GENERATION OF SPINES IN PORCUPINE-LIKE HORSESHOES

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Abstract. We study certain one-parameter families of partially hyperbolic maps $F_t\colon \Sigma_2\times\mathbb{R}\to\Sigma_2\times\mathbb{R}$ of skew-product type generating so-called porcupine-like horseshoes. Such sets are topologically transitive and semiconjugate to the shift map in two symbols. They exhibit a very rich fiber structure characterized by the fact that the set Σ_2 is the disjoint union of two dense and uncountable subsets with opposite behavior: corresponding spines (preimage of a sequence by the semiconjugation) are nontrivial and trivial, respectively, that is, the semiconjugation is noninjective and injective, respectively. We will study the bifurcation process of creation and annihilation of nontrivial spines as the parameter t evolves. In particular, we focus on the Hausdorff dimension of these subsets of Σ_2. This study illustrates the richness of the process.

1. Introduction

We consider one-step skew-products defined over a full shift of two symbols (Σ_2,σ) with one-dimensional fibers,

$$F\colon \Sigma_2\times\mathbb{R}\longrightarrow\Sigma_2\times\mathbb{R},\quad F(\xi,x)=(\sigma(\xi),f_{\xi_0}(x)).$$

This dynamics is "partially hyperbolic" with a hyperbolic part inherited from the shift dynamics and a central part corresponding to the fibers. The two fiber maps f_0 and f_1 have no critical points, see Figure 1. The map f_0 is increasing with two hyperbolic fixed points, say 0 (repelling) and 1 (contracting). The map f_1 is a contraction reversing the orientation satisfying the cycle condition $f_1(1)=0$. Interesting dynamical properties of these skew-products such as occurrence of heterodimensional cycles, transitivity, intermingled contracting and expanding dynamics, and phase transitions associated to the central exponents arise from the reversion of the orientation, the cycle property of f_1, and minimality-like properties of the iterated function systems (IFS) associated to f_0 and f_1. See [5, 7, 9, 8] and the survey [6].

On the one hand, viewing the dynamics of this skew-product as an IFS, one gets a genuinely noncontracting IFS which mixes contracting and expanding behavior. It turns out to be difficult to analyze the dynamics as common approaches are rather limited. Investigating random iterations of general noncontracting IFS, fractal properties and, in particular, relations between Lyapunov exponents, dimension, and entropy have been studied recently (see, e.g. [9, 8] and also the references in [6]). Such approaches focus on properties of measures that are stationary with

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respect to the IFS and are "essentially" contracting, compare the discussion in the introduction of [13].

On the other hand, viewing the dynamics of this skew-product as a partially hyperbolic diffeomorphism (with a central part given by the fiber maps) on a maximal invariant transitive set is not any easier. This is because the transitive set contains periodic points with different (contracting and expanding) behavior in the central direction and the dynamics intermingles these two types of hyperbolicity. Thus, new methods have to be developed.

In this paper we continue the analysis in [10, 5, 7] where a quite simple, but very rich, model family of skew-products F as above is introduced. We study the dynamics of the maximal invariant set Λ of F in $\Sigma_2 \times [0,1]$. The set Λ is *semiconjugate* to the shift map in Σ_2, that is, there is a continuous onto map $\Pi\colon \Lambda \to \Sigma_2$ such that $\Pi \circ F = \sigma \circ \Pi$. For each $\xi \in \Sigma_2$ we consider the set $\Pi^{-1}(\xi) \subset \Lambda$, called the *spine* of ξ. This spine is *nontrivial* if it is not a singleton and *trivial* otherwise. In this way, the set Σ_2 splits into two disjoint invariant sets Σ_2^{non} and Σ_2^{trv} consisting of sequences with nontrivial and trivial spines, respectively.

In some loose sense, the information about the expanding part of the dynamics is encoded in Σ_2^{non}, while the set Σ_2^{trv} is related to the contracting behavior. The occurrence of nontrivial spines also serves as an indicator of the nonhyperbolic behavior in Λ. Thus the topology and dynamics of the transitive set Λ are related to the sets Σ_2^{non} and Σ_2^{trv}. Let us explain this point in more detail.

In our setting, the fiber map f_0 is concave and the fiber map f_1 is affine. It turns out that the spines of Λ are of the form $\{\xi\} \times I_\xi \subset \Lambda$, where $\xi \in \Sigma_2$ and I_ξ is either a point or a nontrivial closed segment. In this context, under very mild assumptions on f_0 and f_1 (see the discussion below), the set Λ is *topologically transitive* (existence of a dense orbit) and is called a *porcupine-like horseshoe.* A naive geometrical idea of a porcupine-like horseshoe is the following: consider a horseshoe in the plane and select two uncountable dense subsets of it, for each point in the first set glue a segment vertically to the plane (a nontrivial spine) and for the second set just glue a point (a trivial spine). The precise definition is the following.

Definition 1.1 (Porcupines)**.** Given an one-step skew product map $F\colon \Sigma_2 \times [0,1] \to \Sigma_2 \times [0,1]$, a compact maximal invariant set Λ of F in $\Sigma_2 \times [0,1]$ is a *porcupine-like horseshoe* (shortly, a *porcupine*) if it is topologically transitive and the subsets Σ_2^{non} and Σ_2^{trv} are both dense and uncountable in Σ_2.[1]

If $\Sigma_2^{\mathrm{non}} = \Sigma_2$ we say that the set Λ is a *completely spiny porcupine.*

Let us say a few words about previous results about porcupines. Sets of such type first appeared (without such a name) in the work [10] about the destruction of hyperbolic sets via heterodimensional cycles. The porcupines in [10] are essentially hyperbolic sets (they only support hyperbolic ergodic measures, [16]) and their spectra of central Lyapunov exponents (those associated to the fiber dynamics) have a gap separating the positive and the negative parts of the spectrum. The results in [16] state some thermodynamical properties of these porcupines. The notion of a porcupine was introduced in [5], where genuinely nonhyperbolic porcupines (supporting nonhyperbolic ergodic measures) are considered.

[1]Porcupines can be defined in more general settings. The key ingredient is the existence of a semiconjugation to some shift map, where the spines are the pre-images by the semiconjugation of the sequences in the set.

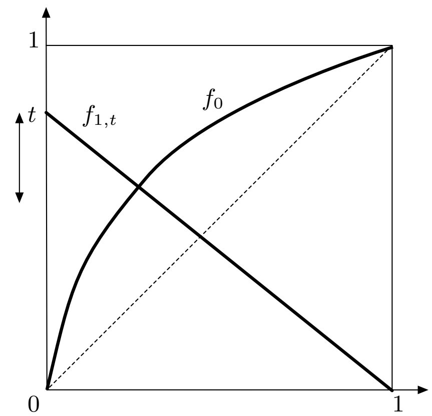

FIGURE 1. The fiber maps f_0 and $f_{1,t}$

Let us observe that a porcupine has the same flavor as the so-called *bony sets*, that is, a set which are the union of the graph of a continuous function (over the shift space) and an uncountable set of vertical segments (so-called *bones*) belonging to the closure of this graph. The bones correspond to the nontrivial spines of the porcupine, see [15].

We study a bifurcation behavior of one-parameter families of maps $(F_t)_{t\in[0,1]}$ of the form

$$F_t\colon \Sigma_2 \times \mathbb{R} \longrightarrow \Sigma_2 \times \mathbb{R}, \qquad (\xi, x) \;\mapsto\; (\sigma(\xi), f_{\xi_0,t}(x)), \tag{1.1}$$

such that for each $t \in (0,1)$ the maximal invariant set Λ_t of F_t in $\Sigma_2 \times [0,1]$ is a porcupine. We would like to understand the scenario of creation and destruction of spines in order to understand better one of the features (occurrence of nontrivial spines) that distinguishes porcupines from hyperbolic sets. For that for each parameter $t \in [0,1]$ we define the subset $\Sigma_{2,t}^{\mathrm{trv}}$ and $\Sigma_{2,t}^{\mathrm{non}}$ of Σ_2 consisting of sequences with trivial and nontrivial spines for F_t, respectively. The goal is to understand how the sets $\Sigma_{2,t}^{\mathrm{trv}}$ and $\Sigma_{2,t}^{\mathrm{non}}$ evolve with t. As a first simple, but still quite cumbersome, approach we study the Hausdorff dimension of the level sets $\Sigma_{2,t}^{\mathrm{trv}}$ and $\Sigma_{2,t}^{\mathrm{non}}$. This can be seen as a first step into a multifractal analysis for porcupines. Further possibilities of finer analysis could involve an investigation of the Hausdorff dimension of the level sets of sequences with spines of a given length or within a given interval.

Let us give more details of the families that we will study. We consider one-parameter families of skew-product maps as in (1.1) where $f_{0,t} = f_0$ is an increasing concave C^2-map independent of t with two fixed hyperbolic points $f_0(0) = 0$ and $f_0(1) = 1$ and $f_{1,t}$ is the affine map $f_{1,t}(x) = t\,(1-x)$, see Figure 1.

We denoted this set of families by $\mathcal{P}$. The maximal invariant set Λ_t of F_t in $\Sigma_2 \times [0,1]$ is defined

$$\Lambda_t \stackrel{\text{def}}{=} \bigcap_{i\in\mathbb{Z}} F_t^i\big(\Sigma_2 \times [0,1]\big).$$

A key property of the families in $\mathcal{P}$ is the *cycle condition* $f_{1,t}(1) = 0$. Another important condition is that the IFS generated by f_0^{-1} and $f_{1,t}^{-1}$ is minimal for every

$t \in (t_c, 1)$, where $t_c \in (0,1)$ is given by $f_0'(t_c) = 1$. This minimality property of the IFS is key for the transitivity of the porcupines Λ_t for $t \in (t_c, 1)$, see [7].

The dynamics for $t = 0$ and $t = 1$ correspond to two "degenerate" cases of different nature: for $t = 0$ the dynamics has no spines at all, while for $t = 1$ there is a completely spine porcupine. The families in $\mathcal{P}$ provide simple models describing a transition from a dynamics without spines ($\Sigma_{2,0}^{\mathrm{trv}} = \Sigma_2$) to a completely spiny dynamics ($\Sigma_{2,1}^{\mathrm{non}} = \Sigma_2$). This bifurcation phenomenon has a similar flavor as (and its study was partially motivated by) the monotonicity of the complexity of the dynamics in the quadratic family, see [2, 18].

One could naively guess that the dynamics of the porcupines Λ_t "gain complexity" as the parameter t increases and approaches the completely spiny porcupine generating new nontrivial spines[2]. Translating this guess to the base dynamics would mean that the set $\Sigma_{2,t}^{\mathrm{non}}$ monotonically "grows" as t approaches 1. However, this does not happen and it will turn out that the process of the generation of nontrivial spines is rather complicated. For instance, a nontrivial spine may disappear after its generation and remains trivial until it revives at $t = 1$ (we will call such a spine *evanescent*), see Theorem 6. This fact has the same flavor of the annihilation of periodic orbits in homoclinic bifurcations in [14].

To state precisely our result we need to fix some notation and facts. Given $\xi = \dots \xi_{-i} \dots \xi_0 \dots \xi_i \in \Sigma_2$ we write

$$\xi \stackrel{\mathrm{def}}{=} \xi^-.\xi^+, \quad \text{where} \quad \xi^- \stackrel{\mathrm{def}}{=} \dots \xi_{-i} \dots \xi_{-1}, \qquad \xi^+ \stackrel{\mathrm{def}}{=} \xi_0\, \xi_1 \dots \xi_i \dots .$$

Let Σ_2^+ and Σ_2^- be the set of sequences ξ^+ and ξ^-, respectively.

In the set Σ_2 consider the canonical distance d defined by

$$(1.2)\quad d(\varpi, \theta) = 2^{1/2}\, 2^{-n(\varpi,\theta)} : \ n(\varpi,\theta) \text{ is the smallest value of } |n| \text{ with } \varpi_n \neq \theta_n.$$

With this distance the Hausdorff dimensions of Σ_2 and Σ_2^- satisfy $\mathrm{HD}(\Sigma_2) = 2$ and $\mathrm{HD}(\Sigma_2^-) = 1$. For the definition of Hausdorff dimension see Section 7.

Theorem 1. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}$. Then for all $t \in (0,1)$ it holds* $\mathrm{HD}(\Sigma_{2,t}^{\mathrm{non}}) < 2$.

Since $f_{i,t}([0,1]) \subset [0,1]$ for every $t \in [0,1]$ and $i = 0,1$, the property whether the spine of $\xi = \xi^-.\xi^+$ is trivial or nontrivial is determined by the negative part ξ^- only: if the spine of $\xi = \xi^-.\xi^+$ is trivial (resp. nontrivial) then the same holds for every $\zeta \in \Sigma_2$ of the form $\zeta = \xi^-.\zeta^+$. Thus we say that $\xi^- \in \Sigma_2^-$ has a *trivial spine* if, and only if, the spine of any sequence of the form $\zeta = \xi^-.\zeta^+$ is trivial. Otherwise, we say that $\xi^- \in \Sigma_2^-$ has a nontrivial spine.

A natural question is whether there exist sequences $\xi \in \Sigma_2^-$ having the same type of spine (trivial or not) for every $t \in (t_0, 1)$ for some $t_0 \in (0,1)$. As in the definitions of $\Sigma_{2,t}^{\mathrm{non}}, \Sigma_{2,t}^{\mathrm{trv}} \subset \Sigma_2$, let $\Sigma_{2,t}^{-,\mathrm{non}}$ and $\Sigma_{2,t}^{-,\mathrm{trv}}$ be the subsets of Σ_2^- of sequences with nontrivial and with trivial spines (for F_t), respectively. Define the subsets

$$\Sigma_2^{-,\mathrm{non}}(t_0) \stackrel{\mathrm{def}}{=} \bigcap_{t\in(t_0,1)} \Sigma_{2,t}^{-,\mathrm{non}} \quad \text{and} \quad \Sigma_2^{-,\mathrm{trv}}(t_0) \stackrel{\mathrm{def}}{=} \bigcap_{t\in(t_0,1)} \Sigma_{2,t}^{-,\mathrm{trv}}$$

consisting of the sequences in Σ_2^- whose spines for all $t \in (t_0, 1)$ are nontrivial and trivial, respectively.

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[2]The topological entropy of $F_t|_{\Lambda_t}$ is constant and equal to $\log 2$ in the parameter range $(0,1]$. For that note that the restriction of $F_t|_{\Lambda_t}$ is semiconjugate to the full shift $\sigma\colon \Sigma_2 \to \Sigma_2$ and the fiber dynamics is noncritical. This implies that no entropy is generated by the fiber dynamics and thus the topological entropy of $F_t|_{\Lambda_t}$ is equal to the one of the shift, see [3, 4].

Let $\mathcal{B}$ be the σ-algebra generated by the cylinders of Σ_2, see Section 2.3 for details. In the set $\mathcal{B}$ we consider the Bernoulli probability measures $\mathfrak{b}_p$, $p \in [0,1]$, where $\mathfrak{b}_p$ gives weight p to the symbol 0 and $(1-p)$ to the symbol 1, and consider the probability space $(\Sigma_2, \mathcal{B}, \mathfrak{b}_{1/2})$.

By [5] the set $\Sigma_{2,t}^{\mathrm{trv}}$ is a residual subset of Σ_2 for all $t \in (t_c, 1)$, recall that $t_c \in (0,1)$ is defined by $f_0'(t_c) = 1$. Recall also that $\beta = f_0'(0) > 1$. Theorem 2 below states that these sets have full $\mathfrak{b}_{1/2}$ measure.

