TRANSVERSE GEOMETRY OF FOLIATIONS CALIBRATED BY NON-DEGENERATE CLOSED 2-FORMS

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Abstract. Codimension one foliated manifolds $(M, \mathcal{F})$ admitting a closed 2-form ω making each leaf symplectic are a natural generalization of 3-dimensional taut foliations. Remarkably, on such closed foliated manifolds $(M, \mathcal{F})$ there exists a class of 3-dimensional transverse closed submanifolds W on which $\mathcal{F}$ induces a taut foliation $\mathcal{F}_W$. Our main result says that the foliated submanifold $(W, \mathcal{F}_W)$ has the same transverse geometry as $(M, \mathcal{F})$. More precisely, the inclusion induces an essential equivalence between the corresponding holonomy groupoids. The proof of our main result relies on a leafwise Lefschetz hyperplane theorem, which is of independent interest.

1. Introduction and Statement of Main Results

Let $\mathcal{F}$ be a foliation by surfaces on a closed 3-dimensional manifold W. The foliation $\mathcal{F}$ is called *taut* if for every leaf there exists a loop C through it such that $C \pitchfork \mathcal{F}$ (C is everywhere transverse to $\mathcal{F}$). This topological definition is equivalent to the following differential geometric characterization: there exists a closed 2-form inducing an area form on each leaf, see [21].

Taut foliations have no Reeb components. The latter are a source of flexibility in the construction of foliations by surfaces on 3-manifolds. Hence, it is not surprising that the existence of a Reebless foliation $\mathcal{F}$ on W has consequences for both the topology of W and the topology of the pair $(W, \mathcal{F})$. The most well-known topological constraints for a Reebless foliation are related to the fundamental group of W: the universal covering space of W is diffeomorphic to $\mathbb{R}^3$ [17]; also, work of Novikov on vanishing cycles ensures that the fundamental group of any leaf injects into the fundamental group of $\mathcal{F}$, and that every loop $C \pitchfork \mathcal{F}$ must be non-trivial in homotopy.

A 3-dimensional closed manifold with a taut foliation has additional remarkable properties: there exist metrics making each compact region of a leaf a minimal hypersurface inside its relative homology class (*topological tautness equals geometric tautness*), and the taut condition can be reformulated in terms of foliation cycles (*topological tautness equals homological tautness*).

The notion of tautness has a straightforward generalization to codimension one foliations on closed manifolds of arbitrary dimension $(M^p, \mathcal{F})$. One requires the existence through any leaf of a loop $C \pitchfork \mathcal{F}$, or, equivalently [20], the existence of a closed $p-1$-form whose restriction to every leaf of $\mathcal{F}$ is a volume form. It is still true for taut foliations on arbitrary dimension that topological tautness is equivalent to both geometric tautness [19] or homological tautness [21, 10]. Likewise, the absence of vanishing cycles in the 3-dimensional case generalizes to the absence of exploding plateaus [1]. However, the lack of exploding plateaus carries no homotopical information. This explains why most of the topological aspects of the rich theory of 3-dimensional taut foliations do not extend to higher dimensions. In fact, taut foliations in high dimensions are very flexible objects, as shown by the h–principle proved in [13].

We have proposed the following generalization of 3-dimensional taut foliations to higher dimensions:

Definition 1. [11] *A codimension one foliation $\mathcal{F}$ of M^{2n+1} is said to admit a 2-calibration if there exists a closed 2-form ω such that the restriction of ω^n to the leaves of $\mathcal{F}$ is nowhere vanishing. A triple $(M, \mathcal{F}, \omega)$, where ω is a 2-calibration for $\mathcal{F}$, is referred to as a 2-calibrated foliation.*

Remark 1. *Recall that a symplectic structure on a manifold of dimension $2n$ is given by a closed 2-form Ω which is everywhere non-degenerate; the non-degeneracy condition is equivalent to Ω^n being nowhere vanishing, that is to say, a volume form. The definition of a 2-calibrated foliation $(M, \mathcal{F}, \omega)$ can be understood as follows: the closed 2-form ω makes every leaf of $\mathcal{F}$ a symplectic manifold.*

In this paper we will study codimension one foliations which admit a 2-calibration. For that purpose we will take advantage of the tools furnished by a 2-calibration ω; we will not be interested in describing the (Poisson) geometry of triples $(M, \mathcal{F}, \omega)$. Foliations will be of class C^3 in the transverse direction unless otherwise stated.

Taut foliations on closed manifolds of arbitrary dimension have, by definition, 1-dimensional closed submanifolds everywhere transverse to the foliation. Remarkably, foliations admitting 2-calibrations have 3-dimensional closed submanifolds everywhere transverse to the foliation, which capture the topology of the ambient leaf space.

Theorem 1 ([11, 12])**.** *Let M be a closed manifold and let $\mathcal{F}$ be a codimension one foliation of M admitting a 2-calibration, which is of class C^3 in the transverse direction. There exists $W \hookrightarrow M$ a 3-dimensional closed submanifold $W \pitchfork \mathcal{F}$, with the following properties:*

1. *W inherits a taut foliation $\mathcal{F}_W$ from $\mathcal{F}$.*
2. *The map between leaf spaces induced by the inclusion*

$$W/\mathcal{F}_W \to M/\mathcal{F}$$

is a homeomorphism.

Following Haefliger's viewpoint, the *transverse geometry* of a foliation $(M, \mathcal{F})$ is described by the group-like structures in which the holonomy parallel transport is encoded. These are either the holonomy pseudogroup or the holonomy groupoid (see section 2 for more details). It is well-known that the leaf space $M/\mathcal{F}$ is encoded in these group-like structures. Therefore, Theorem 1 suggests a relation between the transverse geometries of $(M, \mathcal{F})$ and the 3-dimensional taut foliation $(W, \mathcal{F}_W)$.

The purpose of this article is to relate the transverse geometries of $(M, \mathcal{F})$ and $(W, \mathcal{F}_W)$. Our main result is the following.

Theorem 2. *Let M be a closed manifold and let $\mathcal{F}$ be a foliation of M admitting a 2-calibration. There exists $W \hookrightarrow M$, a 3-dimensional closed submanifold $W \pitchfork \mathcal{F}$, which inherits a taut foliation $\mathcal{F}_W$ from $\mathcal{F}$ with the following (equivalent) properties:*

- *the map between holonomy groupoids induced by the inclusion*

$$\iota\colon \mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F}) \tag{1}$$

  is an essential equivalence,
- *any total transversal T for $(W, \mathcal{F}_W)$ is also a total transversal for $(M, \mathcal{F})$, and the holonomy pseudogroups $\mathcal{H}(\mathcal{F}, T)$ and $\mathcal{H}(\mathcal{F}_W, T)$, induced on T by $\mathcal{F}$ and $\mathcal{F}_W$, respectively, coincide.*

It is a well-known folklore result that Theorem 2 implies that both $(M, \mathcal{F})$ and $(W, \mathcal{F}_W)$ have the same transverse geometry. Since we could not find a precise reference for this statement in the literature, we shall illustrate the equivalence of transverse geometries by stating a few corollaries of Theorem 2.

