0,1]\times K \to \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$, $(s,p) \mapsto (\gamma_s^p, \varphi^p)$, by grafting straight segments linearly with s onto γ^p at $t = 0$, $u_{i,1}(p)$ and $u_{i,2}(p)$ of lengths $L_{i,j}(p) = \rho_i(p)a_{i,j}(p)$ $(j = 0,1,2$, respectively) for all $i \in [l_0]$ and $p \in K$. As before, let $R > 0$ be such that the image of γ^p is contained in $B_R(0)$ for all $p \in K$. By taking the $a_{i,j}$ to be sufficiently large, it can be guaranteed that for each $p \in K$ there exists $i \in [l_0]$ such that

$$\big\langle L_{i,j}(p)e^{i\theta_1^p(u_{i,j})}, e^{i\varphi^p}\big\rangle < -2(R + 2\pi) \text{ for } j = 1,2.$$

In words, γ_1^p "retrocedes" by at least $2(R+2\pi)$ at $t = u_{i,1}$ and $t = u_{i,2}$, with respect to the axis $e^{i\varphi^p}$. Thus if $k_j \in [n_m]$ is such that $u_{i,j} \in J_{k_j}(p)$ $(j = 1,2$ and $J_k(p)$ as in the proof of (5.5)), then

$$\big\langle \gamma_1^p\big(t_{2k_j}(p)\big) - \gamma_1^p\big(t_{2k_j-2}(p)\big)\,,\, e^{i\varphi^p}\big\rangle < -4\pi. \tag{30}$$

Here $t_\nu(p)$ is as in (27); note that $t_{2k_j-2}(p) < J_{k_j}(p) < t_{2k_j}(p)$. The crucial observation here is that (30) implies the existence of $t' \in [t_{2k_j-2}(p), t_{2k_j}(p)]$ such that $(-1)^j\big(\theta^p(t') - \varphi^p\big) > \frac{\pi}{2}$.

Let δ_0 be given by (22) and apply (5.3) to γ_1^p, extending the homotopy to $[0,2]\times K$. (This deformation is necessary to be able to apply (3.10) as in the proof of (5.4).) Because the subarcs of γ_1^p which are stretched in this homotopy all lie in the interior of some $J_k(p)$, and they are stretched in the direction of $\pm e^{i\varphi^p}$, the coordinate $\langle \gamma_s^p(t), e^{i\varphi^p}\rangle$ is the same for all $s \in [1,2]$ provided that $t \notin \bigcup_k J_k(p)$. Hence, (30) is valid for all such s. Now take $R' > 0$ such that the image of γ_2^p is contained in the open disk $B_{R'}(0)$ for all $p \in K$, and take C exactly as in (23), but replacing R by R'. Finally, extend the homotopy to $[0,5]\times K$ by repeating the proofs of (5.4) and (5.5), with R' in place of R. We claim that (30) is sufficient to guarantee that γ_s^p remains diffuse when the constructions in (5.4) and (5.5) are carried out for $s \in [2,5]$. There are three constructions to consider, which will be assumed to take place for $s \in [2,3]$, $[3,4]$ and $[4,5]$, respectively. The first one, in the proof of (5.4), involves stretching subarcs of γ_2^p in the direction of $\pm e^{i\varphi^p}$; as above, this does not affect the validity of (30) since $t_\nu(p) \notin \bigcup_k J_k(p)$ for all even ν. The second, at the beginning of the proof of (5.5), involves flattening each of the subarcs $\gamma_3^p|_{[t_{\nu-1}(p),t_\nu(p)]}$; clearly, this also does not affect (30), because by the definition of flattening, $\gamma_s^p(t)$ remains constant at the endpoints $t = t_{\nu-1}(p)$ and $t_\nu(p)$, as well as outside of $[t_{\nu-1}(p), t_\nu(p)]$. The last step, near the end of the proof of (5.5), is to flatten the restriction of γ_4^p to the intervals (29). This may affect (30), but it can still be guaranteed that

$$\big\langle \gamma_s^p\big(t_{2k_j}(p)\big) - \gamma_s^p\big(t_{2k_j-2}(p)\big)\,,\, e^{i\varphi^p}\big\rangle < 0 \text{ for all } s \in [4,5],\ j = 1,2,$$

because the restriction of γ_4^p to $A_\nu(p) \smallsetminus [t_\nu(p), t_{\nu+1}(p)]$ has length $\frac{\pi}{\kappa_0} < 2\pi$. Thus, for each $p \in K$ and $s \in [0,5]$, there exist $v_j \in [0,1]$ $(j = 1,2)$ such that $(-1)^j\big(\theta_s^p(v_j) - \varphi^p\big) > \frac{\pi}{2}$. ☐

Given any family $(\gamma^p, \varphi^p) \in \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ or $\mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$, indexed by a finite simplicial complex, we have shown that γ^p can be continuously deformed to look like a curve η^p as in Figure 11. To finish the proof of (5.1), it thus suffices to show that any such family is contractible. This is true because any η as in the figure is essentially determined by $p(\eta,\varphi) = (x,\varphi)$, where $x = (x_1,\dots,x_n)$ is obtained as indicated there and $n = |\sigma_m|$. To make this more precise, we begin by describing a construction which implies that each fiber of p is contractible. It will then be shown that p is a quasifibration.

