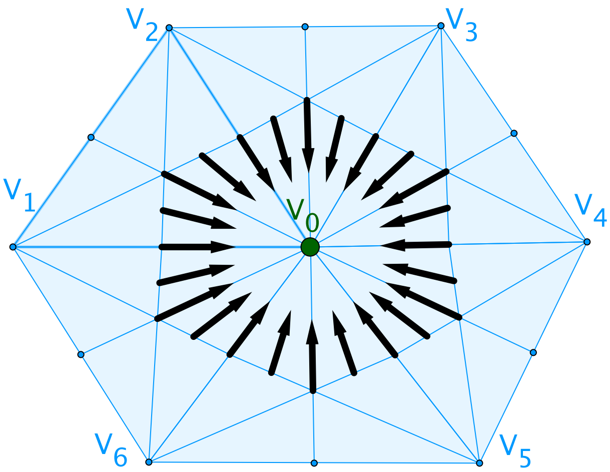
Discrete morse gradient fields with stable matching



João Paixão and Thomas Lewiner

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Discrete morse gradient fields with stable matching

JOÃO PAIXÃO AND THOMAS LEWINER

Department of Matematics — Pontifícia Universidade Católica — Rio de Janeiro — Brazil
zeus.mat.puc--rio.br/{renata,tomlew}.

Abstract. Forman's discrete Morse theory for cell complexes essentially relies on a discrete analogue of gradient fields, defined as an acyclic matching between incident cells. One can easily extract from this object several pieces of topological and geometrical information of a cell complex such as homology or the Morse-Smale decomposition. Most constructions of this discrete gradient are based on some variant of greedy pairing of adjacent cells, similar to a steepest descent algorithm. In this work, an equivalent formulation in terms of stable matching is proposed to study the geometric accuracy of the construction. It further simplifies the proofs of previous results regarding the behavior of the gradient field and the position of critical points.
Keywords: ***Discrete Morse Theory. Stable Matching. Morse-Smale Decomposition. Piecewise-Linear Critical Points. Computational Topology.***

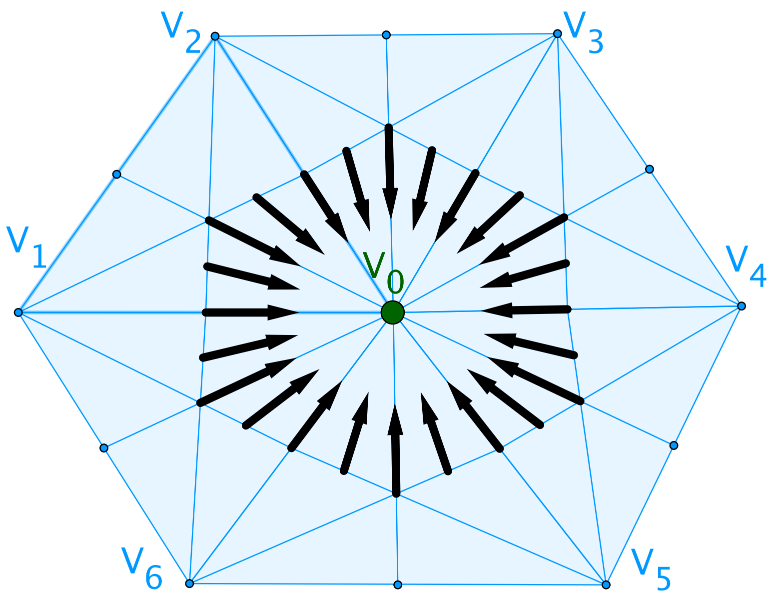

Figure 1: *Critical vertex V_0 (unmatched green node) surrounded by the arcs (black arrows) in the stable matching on the barycentric subdivision (blue) of K. By Theorem 3, V_0 is a critical vertex if and only if $V_i > V_0$ for $1 \leqslant i \leqslant 6$.*

1 Introduction

Forman introduced a combinatorial Morse theory [3] to study the topology of discrete objects such as cell and simplicial complexes. In this theory, the main objects from the smooth theory such as a Morse function, the corresponding gradient field, its critical points, and its flow have a discrete analogue. In particular, the discrete gradient is essentially an alternating cycle-free matching on the Hasse diagram of the complex. The Morse-Smale complex can be seamlessly obtained from the discrete gradient field, by purely combinatorial algorithms [5, 2, 4] avoiding any numerical integration or differentiation. The main issue is how faithful the discrete gradient is to the geometry of a function sampled at the vertices of the complex. This geometric accuracy can be formulated as the position of the critical elements (i.e. the cells of the Morse-Smale decomposition) with respect to Banchoff's critical points, obtained by a simplex-wise interpolation of the function [1].