The first corollary says that the bijection of leaf spaces in Theorem 1 is compatible with dynamical properties of the leaves:

Corollary 1. *Let M be a closed manifold and let $\mathcal{F}$ be a foliation of M admitting a 2-calibration. Let $W \hookrightarrow M$ be a 3-dimensional submanifold as in the statement of Theorem 2.*

The induced homeomorphism between leaf spaces

$$W/\mathcal{F}_W \to M/\mathcal{F}$$

preserves the growth type of the leaves.

Proof. Let T be any total transversal to $\mathcal{F}_W$. The growth type of a leaf [18] of $\mathcal{F}$ (resp. $\mathcal{F}_W$) coincides with the growth type of the corresponding orbit of the compactly generated pseudogroup $\mathcal{H}(\mathcal{F}, T)$ (resp. $\mathcal{H}(\mathcal{F}_W, T)$) [9]. Therefore, the corollary is an immediate consequence of Theorem 2. □

The next corollaries rely on the fact that essential equivalences between etale Lie groupoids identify sheaves on the Lie groupoids, invariant global sections of those sheaves, sheaf valued cohomologies, etc.

Corollary 2. *Let M be a closed manifold admitting a foliation $\mathcal{F}$ which is smooth in the transverse direction and admits a 2-calibration. Let $W \hookrightarrow M$ be a 3-dimensional submanifold as in the statement of Theorem 2.*

The inclusion establishes a bijection between transverse real analytic structures on $(M, \mathcal{F})$ and transverse real analytic structures on $(W, \mathcal{F}_W)$.

Proof. For the foliated space $(M, \mathcal{F})$ its structural sheaf $\mathcal{A}$ is the sheaf of smooth functions constant on leaves. This is a sheaf on $\mathrm{Hol}(\mathcal{F})$, meaning that its stalks have a (continuous right) action of the holonomy groupoid ([16], Chapter 5). A transverse real analytic structure on the codimension one foliated space $(M, \mathcal{F})$ is given by a sheaf $\mathcal{A}^\omega$ on $\mathrm{Hol}(\mathcal{F})$, $\mathcal{A}^\omega \subset \mathcal{A}$, with the following property: for each $x \in M$ there exists a neighborhood U and a function $\tau \in \mathcal{A}^\omega(U)$ such that τ induces a homeomorphism $U/\mathcal{F}_U \to \tau(U)$ and $\mathcal{A}^\omega(U) = \tau^* C^\omega(\tau(U))$, with $\mathcal{F}_U$ the foliation induced by $\mathcal{F}$ on U.

An essential equivalence between etale Lie groupoids establishes an equivalence between sheaves on the groupoids [15]: one of the functors amounts to pulling back the sheaf by the essential equivalence (the inverse image functor). In our specific case, the essential equivalence is $\iota\colon \mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F})$ in (1), and it is clear that $\iota^*\mathcal{A}^\omega$ is a subsheaf of the structural sheaf of $(W, \mathcal{F}_W)$ defining a transverse real analytic structure.

The functor in the other direction $\iota!$ is defined as follows: firstly, it pulls back a given sheaf $\mathcal{B}$ on $\mathrm{Hol}(\mathcal{F}_W)$ to the auxiliary manifold of points in $\mathrm{Hol}(\mathcal{F})$ representing paths with starting point in W, and then pushes the sheaf forward to M using the ending point map. In fact, the functor $\iota!$ has very clear geometric description: it uses the holonomy parallel transport on $(M, \mathcal{F})$ to 'spread' the sheaf from W to M (strictly speaking from $\mathrm{Hol}(\mathcal{F}_W)$ to $\mathrm{Hol}(\mathcal{F})$).

It is easy to check that if $\mathcal{B}$ is a sheaf defining a transverse real analytic structure on $(W, \mathcal{F}_W)$, then $\iota!\mathcal{B}$ also defines a transverse real analytic structure on $(W, \mathcal{F}_W)$.

The two functors above are such that $\iota^* \circ \iota! = \mathrm{Id}$ and, since the holonomy groupoids have unique local bisections ([16], Chapter 5) through any given point, $\iota^! \circ \iota^*\mathcal{C}$ is *canonically* isomorphic to $\mathcal{C}$, where $\mathcal{C}$ is any sheaf on $(M, \mathcal{F})$. Of course, this implies that if $\mathcal{C}$ defines a transverse real analytic structure on $(M, \mathcal{F})$, the sheaf $\iota^! \circ \iota^*\mathcal{C}$ defines the same transverse real analytic structure, and this finishes the proof of the corollary. □

Corollary 3. *Let M be a closed manifold and let $\mathcal{F}$ be a foliation of M admitting a 2-calibration. Let $W \hookrightarrow M$ be a 3-dimensional submanifold as in the statement of Theorem 2.*

The inclusion establishes a bijection between transverse Riemannian metrics on $(M, \mathcal{F})$ and transverse Riemannian metrics on $(W, \mathcal{F}_W)$. More generally, the correspondence extends to transverse invariant measures.

Proof. The normal bundle to the foliation $\nu(\mathcal{F})$ is a bundle on $\mathrm{Hol}(\mathcal{F})$ (the holonomy groupoid acts on it), and so its second symmetric power is as well. Let $\mathcal{R}_{\mathcal{F}}$ be the sheaf on $\mathrm{Hol}(\mathcal{F})$ of sections of $\mathrm{Sym}^2\nu(\mathcal{F})$. Transverse invariant Riemannian metrics on $(M, \mathcal{F})$ are by definition invariant global sections of $\mathcal{R}_{\mathcal{F}}$.

It is clear that ι identifies $\mathcal{R}_{\mathcal{F}}$ with $\mathcal{R}_{\mathcal{F}_W}$ (up to canonical isomorphims). Since an essential equivalence identifies sheaves together with their invariant global sections [15], ι establishes a one-to-one correspondence between transverse Riemannian metrics.

More generally, an invariant transverse (Radon) measure is also an invariant global section of a sheaf $\mathcal{M}_{\mathcal{F}}$ on $\mathrm{Hol}(\mathcal{F})$: for each open subset U, $\mathcal{M}_{\mathcal{F}}(U)$ is defined as the continuous dual of the continuous functions on U which have compact support and are constant on the leaves of the foliation induced by $\mathcal{F}$ on U. One checks that ι identifies $\mathcal{M}_{\mathcal{F}}$ with $\mathcal{M}_{\mathcal{F}_W}$, and this finishes the proof of the corollary. □

For a foliated space $(M, \mathcal{F})$, there are several homology/cohomology theories that can be defined in terms of the holonomy groupoid $\mathrm{Hol}(\mathcal{F})$. One is the differentiable cohomology of the holonomy groupoid $H^*_d(\mathrm{Hol}(\mathcal{F}))$. When $\mathrm{Hol}(\mathcal{F})$ is Hausdorff, $H^*_d(\mathrm{Hol}(\mathcal{F}))$ coincides with Haefliger's cohomology [8] with coefficients in the structural sheaf of the foliation. Others are the homology theories defined out of the convolution algebra of the groupoid, namely, the periodic, Hochschild and periodic cyclic homologies (see for example [3]).