Theorem 2. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}$. Then*

$$\mathfrak{b}_{1/2}(\Sigma_{2,t}^{\mathrm{trv}}) = 1 \quad \textit{for every } t \in (0,1) \quad \textit{and} \quad \mathfrak{b}_{1/2}\Big(\bigcap_{t\in(0,\beta^{-1})} \Sigma_{2,t}^{\mathrm{trv}}\Big) = 1.$$

The above result follows from Theorem 1 and the fact that the Bernoulli measure $\mathfrak{b}_{1/2}$ coincides with the Hausdorff measure m_2 (see Proposition 2.13). A natural question is to estimate the measures of the sets in Theorem 2 for other Bernoulli measures. As a further consequence we obtain that the entropy of the porcupine is "concentrated" in the trivial spines.

Denote by h_{top} the topological entropy.

Corollary 1. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}$. Then $h_{\mathrm{top}}(F|_{\Lambda_t}) = h_{\mathrm{top}}(\sigma|_{\Sigma_{2,t}^{\mathrm{trv}}}) = \log 2$ for every $t \in (0,1)$.*

To prove this corollary recall the comments in the footnote above and note that $h_{\mathrm{top}}(F|_{\Lambda_t}) = \log 2$. On the other hand, Theorem 2 claims that $\mathfrak{b}_{1/2}(\Sigma_{2,t}^{\mathrm{trv}}) = 1$ and thus $h_{\mathrm{top}}(\sigma|_{\Sigma_{2,t}^{\mathrm{trv}}}) = \log 2$ for every $t \in (0,1)$. Thus $h_{\mathrm{top}}(F|_{\Lambda_t}) = h_{\mathrm{top}}(\sigma|_{\Sigma_{2,t}^{\mathrm{trv}}})$, proving the corollary, see [1].

The following theorem implies that the transition to a completely spiny porcupine at $t = 1$ happens suddenly and lots of spines are created instantaneously.

Theorem 3 (Abrupt appearance of spines)**.** *Consider any $(F_t)_{t\in[0,1]} \in \mathcal{P}$. Then it holds $\mathrm{HD}(\Sigma_2^{-,\mathrm{trv}}(0)) > 0$.*

To prove this theorem we exhibit a subset with positive Hausdorff dimension consisting of sequences ξ in $\Sigma_2^{-,\mathrm{trv}}(0)$. More precisely, for each $\ell \in \mathbb{N}$, $\ell \geq 2$, consider the set of words

$$B_\ell \stackrel{\mathrm{def}}{=} \{1^{2\ell}0^\ell,\, 110\}$$

and its associated sets of sequences $\mathbb{E}_{B_\ell}$ defined by

$$\mathbb{E}_{B_\ell} \stackrel{\mathrm{def}}{=} \{\xi^- : \xi_{-1}\cdots\xi_{-k}\cdots \text{ is a concatenation of words in } B_\ell\} \subset \Sigma_2^-.$$

We will see in Proposition 3.6 that $\mathrm{HD}(\mathbb{E}_{B_\ell}) > 0$, thus the next proposition implies Theorem 3.

Proposition 1. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}$. Then there is ℓ_0 such that $\mathbb{E}_{B_\ell} \subset \Sigma_2^{-,\mathrm{trv}}(0)$ for every $\ell \geq \ell_0$ and $t \in (0,1)$.*

In this proposition ℓ_0 depends on the map f_0. A key ingredient of the proof of this proposition is the concavity of f_0 which implies that finite compositions of the maps f_0 and $f_{1,t}$ such that $f_{1,t}$ appears (never/once/several times/an infinite number of times) consecutively an even number of times preserve orientation, are concave and thus have a unique fixed point. This holds for maps "associated" to words in B_ℓ.

We now consider the subset $\mathcal{P}_{\exp}$ of $\mathcal{P}$ consisting of the families $(F_t)_{t\in[0,1]} \in \mathcal{P}$ such that the map f_0 satisfies the additional condition:

$$\frac{\lambda^2}{\beta}\,\frac{(1-\lambda)}{1-\beta^{-1}} > 1, \qquad \text{where} \quad 0<\lambda = f_0'(1) < 1 < \beta = f_0'(0). \tag{1.3}$$

This property implies that the IFS generated by f_0 and $f_{1,t}$ is minimal for every $t \in (0,1]$, which in turn is the key property to show topological transitivity of the porcupine horseshoe Λ_t, see [5].

We now study the set of sequences with nontrivial spines for families in $\mathcal{P}_{\exp}$.

Theorem 4. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}_{\exp}$. Then the following holds*

(1) $\mathrm{HD}(\Sigma_2^{-,\mathrm{non}}(0)) = 0$.
(2) The set $\Sigma_2^{-,\mathrm{non}}(0)$ is uncountable.
(3) For every $t_0 \in (0,1]$ it holds $\mathrm{HD}(\Sigma_2^{-,\mathrm{non}}(t_0)) > 0$.

We next consider the problem of stabilization of nontrivial spines.

Definition 1.2 (Stable spine)**.** The spine of $\xi \in \Sigma_2$ is *stable* at $t_0 \in (0,1]$ if the map $t \mapsto I_{\xi,t}$ is continuous at t_0 (here we consider the Hausdorff distance).

Note that if the sequence ξ has a stable nontrivial spine for t_0 the same holds for all parameter t close to t_0.

We observe that the stabilization of "some" trivial spines is an easy problem. For instance, for every $t_0 \in (0,1)$ there is $\delta(t_0)$ (where $\delta(t_0) \to 0$ as $t_0 \to 0$) such that every sequence with a "proportion" of 1's bigger than $\delta(t_0)$ has a stable trivial spine for every $t \in (0, t_0]$, see Proposition 2.6 and Corollary 2.7. A much more interesting problem concerns the stabilization of nontrivial spines. Note that Theorem 3 implies that there are "many spines" which are not stable at $t=1$. To state a more precise result consider the subset $\Sigma_2^{-,\mathrm{stb}}$ of Σ_2^- defined by

$$\Sigma_2^{-,\mathrm{stb}} \stackrel{\text{def}}{=} \{\xi^- \in \Sigma_2^- \,:\, \xi^-.\xi^+ \text{ is stable a } t=1 \text{ for any choice of } \xi^+ \in \Sigma_2^+\}.$$

Theorem 5. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}_{\exp}$. Then* $\mathrm{HD}(\Sigma_2^{-,\mathrm{stb}}) > 0$.

To prove this theorem we exhibit a subset of $\Sigma_2^{-,\mathrm{stb}}$ with positive Hausdorff dimension. Consider the set

$$C \stackrel{\text{def}}{=} \{0101, 001001\} = \{(01)^2, (001)^2\}$$

and its associated set $\mathbb{E}_C$ of sequences in Σ_2^- with $\mathrm{HD}(\mathbb{E}_C) > 0$ defined by

$$\mathbb{E}_C \stackrel{\text{def}}{=} \{\xi^- : \xi_{-1}\cdots\xi_{-k}\cdots \text{ is a concatenation of words in } C\} \subset \Sigma_2^-.$$

The following result implies Theorem 5.

Proposition 2. *Let $(F_t)_{t\in[0,1]} \in \mathcal{P}_{\exp}$. Then $\mathbb{E}_C \subset \Sigma_2^{-,\mathrm{stb}}$.*

We close this introduction with a result proving the existence of evanescent spines.

Definition 1.3 (Evanescent spine)**.** The spine of $\xi \in \Sigma_2$ is *evanescent* if there are $t_c < t_1 < t_2 < 1$ such that $\{\xi\} \times I_{\xi,t_1}$ is nontrivial and $\{\xi\} \times I_{\xi,t}$ is trivial for all $t \in [t_2, 1)$.[3]

[3]Condition $t_1 > t_c$ implies that the evanescent spine occurs inside a transitive set.

The existence of evanescent spines implies that the appearance of nontrivial spines is not a monotone process.

Theorem 6. *There is a family $(F_t)_{t\in(0,1]} \in \mathcal{P}$ with an evanescent spine.*

To prove this theorem we exhibit a periodic sequence with an evanescent spine. This result is a first step that illustrates the existence of evanescent spines and the richness of the process of generation of spines, but it is still quite unsatisfactory. There are many problems concerning these evanescent spines, for instance, the question of the existence of nonperiodic evanescent spines.

This paper is organized as follows. Theorems 1 and 2 are proved in Section 2. This section also contains some general properties of the spines and a sufficient conditions for a sequence having a trivial spine. In Section 3 we prove Theorem 3. This section also contains auxiliary results on the dynamics of maps associated to sequences where the symbol 1 only appears in groups of even size. In Section 4 we prove Theorem 4 about the existence of sequences whose spines are persistently nontrivial and estimate the Hausdorff dimension of this set of sequences. In Section 5 we prove Theorem 5 about the "stability" of the spines at $t = 1$ for a large subset (positive Hausdorff dimension) of Σ_2^-. In Section 6 we present an example of family of skew product maps with porcupines exhibiting an evanescent spine and prove Theorem 6. Finally, in the appendix in Section 7 we recall the definition of Hausdorff dimension and state some of its properties used throughout the paper.

2. Preponderance of trivial spines: Proofs of Theorems 1 and 2.

In this section we prove Theorems 1 and 2, see Sections 2.3 and 2.4, respectively. Before proving these results we make a brief discussion about properties and characterization of spines in Sections 2.1 and 2.2.

2.1. Properties of spines.

We begging by introducing some notation and definitions. We say that $w = \zeta_1 \dots \zeta_n \in \{0,1\}^n$ is a *word of length* $|w| = n$.

Definition 2.1 (Concatenations)**.** Consider a set of words $W = \{w_1, \dots, w_m\}$. An one-sided sequence $\xi^+ = (\xi_i)_{i\geq 0} \in \Sigma_2^+$ is a concatenation of words in W if there is an increasing infinite sequence of indices $(i_k)_{k\in\mathbb{N}}$ with $i_0 = 0$ such that $\xi_{i_k} \dots \xi_{i_{k+1}-1}$ is a word w_{i_k} in W for every k. In this case we write $\xi^+ = w_{i_0} w_{i_1} \dots w_{i_k} \dots$.

Given a sequence $\xi^- \in \Sigma_2^-$ we define its *conjugate sequence* $\widehat{\xi}^- = (\widehat{\xi}_j^-) \in \Sigma_2^+$ by $\widehat{\xi}_j^- \stackrel{\text{def}}{=} \xi^-_{-j-1}$. For a given a finite set W of words we define the following sets:

$$\begin{aligned}
\mathbb{E}_W &\stackrel{\text{def}}{=} \{\xi^- \in \Sigma_2^- : \widehat{\xi}^- \text{ is a concatenation of words in } W\} \subset \Sigma_2^-, \\
\mathbb{S}_W &\stackrel{\text{def}}{=} \{\xi = \xi^-.\xi^+ : \widehat{\xi}^- \in \mathbb{E}_W\} \subset \Sigma_2.
\end{aligned} \tag{2.1}$$

Remark 2.2. $\mathrm{HD}(\mathbb{S}_W) = 1 + \mathrm{HD}(\mathbb{E}_W)$.

Given a word $w = \zeta_1 \dots \zeta_k$, $\zeta_i = 0, 1$, we let

$$g_{w,t} \stackrel{\text{def}}{=} f_{\zeta_1,t} \circ \dots \circ f_{\zeta_k,t}, \qquad \text{where } f_{0,t} = f_0. \tag{2.2}$$

The geometry of the maps F_t implies that $\Pi_t^{-1}(\xi)$ is of the form $(\xi, I_{\xi,t})$, where $I_{\xi,t}$ is either a point or a closed nontrivial interval. By definition, we have that

$$I_{\xi,t} = \left\{ x \in [0,1] : (f^{-1}_{\xi_{-i},t} \circ \dots \circ f^{-1}_{\xi_{-1},t})(x) \in [0,1] \text{ for every } i \in \mathbb{N} \right\}.$$

Lemma 2.3 (Characterization of spines)**.** *For every $t \in [0,1]$ and $\xi = \xi^-.\xi^+ \in \Sigma_2$, if we write $\widehat{\xi}^- = w_1 w_2 \dots w_n \dots$ as a concatenation of words it holds*

$$I_{\xi,t} = \lim_{n\to\infty} g_{w_1,t} \circ \dots \circ g_{w_n,t}([0,1]).$$

Proof. By definition of a spine, $x \in I_{\xi,t}$ if, and only if, $g^{-1}_{w_n,t} \circ \dots \circ g^{-1}_{w_1,t}(x) \in [0,1]$ for every $n \in \mathbb{N}$, proving the lemma. □

Corollary 2.4. *Consider a set $W = \{u, w\}$ consisting of two words. Suppose that there is an interval $[a,b] \subset [0,1]$ and a parameter $t \in (0,1)$ such that*

$$[a,b] \subset g_{u,t}([a,b]) \cap g_{w,t}([a,b]).$$

Then $[a,b] \subset I_{\xi,t}$ for every $\xi \in \mathbb{S}_W$.

Proof. Given $\xi \in \mathbb{S}_W$ write $\widehat{\xi}^- = u^{h_1} w^{n_1} \dots u^{h_j} w^{n_j} \dots$, with $h_i, n_i \geq 0$. By hypothesis

$$[a,b] \subset g^{h_j}_{u,t} \circ g^{n_j}_{w,t}([a,b]), \quad \text{for every } j \in \mathbb{N}.$$

Lemma 2.3 implies that

$$[a,b] \subset \lim_{r\to\infty} g^{h_1}_{u,t} \circ g^{n_1}_{w,t} \circ \dots \circ g^{h_r}_{u,t} \circ g^{n_r}_{w,t}([a,b]) \subset I_{\xi,t},$$

which proves the corollary. □

A immediate consequence of the corollary above is the following.

Corollary 2.5. *Let w be a word such that $g_{w,t}$ has a repelling fixed point. Then the periodic sequence $w^{\mathbb{Z}}$ has a nontrivial spine (i.e., $I_{w^{\mathbb{Z}},t}$ is a nontrivial closed interval).*

2.2. **A sufficient condition for trivial spines.** For $k \in \{0,1\}$ and $\xi = \xi^-.\xi^+ \in \Sigma_2$ consider the limit frequency of the entry k in ξ^- given by

$$\Phi_k(\xi) \stackrel{\text{def}}{=} \phi_k(\xi^-) \stackrel{\text{def}}{=} \limsup_{n\to\infty} \frac{\#\{i \in [1,n] \colon \xi_{-i} = k\}}{n}. \tag{2.3}$$

Proposition 2.6 (A sufficient condition for trivial spines)**.** *Consider $\delta \in (0,1)$. Then $I_{\xi,t}$ is a singleton for all pair ξ, t such that $\Phi_1(\xi) > \delta$ and $t \in \big(0, \beta^{-\frac{1-\delta}{\delta}}\big)$.*

Corollary 2.7. *Consider $\xi \in \Sigma_2$ such that $\Phi_1(\xi) > 0$. Then there is $t_\xi > 0$ such that ξ has a trivial spine for all $t \in (0, t_\xi]$.*

Intuitively, the previous results mean that the set of sequences with trivial spines "grows" as t goes to 0^+ (note that the spines of a sequences of the form $0^{-\mathbb{N}}.\xi^+$ is nontrivial for all $t \in (0,1]$). On the other hand, there are sequences ξ with $\Phi_1(\xi) = 0$ having trivial spines for every $t \in (t_\xi, 1)$ for some $t_\xi \in (0,1)$:

Proposition 2.8. *There are nonperiodic sequences $\xi \in \Sigma_2$ with $\Phi_1(\xi) = 0$ and numbers $t_\xi \in (0,1)$ such that $I_{\xi,t}$ is a singleton for every $t \in (t_\xi, 1)$.*

By technical reasons we need to postpone the proof of this proposition to Section 3.2. Observe that in this proposition we just exhibit a sequence ξ^- with $\phi_1(\xi^-) = 0$ with a trivial spine. Our method can be used to get more sequences with $\phi_1(\xi^-) = 0$ having trivial spines. Note that the set of these sequences is necessarily "small": Proposition 4.1 claims that $\mathrm{HD}(\{\xi^- \in \Sigma_2^- : \phi_1(\xi^-) = 0\}) = 0$.