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There are numerous constructions proposed for a geometric faithful discrete gradient field. Lewiner et al. [5, 2] propose a modified greedy weighted matching algorithm with weights based on the difference of the function values at incident cells. The algorithm checks if the matching would have an alternating cycle at each step before adding a new edge to the matching. The geometric accuracy of the algorithm was proven robust if the triangulation is subdivided *twice* [6]. The results rely on a complicated proof technique, making it difficult to generalize and refine the results. Gyulassy et al. [4] suggested that a priority-queue based algorithm avoids checking for cycles. Robins et al.[8] developed an algorithm which uses homotopy expansions to build the gradient on the lower star of each vertex, and the authors were able to prove a one–to–one correspondence between the critical cells and Banchoff's critical points.

In this work, we propose a construction of a discrete gradient field for triangulated surfaces using a stable matching algorithm [7]. This construction is equivalent to the greedy one [5] for the same weights. The main advantage is that stable matchings are characterized by a local stability condition, which simplifies the proofs for the geometric accuracy, and gives a better description of the constructed gradient.

2 Stable matching

Given a triangulated surface K, one defines its Hasse diagram H to be a directed graph in which the set of nodes of H is the set of simplexes of K and there is an arc (τ, σ) if and only if simplex τ is contained in simplex σ and $dim(\tau) + 1 = dim(\sigma)$. A scalar function $f : K_0 \to \mathbb{R}$ sampled on the vertices of a triangulated surface, can be extended to all the simplexes $\sigma \in K$ as the mean of the values of f on the vertices of simplex σ. For the stable matching, the arcs of the Hasse diagram of K are sort by the weight function $\mathsf{W}(\tau, \sigma) = f(\sigma) - f(\tau)$.

Given a matching M on H, an arc $(\tau, \sigma) \notin M$ is unstable if there exists $\rho, \rho' \in H$ matched with τ and σ respectively such that:

$$\mathsf{W}(\tau, \sigma) > \mathsf{W}(\tau, \rho) \quad \text{and} \quad \mathsf{W}(\tau, \sigma) > \mathsf{W}(\rho', \sigma) \ .$$

If there is no unstable arc, then M is called stable matching [7]. A stable matching always exists since H is a bipartite graph. In this work, the weights are assumed to be distinct, therefore the stable matching is *unique* and can be found by the classical Gale–Shapley algorithm [7].

A matching M on H defines a discrete vector field $\mathcal{V}$, where $\mathcal{V}$-paths are alternating paths in M. A discrete vector field is a gradient [3] if M is alternating cycle-free. One of the advantages of using stable matchings is that they are characterized by the local stability condition which is used to prove the results of the next section.

3 Results

Gradient paths In Morse theory, the gradient field is $-\nabla f$, and f is thus expected to decrease along the gradient path. Let $\mathcal{V}$ be the discrete vector field on a triangulated surface K, obtained from stable matching M, ordered by a scalar function f, as in the previous section. A $\mathcal{V}$–path is *decreasing* if the value of f at the first vertex of the path is greater than its value at the last vertex.

Theorem 1 *f is globally decreasing on each $\mathcal{V}$–path [6]. In addition $\mathcal{V}$ is a valid discrete gradient vector field, i.e. $\mathcal{V}$ has no closed $\mathcal{V}$–path.*

The proof idea is the following: if there exists a non-decreasing $\mathcal{V}$-path $\blacktriangleleft \rho\ \sigma\ \tau\ \rho' \blacktriangleright$ of length 2, then it is easy to show that (τ, σ) is unstable. Since the paths are non-decreasing, there is no alternating cycle in M.

Relationship to greedy construction When the edges weights are distinct, it is known that the greedy weighted matching is equivalent to the stable matching. Moreover, Theorem 1 ensures that the stability of the matching automatically avoids alternating cycle. Therefore the stable matching construction is equivalent to the greedy construction [5, 2]. The stable matching formulation leads to the following results regarding the geometric accuracy of the greedy construction.

Critical points The following results are concerned with the relation between Forman's critical vertices, edges, and triangles [3] (i.e. unmatched nodes of H) to the Banchoff minima, saddles, and maxima [1] respectively. They are proven solely with the stability condition. Due to space constraints, only the result for critical vertices is presented.

Theorem 2 *A vertex v is a critical vertex of $\mathcal{V}$ if and only if v is minimum over its 2-neighborhood.*

Barycentric subdivision On a subdivision of the triangulated surface, our results about the relation of Forman's theory and Banchoff's theory are more precise. Let K' be the *first* barycentric subdivision of K, with scalar function f' linearly interpolated from f. Let $\mathcal{V}'$ be the discrete gradient on K', obtained from the stable matching ordered by f'.

Theorem 3 *A vertex v is a Banchoff minimum of K' if and only v is critical vertex of $\mathcal{V}'$. If a vertex v is a Banchoff saddle of K', then in $\mathcal{V}'$, there exists a critical edge in the lower star of v. If a vertex v is a Banchoff maximum, then in $\mathcal{V}'$, there exists a critical triangle in the lower star of v.*

This result improves upon previous results on the geometric accuracy of the greedy construction [6] since it requires one less barycentric subdivision and it gives a more precise location of the critical points (Figure 1). Moreover the proof is considerably simpler, which might yield a generalization to higher dimensions.

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