# Envelope of Mid-Planes of Surfaces in $\mathbb{R}^3$

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**Abstract.** Given two points of a smooth convex planar curve, the mid-line is the line passing through the mid-point and the intersection of tangent lines and the Envelope of Mid-Lines is a well-known affine invariant symmetry set. In this paper we generalize this concept to convex surfaces as the Envelope of Mid-Planes (EMP), where a mid-plane is the plane containing the mid-point and the intersection line of the tangent planes at two given points.

In case of non-parallel tangent planes, we show that the EMP is the set of centers of conics with contact of order at least 3 with each surface at the given points, each conic contained in the plane generated by a pair of vectors tangent to the surfaces and orthogonal to the intersection line in the corresponding Blaschke metrics. In case of parallel tangents at non-coincident points, the EMP coincides with a known set, the Mid-Parallel Tangents Surface. The case of coincident points is interesting in itself and connects to very old topics of affine differential geometry, like Transon planes, cones of B.Su and Moutard's quadrics.

## 1. Introduction

Consider a smooth convex planar curve $\gamma$ and let $p_1, p_2$ be points of $\gamma$. The mid-line is the line that passes through the mid-point $M$ of $p_1$ and $p_2$ and the intersection of the tangent lines at $p_1$ and $p_2$. If these tangent lines are parallel, the mid-line is the line through $M$ parallel to both tangents. When $p_1 = p_2$, the mid-line is just the affine normal at the point. The envelope of these mid-lines is an important affine invariant symmetry set associated with the curve. It is important in computer graphics and has been studied by many authors ([1],[2],[3],[4],[10]).

In this paper we generalize this concept to a surface $S$ in $\mathbb{R}^3$ by considering the envelope of its mid-planes. For $p_1, p_2 \in S$, the mid-plane is the plane that passes through the mid-point $M$ of $p_1$ and $p_2$ and the intersection

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line of the tangent planes at $p_1$ and $p_2$. If these tangent planes are parallel, the mid-plane is the plane through $M$ parallel to both tangent planes. When $p_1 = p_2$, we have to consider Transon planes, a classical concept in affine differential geometry.

The *Envelope of Mid-Lines* of planar curves can be divided into 3 parts: The *Affine Envelope Symmetry Set* (AESS), corresponding to pairs $(p_1, p_2)$ with non-parallel tangent planes, the *Mid-Points Parallel Tangent Locus* (MPTL), corresponding to pairs $(p_1, p_2)$, $p_1 \neq p_2$, with parallel tangent planes, and the *Affine Evolute*, corresponding to coincident points. The *Envelope of Mid-Planes* (EMP) is also naturally divided into three subsets: The EMP1, corresponding to pairs $(p_1, p_2)$ with non-parallel tangent planes, the EMP2 corresponding to pairs $(p_1, p_2)$, $p_1 \neq p_2$, with parallel tangent planes, and the EMP3 corresponding to the limit case of coincident points.

For curves, the AESS is very well studied and coincides with the locus of center of conics having contact of order $\geq 3$ with the curve at $p_1$ and $p_2$. Moreover, conditions for the AESS to be a regular at a given point are known ([2],[4]). In this article we prove corresponding results for the set EMP1. The interesting fact here is that all relevant points of this construction belong to one distinguished plane, which is generated by tangent vectors orthogonal in the Blaschke metric to the intersection line of the tangent planes. We prove that a point of the EMP1 is the center of a conic contained in this plane having contact of order $\geq 3$ with the surface. Moreover, under certain generic conditions, the EMP1 is a regular surface. In the case of planar curves, another interesting property is the following: If the AESS is contained in a straight line $l$, then the curve itself is invariant under an affine reflection with axis $l$. We give an example that shows that the correponding property does not remain true for the EMP1.

The *Mid-Parallel Tangent Surface* (MPTS) of a surface is the locus of mid-points of pairs $(p_1, p_2)$ with parallel tangents ([10]). It turns out that the MPTS coincides with the set EMP2. As in the curve case, the EMP1 and MPTS meet at center of conics with contact of order $\geq 3$ and parallel tangents, where both sets are expected to be singular. Based on [10], we give conditions under which, at such a point, the MPTS is equivalent to a cuspidal edge.

The Affine Evolute of a planar curve is the limit of the Envelope of Mid-Lines when $p_1 = p_0$. We study in this paper the corresponding limit set for surfaces in $\mathbb{R}^3$. We verify that if we fix a tangent vector $T$ and make $p_1 \to p_0$ in this direction, the system defining the Envelope of Mid-Planes converges to a system of 4 linearly independent equations. The first equation of this system defines the Transon plane of $T$ at $p_0$, the first two define the line of the cone of B.Su of $T$ at $p_0$, and the first three define the center of the Moutard quadric of $T$ at $p_0$. The notions of Transon Plane, cone of B.Su and Moutard's quadric are very old ([9]), but there are some modern references ([5],[8]). We verify that the tangent vectors $T$ that leads to some solution of this system of 4 equations satisfy a polynomial equation of degree 6, thus they are at most 6. The set of solutions of this system is an affine invariant set that, up to our knowledge, has not yet been considered. We call it the *Affine Mid-Planes Evolute* and give conditions for its regularity.

The paper is organized as follows: In section 2 we review some basic facts of affine differential geometry of surfaces in $\mathbb{R}^3$. In section 3 we study the EMP1, giving necessary and sufficient conditions for a pair $(p_1, p_2)$ to contribute to this set. We give also conditions for the EMP1 to be a regular surface. In section 4 we describe some properties of the MPTS. In section 5, we describe the limit equations for the EMP and relate them with the classical concepts of Transon plane, cone of B.Su and Moutard's quadric. Then we give conditions for the Affine Mid-Planes Evolute to be a regular surface.

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## 2. Affine differential geometry of surfaces in $\mathbb{R}^3$

In this section we review some basic concepts of affine differential geometry of surfaces in $\mathbb{R}^3$ (for details, see [7]). Denote by $D$ the canonical connection and by $\Omega$ the standard volume form in $\mathbb{R}^3$. Let $S \subset \mathbb{R}^3$ be a surface and denote by $\mathfrak{X}(S)$ the tangent bundle of $S$. Given a transversal vector field $\zeta$, write the Gauss equation

$$D_X Y = \nabla_X Y + h(X, Y)\zeta,$$

$X, Y \in \mathfrak{X}(S)$, where $h$ is a symmetric bilinear form and $\nabla$ is a torsion free connection in $S$. We shall assume that $h$ is non-degenerate, which is independent of the choice of $\zeta$. The volume form $\Omega$ induces a volume form in $S$ by the relation

$$\theta(X_1, X_2) = \Omega(X_1, X_2, \zeta).$$

The metric $h$ also defines a volume form in $S$: Given $X_i \in \mathfrak{X}(S)$, $1 \leq i \leq 2$, denote by $H := (h_{ij})$ the $2 \times 2$ matrix whose entries are $h_{ij} := h(X_i, X_j)$ and define

$$\theta_h(X_1, X_2) := |\det H|^{\frac{1}{2}}.$$

Next theorem is fundamental in affine differential geometry ([7], ch.II):

**Theorem 2.1.** *There exists, up to signal, a unique transversal vector field $\zeta$ such that $\nabla\theta = 0$ and $\theta = \theta_h$. The vector field $\zeta$ is called the affine normal vector field and the corresponding metric $h$ the Blaschke metric of the surface.*

The derivative of the Blaschke metric

$$C(X_1, X_2, X_3) = \nabla_{X_1}(h)(X_2, X_3) \tag{1}$$

is called the *cubic form* of the surface. It is symmetric in $(X_1, X_2, X_3)$ and satisfies the apolarity condition

$$tr_h C(X, \cdot, \cdot) = 0, \tag{2}$$

for each $X \in \mathfrak{X}(S)$, where $tr_h$ denotes the trace of the bilinear form with respect to the metric $h$ ([7], ch.II).

Let $\mathbb{R}_3$ denote the dual vector space of $\mathbb{R}^3$. For $x \in S$, let $\nu_x$ be the linear functional in $\mathbb{R}_3$ such that

$$\nu_x(\zeta) = 1 \quad \text{and} \quad \nu_x(X) = 0 \ \forall \ X \in T_xS.$$

The differentiable map $\nu : S \longrightarrow \mathbb{R}_3 - \{0\}$ is called the *conormal map*. It satisfies the following property ([7], ch.II):

**Proposition 2.2.** *Let $S \subset \mathbb{R}^3$ be a non-degenerate surface and $\nu$ the conormal map. Then*

$$D_Y\nu(\zeta) = 0 \ \ \textit{and} \ \ D_Y\nu(X) = -h(Y, X), \quad \forall \ X, Y \in \mathfrak{X}(S).$$

**Corollary 2.3.** *If $X \in \mathfrak{X}(\mathbb{R}^3)$ is any vector field in $\mathbb{R}^3$, then*

$$D_Y\nu(X) = -h(Y, X^T), \quad Y \in \mathfrak{X}(S),$$

*where $X = X^T + \lambda\zeta, \lambda \in \mathbb{R}$ and $X^T$ is the tangent component of $X$.*

*Proof.* We have

$$D_Y\nu(X) = D_Y\nu(X^T + \lambda\zeta) = D_Y\nu(X^T) + \lambda D_Y\nu(\zeta) = -h(Y, X^T),$$

thus proving the corollary. □

## 3. Envelope of Mid-Planes- Non-Parallel Tangent Planes

Let $S$ be a non-degenerate convex surface. Take points $p_1, p_2 \in S$ and let $S_1 \subset S$ and $S_2 \subset S$ be open subsets around $p_1$ and $p_2$, respectively. Denote $h_1$ and $h_2$ the Blaschke metrics of $S_1$ and $S_2$, respectively. In this section we shall assume that the tangent planes at $p_1$ and $p_2$ are non-parallel.