2.2.1. *Proof of Proposition 2.6.* We need the following simple lemma.

Lemma 2.9. *Let $\xi = \xi^-.\xi^+ \in \Sigma_2$ and write $\widehat{\xi^-} = w_1 \dots w_r \dots$ as a concatenation of words w_i. Suppose that there are $\rho > 1$, $C > 0$, and an increasing subsequence $(n_r)_r$ such that*

$$\left| \left(g^{-1}_{w_{n_r},t} \circ \cdots \circ g^{-1}_{w_1,t} \right)'(x) \right| \ge C\, \rho^{n_r}, \quad \text{for every } x \in I_{\xi,t}.$$

Then $I_{\xi,t}$ is a singleton.

Proof. Note that for each r one has $1 \ge \left| \left(g^{-1}_{w_{n_r},t} \circ \cdots \circ g^{-1}_{w_1,t} \right)(I_{\xi,t}) \right| \ge C\, \rho^{n_r}\, |I_{\xi,t}|$. Since $n_r \to \infty$ this implies that $|I_{\xi,t}| = 0$ and thus $I_{\xi,t}$ is a singleton. □

Take $\xi \in \Sigma_2$ and $\delta > 0$ as in Proposition 2.6. By hypothesis, for every $n \in \mathbb{N}$ there exists $m \ge n$ such that

$$\frac{\#\{i \in [1,m] \colon \xi_{-i} = 1\}}{m} > \delta.$$

Recalling that $f_{1,t} = t\,(1-x)$ and $\beta \ge f_0'(x) > 0$ if $x \in (0,1)$, we get

$$\left| (f^{-1}_{\xi_{-m},t} \circ \cdots \circ f^{-1}_{\xi_{-1},t})'(x) \right| \ge t^{-\lfloor \delta\, m \rfloor}\, \beta^{-m + \lfloor \delta\, m \rfloor}, \quad \text{for all } x \in I_{\xi,t},$$

where $\lfloor \alpha \rfloor$ stands for the entire part of $\alpha \in \mathbb{R}$. The proposition follows from Lemma 2.9 taking $t_\delta = (\beta)^{-\frac{1-\delta}{\delta}}$. □

2.3. Proof of Theorem 1.

Fix $t \in (0,1]$ and for $x \in [0,1]$ let

$$\Sigma_{x,t} \stackrel{\text{def}}{=} \{\xi \in \Sigma_2 \text{ such that } x \in I_{\xi,t}\}.$$

The following proposition is the key step of the proof of Theorem 1.

Proposition 2.10. *Given $t \in (0,1)$ there is $\rho_t < 2$ such that $\mathrm{HD}(\Sigma_{x,t}) < \rho_t < 2$ for every $t \in (0,1)$ and $x \in [0,1]$.*

To deduce Theorem 1 from this proposition note that if $I_{\xi,t}$ is a nontrivial interval then it contains some rational number $x \in (0,1)$. Therefore

$$\Sigma^{\mathrm{non}}_{2,t} \subset \bigcup_{x \in \mathbb{Q} \cap (0,1)} \Sigma_{x,t}.$$

Since this union is countable one has that (see Proposition 7.2)

$$\mathrm{HD}(\Sigma^{\mathrm{non}}_{2,t}) \le \sup_{x \in \mathbb{Q} \cap (0,1)} \mathrm{HD}(\Sigma_{x,t}) \le \rho_t < 2,$$

proving the theorem.

Proof of Proposition 2.10. Fix $t \in (0,1)$. Note that $f_0^n(f_{1,t}^2([0,1])) \to 1$ as $n \to \infty$ and that $f_{1,t}^n(f_0^2([0,1]))$ converges to the (attracting) fixed point $\frac{t}{1+t}$ of $f_{1,t}$ as $n \to \infty$. Therefore there is large $N_t \in \mathbb{N}$ ($N_t \to \infty$ as $t \to 1$) such that

$$\left(f_0^{N_t - 2} \circ f_{1,t}^2([0,1]) \right) \bigcap \left(f_{1,t}^{N_t - 2} \circ f_0^2([0,1]) \right) = \emptyset. \tag{2.4}$$

Given $x \in (0,1)$ and $K \in \mathbb{N}$ define the set

$$\Sigma^K_{x,t} \stackrel{\text{def}}{=} \{w \in \{0,1\}^{K N_t} \text{ such that } x \in g_{w,t}([0,1])\}.$$

We now give an upper bound (independent of x) of the cardinality of $\Sigma^K_{x,t}$.

Lemma 2.11. *For every $x \in (0,1)$ it holds $\#\,\Sigma^K_{x,t} \le (2^{N_t} - 1)^K$.*

Proof. Consider $w \in \Sigma^K_{x,t}$ and write $w = w_1 w_2$ where $|w_1| = (K-1)\, N_t$ and $|w_2| = N_t$. Note that

$$x \in g_{w_1,t} \circ g_{w_2,t}([0,1]) \subset g_{w_1,t}([0,1]).$$

Therefore $w_1 \in \Sigma^{K-1}_{x,t}$.

By (2.4), if $g^{-1}_{w_1,t}(x) \in f_0^{N_t-2} \circ f^2_{1,t}([0,1])$ then $g^{-1}_{w_1,t}(x) \notin f_{1,t}^{N_t-2} \circ f_0^2([0,1])$. This implies that there is at least one element $u \in \{0,1\}^{N_t}$ such that $g^{-1}_{u,t} \circ g^{-1}_{w_1,t}(x) \notin [0,1]$. Thus, necessarily, $w \neq w_1 u$. Arguing recursively this implies that

$$\#\,\Sigma^K_{x,t} \leq \#\,\Sigma^{K-1}_{x,t}\,(2^{N_t}-1) \leq (2^{N_t}-1)^K,$$

proving the lemma. □

Recall the definition of the metric $d(\varpi,\theta) = 2^{-n(\varpi,\theta)}$ in (1.2). Consider the *cylinders* of size $m+1$ defined by

$$C(i_0, i_1, \dots, i_m; k_0, k_1 \dots, k_m) \stackrel{\text{def}}{=} \{\theta \in \Sigma_2 \text{ such that } \theta_{i_\ell} = k_\ell,\ 0 \leq \ell \leq m\}.$$

Lemma 2.12. *The set $\Sigma_{x,t}$ has a covering $\mathcal{U}_K$ by cylinders of diameter 2^{-KN_t} with (at most) $2^{KN_t+1}(2^{N_t}-1)^K$ elements.*

Proof. Take $\xi = \xi^-.\xi^+ \in \Sigma_{x,t}$ and write $\widehat{\xi^-} = \zeta_1 \dots \zeta_r \dots, \quad \zeta_i \in \{0,1\}$. Note that $\zeta_1 \dots \zeta_{KN_t} \in \Sigma^K_{x,t}$ for every K. Lemma 2.11 implies that $\Sigma_{x,t}$ has a covering $\mathcal{U}_K$ by cylinders of diameter 2^{-KN_t} with at most $2^{KN_t+1}(2^{N_t}-1)^K$ elements (to see why this is so just note that for a fixed negative tail $\zeta_1 \dots \zeta_{KN_t}$ of a cylinder there are at most 2^{KN_t+1} possibilities for the positive part). □

Let $\mathcal{U}_K$ be a covering of $\Sigma_{x,t}$ as in Lemma 2.12. Then for any $s \in \mathbb{R}$

$$m_s(\mathcal{U}_K) \leq 2^{KN_t+1}\,(2^{N_t}-1)^K\,\left(2^{-KN_t}\right)^s. \tag{2.5}$$

See Section 7 for the standard definitions of the measures $m_s(\Sigma_{x,t})$, $m_{s,\epsilon}(\Sigma_{x,t})$, and $m_s(\mathcal{U})$. We have that

$$m_{s,\epsilon}(\Sigma_{x,t}) = \inf\{m_s(\mathcal{U}) : \mathcal{U} \text{ is a covering of } \Sigma_{x,t} \text{ with } \operatorname{diam}(\mathcal{U}) < \epsilon\}.$$

Thus if $2^{-KN_t} < \epsilon$ then $m_{s,\epsilon}(\Sigma_{x,t}) \leq m_s(\mathcal{U}_K)$. Hence equation (2.5) implies that

$$\begin{aligned} m_s(\Sigma_{x,t}) = \lim_{\epsilon\to 0^+} m_{s,\epsilon}(\Sigma_{x,t}) &\leq \lim_{K\to+\infty} 2^{KN_t+1}\,(2^{N_t}-1)^K\,(2^{-KN_t})^s = \\ &= \lim_{K\to+\infty} 2\left(2^{N_t}\,(2^{N_t}-1)\,(2^{-s\,N_t})\right)^K. \end{aligned}$$

Hence, by definition, $\mathrm{HD}(\Sigma_{x,t})$ is upper bounded by the number $s \in \mathbb{R}$ satisfying

$$2^{N_t}\,(2^{N_t}-1)\,(2^{-s\,N_t}) = 1.$$

Therefore

$$\mathrm{HD}(\Sigma_{x,t}) \leq 1 + \frac{\log(2^{N_t}-1)}{N_t \log 2} = \rho_t < 2,$$

which ends the proof of the proposition. □

2.4. **Proof of Theorem 2.** Let $\mathcal{B}$ be the σ-algebra generated by the cylinders $C(i_0, \dots, i_m; k_0, \dots, k_m)$. Denote by $\mathfrak{b}_{1/2}$ the Bernoulli probability in $(\Sigma_2, \mathcal{B})$ given by

$$\mathfrak{b}_{1/2}\big(C(i_0, \dots, i_m; k_0, \dots, k_m)\big) \stackrel{\text{def}}{=} 2^{-(m+1)}.$$

Next proposition is probably well known, we sketch its proof for completeness.

Proposition 2.13. $m_2 = \mathfrak{b}_{1/2}$.

Proof. Note that any pair of cylinders C and C' with the same size satisfies $m_2(C) = m_2(C')$: just note that associated to any finite covering $\mathcal{U} = (U_i)$ of C there is a covering $\mathcal{U}' = (U_i')$ of C' with the same number of elements and "comparable" diameters. This implies that these cylinders (and thus all cylinders of the same size as C) have the same measure m_2. Thus any cylinder C of size $m+1$ satisfies

$$m_2(C) = 2^{-(m+1)} = \mathfrak{b}_{1/2}(C),$$

proving the proposition. □

By Proposition 2.13, Theorem 2 is a consequence of the following lemma:

Lemma 2.14. *Let $t \in (0,1)$, then $m_2(\Sigma_{2,t}^{\mathrm{trv}}) = 1$ and $m_2\big(\bigcap_{t\in(0,\beta^{-1})} \Sigma_{2,t}^{\mathrm{trv}}\big) = 1$.*

Proof. By Theorem 1, $\mathrm{HD}(\Sigma_{2,t}^{\mathrm{non}}) < 2$ and thus $m_2(\Sigma_{2,t}^{\mathrm{non}}) = 0$. Hence, by Proposition 2.13,

$$1 = m_2(\Sigma_2) = m_2(\Sigma_{2,t}^{\mathrm{non}}) + m_2(\Sigma_{2,t}^{\mathrm{trv}}) = m_2(\Sigma_{2,t}^{\mathrm{trv}}),$$

proving the first part of the lemma.

To prove the second part take the characteristic function $\chi_{[1]}$ of the cylinder $C(0;1) = \{\xi \in \Sigma_2 \colon \xi_0 = 1\}$ and recall the definition of the frequency map Φ_k in (2.3),

$$\Phi_k(\xi) = \limsup_{n\to\infty} \frac{\#\{i \in [1,n] \colon \xi_{-i} = k\}}{n} = \limsup_{n\to\infty} \frac{\sum_{j=0}^{n-1} \chi_{[1]}\big(\sigma^{-j}(\xi)\big)}{n}.$$

Since σ is $\mathfrak{b}_{1/2}$-ergodic, the Birkhoff Ergodic Theorem implies that there is a set $\widehat{\Sigma}_2$ satisfying $\mathfrak{b}_{1/2}(\widehat{\Sigma}_2) = 1$ such that for every $\xi \in \widehat{\Sigma}_2$ it holds

$$\Phi_1(\xi) = \int \chi_{[1]}\, d\mathfrak{b}_{1/2} = \frac{1}{2}.$$

Take a strictly increasing sequence $(\alpha_n)_{n\in\mathbb{N}}$ of real numbers with $\lim_{n\to\infty} \alpha_n = 1/2$. Note that for every $\xi \in \widehat{\Sigma}_2$ and every $n \in \mathbb{N}$ one has $\Phi_1(\xi) > \alpha_n$. Proposition 2.6 now implies that $I_{\xi,t}$ is a singleton if $\xi \in \widehat{\Sigma}_2$ and $t \in (0, \beta^{-\frac{1-\alpha_n}{\alpha_n}}) \subset (0, \beta^{-1})$. Taking $n \to \infty$, we have that $I_{\xi,t}$ is a singleton if $\xi \in \widehat{\Sigma}_2$ and $t \in (0, \beta^{-1})$. Thus $\widehat{\Sigma}_2 \subset \bigcap_{t\in(0,\beta^{-1})} \Sigma_{2,t}^{\mathrm{trv}}$, ending the proof of the lemma. □

The proof of Theorem 2 is now complete. □

3. Abrupt appearance of spines: proof of Theorem 3

In this section we prove Theorem 3. This proof relies on the analysis of the dynamics of maps associated to sequences where the symbol 1 only appears in groups of even size (11-*sequences*). Section 3.1 deals with this sort of sequences. Lemma 3.3 localizes the spines of these 11-sequences. In Section 3.3 we prove Theorem 3. Before, in Section 3.2, we prove Proposition 2.8.

3.1. **Spines of 11-sequences.** A word w is a 11-*word* if it is of the form

$$w = 0^{m_0}\, 1^{2\,n_1}\, 0^{m_1}\, \dots\, 1^{2\,n_r}\, 0^{m_r}, \quad m_0 \geq 0 \text{ and } n_i, m_i \geq 1 \text{ for } i = 1, \dots, r.$$

In this case, the map $g_{w,t}$ associated to w is called a 11-*map* and is concave.

We say that $\xi^- \in \Sigma_2^-$ is an 11-*sequence* if $\widehat{\xi}^-$ can written as an infinite concatenation of 11-words. In particular, ξ^- contains infinitely many 0's.

Proposition 3.1. *The spine of a periodic* 11*-sequence is trivial for all* $t \in (0,1)$.

The subset of Σ_2^- consisting of periodic 11-sequences ξ^- is countable and hence has zero Hausdorff dimension (see Proposition 7.2). Thus to prove Theorem 3 ($\mathrm{HD}(\Sigma_2^{-,\mathrm{trv}}(0)) > 0$) we need to concatenate different types 11-words.

3.1.1. *Proof of Proposition 3.1.* We begin with the following simple lemma.

