Singularities of the Isospectral Hilbert Scheme

Luca Scala

Abstract
We study the singularities of the isospectral Hilbert scheme $B^n$ of $n$ points over a smooth algebraic surface and we prove that they are canonical if $n \leq 5$, log-canonical if $n \leq 7$ and not log-canonical if $n \geq 9$. We describe as well two explicit log-resolutions of $B^3$, one crepant and the other $S_3$-equivariant.

Introduction
The aim of this work is the study of the singularities of the isospectral Hilbert scheme of $n$ points over a smooth complex algebraic surface. The isospectral Hilbert scheme $B^n$ can be defined as the blow-up of the product variety $X^n$ along the big diagonal $\Delta_n$. The isospectral Hilbert scheme has been introduced by Haiman in his works [Hai99] and [Hai01] on Macdonald polynomials; it was proven in [Hai01] that $B^n$ is normal, Cohen-Macauley and Gorenstein. It is an open problem if $B^n$ has canonical or log-canonical singularities. In this work we partially answer these questions.

Apart from being interesting in its own, the investigation about the singularities of $B^n$ is in tight relation with a number of interesting problems. The first and more immediate — which is one of the main motivations of this work — is the potential application to vanishing theorems, since sufficiently good singularities would allow the use of Kawamata-Viehweg or Kodaira vanishing over $B^n$; an example of this use already appeared in [Sca15, Section 5.2].

A second source of interest, which also offers an effective way to address the problem, is the link with the study of log-canonical thresholds of subspace arrangements. Since $B^n$ is the blow-up of the big diagonal in $X^n$, it turns out that the scheme $B^n$ — or, in other words, the pair $(B^n, \emptyset)$ — has exactly the same kind of singularities of the pair $(X^n, I_{\Delta_n})$. Now, one can determine the kind of singularities of the pair $(X^n, I_{\Delta_n})$ by studying its log-canonical threshold at each point. Since this problem is now local in nature, one can take $X$ as the affine plane $\mathbb{C}^2$: in this case the big diagonal $\Delta_n$ can be thought as a subspace arrangement. This problem is similar with that of finding log-canonical thresholds of hyperplane arrangements, already studied and solved in [Mus06]. On the other hand, there are not many examples in literature of computations of log-canonical thresholds of arrangements of subspaces of higher codimension: an exception is the study of configurations of lines through the origin in $\mathbb{C}^3$ by Teitler [Tei07]. An important part of his work deals with the understanding of the embedded components that appear when pulling back the ideal of the configuration of lines to the blow-up of the origin in $\mathbb{C}^3$; the presence of embedded components is the main difficulty that hinders an explicit log-resolution of the ideal of the configuration.

The case of the pair $(X^n, I_{\Delta_n})$ — for $X = \mathbb{C}^2$ — is similar because we deal with an arrangement of codimension 2 subspaces $\Delta_n$ in $\mathbb{C}^{2n}$, but it is more difficult because the complexity of the problem grows very rapidly with $n$. However, for $X = \mathbb{C}^2$, Haiman gave a precise description of a set of generators for the ideal $I_{\Delta_n}$, from which we can deduce the order of the ideal $I_{\Delta_n}$ at each point. As a consequence, we can establish the upper bound (proposition 2.9)

$$\text{lct}(X^n, I_{\Delta_n}) \leq \frac{2n - 2}{d_n}$$

for the log-canonical threshold of the pair $(X^n, I_{\Delta_n})$. Here $d_n$ is the natural number defined in remark 2.7. We actually believe that the above inequality is in fact an equality (Conjecture 1). This would imply that the singularities of $B^n$ are canonical if and only if $n \leq 7$, log-canonical if $n \leq 8$ and not log-canonical if $n \geq 9$ (Conjecture 2). We can actually prove — and this is the main result of this work —
The singularities of the isospectral Hilbert scheme are canonical if \( n \leq 5 \) and log-canonical if \( n \leq 7 \). For \( n \geq 9 \) they are not log-canonical.

Not unexpectedly, this problem is in close relation with the geometry of the Hilbert scheme of points as well. Indeed, after a result by Song in [Son14], results about the pair \((X^n, I_{\Delta_n})\) can be precisely translated into results about the pair \((X^{[n]}, I_{2\Delta X^{[n]}})\), where \( X^{[n]} \) is the Hilbert scheme of \( n \) points over \( X \) and \( 2\Delta X^{[n]} \) is its boundary. In particular the previous upper bound for \( \text{lct}(X^n, I_{\Delta_n}) \) implies the upper bound \( \text{lct}(X^{[n]}, I_{2\Delta X^{[n]}}) \leq (n-2)/d_u \). The mentioned conjecture on \( \text{lct}(X^n, I_{\Delta_n}) \) would imply that the last upper bound is actually an equality.

Finally, the problem of understanding the singularities of the isospectral Hilbert scheme should be a drive to the construction of an explicit \( \mathbb{G}_a \)-equivariant log-resolution of \( B^n \), or — what is equivalent — to an explicit \( \mathbb{G}_a \)-equivariant log-resolution \( f : Y \to X^n \) of the pair \((X^n, I_{\Delta_n})\). This would be a deep and important result on many levels. Firstly, it would provide another important compactification of the configuration space \( F(X,n) := X^n \setminus \Delta_n \) after the celebrated Fulton-MacPherson compactification \( X[n] \) (see [FM94]): the latter is not, unfortunately, a log-resolution of the pair \((X^n, I_{\Delta_n})\), since, when computing the pre-image of the ideal \( I_{\Delta_n} \) to \( X[n] \) embedded components appear. Hence an explicit \( \mathbb{G}_a \)-equivariant log-resolution of \((X^n, I_{\Delta_n})\) might be built by further blowing-up the Fulton-MacPherson compactification in order to get rid of these components; however, it is a very difficult problem to track and control the embedded components that arise in this way.

Secondly, supposing that the stabilizers of the \( \mathbb{G}_a \)-action on the resolution \( Y \) were trivial, then, passing to the quotient would provide an explicit resolution \( \tilde{f} : Y/\mathbb{G}_a \to S^n X \) of the symmetric variety. We mention that, in general, no such explicit resolution is known yet. In [Uly02] Ulyanov made a step forward proposing a refinement of the Fulton-MacPherson compactification in a way that the stabilizers of the natural \( \mathbb{G}_a \)-action are abelian, and not just solvable.

Finally, such a resolution \( f : Y \to X^n \) might be useful in the understanding the geometry of ideal sheaves of subschemes supported in big diagonals of the form \( O(-\lambda \Delta) \), appeared in the work [Sca15].

In the final section of this article we provide two different log-resolutions of the pair \((X^3, I_{\Delta_3})\), and hence of \( B^3 \); one crepant, the other \( \mathbb{G}_a \)-equivariant.

We work over the field of complex numbers. By point we always mean a closed point.

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1 Singularities of pairs and log-canonical thresholds

Definition 1.1. [Kol97] [Laz04] Let \( M \) be an irreducible complex algebraic variety, and \( a \) an ideal sheaf of \( O_M \). A log-resolution of the pair \((M,a)\) is a projective birational map \( f : Y \to M \) such that \( Y \) is nonsingular, the exceptional locus \( \text{exc}(f) \) is a divisor, the ideal sheaf \( f^{-1}a := a \cdot O_Y \) is equal to \( O_Y(-F) \), where \( F \) is an effective divisor on \( Y \) with the property that \( F + \text{exc}(f) \) has simple normal crossing support.

Definition 1.2. Let \( M \) be a complex algebraic variety, normal and irreducible; let \( K_M \) be its canonical divisor. Suppose that \( M \) is \( \mathbb{Q} \)-Gorenstein, that is, for some \( r \in \mathbb{N}^+ \), \( rK_M \) is Cartier. Let \( a \) be an ideal sheaf of \( O_M \). Consider a log-resolution \( f : Y \to M \) of the pair \((M,a)\). Then, as \( \mathbb{Q} \)-Cartier divisors,

\[
K_Y - f^*(K_M) + f^{-1}(a) = \sum_i a_i E_i
\]

where \( E_i \) are irreducible component of a simple normal crossing divisor and \( a_i \in \mathbb{Q} \). We say that the singularities of the pair \((M,a)\) are canonical if \( a_i \geq 0 \); log-canonical if \( a_i \geq -1 \).