### 3.1. Basic definitions

Denote by $M(p_1, P_2)$ the mid-point and by $C(p_1, p_2)$ the mid-chord of $p_1$ and $p_2$, i.e.,

$$M(p_1, p_2) = \frac{p_1 + p_2}{2}, \quad C(p_1, p_2) = \frac{p_1 - p_2}{2}.$$

The mid-plane of $(p_1, p_2)$ is the plane that contains $M(p_1, p_2)$ and the line $r(p_1, p_2)$ of intersection of the tangent planes at $p_1$ and $p_2$.

Let $F : S_1 \times S_2 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by

$$F(p_1, p_2, X) = (\nu_2(C)\nu_1 + \nu_1(C)\nu_2)(X - M). \tag{3}$$

**Lemma 3.1.** *The equation of the mid-plane is given by*

$$F(p_1, p_2, X) = 0.$$

*Proof.* Take $R \in r$. Since $R - M = R - p_1 + C = R - p_2 - C$, we obtain

$$F(p_1, p_2, R) = \nu_2(C)\nu_1(R - p_1 + C) + \nu_1(C)\nu_2(R - p_2 - C) = 0,$$

thus proving the lemma. □

Denote

$$\overline{EPM} = \left\{(p_1, p_2, X) \in S_1 \times S_2 \times \mathbb{R}^3 |\ F = F_{p_1} = F_{p_2} = 0\right\}$$

and

$$\overline{EPM1} = \left\{(p_1, p_2, X) \in \overline{EPM} |\ T_{p_1}S_1 \not\parallel T_{p_2}S_2\right\}.$$

Denoting by $\pi$ the projection in the third coordinate, the envelope EMP of the family of mid-planes is the set $\pi(\overline{EPM})$. In this section we shall study the properties of the set $EPM1 = \pi(\overline{EPM1})$.

Consider a smooth function

$$\begin{array}{cccc} Z: & S_1 \times S_2 & \longrightarrow & \mathbb{R}^3 \\ & (p_1, p_2) & \longmapsto & Z(p_1, p_2) \end{array}$$

such that $Z(p_1, p_2)$ is parallel to $r(p_1, p_2)$. Consider also, for $i = 1, 2$, smooth functions

$$\begin{array}{cccc} Y_i: & S_1 \times S_2 & \longrightarrow & \mathbb{R}^3 \\ & (p_1, p_2) & \longmapsto & Y_i(p_1, p_2) \end{array}$$

such that $Y_i(p_1, p_2)$ is tangent to $S_i$ and $h_i(Y_i, Z) = 0$. We want to find $X$ satisfying $F = F_{p_1} = F_{p_2} = 0$, for some $p_1 \in S_1$, $p_2 \in S_2$. Since $Y_i$ and $Z$ are $h_i$-orthogonals, $\{Y_i, Z\}$ is a basis of $T_{p_i}S_i$, $i = 1, 2$. Thus we have to find $X$ in the following system:

$$\begin{cases} F(p_1, p_2, X) = 0 \\ F_{p_1}(p_1, p_2, X)(Y_1) = 0 \\ F_{p_2}(p_1, p_2, X)(Y_2) = 0 \\ F_{p_1}(p_1, p_2, X)(Z) = 0 \\ F_{p_2}(p_1, p_2, X)(Z) = 0. \end{cases} \tag{4}$$

The notation $F_{p_i}(p_1, p_2, X)(W)$ corresponds to the partial derivative of $F$ with respect $p_i$ in the direction $W \in T_{p_i}S_i$, thus keeping $p_j$, $j \neq i$ and $X$ fixed.

### 3.2. Solutions of the system (4)

We begin with the following simple lemma:

**Lemma 3.2.** *We have that*

$$D_{Y_1}\nu_1 = a\nu_1 + b\nu_2 \quad \textit{and} \quad D_{Y_2}\nu_2 = \bar{a}\nu_1 + \bar{b}\nu_2,$$

*where $a, b, \bar{a}, \bar{b}$ are given by*

$$a = -\frac{h_1(Y_1, X_2)}{\nu_1(X_2)}, \quad b = -\frac{h_1(Y_1, X_1)}{\nu_2(X_1)}, \quad \bar{a} = -\frac{h_2(Y_2, X_2)}{\nu_1(X_2)}, \quad \bar{b} = -\frac{h_2(Y_2, X_1)}{\nu_2(X_1)},$$

*for any $X_1 \in T_{p_1}S_1$, $X_2 \in T_{p_2}S_2$.*

*Proof.* Take a basis $\{\nu_1, \nu_2, \zeta\}$ of the dual space $\mathbb{R}_3$. Thus we can write the linear functional $D_{Y_1}\nu_1$ as a linear combination of the basis vector, i.e., $D_{Y_1}\nu_1 = a\nu_1 + b\nu_2 + c\zeta$. Since $D_{Y_1}\nu_1(Z) = -h_1(Y_1, Z) = 0$ we obtain $c = 0$ and so $D_{Y_1}\nu_1 = a\nu_1 + b\nu_2$. In an analogous way we show that $D_{Y_2}\nu_2 = \bar{a}\nu_1 + \bar{b}\nu_2$. Applying $D_{Y_1}\nu_1$ to any tangent vector field $X_1$ on $S_1$ we get $D_{Y_1}\nu_1(X_1) = b\nu_2(X_1)$, thus proving the formula for $b$. The other formulas are proved similarly. $\square$

**Proposition 3.3.** *The first three equations of the system* (4) *admit a solution if and only if*

$$\nu_1(C) = -\lambda\nu_2(C), \tag{5}$$

*where*

$$\lambda = \left( \frac{\nu_1^2(Y_2)}{\nu_2^2(Y_1)} \frac{h_1(Y_1,Y_1)}{h_2(Y_2,Y_2)} \right)^{\frac{1}{3}}. \tag{6}$$

*Proof.* Since $\nu_1(Y_1) = \nu_2(Y_2) = 0$, it follows that the derivative $F_{p_1}(Y_1)$ is given by

$$D_{Y_1}\nu_1(C)\nu_2(X-M)+\nu_2(C)D_{Y_1}\nu_1(X-M)+\frac{1}{2}\nu_2(Y_1)\nu_1(X-M)-\frac{1}{2}\nu_1(C)\nu_2(Y_1),$$

By lemma 3.2 we obtain

$$F_{p_1}(Y_1) = \left(2b\nu_2(C)\nu_2 + \frac{1}{2}\nu_2(Y_1)\nu_1\right)(X-M) - \frac{1}{2}\nu_1(C)\nu_2(Y_1) + aF.$$

Similarly

$$F_{p_2}(Y_2) = \left(2\bar{a}\nu_1(C)\nu_1 - \frac{1}{2}\nu_1(Y_2)\nu_2\right)(X-M) - \frac{1}{2}\nu_2(C)\nu_1(Y_2) + \bar{b}F.$$

Using that $F = 0$, the equations $F_{p_1}(Y_1) = 0$ and $F_{p_2}(Y_2) = 0$ can be simplified to

$$\left(-\frac{\nu_2(Y_1)\nu_1(C)\nu_2}{\nu_2(C)} + 2b\nu_2(C)\nu_2\right)(X-M) = \frac{1}{2}\nu_1(C)\nu_2(Y_1), \tag{7}$$

$$\left(-\frac{2\bar{a}\nu_1^2(C)\nu_2}{\nu_2(C)} - \nu_1(Y_2)\nu_2\right)(X-M) = \frac{1}{2}\nu_2(C)\nu_1(Y_2).$$

These equations, after some simple calculations, leads to

$$b\nu_2^3(C)\nu_1(Y_2) = -\bar{a}\nu_1^3(C)\nu_2(Y_1),$$

which, together with lemma 3.2, proves the proposition. □

From equation (5), we can write

$$C = A\left(Y_1 - \frac{\lambda\nu_2(Y_1)}{\nu_1(Y_2)}Y_2\right) + \alpha Z, \tag{8}$$

for some $A \in \mathbb{R}$, $\alpha \in \mathbb{R}$. Next lemma is a consequence of the first three equations of system (4):

**Lemma 3.4.** *We have that*

$$X - M = B\left(Y_1 + \frac{\lambda\nu_2(Y_1)}{\nu_1(Y_2)}Y_2\right) + \beta Z, \tag{9}$$

*where* $\beta \in \mathbb{R}$ *and*

$$B = -\frac{\lambda A}{2(\lambda + 2Ab)}. \tag{10}$$

*Proof.* It follows from $F=0$ and equation (5) that $\nu_1(X-M)=\lambda\nu_2(X-M)$. Then equation (9) holds, for some $B\in\mathbb{R}$. From equation (7) we have

$$\nu_2(X-M)=\frac{-\lambda\nu_2(C)\nu_2(Y_1)}{(\lambda\nu_2(Y_1)+4b\nu_2(C))}.$$

We conclude that

$$B=-\frac{\lambda\nu_2(C)}{(\lambda\nu_2(Y_1)+4b\nu_2(C))}=-\frac{\lambda A}{(\lambda+4Ab)},$$

thus proving the lemma. □

Next theorem is the main result of the section and says that the geometry of the envelope of mid-planes occurs in the plane generated by $Y_1$ and $Y_2$ (see figure 1).