Corollary 4. *Let M be a closed manifold and let $\mathcal{F}$ be a foliation of M admitting a 2-calibration. Let $W \hookrightarrow M$ be a 3-dimensional submanifold as in the statement of Theorem 2.*

1. *Assume that* $\mathrm{Hol}(\mathcal{F})$ *is a Hausdorff Lie groupoid. Then the induced homomorphism between differentiable cohomology algebras*
$$H^*_d(\mathrm{Hol}(\mathcal{F}_W)) \to H^*_d(\mathrm{Hol}(\mathcal{F}))$$
is an isomorphism.
2. *Assume that the foliation is smooth in the transverse direction. Then the induced homomorphism between cyclic (resp. Hochschild, periodic cyclic) homologies is an isomorphism.*

Proof. If $\mathrm{Hol}(\mathcal{F})$ is a Hausdorff, then so is $\mathrm{Hol}(\mathcal{F}_W)$, since this is a property stable under essential equivalences. An essential equivalence between Hausdorff Lie groupoids induces an isomorphism between the corresponding differentiable cohomologies [2] (the proof is done for smooth groupoids, but holds in any regularity class). When the Lie groupoids are (non-necessarily Hausdorff) smooth *foliation groupoids* –and this is the case for holonomy groupoids– the essential equivalence also induces an isomorphism between between cyclic, Hochschild and periodic cyclic homologies [3]. Therefore the corollary is an immediate consequence of Theorem 2. □

The structure of the paper is the following: in Section 2 we briefly recall the definitions of holonomy pseudogroup and holonomy groupoid, and we present some basic material on essential equivalences of Lie groupoids. In particular, we easily characterize when the morphism in Theorem 2 between holonomy groupoids

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F}) \tag{2}$$

is an essential equivalence. This characterization leads to a sufficient condition for (2) to be an essential equivalence; the sufficient condition is a Lefschetz-type requirement for the zero-th and first homotopy groups of the pair $(F, F \cap W)$, where F is any leaf of $\mathcal{F}$.

As we shall recall in Section 3, Lefschetz-type theorems are classical results for hyperplane sections in Kähler geometry, which have been extended by Donaldson [4] to the symplectic setting by means of the so called *approximately holomorphic theory*. Very much as in the symplectic setting, approximately holomorphic theory can also be applied to a closed manifold endowed with a 2-calibrated foliation $(M, \mathcal{F}, \omega)$ to produce Lefschetz-type theorems for the pair (M, W), where W is a Donaldson-type submanifold [11] (indeed, the 3-dimensional submanifold appearing in the statement of Theorem 2 is of this kind). Despite the non-compactness of the leaves, we shall prove that Lefschetz type-theorems are also valid for the pair $(F, F \cap W)$.

Theorem 3. *Let $(M^{2n+1}, \mathcal{F}, \omega)$ be a closed manifold of dimension $2n+1$ endowed with a 2-calibrated foliation. Let W be a Donaldson-type submanifold of dimension $2j+1$. Then, for every leaf F of $\mathcal{F}$ it holds that*

$$\pi_k(F, F \cap W) = \{1\},\ 0 \leq k \leq j.$$

Theorem 3 states that the higher the dimension of a Donaldson-type submanifold W is, the more topology of F is captured by $F \cap W$. This should also imply the higher the dimension of W is, the more properties of the geometry of $\mathcal{F}$ are captured by $\mathcal{F}_W$. Indeed, Theorems 1 and 2 are a consequence of Theorem 3 for 3-dimensional Donaldson-type submanifolds. For higher dimensional ones we have:

Theorem 4. *Let M be a closed manifold of dimension at least 7 and let $\mathcal{F}$ be a foliation of M admitting a 2-calibration. Then there exists a 5-dimensional closed submanifold $W \pitchfork \mathcal{F}$ such that the map between homotopy groupoids induced by the inclusion*

$$\Pi(\mathcal{F}_W) \to \Pi(\mathcal{F}) \tag{3}$$

is an essential equivalence.

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2. Transverse submanifolds and essential equivalence of holonomy groupoids

In this section we briefly sketch the construction of the holonomy pseudogroup of a foliation, and recall its relation with the transverse geometry of the foliation. We also introduce the holonomy groupoid, which is an alternative way to encode the transverse geometry. Since most of the results stated in the Introduction use the holonomy groupoid, we will recall some basic material of Lie groupoids. The reader familiar with these constructions may skip them and go directly to subsection 2.3, where we characterize some important essential equivalences between holonomy groupoids.

2.1. **Holonomy pseudogroup.** Let γ be a piecewise smooth path contained in a leaf of the foliated manifold $(M, \mathcal{F})$, starting at x and ending at y. Let T_x and T_y be submanifolds through x and y, respectively, transverse to $\mathcal{F}$, and of dimension complementary to the rank of $\mathcal{F}$. The holonomy parallel transport along γ defines a diffeomorphism

$$h_\gamma \colon (U_x, x) \to (U_y, y),$$

where $U_x \subset T_x$ and $U_y \subset T_y$ are neighborhoods of x and y, respectively; the germ of the diffeomorphism h_γ only depends on the relative homotopy class of γ inside its leaf.

A *total transversal* T to $(M, \mathcal{F})$ is a (not necessarily closed nor connected) submanifold transverse to $\mathcal{F}$, of dimension the co-rank of $\mathcal{F}$, and intersecting every leaf of the foliation; any foliated atlas of $(M, \mathcal{F})$ provides total transversals.

Let T be a total transversal. The representative of the *holonomy pseudogroup* $\mathcal{H}(\mathcal{F}, T)$ is the smallest pseudogroup of diffeomorphisms containing the holonomy parallel transports of paths starting and ending at points of T. There is a natural equivalence relation between pseudogroups: Haefliger's equivalence [9]. Two total transversals T, T' produce Haefliger equivalent pseudogroups $\mathcal{H}(\mathcal{F}, T)$, $\mathcal{H}(\mathcal{F}, T')$. Strictly speaking, the holonomy pseudogroup $\mathcal{H}(\mathcal{F})$ is the Haefliger's equivalence class of any such $\mathcal{H}(\mathcal{F}, T)$.

Following Haefliger's insight, the *transverse geometry* of $(M, \mathcal{F})$ is understood as those properties of the representatives of the holonomy pseudogroups that are stable under Haefliger's equivalence (and which are therefore properties of $\mathcal{H}(\mathcal{F})$). Interestingly enough, there are some properties of a foliated manifold, which despite their original definition, happen to be manifestly dependent only on the the transverse geometry. For example, for compact foliated manifolds, being $\mathcal{H}(\mathcal{F})$ compactly generated [9], growth properties of the leaves of $(M, \mathcal{F})$ correspond to growth properties of the orbits of $\mathcal{H}(\mathcal{F})$.