Lemma 3.2. *Let* $t \in (0,1)$ *and* w *be an* 11*-word. Then* $g_{w,t}$ *is concave and* $0 < g_{w,t}(0) < g_{w,t}(1) < 1$. *In particular,* $g_{w,t}$ *has a unique fixed point* $p_{w,t} \in [0,1]$ *that is attracting.*

Proof. Note that the composition of concave maps with positive derivatives is also concave with positive derivative. As w is an 11-word this holds for the map $g_{w,t}$.

As $g_{w,t}([0,1]) \subset (0,1)$, the map $g_{w,t}$ has at least one fixed point in $[0,1]$. We claim that every fixed point of $g_{w,t}$ is attracting, thus $g_{w,t}$ has exactly one fixed point. Take z with $g_{w,t}(z) = z$. By the mean value theorem and as $g_{w,t}(0) > 0$, there is a $y \in (0,z)$ such that

$$g'_{w,t}(y) = \frac{g_{w,t}(z) - g_{w,t}(0)}{z - 0} = \frac{z - g_{w,t}(0)}{z} < \frac{z}{z} = 1.$$

Hence, by concavity, $0 < g'_{w,t}(z) \leq g'_{w,t}(y) < 1$ and thus z is attracting. $\square$

By Lemma 3.2, for every 11-word w and $t \in (0,1)$ we can associate the unique attracting fixed point $p_{w,t}$ of $g_{w,t}$. We have the following lemma (which will be also used in Section 3.3) that is implies the proposition. Recall the definitions of the sets $\mathbb{E}_W$ and $\mathbb{S}_W$ in (2.1).

Lemma 3.3 (Localization of spines)**.** *Consider a set* $W = \{w_1, \dots, w_r\}$, $r \geq 1$, *consisting of* 11*-words. For each* $t \in (0,1)$ *let* $J_{W,t} \stackrel{\text{def}}{=} [p^-_{W,t}, p^+_{W,t}]$ *where*

$$p^-_{W,t} \stackrel{\text{def}}{=} \min\{p_{w_i,t},\, w_i \in W\} \quad \text{and} \quad p^+_{W,t} \stackrel{\text{def}}{=} \max\{p_{w_i,t},\, w_i \in W\}.$$

Then $I_{\xi,t} \subset J_{W,t}$ *for every* $\xi \in \mathbb{S}_W$ *and* $t \in (0,1)$.

The lemma implies that if $\widehat{\xi}^-$ is obtained concatenating a unique 11-word w then $I_{\xi,t} \subset J_{\{w\},t} = \{p_{w,t}\}$, proving the proposition.

Proof of Lemma 3.3. We argue by contradiction, suppose that there is a point x in $[0, p^-_{W,t}) \cap I_{\xi,t}$ for some $\xi = \xi^-.\xi^+ \in \mathbb{S}_W$ and $t \in (0,1)$. Write $\widehat{\xi}^- = w_{j_1} \cdots w_{j_i} \cdots$, where $w_{j_i} \in W$. Since $x \in I_{\xi,t}$ it holds

$$x_r \stackrel{\text{def}}{=} g^{-1}_{w_{j_r},t} \circ \cdots \circ g^{-1}_{w_{j_1},t}(x) \in [0,1], \quad \text{for all } r \geq 1. \tag{3.1}$$

As $x \leq p_{w_i,t}$ for every $w_i \in W$ and $p_{w_i,t}$ is the attracting fixed point of $g_{w_i,t}$, the concavity of the maps g_{w_i} implies that the sequence $(x_r)_r$ is decreasing and has a limit $x_\infty \in [0,1]$.

Since $f^2_{1,t}(0) = t - t^2$ one has that then $g_{w_i,t}(0) \geq t - t^2$ for every $w_i \in W$. This implies that $x_\infty \in [t - t^2, x]$. As $x < p^-_{W,t}$ there is $\delta > 0$ such that

$$\max\{g^{-1}_{w_i,t}(x_\infty),\, w_i \in W\} < x_\infty - \delta.$$

Therefore for large r we have $x_{r+1} = g^{-1}_{w_{j_{r+1}},t}(x_r) < x_\infty$, which is a contradiction.

A similar argument gives $(p^+_{W,t}, 1] \cap I_{\xi,t} = \emptyset$, proving the lemma. $\square$

3.2. **Proof of Proposition 2.8.** Consider any sequence $\xi = \xi^-.\xi^+$ such that

$$\widehat{\xi}^- = 1101100\dots110^i110^{i+1}11\dots. \tag{3.2}$$

By definition, $\Phi_1(\xi) = 0$. We now see that $I_{\xi,t}$ is a singleton, proving the proposition.

For $i \geq 0$ let $c_i \stackrel{\text{def}}{=} 0^i\,110$ and $p_{c_0,t}$ be the (attracting) fixed point of $g_{c_0,t}$ given by Lemma 3.2. Note that $p_{c_0,t}$ depends continuously on t and that $p_{c_0,t}$ is close to 1^- if t is close to 1^-. Thus there are $\kappa \in (0,1)$ and $t_\star \in (0,1)$ such that

$$f_0'(x) < \kappa, \quad \text{for every } x \in [p_{c_0,t}, 1] \text{ and } t \in [t_\star, 1]. \tag{3.3}$$

Lemma 3.4. *For all $x \in [p_{c_i,t}, 1]$, $i \geq 0$, and $t \in [t_\star, 1]$ it holds $g'_{c_i,t}(x) < \kappa < 1$.*

Proof. Note that $g_{c_0,t}$ is a contraction in $[p_{c_0,t}, 1]$. The lemma follows recalling (3.3) and noting that $g_{c_0,t}([p_{c_0,t}, 1]) \subset [p_{c_0,t}, 1]$, $f_0([p_{c_0,t}, 1]) \subset [p_{c_0,t}, 1]$, and $g_{c_i,t} = f^i_0 \circ g_{c_0,t}$. $\square$

Lemma 3.5. *For every $t \in [t_\star, 1)$ there exists $n_t \in \mathbb{N}$ such that*

$$g^{-1}_{c_n,t} \circ \cdots \circ g^{-1}_{c_1,t} \circ g^{-1}_{c_0,t} \circ g^{-1}_{c_0,t}(I_{\xi,t}) \subset [p_{c_0,t}, 1], \quad \textit{for every } n \geq n_t.$$

Proof. Note that fixed $t \in [t_\star, 1)$ there is $n_t \in \mathbb{N}$ such that for every $n \geq n_t$ and $x \in [0,1]$ we have

$$g_{c_n,t}(x) = f^n_0 \circ g_{c_0,t}(x) \geq f^n_0 \circ g_{c_0,t}(0) > p_{c_0,t}.$$

Therefore

$$g_{c_n,t}\big([0,1]\big) \subset [p_{c_0,t}, 1], \quad \text{for every } n \geq n_t. \tag{3.4}$$

Noting that $g_{c_0,t}\big([p_{c_0,t}, 1]\big) \subset [p_{c_0,t}, 1]$ and that f_0 is increasing it follows

$$g_{c_i,t}\big([p_{c_0,t}, 1]\big) \subset [p_{c_0,t}, 1], \quad \text{for every } i \geq 0. \tag{3.5}$$

Equations (3.4) and (3.5) imply that for $n \geq n_t$ one has that

$$g^2_{c_0,t} \circ g_{c_1,t} \circ \cdots \circ g_{c_n,t}\big([0,1]\big) \subset g^2_{c_0,t} \circ g_{c_1,t} \circ \cdots \circ g_{c_{n-1},t}\big([p_{c_0,t}, 1]\big) \subset [p_{c_0,t}, 1].$$

This inclusion and Lemma 2.3 imply that for every $n \geq n_t$ it holds

$$I_{\xi,t} \subset g^2_{c_0,t} \circ g_{c_1,t} \circ \cdots \circ g_{c_n,t}\big([0,1]\big) \subset [p_{c_0,t}, 1],$$

which implies the lemma. $\square$

Lemmas 3.4 and 3.5 imply that there is $C > 0$ such that for every $t \in [t_\star, 1)$ and $n \geq n_t$ one has

$$\Big(g^{-1}_{c_n,t} \circ \cdots \circ g^{-1}_{c_1,t} \circ g^{-2}_{c_0,t}\Big)'(x) \geq C\,\kappa^{-r}, \quad \text{for every } x \in I_{\xi,t}.$$

By Lemma 2.9 it follows that $I_{\xi,t}$ is a singleton, proving the proposition. $\square$

3.3. **Proof of Theorem 3.** We begin this section with an auxiliary proposition whose proof is given in Section 7.2.

Proposition 3.6. *Consider a set consisting of two words*

$$W = \{w_0 = \theta_1 \dots \theta_k,\ w_1 = \zeta_1 \dots \zeta_m\},$$

such that $\theta_i, \zeta_i \in \{0,1\}$, $k \le m$, *and* $\theta_j \neq \zeta_j$ *for some* $j \le k$. *Then*

$$\frac{1}{m} \le \mathrm{HD}(\mathbb{E}_W) \le \frac{1}{k}.$$

For each $\ell \in \mathbb{N}$, $\ell \ge 2$, consider the set of two independent 11-words

$$B_\ell \stackrel{\text{def}}{=} \{e_\ell \stackrel{\text{def}}{=} 1^{2\ell}0^\ell,\ v \stackrel{\text{def}}{=} 110\}$$

and its associated sets of sequences $\mathbb{E}_{B_\ell}$ and $\mathbb{S}_{B_\ell}$. Note that the set B_ℓ, $\ell \ge 2$, satisfies the hypothesis of Proposition 3.6.

Theorem 3.7. *There is* ℓ_0 *such that for every* $\ell \ge \ell_0$, *every* $t \in (0,1)$, *and every* $\xi \in \mathbb{S}_{B_\ell}$ *the set* $I_{\xi,t}$ *is a singleton.*

This result implies Theorem 3: by Theorem 3.7, $\mathbb{E}_{B_\ell} \subset \Sigma_2^{-,\mathrm{trv}}(0)$ for all $\ell \ge \ell_0$ and by Proposition 3.6, $\mathrm{HD}(\Sigma_2^{-,\mathrm{trv}}(0)) \ge \mathrm{HD}(\mathbb{E}_{B_\ell}) > 0$.

3.3.1. *Proof of Theorem 3.7.* Given $\xi = \xi^-.\xi^+ \in \mathbb{S}_{B_\ell}$ write

$$\widehat{\xi^-} = e_\ell^{h_0} v^{n_1} e_\ell^{h_1} \dots v^{n_r} e_\ell^{h_r} \dots, \text{ where } h_i, n_i \ge 0 \text{ for } i \ge 0. \tag{3.6}$$

Assume first that there is $j \in \mathbb{N}$ such that either $h_i = 0$ for every $i \ge j$ or $n_i = 0$ for every $i \ge j$. Let us consider the first case (the second one is similar and thus omitted). Note that if the spine of ξ is trivial then the spine of any $\sigma^k(\xi)$ is also trivial. Therefore we can assume without loss of generality $j = 0$. In this case ξ^- is periodic and by Proposition 3.1 the set $I_{\xi,t}$ is a singleton.

We now consider the case where $n_i, h_i \ge 1$ for all $i \ge 1$ (note that h_0 may be 0).

Proposition 3.8. *There is* ℓ_0 *such that for every* $\ell \ge \ell_0$ *and for every* $t \in (0,1)$ *there are constants* $C_t > 0$ *and* $\rho_t > 1$ *with the the following property:*

Given any $\xi = \xi^-.\xi^+ \in \mathbb{S}_{B_\ell}$ *write the conjugate* $\widehat{\xi^-}$ *of* ξ^- *as in* (3.6). *If* $h_i, n_i \ge 1$ *for every* $i \ge 1$ *then for all* $r \ge 1$ *and* $x \in I_{\xi,t}$ *it holds*

$$\Big((g_{e_\ell,t}^{h_r})^{-1} \circ (g_{v,t}^{n_r})^{-1} \circ \dots \circ (g_{e_\ell,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1} \circ (g_{e_\ell,t}^{h_0})^{-1}\Big)'(x) \ge C_t\, \rho_t^r. \tag{3.7}$$

By Lemma 2.9, this proposition implies that the set $I_{\xi,t}$ is a singleton for all $t \in (0,1)$ and $\xi \in \mathbb{S}_{B_\ell}$, ending the proof of Theorem 3.7.

Proof of Proposition 3.8. It is enough to see that there are $C_t' > 0$ and $\rho_t > 1$ with

$$\Big((g_{e_\ell,t}^{h_r})^{-1} \circ (g_{v,t}^{n_r})^{-1} \circ \dots \circ (g_{e_\ell,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1}\Big)'(x) \ge C_t'\, \rho_t^r$$

for all $r \ge 1$ and $x \in I_{\sigma^{-\bar h_0}(\xi),t}$, where $\bar h_0 = 3\, h_0\, \ell$.

We need some preliminary constructions. Consider the attracting fixed points $p_{v,t}$ and $p_{e_\ell,t}$ of the maps $g_{v,t}$ and $g_{e_\ell,t}$. Note that for t close to 0 these maps are contractions, while for t close to 1 this is not anymore the case. Define t_1 by

$$g_{v,t_1}'(0) = 1, \qquad t_1 = \beta^{-1/2}. \tag{3.8}$$

The choice of t_1 implies that

$$g_{v,t}'(x) \in (0,1) \quad \text{for all } t \in (0,t_1) \text{ and } x \in [0,1]. \tag{3.9}$$

The definition of t_1 also implies that for $t \geq t_1$ there is (exactly) one point $q_{v,t} \in [0,1]$ (depending continuously on t) with

$$g'_{v,t}(q_{v,t}) = 1, \quad q_{v,t_1} = 0.$$

For $t \in [0, t_1]$ let $q_{v,t} \stackrel{\text{def}}{=} 0$. The points $q_{v,t}$ depend continuously on t.

For $t \in (0,1]$ consider the fixed point $a_t \stackrel{\text{def}}{=} t/(1+t)$ of $f_{1,t}$ and note that

$$g_{v,t}(a_t) = f_{1,t}^2 \circ f_0(a_t) > f_1^2(a_t) = a_t.$$

This fact and the concavity of $g_{v,t}$ immediately imply that

$$p_{v,t} > a_t \quad \text{for all } t \in (0,1]. \tag{3.10}$$

Remark 3.9. Consider an 11-word c, the concave map $g_{c,t}$ (Lemma 3.2) and its fixed point $p_{c,t}$. The calculation above implies that $p_{c,t} > a_t$ for every $t \in (0,1]$.

Choice of ℓ_0. To define ℓ_0 we first define auxiliary constants k_0 and R. The concavity of $g_{v,t}$ and (3.10) imply that $q_{v,t} < p_{v,t}$ and $a_t < p_{v,t}$, respectively. This implies that the number k_0 below is well defined,

$$k_0 \stackrel{\text{def}}{=} \min\{k \geq 0 \text{ such that } g_{v,t}^{-k}(q_{v,t}) < a_t \text{ for all } t \in [t_1, 1]\}. \tag{3.11}$$

Lemma 3.10. *Let $R \stackrel{\text{def}}{=} \max\{g'_{v,t}(a_t),\, t \in [t_1,1]\}$. Then*

$$R^{k_0} \geq \max\{(g_{v,t}^n)'(x) \colon x \in [a_t, p_{v,t}],\, n \in \mathbb{N},\, \text{and } t \in [t_1,1]\} \geq 1. \tag{3.12}$$

Proof. We first see that $R^{k_0} \geq 1$. This is obvious if $k_0 = 0$. If $k_0 \geq 1$ then $a_t < q_{v,t}$ for some $t \in [t_1, 1]$. The concavity of $g_{v,t}$ implies that $1 = g'_{v,t}(q_{v,t}) < g'_{v,t}(a_t) \leq R$. Thus $R^{k_0} \geq 1$ proving the assertion. To prove the lemma tt remains to check the first inequality in (3.12).