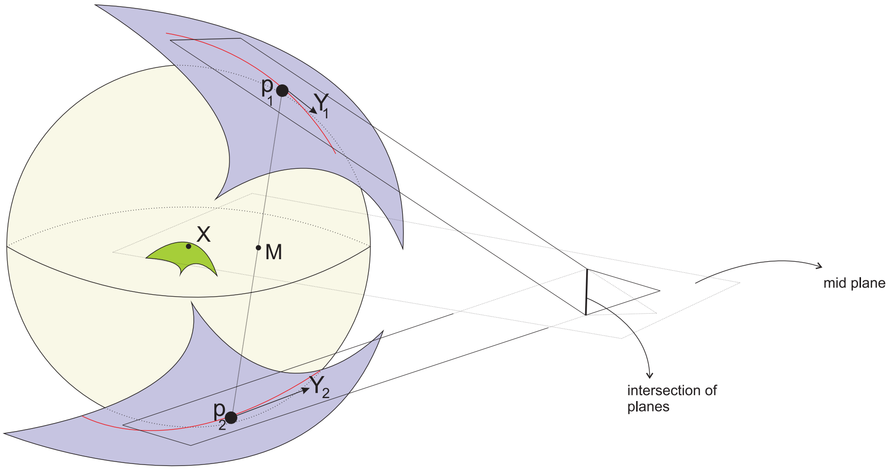


FIGURE 1. The geometry of the EMP1.

**Theorem 3.5.** *The system* (4) *admits a solution if and only if* $C, Y_1$, *and* $Y_2$ *are co-planar and equation* (5) *holds. Moreover, the solution of the system is given by*

$$X-M=B\left(Y_1+\frac{\lambda\nu_2(Y_1)}{\nu_1(Y_2)}Y_2\right), \tag{11}$$

*where* $\lambda$ *and* $B$ *are given by equations* (6) *and* (10), *respectively.*

*Proof.* We must show that $\alpha=\beta=0$ at equations (8) and (9), respectively. For this, we shall consider the last two equations of the system (4). The derivative $F_{p_1}(Z)$ is given by

$$F_{p_1}(Z)=D_Z\nu_1(C)\nu_2(X-M)+\nu_2(C)D_Z\nu_1(X-M).$$

Thus, from lemma 2.3,

$$F_{p_1}(Z)=-h_1(Z,C^T)\nu_2(X-M)-\nu_2(C)h_1(Z,(X-M)^T), \tag{12}$$

where $U^T$ denotes the projection in $T_{p_1}S_1$ along the direction of the affine normal of $S_1$ at $p_1$. We can write

$$C^T = A\left(Y_1 - \frac{\lambda\nu_2(Y_1)}{\nu_1(Y_2)}Y_2^T\right) + \alpha Z$$

and

$$(X-M)^T = B\left(Y_1 + \frac{\lambda\nu_2(Y_1)}{\nu_1(Y_2)}Y_2^T\right) + \beta Z.$$

Substituting these equations in equation (12) we obtain

$$F_{p_1}(Z) = h_1(Z,Z)\nu_2(Y_1)(\alpha B + \beta A).$$

Similarly we obtain

$$F_{p_2}Z = \lambda h_2(Z,Z)\nu_2(Y_1)(-\alpha B + \beta A).$$

Since $S_1$ and $S_2$ are convex, $h_i(Z,Z) > 0$, $i = 1,2$. Moreover, non-parallel tangent planes imply that $A$, $B$, $\nu_2(Y_1)$ and $\lambda$ are non-zero. Thus equations $F_{p_1}Z = F_{p_2}Z = 0$ are equivalent to $\alpha B + \beta A = -\alpha B + \beta A = 0$, which implies that $\alpha = \beta = 0$. □

### 3.3. Conics with $3+3$ contact with the surface

Given two non-degenerate convex surface $S_1$ and $S_2$, consider conics that makes contact of order $\geq 3$ with $S_i$ at points $p_i$, $i = 1,2$, in directions $Y_i$ which are $h_i$-orthogonals to the intersection line $r$ of $T_{p_1}S_1$ and $T_{p_2}S_2$. We shall prove in this section that the set of centers of these $3+3$ conics coincides with the set EMP1.

Along the paper, we shall denote by $O(n)$ terms of degree $\geq n$ in $(x,y)$. We begin with the following lemma:

**Lemma 3.6.** *Let $S$ be the graph of a function $f$ given by*

$$f = f_0 + f_{1,0}x + f_{0,1}y + f_{2,0}x^2 + f_{1,1}xy + f_{0,2}y^2 + O(3).$$

*Then, at $(x,y) = (0,0)$, the tangent vectors $(1,0,f_{1,0})$ and $(0,1,f_{0,1})$ are orthogonal in the Blaschke metric if and only if $f_{1,1} = 0$.*

*Proof.* Let $\psi(x,y) = (x,y,f(x,y))$ be a parameterization of $S$ and denote by $h$ the Blaschke metric of $S$. Then $h(\psi_x,\psi_y)$ is a positive multiple of $[\psi_x,\psi_y,\psi_{xy}]$. Hence, at $(0,0)$, $h(\psi_x,\psi_y) = 0$ if and only if $f_{1,1} = 0$. □

Along this section, we shall assume that $S_i$ is the graph of a function $f_i(x,y)$, $i = 1,2$, and consider the normal vectors

$$N_i = (1,0,(f_i)_x) \times (1,0,(f_i)_y)(p_i). \quad (13)$$

to $S_i$. Let $F : S_1 \times S_2 \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(p_1,p_2,X) = N_1 \cdot C\ N_2 \cdot (X-M) + N_2 \cdot C\ N_1 \cdot (X-M), \quad (14)$$

where $N_i$ is given by equation (13), $M$ is the mid-point of $(p_1,p_2)$ and $C$ is the mid-chord of $(p_1,p_2)$. By lemma 3.1, $F = 0$ is the equation of the mid-plane of $(p_1,p_2)$.

**Lemma 3.7.** *Assume that the pair $(p_1, p_2)$ generates a point of EMP1. Then by an affine change of coordinates, we may assume that $p_1 = (0,0,1)$, $p_2 = (0,0,-1)$ and $S_1$ and $S_2$ are graphs of*

$$f_1(u_1, v_1) = 1 + \frac{p}{c}u_1 - \frac{(p^2+c)}{2c^2}u_1^2 + f_{0,2}v_1^2 + O(3) \tag{15}$$

*and*

$$f_2(u_2, v_2) = -1 - \frac{p}{c}u_2 + \frac{(p^2+c)}{2c^2}u_2^2 + g_{0,2}v_2^2 + O(3), \tag{16}$$

*where $(p,0,0)$ is the corresponding point in EMP1. As a consequence, $(p,0,0)$ is the center of a conic making contact of order $\geq 3$ with $S_i$ at $p_i$ in the direction $Y_i$ $h_i$-orthogonal to the line $r$.*

*Proof.* Consider $p_1 \in S_1$ and $p_2 \in S_2$ with non-parallel tangent planes. By an adequate affine change of variables, we may assume that $p_1 = (0,0,1)$, $p_2 = (0,0,-1)$ and the mid-plane of $(p_1, p_2)$ is $z = 0$. Since, by theorem 3.5, $Y_1, Y_2$ and $(0,0,1)$ are co-planar, we may also assume that $Y_1$ and $Y_2$ are in the $xz$-plane. We may also assume that the line $r$ of intersection of the tangent planes $T_{p_1}S_1$ and $T_{p_2}S_2$ is the $y$-axis. By lemma 3.6, these conditions implies that the coefficients of $u_1v_1$ and $u_2v_2$ are zero. Since the tangent plane at $p_1$ contains $r$, $S_1$ is the graph of a function $f_1$ of the form

$$f_1(u_1, v_1) = 1 + \frac{p}{c}u_1 - \frac{(p^2+c)}{2c^2}u_1^2 + f_{0,2}v_1^2 + O(3),$$

for some $c \in \mathbb{R}$, $p \in \mathbb{R}$. The tangent plane to $S_2$ at $(0,0,-1)$ is the reflection of the tangent plane to $S_1$ at $(0,0,1)$, so $S_2$ is the graph of a function $f_2$ of the form

$$f_2(u_2, v_2) = -1 - \frac{p}{c}u_2 + \delta\frac{(p^2+c)}{2c^2}u_2^2 + g_{0,2}v_2^2 + O(3),$$

for some $\delta \in \mathbb{R}$. Using this model and $F$ given by equation (14), the system $F = F_{u_1} = F_{v_1} = F_{u_2} = F_{v_2} = 0$ at the origin becomes

$$\begin{cases} -2z = 0 \\ x + pz - p = 0 \\ 2f_{0,2}y = 0 \\ (p^2 - p^2\delta - \delta)x + pz + p = 0 \\ 2g_{0,2}y = 0 \end{cases}. \tag{17}$$