2.2. **Holonomy and Lie groupoids.** An alternative way to encode the holonomy of $\mathcal{F}$ is the *holonomy groupoid* $\mathrm{Hol}(\mathcal{F})$ (the graph of the foliation in [22]). This is the collection of germs of all possible holonomy parallel transports. More precisely, the points of the holonomy groupoid correspond to paths inside a leaf modulo the holonomy equivalence relation: two paths γ, γ' inside a leaf, starting at x and ending at y, are holonomy equivalent if they give rise to the same germ of holonomy parallel transport. Concatenation of paths induces a Lie groupoid structure on $\mathrm{Hol}(\mathcal{F})$ with base manifold M.

Definition 2. *A* Lie groupoid $\mathcal{G} \rightrightarrows M$ *over* M *is a (possibly non-Hausdorff) manifold* $\mathcal{G}$ *(of class* C^k*,* $k \geq 1$*) endowed with the following additional structure:*

(1) *two surjective submersions to* M*: the source and target maps* $\mathbf{s}, \mathbf{t}$*,*
(2) *a multiplication map* $\mathcal{G}^{(2)} \to \mathcal{G}$ *defined on the manifold of pairs of elements*

$$\mathcal{G}^{(2)} = \{(g, h) \,|\, \mathbf{s}(g) = \mathbf{t}(h)\},$$

satisfying the usual axioms for the multiplication on a Lie group,
(3) *inversion map given by an involutive diffeomorphism,*
(4) *instead of a unit, an embedding* $M \to \mathcal{G}$ *whose image are the units of the groupoid.*

For a good reference on the notion of Lie groupoids see [16], Chapter 5. We note the following basic facts:

- The set of elements with source and target a fixed $x \in M$ form a Lie group, called the isotropy group of $\mathcal{G}$ at x, which we denote by $\mathcal{G}_x$;

- There is an equivalence relation on M induced by $\mathcal{G}$: two points are related if they are source and target of an element of $\mathcal{G}$; the classes of this equivalence relation fit into a (possibly singular) foliation. More precisely, there is an action of $\mathcal{G}$ in M ([16], Chapter 5) whose orbits are the leaves of the aforementioned foliation.

Remark 2. *It is convenient to think of a Lie groupoid $\mathcal{G} \rightrightarrows M$ as a C^k desingularization of its leaf space $M/\mathcal{G}$.*

Example 1. *For the holonomy groupoid $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$, the source and target maps are the starting and ending point respectively. Multiplication is (as already mentioned) induced by concatenation of paths. The foliation associated to the action of $\mathrm{Hol}(\mathcal{F}) \rightrightarrows M$ is nothing but $\mathcal{F}$. Note that the regularity of the holonomy groupoid coincides with the transverse regularity of the foliation (in our case of class at least C^3).*

Example 2. *In general there might be plenty of Lie groupoids over M besides $\mathrm{Hol}(\mathcal{F})$ whose action on M recovers $\mathcal{F}$. If in the definition of the holonomy groupoid we do not quotient by the holonomy equivalence relation, but by the stronger leafwise homotopy relative to starting and ending points, we obtain the homotopy groupoid $\Pi(\mathcal{F}) \rightrightarrows M$.*

The orbits of the action of the homotopy groupoid on M are precisely the leaves of $\mathcal{F}$. For each point $x \in M$, the isotropy group of $\Pi(\mathcal{F})$ at x is the fundamental group $\pi_1(F, x)$, where F is the leaf containing x.

Let $K(F, x)$ denote the normal subgroup of $\pi_1(F, x)$ of homotopy classes of loops with trivial holonomy parallel transport. By definition of the holonomy groupoid, there is an obvious etale surjection

$$p \colon \Pi(\mathcal{F}) \to \mathrm{Hol}(\mathcal{F}). \tag{4}$$

The inverse image of the units in $\mathrm{Hol}(\mathcal{F})$ is the collection of subgroups $K(F, x)$, where x ranges through M. Equivalently, the isotropy group Hol_x can be identified with the quotient group $\pi_1(F, x)/K(F, x)$.

A *morphism* of Lie groupoids is a map of pairs $(\mathcal{G}', M') \to (\mathcal{G}, M)$ which intertwines source, target, multiplication and unit embedding maps (with the obvious regularity requirements).

Example 3. *Let $(M, \mathcal{F})$ be a foliated manifold and W a submanifold transverse to $\mathcal{F}$, so it inherits a foliation $\mathcal{F}_W$. The inclusion of W in M induces a morphism of Lie groupoids*

$$\Pi(\mathcal{F}_W) \to \Pi(\mathcal{F}). \tag{5}$$

Let $x \in M$ and let F be the leaf through x. The map induced by (5) on isotropy groups

$$\pi_1(F \cap W, x) \to \pi_1(F, x)$$

takes $K(F \cap W, x)$ inside of $K(F, x)$; the reason is that one may choose a transversal to $\mathcal{F}_W$ (of dimension the co-rank of $\mathcal{F}_W$) which is also a transversal to $\mathcal{F}$. Therefore (5) induces a morphism between the holonomy groupoids

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F}). \tag{6}$$

One has the obvious notion of isomorphism of Lie groupoids. Isomorphisms are scarce, and they are not quite the right equivalence relation among Lie groupoids. On the other hand, there is a much broader family of morphism between Lie groupoids which does preserve a great deal of the information that a Lie groupoid encodes.

2.3. **Essential equivalences and holonomy groupoids.** A morphism of Lie groupoids $\phi\colon (\mathcal{G}' \rightrightarrows M') \to (\mathcal{G} \rightrightarrows M)$ is an *essential equivalence* if the following two conditions hold:

(1) The composition

$$\begin{array}{rcl} \mathbf{t}\circ \mathrm{pr}_1 : \mathcal{G}_{\mathbf{s}}\times_{\phi} M' & \longrightarrow & M \\ (h,x) & \longmapsto & \mathbf{t}(h) \end{array} \tag{7}$$

is a surjective submersion;

(2) The square

$$\begin{array}{ccc} \mathcal{G}' & \xrightarrow{\ \phi\ } & \mathcal{G} \\ \big\downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} & & \big\downarrow{\scriptstyle (\mathbf{s},\mathbf{t})} \\ M'\times M' & \xrightarrow{\ \phi\times\phi\ } & M\times M \end{array} \tag{8}$$

is a fibered product of manifolds.

Remark 3. *If one regards $\mathcal{G}' \rightrightarrows M'$ and $\mathcal{G} \rightrightarrows M$ as C^k-desingularizations for $M'/\mathcal{G}'$ and $M/\mathcal{G}$, respectively, then an essential equivalence plays the role of an isomorphism. From this perspective, essential equivalences are the most relevant morphisms in Lie groupoid theory.*

Example 4. *Let T be a total transversal of $(M,\mathcal{F})$. Let $\mathrm{Hol}(\mathcal{F})_{|T}$ denote the points in the holonomy group with source and target in T. Then $\mathrm{Hol}(\mathcal{F})_{|T}$ is a Lie groupoid and the inclusion*

$$\mathrm{Hol}(\mathcal{F})_{|T} \to \mathrm{Hol}(\mathcal{F})$$

is an essential equivalence.