Definition 1.3. Let \( M \) be a smooth algebraic variety and \( a \) an ideal sheaf of \( O_M \). Let \( c \in \mathbb{Q}, \ c > 0 \). Let \( f : Y \to M \) be a log-resolution of the pair \((M,a)\) and let \( F \) be the effective Cartier divisor on \( Y \).
such that \( f^{-1}a = \mathcal{O}_Y(-F) \). Then the multiplier ideal sheaf \( \mathcal{J}(c \cdot a) \) associated to \( c \) and \( a \) is the ideal sheaf of \( \mathcal{O}_M \) defined as

\[
\mathcal{J}(c \cdot a) := f_\ast \mathcal{O}_Y(K_{Y/M} - [c \cdot F]),
\]

where \([c \cdot F]\) is the integral part of the \( \mathbb{Q} \)-divisor \( F \). The definition just given does not depend on the choice of the log-resolution \([Laz04]\). For \( x \in M \), the log-canonical threshold of the pair \((M, a)\) at the point \( x \) is defined as

\[
\text{lct}_x(M, a) := \sup\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot a)_x = \mathcal{O}_{M,x}\} = \inf\{c \in \mathbb{Q} \mid \mathcal{J}(c \cdot a)_x \subset \mathfrak{m}_x\}.
\]

Define, moreover, \( \text{let}(M, a) := \inf_{x \in M} \text{lct}_x(M, a) \).

\begin{remark}
In the above definition of \( \text{lct}_x(M, a) \) the inf are actually minima \([Laz04] \) Example 9.3.16.
\end{remark}

\begin{proposition}
Let \( M \) be a smooth complex algebraic variety and let \( a \) be an ideal sheaf of \( \mathcal{O}_M \). Consider the blow-up \( g : B := \text{Bl}_a M \longrightarrow M \) along the ideal \( a \), with exceptional divisor \( E \). Suppose that \( B \) is irreducible, normal and Gorenstein; suppose moreover that \( K_B = g^\ast K_M + E \). Then \( B \) has (log-) canonical singularities if and only if the pair \((M, a)\) has.
\end{proposition}

\begin{proof}
Let \( h : Y \longrightarrow B \) be a log-resolution of the pair \((B, E)\). Consider the map \( f = g \circ h \). We claim that \( f \) is a log-resolution of the pair \((M, a)\). Indeed \( \text{exc}(f) \) is divisorial, since \( f \) is a birational morphism between smooth varieties. Moreover, set-theoretically, \( \text{exc}(f) = \text{exc}(h) \cup h^{-1} \text{exc}(g) = \text{exc}(h) \cup h^{-1}E \), which — since \( h \) is a log-resolution of \((B, E)\) — is a divisor with snc support. Hence \( \text{exc}(f) \) is a divisor with snc support. Moreover \( f^{-1}a = h^{-1}g^{-1}a = h^{-1}E \) is a Cartier divisor. Finally, as Cartier divisors, \( \text{exc}(f) + h^{-1}E \) coincides with \( \text{exc}(h) + 2h^{-1}E \), which has the same support as \( \text{exc}(f) \) and hence is a divisor with snc support. Then

\[
K_Y - h^\ast K_B = K_Y - h^\ast g^\ast K_M - h^\ast E = K_Y - f^\ast K_M + f^{-1}a
\]

which allows us to conclude.
\end{proof}

2 The isospectral Hilbert scheme

\begin{definition}
Let \( n \in \mathbb{N}^+ \). Let \( X \) be a smooth complex algebraic surface. Let \( \Delta_n \) be the big diagonal in \( X^n \), that is, \( \Delta_n \) is the scheme-theoretic union of pairwise diagonals \( \Delta_{ij}, 1 \leq i < j \leq n \). The isospectral Hilbert scheme \( B^n \) is the blow up of \( X^n \) along the big diagonal \( \Delta_n \).
\end{definition}

\begin{remark}
It is well known that the isospectral Hilbert scheme \( B^n \) is irreducible, normal, Cohen-Macaulay and Gorenstein \([Hai01]\).
\end{remark}

2.1 The big diagonal in \( X^n \)

As an immediate consequence of proposition \([1.5]\) we have a very precise correspondence between the singularities of the isospectral Hilbert scheme \( B^n \) and those of the pair \((X^n, \mathcal{I}_{\Delta_n})\).

\begin{corollary}
The isospectral Hilbert scheme \( B^n \) has (log-) canonical singularities if and only if the pair \((X^n, \mathcal{I}_{\Delta_n})\) has (log-) canonical singularities.
\end{corollary}

\begin{remark}
It is well known \([Laz04] \) Example 9.3.16] that a pair \((M, a)\) has log-canonical singularities if and only if \( \text{lct}(M, a) \geq 1 \). On the other hand, if \( M \) is Gorenstein, then the discrepancies \( a_i \) are necessarily integers; consequently the pair \((M, a)\) is canonical if and only if \( \text{lct}(M, a) = 1 \), that is, if and only if \( \mathcal{J}(M, a) = \mathcal{O}_M \). Hence we have that the isospectral Hilbert scheme \( B^n \) has canonical singularities if and only if \( \text{lct}(X^n, \mathcal{I}_{\Delta_n}) > 1 \) or, equivalently, if \( \mathcal{J}(X^n, \mathcal{I}_{\Delta_n}) \) is trivial; the singularities of \( B^n \) are log-canonical if and only if \( \text{lct}(X^n, \mathcal{I}_{\Delta_n}) \geq 1 \).
\end{remark}

\begin{remark}
The log-canonical threshold \( \text{lct}_x(M, a) \) at the point \( x \in M \) coincides with the complex singularity exponent \( c_\ast(a) \) of \( a \) at the point \( x \) \([DK01]\), which is an holomorphic invariant. As a consequence, the log-canonical threshold \( \text{lct}(X^n, \mathcal{I}_{\Delta_n}) \) of the pair \((X^n, \mathcal{I}_{\Delta_n})\) for an arbitrary surface \( X \) is equal to the log-canonical threshold of the pair \((\mathbb{C}^2)^n, \mathcal{I}_{\Delta_n})\).
\end{remark}
Remark 2.6 (Generators of $\mathcal{I}_{\Delta_n}$ for $X = \mathbb{C}^2$). In [Hai01] Haiman finds an explicit set of generators for ideal of the big diagonal $\Delta_n$ of $(\mathbb{C}^2)^n$. Write $(\mathbb{C}^2)^n$ as $\text{Spec} \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$. If $\bar{p}, \bar{q} \in \mathbb{N}^n$, denote with $\Delta(\bar{p}, \bar{q}, \bar{x}, \bar{y})$ the $\mathfrak{S}_n$-anti-invariant regular function
\[
\Delta(\bar{p}, \bar{q}, \bar{x}, \bar{y}) := \det(x_i^p y_j^q)_{ij}
\]
in the variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. If there is no risk of confusion, we will drop the indication of the variables and we will just write it as $\Delta(\bar{p}, \bar{q})$. Haiman proves that homogeneous polynomials of the form $\Delta(\bar{p}, \bar{q})$ generate the ideal $\mathcal{I}_{\Delta_n}$. Of course the function $\Delta(\bar{p}, \bar{q})$ is non identically zero if and only if the points $(p_i, q_i) \in \mathbb{N} \times \mathbb{N}$ are all distinct.