Since, by hypothesis, this system admit a solution, $(p,0,0)$ satisfies the above system. Thus we conclude that $\delta = 1$. It is not difficult to verify now that there exists a conic centered at $(p,0,0)$ contained in the plane $xz$ and making contact of order $\geq 3$ with $S_1$ at $p_1$ and $S_2$ at $p_2$, thus proving the lemma. □

**Lemma 3.8.** *Consider a conic that makes contact of order $\geq 3$ with $S_i$ at points $p_i$, $i = 1,2$, in directions $Y_i$ which are $h_i$-orthogonals to the intersection line $r$ of $T_{p_1}S_1$ and $T_{p_2}S_2$. Then, by an affine change of coordinates, we may assume that $p_1 = (0,0,1)$, $p_2 = (0,0,-1)$ and $S_1$ and $S_2$ are graphs of functions $f_1$ and $f_2$ given by equations (15) and (16), where $(p,0,0)$ is the center of the conic and $c \in \mathbb{R}$. As a consequence, $(p,0,0)$ belongs to EMP1.*

*Proof.* Consider $p_1 \in S_1$ and $p_2 \in S_2$ with non-parallel tangent planes. By an adequate affine change of variables, we may assume that $p_1 = (0,0,1)$, $p_2 = (0,0,-1)$ and the mid-plane is $z = 0$. We may also assume that the conic is contained in the $xz$-plane and that the line $r$ of intersection of the tangent planes $T_{p_1}S_1$ and $T_{p_2}S_2$ is the $y$-axis. By lemma 3.6, these conditions imply that the coefficients of $u_1v_1$ and $u_2v_2$ are zero.

Now assuming that the center of the conic is the point $(p,0,0)$, $S_1$ and $S_2$ are graphs of functions given by equations (15) and (16), for some $c \in \mathbb{R}$. As a consequence, $(p,0,0)$ satisfies the system (17), which implies that $(p,0,0)$ belongs to EMP1. □

From the above two lemmas we can conclude the main result of this section.

**Proposition 3.9.** *The set of centers of conics which make contact of order* $\geq 3$ *at points* $p_i \in S_i$ *at directions* $Y_i$ *which are* $h_i$*-orthogonals to the intersection* $r$ *of* $T_{p_i}S_i$, $i = 1,2$, *coincides with the set EMP1.*

### 3.4. Regularity of the EPM1

In this section, we shall study the regularity of the set $EPM1$. Let $F$ be given by equation (14) and consider the map

$$\begin{array}{llll} H: & \mathbb{R}^4 \times \mathbb{R}^3 & \longrightarrow & \mathbb{R}^5 \\ & (u_1,v_1,u_2,v_2,X) & \longmapsto & H(u_1,v_1,u_2,v_2,X) = (F, F_{u_1}, F_{v_1}, F_{u_2}, F_{v_2}) \,. \end{array}$$

Then the set $EPM1$ is the projection in $\mathbb{R}^3$ of the set $H = 0$. If $0 \in \mathbb{R}^5$ is a regular value of $H$, then $H^{-1}(0)$ is a 2-dimensional submanifold of $\mathbb{R}^7$. We want to find conditions under which $\pi_2(H^{-1}(0))$ becomes smooth, where $\pi_2(u_1,v_1,u_2,v_2,X) = X$.

The jacobian matrix of $H$ is

$$JH = \left(\begin{array}{cccc|ccc} F_{u_1} & F_{v_1} & F_{u_2} & F_{v_2} & F_x & F_y & F_z \\ \hline F_{u_1u_1} & F_{u_1v_1} & F_{u_1u_2} & F_{u_1v_2} & F_{u_1x} & F_{u_1y} & F_{u_1z} \\ F_{v_1u_1} & F_{v_1v_1} & F_{v_1u_2} & F_{v_1v_2} & F_{v_1x} & F_{v_1y} & F_{v_1z} \\ F_{u_2u_1} & F_{u_2v_1} & F_{u_2u_2} & F_{u_2v_2} & F_{u_2x} & F_{u_2y} & F_{u_2z} \\ F_{v_2u_1} & F_{v_2v_1} & F_{v_2u_2} & F_{v_2v_2} & F_{v_2x} & F_{v_2y} & F_{v_2z} \end{array}\right).$$

Denote by $JH1$ the matrix of second derivatives of $F$ with respect to the parameters $u_1, v_1, u_2, v_2$, which is the $4 \times 4$ matrix consisting of the elements $JH(i,j)$, $2 \leq i \leq 5$, $1 \leq j \leq 4$. Denote $\det(JH1(u_1,v_1,u_2,v_2,X))$ by $\Delta(u_1,v_1,u_2,v_2)$.

**Theorem 3.10.** *If* $\Delta \neq 0$, *then the* $EPM1$ *is smooth at the point* $X$.

*Proof.* Since the mid-plane is non-degenerate, the equalities $F_x = F_y = F_z = 0$ cannot occur simultaneously. Moreover, at points of the envelope, $F_{u_1} = F_{v_1} = F_{u_2} = F_{v_2} = 0$. Thus the hypothesis implies that $JH$ has rank 5 and so $H^{-1}(0)$ is a regular surface in $\mathbb{R}^7$. Moreover, the hypothesis $\Delta \neq 0$ implies that the differential of $\pi$ restricted to $H^{-1}(0)$ is an isomorphism. We conclude that the $EPM1$ is smooth at this point. □

One may ask whether or not the hypothesis $\Delta \neq 0$ occurs frequently. By lemma 3.7, we may assume that $S_1$ and $S_2$ are graphs of functions $f_1$ and $f_2$ given by equations (15) and (16). Straightforward calculations show that the jacobian matrix $JH1$ at point $(0,0,0,0,p,0,0)$ is given by

$$\begin{pmatrix} 3p^2+3p^4-6pf_{3,0} & -2pf_{2,1} & 0 & 0 \\ -2pf_{2,1} & -2f_{0,2}p^2-2f_{2,1}p & 0 & (f_{0,2}+g_{0,2})(p^2+1) \\ 0 & 0 & -3p^4-3p^2-6pg_{3,0} & -2pg_{2,1} \\ 0 & (f_{0,2}+g_{0,2})(p^2+1) & -2pg_{2,1} & -2g_{0,2}p^2-2pg_{1,2} \end{pmatrix}.$$

From this matrix it is easy to see that the hypothesis $\Delta \neq 0$ is in fact generic.

### 3.5. An example

In [1], it is proved that if the AESS of a pair of planar curves is contained in a line, then there exists an affine reflection taking one curve into the other. This fact is not true for the EPM1 of a pair of surfaces as the following example shows us.

Consider $\gamma_1(t) = (t, 0, f(t))$ a smooth convex curve and let $\gamma_2(t) = (t - \lambda f(t), 0, -f(t))$, $\lambda \in \mathbb{R}$, be obtained from $\gamma_1$ by an affine reflection. Let $S_1$ and $S_2$ be rotational surfaces obtained by rotating $\gamma_1$ and $\gamma_2$ around the $z$-axis. $S_1$ and $S_2$ can be parameterized by

$$\phi_1(t,\theta) = (t\cos(\theta), t\sin(\theta), f(t))$$

and

$$\phi_2(t,\theta) = ((t-\lambda f(t))\cos(\theta), (t-\lambda f(t))\sin(\theta), -f(t)).$$

The intersection of the tangent planes at $\phi_1(t,\theta)$ and $\phi_2(t,\theta)$ has direction $Z = (\sin(\theta), -\cos(\theta), 0)$.

Observe that the vectors $Y_1 = (\phi_1)_t$ and $Y_2 = (\phi_2)_t$ are orthogonal to $Z$ in the Blaschke metric. This implies that the EMP of this pair of surfaces is contained in the plane $z = 0$. But it is clear that $S_2$ is not an affine reflection of $S_1$.

## 4. Parallel Tangent Planes

In this section, we shall consider the set EMP2 of points of the EMP obtained from pairs $(p_1, p_2)$, $p_1 \neq p_2$, with parallel tangent planes. In this case, the mid-plane of $(p_1, p_2)$ is the plane through $M$ parallel to the tangent planes at $p_1$ and $p_2$. The *Mid-Parallel Tangents Surface* (MPTS) is the set of mid-points $M$ of all such pairs $(p_1, p_2) \in S_1 \times S_2$. The following proposition is proved in [10].

**Proposition 4.1.** *The tangent plane at any point of the $MPTS$ is parallel to the tangent planes to $T_{p_1}S_1$ and $T_{p_2}S_2$.*

It follows from this proposition that the MPTS is the envelope of the mid-planes with parallel tangents. In other words, the set EMP2 coincides with the MPTS.

In the case of planar curves, one relevant property is that the MPTL and the AESS are both singular at the center of a conic with contact of order $\geq 3$ at $p_1$ and $p_2$, and, under certain conditions, they are ordinary cusps ([4]). Similarly, we expect singularities at points of the EMP2 that are limit of points of the EMP1. In what follows we shall give some conditions under which, at such points, the MPTS is equivalent to a cuspidal edge. Assume that $S_1$ and $S_2$ are graphs of function $f$ and $g$ of the form

$$f(u_1, v_1) = 1 + \sum_{i=0}^{2} f_{2-i,i} u_1^{2-i} v_1^i + \sum_{i=0}^{3} f_{3-i,i} u_1^{3-i} v_1^i + O(4) \tag{18}$$

and

$$g(u_2, v_2) = -1 + \sum_{i=0}^{2} g_{2-i,i} u_2^{2-i} v_2^i + \sum_{i=0}^{3} g_{3-i,i} u_2^{3-i} v_2^i + O(4). \tag{19}$$

The following two propositions are proved in [10].