Remark 4. *The pseudogroup $\mathcal{H}(\mathcal{F},T)$ can be reconstructed from $\mathrm{Hol}(\mathcal{F})_{|T}$ as its pseudogroup of local bisections* [16]. *Therefore $\mathcal{H}(\mathcal{F},T)$ and $\mathrm{Hol}(\mathcal{F})_{|T}$ carry the same information.*

Next, we want to discuss when the morphism in (6) is an essential equivalence. More generally, one has the following result:

Lemma 1. *Let $\phi\colon (\mathcal{G}' \rightrightarrows M') \to (\mathcal{G} \rightrightarrows M)$ be a morphism of Lie groupoids such that $\phi\colon M' \to M$ is an embedding. Let $\mathcal{F}$ (resp. $\mathcal{F}'$) be the possibly singular foliation induced by $\mathcal{G}$ on M (resp. $\mathcal{G}'$ on M'). Then ϕ is an essential equivalence if and only if the following conditions hold:*

(1) *The image $\phi(M')$ has non-empty intersection with every leaf of $\mathcal{F}$, and the restriction $\phi\colon M' \to M$ is transverse to $\mathcal{F}$.*
(2) *For every $x' \in M'$,*
   (i) *The induced map on isotropy groups $\mathcal{G}'_{x'} \to \mathcal{G}_{\phi(x')}$ is an isomorphism;*
   (ii) *The image of the leaf through x' is given by $\phi(M')\cap F$, where F is the leaf through $\phi(x')$.*

Proof. The lemma simply restates the two conditions which define an essential equivalence, when ϕ is an embedding on units:

The map (7) is a surjective submersion if and only if condition (1) holds (this has nothing to do with $\mathcal{G}'$, and is just a property of how ϕ is defined on units). Because ϕ is an embedding on units, the isomorphism of isotropy groups is necessary for (7) to be a fibered product (the isomorphism between isotropy groups is a necessary condition for any morphism to be an essential equivalence; in our setting this is much more transparent); likewise, each leaf of $\mathcal{F}'$ must be identified with the intersection of $\phi(M')$ with a leaf of $\mathcal{F}$. But since ϕ is a morphism, it is also straightforward to check that these conditions are sufficient to show that (7) is a fibered product. $\square$

Corollary 5. *Let $(M,\mathcal{F})$ be a foliated manifold and let W be a submanifold transverse to $\mathcal{F}$, with induced foliation $\mathcal{F}_W$. The induced map on holonomy groupoids*

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F})$$

is an essential equivalence if and only if the inclusion induces isomorphisms

$$\pi_0(F\cap W, x) \to \pi_0(F, x),$$

$$\pi_1(F\cap W, x)/K(F\cap W, x) \to \pi_1(F, x)/K(F, x),$$

where F is the leaf through x.

Corollary 6. *Let $(M,\mathcal{F})$ be a foliated manifold and let W be a submanifold transverse to $\mathcal{F}$, with induced foliation $\mathcal{F}_W$. If every leaf F of $\mathcal{F}$ satisfies*

$$\pi_0(F, F\cap W) = \{1\},\ \pi_1(F, F\cap W) = \{1\},$$

then the induced morphism of holonomy groupoids

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(\mathcal{F})$$

is an essential equivalence.

Proof. By using long exact homotopy sequence for the pair $(F, F\cap W)$, we deduce the isomorphism on zero-th homotopy groups, and the epimorphism of fundamental groups

$$\pi_1(F\cap W, x) \to \pi_1(F, x),$$

for every $x \in F$. The epimorphism automatically implies the surjectivity of

$$h\colon \pi_1(F\cap W, x) \to \pi_1(F, x)/K(F, x). \tag{9}$$

We claim that the kernel of (9) is $K(F\cap W, x)$: by definition, if $\gamma \subset F\cap W$ is such that $h([\gamma]) = 0$, its holonomy parallel transport with respect to $\mathcal{F}$ gives the trivial germ. Since the parallel transport does not depend on the choice of model for the (pullback of) the normal bundle, we may take $\nu(\mathcal{F}_W)$ as model for $\nu(\mathcal{F})$ (because $\gamma \subset W$). Thus we conclude that $[\gamma] \in K(F\cap W, x)$. Reversing the argument one shows that $K(F\cap W, x)$ is in the kernel of (9).

□

The analogous result for the homotopy groupoids is the following:

Corollary 7. *Let $(M,\mathcal{F})$ be a foliated manifold and let W be a submanifold transverse to $\mathcal{F}$, with induced foliation $\mathcal{F}_W$. If every leaf F of $\mathcal{F}$ satisfies*

$$\pi_0(F, F\cap W) = \{1\},\ \pi_1(F, F\cap W) = \{1\},\ \pi_2(F, F\cap W) = \{1\}$$

then the induced morphism of homotopy groupoids

$$\Pi_1(\mathcal{F}_W) \to \Pi_1(\mathcal{F})$$

is an essential equivalence.

3. Applications of approximately holomorphic theory to 2-calibrated foliations

3.1. **Preliminaries.**

Definition 3. *A submanifold $W \hookrightarrow (M,\mathcal{F},\omega)$ is a* 2-calibrated submanifold *if it is everywhere transverse to $\mathcal{F}$ and intersects each leaf of $\mathcal{F}$ in a symplectic submanifold w.r.t. ω. In particular, $(W,\mathcal{F}_W,\omega_{|W})$ is a 2-calibrated foliation.*

These submanifolds are constructed by applying approximately holomorphic techniques, first described by Donaldson [4] in the symplectic setting. Let us summarize some of the needed definitions and results.

Let M^{2n+1} be a closed manifold endowed with a 2-calibrated foliation $(\mathcal{F}, \omega)$. After a small perturbation, we may assume without loss of generality that $[\omega]$ is a rational class; by scaling the class, we may also assume that it is integral. We let $\mathcal{L} \to M$ be the pre–quantum line bundle associated to ω; this is a Hermitian line bundle with a compatible connection ∇ whose curvature is $-2\pi i\omega$.

We let $\nabla^{\mathcal{F}}$ denote the component of ∇ tangential to $\mathcal{F}$. After choosing an almost complex structure J compatible with ω, the tangential connection can be further decomposed into its complex linear and antilinear parts, yielding $\nabla^{\mathcal{F}} = \partial + \bar{\partial}$.

According to [11], Corollary 1.2, upon choosing the almost complex structure J, it is possible to construct a family $s_k : M \to \mathcal{L}^k$ of sections of the k-th tensor powers of $\mathcal{L}$, for k large enough, such that $W_k := s_k^{-1}(0)$ are closed, 2-calibrated submanifolds of codimension two. We call these W_k *Donaldson-type* submanifolds.

To state conditions that are required for the sequence s_k, we need to fix a metric g on M which over the leaves satisfies $g = \omega(\cdot, J\cdot)$. Further, we define a family of scaled metrics $g_k = kg$.