By the concavity of $g_{v,t}^n$ (recall Lemma 3.2),

$$(g_{v,t}^n)'(a_t) \geq (g_{v,t}^n)'(x) \quad \text{for every} \quad x \in [a_t, p_{v,t}] \text{ and } t \in [t_1, 1].$$

This implies that

$$\begin{aligned} \max\{(g_{v,t}^n)'(x) \colon x \in [a_t, p_{v,t}],\, n \in \mathbb{N},\, t \in [0,1]\} = \\ = \max\{(g_{v,t}^n)'(a_t) \colon\, n \in \mathbb{N},\, t \in [t_1,1]\}. \end{aligned}$$

Thus it is enough to to see that

$$R^{k_0} \geq \max\{(g_{v,t}^n)'(a_t) \colon n \in \mathbb{N},\ t \in [t_1,1]\}.$$

For each $t \in [t_1, 1]$ define k_t as the first k with $g_{v,t}^{-k}(q_{v,t}) < a_t$. Note that $k_t \leq k_0$ and $g_{v,t}^j(a_t) \leq q_{v,t}$ for all $0 \leq j \leq k_t - 1$.

Claim 3.11. *It holds $(g_{v,t}^n)'(a_t) \leq (g_{v,t}^{k_t})'(a_t)$ for all $n \geq 0$ and $t \in [t_1, 1]$.*

Proof. Take first $n \geq k_t$. Note that from $(g_{v,t}^{k_t})(a_t) > q_{v,t}$ and the definition of $q_{v,t}$ it follows that $(g_{v,t}^{n-k_t})'(g_{v,t}^{k_t}(a_t)) < 1$. Hence

$$(g_{v,t}^n)'(a_t) = (g_{v,t}^{n-k_t})'(g_{v,t}^{k_t}(a_t))\,(g_{v,t}^{k_t})'(a_t) < (g_{v,t}^{k_t})'(a_t).$$

For the case $0 \leq n < k_t$ note that $(g_{v,t}^j)(a_t) \leq q_{v,t}$ for every $0 \leq j \leq k_t - 1$. Thus

$$(g_{v,t})'(g_{v,t}^j(a_t)) \geq 1 \quad \text{for every } 0 \leq j \leq k_t - 1.$$

This implies that

$$(g_{v,t}^{k_t})'(a_t) = g'_{v,t}(g_{v,t}^{k_t-1}(a_t)) \cdots g'_{v,t}(g_{v,t}^{n}(a_t))\,(g_{v,t}^{n})'(a_t) \geq (g_{v,t}^{n})'(a_t),$$

concluding the proof of the claim. □

We are now ready to end the proof of the lemma. If $k_0 = 0$ then $k_t = 0$ and $R^{k_0} = 1$. In this case $a_t > q_{v,t}$ for all $t \in [t_1, 1]$ and thus $(g_{v,t}^{n})'(a_t) \leq g'_{v,t}(a_t) \leq 1 = R^0$. If $k_0 \geq 1$, as $R^{k_0} \geq 1$ (and thus $R \geq 1$) and $k_t \leq k_0$ one has

$$R^{k_0} \geq R^{k_t} \geq (g_{v,t}^{k_t})'(a_t).$$

The last inequality follows from the definition of R and the concavity of $g_{v,t}$. This ends the proof of the lemma. □

We define ℓ_0 as follows,

$$\ell_0 \stackrel{\text{def}}{=} \min\{\ell \geq 2 \quad \text{such that} \quad (f_0^{\ell})'(x) < (2\,R)^{-(k_0+1)} \quad \text{for all } x \in [a_{t_1}, 1]\}. \tag{3.13}$$

This number is well defined: just note that $f_0'(1) < 1$ and $\lim_{n\to\infty} f_0^n(x) = 1$ for every $x \in [a_{t_1}, 1]$.

Lemma 3.12. *For every $\ell \geq \ell_0$ and $t \in [t_1, 1)$ one has*

$$(f_0^{\ell})'(p_{v,t}) < (2\,R)^{-(k_0+1)} < 1.$$

Proof. As $p_{v,t} > a_t \geq a_{t_1}$ (see (3.10)), from the definition of ℓ_0, $R > 1$, and $t \geq t_1$ we have

$$1 > (2\,R)^{-(k_0+1)} > (f_0^{\ell_0})'(p_{v,t}) = f_0'(f_0^{\ell_0-1}(p_{v,t})) \cdots f_0'(p_{v,t}). \tag{3.14}$$

As $f_0'(f_0^{\ell_0-1}(p_{v,t})) < f_0'(f_0^{\ell_0-2}(p_{v,t})) < \cdots < f_0'(p_{v,t})$ we have that

$$f_0'(f_0^{\ell_0-1}(p_{v,t})) < 1 \quad \text{for all} \quad t \in [t_1, 1].$$

The concavity of f_0 now implies that $f_0'(f_0^{\ell_0}(p_{v,t})) < 1$ and thus

$$(f_0^{\ell-\ell_0})'(f_0^{\ell_0}(p_{v,t})) < 1 \quad \text{for all} \quad t \in [t_1, 1].$$

Now using (3.14) we immediately get that for all $t \in [t_1, 1]$ it holds

$$(f^{\ell})'(p_{v,t}) = (f_0^{\ell-\ell_0})'(f_0^{\ell_0}(p_{v,t}))\,(f_0^{\ell_0})'(p_{v,t}) < (f_0^{\ell_0})'(p_{v,t}) < (2\,R)^{-(k_0+1)},$$

proving the lemma. □

End of the proof of Proposition 3.8. We now see that the expansion in (3.7) in the proposition holds for ℓ_0. We fix $\ell \geq \ell_0$ and, for simplicity, write $e = e_\ell$. Given a sequence $\xi = \xi^-.\xi^+ \in \mathbb{S}_{B_\ell}$ write

$$\widehat{\xi^-} = e^{h_0} v^{n_1} e^{h_1} \dots v^{n_r} e^{h_r} \dots, \quad \text{where } h_0 \geq 0 \text{ and } h_i, n_i \geq 1 \text{ for } i \geq 1.$$

For $r \geq 1$ define $\bar{n}_1 \stackrel{\text{def}}{=} n_1$, $\bar{h}_1 \stackrel{\text{def}}{=} n_1 + h_1$ and for $r \geq 2$ we let

$$\bar{n}_r \stackrel{\text{def}}{=} n_1 + \sum_{2 \leq i \leq r} (h_{i-1} + n_i) \quad \text{and} \quad \bar{h}_r \stackrel{\text{def}}{=} \sum_{1 \leq i \leq r} (n_i + h_i).$$

Given any $j \in \mathbb{N}$ we can write

$$j = j_r + \bar{n}_r \text{ with } 0 \leq j_r < h_r \text{ if } j \in [\bar{n}_r, \bar{h}_r),$$
$$j = j_r + \bar{h}_r \text{ with } 0 \leq j_r < n_{r+1} \text{ if } j \in [\bar{h}_r, \bar{n}_{r+1}).$$

Consider the segment $I_{\sigma^{-3\ell h_0}(\xi),t}$ of the spine of $\sigma^{-3\ell h_0}(\xi)$. Let $I_{0,t} \stackrel{\text{def}}{=} I_{\sigma^{-3\ell h_0}(\xi),t}$ and for $j \geq 1$ let $I_{j,t} \stackrel{\text{def}}{=} I_{\sigma^{-j-3\ell h_0}(\xi),t}$. Note that by definition,

$$I_{j,t} = g_{e,t}^{-j_r} \circ g_{v,t}^{-n_r} \circ \cdots \circ g_{e,t}^{-h_1} \circ g_{v,t}^{-n_1}(I_{0,t}), \text{ if } j \in [\bar{n}_r, \bar{h}_r],$$
$$I_{j,t} = g_{v,t}^{-j_r} \circ g_{e,t}^{-h_r} \cdots \circ g_{e,t}^{-h_1} \circ g_{v,t}^{-n_1}(I_{0,t}), \text{ if } j \in [\bar{h}_r, \bar{n}_{r+1}].$$

Remark 3.13. By Lemma 3.3, the set $I_{j,t}$ is contained in the closed interval $J_{\{e,v\},t}$ bounded by the fixed points $p_{e,t}$ and $p_{v,t}$ of $g_{e,t}$ and $g_{v,t}$, respectively.

We need to consider two cases according to the value of $t \in (0,1)$.

Case 1: $t < t_1 = \beta^{-1/2}$. In this case $g_{v,t}$ is a contraction (recall (3.9)). We claim that $g_{e,t}$ is also a contraction. Note that for any $x \in [0,1]$ it holds

$$g'_{e,t}(x) \leq g'_{e,t}(0) = (f_{1,t}^{2\ell} \circ f_0^{\ell})'(0) = t^{2\ell}(f_0^{\ell})'(0) = t^{2\ell}\beta^{\ell} < t_1^{2\ell}\beta^{\ell} = 1.$$

Hence in this case $g_{v,t}^{-1}$ and $g_{e,t}^{-1}$ are expanding in $[0,1]$ and thus (3.7) holds.

Case 2: $t \geq t_1 = \beta^{-1/2}$. We consider two subcases according to the relative positions of $p_{e,t}$ and $p_{v,t}$.

Case 2.1: $p_{v,t} < p_{e,t}$. By Remark 3.13, $I_{j,t} \subset [p_{v,t}, p_{e,t}]$. To get (3.7) it is enough to check that $g_{e,t}$ and $g_{v,t}$ are uniform contractions in $[p_{v,t}, p_{e,t}]$. To get the contraction of $g_{e,t}$ note that the concavity of $g_{e,t}$ implies that $g_{e,t}(p_{v,t}) > p_{v,t}$. Thus

$$g'_{e,t}(p_{v,t}) = (f_{1,t}^{2\ell} \circ f_0^{\ell})'(p_{v,t}) \leq (f_0^{\ell})'(p_{v,t}) < 1,$$

where the last inequality follows from $t \in [t_1, 1)$ and Lemma 3.12. The concavity of $g_{e,t}$ implies that $g'_{e,t}(x) < 1$ for all $x \in [p_{v,t}, p_{e,t}]$.

The contraction for $g_{v,t}$ follows noting that

$$g'_{v,t}(x) \leq g'_{v,t}(p_{v,t}) < 1 \quad \text{for all } x \in [p_{v,t}, p_{e,t}].$$

This completes the proof in this case.

Case 2.2: $p_{e,t} \leq p_{v,t}$. The expansion in (3.7) is a consequence of the following lemma.

Lemma 3.14. *Let $x \in I_{0,t} \subset [p_{e,t}, p_{v,t}]$, $t \geq t_1$. Then, for every $j \geq 1$, one has*

$$\Big((g_{e,t}^{h_j})^{-1} \circ (g_{v,t}^{n_j})^{-1} \circ \ldots (g_{e,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1}\Big)'(x) \geq 2^j.$$

Proof. We first estimate the derivatives of the maps $g_{e,t}^{h_i}$. By Remark 3.9 and since a_t (the fixed point of $f_{1,t}$) is increasing with t we have

$$[p_{e,t}, p_{v,t}] \subset [a_t, 1] \subset [a_{t_1}, 1].$$

The definition of $g_{e,t} = f_{1,t}^{2\ell} \circ f_0^{\ell}$ and Lemma 3.12 imply that

$$g'_{e,t}(x) \leq (f_0^{\ell})'(x) < (2R)^{-(k_0+1)}, \quad \text{for all } x \in [p_{e,t}, p_{v,t}]. \tag{3.15}$$

Recall that, by Remark 3.13, $I_{r,t} \subset [p_{e,t}, p_{v,t}]$. This implies that if $x \in I_{2i-1,t}$ for some $i \in \mathbb{N}$, then $g_{e,t}^{-m}(x) \in [p_{e,t}, p_{v,t}]$ for every $0 \leq m \leq h_i$. The concavity of $g_{e,t}$ and (3.15) imply that

$$(g_{e,t}^{h_i})'(x) \leq (g'_{e,t})^{h_i}(x) < (2R)^{-h_i(k_0+1)}, \text{ for all } x \in I_{2i-1,t} \text{ and } i \in \mathbb{N}. \tag{3.16}$$

This provides an upper bound for the derivatives of the maps $g_{e,t}^{h_i}$.

To estimate the *complete* product of the derivatives in the lemma consider

$$\min_{[p_{e,t},p_{v,t}]} (g_{v,t}^{-n_i})'(x) = \left(\max_{[p_{e,t},p_{v,t}]} (g_{v,t}^{n_i})'(x) \right)^{-1} \geq R^{-k_0}, ,$$

where the inequality follows from Lemma 3.10. This inequality, (3.15), and $R \geq 1$ imply that for every $x \in I_{0,t}$ and every $j \in \mathbb{N}$ it holds

$$\begin{aligned} \Big((g_{e,t}^{h_j})^{-1} \circ (g_{v,t}^{n_j})^{-1} \circ \dots (g_{e,t}^{h_1})^{-1} \circ (g_{v,t}^{n_1})^{-1} \Big)'(x) &\geq \\ &\geq (2\,R)^{h_j (k_0+1)}\, R^{-k_0} \dots (2\,R)^{h_1 (k_0+1)}\, R^{-k_0} \geq \\ &\geq 2^{h_j+\dots+h_1}\, R^{(h_j+\dots+h_1)(k_0+1)-j\,k_0} \geq 2^j, \end{aligned}$$

where the last inequality follows from $h_i \geq 1$ and thus $h_1 + \dots + h_j \geq j$. This proves the lemma. □

The proof of Proposition 3.8 is now complete. □

4. Persistence of nontrivial spines: Proof of Theorem 4

In this section we prove Theorem 4 about the existence of sequences whose spines are persistently nontrivial and estimate the Hausdorff dimension of this set of sequences.

4.1. **Proof of item** (1) **of Theorem 4:** $\mathrm{HD}(\Sigma_2^{-,\mathrm{non}}(0)) = 0$**.** Recall that by Corollary 2.7 we have

$$\Sigma_2^{-,\mathrm{non}}(0) \subset \{\xi^- \in \Sigma_2^- : \phi_1(\xi^-) = 0\}.$$

Thus it is enough to prove the following:

Proposition 4.1. $\mathrm{HD}\big(\{\xi^- \in \Sigma_2^- : \phi_1(\xi^-) = 0\}\big) = 0.$

Proof. Take $\epsilon < 0$ and let

$$a_n(\varepsilon) \stackrel{\text{def}}{=} \# \left\{ (\xi_{-1} \dots \xi_{-n}) \in \{0,1\}^n : \frac{\xi_{-1} + \dots + \xi_{-n}}{n} < \varepsilon \right\}$$

Recall that $\lfloor y \rfloor$ denotes the entire part of $y \in \mathbb{R}$ and note that

$$a_n(\varepsilon) = \sum_{0 \leq j \leq \lfloor \varepsilon\, n \rfloor} \binom{n}{j}. \tag{4.1}$$

Define

$$\delta(\varepsilon) \stackrel{\text{def}}{=} \mathrm{HD}\big(\{\xi^- \in \Sigma_2^- : \phi_1(\xi^-) < \varepsilon\}\big).$$

By the definition of Hausdorff dimension, the number $\delta(\varepsilon)$ is given by the condition

$$\lim_{n\to\infty} \frac{a_n(\varepsilon)}{2^{n\,r}} = \begin{cases} \infty, & \text{if } r < \delta(\varepsilon), \\ \\ 0, & \text{if } r > \delta(\varepsilon). \end{cases} \tag{4.2}$$

In what follows, we will consider small $\varepsilon \in \mathbb{Q}$ and large numbers $n \in \mathbb{N}$ with $\varepsilon\, n \in \mathbb{N}$.