**Proposition 4.2.** *The MPTS is smooth at* $(0,0,0)$ *if*

$$\det\left(Hess(f) + Hess(g)\right)(0,0) \neq 0.$$

**Proposition 4.3.** *Assume that* $f_{1,1} = 0$*,* $f_{0,2} + g_{0,2} \neq 0$*,* $\zeta = 0$ *and* $\eta \neq 0$*, where*

$$\zeta = g_{1,1}^2 - 4(f_{2,0} + g_{2,0})(f_{0,2} - g_{0,2})$$

*and*

$$\eta = f_{0,3} - g_{0,3} - \left(\frac{f_{1,2} - g_{1,2}}{2(f_{0,2} + g_{0,2})}\right) g_{1,1} + $$
$$+ \left(\frac{f_{2,1} - g_{2,1}}{4(f_{0,2} + g_{0,2})^2}\right) g_{1,1}^2 - \left(\frac{f_{3,0} - g_{3,0}}{8(f_{0,2} + g_{0,2})^3}\right) g_{1,1}^3.$$

*Then the MPTS is locally equivalent to a cuspidal edge at the point* $(0,0)$*.*

We can prove the following corollary:

**Corollary 4.4.** *Assume that* $S_1$ *and* $S_2$ *are given by equations* (18) *and* (19) *and that* $(0,0,0)$ *is a limit point of the set EMP1. If* $f_{0,3} - g_{0,3} \neq 0$ *and* $f_{0,2} + g_{0,2} \neq 0$*, then the MPTS is locally equivalent to a cuspidal edge at* $(0,0,0)$*.*

*Proof.* By a rotation, assume that $f_{1,1} = 0$. Since $(0,0,0)$ is a limit point of the EMP1, there are directions $Y_1 \in T_{p_1}S_1$, $Y_2 \in T_{p_2}S_2$, orthogonal to a direction $Z$ in the corresponding Blaschke metric, such that $\{Y_1, Y_2, C\}$ are coplanar. We may assume that $Z$ is parallel to $(0,1,0)$. Since $f_{1,1} = 0$, it follows from lemma 3.6 that $Y_1$ is parallel to $(1,0,0)$. Now the co-planarity of $Y_1, Y_2$ and $C$ implies that $Y_2$ is also parallel to $(1,0,0)$. Using again lemma 3.6 we conclude that $g_{1,1} = 0$.

Since $(0,0,0)$ is the center of a $3+3$ conic, it follows that $f_{2,0}+g_{2,0}=0$. By proposition 4.3, the MPTS is locally equivalent to a cuspidal edge at this point. □

## 5. Envelope of Mid-Planes at coincident points

In this section we shall describe the limit of the EMP when we make $p_1$ tend to $p_2$. In the case of planar curves, the corresponding subset of the envelope of mid-lines coincides with the affine evolute. For surfaces, we obtain an interesting new set that, under certain conditions, is locally a regular surface with at most 6 branches.

### 5.1. Preliminaries

The following lemma will be useful along this section:

**Lemma 5.1.** *Let $\gamma$ be a planar curve such that $\gamma(0)=(0,0)$. Assume that, close to the origin, $\gamma$ is the graph of*

$$g(x)=\frac{1}{2}x^2+\frac{a_3}{6}x^3+\frac{a_4}{24}x^4+\frac{a_5}{120}x^5+O(6). \tag{20}$$

*Then the affine normal vector at the origin is given by $(-\frac{a_3}{3},1)$, the affine curvature at the origin is*

$$\mu=\frac{1}{9}\left(3a_4-5a_3^2\right) \tag{21}$$

*and the derivative of the affine curvature with respect to affine arc-length is*

$$\mu'=\frac{1}{27}\left(9a_5+40a_3^3-45a_3a_4\right). \tag{22}$$

*Proof.* Differentiate $\gamma(x)=(x,g(x))$ two times with respect to the affine arc-length parameter to obtain $x_s=1$. Differentiate one more to obtain $x_{ss}=-\frac{a_3}{3}$, which implies first claim. Differentiating again we obtain

$$x_{sss}=-\frac{1}{3}a_4+\frac{5}{9}a_3^2,$$

which together with

$$\mu(s)=[\gamma''(s),\gamma'''(s)] \tag{23}$$

implies equation (21). Finally differentiate once more to obtain

$$x_{ssss}=-\frac{1}{3}a_5-\frac{5}{3}a_3^3+\frac{16}{9}a_3a_4.$$

Differentiatiating equation (23) and applying this formula we obtain formula (22). □

Consider a surface $S$ and a point $p_0\in S$. Assume that $p_0=(0,0,0)$ and that the tangent plane at $p_0$ is $z=0$. Assume also that the axes $x$ and $y$ are $h$-orthogonal. Then, close to $p_0$, $S$ is the graph of a function $f$ that can be written as

$$f(x,y)=\frac{1}{2}(x^2+y^2)+f_3(x,y)+f_4(x,y)+O(5), \tag{24}$$

where

$$f_3(x,y)=\sum_{i=0}^{3} f_{3-i,i}x^{3-i}y^i,\quad f_4(x,y)=\sum_{i=0}^{4} f_{4-i,i}x^{4-i}y^i \tag{25}$$

are homogeneous polynomials of degree $k$, $k=3,4$.

We may also assume that the affine normal vector of $S$ at the origin is $(0,0,1)$. In this case, the apolarity condition (equation (2)) implies that $3f_{3,0}+f_{1,2}=0$ and $3f_{0,3}+f_{2,1}=0$ (see [7]). Thus we can write

$$f_3(x,y)=a(x^3-3xy^2)+b(y^3-3yx^2), \tag{26}$$

where $6a=C(U,U,U)$, $6b=C(V,V,V)$, $U=(1,0)$, $V=(0,1)$ and $C$ denotes the cubic form at the origin (see equation (1)). Any other $h$-orthonormal basis, $\{\bar{U},\bar{V}\}$ is related to $\{U,V\}$ by

$$\begin{aligned}\bar{U}&=\cos(\theta)U-\sin(\theta)V\\ \bar{V}&=\sin(\theta)U+\cos(\theta)V,\end{aligned} \tag{27}$$

for some $\theta\in\mathbb{R}$. Then the corresponding cubic form is given by

$$\begin{aligned}a&=\cos(3\theta)\bar{a}-\sin(3\theta)\bar{b}\\ b&=\sin(3\theta)\bar{a}+\cos(3\theta)\bar{b}.\end{aligned} \tag{28}$$

### 5.2. Transon planes, cone of B.Su and Moutard's quadric

We begin with a two-hundred years old result of A.Transon ([9]). For a modern reference, see [5]. For the sake of completeness, we give a proof below.

**Proposition 5.2.** *Consider a regular surface $S\subset\mathbb{R}^3$, $p_0\in S$ and $T\in T_{p_0}S$. Then the affine normal lines of the planar curves obtained as the intersection of $S$ with planes containing $T$ form a plane, which is called the Transon plane of the tangent $T$ at $p_0$.*

*Proof.* We may assume that $T=(1,0)$. Take then a plane of the form $y=\lambda z$, $\lambda\in\mathbb{R}$. Using formulas (24) and (25), the projection of the corresponding section of $S$ on the $xz$ plane is given by

$$z=\frac{1}{2}x^2+f_{3,0}x^3+\left(\frac{\lambda^2}{8}+\frac{\lambda}{2}f_{2,1}+f_{4,0}\right)x^4+O(5). \tag{29}$$

From lemma 5.1, the affine normal direction of this projection is $(-2f_{3,0},1)$. Thus the affine normal of the section is contained in the plane $x+2f_{3,0}z=0$, which is independent of $\lambda$. □

Similar calculations show that when $S$ is the graph of a function given by (24), the Transon plane at the origin in a direction $T=(\xi,\eta)$ is given by $G(\xi,\eta)=0$, where

$$G(\xi,\eta)=\frac{\xi}{2}(\xi^2+\eta^2)x+\frac{\eta}{2}(\xi^2+\eta^2)y+f_3(\xi,\eta)z \tag{30}$$

(see also [5]).

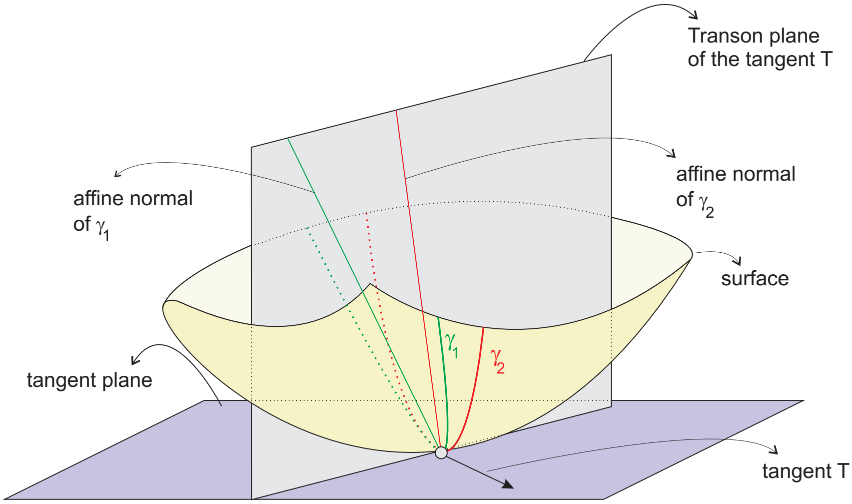


FIGURE 2. The Transon plane of the tangent $T$ is formed by the affine normal lines of the planar sections that contain $T$.