Definition 4.

(1) *A sequence of sections $s_k : M \to \mathcal{L}^k$ is said to be approximately holomorphic if there is a universal constant $C > 0$ such that:*

$$|\nabla^p s_k|_{g_k} < C \qquad |\nabla^p \bar{\partial} s_k|_{g_k} < Ck^{-1/2},\ p = 0, 1,$$

for k large enough.

(2) *A sequence of sections $s_k : M \to \mathcal{L}^k$ is said to be ν-transverse to zero along the foliation $\mathcal{F}$ if at any point either $|s_k|_{g_k} \geq \nu$ or $|\nabla^{\mathcal{F}} s_k|_{g_k} \geq \nu$.*

To every such an approximately–holomorphic transverse to zero sequence s_k one associates a sequence of functions $f_k : M \setminus W_k \to \mathbb{R}$ by $f_k = \log |s_k|^2$. The Lefschetz hyperplane theorem for Donaldson-type submanifolds ([4, 11]) states:

Proposition 1. *Fixing a leaf F, the function $f_k : F \setminus (W_k \cap F) \to \mathbb{R}$, which might not be Morse, has only critical points of index at least n.*

This proposition, when applied to a closed leaf, implies Theorem 3 immediately –as seen in [4, 12]– for codimension two Donaldson-type submanifolds. The higher codimension case can then be proven by an easy induction.

However, since the leaf F is, in general, open, gradient flows may behave badly. The point precisely is that leaves are very special non–compact manifolds and the Morse–theoretical proof found in [4] can be adapted. In the next section we explain the assumptions needed for Morse theory to work in an open manifold and, after that, we show that these assumptions are indeed satisfied by the leaf F after some tweaking.

3.2. **Gradient flows and the topology of open manifolds.** The study of flows which behave well on open manifolds already appears in the literature on foliation theory [6]. For the sake of completeness, we review these facts tailored to the applications we have in mind.

Let f be a Morse function on a manifold M. For any $a \in \mathbb{R}$ set $M_a = \{x \in M \mid f(x) \leq a\}$, and denote by $\mathrm{Crit}_a(f)$ the subset of critical points of f lying in $M \setminus M_a$.

Let a be a regular value for f and let $b > a$. Assume for the moment that M is compact. It is customary to study the relative topology of the pair (M_b, M_a) using minus the gradient flow of f with respect to some fixed metric g. The key point is that the following dichotomy holds: for any $x \in M_b \backslash M_a$ the trajectory of $-\nabla_g f$ starting at x either enters M_a in finite time, or converges to one of the finitely many critical points in $\mathrm{Crit}_a(f)$.

If M is no longer compact but f is proper, then of course the study of the relative topology of the pair (M_b, M_a) goes exactly as in the compact case. There might be cases –as in our setting coming from approximately holomorphic geometry– that the natural Morse functions to be used are not proper, and one needs to impose an appropriate form of the above dichotomy for trajectories of $-\nabla_g f$:

Lemma 2. *Let f be a Morse function on a manifold M and let g be a metric on M so that $\nabla_g f$ is complete. Let a be a regular value, $b > a$, and assume that the following holds:*

1. *For every compact subset $X \subset M_b$, there exist finitely many critical points $c_1, \ldots, c_{i_X}$ in $\mathrm{Crit}_a(f)$ such that the following dichotomy holds: a trajectory of $-\nabla_g f$ starting at $x \in X$ either reaches M_a in finite time, or converges to a critical point in $\{c_1\} \cup \cdots \cup \{c_{i_X}\}$.*
2. *Every $c \in Crit_a(f)$ has index $\geq j$.*

Then we have that $\pi_k(M, M_a) = 0$, for $k = 0, \ldots j-1$.

Proof. Let us start by making the following observation: if X is as in assumption (1) and the collection $\{c_1\} \cup \cdots \cup \{c_{i_X}\}$ is empty, then we claim that X is taken in finite time to M_a by the flow ϕ of $-\nabla_g f$. Indeed, for every $x \in X$ there exists a time $t_x > 0$ such that $f(\phi_{t_x}(x)) < a$; further, since for fixed t, ϕ_t is continuous, there is a small ball $B_g(x, \varepsilon_x)$ centered at x such that $\phi_{t_x}(B_g(x, \varepsilon_x)) \subset M_a$. Then, the result follows by compactness of X.

Now, let N be a compact manifold and $h\colon (N, \partial N) \to (M_b, M_a)$ be a smooth map. Let U be a relatively compact neighborhood of $h(N)$. Then assumption (1) implies that trajectories starting at points in $\bar{U}$ can only enter M_a in finite time or converge to one of the finitely many critical points $\{c_1\}, \ldots, \{c_{i_{\bar{U}}}\}$. In particular, there is a small relatively compact neighborhood V of $h(\partial N)$ such that the flow of $-\nabla_g f$ sends V into M_a: this follows if V is selected so that $f(V)$ lies below the critical values $\{f(c_1)\}, \ldots, \{f(c_{i_{\bar{U}}})\}$.

Let W be a relatively compact open subset of $M_b \backslash M_a$ so that $W \cup V$ covers $h(N)$. We shall construct h', a small perturbation of h relative to V.

Because W is relatively compact, the dichotomy in (1) holds again. Then the usual finite induction argument from [14] applies: the previous finitely many critical points are ordered by decreasing value, and a perturbation h' of h relative to V transverse to the ascending disks is constructed. This perturbation can be taken to be arbitrarily small, so we can still assume that $h'(N \backslash V) \subset W$.

If N has dimension at most $j-1$ then, by hypothesis (2), transversality to the ascending disks means empty intersection. The hypotheses of the claim at the start of the proof are satisfied and it follows that $\pi_k(M, M_a) = 0$, for $k = 0, \ldots j-1$. □

The following result –whose proof we defer to the last section– describes quantitative conditions on the gradient vector field granting the dichotomy in point (1) of Lemma 2.

Proposition 2. *Let f be a Morse function and let g be a complete metric on M. Let $a < b \in \mathbb{R}$ be given and assume that there exist real constants $D, E > 0$ such that:*

(1) *There exist open subsets $\mathcal{C}_i \subset M_b$, $i \in I$, such that for any pair $i, i' \in I$, $i \neq i'$, we have $d_g(\mathcal{C}_i, \mathcal{C}_{i'}) > D$.*
(2) *The diameter of the sets $\mathcal{C}_i$ is at most E.*
(3) *There exist real numbers $\delta_1, \delta_2 > 0$, such that*

$$\delta_2 \geq |\nabla_g(f)(p)| \geq \delta_1, \forall p \in M_b \setminus \left(\bigcup_{i \in I} \mathcal{C}_i \right).$$

Then $-\nabla_g f$ is complete and the dichotomy in point (1) of Lemma 2 for $-\nabla_g f$ holds.

3.3. **Proof of Theorem 3.** Fix some leaf $F \in \mathcal{F}$. In this section all we need to do is to check that for a well chosen Morse function and metric on F the hypotheses of Proposition 2 are satisfied for F. Our candidate is the restriction to the leaf of the function $f_k = \log |s_k|^2$, and the restriction to the leaf of any Riemmanian metric on M.