(5.7) Construction. Let $\gamma_0, \gamma_1 \colon [0,1] \to \mathbf{C}$ be two regular curves parametrized proportionally to arc-length and $\theta_{\gamma_j} \colon [0,1] \to \mathbf{R}$ be continuous functions satisfying $\exp\big(i\theta_{\gamma_j}\big) = \mathbf{t}_{\gamma_j}$ $(j = 0,1)$. Let $\vartheta_0, \vartheta_1 \in \mathbf{R}$ and $\kappa_1 \in (0,1)$. Suppose that $\gamma = \gamma_0, \gamma_1$ satisfies:

(i) $\theta_\gamma(0) = \vartheta_0$ and $\theta_{\gamma(1)} = \vartheta_1$;

(ii) $\kappa_\gamma \colon [0,1] \to [0,\kappa_1]$ is a step function.

Recall that $\dot\theta_\eta = |\dot\eta|\,\kappa_\eta$ for any piecewise C^2 curve (except at finitely many points). Condition (ii) thus implies that θ_γ is an increasing, piecewise linear function. We shall describe a homotopy $s \mapsto \gamma_s$ $(s \in [0,1])$ joining γ_0 to γ_1 through regular curves satisfying (i) and (ii). The idea is to parametrize both γ_j by the argument $\theta \in [\vartheta_0, \vartheta_1]$ and use convex combinations; this works only if both θ_{γ_j} are strictly increasing, but an easy adaptation also covers the general case. See Figure 12.

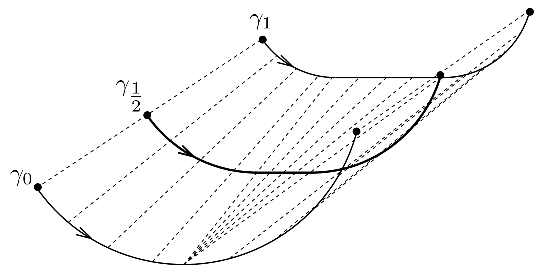


FIGURE 12. An illustration of (5.7).

Let $\{\alpha_1 < \dots < \alpha_n\} \subset [\vartheta_0, \vartheta_1]$ be the union of the set of critical values of θ_{γ_0} and θ_{γ_1}. For each $k \in [n]$, let $[a_j^k, b_j^k] \subset [0,1]$ denote the interval $\theta_{\gamma_j}^{-1}(\{\alpha_k\})$. Define a reparametrization $\eta_j \colon [\vartheta_0, \vartheta_1 + n] \to \mathbf{C}$ of γ_j as follows: The restriction of η_j to an interval of the form

$$\big[\alpha_{k-1} + (k-1)\,,\, \alpha_k + (k-1)\big] \quad (k \in [n+1],\ \alpha_0 := \vartheta_0,\ \alpha_{n+1} := \vartheta_1)$$

is the reparametrization of $\gamma_j|_{[b_j^{k-1}, a_j^k]}$ by the argument $\theta \in [\alpha_{k-1}, \alpha_k]$, where $b_0^j := 0$ and $a_j^{n+1} := 1$. The restriction of η_j to an interval of the form

$$[\alpha_k + (k-1)\,,\, \alpha_k + k] \quad (k \in [n])$$

is the reparametrization of $\gamma_j|_{[a_j^k, b_j^k]}$ proportional to arc-length. Let $\eta_s \colon [\vartheta_0, \vartheta_1 + n] \to \mathbf{C}$ be given by

$$\eta_s(t) = (1-s)\eta_0(t) + s\eta_1(t) \quad (s \in [0,1]).$$

A straightforward computation shows that the radius of curvature $\rho_s = \frac{1}{\kappa_{\gamma_s}}$ satisfies

$$\rho_s = (1-s)\rho_0 + s\rho_1 \in \big[\tfrac{1}{\kappa_1}, +\infty\big) \quad (s \in [0,1])$$

in the interior of intervals of the first type. The restriction of η_s to an interval of the second type is a parametrization of a (possibly degenerate) line segment parallel to $e^{i\alpha_k}$. Thus η_s satisfies (i) and (ii). The desired homotopy $s \mapsto \gamma_s$ is obtained by reparametrizing η_s proportionally to arc-length. Furthermore:

(iii) If $\gamma_0(0) = p = \gamma_1(0)$, then $\gamma_s(0) = p$ for all $s \in [0,1]$; similarly at $t = 1$.

(iv) Let $I_s = \theta_{\gamma_s}^{-1}(\{\vartheta_0\})$. If $\gamma_j|_{I_j}$ is a line segment of length $> L$ $(j = 0,1)$, then $\gamma_s|_{I_s}$ is also a line segment of length $> L$ for all $s \in [0,1]$; similarly for ϑ_1. □

(5.8) Definition. Let $\sigma_1 \prec \dots \prec \sigma_m$ be sign strings, $n = |\sigma_m|$ and $\delta_j > 0$ $(j \in [n])$ satisfy $\delta_{j+1} > 2\delta_j$ for all $j \in [m-1]$. Define $H_d \subset \mathbf{R}^n$ to consist of all $x = (x_1,\dots,x_n) \in \mathbf{R}^n$ such that:

(i) There exist $k_1, k_2 \in [n]$ such that $\sigma_m(k_2) = -\sigma_m(k_1)$ and $\sigma_m(k_i)x_{k_i} > 0$ $(i = 1, 2)$.
(ii) For each $j \in [m]$, if $k_1 < \cdots < k_l$ are all the indices in $[n]$ such that $|x_k| < \delta_j$ (resp. $|x_k| \leq 2\delta_j$), then σ_j is the reduced string of $\tau\colon [l] \to \{\pm\}$, $\tau(i) = \sigma_m(k_i)$;

This space is weakly contractible for any choice of σ_j, δ_j because it is weakly homotopy equivalent to the space

$$X_{(d,\sigma_1,\sigma_1,\ldots,\sigma_m,\sigma_m)}$$

described in (1.17); see (1.18) and (1.19).