Consider now the family of all Transon planes obtained as the direction of $T$ varies. The envelope of this family is called *cone of B.Su* and is obtained by solving the equations $G = G_\xi = G_\eta = 0$, where

$$\begin{cases} G_\xi = \frac{1}{2}(3\xi^2 + \eta^2)x + (\xi\eta)y + ((f_3)_\xi)\, z \\ G_\eta = (\xi\eta)x + \frac{1}{2}(\xi^2 + 3\eta^2)y + ((f_3)_\eta)\, z. \end{cases} \tag{31}$$

Since $G$ is homogeneous of degree 3, we have that $3G = \eta G_\eta + \xi G_\xi$. We conclude that, if $\xi \neq 0$, the cone of B.Su is obtained form the equations $G = G_\eta = 0$. We shall denote this line by $s(p, T)$. We can thus calculate the direction of $s(p, T)$ as the vector product of the normal vectors of $G_\xi = 0$ and $G_\eta = 0$. In particular, for $(\xi, \eta) = (1, 0)$ we obtain

$$(s_1, s_2, s_3)(1, 0) = (-2f_{3,0}, -2f_{2,1}, 1)\,. \tag{32}$$

We shall consider also the osculating conics of all planar sections obtained from planes containing $T$. The following proposition is an old result of T.Moutard ([6]), see also [5]. We give a proof for the sake of completeness.

**Proposition 5.3.** *The union of the osculating conics of all planar sections containing $T$ form a quadric, which is called the Moutard's quadric of the tangent $T$.*

*Proof.* Assume that $S$ is the graph of $f$ given by equation (24), $p_0 = (0, 0, 0)$ and $T = (1, 0)$. Then the projection of the section of $S$ by the plane $y = \lambda z$ is given by equation (29). By lemma 5.1, the affine curvature is given by

$$\mu = \lambda^2 + 4\lambda f_{2,1} + 8f_{4,0} - 20f_{3,0}^2.$$

The projection of the osculating conic of this section in the plane $xz$ is given by

$$(x + 2f_{3,0}z)^2 + \mu z^2 - 2z = 0.$$

Substituting $\lambda = y/z$ in the above equation, after some calculations we obtain that the osculating conic is contained in

$$z = \frac{1}{2}(x^2 + y^2) + 2f_{2,1}yz + 2f_{3,0}xz + 4(f_{4,0} - 2f_{3,0}^2)z^2, \tag{33}$$

thus proving the lemma. For later reference, we remark that the center of this Moutard quadric is

$$X = \frac{1}{4(2f_{4,0} - 5f_{3,0}^2 - f_{2,1}^2)}\,(-2f_{3,0}, -2f_{2,1}, 1)\,. \tag{34}$$

□

We shall call *medial curve* of the tangent $T$ the intersection of $S$ with the plane generated by $T$ and $N$.

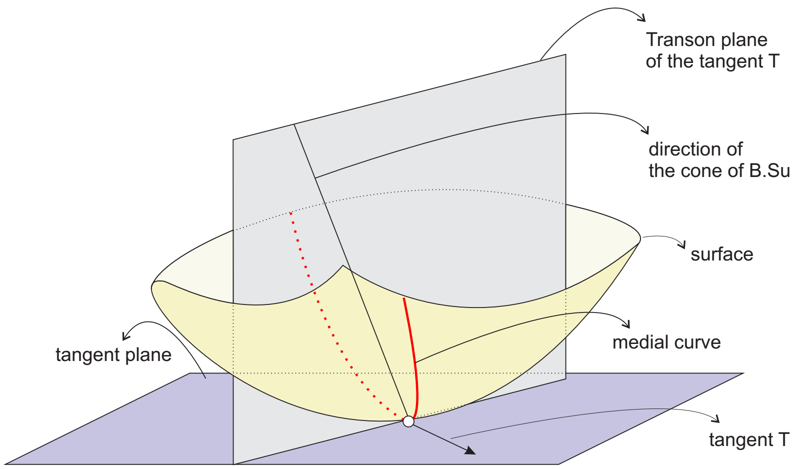


FIGURE 3. The medial curve has affine normal in the direction of the cone of B.Su.

**Proposition 5.4.** *The center of the Moutard's quadric of a tangent $T$ coincides with the center of affine curvature of the medial curve of $T$.*

*Proof.* We may assume that $T = (1, 0)$. The plane generated by $T$ and $N$ has equation $y + 2f_{2,1}z = 0$. Thus, it follows from equation (29) that the projection of the medial curve in the $xz$ plane is given by

$$z = \frac{1}{2}x^2 + f_{3,0}x^3 + \left(f_{4,0} - \frac{f_{2,1}^2}{2}\right)x^4 + O(5).$$

From lemma 5.1, the affine curvature is given $\mu = -4(5f_{3,0}^2 - 2f_{4,0} + f_{2,1}^2)$. Thus the affine center of curvature at the origin is given by

$$\frac{1}{4(2f_{4,0} - 5f_{3,0}^2 - f_{2,1}^2)}\left(-2f_{3,0}, -2f_{2,1}, 1\right),$$

which coincides with formula (34). □

### 5.3. Focal line and affine mid-planes evolute

In the previous section we have discussed the line $s(p,T)$ of the cone of B.Su associated with a point $p$ and a tangent $T \in T_pS$. In fact we can consider the pair $(p,T)$ as a point of the tangent bundle $TS$. Most of the times we choose $T$ satisfying $h(T) = 1$, where $h$ denotes the Blaschke metric. Thus we can write $(p,T) \in T^1S$, where $T^1S$ denotes the unit tangent bundle of $S$.

In this section we shall define another affine invariant line $r(p,T)$ associated to $(p,T) \in T^1S$. Let $\{U(p), V(p)\}$ be an $h$-orthonormal basis of $T_pS$ and $\zeta$ be the affine normal vector of $S$ at $p$. Consider then coordinates $(x,y,z)$ with respect to the basis $\{U(p), V(p), \zeta(p)\}$ of $\mathbb{R}^3$. Then $S$ is the graph of a function $f$ given by equations (24) and (26).

Define the line $r(p,T)$ as the intersection of the planes $H_1 = 0$ and $H_2 = 0$, where

$$H_1 = H_{11}x + H_{12}y + H_{13}z - H_{14}, \quad H_2 = H_{21}x + H_{22}y + H_{23}z - H_{24},$$

are defined by

$$\begin{aligned}
H_{11} &= \frac{1}{2}a\left(5\xi^3 + 3\eta^2\xi\right) - b\left(3\xi^2\eta + 2\eta^3\right), \\
H_{12} &= -3a\eta^3 - \frac{3}{2}b\left(\xi^3 + 3\eta^2\xi\right), \\
H_{13} &= 2f_{4,0}\xi^3 + \frac{3}{2}f_{3,1}\eta\xi^2 + f_{2,2}\eta^2\xi + \frac{1}{2}f_{1,3}\eta^3, \\
H_{14} &= \frac{1}{4}(\xi^2 + \eta^2)\xi,
\end{aligned}$$

$$\begin{aligned}
H_{21} &= -\frac{3}{2}a\left(3\xi^2\eta + \eta^3\right) - 3b\xi^3, \\
H_{22} &= -a\left(2\xi^3 + 3\xi\eta^2\right) + \frac{1}{2}b\left(3\xi^2\eta + 5\eta^3\right), \\
H_{23} &= \frac{1}{2}f_{3,1}\xi^3 + f_{2,2}\xi^2\eta + \frac{3}{2}f_{1,3}\xi\eta^2 + 2f_{0,4}\eta^3, \\
H_{24} &= \frac{1}{4}(\xi^2 + \eta^2)\eta.
\end{aligned}$$

**Lemma 5.5.** *The line $r(p,T)$ is independent of the choice of the orthonormal basis.*

*Proof.* Assume that the basis $\{\bar{U},\bar{V}\}$ and $\{U,V\}$ are related by equation (27) and observe that

$$(\bar{H}_{13},\bar{H}_{23})=(R\cos(\theta_0),R\sin(\theta_0))$$

is the gradient of the homogeneous function $f_4(x,y)$ at $(\bar{\xi},\bar{\eta})$. Then the corresponding gradient in coordinates $(\xi,\eta)$ is given by

$$(H_{13},H_{23})=(R\cos(\theta_0-\theta),R\sin(\theta_0-\theta)).$$

Assume without loss of generality that $(\bar{a},\bar{b})=(1,0)$. Then $(a,b)=(\cos(3\theta),\sin(3\theta))$. Write $(\bar{\xi},\bar{\eta})=(\cos(\alpha),\sin(\alpha))$. Then $(\xi,\eta)=(\cos(\alpha-\theta),\sin(\alpha-\theta))$. Then long but straightforward calculations show that

$$\begin{aligned} H_{11}&=\tfrac{1}{4}\left(9\cos(2\theta+\alpha)+\cos(3\alpha)\right), & H_{12}&=\tfrac{1}{4}\left(-9\sin(2\theta+\alpha)+3\sin(3\alpha)\right),\\ H_{21}&=\tfrac{1}{4}\left(-9\sin(2\theta+\alpha)-3\sin(3\alpha)\right), & H_{22}&=\tfrac{1}{4}\left(-9\cos(2\theta+\alpha)+\cos(3\alpha)\right).\end{aligned}$$