We shall prove a couple of preliminary lemmas, for which we need to recall some notation. Given a function f, defined on a manifold endowed with a codimension one foliation $(M, \mathcal{F})$, the tangential differential $d^{\mathcal{F}} f$ is the composition of the differential with the projection $T^*M \to (T\mathcal{F})^*$. The points in which $d^{\mathcal{F}} f$ vanishes are the *tangential critical points of* f, which we denote by $\Sigma^{\mathcal{F}}(f)$. Of course, $\Sigma^{\mathcal{F}}(f)$ are nothing but the critical points of the restriction of f to each leaf of $\mathcal{F}$.

Lemma 3. *For every k large enough, $W_k \subset M$ has a tubular neighborhood that contains a full regular level set of $f_k = \log |s_k|^2$ and which is also disjoint from $\Sigma^{\mathcal{F}}(f_k)$.*

Proof. It is enough to check that $h_k = ||s_k||^2$ satisfies the Lemma, since log is an increasing monotone function.

We claim that the neighborhood $U = \{x \in M \mid ||s_k(x)|| < \nu\}$ of the submanifold W_k does not intersect $\Sigma^{\mathcal{F}}(f_k)$. Assume that $p \in U$. By the ν–transversality along $\mathcal{F}$ of the section s_k, there is a unitary vector field $v \in T_p\mathcal{F}$ such that $||\nabla_v s_k(p)|| \geq \nu$. By asymptotic holomorphicity, for k large, we have that the unitary vector field $Jv \in T_p\mathcal{F}$ satisfies $||\nabla_{Jv} s_k(p) - i\nabla_v s_k(p)|| = O(k^{-1/2})$. Therefore, the map $\nabla^{\mathcal{F}} s_k(p)$ is surjective. We conclude that $p \notin \Sigma^{\mathcal{F}}(f_k)$.

□

Lemma 4. *Let F, a leaf of $\mathcal{F}$, be fixed. After a perturbation of the sequence s_k, preserving transversality to zero and approximately holomorphicity, it can be assumed that:*

(1) *the restrictions of the f_k to F are Morse functions.*
(2) *$\Sigma^{\mathcal{F}}(f_k)$ is a finite union of disjoint circles in general position with respect to $\mathcal{F}$, and the tangency points are turning points.*

Proof. According to [6], after an arbitrarily small C^r perturbation, $r \geq 2$, the set of tangential critical points $\Sigma^{\mathcal{F}}(f_k)$ can be assumed to fit into a 1-dimensional manifold, such that all but a finite number of its points $c_1, \dots, c_d$ are non-degenerate critical points for the restriction of f_k to the corresponding leaf. These turning points are the points where $\Sigma^{\mathcal{F}}(f_k)$ fails to be transverse to $\mathcal{F}$ and they are birth-death type singularities for the restriction of f_k to the leaf. The property which will be relevant for us is the following: the turning point is a quadratic critical point for the restriction of a local transverse coordinate to $\Sigma^{\mathcal{F}}(f_k)$, so in a small foliated chart a plaque not containing the turning point intersects $\Sigma^{\mathcal{F}}(f_k)$ either in the empty set or in two tangential critical points.

To prove assertion (2) in the Lemma one can as well suppose that none of the $c_1, \dots, c_d$ belong to the fixed leaf F: in [6] a explicit model for displacing the birth–death points to a different leaf is described, using a C^r perturbation. Therefore, we guarantee that $(f_k)_{|F}$ is a Morse function.

These C^r perturbations of f_k can be taken to be the result of a C^r perturbation of s_k. Indeed, let ε_k be a C^r perturbation of f_k. The function ε_k can be assumed to be identically zero away from an arbitrary small neighborhood of $\Sigma^{\mathcal{F}}(f_k)$ so, by lemma 3, the following expression is well defined:

$$\widetilde{s}_k = s_k \sqrt{1 + \varepsilon_k / f_k},$$

since f_k is bounded from below in the support of ε_k. It is clear that

$$||\widetilde{s}_k|| = f_k + \varepsilon_k.$$

The asymptotic holomorphicity of the sequence $\tilde{s}_k$ can be readily checked:

$$\nabla \widetilde{s}_k = \nabla s_k \sqrt{1 + \varepsilon_k / f_k} + s_k \frac{f_k \nabla \varepsilon_k - \varepsilon_k \nabla f_k}{2 f_k^2 \sqrt{1 + \varepsilon_k / f_k}},$$

where the second term is C^r-small and the first is C^r-close to ∇s_k. A similar computation for the higher order derivatives concludes the claim.

□

We can finally address the proof of the theorem.

Proof of Theorem 3. Fix a leaf F and assume that we have all the data needed for developing approximately–holomorphic geometry in M^{2n+1}. The metrics g_k induce complete metrics in F. Given an approximately–holomorphic sequence s_k, with corresponding Donaldson-type submanifolds W_k, an application of Lemma 4 yields a new approximately–holomorphic sequence, still denoted by s_k, that induces Morse functions $(f_k)_{|F}$ in $F \setminus W_k$.

By Lemma 3, W_k has an ε-neighborhood containing a regular level a_k. Lemmata 3 and 4 together mean that $\Sigma^{\mathcal{F}}(f_k)$ has a small tubular neighborhood of positive radius not intersecting the level a_k.

By Lemma 4, the manifold $\Sigma^{\mathcal{F}}(f_k)$ is transverse to $\mathcal{F}$ except in a finite number of turning points $c_1, \dots, c_d$. Fix a closed geodesic arc T_i through each c_i, transverse to the foliation. Let $B^{2n}(0, r) \subset \mathbb{R}^{2n}$ be the closed ball of radius r. For $r > 0$ sufficiently small, the exponential map for the leafwise metric $g_k^{\mathcal{F}}$ yields disjoint foliated charts $\phi_i : U_i \to [0,1] \times B^{2n}(0, r)$ satisfying $\phi_i(T_i) = [0,1] \times \{0\}$. Having fixed r, by taking the T_i sufficiently short – effectively shrinking U_i in the vertical direction – it can be assumed that:

$$\phi_i(\Sigma^{\mathcal{F}}(f_k) \cap U_i) \subset [0,1] \times B^{2n}(0, r/2)$$

Consider the family of closed arcs $I_j \subset \Sigma^{\mathcal{F}}(f_k)$, $j \in [1, 2, .., d]$, whose interiors are precisely $\Sigma^{\mathcal{F}}(f_k) \setminus (\cup_{i=1..d} U_i)$. For sufficiently small $0 < s < r$, the exponential map for the metric $g_k^{\mathcal{F}}$ defines disjoint charts $\psi_j : V_j \to [0,1] \times B^{2n}(0, s)$ with $\phi_j(I_j) = [0,1] \times \{0\}$. The union of the U_i and the V_j covers $\Sigma^{\mathcal{F}}(f_k)$.