(5.9) Definition. Let $\kappa_0 \in (\frac{1}{2}, 1)$, $\sigma_1 \prec \cdots \prec \sigma_m$ be sign strings, $|\sigma_m| = n$ and $\delta_j > 0$ $(j \in [m])$ satisfy $\delta_{j+1} > 2\delta_j$ for all $j \in [m-1]$. Define E_d to be the subspace of $\mathcal{M}(Q) \times R(Q)$ consisting of all (γ, φ) for which there exist $0 = t_0 < \cdots < t_{2n+1} = 1$ and $(x_1, \ldots, x_n) \in H_d$ such that:

(i) $\gamma|_{[t_{2(k-1)}, t_{2k-1}]}$ $(k \in [n+1])$ is an arc of circle of radius $\frac{1}{\kappa_0}$ and amplitude less than π;
(ii) $\gamma|_{[t_{2k-1}, t_{2k}]}$ $(k \in [n])$ is a straight line segment of length greater than 8 and

$$\theta_\gamma\big([t_{2k-1}, t_{2k}]\big) = \big\{\varphi_{\sigma_m(k)} + x_k\big\}\,.$$

The arcs in condition (i) are allowed to be degenerate. Observe that if $(\gamma, \varphi) \in E_d$, then γ is diffuse and (φ, δ_j)-quasicritical of type σ_j for each $j \in [m]$ (for σ_j and δ_j as above). Here $R(Q)$ is the open interval described in (3.12).

(5.10) Lemma. *The space E_d defined above is weakly contractible.*

Proof. Let $p\colon E_d \to H_d \times R(Q)$ be given by $p(\gamma, \varphi) = (x, \varphi)$, where $x = (x_1, \ldots, x_n)$ is as in condition (ii) of (5.9). Fix (x, φ) and $(\gamma_0, \varphi) \in p^{-1}(x, \varphi)$; let $t'_0 = 0 < \cdots < t'_{2n+1} = 1$ be as in condition (ii) for (γ_0, φ). Given $\gamma = \gamma_1 \in p^{-1}(x, \varphi)$ and $t_0 < \cdots < t_{2n+1}$ as above, apply (5.7) to the restrictions $\gamma|_{[t_{\nu-1}, t_\nu]}$ and $\gamma|_{[t'_{\nu-1}, t'_\nu]}$ for each $\nu \in [2n+1]$ to obtain a homotopy $s \mapsto \gamma_s$ joining γ_0 to γ_1. The validity of (iii) and (iv) of (5.7) guarantees that $(\gamma_s, \varphi) \in E_d$ for all $s \in [0, 1]$. Therefore, the fiber $p^{-1}(x, \varphi)$ is either contractible or empty, for any $(x, \varphi) \in H_d \times R(Q)$.

For $x = (x_1, \ldots, x_n) \in H_d$, let

$$\begin{aligned}
\varepsilon_0(x) &= \min\big\{\, |x_k| \;:\; \sigma_m(k)x_k > 0,\;\, k \in [n]\big\},\\
\varepsilon_j(x) &= \min\big\{\delta_j - |x_k| \;:\; |x_k| < \delta_j,\;\, k \in [n]\big\} \quad (j \in [m]),\\
\varepsilon(x) &= \min\{\varepsilon_0(x), \ldots, \varepsilon_m(x)\}\,.
\end{aligned}$$

Then the open ball $B_\varepsilon(x)$ is convex and $B_\varepsilon(x) \subset H_d$ for any $\varepsilon \in (0, \varepsilon(x))$. We claim that p has a section over $B_\varepsilon(x) \times R(Q) \subset H_d \times R(Q)$ for any $x \in H_d$ and $\varepsilon \in (0, \varepsilon(x))$. In particular, p is surjective. Together with contractibility of the fibers and (1.12), this will imply that p is a quasifibration, and hence that E_d is weakly contractible.

Let $x \in H_d$ and $\varepsilon \in (0, \varepsilon(x))$. For each $y = (y_1, \ldots, y_n) \in B_\varepsilon(x)$, consider the (unique) curve $\eta^y\colon [0, 1] \to \mathbf{C}$ of type

$$c\underbrace{lc\ldots lc}_{n}$$

such that $\Phi_{\eta^y}(0) = (0, 1) \in \mathbf{C} \times \mathbf{S}^1$, $\mathbf{t}_{\eta^y}(1) = z$ and $\theta_{\eta^y} = \varphi_{\sigma_m(k)} + y_k$ over the k-th line segment, which we set to be of length 10 for all $k \in [n]$. Then (η^y, φ) satisfies all of the conditions required of elements of E_d, except that $\eta^y(1)$ may not agree with $q \in \mathbf{C}$ as it should.