Lemma 4.2. *For every small $\varepsilon > 0$ and large $n \in \mathbb{N}$ with $\varepsilon\, n \in \mathbb{N}$ it holds*

$$a_n(\varepsilon) \leq \frac{2\,n}{e\,(1-\varepsilon)^{(1-\varepsilon)\,n}\, \varepsilon^{\varepsilon\, n}}.$$

Proof. We need the following estimates.

Claim 4.3. *For every $n \geq 1$ we have*
$$e\left(\frac{n}{e}\right)^n \leq n! \leq n\,e\left(\frac{n}{e}\right)^n.$$

Proof. Clearly, this inequality holds for $n = 1$. We proceed inductively, assume that the inequalities hold for n. Note first that for $\varepsilon \geq 0$ it holds
$$\frac{\varepsilon}{1+\varepsilon} \leq \log(1+\varepsilon) \leq \varepsilon \quad \Longrightarrow \quad e^{\varepsilon} \leq (1+\varepsilon)^{1+\varepsilon} \leq e^{(1+\varepsilon)\,\varepsilon}.$$
Therefore
$$\frac{1}{e}\,(1+\varepsilon)^{\frac{1}{\varepsilon}} \leq 1 \leq \frac{1}{e}\,(1+\varepsilon)^{\frac{1}{\varepsilon}+1}.$$
This implies that for every $n \geq 1$ it holds (take $1/n = \varepsilon$)
$$\begin{aligned}\frac{e\left(\frac{n+1}{e}\right)^{n+1}}{e\left(\frac{n}{e}\right)^n} &= \frac{n+1}{e}\left(\frac{n+1}{n}\right)^n \leq (n+1)\,\frac{1}{e}\left(1+\frac{1}{n}\right)^{1/\frac{1}{n}} \leq n+1 = \\ &= \frac{(n+1)!}{n!} \leq (n+1)\,\frac{1}{e}\left(1+\frac{1}{n}\right)^{1+1/\frac{1}{n}} = \frac{n+1}{e}\left(\frac{n+1}{n}\right)^{n+1} = \\ &= \frac{(n+1)\,e\left(\frac{n+1}{e}\right)^{n+1}}{n\,e\left(\frac{n}{e}\right)^n}.\end{aligned}$$
This inequalities and the induction hypothesis for n imply that
$$e\left(\frac{n+1}{e}\right)^{n+1} \leq n!\,\frac{e\left(\frac{n+1}{e}\right)^{n+1}}{e\left(\frac{n}{e}\right)^n} \leq (n+1)!.$$
A similar argument proves the other inequality in the claim, ending the proof of the claim. $\square$

To get an upper bound for $a_n(\varepsilon)$ note that for $k \in \mathbb{N}$ with $k \leq (n+1)/3$ one has
$$\sum_{0\leq j\leq k}\binom{n}{j} < \binom{n}{k}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^k}+\cdots\right) = 2\binom{n}{k}. \tag{4.3}$$
This inequality follows arguing recursively and noting that for $j \leq k$ it holds
$$\frac{\binom{n}{j}}{\binom{n}{j-1}} = \frac{n-j+1}{j} \geq 2.$$
Recalling (4.1) and that $\varepsilon\,n \in \mathbb{N}$ and using (4.3) and Claim 4.3 we have that
$$\begin{aligned}a_n(\varepsilon) = \sum_{0\leq j\leq \varepsilon\,n}\binom{n}{j} < 2\binom{n}{\varepsilon\,n} = \frac{2\,n!}{(\varepsilon\,n)!\big(n-\varepsilon\,n\big)!} \leq \\ \leq \frac{2\,n\,e\left(\frac{n}{e}\right)^n}{e\left(\frac{\varepsilon\,n}{e}\right)^{\varepsilon\,n} e\left(\frac{(1-\varepsilon)\,n}{e}\right)^{(1-\varepsilon)\,n}} = \frac{2\,n}{e\,(1-\varepsilon)^{(1-\varepsilon)\,n}\,\varepsilon^{\varepsilon\,n}},\end{aligned}$$

proving the lemma. $\square$

Given $r, \varepsilon > 0$ define the map

$$R(r,\varepsilon) \stackrel{\text{def}}{=} (1-\varepsilon)^{1-\varepsilon}\,\varepsilon^{\varepsilon}\,2^{r} \tag{4.4}$$

and define $r_0(\varepsilon)$ implicitly by the condition $R(r_0(\varepsilon),\varepsilon)=1$, that is,

$$r_0(\varepsilon) \stackrel{\text{def}}{=} -\frac{(1-\varepsilon)\,\log(1-\varepsilon)+\varepsilon\,\log\varepsilon}{\log 2}, \qquad \lim_{\varepsilon\to 0} r_0(\varepsilon)=0.$$

Note that with this choice $R(r,\varepsilon)>1$ if $r>r_0(\varepsilon)$.

Lemma 4.4. *For every small rational $\varepsilon>0$ it holds $\delta(\varepsilon)\le r_0(\varepsilon)$.*

As $r_0(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $\delta(\varepsilon)\ge \mathrm{HD}(\Sigma_2^{-,\mathrm{non}}(0))$ this implies the proposition.

Proof of Lemma 4.4. By equation (4.2) it is enough to prove that for every $r>r_0(\varepsilon)$ it holds

$$\lim_{n_k\to\infty}\frac{a_{n_k}(\varepsilon)}{2^{n_k\,r}}=0,$$

where n_k is an increasing sequence of natural numbers with $\varepsilon\,n_k\in\mathbb{N}$ and $n_k\to\infty$. For notational simplicity let us omit in what follows the subscript k.

In order to calculate the limit above let

$$L(n) \stackrel{\text{def}}{=} \left(\frac{2\,n}{e}\right)^{1/n}, \qquad \lim_{n\to\infty} L(n)=1. \tag{4.5}$$

With the notation in equations (4.4) and (4.5) and using Lemma 4.2 we have

$$\frac{a_n(\varepsilon)}{2^{n\,r}} \le \frac{2\,n}{e\,(1-\varepsilon)^{(1-\varepsilon)\,n}\varepsilon^{\varepsilon\,n}\,2^{r\,n}} = \left(\frac{L(n)}{R(r,\varepsilon)}\right)^{n}.$$

Fix $r>r_0(\varepsilon)$. As $R(r,\varepsilon)>1$, the second part of (4.5) implies that for every n large enough it holds $L(n)<\kappa<R(r,\varepsilon)$ for some κ. Thus

$$\frac{L(n)}{R(r,\varepsilon)}<\kappa<1 \quad\Longrightarrow\quad \lim_{n\to\infty}\left(\frac{L(n)}{R(r,\varepsilon)}\right)^{n}\to 0.$$

Hence $\lim_{n\to\infty}\dfrac{a_n(\varepsilon)}{2^{n\,r}}=0$ for all $r>r_0(\varepsilon)$ and thus $\delta(\varepsilon)\le r_0(\varepsilon)$. $\square$

The proof of Proposition 4.1 is now complete. $\square$

4.2. **Proof of item (2) of Theorem 4: $\Sigma_2^{-,\mathrm{non}}(0)$ is uncountable.** Recall first that we are assuming that $\lambda<\beta\,\lambda<1<\beta$. This implies that there are decreasing sequence of parameters $(t_n)_{n\in\mathbb{N}}$ with $t_n\to 0^+$ and increasing sequences of natural numbers $(k_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$ such that

$$t_n>\beta^{-k_n}>\beta^{-(k_n+1)}>(\beta\,\lambda)^{r_n+1}.$$

The previous equation implies that

$$t_n>\beta^{-k_n}>\beta^{-(k_n+1)}>\beta^{-(k_n+r_n+1)}>\beta^{-(k_n+r_n+2)}>\lambda^{r_n+1}. \tag{4.6}$$

For small $\gamma\in(0,1)$, consider the fundamental domains of f_0 in $[0,1]$ given by

$$D^0_\gamma \stackrel{\text{def}}{=} [f_0^{-1}(\gamma),\gamma] \quad\text{and}\quad D^1_\gamma \stackrel{\text{def}}{=} [1-\gamma, f_0(1-\gamma)]$$

and define $\iota(\gamma)>0$ as the first natural number with

$$f_0^{\iota(\gamma)}(D^0_\gamma)\cap D^1_\gamma\neq\emptyset.$$

It is not difficult to check (for details see [9, lemma 2.1]) that there is a decreasing sequence $(\gamma_n)_{n\in\mathbb{N}}$ such that $\iota(\gamma_n)=n-1$ and

$$f_0^{\iota(\gamma_n)+1}(D^0_{\gamma_n}) = f_0^n(D^0_{\gamma_n}) = D^1_{\gamma_n}. \tag{4.7}$$

Fix large $N\in\mathbb{N}$, denote by v_n the word $v_n \overset{\text{def}}{=} 0^n1$, and consider the sequence

$$s_n \overset{\text{def}}{=} k_n + r_n + N, \quad s_n,\, k_n \text{ as in (4.6).}$$

Define now the following subset Γ_N of Σ_2^-,

$$\Gamma_N \overset{\text{def}}{=} \Big\{\xi^- \in \Sigma_2^- : \widehat{\xi}^- = v_{s_1+i_1} v_{s_2+i_2} \dots v_{s_j+i_j} \dots \; i_j \in \{0,1\}\Big\}.$$

By definition the set Γ_N is uncountable. Thus the proposition below implies item (2) of Theorem 4.

Proposition 4.5. *If N is large enough then $\Gamma_N \subset \Sigma_2^{-,\text{non}}(0)$.*

Proof. For simplicity let us omit the subscript N and write $\gamma=\gamma_N$ and $\Gamma=\Gamma_N$. Define for $n\in\mathbb{N}$ the sets

$$E_n = E_n(\gamma) \overset{\text{def}}{=} \bigcup_{i=0}^{r_n} f_0^i(D^1_\gamma) = [1-\gamma, f_0^{r_n+1}(1-\gamma)].$$

Note that if $m<n$ then $r_m \le r_n$ and thus $E_m \subset E_n$. We need the following key lemma:

Lemma 4.6. *For every $n\in\mathbb{N}$ and every $t\in[t_n,1]$*

$$E_n \subset \Big(f_0^{k_n+r_n+N} \circ f_{1,t}(E_n)\Big) \cap \Big(f_0^{k_n+r_n+N+1}\circ f_{1,t}(E_n)\Big).$$

Proof. Let us assume (for simplicity) that f_0 is affine in $[0,\gamma]$ and $[1-\gamma,1]$,

$$f_0(x) = \beta\, x \;\text{ and }\; f_0(1-x) = 1-\lambda x, \quad \text{for every } x\in[0,\gamma].$$

The proof in the general case is analogous. In this case,

$$E_n = [1-\gamma, 1-\lambda^{r_n+1}\gamma].$$

By definition of N we have $f_0^N\big(f_0^{-1}(\gamma)\big) = f_0^N(\beta^{-1}\gamma) = 1-\gamma$, thus

$$f_0^{-(k_n+r_n+N)}(E_n) = f_0^{-(k_n+r_n+N)}\big([1-\gamma, 1-\lambda^{r_n+1}\gamma]\big) = [\beta^{-(k_n+r_n+1)}\gamma, \beta^{-k_n}\gamma].$$

Similarly,

$$f_0^{-(k_n+r_n+N+1)}(E_n) = [\beta^{-(k_n+r_n+2)}\gamma, \beta^{-k_n-1}\gamma].$$

This implies that

$$\begin{aligned} f_{1,t}^{-1}\circ f_0^{-(k_n+r_n+N)}(E_n) &= [1-t^{-1}\beta^{-k_n}\gamma, 1-t^{-1}\beta^{-(k_n+r_n+1)}\gamma], \\ f_{1,t}^{-1}\circ f_0^{-(k_n+r_n+N+1)}(E_n) &= [1-t^{-1}\beta^{-k_n-1}\gamma, 1-t^{-1}\beta^{-(k_n+r_n+2)}\gamma]. \end{aligned} \tag{4.8}$$

Note that for $t\in[t_n,1]$ the inequalities in (4.6) imply

$$t \ge t_n > \beta^{-k_n} > \beta^{-(k_n+1)} > \beta^{-(k_n+r_n+1)} > \beta^{-(k_n+r_n+2)} > \lambda^{r_n+1}.$$

These inequalities immediately imply the following inclusions for every $t\in[t_n,1]$,

$$\begin{aligned} [1-t^{-1}\beta^{-k_n}\gamma, 1-t^{-1}\beta^{-(k_n+r_n+1)}\gamma] &\subset [1-\gamma, 1-\lambda^{r_n+1}\gamma] = E_n, \\ [1-t^{-1}\beta^{-k_n-1}\gamma, 1-t^{-1}\beta^{-(k_n+r_n+2)}\gamma] &\subset [1-\gamma, 1-\lambda^{r_n+1}\gamma] = E_n. \end{aligned}$$

Using the identities in (4.8) and the inclusions above we get

$$f_{1,t}^{-1} \circ f_0^{-(k_n+r_n+N)}(E_n) \subset E_n \quad \text{and} \quad f_{1,t}^{-1} \circ f_0^{-(k_n+r_n+N+1)}(E_n) \subset E_n.$$

These inclusions imply the lemma. □

As $t_n \to 0^+$ and E_n is a nontrivial interval, the following lemma implies the proposition:

Lemma 4.7. *Consider $\xi = \xi^-.\xi^+$, where $\xi^- \in \Gamma$ and $\widehat{\xi^-} = v_{s_1+i_1} \dots v_{s_j+i_j} \dots$, $i_j \in \{0,1\}$. Then for any $t \in (t_n, 1)$, t_n as in (4.6), it holds*

$$g_{v_{s_1+i_1},t} \circ \cdots \circ g_{v_{s_{n-1}+i_{n-1}},t}(E_n) \subset I_{\xi,t}$$

Proof. Applying Lemma 4.6 to $g_{v_{s_n},t}$ ($i_n = 0$) and $g_{v_{s_n+1},t}$ ($i_n = 1$), where

$$g_{v_{s_n},t} = f_0^{k_n+r_n+N} \circ f_{1,t} \quad \text{and} \quad g_{v_{s_n+1},t} = f_0^{k_n+r_n+N+1} \circ f_{1,t},$$

and the nested intervals E_{n+k}, $k \geq 0$, $E_n \subset E_{n+k}$, one has that

$$E_n \subset E_{n+k} \subset g_{v_{s_{n+k}},t}(E_{n+k}) \quad \text{and} \quad E_n \subset E_{n+k} \subset g_{v_{s_{n+k}+1},t}(E_{n+k}). \tag{4.9}$$

This implies that

$$E_n \subset g_{v_{s_n+i_n},t}(E_n) \subset g_{v_{s_n+i_n},t}(E_{n+1}) \subset g_{v_{s_n+i_n},t} \circ g_{v_{s_{n+1}+i_{n+1}},t}(E_{n+1}).$$

Arguing recursively, we get that

$$E_n \subset g_{v_{s_n+i_n},t}(E_n) \subset g_{v_{s_n+i_n},t} \circ \cdots \circ g_{v_{s_{n+k}+i_{n+k}},t}(E_{n+k}).$$

Therefore

$$E_n \subset \lim_{k\to\infty} g_{v_{s_n+i_n},t} \circ \cdots \circ g_{v_{s_{n+k}+i_{n+k}},t}([0,1]).$$

In particular,

$$g_{v_{s_1+i_1},t} \circ \cdots \circ g_{v_{s_{n-1}+i_{n-1}},t}(E_n) \subset \lim_{k\to\infty} g_{v_{s_1+i_1},t} \circ \cdots \circ g_{v_{s_{n+k}+i_{n+k}},t}([0,1]). \tag{4.10}$$

Lemma 2.3 implies that $g_{v_{s_1+i_1},t} \circ \cdots \circ g_{v_{s_{n-1}+i_{n-1}},t}(E_n) \subset I_{\xi,t}$, ending the proof of the lemma. □

The proof of the proposition is now complete. □

4.3. **Proof of item** (3) **of Theorem 4:** $\mathrm{HD}(\Sigma_2^{-,\mathrm{non}}(t_0)) > 0$**.** Fix small $\bar t_0 > 0$ and consider the sequences (t_n), (k_n), and (r_n) in (4.2). Lemma 4.6 holds for all $t \geq t_n$. As $\bar t_0 > t_{n_0}$ for some n_0, the following holds for all $t \geq \bar t_0$ and $n \geq n_0$,

$$E_n \subset \Big(f_0^{k_n+r_n+N} \circ f_{1,t}(E_n)\Big) \cap \Big(f_0^{k_n+r_n+N+1} \circ f_{1,t}(E_n)\Big). \tag{4.11}$$

Consider set A consisting of the words

$$A \stackrel{\text{def}}{=} \{u = 0^{k_{n_0}+r_{n_0}+N}\,1,\ w = 0^{k_{n_0}+r_{n_0}+N+1}\,1\}$$

and its associated set of backward sequences $\mathbb{E}_A$. By Proposition 3.6, $\mathrm{HD}(\mathbb{E}_A) > 0$. Thus item (3) of Theorem 4 follows from the lemma below.