Remark 2.7 (Generators of minimal degree in $\mathcal{I}_{\Delta_n}$). A nonzero homogeneous polynomial of the form $\Delta(\bar{p}, \bar{q})$ is of minimal degree if the set of points $\{p_i, q_i\}, i = 1, \ldots, n$ minimize the weight $\sum_1^h p_i + q_i$. Now for any $n \in \mathbb{N}$ there exist two natural numbers $k$ and $h$, with $h < k$, uniquely determined by $n$, such that $n = k(k+1)/2 + h$. The integers $k$ and $h$ explain how to arrange $n$ distinct points $(p_i, q_i)$ in $\mathbb{N} \times \mathbb{N}$ in such a way that the weight $\sum_1^h (p_i + q_i)$ is the minimum possible: fill in the first antidiagonals in $\mathbb{N} \times \mathbb{N}$, of weight 0 to $k-1$, with $k(k+1)/2$ points of nonnegative integral coordinates and on the antidiagonal of weight $k$ put, in an arbitrary way, $h$ points. Consequently, a generator of minimal degree has degree
\[
d_n = \frac{1}{3} k(k^2 + 3h - 1).
\]

Remark 2.8. Consider the diagonal $\Delta_n$ inside $(\mathbb{C}^2)^n = \text{Spec} \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$ and consider its ideal $\mathcal{I}_{\Delta_n} \subseteq \mathbb{C}[x_1, y_1, \ldots, x_n, y_n]$. We build now a new coordinate system, in the following way. Consider the vector space $(\mathbb{C}^2)^{n-1}$ with coordinates $(z_1, w_1, \ldots, z_{n-1}, w_{n-1})$ and $\mathbb{C}^2$ with coordinates $(\alpha, \beta)$. Consider now the isomorphism
\[
\varphi : (\mathbb{C}^2)^n \longrightarrow (\mathbb{C}^2)^{n-1} \times \mathbb{C}^2
\]
defined by the coordinate change
\[
\begin{align*}
z_i &= x_i - x_{i+1}, & w_i &= y_i - y_{i+1} & \text{for } i = 1, \ldots, n-1 \\
\alpha &= \sum_{i=1}^n x_i, & \beta &= \sum_{i=1}^n y_i
\end{align*}
\]
In the new coordinates the pairwise diagonals in $(\mathbb{C}^2)^n$ are now given by ideals $(z_i, w_i)$ and $(z_i - z_j, w_i - w_j)$, $1 \leq i < j \leq n - 1$ and the ideal $\mathcal{I}_{\Delta_n}$ is the intersection
\[
\mathcal{I}_{\Delta_n} = \cap_{i=1}^{n-1} (z_i, w_i) \cap \cap_{1 \leq i < j \leq n-1} (z_i - z_j, w_i - w_j)
\]
inside $\mathbb{C}[z_1, w_1, \ldots, z_{n-1}, w_{n-1}, \alpha, \beta]$. Since the generators of $\mathcal{I}_{\Delta_n}$ are just polynomials in the $z_i, w_i$, the ideal $\mathcal{I}_{\Delta_n}$ is the extension of an ideal $\mathcal{I}_{\bar{D}_{n-1}} \subseteq \mathbb{C}[z_1, \ldots, z_{n-1}, w_1, \ldots, w_{n-1}]$, generated by the same elements. In other words, we can write
\[
\mathcal{I}_{\Delta_n} \simeq \varphi^*(\mathcal{I}_{\bar{D}_{n-1}} \boxtimes \mathcal{O}_{\mathbb{C}^2}).
\]
Consider now the projection $r : (\mathbb{C}^2)^{n-1} \times \mathbb{C}^2 \longrightarrow (\mathbb{C}^2)^{n-1}$. Under the identification $\varphi$, the small diagonal $\Delta_{1, \ldots, n}$ in $(\mathbb{C}^2)^n$ is the pre-image $r^{-1}(\{0\})$ by $r$ of the origin $\{0\}$ in $(\mathbb{C}^2)^{n-1}$. Consequently, the order of the big diagonal $\Delta_n$ along the small diagonal $\Delta_{1, \ldots, n}$ coincide with the order of $\bar{D}_{n-1}$ at the origin: $\text{ord}_{\Delta_{1, \ldots, n}} \mathcal{I}_{\Delta_n} = \text{ord}_0 \mathcal{I}_{\bar{D}_{n-1}}$; but $\text{ord}_0 \mathcal{I}_{\bar{D}_{n-1}}$ is the minimal degree of generators of $\mathcal{I}_{\bar{D}_{n-1}}$. But $\mathcal{I}_{\Delta_n}$ and $\mathcal{I}_{\bar{D}_{n-1}}$ have the same generators, hence $\text{ord}_{\Delta_{1, \ldots, n}} \mathcal{I}_{\Delta_n} = d_n$. Since the order of a coherent ideal along a subvariety is an holomorphic invariant, we can say in general that, for a smooth algebraic surface $X$,
\[
\text{ord}_{\Delta_{1, \ldots, n}} \mathcal{I}_{\Delta_n} = d_n.
\]
2.2 $F$-pure thresholds

For computational convenience we consider the characteristic $p$ analogue of the log-canonical threshold. Let $k$ be a perfect field of characteristic $p$ and let $R$ be a finitely generated regular $k$-algebra and let $a \subseteq R$ a nonzero ideal; consider $M = \text{Spec } R$ and $x \in V(a)$ a closed point corresponding to a maximal ideal $m_x$. For $e \in \mathbb{N}^*$, define

$$\nu_a(e) := \max \left\{ i \in \mathbb{N} | a^i \not\subseteq m_x^{[p^e]} \right\}$$

where $m_x^{[p^e]}$ is the ideal generated by $p^e$-powers of generators of $m_x$. The inequality $\nu_a(e + 1) \geq p^e \nu_a(e)$ implies that the sequences $\nu_a(e)/p^e$ and $\nu_a(e)/(p^e - 1)$ are nondecreasing [MTW05 Lemma 1.1]. The $F$-pure threshold of the ideal $a$ at the point $x$ is defined as

$$\text{fpt}_x(M,a) := \lim_{e \to +\infty} \frac{\nu_a(e)}{p^e} = \lim_{e \to +\infty} \frac{\nu_a(e)}{(p^e - 1)} = \sup_{e \in \mathbb{N}^*} \frac{\nu_a(e)}{p^e} = \sup_{e \in \mathbb{N}^*} \frac{\nu_a(e)}{(p^e - 1)}.$$  \hspace{1cm} (2.3)

Suppose now that $a$ is principal: we write simply $\nu_f(e)$ instead of $\nu(f)(e)$ and $\text{fpt}_x(M,f)$ instead of $\text{fpt}_x(M,a)$. In this case the sequence $\nu_a(e)/p^e$ is bounded above by 1. Hence, for $e \in \mathbb{N}^*$ we have the inequalities

$$\frac{\nu_f(e)}{(p^e - 1)} \leq \text{fpt}_x(M,f) \leq 1.$$  \hspace{1cm} (2.4)

Suppose now that $M$ is the affine space $\mathbb{A}^n_\mathbb{Z}$ over $\mathbb{Z}$ and $a$ is a nonzero ideal of $R := \mathbb{Z}[x_1,\ldots,x_n]$. For any prime $p$ consider the mod $p$ reduction $M_p := \text{Spec } (R \otimes \mathbb{F}_p)$ and $a_p := a : \mathbb{F}_p[x_1,\ldots,x_n]$. On the other hand, if $K$ is an arbitrary field extension of $\mathbb{Q}$ we can consider the extensions $a_K$ inside $K[x_1,\ldots,x_n]$, respectively and $M_K := \text{Spec } (R \otimes K)$. For varieties defined over arbitrary perfect fields, Zhu recently proved an interpretation of the log-canonical threshold in terms of dimensions of jet-schemes [Zhu13 Theorem B]; this result yields, as a consequence, the inequality $\text{fpt}_x(M_p,a_p) \leq \text{lct}_x(M_0,a_0)$ for every prime $p$ and for every closed point $x \in V(a)$ [Zhu13 Corollary 4.2]. Since the dimension of a scheme does not change upon extension of the field of definition [Gro65 Corollaire 4.1.4], we have, for every prime $p$ and any closed point $x \in V(a)$

$$\text{fpt}_x(M_p,a_p) \leq \text{lct}_x(M_{C},a_{C}).$$  \hspace{1cm} (2.5)

2.3 Singularities of the isospectral Hilbert scheme

We begin with the following upper bound for the log-canonical threshold of the pair $(X^n,\mathcal{I}_{\Delta_n})$.