To correct this, choose k_1, $k_2 \in [n]$ such that

$$\sigma_m(k_1) = +,\; x_{k_1} > 0,\; \sigma_m(k_2) = -,\; x_{k_2} < 0;$$

such indices exist by condition (i) in the definition of H_d. Moreover, by the choice of ε, $y_{k_1} > 0$ and $y_{k_2} < 0$ for any $y \in B_\varepsilon(x)$. Let $t\colon B_\varepsilon(x) \to [0, 1]$ be a continuous function such that $\mathbf{t}_{\eta^y}(t(y)) = e^{i\varphi}$. Then a section $y \mapsto \gamma^y$ of p over $B_\varepsilon(x)$ can be obtained by increasing the length of the k-th line segment to $l_k \geq 10$ for $k = k_1, k_2$ and grafting a straight line segment of length $l_0 \geq 0$ at $t(y)$. More precisely, the origin $0 \in \mathbf{C}$ lies in the interior of the triangle whose vertices are $e^{i\varphi}$, $ie^{i(\varphi + y_{k_1})}$ and $-ie^{i(\varphi + y_{k_2})}$. Therefore, any complex number can be written as

$$a_0 e^{i\varphi} + a_1 i e^{i(\varphi + y_{k_1})} - a_2 i e^{i(\varphi + y_{k_2})} \quad \text{for some } a_0, a_1, a_2 > 10.$$

Consequently the lengths l_0, l_{k_1}, l_{k_2} can be (continuously) chosen to achieve that $\gamma^y(1) = q$. □

Next we establish a version of (5.10) for condensed curves, beginning with the following lemma.

(5.11) Lemma. *Suppose that* $(\gamma,\varphi) \in \mathcal{V}_{(c,\sigma)} \subset \mathcal{N}(Q)$ *for some sign string* σ*. Then there exists a critical curve* $\eta \in \mathcal{M}(Q)$ *of type* σ *for which* $\bar{\varphi}^\eta = \varphi$ *(with* $\bar{\varphi}^\eta$ *as defined in* (12)*).*

Proof. Let $n = |\sigma|$, $J_1 < \cdots < J_n$ and $I_k \subset J_k$ be as in (3.1). Deform each $\gamma|_{I_k}$ to obtain a curve η so that for each $k \in [n]$, $\theta_\eta(t_k) = \varphi_{\sigma(k)}$ for at least one $t_k \in I_k$, but $\theta_\eta([0,1]) \subset [\varphi_-,\varphi_+]$ still holds; this is possible by (2.12) (j). □

(5.12) Definition. Let $\sigma_1 \prec \cdots \prec \sigma_m$ be sign strings, $n = |\sigma_m|$ and $\delta_j > 0$ ($j \in [m]$) satisfy $\delta_{j+1} > 2\delta_j$ for all $j \in [m-1]$. Define $H_c \subset \mathbf{R}^n$ to consist of all $x = (x_1,\dots,x_n) \in \mathbf{R}^n$ such that:

- (i) $\sigma_m(k)x_k < 0$ for each $k \in [n]$;
- (ii) For each $j \in [m]$, if $k_1 < \cdots < k_l$ are all the indices in $[n]$ such that $|x_k| < \delta_j$ (resp. $|x_k| \leq 2\delta_j$), then σ_j is the reduced string of $\tau\colon [l] \to \{\pm\}$, $\tau(i) = \sigma_m(k_i)$.

Again, H_c is weakly contractible for any choice of σ_j, δ_j by (1.15) and (1.19), since it has the same weak homotopy type as

$$X_{(\sigma_1,\sigma_1,\dots,\sigma_m,\sigma_m)}.$$

(5.13) Definition. Let $\sigma_1 \prec \cdots \prec \sigma_m$ be sign strings, $n = |\sigma_m|$ and $\delta_j > 0$ ($j \in [m]$) satisfy $\delta_{j+1} > 2\delta_j$ for all $j \in [m-1]$. Let $J(Q)$ denote the open interval consisting of all $\varphi \in \mathbf{R}$ such that $\mathcal{M}(Q)$ contains critical curves η of type σ_m with $\bar{\varphi}^\eta = \varphi$ (cf. [4], Corollary 5.4). For S a closed subinterval of $J(Q)$, define $E_c \subset \mathcal{M}(Q) \times S$ as in (5.9), replacing $R(Q)$ by S and H_d by H_c.

(5.14) Lemma. *Let* S *be a closed subinterval of* $J(Q)$*. Then for all sufficiently small* $\delta_m > 0$*, the space* E_c *defined above is weakly contractible.*

Proof. It was established in the proof of Proposition 5.3 of [4] that there exists a critical curve $\eta \in \mathcal{M}(Q)$ of type σ_m with $\bar{\varphi}^\eta = \varphi$ if and only if $\varphi \in \overline{R(Q)}$ and q lies in the open region to the right of the tangent T_φ of direction $ie^{i\varphi}$ to a certain circle. The set of all such φ is the open interval $J(Q)$, and if $S \subset J(Q)$ is a closed interval, then there exists a lower bound for the distance from q to T_φ for $\varphi \in S$.