Denoting $\Delta=\det(H_{ij})$ we obtain that

$$\Delta=\frac{1}{16}\left(-81+\cos^2(3\alpha)+9\sin^2(3\alpha)\right)$$

is independent of $\theta$. We can obtain the direction of $r(p,T)$ as the vector product of $(H_{11},H_{12},H_{13})$ with $(H_{21},H_{22},H_{23})$. Thus this direction is $(r_1,r_2,\Delta)$, where

$$\begin{bmatrix} r_1\\ r_2\end{bmatrix}=\frac{R}{4}\begin{bmatrix}\cos(\theta_0-\theta) & -\sin(\theta_0-\theta)\\ \sin(\theta_0-\theta) & \cos(\theta_0-\theta)\end{bmatrix}.\begin{bmatrix}-\cos(3\alpha)+9\cos(2\theta_0+\alpha)\\ -3\sin(3\alpha)-9\sin(2\theta_0+\alpha)\end{bmatrix}.$$

The intersection of $r(p,T)$ with the $xy$ plane is obtained as the solution of the system

$$\begin{aligned} H_{11}x+H_{12}y&=H_{14}\\ H_{21}x+H_{22}y&=H_{24}.\end{aligned}$$

Thus this intersection is given by

$$\frac{1}{4\Delta}\begin{bmatrix}\cos(\alpha-\theta) & -\sin(\alpha-\theta)\\ \sin(\alpha-\theta) & \cos(\alpha-\theta)\end{bmatrix}.\begin{bmatrix}-2\cos(3\alpha)\\ 3\sin(3\alpha)\end{bmatrix}.$$

We conclude that the equations of $r(p,T)$ in the system $\{U,V\}$ is obtained from the equations of $r(p,T)$ in the system $\{\bar{U},\bar{V}\}$ by a rotation of angle $-\theta$ around the $z$-axis, thus proving the lemma. □

From the above lemma, we conclude that the line $r(p,T)$ is an affine invariant line associated with $(p,T)\in T^1S$. In next section we give some geometric meaning to this line. We shall call this line the *focal line* and next lemma justifies this name.

**Lemma 5.6.** *Assume that $r(p,T)$ and $s(p,T)$ intersect. Then the intersection point is the center of the Moutard's quadric of $T=(\xi,\eta)$.*

*Proof.* We may assume that the tangent $(\xi, \eta)$ is $(1, 0)$. Taking $\xi = 1, \eta = 0$ in the system $G = G_\eta = H_1 = 0$ we get

$$\begin{cases} \frac{1}{2}x + az &= 0 \\ \frac{1}{2}y - 3bz &= 0 \\ \frac{5}{2}ax - \frac{3}{2}by + 2f_{4,0}z &= \frac{1}{4} \end{cases}.$$

The solution of this system is

$$(x, y, z) = \frac{1}{4(2f_{4,0} - 5a^2 - 9b^2)}\,(-2a, 6b, 1)\,,$$

which is exactly the center of the Moutard's quadric of $T = (1, 0)$ at the origin (see formula (34)). ☐

We are interested in finding pairs $(p, T)$ for which the lines $s(p, T)$ and $r(p, T)$ intersect. This is equivalent to solve the system

$$G = G_\xi = G_\eta = H_1 = H_2 = 0 \tag{35}$$

at a point $(p, T) \in T^1S$. Since $G = G_\xi = G_\eta = 0$ are linearly dependent, we shall discard the equation $G = 0$. Denote by $D : TS \longrightarrow \mathbb{R}$ the determinant of the extended matrix of

$$G_\xi = G_\eta = H_1 = H_2 = 0.$$

Then the system (35) admits a solution if and only if $D(p, T) = 0$.

**Lemma 5.7.** *For each $p \in S$, the system (35) admit solutions for at most 6 values of $T$.*

*Proof.* We may assume $p = (0, 0)$ and let $T = (\xi, \eta)$. Straightforward calculations show that

$$D(p, T) = \frac{3}{32}(\xi^2 + \eta^2)^2 \cdot q(\xi, \eta),$$

where $q(\xi, \eta) = 12q_3(\xi, \eta) + q_4(\xi, \eta)$ is a homogeneous polynomial of degree 6, where

$$q_3 = ab\xi^6 + 3(a^2 - b^2)\xi^5\eta - 15ab\xi^4\eta^2 + 10(b^2 - a^2)\xi^3\eta^3 + 15ab\xi^2\eta^4 + 3(a^2 - b^2)\xi\eta^5 - ab\eta^6$$

and

$$q_4 = -f_{3,1}\xi^6 + (4f_{4,0} - 2f_{2,2})\xi^5\eta + (2f_{3,1} - 3f_{1,3})\xi^4\eta^2 + 4(f_{4,0} - f_{0,4})\xi^3\eta^3 \\ + (3f_{3,1} - 2f_{1,3})\xi^2\eta^4 + (2f_{2,2} - 4f_{0,4})\xi\eta^5 + f_{1,3}\eta^6.$$

Thus $D = 0$ admit at most 6 solutions. ☐

**Corollary 5.8.** *For each $p \in S$, the system (35) admits a solution for at most 6 values of the direction $T$.*

**Example 5.9.** *Consider a point $p \in S$ with $f_4 = 0$. We may assume, by a rotation of the tangent plane, that $b = 0$. Then*

$$q_3 = 3a^2\xi^5\eta - 10a^2\xi^3\eta^3 + 3a^2\xi\eta^5.$$

*This polynomial has exactly six roots, $(\xi, \eta) = (\cos\frac{k\pi}{6}, \sin\frac{k\pi}{6})$, $0 \le k \le 5$.*

For each $p \in S$ and $T_i(p)$, $1 \leq i \leq 6$, given by the above corollary, define $X_i(p)$ as the solution of the system (35) and write $E(p) = \cup_{i=1}^{6} X_i(p)$. Up to our knowledge, the set $E(p)$ has not yet been considered. We call the set $E(p)$, $p \in S$, the *Affine Mid-Planes Evolute* of $S$.

### 5.4. Affine mid-planes evolute as limit of the EMP

Let $S$ be a surface and consider $F : S \times S \times \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(p_1, p_2, X) = N_1(C)N_2(X - M) + N_2(C)N_1(X - M), \tag{36}$$

where $M$ is the mid-point of $(p_1, p_2)$, $C$ is the mid-chord of $(p_1, p_2)$ and $N_i$ is any functional that is zero in $T_{p_i}S$, $i = 1, 2$. Then lemma 3.1 says that the equation of the mid-plane of $(p_1, p_2)$ is $F = 0$.

Along this section, we shall assume that $S$ is the graph of a function $f$ defined by equations (24) and (25), and take

$$N_i = (1, 0, f_x) \times (1, 0, f_y)(p_i)$$

as normal vectors, where $p_i = (u_i, v_i, f(u_i, v_i))$, $i = 1, 2$. Write $\Delta u = u_1 - u_2$, $\Delta v = v_1 - v_2$. Next lemma is the main tool of this section:

**Lemma 5.10.** *We have that*

$$F(u_1, v_1, u_2, v_2, X) = G(\Delta u, \Delta v)+H_1(\Delta u, \Delta v)(u_1+u_2)+H_2(\Delta u, \Delta v)(v_1+v_2)+O(5), \tag{37}$$

*where $O(5)$ denotes fifth order terms in $(u_1, u_2, v_1, v_2)$.*

*Proof.* Long but straightforward calculations. □

This lemma has some important consequences, one of them is the following proposition:

**Proposition 5.11.** *Take $p_1 = \psi(u_1, v_1)$ and $p_2 = \psi(u_2, v_2)$ such that $(\Delta u, \Delta v) = t(\xi, \eta)$. Then the limit of the mid-planes of $(p_1, p_2)$ when $t$ goes to $0$ is the Transon plane of $(\xi, \eta)$ at $p_0$.*

*Proof.* It follows from lemma 5.10 that $t^3 G(\xi, \eta) = O(t^4)$. When $t \to 0$, this equation is converging to $G(\xi, \eta) = 0$, which is the equation of the Transon plane. □

We have to consider the system

$$\begin{cases} F(u_1, v_1, u_2, v_2, X) = 0 \\ F_{u_1}(u_1, v_1, u_2, v_2, X) = 0 \\ F_{v_1}(u_1, v_1, u_2, v_2, X) = 0 \\ F_{u_2}(u_1, v_1, u_2, v_2, X) = 0 \\ F_{v_2}(u_1, v_1, u_2, v_2, X) = 0 \end{cases},$$

which is equivalent to

$$\begin{cases} F(u_1, v_1, u_2, v_2, X) = 0 \\ F_{v_1}(u_1, v_1, u_2, v_2, X) - F_{v_2}(u_1, v_1, u_2, v_2, X) = 0 \\ F_{u_1}(u_1, v_1, u_2, v_2, X) - F_{u_2}(u_1, v_1, u_2, v_2, X) = 0 \\ F_{u_1}(u_1, v_1, u_2, v_2, X) + F_{u_2}(u_1, v_1, u_2, v_2, X) = 0 \\ F_{v_1}(u_1, v_1, u_2, v_2, X) + F_{v_2}(u_1, v_1, u_2, v_2, X) = 0 \end{cases}. \tag{38}$$

Proposition 5.11 says that the first equation of the system (38) is converging to $G = 0$, the equation of the Transon plane. The second equation of system (38) can be written as $G_\xi(\Delta u, \Delta v) = O(4)$. Taking pairs $(p_1, p_2)$ such that $(\Delta u, \Delta v) = t(\xi, \eta)$, the same argument of proposition 5.11 shows that the second equation of the system is converging to $G_\xi(\xi, \eta) = 0$. The same reasoning applied to the third equation of system (38) leads to the equation $G_\eta(\xi, \eta) = 0$. It follows from section 5.2 that $G = G_\xi = G_\eta = 0$ define the line $s(p, T)$ of the cone of B.Su associated to $T = (\xi, \eta)$.