Proposition 2.9. The log-canonical threshold of $(X^n,\mathcal{I}_{\Delta_n})$ is bounded above by $(2n - 2)/d_n$:

$$\text{lct}(X^n,\mathcal{I}_{\Delta_n}) \leq \frac{2n - 2}{d_n}.$$  \hspace{1cm} (2.6)

Proof. By remark 2.5 it is sufficient to prove the inequality when $X = C^2$. By remark 2.8, for $c \in \mathbb{Q}$, $c > 0$, the order of $x \cdot \mathcal{I}_{\Delta_n}$ along the small diagonal $\Delta_{1,\ldots,n}$ is $cd_n$; as soon as $cd_n \geq \text{codim}_X \Delta_{1,\ldots,n} + 1 - 1 = 2n - 2$, that is, if $c \geq (2n - 2)/d_n$, by [Laz04 Example 9.3.7] we have that $\mathcal{I}(X,c \cdot \mathcal{I}_{\Delta_n}) \subseteq \mathcal{I}_{\Delta_{1,\ldots,n}}$. By definition of log-canonical threshold $\text{lct}_0(X^n,\mathcal{I}_{\Delta_n})$ as infimum, we get the desired inequality $\text{lct}(X^n,\mathcal{I}_{\Delta_n}) \leq \text{lct}_0(X^n,\mathcal{I}_{\Delta_n}) \leq (2n - 2)/d_n$. \hspace{1cm} \qed

Remark 2.10. Consider the symmetric variety $S^nX$, where $X$ is a smooth complex algebraic surface; we will indicate with $\pi : X^n \longrightarrow S^nX$ the quotient projection. It is well known that $S^nX$ admits a stratification in strata $S^n_\lambda X$, where $\lambda$ is a partition of $n$. The stratum $S^n_\lambda X$ is the locally closed subset of 0-cycles of the form $\sum_{i=1}^{l(\lambda)} \lambda_i x_i$, where $l(\lambda)$ is the length of the partition $\lambda$ and $x_i$ are $l(\lambda)$ distinct points in $X$. By means of this stratification of $S^nX$ we can define a stratification of $X^n$ setting the stratum $X^n_\lambda$ as the locally closed subset $\pi^{-1}(S^n_\lambda X)$. It is clear that if $x \in X^n_\lambda$ then a sufficiently small open set $V_1$ of $x$ in $X^n$ in the standard topology is biholomorphic to a sufficiently small open set $V_2$ of the origin in $(\mathbb{C}^2)^n$ of the form $V_2 = U_{l(\lambda)}^1 \times \cdots \times U_{l(\lambda)}^l$, where $U_i$ are adequate small open sets of the origin in $\mathbb{C}^2$, such that, via the biholomorphic map, the ideal $\mathcal{I}_{\Delta_n}$ over $V_1$ is sent to $\mathcal{I}_{\Delta_{l(\lambda)}} \otimes \mathbb{C} \otimes \mathbb{C} \mathcal{I}_{\Delta_{l(\lambda)}}$ over $V_2$. Therefore, if $x \in X^n_\lambda$, we have, by proposition 2.9 and by [Laz04 Proposition 9.5.22] that

$$\text{lct}_x(X^n,\mathcal{I}_{\Delta_n}) = \min \{ \text{lct}_0((\mathbb{C}^2)^{\lambda_i},\mathcal{I}_{\Delta_{\lambda_i}}) | i = 1,\ldots,l(\lambda) \} \leq \frac{2\lambda_1 - 2}{d_{\lambda_1}}.$$  \hspace{1cm} (2.6)
We now make the following conjecture

**Conjecture 1.** If a point \( x \) of \( X^n \) lies in the stratum \( X^n_\lambda \), where \( \lambda \) is a partition of \( n \), then \( \text{lct}(X^n, \mathcal{I}_{\Delta_n}) = (2\lambda_1 - 2)/d_\lambda \). Therefore

\[
\text{let}(X^n, \mathcal{I}_{\Delta_n}) = \frac{2n - 2}{d_n}.
\]

This conjecture would immediately follow the fact about the singularities of the isospectral Hilbert scheme \( B^n \).

**Conjecture 2.** The singularities of the isospectral Hilbert scheme \( B^n \) are canonical if and only if \( n \leq 7 \), log-canonical if \( n \leq 8 \), not log-canonical if \( n \geq 9 \).

We are able to partially prove conjecture 2.

**Theorem 2.11.** The singularities of the isospectral Hilbert scheme are canonical if \( n \leq 5 \), log-canonical if \( n \leq 7 \). For \( n \geq 9 \) they are not log-canonical.

**Proof.** By corollary 2.3 and by remark 2.4, the singularities of the isospectral Hilbert scheme are log-canonical if and only if \( \text{let}(X^n, \mathcal{I}_{\Delta_n}) \geq 1 \) and canonical if and only if \( \text{let}(X^n, \mathcal{I}_{\Delta_n}) = 1 \). For \( n \geq 9 \), by proposition 2.9, \( \text{let}(X^n, \mathcal{I}_{\Delta_n}) \leq (2n - 2)/d_n \leq 16/17 \). Hence they can’t be log-canonical.

Let’s now prove the first statement. Using corollary 2.3 and remark 2.4 it is sufficient to prove that the singularities of the pair \((X^n, \mathcal{I}_{\Delta_n})\) are canonical for \( n \leq 5 \) and that \( \text{let}(X^n, \mathcal{I}_{\Delta_n}) \geq 1 \) for \( n = 6, 7 \).

By remark 2.5, it is sufficient to prove these facts for \( X = \mathbb{C}^2 \). By [2], it is then sufficient to prove that the pair \((\mathbb{C}^{2n-2}, \mathcal{I}_{\Delta_{n-1}})\) has canonical singularities for \( n \leq 5 \) and is log-canonical for \( n = 6, 7 \).

To prove that the pair \((\mathbb{C}^{2n-2}, \mathcal{I}_{\Delta_{n-1}})\) is canonical for \( n \leq 4 \) we will use Kollár-Bertini theorem [Ko07, Theorems 4.5, 4.5.1], [Laz04, Example 9.3.50]: in other words we will find an \( g \in \mathcal{I}_{\Delta_{n-1}} \) such that \( \text{div} \ g \) has rational (or canonical) singularities; then Kollár-Bertini theorem implies that the pair \((\mathbb{C}^{2n-2}, \mathcal{I}_{\Delta_{n-1}})\) is canonical. For \( n = 3 \) such a \( g \) can be chosen as the generator of minimal degree of \( \mathcal{I}_{\Delta_2} \), that is, \( g = z_1w_2 - z_2w_1 \); it defines an affine quadric cone of \( \mathbb{C}^4 \) projecting a smooth quadric in \( \mathbb{P}^3 \) from the origin of \( \mathbb{C}^4 \). Hence, by [BM74, Example 1.2], it has rational singularities. For \( n = 4 \) we can use the generator of minimal degree of \( \mathcal{I}_{\Delta_3} \), given by the polynomial \( g = \Delta_5((1, 0, 1), (0, 1, 1), (0, 1, 2)) \). We can check, using Macaulay2 and passing modulo \( p = 7 \), that the class of \( g^2h^5 \) is nonzero in \( \mathbb{F}_7[z_1, \ldots, z_4, w_1, \ldots, w_4]/m_7^2 \), thus proving that \( \nu_q(1) \geq 7 \), where \( q = (\mathcal{I}_{\Delta_3})_7 \), and hence that \( \text{fpt}_q((\mathbb{F}_7)^4, (\mathcal{I}_{\Delta_3})_7) \geq 7/6 > 1 \), by (2.3). Therefore the pair \((X^5, \mathcal{I}_{\Delta_3})\) has canonical singularities.