The proof of the present lemma is analogous to that of (5.10) except for the last paragraph. Retaining the notation used there, choose $k_1, k_2 \in [n]$ such that $\sigma_m(k_1) = -$, $\sigma_m(k_2) = +$ and $|y_{k_i}| < \delta_1$ ($i = 1,2$) for all $y \in B_\varepsilon(x)$, where $\varepsilon < \varepsilon(x) = \min\{\varepsilon_1(x),\dots,\varepsilon_m(x)\}$. By the preceding remark, if $\delta_m > 0$ is sufficiently small, then q lies to the right of the line through $\eta^y(1)$ of direction $ie^{i\varphi}$. By further reducing $\delta_m > 0$ if necessary, it can be guaranteed that q lies in the cone with vertex at $\eta^y(1)$ and sides parallel to

$$i\exp\big(i(\varphi + y_{k_1})\big) \quad \text{and} \quad -i\exp\big(i(\varphi + y_{k_2})\big),$$

but does not lie in the triangle with vertices

$$\eta^y(1), \quad \eta^y(1) + 10i\exp\big(i(\varphi + y_{k_1})\big) \quad \text{and} \quad \eta^y(1) - 10i\exp\big(i(\varphi + y_{k_2})\big)$$

for any $\varphi \in S$, $y \in B_\varepsilon(x)$. This implies that q can be written as

$$a_1 ie^{i(\varphi + y_{k_1})} - a_2 ie^{(\varphi + y_{k_2})} \quad \text{for some } a_1, a_2 > 10.$$

A section $y \mapsto \gamma^y$ for p over $B_\varepsilon(x) \times S$ can thus be obtained by increasing the lengths l_{k_1}, l_{k_2} of the line segments of η^y to ensure that $\gamma^y(1) = q$. □

The proof of (5.1) is obtained by assembling the results of this section.

Proof of (5.1). It suffices to show that each of $\mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$, $\mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ and $\mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$ is weakly contractible. By (5.2), the case of $\mathcal{V}_{(\sigma_1,\dots,\sigma_m)}$ can be reduced to that of $\mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$. Let $k \geq 0$ and $g\colon \mathbf{S}^k \to \mathcal{V}$, $g(p) = (\gamma^p,\varphi^p)$, be a continuous map, where $\mathcal{V} = \mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ or $\mathcal{V} = \mathcal{V}_{(d,\sigma_1,\dots,\sigma_m)}$.

In the former case, let $S = \big\{\varphi^p \in \mathbf{R} : p \in \mathbf{S}^k\big\}$. By (5.11), S is a closed subinterval of $J(Q)$. By (5.4) and (5.5), g can be deformed within $\mathcal{V}_{(c,\sigma_1,\dots,\sigma_m)}$ to have image contained in E_c, with $\delta_m > 0$ as small as desired. Hence g is nullhomotopic by (5.14).

In the latter case, (5.6), (5.6) and (5.10) immediately imply that g is nullhomotopic. □

(5.15) Corollary. *Let τ be a top sign string for $\mathcal{M}(Q)$ and $n = |\tau|$. If there exist critical curves of type $-\tau$ in $\mathcal{M}(Q)$, then $\mathcal{N}(Q) \approx \mathbf{E} \times \mathbf{S}^{n-1}$. Otherwise $\mathcal{N}(Q) \approx \mathbf{E}$, for $\mathbf{E}$ the separable Hilbert space.*

Proof. Immediate from (4.3), (4.8) and (5.1). □

6. Homotopy equivalence between $\mathcal{M}(Q)$ and a sphere

(6.1) Lemma. *Suppose that $\pm\tau$ are both top sign strings for $\mathcal{M}(Q)$, where $|\tau| = n$. If $f\colon \mathbf{S}^{n-1} \to \mathcal{M}(Q)$ and $g\colon \mathcal{M}(Q) \to \mathbf{S}^{n-1}$ satisfy $\deg(gf) = 1$, then f and g are homotopy equivalences. In particular, $\mathcal{M}(Q)$ is homeomorphic to $\mathbf{E} \times \mathbf{S}^{n-1}$ and f represents a generator of $\pi_{n-1}\mathcal{M}(Q)$.*

Proof. According to (3.13), (3.15) and (5.15), under the present hypothesis $\mathcal{M}(Q)$ is either weakly contractible or a homology sphere of dimension $n-1$. The fact that $\deg(gf) = 1$ implies that the latter must hold, and that f and g induce isomorphisms on all (co)homology groups.

When $n = 2$, it follows directly from (3.13) that all higher homotopy groups of $\mathcal{M}(Q)$ are trivial, so that f and g are weak homotopy equivalences.

When $n > 2$, $\mathcal{M}(Q)$ and $\mathbf{S}^{n-1}$ are simply-connected. Passing to mapping cylinders and applying the relative version of the Hurewicz theorem, we again conclude that f and g induce isomorphisms on all homotopy groups.

The result follows by replacing $\mathbf{S}^{n-1}$ with $\mathbf{E} \times \mathbf{S}^{n-1}$ and using the fact that a weak homotopy equivalence between Hilbert manifolds is a homotopic to a homeomorphism (cf. [4], Lemma 1.7). □

Our next objective is to show that (under the hypothesis of the lemma) such f and g always exist. In fact, they can be constructed explicitly.