Lemma 4.8. $\mathbb{E}_A \subset \Sigma_2^{-,\mathrm{non}}(t_0)$.

Proof. Take $\xi^- \in \mathbb{E}_A$ and any sequence ξ of the form $\xi = (\xi^-, \xi^+)$. By (4.11)

$$E_n \subset g_{u,t}(E_n) \quad \text{and} \quad E_n \subset g_{w,t}(E_n), \quad \text{for all } t \in [t_0, 1).$$

By Corollary 2.4 the interval E_n is contained in $I_{\xi,t}$, proving the lemma. □

5. Stabilization of spines. Proof of Theorem 5

In Section 3 we described a large subset of Σ_2 whose spines are abruptly created at $t = 1$. In this section, we prove Theorem 5 that is a result in the opposite direction: there is also a subset of Σ_2 with Hausdorff dimension bigger than one consisting of sequences whose spines depend continuously on the parameter t for $t = 1$. In particular, these nontrivial spines are created before $t = 1$.

We now go to the details of the proof of Theorem 5. Consider the set

$$C \stackrel{\text{def}}{=} \{u \stackrel{\text{def}}{=} 0101,\ s \stackrel{\text{def}}{=} 001001\},$$

its associated maps $g_{u,t}$ and $g_{s,t}$, and the set $\mathbb{E}_C \subset \Sigma_2^-$. By Proposition 3.6, $0 < \mathrm{HD}(\mathbb{E}_C)$.

Consider the set

$$\Sigma_2^{-,\mathrm{stb}} \stackrel{\text{def}}{=} \{\xi^- \in \Sigma_2^- \ :\ \xi^-.\xi^+ \text{ is stable a } t = 1 \text{ for any choice of } \xi^+ \in \Sigma_2^+\}.$$

Theorem 5.1. $\mathbb{E}_C \subset \Sigma_2^{-,\mathrm{stb}}$.

This result implies that $0 < \mathrm{HD}(\mathbb{E}_C) \le \mathrm{HD}(\Sigma_2^{-,\mathrm{stb}})$, proving Theorem 5

Proof of Theorem 5.1. Note first that condition (1.3) implies that $f_0'(0)\, f_0'(1) = \beta\,\lambda < 1$. We begin with a simple claim that follows by a straightforward calculation that we omit.

Claim 5.2. *The points* 0 *and* 1 *are hyperbolic attracting fixed points of* $g_{u,1}$ *and* $g_{s,1}$.

For $j = u, s$ and $t \in [0, 1]$ close to 1, denote by $p^0_{j,t}$ the continuation for $g_{j,t}$ of the hyperbolic fixed point 0 of $g_{j,1}$. Similarly, $p^1_{j,t}$ is the continuation of 1 for $g_{j,t}$.

Consider the sets of words consisting of sub-words of u and s

$$\mathcal{U} \stackrel{\text{def}}{=} \{0, 01, 010, 0101\} \quad \text{and} \quad \mathcal{S} \stackrel{\text{def}}{=} \{0, 00, 001, 0010, 00100, 001001\}.$$

Lemma 5.3. *There exists* $\bar t \in (0, 1)$ *such that*

$$p^0_{u,t}, p^0_{s,t}, p^1_{u,t}, p^1_{s,t} \in [0, 1], \quad \textit{for every } t \in [\bar t, 1].$$

Moreover,

$$g_{\bar u,t}(p^i_{u,t}) \in [0, 1], \quad \textit{for every } \bar u \in \mathcal{U} \textit{ and } i = 0, 1$$

and

$$g_{\bar s,t}(p^i_{s,t}) \in [0, 1] \quad \textit{for every } \bar s \in \mathcal{S} \textit{ and } i = 0, 1.$$

Proof. The second part of the lemma follows from the first part noting that $p^i_{j,t} \in [0, 1]$ for $i = 0, 1$ and $j = s, u$, and $f_{i,t}([0, 1]) \subset [0, 1]$, $i = 0, 1$.

We now prove the first part of the lemma for the continuations $p^0_{u,t}$ and $p^1_{u,t}$. By Claim 5.2 there is small $\delta > 0$ such that $(g_{u,1})'(x) < 1$ for every $x \in I_\delta \stackrel{\text{def}}{=} [-\delta, \delta] \cup [1 - \delta, 1 + \delta]$ and

$$g_{u,1}(-\delta) > -\delta, \quad g_{u,1}(\delta) < \delta, \quad g_{u,1}(1 - \delta) > 1 - \delta, \quad g_{u,1}(1 + \delta) < 1 + \delta.$$

Note that $g_{u,t}(0), g_{u,t}(1) \in (0, 1)$ for every $t \in (0, 1)$. This fact, the continuous dependence on t of $g_{u,t}$, and $g'_{u,t}(x) < 1$ for $x \in I_\delta$ for t close to 1 imply that there exist $t_u \in (0, 1)$ and small $\delta > 0$ such that for every $t \in [t_u, 1]$ we have

$$g_{u,t}(\delta) < \delta\ ,\ g_{u,t}(x) > x, \text{ for all } x \in [-\delta, 0),$$
$$g_{u,t}(1 - \delta) > 1 - \delta\ ,\ g_{u,t}(x) < x, \text{ for all } x \in (1, 1 + \delta].$$

These inequalities imply that $p^0_{u,t} \in (0,\delta)$ and $p^1_{u,t} \in (1-\delta,1)$ for all $t \in [t_u,1)$, proving the lemma for the continuations $p^1_{u,t}$ and $p^0_{u,t}$.

Arguing similarly, we get t_s such that $p^0_{s,t} \in (0,\delta)$ and $p^1_{s,t} \in (1-\delta,1)$ for all $t \in [t_s,1)$. The lemma follows taking $\bar t = \max\{t_s,t_u\}$. □

Let $\bar t$ as in Lemma 5.3 and for each $t \in [\bar t,1]$ define

$$p^0_t \stackrel{\text{def}}{=} \max\{p^0_{u,t},p^0_{s,t}\} \quad \text{and} \quad p^1_t \stackrel{\text{def}}{=} \min\{p^1_{u,t},p^1_{s,t}\}.$$

Note that $p^0_t < p^1_t$.

Proposition 5.4. *There is $\hat t \in [\bar t,1)$ such that $[p^0_t,p^1_t] \subset I_{\xi,t}$ for all $\xi = \xi^-.\xi^+$ with $\xi^- \in \mathbb{E}_C$ and $t \in [\hat t,1]$.*

This proposition implies Theorem 5.1. To see why this is so, fix small $\epsilon > 0$. As the points p^0_t and p^1_t depends continuously on t and $p^0_1 = 0$ and $p^1_1 = 1$, there is $t_\epsilon \in [\bar t,1)$ such that:

$$p^0_t < \frac{\epsilon}{2} < 1 - \frac{\epsilon}{2} < p^1_t \quad \text{for every} \quad t \in [t_\epsilon,1].$$

Proposition 5.4 implies that $1-\epsilon < \big|[p^0_t,p^1_t]\big| \le |I_{\xi,t}|$, which implies the theorem.

Proof of Proposition 5.4. Given $\xi^- \in \mathbb{E}_C$ consider its conjugate

$$\widehat{\xi}^- = u^{h_1}\, s^{n_1}\, u^{h_2}\, s^{n_2} \dots, \quad \text{where } h_i, n_i \ge 0 \text{ for } i \ge 1.$$

By the characterization of the spines in Lemma 2.3,

$$I_{\xi,t} = \lim_{r\to\infty} g^{h_1}_{u,t} \circ g^{n_1}_{s,t} \circ \dots \circ g^{h_r}_{u,t} \circ g^{n_r}_{s,t}\big([0,1]\big).$$

Therefore to prove the proposition it is enough to see that

$$g^{h_1}_{u,t} \circ g^{n_1}_{s,t} \circ \dots \circ g^{h_r}_{u,t} \circ g^{n_r}_{s,t}(0) \le p^0_t < p^1_t \le g^{h_1}_{u,t} \circ g^{n_1}_{s,t} \circ \dots \circ g^{h_r}_{u,t} \circ g^{n_r}_{s,t}(1).$$

These inequalities are consequence of the following lemma.

Lemma 5.5. *There is $\hat t \in [\bar t,1)$ such that for every $t \in [\hat t,1]$ the following holds:*

- $g^{h_j}_{u,t} \circ g^{n_j}_{s,t}(x) \in [p^1_t,1]$ *for every* $x \in [p^1_t,1]$ *and*
- $g^{h_j}_{u,t} \circ g^{n_j}_{s,t}(x) \in [0,p^0_t]$ *for every* $x \in [0,p^0_t]$.

Proof. We prove the first item of the lemma, the second one follows analogously. For each $t \in [\bar t,1]$ and $i = s,u$ consider the subsets of $[0,1]$ defined by

$$F_{i,t} \stackrel{\text{def}}{=} \{r \in [0,1]\colon g_{i,t}(r) = r \text{ and } r \text{ is not an attractor}\}$$

and select the subset of parameters

$$L \stackrel{\text{def}}{=} \{t \in [\bar t,1]\colon \big(F_{u,t} \cup F_{s,t}\big) \cap \big([0,p^0_t] \cup [p^1_t,1]\big) = \emptyset\}.$$

Claim 5.6. *There is $\hat t \in [\bar t,1)$ such that $[\hat t,1] \subset L$.*

Proof. Note that for $t = 1$ one has

$$0 = p^0_1 = p^0_{u,1} = p^0_{s,1} \quad \text{and} \quad 1 = p^1_1 = p^1_{u,1} = p^1_{s,1}$$

and these points are hyperbolic attractors of $g_{u,1}$ and $g_{s,1}$. Thus there is small $\epsilon > 0$ such that for t close to 1, $t < 1$, the only fixed point of $g_{u,t}$ in $[1-\epsilon,1+\epsilon]$ (resp. $g_{s,t}$) is $p^1_{u,t}$ (resp. $p^1_{s,t}$) which is the continuation of 1 and is attracting. Similarly, the only fixed point of $g_{u,t}$ (resp. $g_{s,t}$) in $[-\epsilon,\epsilon]$ is $p^0_{u,t}$ (resp. $p^0_{s,t}$) which is attracting. This completes the proof of the claim. □

To prove the lemma let us assume that $p^1_t = p^1_{u,t}$ (the case $p^1_t = p^1_{s,t}$ is analogous). As $p^1_{s,t} \geq p^1_{u,t}$, by the definition of L, if $t \in L$ the map $g_{s,t}$ has no repelling points in $[p^1_{u,t}, p^1_{s,t}]$. Thus one has

$$g^k_{s,t}(p^1_{u,t}) \geq p^1_{u,t} \quad \text{for every} \quad k \geq 0. \tag{5.1}$$

As that $g_{s,t}$ and $g_{u,t}$ preserve the orientation we have that for for every $x \in [p^1_t, 1]$ it holds

$$g^{h_j}_{u,t} \circ g^{n_j}_{s,t}(x) \geq g^{h_j}_{u,t} \circ g^{n_j}_{s,t}(p^1_{u,t}) \geq g^{h_j}_{u,t}(p^1_{u,t}) = p^1_{u,t}.$$

This ends the proof of the lemma. □

The proof of the proposition is now complete. □

The proof of Theorem 5.1 is now completed. □

6. Proof of Theorem 6: Porcupines with evanescent spines

In this section we study the persistence of nontrivial spines after their generation. We prove Theorem 6 claiming the existence of fiber maps f_0 such that the porcupines associated to the corresponding one-parameter families of skew-product maps have evanescent spines: there are a sequence $\xi \in \Sigma_2$ and parameters $0 < t_1 < t_2 < 1$ such that I_{ξ,t_1} is a nontrivial interval and $I_{\xi,t}$ is a singleton for every $t \in [t_2, 1)$.

We first construct an auxiliary family of porcupines with an evanescent spine where the fiber map f_0 is piecewise affine. Thereafter we will modify this construction to obtain a map $\widehat{f}_0$ that is C^∞.

6.1. **An evanescent spine: a piecewise affine model.** Consider the skew-product maps F_t defined as in (1.1) whose fiber maps are

$$f_{1,t}(x) = t\,(1-x), \quad f_{0,t}(x) = f_0(x) = \begin{cases} \dfrac{5}{2}x, & \text{if } x \leq \dfrac{1}{4}, \\[2mm] \dfrac{1}{2} + \dfrac{x}{2}, & \text{if } \dfrac{1}{4} < x \leq 1. \end{cases}$$

Note that for $t > 1/4$ one has $f_0'(x) < 1$ for every $x > t > 1/4$. This implies that for $t > 1/4$ the set Λ_t is transitive[4].

Proposition 6.1. *Consider the family $(F_t)_{t\in[0,1]}$ above. Then the spine of $\varpi = \overline{10}^Z$ is nontrivial for $t = \frac{1}{2}$ and is trivial for $t \in (\frac{2}{3}, 1)$.*

Proof. To see that the spine of $\varpi = \overline{10}^Z$ is nontrivial for $t = \frac{1}{2}$ note that the restriction of $g_{10,\frac{1}{2}}$ to $[0, 1/4)$ is of the form $g_{10,\frac{1}{2}}(x) = \frac{1}{2}\left(1 - \frac{5}{2}\,x\right)$. Thus $\frac{2}{9} < \frac{1}{4}$ is a fixed point of $g_{10,\frac{1}{2}}$. As $g'_{10,\frac{1}{2}}(\frac{2}{9}) = -\frac{5}{4}$, this point is repelling. By Lemma 2.5 the spine $I_{\varpi,\frac{1}{2}}$ is nontrivial.