We now make the following conjecture

Let now \( n = 6, 7 \). By the equality in (2.6) and by what we just proved, we know that for any point \( x \) in a strata \( X_\lambda^n \), with \( \lambda \neq (6) \) — in the case \( n = 6 \) — or \( \lambda \neq (7) \) and \( \lambda \neq (6,1) \) — in the case \( n = 7 \) — we have \( \text{let}_x(X^n, \mathcal{I}_{\Delta_n}) \geq \text{let}(X^5, \mathcal{I}_{\Delta_3}) > 1 \). For \( n = 6 \) it is sufficient to prove that \( \text{let}_x(X^n, \mathcal{I}_{\Delta_n}) > 1 \) when \( x \in \Delta_4, \ldots, \Delta_6 \); by the isomorphism (2.2), it is sufficient to prove that \( \text{let}_0(\mathbb{C}^{12}, \mathcal{I}_{\Delta_4}) \geq 1 \); once we prove it, it is sufficient to prove that \( \text{let}_x(\mathbb{C}^{12}, \mathcal{I}_{\Delta_4}) \geq 1 \) for \( x \in \Delta_4, \ldots, \Delta_7 \), or equivalently, after (2.2), that \( \text{let}_0(\mathbb{C}^{12}, \mathcal{I}_{\Delta_4}) \geq 1 \). By (2.5), it is sufficient to prove, for some prime \( p \), that \( \text{fpt}_p((\mathbb{F}_p)^{n-1}, (\mathcal{I}_{\Delta_{n-1}})_p) \geq 1 \). By the first of the inequalities (2.4) it is then sufficient to find a polynomial \( g \in \mathcal{I}_{\Delta_{n-1}} \) with integral coefficients, such that, for some prime \( p, \nu_p(1) = p - 1 \) at the origin: here, for a polynomial \( g \) with integral coefficients, we denote with \( g_p \) its mod \( p \) reduction in in \( (\mathcal{I}_{\Delta_{n-1}})_p \). Consider the polynomials with integral coefficients \( g = \Delta((1, 0, 2, 1, 0), (0, 1, 0, 1, 2), \xi, \bar{w}) \), for \( n = 6 \), and \( h = \Delta((1, 0, 2, 1, 0, 2), (0, 1, 0, 1, 2, 1), \xi, \bar{w}) \), for \( n = 7 \). Then, using Macaulay2 and passing modulo 7, we checked that the class of \( g^2 \) in \( \mathbb{F}_7[z_1, \ldots, z_6, w_1, \ldots, w_6]/m_7^2 \) and \( h^2 \) in \( \mathbb{F}_7[z_1, \ldots, z_6, w_1, \ldots, w_6]/m_7^2 \) are both non zero. This proves that, choosing the prime 7, \( \nu_7, (1) = 6 = \nu_7, (1) \) and hence \( \text{fpt}_7((\mathbb{F}_7)^5, g_7) = 1 \), in case \( n = 6 \), and \( \text{fpt}_7((\mathbb{F}_7)^6, h_7) = 1 \), in case \( n = 7 \), and we can conclude. □
2.4 Relation with the geometry of the Hilbert scheme of points

The geometry of the pair $(X^n, I_{\Delta_n})$ is not only directly related to the geometry of the isospectral Hilbert scheme $B^n$, but also to the geometry of the Hilbert scheme of $n$ points $X^{[n]}$ over the surface $X$. Consider the boundary $\partial X^{[n]}$ of $X^{[n]}$. Song proved in [Son14] Proposition 4.3.5 that

$$\lct(X^{[n]}, I_{\partial X^{[n]}}) = \lct(S^n X, I_{\Delta_n}) = \frac{1}{2} \lct(X^n, I_{\Delta_n}).$$

Hence proposition 2.9 implies immediately the

**Corollary 2.12.** The log-canonical threshold of the pair $(X^{[n]}, I_{\partial X^{[n]}})$ is bounded above by $(n-1)/d_n$.

Moreover, conjecture 1 would imply

**Conjecture 3.** The log-canonical threshold of the pair $(X^{[n]}, I_{\partial X^{[n]}})$ is precisely given by $(n-1)/d_n$.

3 Two resolutions of $B^3$

The aim of this subsection is two provide two explicit resolutions of singularities of $B^3$; the first will be crepant, the second will be $\mathbb{G}_m$-equivariant. We begin with some remarks and technical lemmas.

**Remark 3.1.** Let $M$ a smooth algebraic variety and let $F$ be a coherent sheaf over $M$. We recall that an integral subscheme $V$ of $M$ is called a **prime cycle associated to** $F$ if there exists an invertible coherent $O_V$-module $L$ and an embedding $L \hookrightarrow F$ of coherent $O_M$-modules.

**Remark 3.2.** Let $M$ be a smooth algebraic variety and $Y$ a smooth subvariety. Let $Z \subseteq M$ be a closed subscheme, defined by the ideal sheaf $I_Z$. Let $r = \ord_Y I_Z$ the order of $Z$ along $Y$. Consider the blow-up $f : \text{Bly} M \longrightarrow M$ of $Y$ in $M$ and denote with $E$ its exceptional divisor. The weak transform $\tilde{Z}$ of $Z$ in $\text{Bly} M$ is defined by the residual ideal $I_{\tilde{Z}} := (I_{f^{-1}(Z)} : I_E^r)$. The ideal of the total transform $f^{-1}(Z)$ is then given by the product

$$I_{f^{-1}(Z)} = I_E \cdot I_{\tilde{Z}}.$$

It is well known that the weak transform does not necessarily coincide with the strict transform $\tilde{Z}$; in general one just has that $I_{\tilde{Z}} \subseteq I_{\tilde{Z}}$, and that the two ideals coincide outside the exceptional divisor. Indeed the weak transform $\tilde{Z}$ could contain embedded components, while the strict transform doesn’t. This is, in any case, the only possible difference between $\tilde{Z}$ and $\hat{Z}$, as the next criterion proves.

**Proposition 3.3.** Let $M$ be a smooth algebraic variety and $Y$ a smooth subvariety. Let $Z \subseteq M$ be a closed subscheme. Consider the blow-up map $f : \text{Bly} M \longrightarrow M$ and let $E$ be the exceptional divisor. Then the weak transform $\tilde{Z}$ of $Z$ coincide with the strict transform $\hat{Z}$ if and only if $E$ does not contain any prime cycle associated to $\hat{Z}$. In this case, for any positive integer $l$, the subschemes $lE$ and $\hat{Z}$ are transverse.

**Proof.** The necessity of the condition is clear. We just have to prove the sufficiency. Recall that the strict transform $\hat{Z}$ can be identified with the blow-up $\text{Bly}_{Y\cap Z} Z$: this is a consequence, for example, of [EH00] Proposition IV.21. Indicate with $\lambda$ the canonical section of $O_{\text{Bly} M}(E)$. We have that $E$ does not contain prime cycles associated to $\hat{Z}$ if and only if the morphism $\lambda : O_{\hat{Z}}(-E) \longrightarrow O_{\hat{Z}}$ is injective. In the case the ideal $I_{Z \cap E \cap \hat{Z}}$ of $\hat{Z} \cap E$ in $\hat{Z}$ is an invertible ideal of $O_{\hat{Z}}$. Hence the map $f|_{\hat{Z}} : \hat{Z} \longrightarrow Z$ factors via the blow-up $\text{Bly}_{Y\cap Z} Z$, that is, via the strict transform $\tilde{Z}$. Hence we have the injection of schemes $\tilde{Z} \hookrightarrow \hat{Z}$. But it is always true that $\tilde{Z} \subseteq \hat{Z}$. Hence the weak transform coincides with the strict one. In this case, for any fixed positive integer $l$, the morphism $\lambda^l : O_{\hat{Z}}(-lE) \longrightarrow O_{\hat{Z}}$ is injective. Since $R^j := 0 \longrightarrow O_{\text{Bly} M}(-jE) \longrightarrow O_{\text{Bly} M}$ is a locally free resolution of $O_{E}$, we can compute $\text{Tor}_j(O_{E}, O_{\hat{Z}})$ as of the $(-j)$-cohomology of the complex $R^j \otimes O_{\hat{Z}}$, which is $0 \longrightarrow O_{\hat{Z}}(-lE) \longrightarrow O_{\hat{Z}}$ injective. Hence $\text{Tor}_j(O_{E}, O_{\hat{Z}}) = 0$ for $j > 0$. □

**Remark 3.4.** Let $M$ be a smooth algebraic variety, and $Y$ a smooth subvariety. Consider the blow-up map $f : \text{Bly} M \longrightarrow M$. Let $H$ be an hypersurface in $M$. Then its weak and strict transform in $\text{Bly} M$ coincide.
Proof. Let $E$ be the exceptional divisor. The weak transform $\tilde{H}$ is a divisor whose associated prime cycles are the irreducible components of $\tilde{H}$. Since, by definition of $\tilde{H}$, one has that $E \not\subset \tilde{H}$, then \text{codim}_{\tilde{H},E} E \cap \tilde{H} = 2$ and hence the local equations of $E$ and $\tilde{H}$ define a regular sequence; hence $E$ does not contain any prime cycles relative to $\tilde{H}$. Hence $\tilde{H} = H$. \hfill $\Box$

**Lemma 3.5.** Let $M$ be a smooth algebraic variety and let $Y, W, Z$ three subschemes of $M$, such that $Y$ is closed, $W$ is integral and that $Y \not\subset W$. Let $\tilde{W}, \tilde{Z}$ be the strict transforms of $W$ and $Z$ inside $\text{Bl}_Y M$. Then \text{ord}_W \mathcal{I}_Z = \text{ord}_{\tilde{W}} \mathcal{I}_{\tilde{Z}}$.