Briefly, the map g defined below measures the extent to which curves in $\mathcal{M}(Q)$ fail to be critical of type τ. Its definition is a slight variation of that of the map h in (3.17); cf. also Figure 11.

(6.2) Construction. Let $\mathcal{U}_\tau \subset \mathcal{M}(Q)$ consist of all curves which are $(\bar{\varphi}^\gamma, \varepsilon)$-quasicritical of type τ for some $\varepsilon \in (0, \frac{\pi}{4})$, where $\bar{\varphi}^\gamma$ is given by (12). Then $\mathcal{U}_\tau$ is an open subset of $\mathcal{M}(Q)$ containing the set $\mathcal{C}_\tau$ of all critical curves of type τ. Moreover, $\mathcal{C}_\tau$ is closed in $\mathcal{M}(Q)$; here the hypothesis that τ is a *top* sign string for $\mathcal{M}(Q)$ is essential.

Given $\gamma \in \mathcal{U}_\tau$ and intervals $J_1 < \dots < J_n$ satisfying the conditions of (3.1) for the quadruple $(\gamma, \bar{\varphi}^\gamma, \varepsilon, \tau)$, define

$$\alpha_k(\gamma) = \begin{cases} \sup_{t \in J_k} \{\theta_\gamma(t)\} - \frac{\pi}{2} & \text{if } \tau(k) = +; \\ \inf_{t \in J_k} \{\theta_\gamma(t)\} + \frac{\pi}{2} & \text{if } \tau(k) = -; \end{cases} \quad (k \in [n]) \quad \text{and}$$

$$\alpha(\gamma) = \frac{1}{n}\big[\alpha_1(\gamma) + \dots + \alpha_n(\gamma)\big].$$

It follows from (3.5) (a) that the maps $\alpha_k\colon \mathcal{U}_\tau \to \mathbf{R}$ are well-defined (i.e., they do not depend on the choice of ε and the J_k) and continuous; compare (3.18). Let

$$\Sigma = \big\{(x_1, \dots, x_n) \in \mathbf{R}^n : \textstyle\sum_k x_k = 0\big\} \approx \mathbf{R}^{n-1}$$

and define

$$A\colon \mathcal{U}_\tau \to \Sigma,\ A(\gamma) = \big(\alpha_1(\gamma) - \alpha(\gamma), \dots, \alpha_n(\gamma) - \alpha(\gamma)\big).$$

Clearly, $A(\gamma) = 0$ if and only if $\gamma \in \mathcal{C}_\tau$. Let $\mathcal{W} \subset \mathcal{M}(Q)$ be an open set such that

$$\mathcal{C}_\tau \subset \mathcal{W} \subset \overline{\mathcal{W}} \subset \mathcal{U}_\tau$$

and let $\lambda\colon \mathcal{M}(Q) \to [0,1]$ be a continuous function such that $\lambda^{-1}(1) = \mathcal{M}(Q) \smallsetminus \mathcal{W}$ and $\lambda^{-1}(0)$ is a neighborhood of $\mathcal{C}_\tau$. Let $r\colon \Sigma \to \mathbf{S}^{n-1}$ denote the map which collapses the complement of $B_1(0) \cap \Sigma$ to a single point, which will be identified with the south pole $-N \in \mathbf{S}^{n-1}$, with 0 mapping to the north pole N. Finally, define

$$g\colon \mathcal{M}(Q) \to \mathbf{S}^{n-1},\ g(\gamma) = \begin{cases} r\Big((1 - \lambda(\gamma))A(\gamma) + \lambda(\gamma)\frac{A(\gamma)}{|A(\gamma)|}\Big) & \text{if } \gamma \in \overline{\mathcal{W}}; \\ -N & \text{if } \gamma \notin \mathcal{W}. \end{cases} \tag{31}$$

Observe that $g^{-1}(N) = \mathcal{C}_\tau$. □

We shall now construct a generator f for $\pi_{n-1}\mathcal{M}(Q)$.

(6.3) Construction. Let C denote the cube $\big[-\frac{\pi}{2},\frac{\pi}{2}\big]^n \subset \mathbf{R}^n$, ∂C its boundary and

$$S = \big\{(x_1,\dots,x_n) \in C : x_{k_1} = \tfrac{\pi}{2} \text{ and } x_{k_2} = -\tfrac{\pi}{2} \text{ for some } k_1,\, k_2 \in [n]\big\}.$$

Note that $S \subset \partial C$ is the complement of the union of the open stars of the opposite vertices of C whose coordinates are given by $x_k = \frac{\pi}{2}$ and $x_k = -\frac{\pi}{2}$ for each $k \in [n]$, respectively. (These vertices are labeled by $+++$ and $---$ in Figure 13 (b).) We shall identify ∂C with $\mathbf{S}^{n-1}$ and S with its equator $\mathbf{S}^{n-2}$ when convenient.