The subsets $\mathcal{C}_i$, as in Proposition 2, can be defined and they come in two families:

(1) $s/2$–neighborhoods, in the metric $g_k^{\mathcal{F}}$, of the points $x \in \mathring{I}_j \cap F$, for any j,
(2) $r/2$–neighborhoods, in the metric $g_k^{\mathcal{F}}$, of the points $x \in T_i \cap F$, for any i.

By construction, the $g_k^{\mathcal{F}}$–diameter of the $\mathcal{C}_i$ is bounded above by $r/2$. Further, the $g_k^{\mathcal{F}}$–distance between any two sets $\mathcal{C}_i$ and $\mathcal{C}_{i'}$ is bounded below by s. Therefore conditions (1) and (2) in Proposition 2 hold. Condition (3) follows immediately from the fact that the union of the $\mathcal{C}_i$ is the intersection of a neighborhood of

$\Sigma^{\mathcal{F}}(f_k)$ with the leaf F.

An application of Lemma 2 shows that the relative homotopy groups $\pi_j(F, F \cap W_k)$ vanish for $j < n$ and for k large enough, since we already did the index computation in Proposition 1. This proves the theorem when W_k has codimension two. The general case follows iterating the previous construction (alternatively, we could have used approximately holomorphic sections of $\mathcal{L}^k \otimes \mathbb{C}^j$)

□

Proof of Theorem 2. Combining Theorem 3 and Corollary 6, we construct a 3-dimensional Donaldson-type submanifold $W \pitchfork \mathcal{F}$ so that the natural map

$$\mathrm{Hol}(\mathcal{F}_W) \to \mathrm{Hol}(W)$$

is an essential equivalence.

As for the pseudogroup approach, the vanishing of $\pi_0(F, F \cap W)$ implies that any total transversal T to $\mathcal{F}_W$ is a total transversal to $\mathcal{F}$, and the vanishing of $\pi_1(F, F \cap W)$ easily gives $\mathcal{H}(\mathcal{F}_W, T) = \mathcal{H}(\mathcal{F}, T)$. □

Proof of Theorem 4. It follows from combining Theorem 3 and Corollary 7. □

4. Proof of Proposition 2

Under the assumptions of Proposition 2, given any $x \in M$, we denote by γ_x the positive half of the flow line that contains x. Let γ_x^t designate the segment of the curve γ_x between x and $\phi_t(x)$.

Lemma 5. *Let $a < b \in \mathbb{R}$ be given. There is a constant R, independent of $t \in \mathbb{R}$ and $x \in M_b$, such that $d_g(\phi_t(x), x) > R$ implies $f(\phi_t(x)) < a$.*

Proof. For every curve γ we denote by $\widetilde{\gamma}$ the (possibly disconnected) curve:

$$\widetilde{\gamma} = \left\{ p \in \gamma : p \notin \bigcup_{i \in I} \mathcal{C}_i \right\},$$

that is, the segments of γ that are disjoint from the sets $\mathcal{C}_i$.

Given any curve $\gamma \subset B(x, R)$ starting at x and intersecting the boundary of $B(x, R)$ at y, we can associate to it another curve, which we denote by $\eta = \eta_\gamma$, using the following procedure:

(1) list, in order, all the sets $\mathcal{C}_i$ that γ intersects. Remove all the consecutive repetitions of the same $\mathcal{C}_i$, listing just the first one in each series of repetitions. Write $\{\mathcal{C}_{i_j}\}_{j \in [1,..k]}$ for this finite list,
(2) mark the entry and exit points e_j and f_j of γ into each $\mathcal{C}_{i_j}$. In the case of consecutive repetitions of the same $\mathcal{C}_i$, just mark the first entry point and the last exit point of the series. For simplicity, denote $f_0 = x$ and $e_{k+1} = y$,
(3) call η the piecewise smooth curve formed by connecting these marked points in the order they appear. From e_j to f_j, take the shortest geodesic between the two points. From f_j to e_{j+1}, take the shortest path not intersecting any C_i. Denote these paths by $l(e_j, f_j)$ and $l(f_j, e_{j+1})$ respectively.

Assume $R > E + D$. If $k = 0, 1$, it is immediate that

$$\frac{\mathrm{length}(\widetilde{\eta})}{\mathrm{length}(\eta)} \geq \frac{D}{E + D},$$

otherwise, the following estimate holds:

$$\frac{\text{length}(\widetilde{\eta})}{\text{length}(\eta)} = \frac{\sum_{j=0}^{k} \text{length}(l(f_j, e_{j+1}))}{\sum_{j=0}^{k} \text{length}(l(f_j, e_{j+1})) + \sum_{j=1}^{k} \text{length}(l(e_j, f_j))} \geq$$

$$\frac{\sum_{j=1}^{k-1} \text{length}(l(f_j, e_{j+1}))}{\sum_{j=1}^{k-1} \text{length}(l(f_j, e_{j+1})) + kE} \geq \frac{(k-1)D}{(k-1)D + kE} \geq \frac{D}{2(E+D)}.$$

For any radius $r > E + D$, denote by τ the time at which the curve γ_x first intersects the ball $B(x, r)$. Denote this intersection point by y. Consider the segment γ_x^τ and its associated curve $\eta = \eta_{\gamma_x^\tau}$. Use the fact that over $\widetilde{\gamma}_x^\tau$ we have a lower bound for the gradient $|\nabla_g f| > \delta_1 > 0$:

$$|f(y) - f(x)| \geq \delta_1 \,\text{length}(\widetilde{\gamma}_x^\tau) \geq \delta_1 \,\text{length}(\widetilde{\eta}) \geq \delta_1 \,\text{length}(\eta) \frac{D}{2(E+D)} \geq r \frac{\delta_1 D}{2(E+D)}$$

which implies that, if r is taken to be large enough, $|f(y) - f(x)| > b - a$, and hence $y \in M_a$.

□

Proof of Proposition 2. Let $X \subset M_b$ be a compact set. Let R be the universal constant given by Lemma 5. Denote by $X(R)$ the R-neighborhood of X, which is a relatively compact set. Lemma 5 implies that any trajectory starting at X either reaches the interior of M_a – which is equivalent to saying that it reaches M_a in finite time – or it remains in $X(R)$ for all time.

It must be shown that if a trajectory γ_x remains within $X(R)$ for all times then it must converge to a critical point. Since $X(R)$ is relatively compact and f is a Morse function, there is a finite number k of critical points in its closure. Each of those critical points $\{c_i\}_{i=1}^k$ has an arbitrarily small neighborhood V_i which corresponds to a ball in the standard Morse model around c_i. In particular, a trajectory that intersects V_i must intersect just once, either converging to c_i or escaping from V_i eventually. From this it follows that there is a time $t_0 > 0$ such that $\gamma_x(t) \notin V_i$, for all $t > t_0$ and every i. Since the gradient $|\nabla_g f| > \delta > 0$ is bounded from below in $X(R) \setminus \cup_{i=1..k} V_i$, this shows that $f(\gamma_x(t)) < a$ for t large enough, which is a contradiction.

□

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