Proof. Note that if $S, T$ are two subschemes of a smooth algebraic variety $V$, with $T$ integral, then \text{ord}_T \mathcal{I}_S$ can be characterized as \text{ord}_T \mathcal{I}_S = \max \{ n \in \mathbb{N} \mid \mathcal{I}_{S,T} \subseteq \mathfrak{m}_T^n \}$ where $\mathfrak{m}_T$ is the maximal ideal of the local ring $\mathcal{O}_{V,T}$ — that is, the ring of regular functions $g$ defined on some open set $U$ intersecting $T$ (Har77 Exercise 3.13) — and where $\mathcal{I}_{S,T}$ is the ideal of functions $g$ in $\mathcal{O}_{V,T}$ vanishing over $S \cap U$, if $U$ is the open set of definition of $g$. Now the blow-up map $f : \text{Bl}_Y M \longrightarrow M$ induces an isomorphism of local rings $f^*_W : \mathcal{O}_{M,W} \longrightarrow \mathcal{O}_{\text{Bl}_Y M,\tilde{W}}$ under which $\mathcal{I}_{Z,W}$ is sent onto $\mathcal{I}_{\tilde{Z},\tilde{W}}$, hence the statement. \hfill $\Box$

**Lemma 3.6.** Let $M$ be a smooth algebraic variety of dimension at least 3; let $H$ be a smooth hypersurface in $M$ and $W_1, W_2$ two smooth subvarieties of $M$ contained in $H$ and transverse inside $H$. Consider now the composition $f$ of blow-ups

$$f : B := \text{Bl}_{W_2} \text{Bl}_{W_1} M \longrightarrow \text{Bl}_{W_1} M \longrightarrow M,$$

where $\tilde{W}$ is the strict transform of $W$ inside $\text{Bl}_{W_1} M$. Denote with $E_{W_1}$ the exceptional divisor of $\text{Bl}_W M$ and with $E_{W_2}$ that of $\text{Bl}_{W_2} \text{Bl}_{W_1} M$. Then $f$ is an isomorphism outside $f^{-1}(W_1 \cup W_2)$; moreover

$$f^{-1}(\mathcal{I}_{W_1 \cup W_2}) = \mathcal{I}_{E_{W_1}} \cdot \mathcal{I}_{E_{W_2}} = \mathcal{O}_B(-E_{\tilde{W}_1} - E_{\tilde{W}_2}).$$

Finally the relative canonical bundle $K_{B/M}$ is isomorphic to $\mathcal{O}_B(E_{\tilde{W}_1} + E_{\tilde{W}_2})$.

Proof. In the particular case in which $M = \mathbb{C}^3$; $\mathcal{I}_H = (x)$; $\mathcal{I}_{W_1} = (x,y)$; $\mathcal{I}_{W_2} = (x,z)$ and hence $\mathcal{I}_{W_1 \cup W_2} = (x,yz)$, the statement can be proved by an explicit computation in coordinates, which we leave to the reader.

Let now pass to the general case. Consider a point $p$ in the intersection $W_1 \cap W_2$. Over an adequate open neighbourhood $U$ of $p$ in the standard complex topology, we can find local holomorphic coordinates $x, y, z$ such that $H$ is defined (over $U$) by the zeros of $x$, and $W_1$ and $W_2$ by the ideals $(x, y)$ and $(x, z)$, respectively. Alternatively, one can find an adequate affine neighbourhood $U$ of $p$ and regular function $x, y, z$ over $U$ such that the differentials $dx, dy, dz$ are independent in $m_q/m_q^2$ for all $q \in U$ and such that $H$, $W_1$, $W_2$ are defined by ideals of the regular functions $(x)$, $(x, y)$ and $(x, z)$ as in the holomorphic case. Hence the general situation can be obtained locally from the particular one above by a smooth base change: the statement follows. \hfill $\Box$

**Lemma 3.7.** Let $M$ be a smooth algebraic variety, $H$ a smooth hypersurface of $M$, and $W$ and $Q$ two codimension 2 smooth subvarieties of $M$ such that $Q \subseteq H$, $W \cap H \subseteq Q$ and $W \cap H$ is a smooth codimension 3 subvariety of $M$. Consider the blow-up $f : \text{Bl}_W M \longrightarrow M$ of $W$ in $M$, with exceptional divisor $E_W$. Then

$$f^{-1}(\mathcal{I}_W \cap \mathcal{I}_Q) = \mathcal{I}_{E_W} \cdot \mathcal{I}_{\tilde{Q}} = \mathcal{I}_{E_W} \cap \mathcal{I}_{\tilde{Q}}$$

where $\tilde{Q}$ denote the strict transform of $Q$ in $\text{Bl}_W M$.

Proof. The statement is local in nature, over the base $M$: hence, by placing ourselves on a small open neighbourhood of a point $p \in W \cap H$ in the complex topology, equipped with some holomorphic coordinates $(x, y, z, w_1, \ldots, w_r)$, we can suppose that the ideals of $H$, $W$ and $Q$ are given locally by $\mathcal{I}_H = (z)$, $\mathcal{I}_W = (x, y)$, $\mathcal{I}_Q = (x, z)$. Then $\mathcal{I}_W \cap \mathcal{I}_Q = (x, yz)$; the proof of the statement is now achieved through an easy computation in coordinates. \hfill $\Box$
3.1 A crepant resolution of $B^3$.

Conjecture $3.8$ states that the log-canonical threshold of the pair $(X^3, \mathcal{I}_{\Delta_n})$ is 2. This fact suggests that $B^3$ might admit a crepant resolution. This is indeed the case, as we will prove in this subsection.

**Remark 3.8.** Let $X$ be a smooth algebraic surface. If $Y$ is any smooth variety admitting a projective birational morphism $f : Y \rightarrow X^n$ over $X^n$ such that such that $f^{-1}(\mathcal{I}_{\Delta_n})$ is an invertible ideal sheaf of $\mathcal{O}_Y$, then, by the universal property of the blow-up, the map $f$ factors via the isospectral Hilbert scheme $B^n$ as

$$
\begin{array}{ccc}
Y & \xrightarrow{h} & X^n \\
\downarrow & & \downarrow \\
B^n & \xrightarrow{p} & X^n
\end{array}
$$

providing a resolution $h$ of $B^n$ such that

$$K_Y - h^*K_{B^n} = K_Y - h^*(p^*K_{X^n} + E) = K_Y - f^*K_{X^n} - h^*E = K_Y - f^*K_{X^n} + f^{-1}(\mathcal{I}_{\Delta_n}) .$$

**Remark 3.9.** By the previous remark, in order to find a crepant resolution of $B^n$, it is sufficient to build a smooth variety $Y$ and a projective birational map $f : Y \rightarrow X^n$ such that $f^{-1}(\mathcal{I}_{\Delta_n})$ is an invertible ideal isomorphic to the relative anticanonical $−K_{Y/X^n} = f^*K_{X^n} - K_Y$.