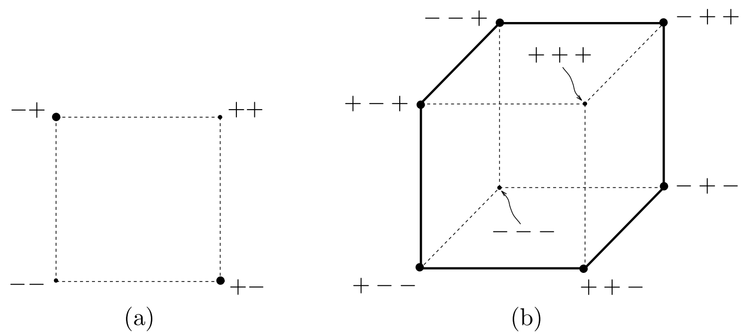


FIGURE 13. The subset $S \approx \mathbf{S}^{n-2}$ of C (in thick) for $n = 2$ and 3.

To simplify the explanation, let us assume first that there exists $\varphi \in \mathbf{R}$ such that it is possible to find critical curves $\gamma_1, \gamma_2 \in \mathcal{M}(Q)$ of types τ and $-\tau$ such that $\bar{\varphi}^{\gamma_1} = \varphi = \bar{\varphi}^{\gamma_2}$. (It is not hard to show that this is always the case if n is even, but this fact will not be used.) This implies that it is possible to find a critical curve $\gamma \in \mathcal{M}(Q)$ of type σ with $\bar{\varphi}^{\gamma} = \varphi$ for any σ with $|\sigma| \leq n$. Let κ_0, $\delta \in (0,1)$. Let $T = S \times [-\delta,\delta]$ be identified with a tubular neighborhood of $\mathbf{S}^{n-2} \equiv S$ in $\mathbf{S}^{n-1} \equiv \partial C$, with a point $(x,s) \in T$ lying in the hemisphere $H_{\operatorname{sign}(s)}$ at distance $|s|$ from $\mathbf{S}^{n-2}$ and $x \in \mathbf{S}^{n-2}$ realizing this distance (here $H_\pm$ are the two hemispheres determined by $\mathbf{S}^{n-2}$).

For each $(x,s) \in T$, let $\eta^{(x,s)}$ denote the unique curve of type

$$\underbrace{c\dots c}_{n+1}$$

such that $\Phi_{\eta^{(x,s)}}(0) = (0,1) \in \mathbf{C} \times \mathbf{S}^1$, $\mathbf{t}_{\eta^{(x,s)}}(1) = z$ and

$$\theta_{\eta^{(x,s)}} = \varphi + (1+s)x_k \tag{32}$$

at the point where the k-th circle is concatenated with the $(k+1)$-th circle, for all $k \in [n]$, where each of the circles has radius $\frac{1}{\kappa_0}$. Observe that for all $x \in S$, $\eta^{(x,s)}$ is critical, condensed or diffuse according as $s = 0$, $s < 0$ or $s > 0$, respectively.

The curves $\eta^{(x,s)}$ do not in general satisfy $\eta^{(x,s)} = q$, but this can be corrected as follows. Because of the hypothesis on φ, if $\kappa_0 \in (0,1)$ is sufficiently close to 1 and $\delta \in (0,1)$ sufficiently close to 0, then

$$\big\langle \eta^{(x,s)} - q\,,\, e^{i\varphi}\big\rangle < 0 \quad \text{for all } x \in S,\ s \in [-\delta,\delta].$$

For fixed $x \in S$, choose $t_0, t_1, t_2 \in [0,1]$ such that $\theta_{\eta^{(x,0)}}(t_i) = \varphi$, $\varphi + \frac{\pi}{2}$ and $\varphi - \frac{\pi}{2}$ for $i = 0,1,2$, respectively. By grafting line segments at $\eta^{(x,0)}(t_i)$ $(i = 0,1,2)$, a curve $\gamma^{(x,0)}$ with $\gamma^{(x,0)}(1) = q$ as desired is obtained. Clearly, the same procedure will work in a neighborhood of $(x,0)$, for the same choices of t_i. Using a partition of unity and reducing $\delta > 0$ further if necessary, this yields a family $\gamma^{(x,s)} \in \mathcal{M}(Q)$ $(x \in S,\ s \in [-\delta,\delta])$. The chosen open sets, the corresponding t_i and the lengths of the segments do not change the homotopy class of f and are irrelevant for the calculation of $\deg(gf)$.

The correspondence $(x,s) \mapsto \gamma^{(x,s)} \in \mathcal{M}(Q)$ can be extended to a map $f\colon \mathbf{S}^{n-1} \to \mathcal{M}(Q)$ through nullhomotopies of the families $\gamma^{(x,\delta)}$ and $\gamma^{(x,-\delta)}$ $(x \in S)$ within $\mathcal{U}_d$ and $\mathcal{U}_c$, respectively. The latter two sets are contractible by Theorems 3.3 and 4.19 of [4]. This completes the construction of f under the initial assumption on φ.

In the general case, let $\gamma_{\pm\tau} \in \mathcal{M}(Q)$ be arbitrary critical curves of type $\pm\tau$, and set $\varphi_{\pm\tau} = \bar{\varphi}^{\gamma_{\pm\tau}} \in \mathbf{R}$. Let $U_{\pm\tau}$ denote the open star in S of the vertices $p = \frac{\pi}{2}(\tau(1),\dots,\tau(n))$ and $-p$, respectively.