We now see that the spine of $\varpi = \overline{10}^Z$ is trivial for every $t \in (\frac{2}{3}, 1)$. Note that the restriction of $g_{10,t}$ to $[0, 1/4)$ is of the form $g_{10,t}(x) = t\left(1 - \frac{5}{2}\,x\right)$. A direct calculation gives that for $x \in (0, \frac{1}{4})$ and $t > 2/3$ it holds

$$g_{10,t}(x) > t\left(\frac{3}{8}\right) > x.$$

[4] In this nondifferentiable case the parameter $t = 1/4$ plays the role of the parameter t_c in the differentiable case ($f_0'(t_c) = 1$), arguing as in [8] one gets the transitivity of Λ_t.

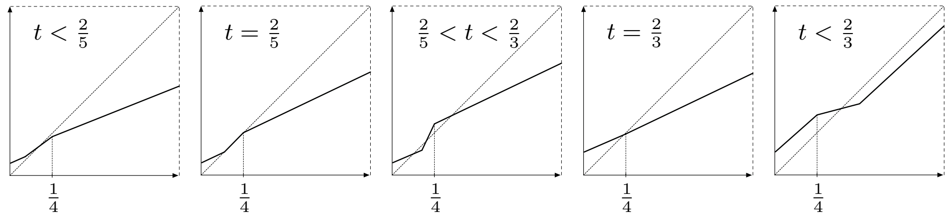

FIGURE 2. The maps $g^2_{10,t}$

Thus the fixed points of $g_{10,t}$ are in $(1/4, 1]$. As $g_{10,t}(x)$ is a contraction in $(1/4, 1]$ (the derivative is $-t/2$) this fixed point is unique and equal to $q = \frac{t}{2+t}$.

We argue by contradiction assuming that the spine of ϖ is nontrivial. In this case $g_{10,t}$ necessarily has a periodic point $q' < q$ of period two. Thus $g^2_{10,t}$ has at least 3 fixed points $q' < q < g_{10,t}(q')$. Observe also that $g^2_{10,t}(0) > 0$ and $g^2_{10,t}(1) < 1$. These conditions imply that the derivative of $g^2_{10,t}$ has at least five different values. These maps are depicted in Figure 6.1

On the other hand, by definition of $g_{10,t}$, there are closed intervals $I_1, \dots, I_4$ such that $[0, 1] = I_1 \cup I_2 \cup I_3 \cup I_4$ and $g^2_{10,t}$ is affine in each interval I_j. Thus the derivative of $g^2_{10,t}$ has (at most) four different values, getting a contradiction.

The proof of the proposition is now complete. □

We close this subsection with a remark about the affine model. Recall that $\frac{2}{9}$ is the fixed expanding point of $g_{10,\frac{1}{2}}$ and that for $t > 2/3$, $\frac{t}{2+t}$ is the attracting fixed point for $g_{10,t}$. Given $t > \frac{2}{3}$ consider the "orbits"

$$O_{\frac{2}{9},\frac{1}{2}} = \left\{ \frac{2}{9}, f_0\left(\frac{2}{9}\right) \right\} \quad \text{and} \quad O_{\frac{t}{2+t},t} = \left\{ \frac{t}{2+t}, f_0\left(\frac{t}{2+t}\right) \right\}.$$

Claim 6.2. There is small $\epsilon > 0$ such that the ϵ-neighborhood V_ϵ of $\frac{1}{4}$ is disjoint from $O_{\frac{2}{9},\frac{1}{2}} \cup O_{\frac{t}{2+t},t}$ for every $t \in [\frac{2}{3} + \epsilon, 1)$.

Proof. Note that $\frac{1}{4} \notin O_{\frac{2}{9},\frac{1}{2}}$ and for small $\epsilon > 0$ and $t \in [\frac{2}{3} + \epsilon, 1)$ one has $\frac{t}{2+t} > \frac{1}{4}$. As the fixed point $\frac{t}{2+t}$ of $g_{10,t}$ increases with t and $f_0(x) > x$ for every $x \in (0, 1)$, the ϵ-neighborhood V_ϵ of $\frac{1}{4}$ is disjoint from $O_{\frac{2}{9},\frac{1}{2}} \cup O_{\frac{t}{2+t},t}$ for every $t \in [\frac{2}{3} + \epsilon, 1)$. □

6.2. **An evanescent spine: general case.** To construct evanescent spines in the general differentiable case we modify f_0 in an small neighborhood of the nondifferentiability point $x = 1/4$ and denote by $\widehat{f}_0$ the resulting map. We let $\widehat{f}_{1,t} = f_{1,t}$ and write $\widehat{g}_{u,t}$ for the corresponding compositions. We denote the skew product map associated to $\widehat{f}_0$ and $\widehat{f}_{1,t}$ by $\widehat{F}_t$.

Consider the set V_ϵ in Claim 6.2 and any C^∞ concave map $\widehat{f}_0$ whose restriction to $[0, 1] - V_\epsilon$ coincides with f_0 and such that $\widehat{f}_0'(\frac{1}{4}) = 1$. Theorem 6 follows from the next proposition.

Proposition 6.3. *Consider the skew-product family $(\widehat{F}_t)_{t\in[0,1]}$. The spine of $\varpi = \overline{10}^Z$ is nontrivial for $t = \frac{1}{2}$ and is trivial for every $t \in \left[\frac{2}{3} + \epsilon, 1\right)$.*

Proof. Since $O_{\frac{2}{9},\frac{1}{2}} \cap V_\epsilon = \emptyset$ one has that $\widehat{g}_{10,\frac{1}{2}}(\frac{2}{9}) = \frac{2}{9}$ and $\widehat{g}'_{10,\frac{1}{2}}(\frac{2}{9}) = -\frac{5}{4}$. Thus $\frac{2}{9}$ is an expanding fixed point for $\widehat{g}_{10,\frac{1}{2}}$. By Lemma 2.5 the set $I_{\varpi,\frac{1}{2}}$ is not a singleton. This proves the first part of the proposition.

Analogously, we see that $\frac{t}{2+t}$ is an attracting fixed point for $g_{10,t}$ for $t \in [\frac{2}{3}+\epsilon, 1)$.

We now see that $I_{\varpi,t} = \{\frac{t}{2+t}\}$ for every $t \in [\frac{2}{3}+\epsilon, 1)$. This follows as in the piecewise affine case arguing by contradiction. If $I_{\varpi,t} \neq \{\frac{t}{2+t}\}$ then $\widehat{g}_{10,t}$ has at least one periodic point q' of period two that not is attracting. This implies that $\widehat{g}^2_{10,t}$ has three fixed points, $q' < q < g^2_{10,t}(q')$. Since that $\widehat{g}^2_{10,t}(0) > 0$ and $\widehat{g}^2_{10,t}(1) < 1$ this implies that $\widehat{g}^2_{10,t}$ changes its concavity at least four times in $[0,1]$.

On the other hand, by construction, the map $\widehat{g}^2_{10,t}$ changes the concavity at most three times in $[0,1]$. This gives a contradiction.

The proof of the proposition is now complete. □

7. Appendix: Hausdorff dimension

In this section we state some properties of the Hausdorff measure and dimension that we used throughout the paper.

7.1. Hausdorff dimension and Hausdorff s-measures.

Let (M,d) be a compact metric space and K a subset of M. The *diameter* of a covering $\mathcal{U} = (U_j)_{j\in J}$ of K is defined by

$$\mathrm{diam}(\mathcal{U}) \stackrel{\text{def}}{=} \sup\{\mathrm{diam}(U_j),\, j \in J\},$$

where $\mathrm{diam}(U)$ denotes the diameter of the set U.

The *s-measure* of a finite covering $\mathcal{U} = (U_j)_{j\in J}$ of K is defined by

$$m_s(\mathcal{U}) \stackrel{\text{def}}{=} \sum_{j\in J} \big(\mathrm{diam}(U_j)\big)^s.$$

For each pair $s, \epsilon > 0$ the (s,ϵ)-measure of K is defined by

$$m_{s,\epsilon}(K) \stackrel{\text{def}}{=} \inf\{m_s(\mathcal{U}) \,:\, \mathcal{U} \text{ a finite covering of } K \text{ with } \mathrm{diam}(\mathcal{U}) < \epsilon\}.$$

Note that $m_{s,\epsilon}(K)$ decreases with ϵ. The *Hausdorff s-measure* of K is defined by

$$m_s(K) \stackrel{\text{def}}{=} \lim_{\epsilon\to 0^+} m_{s,\epsilon}(K).$$

The map $m_s(K)$ is decreasing with s and there is a value $\mathrm{HD}(K)$, called the *Hausdorff Dimension* of K, such that

$$\mathrm{HD}(K) \stackrel{\text{def}}{=} \inf\{s \in \mathbb{R} : m_s(K) = 0\} = \sup\{s \in \mathbb{R} : m_s(K) = \infty\}.$$

For further details see [12]. The *Hausdorff Measure* of K is the number $m_{\mathrm{HD}(K)}(K)$.

Remark 7.1. The definitions of $\mathrm{HD}(K)$ and $m_s(K)$ depend on the metric considered in the set K. However, the Hausdorff dimension of a set is the same for equivalent metrics and isometric sets have the same dimensions (see [12], p. 32-33 for details).

Another classical result is the following, see [11, page 1041].

Proposition 7.2. *Let $K = \bigcup_{n\in\mathbb{N}} K_n$. Then $\mathrm{HD}(K) = \sup_{n\in\mathbb{N}}\{\mathrm{HD}(K_n)\}$. In particular, every countable set has zero Hausdorff dimension.*

7.2. **Proof of Proposition 3.6.** Define the "conjugate" set of $\mathbb{E}_W$ by $\mathbb{E}'_W = \{w_0, w_1\}^{\mathbb{N}}$. Note that $\mathrm{HD}(\mathbb{E}'_W) = \mathrm{HD}(\mathbb{E}_W)$. Thus to prove the proposition is enough to see that $\frac{1}{m} \le \mathrm{HD}(\mathbb{E}'_W) \le \frac{1}{k}$.

Note that the definition of W implies that for any sequence $\xi \in \mathbb{E}'_W$ there is a unique sequence $w_i(\xi)$ of words in $W = \{w_0, w_1\}$ with $\xi = w_1(\xi)w_2(\xi)\dots w_i(\xi)\dots$.

Given $\xi, \eta \in \mathbb{E}'_W$ with $\xi = w_1(\xi)w_2(\xi)\dots$ and $\eta = w_1(\eta)w_2(\eta)\dots$ we let

$$\rho(\xi, \eta) \stackrel{\text{def}}{=} \min\{n \colon\ w_n(\xi) \neq w_n(\eta)\}$$

and define the metric d_1 by

$$d_1(\xi, \eta) \stackrel{\text{def}}{=} 2^{1/2}\, 2^{-\rho(\xi,\eta)}.$$

Consider the "substitution" map

$$h \colon (\Sigma_2^-, d) \to (\mathbb{E}'_W, d_1), \quad h(\dots\xi_{-k}\dots\xi_{-1}) \stackrel{\text{def}}{=} w_{\xi_{-1}}\dots w_{\xi_{-k}}\dots.$$

By construction, the map h is an isometry. This implies that $\mathrm{HD}(\mathbb{E}'_W, d_1) = \mathrm{HD}(\Sigma_2^-, d) = 1$, see Remark 7.1.

Recall that $w_0 = \theta_1\dots\theta_k$ and $w_1 = \zeta_1\dots\zeta_m$ and let s be the first j with $\theta_j \neq \zeta_j$. Recall that the number $n(\xi,\eta)$ in the definition of the metric d is the first j with $\xi_j \neq \eta_j$. As $k \le m$ we have that for every $\xi, \eta \in \mathbb{E}'_W$, $\eta \neq \xi$, it holds

$$\big(\rho(\xi,\eta) - 1\big)\,k + s \le n(\xi,\eta) \le \big(\rho(\xi,\eta) - 1\big)\,m + s.$$

Therefore

$$2^{-\rho(\xi,\eta)\,m + m - s} \le 2^{-n(\xi,\eta)} \le 2^{-\rho(\xi,\eta)\,k + k - s}.$$

Rewriting these inequalities using the distances d_1 and d we get

$$2^{m-s}\, d_1(\xi,\eta)^m \le d(\xi,\eta) \le 2^{k-s}\, d_1(\xi,\eta)^k.$$

Using the definition of Hausdorff dimension one immediately gets

$$\frac{1}{m} = \frac{1}{m}\,\mathrm{HD}(\mathbb{E}'_W, d_1) \le \mathrm{HD}(\mathbb{E}'_W, d) \le \frac{1}{k}\,\mathrm{HD}(\mathbb{E}'_W, d) = \frac{1}{k},$$

ending the proof of the proposition. □

References

1. R. Bowen, *Topological entropy for noncompact sets*, Trans. Amer. Math. Soc. **184** (1973), 125–136.
2. H. Bruin and S. van Strien, *Monotonicity of entropy for real multimodal maps*, J. Amer. Math. Soc. **28** (2015), no. 1, 1–61.
3. W. Cowieson and L. S. Young, *SRB measures as zero-noise limits*, Ergodic Theory Dynam. Systems **25** (2005), no. 04, 1115–1138.
4. L. J. Díaz and T. Fisher, *Symbolic extensions for partially hyperbolic diffeomorphisms*, Discrete Contin. Dyn. Syst. **29** (2011), no. 4, 1419–1441.
5. L. J. Díaz and K. Gelfert, *Porcupine-like horseshoes: Transitivity, Lyapunov spectrum, and phase transitions*, Fund. Math. **216** (2012), no. 1, 55–100.
6. L.J. Díaz and K. Gelfert, *Porcupine-like horseshoes: Topological and ergodic aspects*, Progress and Challenges in Dynamical Systems, 199–219, Springer Proc. Math. Stat., Springer, 2013.
7. L. J. Díaz, K. Gelfert, and M. Rams, *Rich phase transitions in step skew products*, Nonlinearity **24** (2011), no. 12, 3391–3415.
8. L. J. Díaz, K. Gelfert, and M Rams, *Abundant rich phase transitions in step skew products*, Nonlinearity **27** (2014), no. 9, 2255–2280.
9. L. J. Díaz, K. Gelfert, and M. Rams, *Almost complete lyapunov spectrum in step skew-products*, Dyn. Syst. **28** (2013), no. 1, 76–110.

10. L. J. Díaz, V. Horita, I. Rios, and M. Sambarino, *Destroying horseshoes via heterodimensional cycles: generating bifurcations inside homoclinic classes*, Ergodic Theory Dynam. Systems **29** (2009), no. 2, 433–474.
11. K. Falconer, *Fractal measures*, Handbook of measure theory, Vol. I, II, 1037Ð1054, North-Holland, Amsterdam, 2002.
12. K. Falconer, *Fractal geometry: mathematical foundations and applications*, John Wiley & Sons, Inc., Hoboken, NJ, 2003
13. A. J. Homburg, *Synchronization in iterated function systems*, arXiv:1303.6054 (2013).
14. I. Kan, H. Koçak, and J. A. Yorke, *Antimonotonicity: concurrent creation and annihilation of periodic orbits*, Ann. of Math. (2) **136** (1992), no. 2, 219–252.
15. Y. Kudryashov, *Bony attractors*, Funct. Anal. Appl. **44** (2010), no. 3, 219–222.
16. R. Leplaideur, K. Oliveira, and I. Rios, *Equilibrium states for partially hyperbolic horseshoes*, Ergodic Theory Dynam. Systems **31** (2011), no. 1, 179–195.
17. T. Marcarini, *Generation of spines in porcupine-like horseshoes*, Tese de Doutorado PUC-Rio, 2014.
18. S. van Strien, *One-dimensional dynamics in the new millennium*, Discrete Contin. Dyn. Syst. **27** (2010), no. 2, 557–588.

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