**Remark 3.10.** The questions posed in the previous two remarks are local over the base and analytical in nature. Hence, to find a resolution of $B^n$ in general, it is sufficient to find a smooth variety $Y$ and a birational map as in the remark $3.8$ for $X = \mathbb{C}^2$. Moreover, since in the identification (2.2), the ideal sheaf $\mathcal{I}_{\Delta_n}$ corresponds to $\mathcal{I}_{\tilde{D}_n-1} \boxtimes \mathcal{O}_{\mathbb{C}^2}$, by flat base change it is sufficient to find a smooth variety $Y$ and a projective birational morphism $f : Y \rightarrow (\mathbb{C}^2)^{n-1}$ such that $f^{-1}(\mathcal{I}_{\tilde{D}_{n-1}})$ is an invertible ideal. The resolution thus built will be crepant if and only if $f^{-1}(\mathcal{I}_{\tilde{D}_{n-1}})$ is isomorphic to the anticanonical $−K_Y$.

For brevity’s sake, in what follows, we will indicate the affine space $(\mathbb{C}^2)^2$ with $V$, the subscheme $\tilde{D}_2$ with $W$. Fix coordinates $(x, y, z, w)$ over $V$. The irreducible components of the subscheme $W$ are linear subspaces $W_1, W_2, W_3$, defined by the ideals $I_1 = (x, y)$, $I_2 = (z, w)$, $I_3 = (x - z, y - w)$. The ideal $\mathcal{I}_W$ is then given by $\langle q, I_1I_2I_3 \rangle$, where $q$ is the quadric $q = xw - yz$.

**Proposition 3.11.** The projective birational morphism $f : Y \rightarrow V$, defined as the composition of smooth blow-ups

$$
Y = Y_3 \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} V
$$

where $Y_1 = \text{Bl}_{W_1}V$, $Y_2 = \text{Bl}_{\tilde{W}_2}Y_1$, $Y_3 = \text{Bl}_{\tilde{W}_3}Y_2$, where $\tilde{W}_2, \tilde{W}_3$ are the strict transforms of $W_2, W_3$ in $Y_1, Y_2$, respectively, is an isomorphism outside the locus $f^{-1}(W)$. Moreover, the ideal sheaf $f^{-1}(\mathcal{I}_W)$ is invertible and isomorphic to $-K_Y$.

**Proof.** As generators of the ideal $\mathcal{I}_W$ we can choose the polynomials $q, xz(x - z), xw(y - w), yw(x - z), yw(y - w)$. Consider the first blow-up $Y_1 = \text{Bl}_{W_1}V \simeq \text{Bl}_0(\mathbb{C}^2) \times \mathbb{C}^2$ and denote with $E_1$ the exceptional divisor. We can write globally

$$
x = \lambda u , \quad y = \lambda v$$

where $\lambda$ is the canonical section of $\mathcal{O}_{Y_1}(E_1)$ and $u, v$ are homogeneous coordinates, thought as a basis in $H^0(\mathcal{O}_{Y_1}(-E_1))$. By definition of weak transform we have $\mathcal{I}_{f_1^{-1}(W)} = \mathcal{I}_{E_1} \cdot \mathcal{I}_{\tilde{W}}$. The weak transform $\tilde{W}$ is given by the equations

$$
\begin{align*}
uw - vz &= 0 \\
uz(\lambda u - z) &= 0 \\
uw(\lambda v - w) &= 0 \\
vw(\lambda u - z) &= 0 \\
vw(\lambda v - w) &= 0
\end{align*}
$$

9
We prove now that the weak transform $\hat{W}$ coincides with the strict transform $\hat{W}$. By proposition 3.3 and its proof we just have to show that the morphism $\lambda : \mathcal{O}_{\hat{W}}(-E_1) \longrightarrow \mathcal{O}_{\hat{W}}$ is injective. Now, $\hat{W}$ is contained in the hypersurface $H$ of $Y_1$ defined by the equation $uw - vz = 0$. Over $H$ we can globally write $z = \mu w$, $w = \mu v$, where $\mu$ can be seen as a section in $H^0(\mathcal{O}_H(E_1))$. Then $\hat{W}$ is given, inside $H$, by the equations
\[
\begin{align*}
u^2s^3(\lambda - \mu) &= 0 \\
u^2v^2(\lambda - \mu) &= 0 \\
u^2w^2(\lambda - \mu) &= 0 \\
u^3v^3(\lambda - \mu) &= 0
\end{align*}
\]
Since $u$ and $v$ do not vanish at the same time, the weak transform is given by the equation $\mu(\lambda - \mu) = 0$ inside the hypersurface $H$, with respect to the coordinates $([u,v], \lambda, \mu)$. Hence $\lambda$ is not zero divisor in $\hat{W}$ and $\hat{W}$ is given by $\hat{W}$. Hence
\[
\mathcal{I}_{\hat{W}}^{-1}(\hat{W}) = \mathcal{I}_{\hat{E}_1} \cdot \mathcal{I}_{\hat{W}}.
\]
Now $\hat{W}$ is clearly the union, inside $H$, of the two smooth surfaces $\hat{W}_2$ and $\hat{W}_3$ intersecting transversally along a smooth curve inside the exceptional divisor $E_1$. Consider now the blow-ups $f_2 : \text{Bl}_{\hat{W}_2} Y_1 \longrightarrow Y_1$, with exceptional divisor $E_2$, and $f_3 : \text{Bl}_{\hat{W}_3} Y_2 \longrightarrow Y_2$, with exceptional divisor $E_3$; denote with $\hat{E}_1$ and $\hat{E}_2$ the strict transforms of $E_1$ and $E_2$ in $Y_3$, respectively. Let now $g := f_2 \circ f_3$ and let $f := f_1 \circ g$. Then by lemma 3.6 we have
\[
f^{-1}(\mathcal{I}_W) = g^{-1}(\mathcal{I}_{f_1^{-1}(W)}) = g^{-1}(\mathcal{I}_{E_1}) \cdot g^{-1}(\mathcal{I}_W) = \mathcal{I}_{\hat{E}_1} \cdot \mathcal{I}_{\hat{E}_2} \cdot \mathcal{I}_{E_3},
\]
where we used that $\hat{E}_1 = \hat{E}_1$ and $\hat{E}_1 = \hat{E}_1$ by remark 3.4. Hence $f^{-1}(\mathcal{I}_W)$ is invertible and isomorphic to $\mathcal{O}_Y(-\hat{E}_1 - \hat{E}_2 - E_3)$; it is now easy to show that the latter coincides with the anticanonical divisor $-K_Y$. \hfill $\Box$

As an immediate consequence of remarks 3.8, 3.9 and 3.10 we deduce the

**Corollary 3.12.** The map $f : Y \longrightarrow V$ factors through a crepant resolution $h : Y \longrightarrow \text{Bl}_W V$. Consequently the map $h \times \text{id} : Y \times \mathbb{C}^2 \longrightarrow \text{Bl}_W V \times \mathbb{C}^2 \simeq B^3$ identifies to a crepant resolution of $B^3$.

Let now $X$ be an arbitrary smooth algebraic surface and let $\Delta_{I_1}, \Delta_{I_2}, \Delta_{I_3}$ be the pairwise diagonals $\Delta_I, |I| = 2$, taken in whatever order. We have the following

**Theorem 3.13.** The composition of blow-ups $s := s_1 \circ s_2 \circ s_3$
\[
Y := \text{Bl}_{\Delta_{I_3}} Y_2 \xrightarrow{s_3} Y_2 := \text{Bl}_{\Delta_{I_2}} Y_1 \xrightarrow{s_2} Y_1 := \text{Bl}_{\Delta_{I_1}} X^3 \xrightarrow{s_1} X^3
\]
where $\hat{\Delta}_{I_2}$ and $\hat{\Delta}_{I_3}$ are the strict transforms of $\Delta_{I_2}$ and $\Delta_{I_3}$ in $Y_1$ and $Y_2$, respectively, is a log-resolution of the pair $(X^3, \mathcal{I}_{\Delta_{I_3}})$ such that $s^{-1}(\mathcal{I}_{\Delta_{I_3}})$ is an invertible ideal isomorphic to the relative anticanonical $-K_{X/P^3}$. Hence $s$ factors through a crepant resolution $g : Y \longrightarrow B^3$ of the isospectral Hilbert scheme $B^3$.