Since $\overline{U}_\tau \cap \overline{U}_{-\tau} = \emptyset$, we can find a continuous function $S \to \mathbf{R}$, $x \mapsto \varphi^x$, taking values in the closed interval with endpoints $\varphi_{\pm\tau}$, such that $\varphi^x = \varphi_{\pm\tau}$ if $x \in U_{\pm\tau}$. By Proposition 5.3 in [4], if $|\sigma| < n$, then there exist critical curves γ of type σ with $\bar{\varphi}^\gamma = \psi$ for all ψ in this interval. Hence, the preceding definition of $\gamma^{(x,s)}$ works for every $x \in S$ if φ is replaced by φ^x in (32). $\square$

(6.4) Lemma. *Suppose that $\pm\tau$ are both top sign strings for $\mathcal{M}(Q)$, where $|\tau| = n$. Let $g\colon \mathcal{M}(Q) \to \mathbf{S}^{n-1}$ and $f\colon \mathbf{S}^{n-1} \to \mathcal{M}(Q)$ be the maps described in Constructions 6.2 and 6.3. Then* $\deg(gf) = \pm 1$.

Proof. Let N denote the north pole of $\mathbf{S}^{n-1}$ and $p = \frac{\pi}{2}(\tau(1), \dots, \tau(n)) \in S$. Then $(gf)^{-1}(N) = \{p\}$, hence the result will follow if gf is a homeomorphism near p. By Brouwer's invariance of domain, it suffices to show that gf is injective on a neighborhood of p in ∂C. Finally, by the definition of $f|_T$, it actually suffices to show that gf is injective on a neighborhood of p in S. Let $U \subset S$ be an open set containing p such that $\lambda(U) = \{0\}$ and $A(U) \subset B_1(0)$, where A and λ are as in (6.2). For $x, \bar{x} \in U$,

$$\alpha_k(\gamma^x) - \alpha_k(\gamma^{\bar{x}}) = x_k - \bar{x}_k \ (k \in [n]) \quad \text{and} \quad \alpha(\gamma^x) - \alpha(\gamma^{\bar{x}}) = \frac{1}{n}\sum_{k=1}^{n}(x_k - \bar{x}_k).$$

Therefore, $A(\gamma^x) = A(\gamma^{\bar{x}})$ if and only if $(x - \bar{x})$ is a multiple of $(1, 1, \dots, 1)$. In a small neighborhood of p in S, this occurs if and only if $x = \bar{x}$. Thus $gf|_S$ is injective near p, and $\deg(gf) = \pm 1$. $\square$

Obviously, one can achieve that $\deg(gf) = +1$ by composing g with a reflection if necessary. The next result is a corollary of (6.1) and (6.4).

(6.5) Corollary. *Let* $\mathrm{pr}\colon \mathcal{N}(Q) \to \mathcal{M}(Q)$ *be the restriction of the projection* $\mathcal{M}(Q) \times \mathbf{R} \to \mathcal{M}(Q)$. *Then* pr *is a homotopy equivalence and* $\mathcal{M}(Q)$ *is homeomorphic to* $\mathcal{N}(Q)$.

Proof. By (3.13), the induced map $\mathrm{pr}_*\colon H_*(\mathcal{N}(Q)) \to H_*(\mathcal{M}(Q))$ is surjective. Since $\mathcal{M}(Q)$ and $\mathcal{N}(Q)$ are either simultaneously contractible or simultaneously homotopy equivalent to a sphere, pr_* must actually be an isomorphism. We conclude that pr is a homotopy equivalence using the same argument as in the proof of (6.1). $\square$

The proof of the main theorem (stated in the introduction) is now straightforward.

(6.6) Theorem. *Let $\theta_1 \in (-\pi, \pi)$, $z = e^{i\theta_1}$ and $Q = (q, z) \in \mathbf{C} \times \mathbf{S}^1$. Then $\mathcal{M}(Q)$ is homeomorphic to $\mathbf{E} \times \mathbf{S}^{2k}$ or $\mathbf{E} \times \mathbf{S}^{2k+1}$ $(k \geq 0)$ for q in the open region not intersecting the negative real axis bounded by the three circles*

$$\begin{cases} C_{4k+4}(iz - i) \ \text{ and } \ C_{4k+2}(\pm(i + iz)), & \text{or} \\ C_{4k+4}(i - iz) \ \text{ and } \ C_{4k+6}(\pm(i + iz)), & \text{respectively.} \end{cases}$$

If q does not lie in the closure of any of these regions, then $\mathcal{M}(Q) \approx \mathbf{E}$. If q lies on the boundary of one of them, then $\mathcal{M}(Q) \approx \mathcal{M}\big((q - \delta, z)\big)$ for all sufficiently small $\delta > 0$.

Proof. Proposition 5.3 of [4] describes precisely when $\mathcal{M}(Q)$ contains critical curves of any given type. If $\mathcal{M}(Q)$ does not contain any critical curves (or, equivalently, if it does not admit a top sign string), then $\mathcal{M}(Q) \approx \mathbf{E}$ or $\mathbf{E} \times \mathbf{S}^0$ according as $\mathcal{U}_c$ is empty or not, as described in Theorem 6.1 of [4]. If it does admit a top sign string, then we conclude from (5.15) and Proposition 5.3 of [4] that the theorem holds if $\mathcal{M}(Q)$ is replaced by $\mathcal{N}(Q)$ in the statement. But $\mathcal{M}(Q) \approx \mathcal{N}(Q)$ by (6.5). $\square$

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