Now let us consider the fourth and fifth equations of system (38). As above, take pairs $(p_1, p_2)$ such that $(\Delta u, \Delta v) = t(\xi, \eta)$. Then lemma 5.10 implies that the limit of the fourth equation and fifth equations when $t$ goes to zero are $H_1(\xi, \eta, X) = 0$ and $H_2(\xi, \eta, X) = 0$, respectively, where $H_1$ and $H_2$ are defined in section 5.3. As a conclusion, we have the following proposition:

**Proposition 5.12.** *The limit of the system* (38) *defining the envelope of mid-planes is the system* (35) *that defines the affine mid-planes evolute.*

### 5.5. Regularity of the Affine Mid-Planes Evolute

In this section we study the regularity of the branches of the Affine Mid-Planes Evolute.

We begin by showing that, under certain conditions, the vector fields $T_i(p)$, $1 \le i \le 6$, and the corresponding map $X_i(p)$ are smooth.

**Lemma 5.13.** *Let $p_0 \in S$ and take $(\xi_0, \eta_0) \in T_{p_0}S$ with $q(p_0)(\xi_0, \eta_0) = 0$. Assume that $(\xi_0, \eta_0)$ is a simple root of $q(p_0)$. Then there exist a neighborhood $U$ of $p_0$ and a map $(Id, (\xi, \eta)) : U \to T^1 S$ such that $(\xi, \eta)(p_0) = (\xi_0, \eta_0)$ and $(\xi, \eta)(p)$ is a simple root of $q(p)$. Denoting by $X(p)$ the center of the Moutard quadric of $(\xi, \eta)(p)$, the map $X : U \to \mathbb{R}^3$ is differentiable.*

*Proof.* The first claim follows from the implicit function theorem and the second one from the formula of the center of the Moutard quadric. □

From now on, we shall assume that, for a given branch of the Affine Mid-Planes Evolute, $T = T(p)$ and $X = X(p)$ are smooth functions of $p$. We now look for conditions under which this branch of the Affine Mid-Planes Evolute is a regular surface.

Assume that the frame $\{U(p), V(p)\}$ is $h$-orthonormal and that $U(p)$ vanishes the cubic form at each $p$. Then $S \times S$ is locally parameterized by

$$(p, x, y) \to (p, (xU(p) + yV(p) + z(p, x, y)\zeta(p))),$$

where $\zeta(p)$ is the affine normal vector at $p$. We may write

$$z(p, x, y) = \frac{1}{2}(x^2 + y^2) + b(p)(y^3 - 3yx^2) + O(4)(p)(x, y),$$

where $b^2(p)$ is the Pick invariant of $S$ at $p$. Consider $\mathbf{G} : TS \to \mathbb{R}^3$ given by

$$\mathbf{G}(p, \xi, \eta) = \left( \frac{\xi}{2}(\xi^2 + \eta^2), \frac{\eta}{2}(\xi^2 + \eta^2), b(p)(\eta^3 - 3\eta\xi^2) \right).$$

Then, for fixed $(p,\xi,\eta)$, denoting $X=(x,y,z)$, $\mathbf{G}(p,\xi,\eta)\cdot X=0$ is the equation of the Transon plane at $p$ in the direction $(\xi,\eta)$.

Consider a curve $p=p(t)$, $p(0)=p_0$, along the surface $S$ and denote by $T(t)=(\xi(t),\eta(t))$ and $X(t)$ the corresponding values of $T$ and $X$ along the branch.

**Lemma 5.14.** *Assume that $b'(0)\neq 0$. Then $X'(0)$ belong to the Transon plane if and only if $(\xi,\eta)(0)$ vanishes the cubic form at $p$. Moreover, if $(\xi,\eta)(0)$ vanishes the cubic form at $p$, then $X'(0)$ is not in the cone of B.Su.*

*Proof.* Write $\mathbf{G}(t)=\mathbf{G}(p(t),\xi(t),\eta(t))$. Differentiating $\mathbf{G}(t)\cdot X=0$ at $t=0$ we obtain

$$\mathbf{G}_\xi\cdot X\ \xi'+\mathbf{G}_\eta\cdot X\ \eta'+\mathbf{G}\cdot X'+b'(\eta^3-3\eta\xi^2)z=0.$$

For $X=(x,y,z)$ center of Moutard's quadric, $\mathbf{G}_\xi\cdot X=\mathbf{G}_\eta\cdot X=0$. Thus $\mathbf{G}\cdot X'=0$ if and only if $\eta^3-3\eta\xi^2=0$. We conclude that $X'$ belongs to the Transon plane if and only if $\eta^3-3\eta\xi^2=0$, thus proving the first claim of the lemma.

For the second claim, we may assume that $(\xi(0),\eta(0))=(1,0)$. Thus, at $t=0$, $X$ is given by equation (34) with $f_{3,0}=0$, $f_{2,1}=-3b(0)$, i.e.,

$$(x,y,z)(0)=-\frac{1}{4(9b^2-2f_{4,0})}\,(0,6b(0),1)\,.$$

Observe that

$$\mathbf{G}_\eta(p,\xi,\eta)=\left(\xi\eta,\frac{1}{2}(\xi^2+3\eta^2),3b(p)(\eta^2-\xi^2)\right).$$

Differentiating $\mathbf{G}_\eta(t)\cdot X(t)=0$ we obtain

$$\mathbf{G}_{\eta\xi}\cdot X\ \xi'+\mathbf{G}_{\eta\eta}\cdot X\ \eta'+\mathbf{G}_\eta\cdot X'+3b'(\eta^2-\xi^2)z=0.$$

Now take $t=0$ and use $\xi(0)=1$, $\eta(0)=0$ to obtain

$$(0,1,-6b(0))\cdot X\ \xi'+(1,0,0)\cdot X\ \eta'+\mathbf{G}_\eta\cdot(X')-3b'(0)z=0.$$

Using the above formula for $(x,y,z)$ we get

$$\mathbf{G}_\eta\cdot X'+\frac{3b'(0)}{4(9b^2-2f_{4,0})}=0.$$

We conclude that $G_\eta(X')\neq 0$, which implies that $X'$ is not on the cone of B.Su. □

**Lemma 5.15.** *Assume $p'(0)=T$ and denote by $\gamma$ the medial curve. If $\mu_\gamma'(0)\neq 0$, where $\mu_\gamma$ denotes the affine curvature of $\gamma$, $X'(0)$ is a non-zero vector in the direction of the cone of B.Su.*

*Proof.* Since $X$ is a point of the affine evolute of the medial curve and $T$ is tangent to this curve, $X'(0)$ is in the direction of the its affine normal, which belongs to the cone of B.Su. □

If we assume that $S$ is given by (24), $p=(0,0)$ and $T(p)=(1,0)$, the condition $\mu'_\gamma(0)\neq 0$ can be explicitly described. Expanding $f$ until order 5, it is straightforward to verify that the projection of $\gamma$ in the $xz$ plane is given by

$$z=\frac{1}{2}x^2+ax^3+\left(f_{4,0}-\frac{9}{2}b^2\right)x^4+\left(-27ab^2+3bf_{3,1}+f_{5,0}\right)x^5+O(6).$$

From lemma 5.1, it follows that

$$\mu'_\gamma(0)=\frac{1}{27}\left(9a_5+40a_3^3-45a_3a_4\right),$$

where $a_3=6a$, $a_4=24\left(f_{4,0}-\frac{9}{2}b^2\right)$ and $a_5=120\left(-27ab^2+3bf_{3,1}+f_{5,0}\right)$.

**Proposition 5.16.** *Assume that $p$ is not critical for the Pick invariant and that $\mu'_\gamma(0)\neq 0$. Then the corresponding branch of the affine mid-planes evolute is smooth at $p$.*

*Proof.* From lemma 5.15, $X_T$ is a non-zero vector in the direction of the cone of B.Su. Take a direction $W\neq T$ such that the derivative of $b$ in this direction in non-zero. From lemma 5.14, $X_W$ is not a multiple of $X_T$, thus proving the proposition. □

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