**Proof.** Locally over $X^3$, the map $s$ coincides precisely with $\varphi^{-1} \circ (f \times \text{id}_{\mathbb{C}^2})$, where $f$ is the birational map built in theorem 3.11 and $\varphi$ is the map (2.1). The theorem is then an immediate consequence of proposition 3.11 and remarks 3.8 and 3.9. \hfill $\Box$

### 3.2 An $\mathbb{G}_m$-equivariant resolution of $B^3$

Consider the 4-dimensional vector space $V = (\mathbb{C}^2)^2$ with coordinates $(x, y, z, w)$ and the subscheme $W = W_1 \cup W_2 \cup W_3$ introduced in subsection 3.1. Consider the blow-up $f_1 : Y_1 := \text{Bl}_0(V) \longrightarrow V$ of $V$ at the origin and let $E_0$ be its exceptional divisor; since it can be identified with the total space of the Hopf line bundle over the projective space $\mathbb{P}(V)$, the variety $Y_1$ is equipped with a fibration $Y_1 \longrightarrow \mathbb{P}(V)$. Now, the polynomial $q = xw - yz$ defines a smooth quadric $Q$ in $\mathbb{P}(V)$, which can be seen as a smooth subvariety of $Y_1$ inside $E_0$, thanks to the embedding of $\mathbb{P}(V)$ into $Y_1$ given by the zero section of the Hopf bundle.
**Proposition 3.14.** The birational morphism \( f : Y \longrightarrow V \) defined as the composition of smooth blow-ups
\[
Y = Y_3 \xrightarrow{f_3} Y_2 \xrightarrow{f_2} Y_1 \xrightarrow{f_1} V
\]
where \( Y_2 = \text{Bl}_{\hat{W}}(Y_1), Y_3 = \text{Bl}_{\hat{Q}}(Y_2), \) where \( \hat{W} \) and \( \hat{Q} \) are the strict transforms of \( W \) and \( Q \) in \( Y_1 \) and \( Y_2 \), respectively, is an isomorphism outside \( f^{-1}(W) \). Moreover the ideal sheaf \( f^{-1}(I_W) \) is given by
\[
f^{-1}(I_W) = O_Y(-2\hat{E}_0 - \hat{E}_{\hat{W}} - 3E_{\hat{Q}})
\]
where \( E_{\hat{W}} \) and \( E_{\hat{Q}} \) are the exceptional divisors in \( Y_2 \) and \( Y_3 \), respectively, and where \( \hat{E}_{\hat{W}} \) and \( \hat{E}_0 \) are the strict transforms of \( E_{\hat{W}} \) and \( E_0 \) in \( Y \).

**Proof.** Since \( \text{ord}_0 I_W = 2 \), we have
\[
I_{f^{-1}(W)} = I_{E_0}^2 \cdot I_{\hat{W}}
\]
where \( \hat{W} \) is the weak transform of \( W \) in \( Y_1 \). By a computation in coordinates, using the same generators for \( I_W \) we used in the proof of theorem 3.11 one gets
\[
I_{\hat{W}} = I_Q \cap I_{\hat{W}}
\]
that is, the weak transform \( \hat{W} \) is the scheme-theoretic union of the quadric \( Q \) and the strict transform \( \hat{W} \) of \( W \) in \( Y_1 \), which is a smooth codimension 2 subvariety with three irreducible components \( \hat{W}_i \), \( i = 1, \ldots, 3 \). Moreover \( \hat{W} \cap E_0 \) is contained in \( Q \) and is precisely the union of three skew lines in \( E_0 \approx \mathbb{P}(V) \); hence \( \hat{W} \cap E_0 \) is a smooth codimension 3 subvariety of \( Y_1 \). Therefore the hypothesis of lemma 3.13 are satisfied; this means that, when blowing up the strict transform \( \hat{W} \) in \( Y_1 \) one gets
\[
f^{-1}_2(I_Q \cap I_{\hat{W}}) = I_{E_0} \cdot I_{\hat{W}}
\]
Since \( \text{ord}_{\hat{W}} E_0 = 0 \), we get
\[
(f_1 \circ f_2)^{-1}(I_W) = I_{E_0}^2 \cdot I_{E_0} \cdot I_{\hat{Q}}
\]
Remembering that \( \text{ord}_{\hat{Q}} E_0 = 1 \), the last blow-up now yields the formula in the statement. \( \square \)

**Corollary 3.15.** The map \( f : Y \longrightarrow V \) factors through a resolution \( h : Y \longrightarrow \text{Bl}_W V \). Consequently the map \( h \times \text{id} : Y \times \mathbb{C}^2 \longrightarrow \text{Bl}_W V \times \mathbb{C}^2 \cong B^3 \) identifies to an \( \mathcal{G}_3 \)-equivariant log-resolution of \( B^3 \).

Consider now the case of an arbitrary smooth algebraic surface \( X \). Consider the blow-up \( s_1 : Y_1 := \text{Bl}_{\Delta_{123}} \longrightarrow X^3 \) of the small diagonal \( \Delta_{123} \) in \( X^3 \) and let \( E_0 \) be its exceptional divisor. The situation is locally, over \( X^3 \), analogous to the one just studied. Hence it is now clear that \( s_1^{-1}(I_{\hat{Q}}) = I_{E_0}^2 \cdot (I_Q \cap I_{\hat{Q}}) \), where \( \hat{Q} \) is the strict transform of \( Q \) in \( Y_1 \) and where \( Q \) is a quadric subbundle of \( \mathbb{P}(N_{\Delta_{123}/X^3}) \) over \( \Delta_{123} \) and hence a smooth subvariety of \( Y_1 \) inside \( E_0 \). We have the following theorem

**Theorem 3.16.** The composition of smooth blow-ups \( s := s_1 \circ s_2 \circ s_3 : \)
\[
Y := \text{Bl}_{\hat{Q}} Y_2 \xrightarrow{s_3} Y_2 := \text{Bl}_{\hat{Q}} Y_1 \xrightarrow{s_2} Y_1 \xrightarrow{s_1} X^3
\]
where \( \hat{Q} \) and \( \hat{Q} \) are the strict transforms of \( \Delta_3 \) and \( Q \) in \( Y_1 \) and \( Y_2 \), respectively, defines an \( \mathcal{G}_3 \)-equivariant log-resolution of the pair \( (X^3, I_{\hat{Q}}) \) and hence factors through a \( \mathcal{G}_3 \)-equivariant log-resolution \( g : Y \longrightarrow B^3 \) of the isospectral Hilbert scheme \( B^3 \).

**Proof.** The map \( s \) is clearly \( \mathcal{G}_3 \)-equivariant and, locally over \( X^3 \), coincides with the map \( \varphi^{-1} \circ (f \times \text{id}_{\mathbb{C}^2}) \), where \( f \) is the map introduced in proposition 3.14 and where \( \varphi \) is the map (2.1). The content of the theorem is then a consequence of proposition 3.14 corollary 3.15 and remarks 3.8 and 3.9 \( \square \)

**Remark 3.17.** This resolution is not crepant, as one gets easily \( K_{Y/X^3} + s^{-1}(I_{\hat{Q}}) = O_{\hat{E}_0} + E_{\hat{Q}} \), where \( E_{\hat{Q}} \) is the exceptional divisor in \( Y_3 \) and where \( \hat{E}_0 \) is the strict transform of \( E_0 \) in \( Y \).

**Remark 3.18.** The step \( Y_2 \) coincides with the Fulton-MacPherson compactification \( X[3] \) of \( X^3 \setminus \Delta_3 \) (see [FM04]).

**Remark 3.19.** By construction, the resolution \( Y \) is equipped with a \( \mathcal{G}_3 \)-action. The stabilizer of any point for this action is trivial. Hence, passing to the quotient modulo \( \mathcal{G}_3 \), the induced map \( \hat{f} : Y/\mathcal{G}_3 \longrightarrow S^3 X \) provides an explicit resolution of \( S^3 X \) which factors through the Hilbert scheme of points \( X[3] = B^3/\mathcal{G}_3 \).
References


Departamento de Matemática, Puc-Rio, Rua Marquês São Vicente 225, 22451-900 Gávea, Rio de Janeiro, RJ, Brazil

Email address: lucascalesmat.puc-rio.br